Rhaly operators on small weighted hardy spaces

Lin, Tan Pin; Doan, Minh Luan; Khoi, Le Hai

2018

Lin, T. P., Doan, M. L., & Khoi, L. H. (2018). Rhaly operators on small weighted hardy spaces. *Acta Math Vietnam*, 43, 749-759. doi:10.1007/s40306-018-0277-9

<https://hdl.handle.net/10356/137246>

<https://doi.org/10.1007/s40306-018-0277-9>

This is a post-peer-review, pre-copyedit version of an article published in *Acta Mathematica Vietnamica*. The final authenticated version is available online at:
<http://dx.doi.org/10.1007/s40306-018-0277-9>

Downloaded on 26 Apr 2023 17:01:05 SGT

RHALY OPERATORS ON SMALL WEIGHTED HARDY SPACES

TAN PIN LIN, MINH LUAN DOAN & LE HAI KHOI

ABSTRACT. We study several properties of Rhaly operators on small weighted Hardy spaces of holomorphic functions in the unit disc $\mathbb{D}$. In particular, we obtain criteria for boundedness, compactness, Hilbert-Schmidt norm and p -Schatten class membership of such operators. A closedness of Rhaly operator is also studied.

1. INTRODUCTION

Consider a sequence $\mathbf{c} = (c_n) \subseteq \mathbb{C}$, not necessarily summable. Let $s_n(\mathbf{c}) = \sum_{k=0}^n c_k$ be the n th partial sum of entries of $\mathbf{c}$. The sequence $\mathbf{c}$ is said to be *Cesàro summable* if the sequence

$$C(\mathbf{c}) = \left(\frac{s_n}{n+1} \right)_{n=0}^{\infty}$$

is summable. Note that the n th entry of $C(\mathbf{c})$ is the arithmetic mean of the first n entries of $\mathbf{c}$. The *Cesàro operator* C defined above is known to be bounded on the sequence space ℓ^2 of square-summable sequences. It was first introduced in [1].

In [8] Rhaly extended the study of Cesàro operator on ℓ^2 by defining a new operator, which he called *p -Cesàro matrices*

$$C_p = \begin{bmatrix} \frac{1}{1^p} & 0 & 0 & 0 & \cdots \\ \frac{1}{2^p} & \frac{1}{2^p} & 0 & 0 & \cdots \\ \frac{1}{3^p} & \frac{1}{3^p} & \frac{1}{3^p} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is proved that the operator C_p is invariant and bounded on ℓ^2 for any $p \geq 1$, and that when $p > 1$, C_p is Hilbert-Schmidt (and hence is compact), is not hyponormal (and hence is not subnormal). Some nice properties of the spectral properties C_p and estimated the spectral radius are provided.

Date: April 14, 2018.

2010 Mathematics Subject Classification. 30 D15, 47 B37.

Key words and phrases. Hilbert space, entire function, Rhaly operator.

Supported in part by MOE's AcRF Tier 1 grants M4011166.110 (RG24/13) and M4011724.110 (RG128/16).

Rahly also remarked that some techniques for C_p could be applied to any *terraced matrix* $R_{\mathbf{a}}$, which is the lower triangular matrix with constant row-segments

$$R_{\mathbf{a}} = \begin{bmatrix} a_0 & 0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & 0 & \dots \\ a_2 & a_2 & a_2 & 0 & \dots \\ a_3 & a_3 & a_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $\mathbf{a} = (a_n)_{n \geq 0}$ is a sequence of complex numbers.

Terraced matrices (also known as *Rhaly matrix* $R_{\mathbf{a}}$), therefore, are a generalization of p -Cesàro matrices, and the name Rhaly matrices came from [5].

The action of Rhaly matrices on sequence spaces ℓ^p and other sequence spaces is studied in many papers (see, e.g., [5, 8, 9] and the references therein). However, as far as we know, the boundedness of $R_{\mathbf{a}}$ on ℓ^p , or even in the Hilbert space ℓ^2 , is still open. In fact the problem whether $R_{\mathbf{a}}$ maps from ℓ^2 to itself remains unanswered.

In addition, there has been little research on Rhaly matrices acting on function spaces.

The aim of this paper is to study several properties of Rhaly operators acting on some classes of holomorphic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. This requires rigorous definitions of both the function spaces and the operators.

In a general context, we suppose $\mathbf{a} = (a_n) \subset \mathbb{C}$ is a sequence of complex numbers, and define the *Rhaly operator induced by $\mathbf{a}$* as

$$R_{\mathbf{a}}(f)(z) = \sum_{n=0}^{\infty} \left(a_n \sum_{j=0}^n c_j \right) z^n,$$

where $f(z) = \sum_{n=0}^{\infty} c_n z^n$ represents a formal power series (i.e. f needs not be convergent), and $(c_n) \subset \mathbb{C}$.

We study the relation between the sequence $\mathbf{a}$ and the boundedness, compactness and Schatten class properties of $R_{\mathbf{a}}$ on a specific class of Hilbert spaces of holomorphic functions on $\mathbb{D}$, namely the small weighted Hardy spaces.

The structure of the paper is as follows. Section 2 provides preliminary results on weighted Hardy spaces $H(\beta)$. In Section 3, criteria on invariance, boundedness, compactness, Hilbert-Schmidt norm, and Schatten class membership of Rhaly operator R_a are proved. Section 4 studies closedness of the range of R_a .

2. SMALL WEIGHTED HARDY SPACES AND REPRODUCING KERNEL

A Hilbert space $\mathcal{H}$ of functions from a non-empty set X to $\mathbb{C}$ is called a *reproducing kernel Hilbert space* (RKHS) if for every $y \in X$, the evaluation functional $\delta_y : f \mapsto f(y)$ is bounded. By the Riesz Representation Theorem, there exists a unique element $K_y \in \mathcal{H}$ such that $f(y) = \langle f, K_y \rangle$ for every $f \in \mathcal{H}$. We call K_y the *reproducing kernel at the point y* . The function $K : X \times X \rightarrow \mathbb{C}$ defined by

$$K(x, y) = \langle K_y, K_x \rangle = K_y(x), \quad x, y \in X$$

is called the *reproducing kernel for $\mathcal{H}$* . It is well known that if a collection of elements $\{e_j : j \in J\}$ is an orthonormal basis for $\mathcal{H}$, then

$$K(x, y) = \sum_{j \in J} e_j(x) \bar{e}_j(y),$$

where the convergence is pointwise for $x, y \in X$.

Now to each sequence of positive real numbers $\beta = (\beta_n)$ we associate the set of complex sequences

$$(1) \quad \ell_\beta^2 = \left\{ \mathbf{c} = (c_n) : \|\mathbf{c}\| = \left(\sum_{n=0}^{\infty} |c_n|^2 \beta_n^2 \right)^{1/2} < +\infty \right\},$$

which is a Hilbert space with the inner product

$$\langle \mathbf{b}, \mathbf{c} \rangle = \sum_{n=0}^{\infty} b_n \bar{c}_n \beta_n^2, \quad \mathbf{b} = (b_n), \mathbf{c} = (c_n) \in \ell_\beta^2.$$

This type of sequence spaces has many important applications in studying operators on function spaces. We refer the reader to [2] for more detailed information.

As it is well-known by Cauchy–Hadamard theorem, a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ represents a homomorphic function on $\mathbb{D}$ if f has radius of convergence at least 1, i.e.,

$$(2) \quad \limsup_{n \rightarrow \infty} |c_n|^{1/n} \leq 1.$$

If we denote by ℓ_D the set of all complex sequences (c_n) that satisfy condition (2), then ℓ_D is a complex vector space.

To obtain a Hilbert space of analytic functions on $\mathbb{D}$, it is natural to consider all coefficient sequences (a_n) in the intersection $\ell_\beta^2 \cap \ell_D$. In particular, the function space

$$(3) \quad \mathcal{H}(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n, (a_n) \in \ell_\beta^2 \cap \ell_D \right\}$$

is an inner product space, equipped with norm $\|f\|_{\mathcal{H}(\beta)} = \|(a_n)\|$. Without any ambiguity we shall denote $\|f\|$ in place of $\|f\|_{\mathcal{H}(\beta)}$

Depending on (β_n) , the space $\ell_\beta^2 \cap \ell_D$ may not be complete in norm (1), so $\mathcal{H}(\beta)$ is not necessarily complete, and thus not a Hilbert space. The following lemma shows the relationship between ℓ_D and ℓ_β^2 , whose proof is quite analogous to the case of entire functions in [3].

Lemma 1. *Let $\beta_* = \liminf_{n \rightarrow \infty} (\beta_n)^{1/n}$ and $\beta^* = \limsup_{n \rightarrow \infty} (\beta_n)^{1/n}$. Exactly one of the following cases can happen:*

- (i) $\ell_\beta^2 \subsetneq \ell_D$ if and only if $\beta_* \geq 1$.
- (ii) $\ell_D \subsetneq \ell_\beta^2$ if and only if $\beta^* < 1$.
- (iii) $\ell_D \setminus \ell_\beta^2 \neq \emptyset$ and $\ell_\beta^2 \setminus \ell_D \neq \emptyset$ if and only if $\beta_* < 1 \leq \beta^*$.

Proof. (i). Suppose $\ell_\beta^2 \subsetneq \ell_D$, then every $\mathbf{c} = (c_n) \in \ell_\beta^2$ generates a holomorphic function on $\mathbb{D}$.

Consider the sequence $\mathbf{b} = (b_n)$ where $b_n = ((n+1)\beta_n)^{-1}$. We can easily verify that $\mathbf{b} \in \ell_\beta^2$ and so by (2), we have

$$\begin{aligned} 1 &\geq \limsup_{n \rightarrow \infty} |b_n|^{1/n} = \limsup_{n \rightarrow \infty} \left(\frac{1}{(n+1)\beta_n} \right)^{1/n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right)^{1/n} \cdot \frac{1}{\liminf_{n \rightarrow \infty} (\beta_n)^{1/n}} = 1 \cdot \frac{1}{\beta_*}, \end{aligned}$$

which shows that $\beta_* \geq 1$.

Conversely, suppose $\beta_* \geq 1$. For any $\mathbf{c} \in \ell_\beta^2$, we note that $\|\mathbf{c}\|^2 < +\infty$ and so $|c_n|\beta_n < 1$ for all n large enough. As a result,

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{1}{(\beta_n)^{1/n}} = \frac{1}{\liminf_{n \rightarrow \infty} (\beta_n)^{1/n}} = \frac{1}{\beta_*} \leq 1.$$

Thus $\mathbf{c}$ satisfies (2), and by Cauchy–Hadamard theorem, the power series $\sum_{n=0}^{\infty} c_n z^n$ represents a holomorphic on $\mathbb{D}$. This shows that $\ell_\beta^2 \subseteq \ell_D$.

Consider the sequence $\mathbf{d} = (d_n)$ where $d_n = \beta_n^{-1}$. We can easily check that $\mathbf{d} \in \ell_D \setminus \ell_\beta^2$, so $\ell_\beta^2 \subsetneq \ell_D$.

(ii). Suppose $\ell_D \subsetneq \ell_\beta^2$. Assume in contrary that $\beta^* \geq 1$. Then either $\beta^* > 1$ or $\beta^* = 1$.

If $\beta^* = 1$, then there is a subsequence (β_{n_k}) such that $\lim_{k \rightarrow \infty} (\beta_{n_k})^{1/n_k} = 1$. Define $\mathbf{d} = (d_n)$, where

$$d_n = \begin{cases} \beta_{n_k}^{-1} & \text{if } n = n_k \ (k = 0, 1, 2, \dots), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\limsup_{n \rightarrow \infty} |d_n|^{1/n} = \lim_{k \rightarrow \infty} (\beta_{n_k})^{-1/n_k} = 1$, and so $\mathbf{d} \in \ell_D$. However, $|d_{n_k}|\beta_{n_k} = 1 \neq 0$, for all $k \geq 0$, which shows that $\mathbf{d} \notin \ell_\beta^2$. This contradicts $\ell_D \subsetneq \ell_\beta^2$.

If $\beta^* > 1$, then there is a subsequence (β_{n_k}) such that $(\beta_{n_k})^{1/n_k} > r > 1$ for some r , which gives $\beta_{n_k} > 1$, for all $k \geq 0$. Consider $\mathbf{e} = (e_n)$, where $e_n = 1$ for all $n \geq 0$. It is clear that $\mathbf{e} \in \ell_D$, and since $\ell_D \subsetneq \ell_\beta^2$,

$\|\mathbf{e}\|^2 = \sum_{n=0}^{\infty} \beta_n^2 < +\infty$. But it is impossible, because (β_{n_k}) cannot have the zero limit as $k \rightarrow \infty$.

Conversely, suppose $\beta^* < 1$. Then there is $r \in (0, 1)$ and $N_1 \in \mathbb{N}$ such that $(\beta_n)^{1/n} < r$ for all $n > N_1$. Take arbitrarily $\mathbf{c} = (c_n) \in \ell_D$. By (2), we can find $N_2 \in \mathbb{N}$ so that $|c_n|^{1/n} \leq 1$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$, we then have

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n|^2 \beta_n^2 &= \sum_{n=0}^N |c_n|^2 \beta_n^2 + \sum_{n=N+1}^{\infty} |c_n|^2 \beta_n^2 \\ &\leq \sum_{n=0}^N |c_n|^2 \beta_n^2 + \sum_{n=N+1}^{\infty} r^{2n} = \sum_{n=0}^N |c_n|^2 \beta_n^2 + \frac{r^{2N+2}}{1-r^2} < +\infty, \end{aligned}$$

which shows that $\mathbf{c} \in \ell_\beta^2$.

Now consider $\mathbf{b} = (b_n)$, where $b_n = ((n+1)\beta_n)^{-1}$. Then $\mathbf{b} \in \ell_\beta^2$, and we also have

$$\begin{aligned} \limsup_{n \rightarrow \infty} |b_n|^{1/n} &\geq \liminf_{n \rightarrow \infty} |b_n|^{1/n} = \liminf_{n \rightarrow \infty} \left(\frac{1}{(n+1)\beta_n} \right)^{1/n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right)^{1/n} \cdot \frac{1}{\limsup_{n \rightarrow \infty} (\beta_n)^{1/n}} = 1 \cdot \frac{1}{\beta^*} > 1. \end{aligned}$$

Hence $\mathbf{b} \in \ell_\beta^2 \setminus \ell_D$, and so $\ell_D \subsetneq \ell_\beta^2$.

(iii). It follows from (i) and (ii). $\square$

Using Lemma 1, we obtain the following theorem which characterizes for which sequence β the space $\mathcal{H}(\beta)$ is complete in its norm.

Theorem 2. *The function space $\mathcal{H}(\beta)$ defined in (3) is a Hilbert space of holomorphic functions on the unit disc $\mathbb{D}$ if and only if*

$$(4) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

Moreover, if (4) is satisfied, then $\mathcal{H}(\beta)$ is a reproducing kernel Hilbert space with the kernel $K : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ defined as

$$K(z, w) = K_w(z) = \sum_{n=0}^{\infty} \frac{\overline{w}^n z^n}{\beta_n^2}, \quad z, w \in \mathbb{D}.$$

The space $\mathcal{H}(\beta)$ induced by the weight β that satisfies (4) is known as a *weighted Hardy space*. This name comes from the observation that if $\beta_n \equiv 1$, then $\mathcal{H}(\beta)$ becomes the classical Hardy space $H^2(\mathbb{D})$.

A noteworthy class of the weight sequences is where a sequence (β_n) satisfies

$$(5) \quad \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < +\infty.$$

It is easy to verify that condition (5) is stronger than (4), but the inverse is not true in general. For instance, the weight of the classical Hardy space $\beta_n \equiv 1$ satisfies (4) but not (5).

Any space $\mathcal{H}(\beta)$ with β satisfying (5) is called a *small weighted Hardy space* (see, e.g. [6]). One important properties of these spaces $\mathcal{H}(\beta)$ is that functions in $\mathcal{H}(\beta)$ extend continuously to $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Example 3. The space $\mathcal{H}(\beta)$ is a Hilbert space of entire functions if and only if $\beta_* = +\infty$ (see, e.g. [3]). In particular, $\beta_* > 1$. Clearly any sequence (β_n) that satisfies $\beta_* > 1$ also satisfies $\sum_{n=0}^{\infty} \beta_n^{-2} < +\infty$. This means that the induced $\mathcal{H}(\beta)$ space is a small weighted Hardy space.

The following example is also worth to note.

Remark 4. The Fock space $\mathcal{F}^2(\mathbb{C})$ is a specific example of a Hilbert space $\mathcal{H}(\beta)$ of entire functions with weight $\beta_n = \sqrt{n!}$.

3. INVARIANCE, BOUNDEDNESS AND COMPACTNESS OF $R_{\mathbf{a}}$

In this section, we limit our study of Rhaly operators to small weighted Hardy spaces. We establish some properties such as invariance, boundedness and compactness of Rhaly operators acting on any small weighted Hardy spaces.

In the sequel, $\beta = (\beta_n)$ is a given weight sequence that satisfies (5) and $\mathcal{H}(\beta)$ be the corresponding small weighted Hardy space. We keep in mind, from the previous section, that the relation $\ell_\beta^2 \subsetneq \ell_D$ holds. Also let us define the following positive constant

$$(6) \quad C_\beta = \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < +\infty.$$

Let $\mathbf{a} = (a_n) \subset \mathbb{C}$ and $R_{\mathbf{a}}$ be the Rhaly operator induced by $\mathbf{a}$. We call $\mathbf{a}$ the *symbol* that induces $R_{\mathbf{a}}$.

3.1. Invariance and boundedness.

3.1.1. Invariance. A linear operator T acting on a space H is said to be *invariant* if T maps H into itself, i.e. $T(H) \subseteq H$.

Let $\mathcal{O}(\mathbb{D})$ be the space of all holomorphic functions in the unit disc. We are first interested in the conditions for which $R_{\mathbf{a}}$ is invariant on $\mathcal{O}(\mathbb{D})$.

We have the following result.

Proposition 5. *The Rhalý operator $R_{\mathbf{a}}$ is invariant on $\mathcal{O}(\mathbb{D})$ if and only if $\mathbf{a} \in \ell_D$.*

Proof. Necessity. Suppose $R_{\mathbf{a}}$ is invariant on $\mathcal{O}(\mathbb{D})$. This implies, in particular, that for the constant function $\mathbf{1}(z) \equiv 1$ from $\mathcal{O}(\mathbb{D})$, we have

$$R_{\mathbf{a}}(\mathbf{1})(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(\mathbb{D}),$$

which shows that $\mathbf{a} \in \ell_D$.

Sufficiency. Suppose $\mathbf{a} \in \ell_D$, i.e., $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(\mathbb{D})$. This gives

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq 1.$$

Take arbitrarily $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{O}(\mathbb{D})$. Since

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} \leq 1,$$

for every $r > 1$, there exists $M_r > 0$ such that for any $n \in \mathbb{N}$,

$$|c_n| \leq M_r r^n.$$

This implies that

$$\begin{aligned} \left| a_n \sum_{j=0}^n c_j \right|^{1/n} &\leq \left(|a_n| \sum_{j=0}^n |c_j| \right)^{1/n} \leq \left(|a_n| M_r \sum_{j=0}^n r^j \right)^{1/n} \\ &= \left(|a_n| M_r \frac{r^{n+1} - 1}{r - 1} \right)^{1/n} < \left(\frac{|a_n| M_r r}{r - 1} \right)^{1/n} r. \end{aligned}$$

Hence, for every $r > 1$,

$$\limsup_{n \rightarrow \infty} \left| a_n \sum_{j=0}^n c_j \right|^{1/n} \leq \limsup_{n \rightarrow \infty} \left(\frac{|a_n| M_r r}{r - 1} \right)^{1/n} r \leq r.$$

Consequently, due to arbitrariness of $r > 1$, we get

$$\limsup_{n \rightarrow \infty} \left| a_n \sum_{j=0}^n c_j \right|^{1/n} \leq 1,$$

and hence $R_{\mathbf{a}} f \in \mathcal{O}(\mathbb{D})$. □

The invariance of $R_{\mathbf{a}}$ on the space $\mathcal{H}(\beta)$ behaves in a similar manner.

Proposition 6. *The Rhalý operator $R_{\mathbf{a}}$ is invariant on $\mathcal{H}(\beta)$ if and only if $\mathbf{a} \in \ell_{\beta}^2 \subsetneq \ell_D$.*

Proof. Necessity. Suppose $R_{\mathbf{a}}$ is invariant on $\mathcal{H}(\beta)$. Then $R_{\mathbf{a}}(\mathbf{1})(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta)$, which gives $\mathbf{a} \in \ell_{\beta}^2$.

Sufficiency. Suppose that $\mathbf{a} \in \ell_{\beta}^2$. Take arbitrarily $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{H}(\beta)$. Then for each $n \in \mathbb{N}$, by Cauchy-Schwarz's inequality, recalling the constant C_{β} defined in (6), we have

$$\left| \sum_{j=0}^n c_j \right|^2 \leq \sum_{j=0}^n |c_j|^2 \beta_j^2 \cdot \sum_{j=0}^n \frac{1}{\beta_j^2} \leq C_{\beta} \sum_{j=0}^{\infty} |c_j|^2 \beta_j^2 = C_{\beta} \|f\|^2.$$

From this it follows that

$$\begin{aligned} \|R_{\mathbf{a}}f\|^2 &= \sum_{n=0}^{\infty} \left| a_n \sum_{j=0}^n c_j \right|^2 \beta_n^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \left| \sum_{j=0}^n c_j \right|^2 \\ &\leq C_{\beta} \|f\|^2 \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 = C_{\beta} \|\mathbf{a}\|^2 \|f\|^2 < \infty, \end{aligned}$$

which shows that $R_{\mathbf{a}}f \in \mathcal{H}(\beta)$. Thus $R_{\mathbf{a}}$ is invariant on $\mathcal{H}(\beta)$. $\square$

3.1.2. Boundedness. A linear operator T acting on a normed space H is said to be *bounded* if there is $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in H$. The smallest such M is called the *operator norm* $\|T\|_{\text{op}}$ of T .

It is well-known that for a wide class of linear operators, for example, composition operators, acting on some specific function spaces, for example, functional Banach and reproducing kernel Hilbert spaces, the invariance is equivalent to the boundedness, while in general, it is not true.

In our case, Rhaly operators are not composition operators. However, as we see in the proof of Proposition 6, $R_{\mathbf{a}}$ is bounded on $\mathcal{H}(\beta)$ whenever it is invariant on this space.

Thus we have the following result.

Proposition 7. *$R_{\mathbf{a}}$ is bounded on $\mathcal{H}(\beta)$ if and only if $\mathbf{a} \in \ell_{\beta}^2$.*

3.2. Compactness, Hilbert–Schmidt norm and Schatten classes. A bounded linear operator T acting on a Banach space H is said to be *compact* if it maps the unit ball of H into a relatively compact subset of H (that is, a subset of H with compact closure).

We study a compactness of $R_{\mathbf{a}}$ through the study of Schatten classes, whose definitions are given below.

First of all, we recall the definition of approximation numbers of an operator acting on Banach spaces.

Let X and Y be Banach spaces and $T : X \rightarrow Y$ be a linear operator. We denote by $\|T\|_{\text{op}}$ the operator norm of T . The *n th approximation*

number a_n of T is defined as

$$\mathfrak{a}_n(T) = \inf\{\|T - R\|_{op}, R \in \mathcal{L}(X, Y), \text{rank}(R) < n\}, n \geq 1,$$

where $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators acting from X into Y .

It is straightforward that

$$\|T\|_{op} = \mathfrak{a}_1(T) \geq \mathfrak{a}_2(T) \geq \cdots \geq 0,$$

and that

$$\lim_{n \rightarrow \infty} \mathfrak{a}_n = 0 \implies T \text{ is compact.}$$

We also recall the concept of *singular values* of T , that is, the eigenvalues ($s_n(T)$) of the Hermitian operator $|T| := \sqrt{T^*T}$ that are usually listed in decreasing order, i.e. $s_1(T) \geq s_2(T) \geq \cdots \geq 0$. A well-known theorem states that if $X = Y$ and both spaces are separable Hilbert spaces, then the n th approximation number is precisely the n th singular value of T , i.e., $\mathfrak{a}_n = s_n$ (see, e.g., [4]).

We remind the reader of the notion of Hilbert–Schmidt and Schatten class operators. A bounded linear operator T on a Hilbert space H is called a *Hilbert–Schmidt* operator if it has finite Hilbert–Schmidt norm (also known as Frobenius norm) $\|T\|_{HS}$, which is defined as follows. Let $\{e_j\}_{j \in J}$ be an orthonormal basis for H , then

$$\|T\|_{HS} = \left(\sum_j \|Te_j\|^2 \right)^{1/2} < +\infty.$$

It is known that $\|T\|_{HS}$ does not depend on the choice of orthonormal basis. In addition, if T is a Hilbert–Schmidt operator, then T is compact.

For $0 < p < \infty$, the Schatten p -class consists of all bounded linear operators T on H for which $(T^*T)^{p/4}$ is a Hilbert–Schmidt operator. In particular, for separable H , the Schatten p -norm $\|T\|_p$ is finite, i.e.

$$\|T\|_p := \left(\sum_{n=1}^{\infty} s_n^p \right)^{1/p} < +\infty.$$

When $p = \infty$, we define $\|T\|_{\infty} = \|T\|_{op}$.

The Schatten 2-class is the Hilbert–Schmidt class, and $\|T\|_{HS} = \|T\|_2$. Another important Schatten class is the Schatten 1-class, or *trace class*, where $\|T\|_1 = \text{trace}(\sqrt{T^*T})$. Schatten norms are known to be non-increasing, i.e., for a compact linear operator T on a Hilbert space H , if $1 \leq p \leq q \leq \infty$, then

$$\|T\|_p \geq \|T\|_q.$$

We refer the reader to the books [4] and [10] for having more detailed information on these classes of operators.

We have the following result about compactness of $R_{\mathbf{a}}$.

Proposition 8. *The following statements for the operator $R_{\mathbf{a}}$ are equivalent:*

- (i) $R_{\mathbf{a}}$ belongs to Schatten p -class for all $p \geq 2$,
- (ii) $R_{\mathbf{a}}$ is Hilbert–Schmidt,
- (iii) $R_{\mathbf{a}}$ is compact
- (iv) $\mathbf{a} \in \ell_{\beta}^2$.

Proof. (i) $\iff$ (ii). It is due to the non-increasing property of Schatten p -norm.

(ii) $\implies$ (iii). It is obvious.

(iii) $\implies$ (iv). It follows from Proposition 7.

(iv) $\implies$ (ii). Suppose $\mathbf{a} \in \ell_{\beta}^2$. Consider the orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ for $\mathcal{H}(\beta)$, where $e_k(z) = \beta_k^{-1} z^k$ ($k \geq 0$). Then for each k , we have

$$R_{\mathbf{a}} e_k(z) = \sum_{n=k}^{\infty} a_n \frac{1}{\beta_k} z^n,$$

which gives

$$\|R_{\mathbf{a}} e_k\|^2 = \sum_{n=k}^{\infty} |a_n|^2 \frac{1}{\beta_k^2} \beta_n^2 = \frac{1}{\beta_k^2} \sum_{n=k}^{\infty} |a_n|^2 \beta_n^2.$$

Consequently,

$$\begin{aligned} \|R_{\mathbf{a}}\|_{HS} &= \sum_{k=0}^{\infty} \|R_{\mathbf{a}} e_k\|^2 = \sum_{k=0}^{\infty} \frac{1}{\beta_k^2} \sum_{n=k}^{\infty} |a_n|^2 \beta_n^2 \\ &\leq \sum_{k=0}^{\infty} \frac{1}{\beta_k^2} \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 = C_{\beta} \|\mathbf{a}\|^2 < \infty. \end{aligned}$$

This shows that $R_{\mathbf{a}}$ is in the Hilbert–Schmidt class. $\square$

More can be said about Schatten p -class when the weight $\beta = (\beta_n)$ grows sufficiently large, i.e. $\beta_* > 1$, for instance, in the case of spaces $\mathcal{H}(\beta)$ of entire functions where $\beta_* = +\infty$. We need the following lemma.

Lemma 9 ([4, Proposition 1.b.16]). *Suppose T is a bounded linear operator on a Hilbert space H and $\{e_j\}_{j \in J}$ is an orthonormal basis for H . For all $0 < p < 2$, if $\sum_j \|Te_j\|^p < +\infty$, then T is in Schatten p -class and*

$$\|T\|_p < \sum_j \|Te_j\|^p.$$

We have the following result.

Proposition 10. *Suppose $\beta_* = \liminf_{n \rightarrow \infty} (\beta_n)^{1/n} > 1$. Then the operator $R_{\mathbf{a}}$ belongs to all Schatten p -class ($p > 0$) if and only if $\mathbf{a} \in \ell_{\beta}^2$.*

Proof. It suffices to prove that if $\mathbf{a} \in \ell_\beta^2$ then $R_{\mathbf{a}}$ belongs to any Schatten p -class with $0 < p < 2$.

Let $p \in (0, 2)$. Then there is some $r > 0$ such that $\beta_* > r^{1/p} > 1$. In this case, $\beta_n > r^{n/p}$ for all n large enough, which gives $\beta_n^{-p} < r^{-n}$. As a result, $\sum_{n=0}^{\infty} \beta_n^{-p} < +\infty$ by the comparison test.

With $\{e_k\}$ is chosen as in Proposition 8, we have

$$\sum_{k=0}^{\infty} \|R_{\mathbf{a}} e_k\|^p = \sum_{k=0}^{\infty} \frac{1}{\beta_k^p} \left(\sum_{n=k}^{\infty} |a_n|^2 \beta_n^2 \right)^{p/2} \leq \|\mathbf{a}\|^p \sum_{k=0}^{\infty} \frac{1}{\beta_k^p} < +\infty.$$

The proof is complete by Lemma 9. $\square$

Summarizing all the result obtained above, we can formulate the following theorem.

Theorem 11. *Let $R_{\mathbf{a}}$ be the Rhaly operator induced by a sequence $\mathbf{a} = (a_n) \subset \mathbb{C}$. The following assertions are equivalent:*

- (i) $\mathbf{a} \in \ell_\beta^2$,
- (ii) $R_{\mathbf{a}}$ is invariant on $\mathcal{H}(\beta)$,
- (iii) $R_{\mathbf{a}}$ is bounded on $\mathcal{H}(\beta)$,
- (iv) $R_{\mathbf{a}}$ is compact on $\mathcal{H}(\beta)$.
- (v) $R_{\mathbf{a}}$ is in Schatten p -class of $\mathcal{H}(\beta)$ for all $p \geq 2$.

In particular, if $\liminf_{n \rightarrow \infty} (\beta_n)^{1/n} > 1$, then all of the assertions (i) – (v) are equivalent to the following statement:

- (vi) $R_{\mathbf{a}}$ is in Schatten p -class of $\mathcal{H}(\beta)$ for all $p > 0$.

Example 12. For any Hilbert space of entire functions (Remark 4), specifically the Fock space $\mathcal{F}^2$, the Rhaly operator $R_{\mathbf{a}}$ belongs to the Schatten p -class for all $p > 0$.

4. CLOSED RANGE

We have the following result on the range of $R_{\mathbf{a}}$.

Theorem 13. *The Rhaly operator $R_{\mathbf{a}}$ has a closed range (that is $R_{\mathbf{a}}(\mathcal{H}(\beta)) = \overline{R_{\mathbf{a}}(\mathcal{H}(\beta))}$) if and only if $\mathbf{a}$ is a finite sequence.*

Proof. Sufficiency. It is obvious.

Necessity. Suppose $R_{\mathbf{a}}$ is bounded and has closed range. By Theorem 11, $R_{\mathbf{a}}$ is compact, and hence it is finite dimensional (see, e.g. [7, Theorem 4.18b]).

Assume in contrary that $\mathbf{a}$ is not a finite sequence. Then with the sequence (e_k) chosen as in the proof of Proposition 8, we see that

$$\dim(R_{\mathbf{a}}(\mathcal{H}(\beta))) \geq \dim(\text{span}\{R_{\mathbf{a}} e_k, k \in \mathbb{N}\}) = +\infty,$$

which is a contradiction. So $\mathbf{a}$ is a finite sequence. $\square$

The final conclusion is that as already stated in Remark 4, Fock space $\mathcal{F}^2(\mathbb{C})$ is a special case of small weighted Hardy spaces. Then our results in Section 3 are valid for $\mathcal{F}^2(\mathbb{C})$ too.

Acknowledgments. The authors would like to thank the referee for useful remarks. Thanks also go to Hervé Queffélec for valuable comments on the first draft of this paper.

REFERENCES

- [1] A. Brown, P. R. Halmos, and A. L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged) **26** (1965), 125–137.
- [2] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.
- [3] M. L. Doan and L. H. Khoi, *Hilbert spaces of entire functions and composition operators*, Complex Anal. Oper. Theory **10** (2016), no. 1, 213–230.
- [4] H. König, *Eigenvalue Distribution of Compact Operators*, Operator Theory: Advances and Applications, vol. 16, Birkhäuser Basel, 1986.
- [5] G. Leibowitz, *Rhaly matrices*, J. Math. Anal. Appl. **128** (1987), 272–286.
- [6] B. D. MacCluer, X. Zeng, and N. Zorboska, *Composition operators on small weighted Hardy spaces*, Illinois J. Math. **40** (1996), no. 4, 662–677.
- [7] Walter Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, 1991.
- [8] H. C. Jr. Rhaly, *p-Cesàro matrices*, Houston Jour. Math. **15** (1989), no. 1, 137–146.
- [9] M. Yildirim, *On the spectrum of the Rhaly operators on bv_0* , Commun. Korean Math. Soc. **18** (2003), no. 4, 669–676.
- [10] K. Zhu, *Operator theory in function spaces*, Amer. Math. Soc., Providence, RI., 1990.

(TAN) DIVISION OF MATHEMATICAL SCIENCES, SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES, NANYANG TECHNOLOGICAL UNIVERSITY (NTU), 637371 SINGAPORE

E-mail address: M140141@e.ntu.edu.sg

(DOAN) DIVISION OF MATHEMATICAL SCIENCES, SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES, NANYANG TECHNOLOGICAL UNIVERSITY (NTU), 637371 SINGAPORE

CURRENT ADDRESS: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, IN 46556, USA

E-mail address: DOAN0014@e.ntu.edu.sg

(KHOI) DIVISION OF MATHEMATICAL SCIENCES, SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES, NANYANG TECHNOLOGICAL UNIVERSITY (NTU), 637371 SINGAPORE

E-mail address: lhkhoei@ntu.edu.sg