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Generalized Stability Criterion of 3-D FDTD Schemes for Doubly Lossy Media

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***Abstract*—This paper presents the generalized stability criterion of 3-D finite-difference time-domain (FDTD) schemes for doubly lossy media, where both electric and magnetic conductivities coexist. The generalized stability criterion is applicable for all 3-D FDTD schemes, such as time-average (TA), time-forward (TF), time-backward (TB) and exponential time differencing (ETD). It is reducible to either electrically lossy, magnetically lossy or lossless media. The stability criterion for perfectly matched layer (PML) matching condition can also be obtained as a special case to the doubly lossy media. It is shown that, for doubly lossy media, the stability criterion for ETD and TF becomes even more relaxed, and for TB, even more stringent compared to either electrically lossy, magnetically lossy or lossless media. On the other hand, the stability criterion for TA remains unchanged even in doubly lossy media. As numerical demonstration, the tunneling of electromagnetic wave through a very thin doubly lossy conductor is simulated. Numerical experiments further show the maximum allowed time step as dictated by the derived stability criterion for different schemes.**

***Index Terms*—Doubly lossy media, finite-difference time-domain, stability criterion.**

I. Introduction

The finite-difference time-domain (FDTD) method [1] has been widely used for simulating various electromagnetics problems. Since then, four different schemes have existed to treat lossy media, i.e. time-average (TA), time-forward (TF), time-backward (TB) and exponential time differencing (ETD). The stability criterion of FDTD scheme in lossless media was first shown in [2]. Following that, in [3], [4], the stability analysis for TA, TF and TB was shown in 3-D, while the stability criterion for ETD was derived only for 1-D in [5]. Using transformation method, a new stability criterion was further developed in [6] for ETD in 3-D under the perfectly matched layer (PML) matching condition ($\sigma/\epsilon = \sigma^*/\mu$), where ϵ, σ, and μ, σ^* are the permittivity, electric conductivity and permeability, magnetic conductivity, respectively.

In all the above mentioned works, the stability analysis is performed either for only electrically lossy media ($\sigma^* = 0$), or PML matching condition. Moreover, the stability criterion obtained is only restricted to a particular scheme and not easily applicable to other different schemes or magnetically lossy media. For instance, the generalized stability criterion obtained in [4] for electrically lossy media is applicable for weighted average related schemes such as TA, TF or TB. However, finding the stability criterion for ETD remains elusive. More importantly, we are interested to find the stability criterion for the most general case of doubly lossy media where both electric and magnetic conductivities coexist (i.e. $\sigma \neq 0, \sigma^* \neq 0$). In this paper, we present the generalized stability criterion of 3-D FDTD schemes for doubly lossy media. The stability criterion is expressed in terms of generalized update coefficients and thus, applicable for all aforementioned schemes. Furthermore, it is reducible to the case of either electrically lossy, magnetically lossy or even lossless media. The stability criterion for PML matching condition can also be obtained as a special case to the doubly lossy media.

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II. Generalized Stability Criterion For Doubly Lossy Media

Using von-Neumann analysis and normalized fields components $\boldsymbol{e} = \sqrt{\epsilon}\boldsymbol{E}$ and $\boldsymbol{h} = \sqrt{\mu}\boldsymbol{H}$, we can cast the whole update equations of 3-D FDTD schemes for doubly lossy media into

$$\boldsymbol{u}^{n+1} = (\mathbf{I}_6 - \mathbf{B})^{-1}(\mathbf{D} + \mathbf{A})\boldsymbol{u}^n = \mathbf{M}\boldsymbol{u}^n \tag{1}$$

where

$$\boldsymbol{u} = [e_x, e_y, e_z, h_x, h_y, h_z]^T \tag{2}$$

$$\mathbf{A} = c_{b,e}\begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 \\ -\mathbf{C} & \mathbf{O}_3 \end{bmatrix} \quad \mathbf{B} = c_{b,h}\begin{bmatrix} \mathbf{O}_3 & \mathbf{C} \\ \mathbf{O}_3 & \mathbf{O}_3 \end{bmatrix} \tag{3}$$

$$\mathbf{C} = \frac{1}{\sqrt{\mu\epsilon}}\begin{bmatrix} 0 & jK_z & -jK_y \\ -jK_z & 0 & jK_x \\ jK_y & -jK_x & 0 \end{bmatrix} \tag{4}$$

$$\mathbf{D} = \begin{bmatrix} c_{a,e}\mathbf{I}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & c_{a,h}\mathbf{I}_3 \end{bmatrix} \tag{5}$$

$$K_\xi = 2\sin\left(k_\xi \Delta\xi/2\right)/\Delta\xi \tag{6}$$

$c_{a,e}, c_{b,e}$ are the generalized update coefficients for normalized electric fields, and their dual $c_{a,h}, c_{b,h}$ are for normalized magnetic fields. These generalized update coefficients include nonzero σ and σ^* for doubly lossy media, and will be provided later for different FDTD schemes. k_ξ are the wavenumbers in x, y and z directions. $\mathbf{I}_r$ and $\mathbf{O}_r$ are the identity and null matrices with dimensions $r \times r$, respectively. The eigenvalues of the overall updating matrix $\mathbf{M}$ can be solved as

$$\lambda_1 = c_{a,e}, \quad \lambda_2 = c_{a,h}, \tag{7}$$

$$\lambda_3 = \lambda_4 = p + q, \tag{8}$$

$$\lambda_5 = \lambda_6 = p - q \tag{9}$$

where

$$p = \frac{1}{2}(c_{a,e} + c_{a,h}) - \frac{1}{2}c_{b,e}c_{b,h}W^2 \tag{10}$$

$$q = \frac{1}{2}\sqrt{(W^2 - r_+)(W^2 - r_-)} \tag{11}$$

$$W^2 = \frac{1}{\mu\epsilon}(K_x^2 + K_y^2 + K_z^2) \tag{12}$$

$$r_\pm = \frac{c_{a,e} + c_{a,h} \pm 2\sqrt{c_{a,e}c_{a,h}}}{c_{b,e}c_{b,h}} \tag{13}$$

For stable algorithm, all magnitudes of eigenvalues have to be less than or equal to one, $|\lambda_i| \leq 1$. From the first two eigenvalues, we obtain

$$|c_{a,e}| \leq 1, \quad |c_{a,h}| \leq 1 \tag{14}$$

For the remaining eigenvalues, we note that q can only be either positively real or purely imaginary. Consider these two possibilities separately. When q is purely imaginary, we find that two complex-conjugate pairs of eigenvalues exist, with magnitudes

$$|\lambda_3| = |\lambda_4| = |\lambda_5| = |\lambda_6| = \sqrt{c_{a,e}c_{a,h}} \tag{15}$$

If (14) holds, then $\sqrt{c_{a,e}c_{a,h}} \leq 1$ and the stability for purely imaginary q is guaranteed.

On the other hand, to ensure stability for positive real q, we ought to have

$$-1 \leq p - q \leq p + q \leq 1 \tag{16}$$

along with (14). It can be shown, upon some manipulations that (16) is satisfied if and only if

$$\frac{c_{a,e} + c_{a,h} - c_{a,e}c_{a,h} - 1}{c_{b,e}c_{b,h}} \leq W^2 \leq \frac{1 + c_{a,e} + c_{a,h} + c_{a,e}c_{a,h}}{c_{b,e}c_{b,h}} \tag{17}$$

Since (14) holds, we find that

$$\frac{c_{a,e} + c_{a,h} - c_{a,e}c_{a,h} - 1}{c_{b,e}c_{b,h}} \leq 0 \tag{18}$$

For nonnegative W^2, (17) can be further reduced into

$$W^2 \leq \frac{1 + c_{a,e} + c_{a,h} + c_{a,e}c_{a,h}}{c_{b,e}c_{b,h}} \tag{19}$$

The overall necessary and sufficient condition to guarantee algorithm stability for all q is therefore given by

$$|c_{a,e}| \leq 1, \quad |c_{a,h}| \leq 1 \tag{20a}$$

$$\frac{4c_{b,e}c_{b,h}}{1 + c_{a,e} + c_{a,h} + c_{a,e}c_{a,h}} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}} \tag{20b}$$

Equation (20) represents the generalized stability criterion of 3-D FDTD schemes for doubly lossy media. This generalized stability criterion is very versatile and convenient in a sense that, for any particular scheme, one would only need to replace $c_{a,e}$, $c_{b,e}$, $c_{a,h}$ and $c_{b,h}$ with their corresponding update coefficients.

Table I provides the stability criterion for 3-D FDTD schemes, as well as their respective update coefficients. For nonnegative Δt, τ and τ^*, the conditions in (20a) are redundant for ETD, TA and TF because $|c_{a,e}|$ and $|c_{a,h}|$ are always less than or equal to absolute one. For TB, $\Delta t \leq 2\min(\tau, \tau^*)$ and $\Delta t \leq 2\tau$ are needed for doubly lossy and electrically lossy media, respectively, in order to ensure (20a). For completeness, TA, TF and TB can be further generalized into weighted average scheme, and its entries are also included with coefficients a and b defined in [4]. The stability criterion for 1 or 2-D is very straight forward by removing either one or two of the spatial steps. Three types of media are shown, the doubly lossy, electrically lossy and lossless. The magnetically lossy media are the dual of electrically lossy media, and the stability criterion can be obtained simply by replacing τ with τ^*. The stability criterion of TA, TF, TB and weighted average schemes obtained for electrically lossy media is consistent with those found in [3], [4]. The stability criterion of ETD in Table I is also consistent with the one found in [5] for 1-D electrically lossy media. Furthermore, we note that the stability criterion (equation (8) in [6]) obtained by transformation method is only applicable for PML matching condition ($\tau = \tau^*$) and belongs to the special case of our generalized doubly lossy ETD scheme.

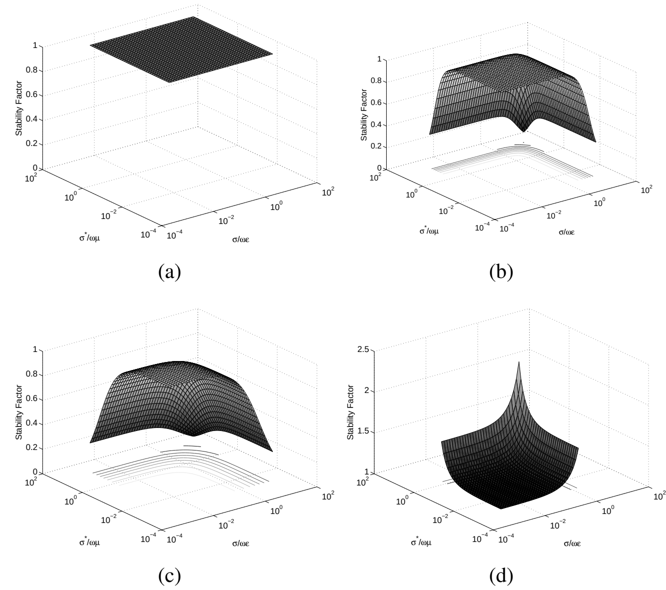

Fig. 1. 3-D plots of stability factor with respect to $\sigma/\omega\epsilon$ and $\sigma^*/\omega\mu$ for (a) TA, (b) ETD, (c) TF, (d) TB.

III. Numerical Results

It has been shown in previous studies [3], [4], [5] that the stability criterion for electrically lossy media remains unchanged for TA, is more relaxed for TF, ETD and is more stringent for TB compared to FDTD scheme in lossless media. Here, we further show that for doubly lossy media, stability criterion for TA still remains unchanged. For TF and ETD, the stability criterion is even more relaxed than that of electrically lossy media, while it is even more stringent for TB. We first introduce the stability factor, f by bringing all terms in (20b) to the left hand side:

$$f = \frac{4c_{b,e}c_{b,h}(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2})}{(1 + c_{a,e} + c_{a,h} + c_{a,e}c_{a,h})\mu\epsilon} \tag{21}$$

The stability factor will serve as a measure as to how stable a particular scheme is at any arbitrary $\Delta x, \Delta y, \Delta z, \Delta t, \tau$ and τ^*. A stability factor larger than one indicates instability and vice-versa. Fig. 1 shows the 3-D plots of stability factor with respect to $\sigma/\omega\epsilon$ and $\sigma^*/\omega\mu$ for (a) TA, (b) ETD, (c) TF and (d) TB. Uniform cells ($\Delta x = \Delta y = \Delta z$) are assumed and the frequency is arbitrarily fixed at 10 GHz. CPW is set at 10 at that particular frequency and Δt is chosen as the 3-D lossless Courant limit. Note that all these parameters are fixed throughout with only σ and σ^* varied. We observe that for TA, stability factor remains at one throughout, which indicates that the stability criterion is unperturbed by σ, σ^* and is still the same as the lossless case. For ETD and TF, a negative gradient for stability factor is seen with increasing σ and σ^*. This suggests that toward increasing σ and σ^*, both schemes have more increment room for their Δt before reaching instability (i.e. more relaxed criterion). Also, TF has slightly smaller stability factor (slightly more relaxed stability criterion) throughout compared to ETD. On the other hand, we observe a positive gradient for stability factor of TB, which is

TABLE I
STABILITY CRITERION AND UPDATE COEFFICIENTS FOR 3-D FDTD SCHEMES

Scheme		$c_{a,e}$	$c_{b,e}$	$c_{a,h}$	$c_{b,h}$	Stability Criterion
Doubly Lossy	ETD	$e^{-\frac{\Delta t}{\tau}}$	$\tau(1-e^{-\frac{\Delta t}{\tau}})$	$e^{-\frac{\Delta t}{\tau^*}}$	$\tau^*(1-e^{-\frac{\Delta t}{\tau^*}})$	$\frac{4\tau\tau^*(1-e^{-\frac{\Delta t}{\tau}})(1-e^{-\frac{\Delta t}{\tau^*}})}{(1+e^{-\frac{\Delta t}{\tau}})(1+e^{-\frac{\Delta t}{\tau^*}})} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	TA ($a=\frac{1}{2}, b=\frac{1}{2}$)	$\frac{1-\frac{\Delta t}{2\tau}}{1+\frac{\Delta t}{2\tau}}$	$\frac{\Delta t}{1+\frac{\Delta t}{2\tau}}$	$\frac{1-\frac{\Delta t}{2\tau^*}}{1+\frac{\Delta t}{2\tau^*}}$	$\frac{\Delta t}{1+\frac{\Delta t}{2\tau^*}}$	$\Delta t^2 \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	TF ($a=1, b=0$)	$\frac{1}{1+\frac{\Delta t}{\tau}}$	$\frac{\Delta t}{1+\frac{\Delta t}{\tau}}$	$\frac{1}{1+\frac{\Delta t}{\tau^*}}$	$\frac{\Delta t}{1+\frac{\Delta t}{\tau^*}}$	$\frac{\Delta t^2}{(1+\frac{\Delta t}{2\tau})(1+\frac{\Delta t}{2\tau^*})} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	TB ($a=0, b=1$)	$1-\frac{\Delta t}{\tau}$	Δt	$1-\frac{\Delta t}{\tau^*}$	Δt	$\Delta t \leq 2\min(\tau,\tau^*)$ $\frac{\Delta t^2}{\left(1-\frac{\Delta t}{2\tau}\right)\left(1-\frac{\Delta t}{2\tau^*}\right)} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	Weighted Average (any $a+b=1$)	$\frac{1-b\frac{\Delta t}{\tau}}{1+a\frac{\Delta t}{\tau}}$	$\frac{\Delta t}{1+a\frac{\Delta t}{\tau}}$	$\frac{1-b\frac{\Delta t}{\tau^*}}{1+a\frac{\Delta t}{\tau^*}}$	$\frac{\Delta t}{1+a\frac{\Delta t}{\tau^*}}$	$(b-a)\Delta t \leq 2\min(\tau,\tau^*)$ $\frac{\Delta t^2}{\left(1+(a-b)\frac{\Delta t}{2\tau}\right)\left(1+(a-b)\frac{\Delta t}{2\tau^*}\right)} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
Electrically Lossy ($\sigma^*=0$)	ETD	$e^{-\frac{\Delta t}{\tau}}$	$\tau(1-e^{-\frac{\Delta t}{\tau}})$	1	Δt	$\frac{2\tau(1-e^{-\frac{\Delta t}{\tau}})\Delta t}{(1+e^{-\frac{\Delta t}{\tau}})} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	TA ($a=\frac{1}{2}, b=\frac{1}{2}$)	$\frac{1-\frac{\Delta t}{2\tau}}{1+\frac{\Delta t}{2\tau}}$	$\frac{\Delta t}{1+\frac{\Delta t}{2\tau}}$			$\Delta t^2 \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	TF ($a=1, b=0$)	$\frac{1}{1+\frac{\Delta t}{\tau}}$	$\frac{\Delta t}{1+\frac{\Delta t}{\tau}}$			$\frac{\Delta t^2}{(1+\frac{\Delta t}{2\tau})} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	TB ($a=0, b=1$)	$1-\frac{\Delta t}{\tau}$	Δt			$\Delta t \leq 2\tau$ $\frac{\Delta t^2}{\left(1-\frac{\Delta t}{2\tau}\right)} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
	Weighted Average (any $a+b=1$)	$\frac{1-b\frac{\Delta t}{\tau}}{1+a\frac{\Delta t}{\tau}}$	$\frac{\Delta t}{1+a\frac{\Delta t}{\tau}}$			$(b-a)\Delta t \leq 2\tau$ $\frac{\Delta t^2}{\left(1+(a-b)\frac{\Delta t}{2\tau}\right)} \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$
Lossless ($\sigma=\sigma^*=0$)		1	Δt	1	Δt	$\Delta t^2 \leq \frac{\mu\epsilon}{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}$

$\tau = \epsilon/\sigma, \quad \tau^* = \mu/\sigma^*$

TABLE II
CPU TIME INCURRED BY DIFFERENT FDTD SCHEMES

Scheme	Δt	Total no. of Iterations	CPU Time
TA	8.339×10^{-16}	115711	1.666s
ETD	3.927×10^{-15}	24572	0.353s
TF	4.097×10^{-15}	23550	0.339s

a clear indication of more stringent criterion toward increasing σ and σ^*.

As numerical demonstration, we simulate the tunneling phenomena of 1-D electromagnetic wave propagation through a very thin conductor with doubly lossy nature. A domain of 150 cells with nonuniform cell size are adopted. Two cells at the centre are filled with a doubly lossy conductor with $\sigma = 10^5$S/m, $\sigma^* = 10^5\Omega$/m and the rest are free space. Each cell size of the conductor is set at 0.25μm, while for the rest of the cell, cell per wavelength (CPW) is chosen as 10 at 150 GHz. An electric field gaussian pulse $g(t) = \exp\left(-\left(\frac{(t-t_d)}{t_w}\right)^2\right)$ where $t_d = 18.75$ps, $t_w = 6.25$ps is launched at initial point as a hard source impinging upon the conductor, and the simulation is terminated after 96.5ps. Since TB has an even more stringent stability criterion in doubly lossy media compared to TA, it shall be omitted here due to its inefficiency. The experiment is run using three different schemes, TA, ETD and TF with time step, Δt set at their respective maximum allowed time step, listed in Table II. The maximum allowed time step is solved from the stability criterion in Table I with minimum cell size 0.25μm. It can be seen from Table II that ETD and TF have more relaxed Δt, and therefore, less total number of iterations and CPU time needed

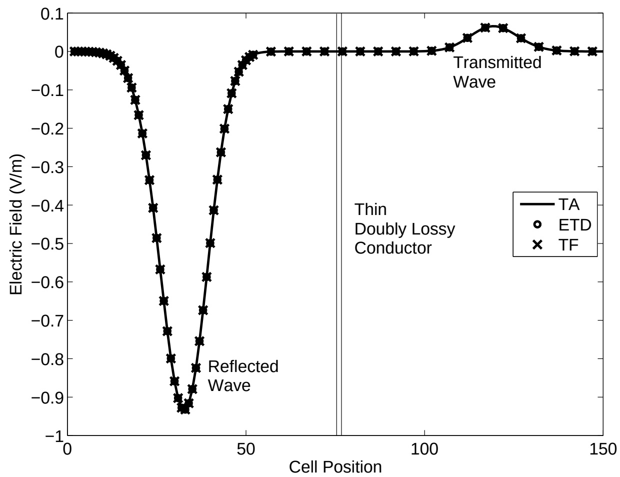

Fig. 2. Snapshot of reflected and transmitted electric fields through thin conductor ($\sigma = 10^5$S/m, $\sigma^* = 10^5\Omega$/m) after 96.5ps.

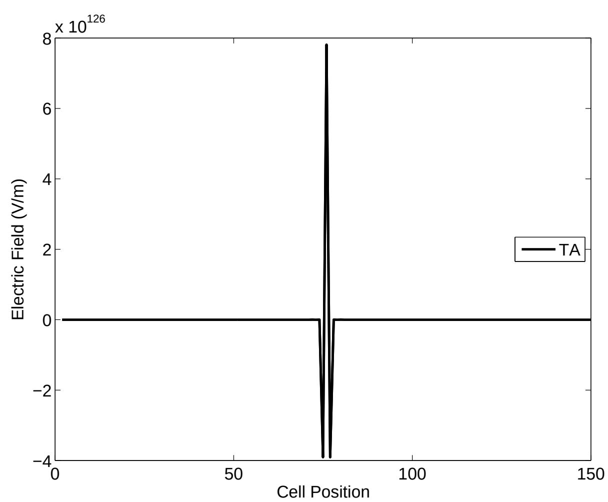

Fig. 3. Instability shown by TA around thin conductor after 1000 iterations (Δt set at maximum allowed Δt of ETD instead of its own).

for the simulation compared to TA. We can also see that TF has a slightly more relaxed Δt compared to ETD. Fig. 2 shows the snapshot of electric fields after 96.5ps. As long as the doubly lossy conductor is sufficiently thin, we observe a portion of electric field being able to tunnel through the thin conductor, and the rest being reflected. Fig. 3 further shows the instability exhibited by TA around the thin conductor just after 1000 iterations if Δt is set at ETD's more relaxed Δt instead of its own maximum allowed time step.

To further verify the stability criterion provided in Table I, we consider a 50 × 50 × 50 meshed cavity with PEC walls and uniform cell size of 35mm, being excited by a point current source at the centre. The cavity is filled with doubly lossy medium, where σ and σ^* are arbitrarily set as 0.1S/m and 100Ω/m, respectively. Simulations are performed using the schemes of ETD, TA, TF and TB, with Δt set at their maximum allowed time step as 69.0757ps, 67.4041ps, 81.5990ps and 55.7355ps (solved from Table I), respectively. No instability is observed under such circumstances. The experiments are then repeated, with the respective Δt slightly amplified by a factor of 1.001. In this case, exponential growth of fields originating from the point source are visible as the time marches. These further confirm and ascertain the stability criterion of FDTD schemes derived previously.

IV. Conclusion

This paper has presented the generalized stability criterion for 3-D FDTD schemes for doubly lossy media, where both electric and magnetic conductivities coexist. The generalized stability criterion is applicable for all FDTD schemes, such TA, TF, TB and ETD. It is reducible to either electrically lossy, magnetically lossy or lossless media. The stability criterion for perfectly matched layer (PML) matching condition can also be obtained as a special case to the doubly lossy media. It has been shown that, for doubly lossy media, the stability criterion for ETD and TF becomes even more relaxed, and for TB, even more stringent compared to either electrically lossy, magnetically lossy or lossless media. On the other hand, the stability criterion for TA remains unchanged even in doubly lossy media. The tunneling of electromagnetic wave through a very thin doubly lossy conductor has been numerically demonstrated. Numerical experiments further show the maximum allowed time step as dictated by the derived stability criterion for different schemes.

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