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2019

Heh, D. Y., & Tan, E. L. (2020). Multiple LOD-FDTD method for inhomogeneous coupled transmission lines and stability analyses. IEEE Transactions on Antennas and Propagation, 68(3), 2198-2205. doi:10.1109/tap.2019.2943428

<https://hdl.handle.net/10356/137214>

<https://doi.org/10.1109/TAP.2019.2943428>

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Multiple LOD-FDTD Method for Inhomogeneous Coupled Transmission Lines and Stability Analyses

Ding Yu Heh, *Member, IEEE* and Eng Leong Tan, *Senior Member, IEEE*,

Abstract—A multiple locally one-dimensional (MLOD) finite-difference time-domain (FDTD) method for inhomogeneous coupled transmission lines and stability analyses are presented. The method is aptly called the MLOD coupled line (CL)-FDTD method. Various split matrices are proposed, and the corresponding update equations are formulated and discussed. All the proposed split matrices yield implicit electric field update equations with tridiagonal or block tridiagonal matrices on the left-hand sides. For more efficiency, the block tridiagonal matrices for implicit electric field may be reformulated and replaced by tridiagonal matrices for implicit magnetic field. The stability analysis is first performed using the von Neumann method in the Fourier domain. It is shown that the von Neumann method alone may not be sufficient to ascertain stability for inhomogeneous media. To include media inhomogeneity, the two-media reduced-matrix stability analysis is proposed. It allows us to analyze efficiently the key stability characteristics in inhomogeneous media, and is useful for quick detection of any potential instability that is not apparent via the von Neumann method. The stability characteristics with variation of media parameters are also investigated and discussed. Numerical results are provided to validate the stability and accuracy of the proposed MLOD CL-FDTD method with various split matrices.

Index Terms—Multiple LOD FDTD method, coupled transmission lines, split matrices, stability

I. INTRODUCTION

Three-dimensional (3-D) finite-difference time-domain (FDTD) methods [1], [2] are generally computationally intensive and incur large computational resources for simulation of electromagnetic (EM) wave propagation. To overcome the inefficiency of the 3-D FDTD methods, the multiple one-dimensional (M1-D) FDTD method has been proposed for simulation of EM wave propagation in multiple transmission lines and stubs [3]. It has been extended to incorporate coupled transmission lines [4]. However, the time step is still constrained by its own Courant-Friedrichs-Lewy (CFL) stability condition. In [5], the M1-D alternating direction implicit (ADI)-FDTD method is proposed for coupled transmission lines. The method yields unconditionally stable update equations with tridiagonal matrices on the left-hand sides (LHS) that can be solved efficiently. Note that the formulated split matrices are different from those of the 3-D ADI-FDTD method [6]–[8] for Maxwell's equations. Alternative to the ADI-FDTD method, the locally one-dimensional (LOD)-FDTD method [9]–[11] is another popular implicit scheme which features unconditional stability. It is more efficient than

the ADI-FDTD method due to fewer terms and arithmetic operations. Hence, it would be desirable to formulate the M1-D LOD-FDTD method for coupled transmission lines. Furthermore, the stability has not been proven rigorously in [5] when media inhomogeneity is included. In [12], initial numerical stability analysis of M1-D LOD-FDTD method for inhomogeneous coupled transmission lines has been presented. However, only one set of split matrices is considered and the stability analysis is merely performed numerically. There are other split matrices to be explored and analyzed further in regard of their implicit update equations and stability characteristics. Also, the stability characteristics with variation of media parameters and time step size have not been investigated thoroughly. Note that proper split matrices are of utmost importance in maintaining the stability and efficiency of numerical methods. Indeed, the path to them is usually not easy [13] and they form the key factors leading to the success and popularity of 3-D ADI-FDTD in [6]–[7].

In this paper, we present the multiple LOD (MLOD) FDTD method for inhomogeneous coupled transmission lines and stability analyses. The method is hereby aptly called the MLOD coupled line (CL)-FDTD method. Various split matrices will be proposed and analyzed in regard to their implicit update equations and stability characteristics. Due to its simplicity, the von Neumann method in Fourier domain is often utilized for stability analysis. However, the method alone may not be sufficient to ascertain stability when media inhomogeneity is present. On the other hand, the stability analysis using full transition matrix for whole computational domain is more rigorous. However, it is often too laborious to formulate and too complex to analyze due to huge matrix sizes. Determining the eigenvalues of huge matrices for stability analysis is also very computationally intensive. To reduce the complexity, we propose the reduced transition matrix for a smaller domain comprising two inhomogeneous media. Such a two-media reduced-matrix allows us to analyze efficiently the key stability characteristics in inhomogeneous media, and is useful for quick detection of any potential instability that is not apparent via the von Neumann method.

The paper will be organized as follows. In Section II, various sets of split matrices will be proposed, and the corresponding update equations will be formulated and discussed. It will be found that all the proposed split matrices yield implicit electric field update equations with tridiagonal or block tridiagonal matrices on the LHS. For more efficiency, the block tridiagonal matrices for implicit electric field may be reformulated and replaced by tridiagonal matrices for implicit magnetic field. In Section III, the stability analysis will be

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The authors are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: dy-heh@ntu.edu.sg; eeltan@ntu.edu.sg).

Fig. 1. Schematic of coupled transmission lines.

first performed using the von Neumann method in the Fourier domain. To include media inhomogeneity, the two-media reduced-matrix stability analysis will be carried out. It will be shown that for certain split matrices, while the von Neumann method indicates stability for homogeneous media, the two-media reduced-matrix stability analysis may indicate potential instability for inhomogeneous media. Numerical results will be provided in Section IV to further validate the stability and accuracy of the MLOD CL-FDTD method with various split matrices.

II. SPLIT MATRICES FOR MLOD CL-FDTD METHOD

Fig. 1 shows the schematic of coupled transmission lines, along with electric and magnetic fields. The differential equations for coupled transmission lines can be written in compact form as [5]

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{W} \mathbf{u} \quad (1)$$

where

$$\mathbf{u} = [E_{x1}, E_{x2}, H_{y1}, H_{y2}]^T \quad (2a)$$

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} \\ 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} \\ \frac{-1}{\mu_s} \frac{\partial}{\partial z} & \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 \\ \frac{1}{\mu_m} \frac{\partial}{\partial z} & \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}. \quad (2b)$$

E_{x1} , H_{y1} are the electric and magnetic fields along line 1, E_{x2} , H_{y2} are the electric and magnetic fields along line 2, ϵ_s , ϵ_m and μ_s , μ_m are the self and mutual permittivities and permeabilities, expressed in terms of even- and odd-mode characteristic impedances (Z_{0e} , Z_{0o}), phase velocities (v_e , v_o) and effective permittivities (ϵ_{eff}^e , ϵ_{eff}^o). Z_{0e} and Z_{0o} are the characteristic impedances of the coupled transmission lines when they are excited at even- (equal polarity between lines) and odd-mode (opposite polarity between lines) respectively. v_e and v_o are the corresponding phase velocities when the coupled transmission lines are excited at even- and odd-mode respectively. These parameters are related to the per-unit-length self and mutual capacitance (C_s , C_m) and inductance (L_s , L_m) of the coupled transmission lines by

$$C_s = \frac{1}{2} \left(\frac{1}{v_o Z_{0o}} + \frac{1}{v_e Z_{0e}} \right), C_m = \frac{1}{2} \left(\frac{1}{v_o Z_{0o}} - \frac{1}{v_e Z_{0e}} \right), \quad (3a)$$

$$L_s = \frac{1}{2} \left(\frac{Z_{0e}}{v_e} + \frac{Z_{0o}}{v_o} \right), L_m = \frac{1}{2} \left(\frac{Z_{0e}}{v_e} - \frac{Z_{0o}}{v_o} \right), \quad (3b)$$

$$\frac{1}{\epsilon_s} = \frac{C_s}{C_s^2 - C_m^2} = \frac{v_e Z_{0e} + v_o Z_{0o}}{2}, \quad (3c)$$

$$\frac{1}{\epsilon_m} = \frac{C_m}{C_s^2 - C_m^2} = \frac{v_e Z_{0e} - v_o Z_{0o}}{2}, \quad (3d)$$

$$\frac{1}{\mu_s} = \frac{L_s}{L_s^2 - L_m^2} = \frac{1}{2} \left(\frac{v_o}{Z_{0o}} + \frac{v_e}{Z_{0e}} \right), \quad (3e)$$

$$\frac{1}{\mu_m} = \frac{L_m}{L_s^2 - L_m^2} = \frac{1}{2} \left(\frac{v_o}{Z_{0o}} - \frac{v_e}{Z_{0e}} \right), \quad (3f)$$

$$v_e = c / \sqrt{\epsilon_{eff}^e} \quad \text{and} \quad v_o = c / \sqrt{\epsilon_{eff}^o}. \quad (3g)$$

The differential equations and parameters above are derived based on distributed equivalent circuit of coupled lines along with even- and odd-mode approach [14], [15]. In general, all parameters may vary along z for inhomogeneous coupled lines.

The MLOD CL-FDTD method is proposed here to solve (1) through proper splitting of $\mathbf{W} = \mathbf{A} + \mathbf{B}$. The method calls for the following two LOD update procedures

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right) \mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) \mathbf{u}^n \quad (4a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{u}^{n+\frac{1}{2}} \quad (4b)$$

where $\mathbf{A}$ and $\mathbf{B}$ are the 4×4 split matrices. It should be emphasized that proper split matrices are required to ensure that the method is stable. Not only that, the split matrices should be such that the LHS of (4) would result in tridiagonal matrices that can be solved efficiently. In what follows, we propose various sets of split matrices and analyze their corresponding update equations and stability in detail.

A. 1st Set of Split Matrices

We first consider the set of split matrices proposed in [5]. They consist of only column elements and are henceforth regarded as the 1st set of split matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} & 0 \\ \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 & 0 \\ \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 & 0 \end{bmatrix}, \quad (5a)$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} \\ 0 & 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} \\ 0 & \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}. \quad (5b)$$

Substituting (5) into (4), the update equations of MLOD CL-FDTD method for the first procedure can be obtained as follows:

$$E_{x1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} = E_{x1}^n - \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} \quad (6a)$$

$$E_{x2}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} = E_{x2}^n - \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y1}^n}{\partial z} \quad (6b)$$

$$H_{y1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} = H_{y1}^n - \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \quad (6c)$$

$$H_{y2}^{n+\frac{1}{2}} - \frac{\Delta t}{2\mu_m} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} = H_{y2}^n + \frac{\Delta t}{2\mu_m} \frac{\partial E_{x1}^n}{\partial z}. \quad (6d)$$

The update equations for the second procedure can be obtained by permuting the subscripts of fields in lines 1 and 2, e.g. $E_{x1} \leftrightarrow E_{x2}$. In the first procedure, E_{x1} is solved implicitly by substituting (6c) into (6a) as

$$\begin{aligned} E_{x1}^{n+\frac{1}{2}} - \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} \right) \\ = E_{x1}^n - \frac{\Delta t}{\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} + \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \right). \end{aligned} \quad (7)$$

Upon discretizing the spatial derivatives using central differencing operated on standard staggered cells, one can find that the LHS of (7) yields tridiagonal matrices which can be solved efficiently. Compared to M1-D ADI-FDTD method [5], the implicit update equation of MLOD CL-FDTD method has fewer terms and arithmetic operations. In the second procedure, E_{x2} is also solved implicitly in the similar manner. All other field components are updated explicitly.

B. 2nd Set of Split Matrices

There are other viable choices of split matrices which would also yield tridiagonal matrices on the LHS of the update equations. We hereby propose the 2nd set of split matrices, which are chosen to consist of only row elements as

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\mu_s} \frac{\partial}{\partial z} & \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (8a)$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\mu_m} \frac{\partial}{\partial z} & \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}. \quad (8b)$$

Substituting (8) into (4), we arrive at the following update equations for the first procedure:

$$\begin{aligned} E_{x1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} + \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y2}^{n+\frac{1}{2}}}{\partial z} \\ = E_{x1}^n - \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} - \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y2}^n}{\partial z} \end{aligned} \quad (9a)$$

$$E_{x2}^{n+\frac{1}{2}} = E_{x2}^n \quad (9b)$$

$$\begin{aligned} H_{y1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} - \frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^{n+\frac{1}{2}}}{\partial z} \\ = H_{y1}^n - \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} + \frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^n}{\partial z} \end{aligned} \quad (9c)$$

$$H_{y2}^{n+\frac{1}{2}} = H_{y2}^n. \quad (9d)$$

The update equations for the second procedure can be obtained by permuting the subscripts of fields in lines 1 and 2. In the first procedure, E_{x1} is solved implicitly by substituting (9c)

into (9a). Taking into account (9b) and (9d), the implicit update equation reads

$$\begin{aligned} E_{x1}^{n+\frac{1}{2}} - \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} \right) \\ = E_{x1}^n - \frac{\Delta t}{\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} - \frac{\Delta t}{\epsilon_m} \frac{\partial H_{y2}^n}{\partial z} \\ + \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \right) - \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{\mu_m} \frac{\partial E_{x2}^n}{\partial z} \right). \end{aligned} \quad (10)$$

Using the same discretization scheme for the spatial derivatives, the LHS of (10) also yields tridiagonal matrices which can be solved efficiently. In the second procedure, E_{x2} is solved implicitly in the similar manner. All other field components are again updated explicitly.

C. 3rd Set of Split Matrices

We next consider the 3rd set of split matrices, which comprise combination of row and column elements as

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 & 0 \\ \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 & 0 \end{bmatrix}, \quad (11a)$$

$$\mathbf{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} \\ 0 & \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}. \quad (11b)$$

Substituting (11) into (4), the update equations for the first procedure are obtained as

$$\begin{aligned} E_{x1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} + \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y2}^{n+\frac{1}{2}}}{\partial z} \\ = E_{x1}^n - \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} - \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y2}^n}{\partial z} \end{aligned} \quad (12a)$$

$$E_{x2}^{n+\frac{1}{2}} = E_{x2}^n \quad (12b)$$

$$H_{y1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} = H_{y1}^n - \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \quad (12c)$$

$$H_{y2}^{n+\frac{1}{2}} - \frac{\Delta t}{2\mu_m} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} = H_{y2}^n + \frac{\Delta t}{2\mu_m} \frac{\partial E_{x1}^n}{\partial z}. \quad (12d)$$

The update equations for second procedure are again obtained by permuting the subscripts of fields in lines 1 and 2. To solve for E_{x1} implicitly in the first procedure, (12c) and (12d) are substituted into (12a) to yield

$$\begin{aligned} E_{x1}^{n+\frac{1}{2}} - \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} \right) + \frac{\Delta t}{2\epsilon_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_m} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} \right) \\ = E_{x1}^n - \frac{\Delta t}{\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} - \frac{\Delta t}{\epsilon_m} \frac{\partial H_{y2}^n}{\partial z} + \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \right) \\ - \frac{\Delta t}{2\epsilon_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_m} \frac{\partial E_{x1}^n}{\partial z} \right). \end{aligned} \quad (13)$$

Applying central differencing to the spatial derivatives, the LHS of (13) also yields tridiagonal matrices.

D. 4th Set of Split Matrices

Finally, we consider the 4th set of split matrices, which also comprise combination of row and column elements as

$$\mathbf{A}_4 = \begin{bmatrix} 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} & 0 \\ \frac{-1}{\mu_s} \frac{\partial}{\partial z} & \frac{1}{\mu_m} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (14a)$$

$$\mathbf{B}_4 = \begin{bmatrix} 0 & 0 & 0 & \frac{-1}{\epsilon_m} \frac{\partial}{\partial z} \\ 0 & 0 & 0 & \frac{-1}{\epsilon_s} \frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\mu_m} \frac{\partial}{\partial z} & \frac{-1}{\mu_s} \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}. \quad (14b)$$

The update equations for the first procedure after expanding (4) with (14) yield

$$E_{x1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} = E_{x1}^n - \frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} \quad (15a)$$

$$E_{x2}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} = E_{x2}^n - \frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y1}^n}{\partial z} \quad (15b)$$

$$H_{y1}^{n+\frac{1}{2}} + \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} - \frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^{n+\frac{1}{2}}}{\partial z} = H_{y1}^n - \frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} + \frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^n}{\partial z} \quad (15c)$$

$$H_{y2}^{n+\frac{1}{2}} = H_{y2}^n. \quad (15d)$$

In the first procedure, E_{x1} and E_{x2} are solved implicitly by substituting (15c) into (15a) and (15b) as

$$E_{x1}^{n+\frac{1}{2}} - \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} \right) + \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^{n+\frac{1}{2}}}{\partial z} \right) = E_{x1}^n - \frac{\Delta t}{\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} + \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \right) - \frac{\Delta t}{2\epsilon_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^n}{\partial z} \right) \quad (16a)$$

$$E_{x2}^{n+\frac{1}{2}} + \frac{\Delta t}{2\epsilon_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^{n+\frac{1}{2}}}{\partial z} \right) - \frac{\Delta t}{2\epsilon_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^{n+\frac{1}{2}}}{\partial z} \right) = E_{x2}^n - \frac{\Delta t}{\epsilon_m} \frac{\partial H_{y1}^n}{\partial z} - \frac{\Delta t}{2\epsilon_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_m} \frac{\partial E_{x2}^n}{\partial z} \right) + \frac{\Delta t}{2\epsilon_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\mu_s} \frac{\partial E_{x1}^n}{\partial z} \right). \quad (16b)$$

Note that unlike the previous sets of split matrices, the LHS of (16) constitute block tridiagonal matrices from which E_{x1} and E_{x2} are to be solved simultaneously. Although block tridiagonal matrices could still be solved practically, one could alternatively reformulate and substitute (15a)-(15b) into (15c) in the first procedure to obtain

$$H_{y1}^{n+\frac{1}{2}} - \frac{\Delta t}{2\mu_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} \right) + \frac{\Delta t}{2\mu_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y1}^{n+\frac{1}{2}}}{\partial z} \right) = H_{y1}^n - \frac{\Delta t}{\mu_s} \frac{\partial E_{x1}^n}{\partial z} + \frac{\Delta t}{\mu_m} \frac{\partial E_{x2}^n}{\partial z} + \frac{\Delta t}{2\mu_s} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\epsilon_s} \frac{\partial H_{y1}^n}{\partial z} \right) - \frac{\Delta t}{2\mu_m} \frac{\partial}{\partial z} \left(\frac{\Delta t}{2\epsilon_m} \frac{\partial H_{y1}^n}{\partial z} \right). \quad (17)$$

In doing so, the LHS of (17) has been replaced by tridiagonal matrices which could be solved more efficiently. Note however that the implicit update equations here involve magnetic fields instead of electric fields as in other sets of split matrices.

III. STABILITY ANALYSES

A. Von Neumann Stability Analysis

In this section, we shall analyze the stability of various split matrices proposed above. Using the von Neumann method, the Fourier modes are applied to the field components and the Fourier amplification matrix is expressed as

$$\tilde{\mathbf{M}} = \left(\mathbf{I} - \frac{\Delta t}{2} \tilde{\mathbf{B}} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \tilde{\mathbf{B}} \right) \left(\mathbf{I} - \frac{\Delta t}{2} \tilde{\mathbf{A}} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \tilde{\mathbf{A}} \right) \quad (18)$$

where $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are split matrices in the Fourier domain. For instance, the 2nd set of split matrices in Fourier domain can be written as

$$\tilde{\mathbf{A}}_2 = \begin{bmatrix} 0 & 0 & \frac{j\bar{K}_z}{\epsilon_s} & \frac{j\bar{K}_z}{\epsilon_m} \\ 0 & 0 & 0 & 0 \\ \frac{j\bar{K}_z}{\mu_s} & \frac{-j\bar{K}_z}{\mu_m} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (19a)$$

$$\tilde{\mathbf{B}}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{j\bar{K}_z}{\epsilon_m} & \frac{j\bar{K}_z}{\epsilon_s} \\ 0 & 0 & 0 & 0 \\ \frac{-j\bar{K}_z}{\mu_m} & \frac{j\bar{K}_z}{\mu_s} & 0 & 0 \end{bmatrix} \quad (19b)$$

where $\bar{K}_z = \frac{2}{\Delta z} \sin\left(\frac{k_z \Delta z}{2}\right)$ and k_z is the wave number along the z direction. Other split matrices in the Fourier domain can be similarly written down. To that end, the eigenvalues of the Fourier amplification matrix are obtained as

$$\lambda_{\xi,1,2} = \frac{P_\xi + 2Q_\xi \pm \sqrt{R_\xi + 4P_\xi Q_\xi}}{S_\xi} \quad (20a)$$

$$\lambda_{\xi,3,4} = \frac{P_\xi - 2Q_\xi \pm \sqrt{R_\xi - 4P_\xi Q_\xi}}{S_\xi} \quad (20b)$$

where the subscript $\xi = 1, 2, 3, 4$ is associated with the 1st, 2nd, 3rd and 4th sets of split matrices, respectively. It can be shown that the Fourier amplification matrices for the 1st and 2nd sets of split matrices share the same characteristic polynomial, and similarly for the 3rd and 4th sets of split matrices. Hence,

$$\lambda_{1,i} = \lambda_{2,i}, \quad \lambda_{3,i} = \lambda_{4,i} \quad (21)$$

for all eigenvalue indices $i = 1, 2, 3, 4$. The analytical expressions of various terms above are given as

$$P_1 = P_2 = 2\bar{K}_z^4 [\epsilon_m^2 \mu_s^2 - \mu_m^2 (\epsilon_m^2 - \epsilon_s^2)] + 4\bar{K}_z^2 \epsilon_m \epsilon_s^2 \mu_m \mu_s^2 + 2\epsilon_m^2 \epsilon_s^2 \mu_m^2 \mu_s^2 \quad (22a)$$

$$Q_1 = Q_2 = \{ \bar{K}_z^8 (\epsilon_m^2 \mu_s^2 - \epsilon_s^2 \mu_m^2)^2 + 4\bar{K}_z^4 (\epsilon_m \mu_s - \epsilon_s \mu_m)^2 \times [\bar{K}_z^2 \epsilon_m \epsilon_s \mu_m \mu_s (\epsilon_m \mu_m + \epsilon_s \mu_s) + \epsilon_m^2 \epsilon_s^2 \mu_m^2 \mu_s^2] \}^{\frac{1}{2}} \quad (22b)$$

$$R_1 = R_2$$

$$\begin{aligned}
&= 8\bar{K}_z^8 [\mu_m^4 (\epsilon_s^4 - \epsilon_m^2 \epsilon_s^2) + \epsilon_m^4 \mu_s^2 (\mu_s^2 - \mu_m^2)] \\
&\quad - 16\bar{K}_z^6 \epsilon_m \epsilon_s \mu_m \mu_s (\epsilon_m \mu_m + 2\epsilon_s \mu_s) \\
&\quad \times (\epsilon_m^2 \mu_m^2 - \epsilon_m^2 \mu_s^2 + \epsilon_m \epsilon_s \mu_m \mu_s - \epsilon_s^2 \mu_m^2) \\
&\quad + 8\bar{K}_z^4 \epsilon_m^2 \epsilon_s^2 \mu_m^2 \mu_s^2 (-4\epsilon_m^2 \mu_m^2 + 3\epsilon_m^2 \mu_s^2 - 4\epsilon_m \epsilon_s \mu_m \mu_s \\
&\quad \quad + 3\epsilon_s^2 \mu_m^2 + 2\epsilon_s^2 \mu_s^2) \\
&\quad - 16\bar{K}_z^2 \epsilon_m^3 \epsilon_s^3 \mu_m^3 \mu_s^3 (\epsilon_m \mu_m - \epsilon_s \mu_s) \quad (22c)
\end{aligned}$$

$$S_1 = S_2 = 2\epsilon_m^2 \mu_m^2 (\bar{K}_z^2 + \epsilon_s \mu_s)^2. \quad (22d)$$

$$\begin{aligned}
P_3 &= P_4 \\
&= 2\bar{K}_z^4 [\epsilon_m^2 (\mu_s^2 - \mu_m^2) + \epsilon_s^2 (\mu_m^2 - \mu_s^2)] + 2\epsilon_m^2 \epsilon_s^2 \mu_m^2 \mu_s^2 \quad (22e)
\end{aligned}$$

$$\begin{aligned}
Q_3 &= Q_4 \\
&= \{ \bar{K}_z^8 (\epsilon_m \mu_s - \epsilon_s \mu_m)^4 + 4\bar{K}_z^4 (\epsilon_m \mu_s - \epsilon_s \mu_m)^2 \\
&\quad \times [\bar{K}_z^2 \epsilon_m \epsilon_s \mu_m \mu_s (\epsilon_m \mu_m + \epsilon_s \mu_s) + \epsilon_m^2 \epsilon_s^2 \mu_m^2 \mu_s^2] \}^{\frac{1}{2}} \quad (22f)
\end{aligned}$$

$$\begin{aligned}
R_3 &= R_4 \\
&= -8\bar{K}_z^8 (\epsilon_m^2 - \epsilon_s^2) (\mu_m^2 - \mu_s^2) (\epsilon_m \mu_s - \epsilon_s \mu_m)^2 \\
&\quad - 16\bar{K}_z^6 \epsilon_m \epsilon_s \mu_m \mu_s (\epsilon_m^2 - \epsilon_s^2) (\mu_m^2 - \mu_s^2) (\epsilon_m \mu_m - \epsilon_s \mu_s) \\
&\quad + 8\bar{K}_z^4 \epsilon_m^2 \epsilon_s^2 \mu_m^2 \mu_s^2 (-4\epsilon_m^2 \mu_m^2 + 3\epsilon_m^2 \mu_s^2 + 2\epsilon_m \epsilon_s \mu_m \mu_s \\
&\quad \quad + 3\epsilon_s^2 \mu_m^2 - 4\epsilon_s^2 \mu_s^2) \\
&\quad - 16\bar{K}_z^2 \epsilon_m^3 \epsilon_s^3 \mu_m^3 \mu_s^3 (\epsilon_m \mu_m - \epsilon_s \mu_s) \quad (22g)
\end{aligned}$$

$$S_3 = S_4 = 2 [\bar{K}_z^2 (\epsilon_m \mu_m - \epsilon_s \mu_s) + \epsilon_m \epsilon_s \mu_m \mu_s]^2. \quad (22h)$$

All the terms in (22) are real.

As is evident from (3), we note that $\epsilon_m > \epsilon_s$ and $\mu_m > \mu_s$ for nonnegative permittivities and permeabilities. It can then be shown that the terms within the square roots in (20) are negative. Subsequently, the magnitude of eigenvalues can be found as

$$|\lambda_{1,i}| = |\lambda_{2,i}| = \frac{\sqrt{P_1^2 + 4Q_1^2 - R_1}}{S_1}, \quad (23a)$$

$$|\lambda_{3,i}| = |\lambda_{4,i}| = \frac{\sqrt{P_3^2 + 4Q_3^2 - R_3}}{S_3}. \quad (23b)$$

for all eigenvalue indices $i = 1, 2, 3, 4$. Substituting the terms in (22) into (23) and upon some manipulations, we obtain

$$|\lambda_{1,i}| = |\lambda_{2,i}| = \frac{\sqrt{4\epsilon_m^4 \mu_m^4 (\bar{K}_z^2 + \epsilon_s \mu_s)^4}}{2\epsilon_m^2 \mu_m^2 (\bar{K}_z^2 + \epsilon_s \mu_s)^2} = 1 \quad (24a)$$

$$|\lambda_{3,i}| = |\lambda_{4,i}| = \frac{\sqrt{4 [\bar{K}_z^2 (\epsilon_m \mu_m - \epsilon_s \mu_s) + \epsilon_m \epsilon_s \mu_m \mu_s]^4}}{2 [\bar{K}_z^2 (\epsilon_m \mu_m - \epsilon_s \mu_s) + \epsilon_m \epsilon_s \mu_m \mu_s]^2} = 1 \quad (24b)$$

for all eigenvalue indices $i = 1, 2, 3, 4$. Hence, all eigenvalues have magnitudes of one regardless of the chosen time step Δt . Based on the von Neumann method, it would be commonly deduced that the MLOD CL-FDTD method is unconditionally stable for every set of split matrices. However, as to be shown in the next subsection, this common deduction may not always hold true if media inhomogeneity is present.

B. Two-Media Reduced-Matrix Stability Analysis

Recall that the stability analysis based on von Neumann method assumes homogeneous media when the Fourier modes

are applied to the field components. For more rigorous analysis, the transition matrix in the spatial domain should be considered, which includes media inhomogeneity. The transition matrix is expressed as

$$\bar{\mathbf{M}} = \left(\mathbf{I} - \frac{\Delta t}{2} \bar{\mathbf{B}} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \bar{\mathbf{B}} \right) \left(\mathbf{I} - \frac{\Delta t}{2} \bar{\mathbf{A}} \right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{2} \bar{\mathbf{A}} \right) \quad (25)$$

where the split matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are now expressed in the spatial domain. While the analysis using the full transition matrix is more rigorous, it is often too laborious to formulate and too complex to analyze due to huge matrix size. Determining the eigenvalues of a huge matrix for stability analysis is also very computationally intensive. To reduce the complexity, we propose the reduced transition matrix for a smaller domain comprising two inhomogeneous media. Note that in the minimal case at least two media will be required to exhibit inhomogeneity, which will be useful for stability analysis. The two-media reduced-matrix allows us to analyze efficiently the key stability characteristics in inhomogeneous media, and is useful for quick detection of any potential instability. In addition, for the simplest case of reduced matrix, a two-cell domain is to be utilized. The two-cell domain is judiciously terminated with perfect electric and magnetic conducting boundary conditions on the left and right boundaries respectively. This ensures the number of electric and magnetic fields within the domain is equal. Then the field vector components can be arranged as

$$\bar{\mathbf{u}} = [E_{x1}|_1, E_{x2}|_1, H_{y1}|_1, H_{y2}|_1, E_{x1}|_2, E_{x2}|_2, H_{y1}|_2, H_{y2}|_2]^T \quad (26)$$

where $E_{x1}|_k$, $E_{x2}|_k$, $H_{y1}|_k$ and $H_{y2}|_k$ are the electric and magnetic fields of lines 1 and 2 located at the k -th cell, with $k = 1, 2$. Based on the field arrangement in (26), the 2nd set of split matrices can be expressed in the spatial domain for exemplification as

$$\bar{\mathbf{A}}_2 = \frac{1}{\Delta z} \begin{bmatrix} 0 & 0 & \frac{1}{\epsilon_s|_1} & \frac{1}{\epsilon_m|_1} & 0 & 0 & \frac{-1}{\epsilon_s|_1} & \frac{-1}{\epsilon_m|_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\mu_s|_1} & \frac{1}{\mu_m|_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon_s|_2} & \frac{1}{\epsilon_m|_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\mu_s|_2} & \frac{-1}{\mu_m|_2} & 0 & 0 & \frac{-1}{\mu_s|_2} & \frac{1}{\mu_m|_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (27a)$$

$$\bar{\mathbf{B}}_2 = \frac{1}{\Delta z} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon_m|_1} & \frac{1}{\epsilon_s|_1} & 0 & 0 & \frac{-1}{\epsilon_m|_1} & \frac{-1}{\epsilon_s|_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\mu_m|_1} & \frac{-1}{\mu_s|_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon_m|_2} & \frac{1}{\epsilon_s|_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\mu_m|_2} & \frac{1}{\mu_s|_2} & 0 & 0 & \frac{1}{\mu_m|_2} & \frac{-1}{\mu_s|_2} & 0 & 0 \end{bmatrix} \quad (27b)$$

Fig. 2. Eigenvalues of two-media reduced-matrices for various split matrices and CFLNs around unit semicircle.

where $\epsilon_s|_k$, $\epsilon_m|_k$, $\mu_s|_k$ and $\mu_m|_k$ are the permittivities and permeabilities located at the k -th cell and Δz is the cell size. Other sets of split matrices in the spatial domain can also be written down in a similar fashion. Note that the upper and lower block diagonal parts in (27a) and (27b) [highlighted in boxes] correspond to the self finite-difference terms. They are consistent with the split matrices in (8a) and (8b) at media 1 and 2, apart from some sign changes due to finite-differencing operated on different cells. The off-diagonal parts in (27a) and (27b) correspond to the neighboring finite-difference terms. Notice that the field arrangement in (26) is consistent with (2a), and one could also easily extend the split matrices in the spatial domain to more than two cells.

The eigenvalues of the two-media reduced-matrices are to be computed next. To that end, the two media parameters are set as $\epsilon_s|_1 = 4.39\epsilon_0$, $\epsilon_m|_1 = 9.05\epsilon_0$, $\mu_s|_1 = 0.40\mu_0$, $\mu_m|_1 = 0.73\mu_0$, $\epsilon_s|_2 = 5.52\epsilon_0$, $\epsilon_m|_2 = 139.34\epsilon_0$, $\mu_s|_2 = 0.46\mu_0$, $\mu_m|_2 = 10.28\mu_0$, where ϵ_0 and μ_0 are the free space permittivity and permeability. The media parameters are selected such that there is sufficient inhomogeneity contrast in the mutual permittivities and permeabilities between the two media. The time step is specified as $CFLN = \Delta t / \Delta t_{CFL}$, where Δt_{CFL} is the CFL time step in [4]. Fig. 2 plots the eigenvalues of reduced-matrices for various split matrices and CFLNs around a unit semicircle. It is observed that the two-media reduced-matrices for the 1st and 2nd sets of split matrices have the same eigenvalues, and similarly for the 3rd and 4th sets of split matrices. Moreover, all eigenvalues of the 1st and 2nd sets of split matrices lie on the unit semicircle for $CFLN = 1$ and 10, thus implying stability. However, while the eigenvalues of the 3rd and 4th sets of split matrices lie on the unit semicircle for $CFLN = 1$, there are eigenvalues lying outside for $CFLN = 10$, implying that the method is not unconditionally stable with these split matrices. This finding is in contrary to the von Neumann stability analysis in the Fourier domain used for homogeneous media earlier, i.e. the eigenvalues of Fourier amplification matrix for all split matrices have unity magnitudes [c.f. (24)]. These results indicates that the stability analysis based on von Neumann method alone may not be sufficient when media inhomogeneity is present. Here, the two-media reduced-matrix stability analysis is useful for quick detection of any potential instability that is not apparent via the von Neumann method.

To investigate further the stability characteristics with variation of media parameters, we shall modify the mutual permittivity and permeability of medium 2 in relation to that of

Fig. 3. Maximum absolute eigenvalues of reduced-matrices for various split matrices and CFLNs versus m in (28).

medium 1 according to

$$\epsilon_m|_2 = \epsilon_m|_1 + m \times 0.25\epsilon_0 \quad (28a)$$

$$\mu_m|_2 = \mu_m|_1 + m \times 0.25\mu_0 \quad (28b)$$

where $m \geq 0$. When $m = 0$, it means a homogeneous medium while larger m indicates that the inhomogeneity contrast between $\epsilon_m|_2$ and $\epsilon_m|_1$ is greater. Fig. 3 plots the maximum absolute eigenvalues of reduced-matrices for various split matrices and CFLNs versus m in (28). We notice that for $CFLN = 1$, the maximum absolute eigenvalues of the 3rd and 4th sets of split matrices remain one (stable) as m increases. However, for $CFLN = 10$ and 20, the maximum absolute eigenvalues begin to increase beyond one and thus the method becomes unstable after certain m . Furthermore, for $CFLN = 20$, the 3rd and 4th sets of split matrices appear to exhibit instability relatively faster compared to $CFLN = 10$, i.e., maximum absolute eigenvalues increase beyond one after smaller m . For the 1st and 2nd sets of split matrices, the maximum absolute eigenvalues remain one for various m and CFLNs. (We also carried out exhaustive tests for various media parameters and CFLNs, they remain stable.) Based on the two-media reduced-matrix stability analysis, it can be concluded that the MLOD CL-FDTD method with the 3rd and 4th sets of split matrices may not be stable after all for inhomogeneous coupled transmission lines, especially those with great contrast between media parameters $\epsilon_m|_2$ and $\epsilon_m|_1$. The instability may be attributed to the fact that the LHS of (13) and (17) of the implicit field update equations (the coefficients of tridiagonal matrices) consist of mutual permittivities and permeabilities terms. These mutual terms introduce fields that are coupled from the other line and thus could be more likely to cause instability. On the other hand, for the 1st and 2nd sets of split matrices, their LHS in (7) and (10) consist of only self terms and they maintain stability for inhomogeneous media.

IV. NUMERICAL RESULTS

We first verify the stability of the MLOD CL-FDTD method for various split matrices with numerical simulations. For consistency, only implicit update equations with tridiagonal LHS involving electric fields are considered. Hence, the 4th set of split matrices is excluded here. The electric field

Fig. 4. Time domain E field at the center of line 2 for homogeneous coupled transmission lines computed using MLOD CL-FDTD method with (a) 1st, (b) 2nd and (c) 3rd sets of split matrices at CFLN = 4, 10. Inset shows the schematic of coupled transmission lines with short circuit terminations.

is computed for homogeneous and inhomogeneous coupled transmission lines. The schematic of the coupled transmission lines with short circuit terminations is shown in inset of Fig. 4.

For homogeneous coupled transmission lines, the parameters are set to $\epsilon_s|_1$, $\epsilon_m|_1$, $\mu_s|_1$ and $\mu_m|_1$ throughout the whole line. They correspond to those used in the previous section. Time step is set as CFLN = 4 and 10. Fig. 4 plots the time domain E field at the center of line 2 for homogeneous coupled transmission lines computed using MLOD CL-FDTD method with (a) 1st, (b) 2nd and (c) 3rd sets of split matrices at CFLN = 4, 10. It can be seen that the E fields for the 1st, 2nd and 3rd sets of split matrices are all bounded for homogeneous coupled transmission lines. For inhomogeneous coupled transmission lines, a half-space medium is chosen where the parameters are set to $\epsilon_s|_1$, $\epsilon_m|_1$, $\mu_s|_1$, $\mu_m|_1$ and $\epsilon_s|_2$, $\epsilon_m|_2$, $\mu_s|_2$, $\mu_m|_2$ divided at the center of the line. The parameters are also the same as those used in the previous section. The time domain E fields for various split matrices are plotted in Fig. 5. For inhomogeneous coupled transmission lines, it is observed that the E field for the 1st and 2nd sets of split matrices remains bounded for CFLN = 4, 10. However, the E field for the 3rd set of split matrices goes unbounded after certain time for CFLN = 10. Hence, the numerical results here are in consistent with the conclusions inferred from the von Neumann and two-media reduced-matrix stability analyses. Since there exists potential instability for the 3rd and 4th sets of split matrices, they shall be excluded in the subsequent analysis for inhomogeneous coupled transmission lines.

Fig. 5. Time domain E field at the center of line 2 for inhomogeneous coupled transmission lines computed using MLOD CL-FDTD method with (a) 1st, (b) 2nd and (c) 3rd sets of split matrices at CFLN = 4, 10.

Fig. 6. Time domain E field at (a) port 2 and (b) port 3 of tapered coupled lines computed using various methods.

To validate the accuracy of the MLOD CL-FDTD method, we shall consider an example of tapered coupled lines having the same schematic as in Fig. 1. Note that in our proposed MLOD CL-FDTD method, the complete 3-D discretizations of the coupled lines are not required. Only the discretizations in the z -direction, along with Z_{0e} , Z_{0o} , ϵ_{eff}^e and ϵ_{eff}^o are needed. In this case, Z_{0e} and Z_{0o} taper linearly from 100 to 50 Ω and 25 to 50 Ω , respectively, while $\epsilon_{eff}^e = \epsilon_{eff}^o = 2.95$. For common coupled microstrip or strip lines, the physical dimensions such as line width and spacing can be converted to even- and odd-mode parameters Z_{0e} , Z_{0o} , ϵ_{eff}^e and ϵ_{eff}^o , or vice-versa, using some commercial software tools, e.g. Advanced Design System (ADS). The length of the coupled lines is one wavelength, $l = 1\lambda$, at 1 GHz. The reference characteristic impedance Z_0 is 50 Ω . The cell size Δz is set as 0.8735 mm. For source excitation, it is a vertical electric

current density J_x with modulated Gaussian pulse given as

$$J_x = 50 \sin \{2\pi f_c(t - t_d)\} e^{\left(\frac{t-t_d}{\tau}\right)^2} \quad (29)$$

where $f_c = 1$ GHz, $\tau = 1$ ns, $t_d = 3\tau$. The pulse is injected from port 1 and the time domain E fields at ports 2 and 3 are recorded. Fig. 6(a) shows the time domain E field at port 2 of the tapered coupled lines computed using various methods. The fields computed by using MLOD CL-FDTD method with the 1st and 2nd sets of split matrices are compared with those of M1-D ADI-FDTD and the M1-D explicit FDTD methods. Good agreement is observed even at high CFLN = 10 and 20, thus validating the proposed method. Fig. 6(b) shows the time domain E field at port 3 of the tapered coupled lines. Again, good agreement is observed among the methods. Furthermore, it can be observed that the field at port 3 is mainly coupled from line 1 while the field at port 2 directly propagates from input port 1. At CFLN = 20, the MLOD CL-FDTD method runs around 4.5 times faster than the M1-D explicit FDTD method. The time step should be chosen to maintain acceptable accuracy. Generally, it should not exceed CFLN = 20. The optimal CFLN is chosen by considering the trade-off between accuracy and efficiency. From this example, it is shown that the time domain response of tapered coupled lines can be simulated using the proposed MLOD CL-FDTD method at time step larger than the CFL stability limit with good accuracy.

V. CONCLUSION

This paper has presented the MLOD CL-FDTD method for inhomogeneous coupled transmission lines and stability analyses. Various split matrices have been proposed, and the corresponding update equations have been formulated and discussed to yield implicit update equations with tridiagonal matrices on the LHS. The stability analysis has been first performed using the von Neumann method in the Fourier domain. It has been shown that the von Neumann method alone may not be sufficient to ascertain stability for inhomogeneous media. To include media inhomogeneity, the two-media reduced-matrix stability analysis has been proposed. It allows us to analyze efficiently the key stability characteristics in inhomogeneous media, and is useful for quick detection of any potential instability that is not apparent via the von Neumann method. Numerical results have further validated the stability and accuracy of the proposed MLOD CL-FDTD method with various split matrices.

Thus far, the even- and odd-mode approach has been applied for symmetric coupled transmission lines in this paper. For asymmetric coupled transmission lines, the even and odd-mode parameters can no longer be utilized. However, this could be circumvented by introducing different self parameters (e.g. ϵ_{s1} , ϵ_{s2} , μ_{s1} , μ_{s2}) for lines 1 and 2 of the asymmetric lines. The work herein can be extended for asymmetric and even multiple coupled transmission lines through proper formulation of stable and efficient split matrices.

REFERENCES

- [1] K. S. Yee, "Numerical solution of initial boundary value problems involving maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. 14, no. 3, pp. 302-307, May 1966.
- [2] A. Taflov and S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, Boston, M. A.: Artech House, 2005.
- [3] Z. Yang and E. L. Tan, "Microstrip circuit tool kit app with FDTD analysis including lumped elements," *IEEE Microw. Mag.*, vol. 16, no. 1, pp. 74-80, Feb. 2015.
- [4] Z. Yang and E. L. Tan, "Multiple one-dimensional FDTD method for coupled transmission lines and stability condition," *IEEE Microw. Wireless Compon. Lett.*, vol. 26, no. 11, pp. 864-866, Nov. 2016.
- [5] D. Y. Heh and E. L. Tan, "Unconditionally stable multiple one-dimensional ADI-FDTD method for coupled transmission lines," *IEEE Trans. Antennas Propagat.*, vol. 66, no. 12, pp. 7488-7492, Dec. 2018.
- [6] F. Zheng, Z. Chen and J. Zhang, "Toward the development of a three-dimensional unconditionally stable finite-difference time-domain method," *IEEE Trans. Microw. Theory Tech.*, vol. 48, no. 9, pp. 1550-1558, Sep. 2000.
- [7] T. Namiki, "3-D ADI-FDTD method- unconditionally stable time-domain algorithm for solving full vector Maxwell's equations," *IEEE Trans. Microw. Theory Tech.*, vol. 48, no. 10, pp. 1743-1748, Oct. 2000.
- [8] E. L. Tan, "Fundamental schemes for efficient unconditionally stable implicit finite-difference time-domain methods," *IEEE Trans. Antennas Propagat.*, vol. 56, no. 1, pp. 170-177, Jan. 2008.
- [9] W. Fu and E. L. Tan, "Development of split-step FDTD method with higher order spatial accuracy," *Electron. Lett.*, vol. 40, no. 20, pp. 1252-1254, Sep. 2004.
- [10] J. Shibayama, M. Muraki, J. Yamauchi, and H. Nakano, "Efficient implicit FDTD algorithm based on locally one-dimensional scheme," *Electron. Lett.*, vol. 41, no. 19, pp. 1046-1047, Sept. 2005.
- [11] E. L. Tan, "Unconditionally stable LOD-FDTD method for 3-D Maxwell's equations," *IEEE Microw. Wireless Compon. Lett.*, vol. 17, no. 2, pp. 85-87, Feb. 2007.
- [12] D. Y. Heh and E. L. Tan, "Numerical stability analysis of M1-D LOD-FDTD method for inhomogeneous coupled transmission lines," in *2019 IEEE Int. Symp. Antennas Propag. and USNC-URSI Rad. Sci. Meeting, AP-S/URSI 2019*, Georgia, USA, Jul. 2019.
- [13] R. Holland, "Implicit three-dimensional finite differencing of Maxwell's equations," *IEEE Trans. Nucl. Sci.*, vol. NS-31, pp. 1322-1326, 1984.
- [14] C. R. Paul, *Analysis of Multiconductor Transmission Lines*, 2nd ed., New York, NY, USA: Wiley, 2008.
- [15] R. K. Mongia, I. J. Bahl, P. Bhartia, and J. Hong, *RF and Microwave Coupled-Line Circuits*, 2nd ed., Norwood, MA, USA: Artech House, 2007.