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Further Reinterpretation of Multi-Stage Implicit FDTD Schemes

Ding Yu Heh and Eng Leong Tan

Abstract—This communication presents further reinterpretation of multi-stage implicit finite-difference time-domain (FDTD) schemes. The equivalence of several multi-stage split-step (SS) schemes is shown based on amplification matrices and power property of eigenvalues in dispersion relations. In doing so, the explicit expressions for dispersion relations need not be derived, thus amenable to 3D and generalized multi-stage SS schemes conveniently. The improvement of temporal accuracy for generalized multi-stage SS schemes using generalized input and output processing matrices is also provided. It is further shown that the SS, alternating-direction-implicit (ADI) as well as the recently proposed divergence-preserving (DP) ADI schemes can be interpreted based on the more concise matrix exponential.

Index Terms—finite-difference time-domain (FDTD), multi-stage implicit schemes, amplification matrices, dispersion relations, matrix exponential

I. INTRODUCTION

In recent years, various multi-stage split-step (SS) finite-difference time-domain (FDTD) schemes [1]–[6] have been proposed which may feature better performances such as higher accuracy, lower anisotropy, etc compared to the conventional two-stage SS [or locally-one-dimensional (LOD)] scheme [7]. In [8], it was shown that certain four-stage SS schemes [3], [5], [6] can be reinterpreted as two-stage ones [7], [9] with different time step scaling. The explicit expressions for dispersion relations and truncation error matrices were provided for these schemes in 2D to show the equivalence. Nevertheless, to derive these explicit expressions for dispersion relations and truncation error matrices may be rather cumbersome, and the analysis is often limited to small number of stages and 2D only. Furthermore, for generalized multi-stage SS scheme with arbitrary number of stages and dimensions, the derived explicit expressions are not general enough to be directly applicable.

In this communication, we present further reinterpretation of multi-stage implicit FDTD schemes in [8]. In particular, the equivalence of several multi-stage SS schemes will be shown based on amplification matrices and power property of eigenvalues in dispersion relations. In doing so, the explicit expressions for dispersion relations need not be derived, thus amenable to 3D and generalized multi-stage SS schemes conveniently. The improvement of temporal accuracy for generalized multi-stage SS schemes using generalized input and output processing matrices will also be provided. In addition to SS scheme, other implicit schemes such as the alternating-direction-implicit (ADI) scheme [10], [11] and the recently proposed divergence-preserving (DP) ADI scheme [12] are also considered. All this while, the ADI scheme is often derived and interpreted as Crank-Nicolson's perturbation [13], [14]. In this communication, we further show that the SS

and ADI schemes, as well as the DP-ADI scheme can be interpreted based on the more concise matrix exponential, which represents the exact solution to the partial differential Maxwell's equations.

II. FURTHER REINTERPRETATION OF MULTI-STAGE IMPLICIT SCHEMES

In this section, several multi-stage SS schemes will be first reviewed and categorized based on the matrix operators sequence in their amplification matrices. Subsequently, the equivalence of schemes based on amplification matrices and power property of eigenvalues in dispersion relations shall be shown and discussed.

A. Schemes

1) *AB*: The conventional SS or LOD scheme [7] calls for the following update procedures:

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)\mathbf{u}^n \quad (1a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B}\right)\mathbf{u}^{n+\frac{1}{2}} \quad (1b)$$

where

$$\mathbf{u} = [E_x \ E_y \ E_z \ H_x \ H_y \ H_z]$$

and $\mathbf{A}$ and $\mathbf{B}$ are the standard curl splitting operators that can be found in [15]. Since the first procedure involves only matrix operator $\mathbf{A}$, followed by $\mathbf{B}$ in the second procedure, it is categorized here as scheme *AB*.

The amplification matrix Λ of the scheme can be defined as

$$\Lambda = \mathbf{M}_B \left(\frac{\Delta t}{2}\right) \mathbf{M}_A \left(\frac{\Delta t}{2}\right) \quad (2)$$

where

$$\mathbf{M}_A \left(\frac{\Delta t}{2}\right) = \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)$$

$$\mathbf{M}_B \left(\frac{\Delta t}{2}\right) = \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B}\right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B}\right).$$

The amplification matrix shall be used to prove the equivalence of schemes in the subsequent subsection.

In [3], [5], a four-stage SS scheme was proposed with update procedures given by

$$\left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{4}} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^n \quad (3a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n+\frac{1}{4}} \quad (3b)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^{n+\frac{3}{4}} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \quad (3c)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n+\frac{3}{4}}. \quad (3d)$$

The amplification matrix Λ of the scheme is found to be

$$\Lambda = \mathbf{M}_B \left(\frac{\Delta t}{4}\right) \mathbf{M}_A \left(\frac{\Delta t}{4}\right) \mathbf{M}_B \left(\frac{\Delta t}{4}\right) \mathbf{M}_A \left(\frac{\Delta t}{4}\right) \quad (4)$$

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where the arguments of $\mathbf{M}_A$ and $\mathbf{M}_B$ are now $\Delta t/4$. Moreover, Λ can be factorized to yield

$$\begin{aligned} & \mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_A \left(\frac{\Delta t}{4} \right) \mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_A \left(\frac{\Delta t}{4} \right) \\ &= \left[\mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_A \left(\frac{\Delta t}{4} \right) \right]^2. \end{aligned} \quad (5)$$

In fact, for generalized multi-stage SS scheme with arbitrary number of stages and dimensions, the amplification matrix can be factorized and generalized as

$$\Lambda = \left[\mathbf{M}_B \left(\frac{\Delta t}{2m} \right) \mathbf{M}_A \left(\frac{\Delta t}{2m} \right) \right]^m \quad (6)$$

where m is an integer. The generalization is applicable to multi-stage SS schemes having the same time step Δt in each procedure, such as the six-stage SS scheme denoted by SS6-1 in [4]. These schemes are categorized here as *AB* due to similar matrix operators sequence in the (factorized) amplification matrices (2), (5) and (6). Note however that the generalization is not applicable for higher-order multi-stage SS schemes having potentially different time step in certain procedures [1], [16].

2) *ABBA*: Another multi-stage SS scheme, originally proposed by [9], calls for the following update procedures:

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right) \mathbf{u}^{n-\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) \mathbf{u}^{n-1} \quad (7a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{u}^n = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{u}^{n-\frac{1}{2}} \quad (7b)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{u}^n \quad (7c)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right) \mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) \mathbf{u}^{n+\frac{1}{2}} \quad (7d)$$

Here, it is categorized as scheme *ABBA* based on the sequence of the matrix operator used in each procedure with amplification matrix Λ

$$\Lambda = \mathbf{M}_A \left(\frac{\Delta t}{2} \right) \mathbf{M}_B \left(\frac{\Delta t}{2} \right) \mathbf{M}_B \left(\frac{\Delta t}{2} \right) \mathbf{M}_A \left(\frac{\Delta t}{2} \right). \quad (8)$$

It should be pointed out that the above scheme is on double time interval because the fields are marched from time index $n-1$ to $n+1$.

The authors in [6] further proposed another scheme, also categorized as *ABBA* but based on single time interval with the following update procedures:

$$\left(\mathbf{I} - \frac{\Delta t}{4} \mathbf{A} \right) \mathbf{u}^{n+\frac{1}{4}} = \left(\mathbf{I} + \frac{\Delta t}{4} \mathbf{A} \right) \mathbf{u}^n \quad (9a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4} \mathbf{B} \right) \mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{4} \mathbf{B} \right) \mathbf{u}^{n+\frac{1}{4}} \quad (9b)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4} \mathbf{B} \right) \mathbf{u}^{n+\frac{3}{4}} = \left(\mathbf{I} + \frac{\Delta t}{4} \mathbf{B} \right) \mathbf{u}^{n+\frac{1}{2}} \quad (9c)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4} \mathbf{A} \right) \mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{4} \mathbf{A} \right) \mathbf{u}^{n+\frac{3}{4}} \quad (9d)$$

and amplification matrix Λ

$$\Lambda = \mathbf{M}_A \left(\frac{\Delta t}{4} \right) \mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_A \left(\frac{\Delta t}{4} \right). \quad (10)$$

B. Equivalence based on amplification matrices

Although by visual inspection, one may still observe some similarities between update procedures (1) and (3), or between (7) and (9), a more systematic and concise proof of equivalence shall be provided below based on amplification matrices.

1) *AB*: We start from generalized multi-stage SS scheme in (6) and define the overall amplification matrix, Λ^n , which is used to obtain the field solution $\mathbf{u}^n$ via

$$\begin{aligned} \mathbf{u}^n &= \Lambda^n \cdot \mathbf{u}^0 \\ &= \left[\mathbf{M}_B \left(\frac{\Delta t}{2m} \right) \mathbf{M}_A \left(\frac{\Delta t}{2m} \right) \right]^{m \times n} \cdot \mathbf{u}^0 \end{aligned} \quad (11)$$

where $\mathbf{u}^0$ is the initial field and n is the number of iterations.

If we let $\Delta t' = \Delta t/m$ and define $n = T/\Delta t$, where T is the physical time, the overall amplification matrix in (11) can be rewritten as

$$\begin{aligned} & \left[\mathbf{M}_B \left(\frac{\Delta t'}{2} \right) \mathbf{M}_A \left(\frac{\Delta t'}{2} \right) \right]^{\frac{\Delta t}{\Delta t'} \times \frac{T}{\Delta t'}} \\ &= \left[\mathbf{M}_B \left(\frac{\Delta t'}{2} \right) \mathbf{M}_A \left(\frac{\Delta t'}{2} \right) \right]^{\frac{T}{\Delta t'}}. \end{aligned} \quad (12)$$

It is evident that (12) represents the overall amplification matrix of conventional SS scheme in (2) but with arbitrary time step scaling of $\Delta t'$. Different $\Delta t'$ will result in different number of iterations with the end physical time T remains unchanged. Hence the equivalence between schemes (2) and (6) is proven readily. In fact, as pointed out by [8], the improved accuracy of multi-stage SS schemes in [3], [5] and SS6-1 of [4] is just a consequence of using a reduced time step. The temporal accuracy of the overall amplification matrix for these multi-stage SS schemes is still the same as the conventional SS scheme which is of first-order only.

To improve the temporal accuracy to second-order, one may apply input and output processings to the overall amplification matrix in (11) as

$$\mathbf{M}_B \left(\frac{-\Delta t}{4m} \right) \cdot \left[\mathbf{M}_B \left(\frac{\Delta t}{2m} \right) \mathbf{M}_A \left(\frac{\Delta t}{2m} \right) \right]^{m \times n} \cdot \mathbf{M}_B \left(\frac{\Delta t}{4m} \right) \quad (13)$$

where $\mathbf{M}_B \left(\frac{\Delta t}{4m} \right)$ and $\mathbf{M}_B \left(\frac{-\Delta t}{4m} \right)$ are the generalized input and output processing matrices. Interestingly, the negative argument of output processing matrix indicates a time reversal stepping, which is necessary to achieve second-order temporal accuracy. It should be noted that the input and output processing matrices need only be invoked once at the initial and final steps of the scheme respectively and are independent of the number of iterations n . Therefore, stability will not be affected and the scheme still remains unconditionally stable. Equation (13) now represents the overall amplification matrix for generalized multi-stage SS scheme with second-order temporal accuracy. For $m = 1$, (13) reduces to the second-order LOD2 scheme in [2].

2) *ABBA*: From scheme *ABBA* in (10), the overall amplification matrix is

$$\left[\mathbf{M}_A \left(\frac{\Delta t}{4} \right) \mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_B \left(\frac{\Delta t}{4} \right) \mathbf{M}_A \left(\frac{\Delta t}{4} \right) \right]^{\frac{T}{\Delta t}}. \quad (14)$$

Letting $\Delta t' = \Delta t/2$, we can see that the overall amplification in (14) is now

$$\left[\mathbf{M}_A \left(\frac{\Delta t'}{2} \right) \mathbf{M}_B \left(\frac{\Delta t'}{2} \right) \mathbf{M}_B \left(\frac{\Delta t'}{2} \right) \mathbf{M}_A \left(\frac{\Delta t'}{2} \right) \right]^{\frac{T}{2\Delta t'}} \quad (15)$$

which corresponds to the overall amplification matrix in (8) with different time step scaling $\Delta t'$, thus proving their equivalence. Note that the physical time in (15) is halved to $T/2$ due to the double time interval used in the scheme. The temporal accuracy of scheme *ABBA* is of second-order [9].

C. Power property of eigenvalues in dispersion relations

Next, we shall prove the equivalence based on power property of eigenvalues in dispersion relations. The power property of eigenvalues states that if the eigenvalues of amplification matrix, $\mathbf{A}$ with dimensions $i \times i$ are $\lambda_1, \lambda_2 \dots \lambda_i$, the eigenvalues for $\mathbf{A}^m$ are $\lambda_1^m, \lambda_2^m \dots \lambda_i^m$ for positive integer m .

1) *AB*: To begin, we first write down the general dispersion relation of the conventional SS scheme in (2) as

$$e^{j\omega\Delta t} = \lambda_i \left(\frac{\Delta t}{2} \right) \quad \forall i \quad (16)$$

where $\lambda_i \left(\frac{\Delta t}{2} \right)$ are the eigenvalues for amplification matrix $\mathbf{A}$ in (2) with argument $\Delta t/2$ and ω is the angular frequency.

If we were to scale the time step Δt in (16) to $\Delta t/m$, the dispersion relation now reads

$$e^{j\omega \frac{\Delta t}{m}} = \lambda_i \left(\frac{\Delta t}{2m} \right) \quad \forall i. \quad (17)$$

Raising both sides of (17) to power m , we finally arrive at

$$\begin{aligned} \left(e^{j\omega \frac{\Delta t}{m}} \right)^m &= \lambda_i \left(\frac{\Delta t}{2m} \right)^m \quad \forall i \\ e^{j\omega\Delta t} &= \lambda_i \left(\frac{\Delta t}{2m} \right)^m \quad \forall i. \end{aligned} \quad (18)$$

Based on the power property of the eigenvalues above, (18) is indeed the dispersion relation of the generalized multi-stage SS scheme in (6), which further proves that the two schemes are equivalent. It should be emphasized that the dispersion relation expressions of the two schemes need not be derived explicitly as in [8], thus (18) is amenable to 3D and generalized multi-stage SS schemes conveniently.

2) *ABBA*: For scheme *ABBA*, the dispersion relation of (8) is

$$e^{j\omega 2\Delta t} = \lambda_i \left(\frac{\Delta t}{2} \right) \quad \forall i \quad (19)$$

where $2\Delta t$ on the left-hand side is due to the double time interval.

If Δt is now scaled to $\Delta t/2$, one arrives at

$$e^{j\omega\Delta t} = \lambda_i \left(\frac{\Delta t}{4} \right) \quad \forall i, \quad (20)$$

which is exactly the dispersion relation of (10), thus proving both schemes to be equivalent.

III. INTERPRETATION BASED ON MATRIX EXPONENTIAL

All this while, the ADI scheme is often derived and interpreted as Crank-Nicolson's perturbation [13], [14]. In this section, we further show that the SS and ADI schemes, as well as the DP-ADI scheme can be interpreted based on the more concise matrix exponential, which represents the exact solution to the partial differential Maxwell's equations.

A. SS

The interpretation of SS scheme begins from the matrix exponential solution $e^{(\mathbf{A}+\mathbf{B})\Delta t}$, which is the exact solution to the partial differential Maxwell's equations $\frac{\partial \mathbf{u}}{\partial t} = (\mathbf{A} + \mathbf{B}) \mathbf{u}$. It is first approximated by

$$e^{(\mathbf{A}+\mathbf{B})\Delta t} = e^{\mathbf{B}\Delta t} e^{\mathbf{A}\Delta t} + O(\Delta t^2). \quad (21)$$

Applying the following Padé approximations to the matrix exponential terms on the right-hand side

$$e^{\mathbf{B}\Delta t} = \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right) + O(\Delta t^3) \quad (22)$$

$$e^{\mathbf{A}\Delta t} = \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) + O(\Delta t^3), \quad (23)$$

the SS scheme can be derived as

$$\begin{aligned} e^{(\mathbf{A}+\mathbf{B})\Delta t} &= e^{\mathbf{B}\Delta t} e^{\mathbf{A}\Delta t} + O(\Delta t^2) \\ &= \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right) \\ &\quad \times \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) + O(\Delta t^2). \end{aligned} \quad (24)$$

It can be seen that the overall approximation of SS scheme in (24) has first-order temporal accuracy [$O(\Delta t^2)$ in error].

B. ADI

The interpretation of ADI scheme [10], [11] also begins from the same matrix exponential solution $e^{(\mathbf{A}+\mathbf{B})\Delta t}$, but is now approximated by

$$e^{(\mathbf{A}+\mathbf{B})\Delta t} = e^{\frac{1}{2}\mathbf{B}\Delta t} e^{\mathbf{A}\Delta t} e^{\frac{1}{2}\mathbf{B}\Delta t} + O(\Delta t^3). \quad (25)$$

Note that the above approximation is also known as Strang splitting which is second-order accurate in time.

To derive ADI scheme, the first $e^{\frac{1}{2}\mathbf{B}\Delta t}$ is approximated using

$$e^{\frac{1}{2}\mathbf{B}\Delta t} = \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right)^{-1} - \frac{\Delta t^2}{8} \mathbf{B}^2 + O(\Delta t^3). \quad (26)$$

This is followed by $e^{\mathbf{A}\Delta t}$, which is approximated using the commutative property of (23) as

$$e^{\mathbf{A}\Delta t} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right)^{-1} + O(\Delta t^3) \quad (27)$$

and first-order Taylor approximation of second $e^{\frac{1}{2}\mathbf{B}\Delta t}$ using

$$e^{\frac{1}{2}\mathbf{B}\Delta t} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right) + \frac{\Delta t^2}{8} \mathbf{B}^2 + O(\Delta t^3). \quad (28)$$

Substituting (26)-(28) into (25), one first obtains

$$\begin{aligned}
& e^{(\mathbf{A}+\mathbf{B})\Delta t} \\
&= e^{\frac{1}{2}\mathbf{B}\Delta t} e^{\mathbf{A}\Delta t} e^{\frac{1}{2}\mathbf{B}\Delta t} + O(\Delta t^3) \\
&= \left[\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B} \right)^{-1} - \frac{\Delta t^2}{8}\mathbf{B}^2 + O(\Delta t^3) \right] \\
&\quad \times \left[\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \right) \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} \right)^{-1} + O(\Delta t^3) \right] \\
&\quad \times \left[\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B} \right) + \frac{\Delta t^2}{8}\mathbf{B}^2 + O(\Delta t^3) \right] + O(\Delta t^3).
\end{aligned} \tag{29}$$

Expanding (29) and upon some manipulations, one finally arrives at

$$\begin{aligned}
& e^{(\mathbf{A}+\mathbf{B})\Delta t} \\
&= \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \right) \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B} \right) \\
&\quad - \frac{\Delta t^2}{8}\mathbf{B}^2 + \frac{\Delta t^2}{8}\mathbf{B}^2 + O(\Delta t^3) \\
&= \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \right) \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B} \right) \\
&\quad + O(\Delta t^3).
\end{aligned} \tag{30}$$

It is interesting to note that even though the approximation of (26) and (28) has only first-order temporal accuracy, the overall approximation of ADI scheme in (30) has second-order temporal accuracy due to cancellation of second-order error terms.

C. DP-ADI

For DP-ADI scheme [12], the matrix exponential solution $e^{(\mathbf{A}+\mathbf{B})\Delta t}$ shall also be approximated using Strang splitting in (25).

The first $e^{\frac{1}{2}\mathbf{B}\Delta t}$ is replaced by first-order Taylor's approximation

$$e^{\frac{1}{2}\mathbf{B}\Delta t} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B} \right) + \frac{\Delta t^2}{8}\mathbf{B}^2 + O(\Delta t^3), \tag{31}$$

followed by Padé approximation of $e^{\mathbf{A}\Delta t}$

$$e^{\mathbf{A}\Delta t} = \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \right) + O(\Delta t^3) \tag{32}$$

while the second $e^{\frac{1}{2}\mathbf{B}\Delta t}$ is approximated using

$$e^{\frac{1}{2}\mathbf{B}\Delta t} = \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B} \right)^{-1} - \frac{\Delta t^2}{8}\mathbf{B}^2 + O(\Delta t^3). \tag{33}$$

Following the procedure in ADI scheme, one would finally arrive at

$$\begin{aligned}
e^{(\mathbf{A}+\mathbf{B})\Delta t} &= e^{\frac{1}{2}\mathbf{B}\Delta t} e^{\mathbf{A}\Delta t} e^{\frac{1}{2}\mathbf{B}\Delta t} + O(\Delta t^3) \\
&= \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B} \right) \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} \right)^{-1} \\
&\quad \times \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \right) \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B} \right)^{-1} + O(\Delta t^3).
\end{aligned} \tag{34}$$

Note that the DP-ADI scheme in (34) has also second-order temporal accuracy.

IV. CONCLUSION

This communication has presented further reinterpretation of multi-stage implicit FDTD schemes. The equivalence of several multi-stage SS schemes has been shown based on amplification matrices and power property of eigenvalues in dispersion relations. In doing so, the explicit expressions for dispersion relations need not be derived, thus amenable to 3D and generalized multi-stage SS schemes conveniently. The improvement of temporal accuracy for generalized multi-stage SS schemes using generalized input and output processing matrices has also been provided. It has been further shown that the SS, ADI as well as the recently proposed DP-ADI schemes can be interpreted based on more concise matrix exponential.

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