Isogeometric analysis-based approaches for fracture and contact problems of structures and materials

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**ISOGEOMETRIC ANALYSIS-BASED APPROACHES
FOR FRACTURE AND CONTACT PROBLEMS
OF STRUCTURES AND MATERIALS**

HUANG JIAZHAO

**SCHOOL OF MECHANICAL AND AEROSPACE
ENGINEERING**

2020

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SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

A thesis submitted to the Nanyang Technological University in
partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2020

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

12 Mar. 2020

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Huang Jiazhao

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Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

12 Mar. 2020

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Date

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Assoc Prof Zhou Kun

Authorship Attribution Statement

This thesis contains material from 2 paper(s) published in the following peer-reviewed journal(s) / from papers accepted at conferences in which I am listed as an author.

Chapter 3 is published as **J. Huang**, N. Nguyen-Thanh and K. Zhou. Extended isogeometric analysis based on Bézier extraction for the buckling analysis of Mindlin–Reissner plates, *Acta Mechanica*, 2017. **228**(9):3077-3093.

The contributions of the co-authors are as follows:

- Associate Professor Zhou proposed the research idea, analysed the simulation results and revised the manuscript.
- I developed the simulation method, investigated a series of benchmark problems, analysed the simulation results and drafted the manuscript.
- Dr. Nguyen-Thanh provided technique support of the research and proofread the draft of manuscript.

Chapter 4 is published as **J. Huang**, N. Nguyen-Thanh and K. Zhou. An isogeometric-meshfree coupling approach for contact problems by using the third medium method, *International Journal of Mechanical Sciences*, 2018. **148**:327-336.

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- Dr. Nguyen-Thanh provided technique support of the research and proofread the draft of manuscript.

Chapter 5 is published as **J. Huang**, N. Nguyen-Thanh, W. Li, K. Zhou. An adaptive isogeometric-meshfree coupling approach for the limit analysis of cracked structures, *Theoretical and Applied Fracture Mechanics*, 2019. **106**:10242.

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- Dr. Nguyen-Thanh provided technique support of the research and proofread the draft of manuscript.

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Table of Contents

Statement of Originality	I
Supervisor Declaration Statement	III
Authorship Attribution Statement	V
Acknowledgements.....	VII
Table of Contents	IX
Abstract	XII
List of Publications.....	XV
List of Figures	XVII
List of Table	XX
List of Symbols	XXI
Chapter 1 Introduction.....	1
1.1. Background	1
1.2. Motivations.....	7
1.3. Objectives.....	8
1.4. Dissertation layout.....	9
Chapter 2 Literature Review	10
2.1. Fracture problems.....	10
2.1.1. Fracture mechanics of plates.....	13
2.1.2. Plastic collapse of cracked structures	19
2.2. Contact problems.....	23
2.3. Numerical solutions.....	26
2.3.1. Finite element method.....	27
2.3.2. Meshfree methods.....	31
2.3.3. Isogeometric analysis.....	34
2.4. Summary	39
Chapter 3 A Bézier Extraction-Based XIGA Approach for the Buckling of Cracked Plates.....	40
3.1. Problem description and solution approach	40
3.1.1. Buckling of Mindlin–Reissner plates.....	40
3.1.2. XIGA formulations for cracked plates.....	44

3.2. Numerical results for the convergence test	45
3.3. Numerical results for buckling of plates	46
3.3.1. Plate under uniaxial compression.....	48
3.3.2. Plate under biaxial compression.....	50
3.3.3. Plate under in-plane shear	50
3.4. Numerical results for buckling of cracked plates	51
3.4.1. Central cracked plate under uniaxial compression.....	52
3.4.2. Edge-cracked plate under uniaxial compression.....	55
3.4.3. Central cracked circular plate under uniaxial compression	57
3.5. Summary.....	59
Chapter 4 An Isogeometric-Meshfree Coupling Approach for Contact Problems...	60
4.1. Problem description and solution approach	60
4.1.1. Third medium method.....	60
4.1.2. Discretisation based on an IMF coupling approach.....	64
4.2. Numerical results.....	68
4.2.1. Material parameter test for the third medium	68
4.2.2. Hertzian contact	71
4.2.3. Contact problem with sliding	75
4.2.4. Contact problem with large deformation	77
4.3. Summary.....	80
Chapter 5 An Adaptive Isogeometric-Meshfree Coupling Approach for the Limit Analysis of Cracked Structures	81
5.1. Problem description and solution approach	81
5.1.1. Kinematic formulation	81
5.1.2. Discretisation based on an adaptive IMF coupling approach	83
5.2. Numerical results of structures	87
5.2.1. Square plate with a central hole	88
5.2.2. Grooved rectangular plate	93
5.2.3. Thin plate	95
5.3. Numerical results of cracked structures	100
5.3.1. Central cracked plate.....	100
5.3.2. Edge-cracked plate	104

5.3.3. Grooved cracked plate	108
5.4. Summary	110
Chapter 6 Conclusions and Recommendations	111
6.1. Conclusions	111
6.2. Recommendations	113
Appendix A	115
A.1. Bézier extraction formulation.....	115
A.2. Discrete shear gap method	117
Appendix B.....	119
B.1. Constitutive behaviours of hyper-elastic materials	119
B.2. Reproducing conditions for an IMF coupling approach	119
Appendix C	123
References	125

Abstract

The field of fracture mechanics focuses on investigating the initiation and propagation of cracks in materials. The mechanical and physical properties of engineered structures are significantly affected by the presence of cracks, which may eventually lead to structural failure. Another important field of engineering is contact mechanics, in which many mechanical components operate under conditions of contact loading. However, analysis of the stresses and deformations induced by contact loading is complicated due to their multi-scale and multi-physical nature. To address both fracture and contact problems, isogeometric analysis (IGA) has been developed as a promising computational tool. It exhibits significant advantages over traditional numerical methods in terms of geometrical exactness, high-order approximation and tailorable inter-element connectivity. Therefore, this Ph.D. study develops IGA-based solutions for the fracture and contact problems of structures and materials.

Firstly, this study investigates the effectiveness of IGA on predicting the buckling factors of cracked Mindlin–Reissner plates. Enrichment functions are introduced into IGA basis functions to form an extended IGA (XIGA) approach for fracture analysis, while the Bézier extraction method is incorporated into the formulation for the direct integration of XIGA basis functions across C^0 Bézier elements. The buckling factors of plates under various crack and loading conditions are calculated and are found to be in good agreement with the analytical solutions provided by Stahl and Keer [1]. When compared with traditional finite element methods for crack modelling, IGA not only simplifies the overall simulation process by circumventing the time-consuming remeshing process but also

preserves the geometrical exactness of the complex crack surfaces. A high convergence rate can be further obtained by increasing the polynomial order of the XIGA basis functions.

Secondly, a novel contact treatment is developed by incorporating the third medium method into an isogeometric-meshfree (IMF) coupling approach. IMF coupling basis functions inherit advantages from both isogeometric and meshfree approximations in contact modelling. The parametric domain is divided into three sub-domains represented by the isogeometric, IMF coupling and meshfree approximations, respectively. IGA is applied to a thin region along the body's boundary, while the moving least-squares meshfree method is utilised in the interior domain. Continuum formulations of an entire contact system are formed via the third medium method, which avoids the expensive computational gap-search algorithm required by conventional contact treatments. Through the analyses of benchmark contact problems, the proposed method delivers evident advantages, including the definition of geometry directly in a computer-aided design environment, the achievement of higher orders of spatial convergence and the treatment of extreme element distortions within the third medium while undergoing large deformation and sliding.

Finally, an adaptive local refinement strategy is introduced into the IMF coupling basis functions to form an adaptive IMF coupling approach for the limit analysis problems of cracked structures. This approach relies on developing a correspondence between IGA and meshfree approximations based on a reproducing condition. Thus, the entire parametric domain can be represented by a unified basis function that allows an adaptive local refinement for the performance of IGA in a straightforward meshfree fashion. The

adaptivity of refinement is achieved via an indicator, which automatically identifies the meshes associated with a plastic dissipation greater than a predefined threshold. The marked meshes are subsequently refined through the insertion of linear reproducing points, and the resulting optimisation formulation can be rewritten as a second-order cone programming problem. The simulation results obtained by the present method agree well the reference solutions. Aside from generating accurate results, an adaptive IMF coupling approach also achieves a significantly higher convergence rate than those offered by reference methods with uniform refinement.

The development of these IGA-based solutions advances the understanding of the fracture and contact behaviours of structures and materials and is critical to improving the function and safety designs of various engineering applications.

List of Publications

- [1] **J. Huang**, N. Nguyen-Thanh, W. Li, K. Zhou. An adaptive isogeometric-meshfree coupling approach for the limit analysis of cracked structures, *Theoretical and Applied Fracture Mechanics*, 2019. **106**:10242.
- [2] **J. Huang**, N. Nguyen-Thanh, K. Zhou. An isogeometric-meshfree coupling approach for contact problems by using the third medium method, *International Journal of Mechanical Sciences*, 2018. **148**:327-336.
- [3] **J. Huang**, N. Nguyen-Thanh, K. Zhou. Extended isogeometric analysis based on Bézier extraction for the buckling analysis of Mindlin–Reissner plates, *Acta Mechanica*, 2017. **228**(9):3077-3093.
- [4] N. Nguyen-Thanh, **J. Huang**, K. Zhou. An isogeometric-meshfree coupling approach for analysis of cracks, *International Journal for Numerical Methods in Engineering*, 2017. **113**(10):1630-1651.
- [5] N. Nguyen-Thanh, W. Li, **J. Huang**, N. Srikanth, K. Zhou. An adaptive isogeometric analysis meshfree collocation method for elasticity and frictional contact problems, *International Journal for Numerical Methods in Engineering*, 2019. **120**(2): 209-230.

List of Figures

Fig. 1.1. Fracture of plate-like structures: (a) a rail tie plate and (b) a shipbuilding plate..	2
Fig. 1.2. The schematic diagram of a plate.	3
Fig. 1.3. The load-strain curve for the structural steel.	4
Fig. 1.4. The stress-strain diagrams of materials: (a) an elasto-plastic material and (b) an elastic-perfectly plastic material.....	4
Fig. 2.1. Three types of crack mode.....	11
Fig. 2.2. An edge-cracked plate subjected to four types of loading conditions.	13
Fig. 2.3. Stress components around one tip of a crack in an infinite plane subjected to in-plane tension.....	14
Fig. 2.4. A coordinate system located at the crack tip on the mid-plane of a plate.	17
Fig. 2.5. Limit analysis systems: (a) a lower bound system and (b) an upper bound system.	21
Fig. 2.6. (a) 1D and (b) 2D diagrams of quadratic B-Spline basis functions associated with the knot vectors Ξ given by $\{0, 0, 0, 1/3, 2/3, 1, 1, 1\}$..	35
Fig. 3.1. A Mindlin–Reissner plate.	41
Fig. 3.2. The convergence of the buckling factor λ_n for a central cracked square plate with $l_c/l_2 = 0.5$	46
Fig. 3.3. Loading conditions of Mindlin–Reissner plates: (a) uniaxial compression, (b) biaxial compression and (c) in-plane shear.	47
Fig. 3.4. Crack configurations of rectangular Mindlin–Reissner plates: (a) a central cracked plate and (b) an edge-cracked plate.....	52
Fig. 3.5. The first four buckling modes of a central cracked square plate with $l_c/l_2 = 0.9$	53
Fig. 3.6. The first four buckling modes of a central cracked rectangular plate with $l_c/l_2 = 0.9$	53
Fig. 3.7. The first four buckling modes of an edge-cracked rectangular plate with $l_c/l_2 = 0.5$	55
Fig. 3.8. Crack configurations of circular Mindlin–Reissner plates: (a) a central cracked plate and (b) an edge-cracked plate.....	57
Fig. 3.9. The first four buckling modes of a central cracked circular plate with $l_c/l_2 = 0.9$	58

Fig. 3.10. The first four buckling modes of an edge-cracked circular plate with $l_c/l_2 = 0.9$	58
Fig. 4.1. Contacting bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ embedded in the third medium $\mathcal{B}_3$	61
Fig. 4.2. Definition of the surface normal $\mathbf{n}$, gap g_n and threshold gap g_c	63
Fig. 4.3. (a) 1D and (b) 2D diagrams of quadratic IMF coupling basis functions.	66
Fig. 4.4. Schematics of the material parameter test: (a) the setup diagram and (b) the discretisation configuration.	69
Fig. 4.5. The gap between contacting bodies verses increments of applied force with respect to different a_5	69
Fig. 4.6. Stress distribution diagrams for $a_5 = 0.0001$: (a) Horizontal stress σ_{xx} , (b) vertical stress σ_{yy} and (c) shear stress σ_{xy} distribution.	70
Fig. 4.7. Schematics of the Hertzian contact simulation: (a) the setup diagram and (b) the discretisation configuration.	72
Fig. 4.8. Convergence test of an IMF coupling approach for Hertzian contact problem with respect to different polynomial orders of shape functions.	73
Fig. 4.9. Comparison of simulation results from different approaches for the Hertzian contact.	74
Fig. 4.10. Vertical stress σ_{yy} distribution of Hertzian contact system.	74
Fig. 4.11. Vertical stress σ_{yy} distribution diagrams associated with the horizontal movement in steps (a) 30, (b) 50 and (c) 70.	76
Fig. 4.12. Horizontal and vertical force curves obtained by contact simulation.	77
Fig. 4.13. Schematics of the contact problem with large deformation: (a) the setup diagram and (b) the discretisation configuration.	78
Fig. 4.14. Deformed configurations of the third medium and contact bodies in step (a) 5, (b) 20, (c) 35 and (d) 50.	79
Fig. 5.1. Schematics of a square plate with a central hole: (a) the full mode and (b) its quarter mode with symmetric boundary condition.	88
Fig. 5.2. Mesh refinements for a square plate with a central hole ($r = 0.2l$): (a) the initial mesh and (b) the mesh after 3 steps of refinement.	89
Fig. 5.3. A companion of the convergence rates.	90
Fig. 5.4. Plastic dissipation distribution associated to $\sigma_2 = 0$	91
Fig. 5.5. Schematics of a grooved rectangular plate: (a) the full mode and (b) its quarter mode with symmetric boundary condition.	93

Fig. 5.6. Mesh refinements for a grooved rectangular plate: (a) the initial mesh and (b) the mesh after 3 steps of refinement.....	94
Fig. 5.7. Plastic dissipation distribution.....	95
Fig. 5.8. Schematics of a thin plate subjected to the simply supported boundary condition: (a) the top view and (b) the front view.....	96
Fig. 5.9. Mesh refinements for a square thin plate: (a) the initial mesh and (b) the mesh after 4 steps of refinement.....	97
Fig. 5.10. Plastic dissipation distribution.....	98
Fig. 5.11. Mesh refinements for a rectangular thin plate: (a) the initial mesh and (b) the mesh after 4 steps of refinement.....	99
Fig. 5.12. Plastic dissipation distribution.....	100
Fig. 5.13. A central cracked rectangular plate subjected to uniaxial tension.....	101
Fig. 5.14. Mesh refinements for a central cracked rectangular plate with $l_c/l = 0.5$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.	102
Fig. 5.15. Plastic dissipation distribution associated to $l_c/l = 0.5$	102
Fig. 5.16. A companion of the convergence rates.....	103
Fig. 5.17. An edge-cracked rectangular plate subjected to uniaxial tension.....	105
Fig. 5.18. Mesh refinements for an edge-cracked rectangular plate with $l_c/l = 0.06$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.	106
Fig. 5.19. Mesh refinements for an edge-cracked rectangular plate with $l_c/l = 0.3$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.	106
Fig. 5.20. Plastic dissipation distribution associated to edge-cracked rectangular plates: (a) the crack length ratio $l_c/l = 0.06$ and (b) the crack length ratio $l_c/l = 0.3$	107
Fig. 5.21. A grooved cracked plate subjected to uniaxial tension.	108
Fig. 5.22. Mesh refinements for a grooved cracked rectangular plate with $l_c/l = 0.3$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.	109
Fig. 5.23. Plastic dissipation distribution associated with $l_c/l = 0.3$	109

List of Table

Table 3.1. Axial buckling load factors λ_n of square plates.	48
Table 3.2. Axial buckling load factors λ_n of rectangular plates.	49
Table 3.3. Biaxial buckling load factors λ_n of a square plate with $h/l_1 = 0.01$	50
Table 3.4. Shear buckling load factors λ_n of square plates with $h/l_1 = 0.01$	51
Table 3.5. Axial buckling load factors λ_n of central cracked rectangular plates.....	54
Table 3.6. Axial buckling load factors λ_n of edge-cracked rectangular plates.....	56
Table 5.1. Limit load factors λ_p of a square plate with a central hole ($r = 0.2l$) under various loading conditions with $\sigma_1 = \sigma_0$	91
Table 5.2. Limit load factors λ_p of a square plate with various radius-to-length ratios r/l under uniaxial tension.....	92
Table 5.3. Limit load factors λ_p of a grooved rectangular plate.....	95
Table 5.4. Limit load factors λ_p of a square thin plate.	98
Table 5.5. Limit load factors λ_p of a rectangular thin plate.....	99
Table 5.6. Limit load factors λ_p of a central cracked rectangular plate with various crack lengths l_c	104
Table 5.7. Limit load factors λ_p of an edge-cracked rectangular plate with various crack lengths.....	108
Table 5.8. Limit load factors λ_p of a grooved cracked rectangular plate.	110

List of Symbols

Symbol	Physical meaning
K_I, K_{II}, K_{III}	Stress intensity factor of the crack in Modes I, II or III
E	Young's modulus
μ	Shear modulus
ν	Poisson's ratio
k_s	Shear correction factor
u	Surface displacement
f	Load
σ	Stress
ε	Strain
r	Radius
h	Thickness
β	Rotational degree of freedom
w	Displacement along z-axis
ϖ	Weights of NURBS
Ξ	Set of knot vectors
H	Heaviside function
χ	Enrichment function

Chapter 1 Introduction

This Ph.D. dissertation aims to develop innovative isogeometric analysis-based approaches for fracture and contact problems of structures and materials. In this chapter, the research background is firstly elaborated in Section 1.1. Thereafter, the motivations and objectives of this research are stated in Section 1.2 and Section 1.3, respectively. The organisation of the whole dissertation is outlined in Section 1.4.

1.1. Background

The fracture behaviour of cracked linear elastic and elastic-plastic materials is a vital issue in the context of engineering problems. Since no material is perfect, and fabrication flaws cannot be eliminated, the cracks induced by these pre-existing flaws can become a major cause of component fracture and subsequent structural failure. This is because the presence of cracks can reduce structural strength to a critical point at which fractures can occur. Therefore, the comprehensive understanding and characterisation of cracks and their effects are beneficial for the safe design of engineering structures. In 1921, Griffith [2] proposed an energy-based failure theory that states that the growth of cracks is determined by the global energy balance. Crack initiation is attributed to a large amount of energy stored in the fracture that exceeds the surface energy of materials. As the fundamental basis of linear elastic fracture mechanics, this theory is valid for materials with little to no plastic deformation within their crack-tip areas. The Dugdale [3] and Irwin [4] models were further proposed to take the plastic zone of crack tips into consideration by modifying crack-tip stresses; the two approaches are widely applied to solving two-dimensional (2D) crack problems [5-7].

The fracture of structural materials is of great theoretical and practical interest due to the inherent structural characteristics of materials used in the design of engineering applications. Since plate-like structures are one of the most basic structural elements, the analysis of the fracture behaviour of plates is critical to the effective and safety design of many engineering applications in the aerospace, marine and railway industries, as shown in Fig. 1.1.

Fig. 1.1. Fracture of plate-like structures: (a) a rail tie plate and (b) a shipbuilding plate.

Fig. 1.2 shows that a plate is regarded as a three-dimensional (3D) solid with a relatively small thickness compared with its other dimensions. Stresses generated by applied loads are predominantly assumed to be perpendicular to a plate. Furthermore, the presence of cracks can significantly affect the stress fields within a plate, although many studies have only considered in-plane tension for intra-plate cracks. However, the practical fracture problems of cracked plates usually address mixed-mode crack-tip stress fields generated both by in-plane tension and out-of-plane shear. These stresses can be expressed in terms of the Kirchhoff–Love plate theory or the Mindlin–Reissner plate theory. The Kirchhoff–Love theory is the most basic approach for the analysis of plates, which neglects

both transverse shear and transverse effects, while the Mindlin–Reissner theory is further developed as a shear deformation plate approach.

Fig. 1.2. The schematic diagram of a plate.

The failure mode of a cracked structure can be a combination of fracture and plastic collapse. The dominated failure mode for a cracked structure is fracturing when the structure is made of brittle materials. For materials with increasing toughness, the plastic flow would become a more significant factor which replaces the plastic collapse as the major failure mode. Therefore, consideration should also be given to plastic collapse to prevent potential material failure. From the perspective of continuum mechanics, a fracture is generally initiated by the crack-tip hydrostatic stress that leads to crack tearing, while plastic collapse is controlled by the deviatoric stress and crack ligament yielding. Consequently, failure commonly occurs through the plastic flow without any noticeable crack tearing for plastic collapse. As plotted in Fig. 1.3, this deformation process is generally complex for materials such as structural steel which is widely used in

manufacturing and construction. Other materials could miss one or more stages when they are subjected to external loads.

Fig. 1.3. The load-strain curve for the structural steel.

Fig. 1.4. The stress-strain diagrams of materials: (a) an elasto-plastic material and (b) an elastic-perfectly plastic material.

Fig. 1.4(a) plots a typical strain-stress curve for an elasto-plastic material. Once the applied load exceeds the elastic limit, this part of the curve is considered as the plastic region where material starts to deform permanently. To be more specific, a slight increment in load would lead to a considerable increase in the deformation of a material. If the material is idealised as an elastic-perfectly plastic medium by neglecting the changes in its geometry, displacements will increase infinitely under a constant load as shown in Fig. 1.4(b). Based on this ideal situation, the plastic limit load of materials can be calculated. This hypothetical load is commonly employed to estimate the physical plastic limit load of materials or the load at which excessive deformations would take place.

Predicting the mechanical response of materials under different loading conditions such as contact loads is also of great interest in engineering problems. When materials are manufactured into gears and bearings that come into contact with other components during operations, contact surfaces are not perfectly flat in general. This surface roughness could result in extremely small contact areas that lead to high contact pressure and stress. Consequently, failure and yielding might occur in the region of concentrated stresses. Such contact stress fields could be affected by many factors, including the surface geometries, loading conditions and material properties. This issue becomes more complicated if cracks are considered. This is because the presence of cracks would substantially influence the contact area, surface deformation and pressure distribution. Yang *et al.* [8] found that the peak value of the surface pressure was inversely proportional to the length of the pre-existing crack, resulting in more flat surface contact pressure profiles.

Analytical and numerical approaches are two major types of solutions for fracture and contact problems. Analytical approaches are sometimes only limited to solving 2D fracture

or contact problems containing simple geometries and specific boundary conditions. Nonetheless, such problems would become highly complicated in industrial applications due to many factors such as nonlinear materials and intricate structural geometries. Numerical methods alternately show great capability in handling complex geometries and boundary conditions with decent convergence and accuracy. As one of the most commercialised numerical tools, finite element method (FEM) is widely applied to fracture and contact simulations. For fracture modelling, FEM generally depends on tedious remeshing to ensure the alignment between edges of the finite elements and the crack path. The extended finite element method (XFEM) was consequently introduced by Belytschko and Black [9] to avoid the computationally expensive remeshing around the cracks. Sukumar *et al.* [10] extended XFEM to 3D crack modelling by taking discontinuous displacement fields and asymptotic crack-tip displacement fields into consideration.

Although the remeshing burden in fracture modelling is alleviated by XFEM, its further applications to complex structures such as plates are limited by its approximation of the material geometry and low-order continuity. As for contact problems, numerical solutions are developed based on the h -version of the FEM using linear or quadratic ansatz functions for spatial discretisation [11]. Contact constraints are regularly enforced along the contact interfaces that can be discretised by node-to-node or node-to-segment contact elements [12-14]. Nevertheless, non-smooth surfaces stemming from the Lagrange discretisation would cause the trapping of nodes at re-entrant corners, especially for problems involving large deformation or sliding. As a potential alternative, isogeometric analysis (IGA) has been proposed by Hughes *et al.* [15] to allow a direct import of CAD geometry data into the finite element analysis through B-Spline or non-uniform rational B-

Spline (NURBS). Unlike the Lagrangian FEM shape function, B-Spline and NURBS are two basis functions which can be directly defined within the CAD environment to plot the exact geometry of models. Therefore, the data generated from CAD can be directly inputted into IGA without the redundant data conversion progress required by FEM. Furthermore, NURBS allows IGA elements with arbitrary continuities to be formed for high-order polynomial approximations, while the conventional FEM elements show limitations and deficiencies in dealing with problems involving conic sections. Ghorashi *et al.* [16] further developed the extended IGA (XIGA) by integrating enrichment functions with IGA basis functions for fracture simulations. As for contact problems, IGA avoids the drawbacks of FEM by achieving controllable smoothness for contact surfaces undergoing large deformation and sliding. Therefore, many IGA-based contact formulations have been developed in the past decade [17-19].

1.2. Motivations

The development of analytical and numerical solutions for fracture and contact problems facilitates the safe design of modern engineering applications. Analytical solutions are generally insufficient for tackling practical industrial problems that can be effectively solved by numerical approaches. For fracture problems, IGA exhibits the advantages of exact geometry and high-order smoothness over other numerical approaches such as FEM and the meshfree approximations. Nevertheless, the IGA-based fracture modelling generally lacks local mesh refinement around the crack path induced by the tensor product structure of IGA basis functions. An effective local refinement strategy also needs to be developed for IGA-based contact modelling. Although IGA discretizations

eliminate the geometry discontinuities across the contact surfaces and are insensitive to extreme element distortions, an excessive overhead of control points could be generated by the increase in mesh refinement. This phenomenon will lead to a significant decrease in the computational efficiency of IGA simulations if discontinuities such as cracks are further considered in the contact problems.

Several techniques have been developed from modified IGA basis functions to offer a robust local refinement strategy within the IGA scheme, including T-Splines [20], polynomial splines over hierarchical T-meshes [21], hierarchically refined B-Splines [22] and locally refined B-Splines [23]. However, these approaches are difficult to be represented and computed due to their complex formulations. Zhang and Wang [24] have recently proposed a novel local refinement strategy for IGA in a direct meshfree manner by combining IGA with meshfree approximations under the reproducing condition. The applications of isogeometric-meshfree (IMF) coupling methods should be further explored.

1.3. Objectives

Developing robust and effective numerical solutions for fracture and contact problems has both significant theoretical and practical contributions to modern engineering. Fracture simulation requires an accurate and efficient representation of deformation and stress fields surrounding the crack path. Contact modelling can be enhanced by preserving the exact geometry and high-order discretisation of contact surfaces. IGA is developed as a promising candidate for addressing all these issues above. The present Ph.D. study aims to develop IGA-based approaches that offer an exact representation of geometry and high

flexibility of local refinement for both fracture and contact problems. The main objectives of the present research are as follows:

- To investigate the fracture behaviours of cracked plates subjected to imposed pressure through IGA-based approaches;
- To study the elastic behaviours of two bodies in contact by developing an isogeometric-meshfree coupling method;
- To predict the plastic limit load of cracked structures by introducing an effective adaptive local refinement strategy into an IMF coupling approach;
- To provide guidance for the applications of structures and materials containing cracks under given loadings.

1.4. Dissertation layout

This Ph.D. dissertation is organised into six chapters. Following this introductory chapter, Chapter 2 provides a literature review on the fundamental concepts of fracture and contact problems and their numerical solutions, including FEM, meshfree methods, IGA and isogeometric-meshfree coupling approach. In Chapter 3, a Bézier extraction-based XIGA approach is developed to analyse the buckling behaviours of cracked plates. Chapter 4 focuses on solving contact problems by integrating an IMF coupling approach with the third medium method. In Chapter 5, an adaptive IMF coupling approach is further developed for the plastic collapse mechanisms of cracked structures. In Chapter 6, the main achievements of this dissertation are concluded, and future works are recommended.

Chapter 2 Literature Review

In this chapter, the fundamentals of fracture mechanicals of structures are firstly introduced, followed by another review of previous studies on contact mechanics. Afterwards, different numerical solutions for fracture and contact problems are presented. Finally, a summary is given at the end of this chapter.

2.1. Fracture problems

Failure of structural components could be triggered by many factors such as uncertainties in loading and defects in materials. Among these factors, crack is one of the priorities for the fracture analysis. The presence of cracks might cause the fracture of components when the stresses are below the yield strength of materials.

Since cracks cannot be avoided in materials during their manufacturing or utilisation processes, the linear elastic fracture mechanics (LEFM) is developed to take the stress, geometry and crack size into consideration for fracture problems. In 1921, Griffith [2] was the first one who performed a systematic investigation of the fracture of brittle materials by developing a theory based on a simple energy balance. Griffith's fracture criteria state that the fracture will occur when the negative energy release rate G equals that required for forming new crack surfaces as

$$-G = \frac{dW_s}{dA}, \quad (2.1)$$

where W_s denotes the surface energy, and A denotes area of the crack. energy release rate G can be further defined in terms of the total potential energy Π as

$$G = \frac{d\Pi}{dA}. \quad (2.2)$$

However, LEFM can only be applied to materials such as glass and ceramics that behave in a linear elastic manner. Ductile materials such as metals and alloys usually undergo plastic deformation due to the increase in the applied load. By taking plasticity effects into account for the fracture analysis, Eq. (2.3) should be reformulated as [25]

$$-G = \frac{dW_s}{dA} + \frac{dW_p}{dA}, \quad (2.3)$$

where W_p denotes the plastic energy.

Fig. 2.1. Three types of crack mode.

Cracks can be generally classified into three types according to their loading conditions: opening mode (Mode I), in-plane shear mode (Mode II) and out-of-plane shear mode (Mode III) as illustrated in Fig. 2.1. For a Mode I crack, applied tension is vertical to the crack plane. For a Mode II crack, applied shear is parallel to the crack plane while being vertical to the crack front. For a Mode III crack, applied shear is simultaneously parallel to the crack plane and crack front. Practically, more than one types of the above loading

conditions are generally applied to engineering components. Cracks within these components can be defined as mixed-mode cracks [26]. In order to model the displacement and stress fields around crack tips for Mode I, II and III cracks, Irwin [25] developed their corresponding stress intensity factors (SIFs) K_I , K_{II} and K_{III} . The energy release rate G is further expressed by SIFs as

$$G = \frac{1}{E} (K_I^2 + K_{II}^2) + \frac{K_{III}^2}{2\mu} \text{ for the plane stress condition} \quad (2.4)$$

and

$$G = \frac{1-\nu^2}{E} (K_I^2 + K_{II}^2) + \frac{K_{III}^2}{2\mu} \text{ for the plane strain condition,} \quad (2.5)$$

where μ denotes the shear modulus, E denotes the Young's modulus and ν denotes the Poisson's ratio of the matrix material.

It should be noticed that the crack-tip stress fields are assumed to be singular based on LEFM, whereas stresses are practically bounded by the plastic deformation of materials. This plasticity effect is taken into consideration in the Dugdale model [3] and Irwin model [4]. Dugdale model adopted a concept of the thin strip to describe the plastic zone of crack tips. The size of a thin strip is calculated by neglecting the SIFs caused by the given load and the closure stress. Irwin model focused on estimating the extent ahead of a crack tip. This extent was regarded as the boundary between elastic and plastic regions around the crack tip, and the size of extent was determined based upon the assumption that crack-tip stress equals to the yield stress of the material. These two theoretical models are the foundations of elastic-plastic fracture mechanics and used for the analysis of different fracture configurations in ductile materials [27-30].

2.1.1. Fracture mechanics of plates

Plates are applied in many architectural structures, including bridges, containers, aeroplanes, ships and machine parts. Plates are defined as flat structural elements with a small thickness compared to other dimensions. Many engineering problems involve the crack loaded in a combination of in-plane tension and out-of-plane shear [31]. As a result, a cracked plate generally contains mixed-mode crack tip stress fields in real cases. Fig. 2.2 illustrates four different types of loading conditions for a central cracked plate based upon different combinations of tension and shear. The crack within this plate cannot be considered only subjected to Mode I load. The Kirchhoff–Love plate or the Mindlin–Reissner plate theories can be adopted to describe the out-of-plane crack tip stresses for elastic plates.

Fig. 2.2. An edge-cracked plate subjected to four types of loading conditions.

(a) The Kirchhoff–Love theory-based crack-tip fields

Based on the assumption of small strain, linear elastic, isotropic, homogeneous materials, Love and Darwin [32] developed the Kirchhoff–Love theory based on the research by Kirchhoff [33]. It is also the simplest solution for the fracture problems shown in Fig. 2.2 when the transverse shear deformation effects are neglected. Crack-tip stress fields generated by in-plane and out-of-plane loading can be obtained through integration of plate theory-based stress fields and plane stresses [34]. Based on the model of an infinite plate under the in-plane tension, Williams [35] firstly investigated the near crack-tip displacement and stress fields around a sharp crack on the mid-plane of the plate and found that the maximum principal tension stress could take place on $\pm 60^\circ$ to the crack path. Stahl and Keer [1] utilised the homogeneous Fredholm integral equations to develop analytical solutions for cracked rectangular plates under mixed boundary conditions. Their analytical solutions are adopted in this study for the validation of the proposed IGA-based approaches.

Fig. 2.3. Stress components around one tip of a crack in an infinite plane subjected to in-plane tension.

Through the formation of a polar coordinate system (r, θ) at the crack tip shown in Fig. 2.3, the detailed expressions of the crack-tip stress and displacement fields are computed by [34]

$$\begin{aligned} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \begin{Bmatrix} 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \\ 1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \end{Bmatrix} \\ &+ \frac{K_{II}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin\left(\frac{\theta}{2}\right) \left[2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right] \\ \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] \end{Bmatrix} \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} &= \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos\left(\frac{\theta}{2}\right) \left[\frac{1-\nu}{1+\nu} + \sin^2\left(\frac{\theta}{2}\right) \right] \\ \sin\left(\frac{\theta}{2}\right) \left[\frac{2}{1+\nu} - \cos^2\left(\frac{\theta}{2}\right) \right] \end{Bmatrix} \\ &+ \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin\left(\frac{\theta}{2}\right) \left[\frac{2}{1+\nu} + \cos^2\left(\frac{\theta}{2}\right) \right] \\ -\cos\left(\frac{\theta}{2}\right) \left[\frac{1-\nu}{1+\nu} - \sin^2\left(\frac{\theta}{2}\right) \right] \end{Bmatrix}, \end{aligned} \quad (2.7)$$

where μ is the shear modulus, ν is the Poisson's ratio, K_I is the tensile SIF given by

$$\lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{\theta\theta}(r, 0) \text{ and } K_{II} \text{ is the shear SIF given by } \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{r\theta}(r, 0).$$

Williams [35] further calculated the stress and deflection fields for the bending problem. Based on this work, Hui and Zehnder [36] computed the crack-tip stress and displacement field around the crack within a plate shown in Fig. 2.4. The detailed expressions based upon the crack tip polar coordinates (r, θ) are given by

$$\begin{aligned} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{Bmatrix} &= \frac{k_1 z}{2h\sqrt{2r}(3+\nu)} \begin{Bmatrix} (3+5\nu)\cos\left(\frac{\theta}{2}\right) - (7+\nu)\cos\left(\frac{3\theta}{2}\right) \\ (5+3\nu)\cos\left(\frac{\theta}{2}\right) + (7+\nu)\cos\left(\frac{3\theta}{2}\right) \\ -(1-\nu)\sin\left(\frac{\theta}{2}\right) + (7+\nu)\sin\left(\frac{3\theta}{2}\right) \end{Bmatrix} \\ &+ \frac{k_2 z}{2h\sqrt{2r}(3+\nu)} \begin{Bmatrix} -(3+5\nu)\sin\left(\frac{\theta}{2}\right) + (5+3\nu)\sin\left(\frac{3\theta}{2}\right) \\ -(5+3\nu)\left[\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\right] \\ -(1+\nu)\cos\left(\frac{\theta}{2}\right) + (5+3\nu)\cos\left(\frac{3\theta}{2}\right) \end{Bmatrix}, \end{aligned} \quad (2.8)$$

$$\begin{Bmatrix} \sigma_{rz} \\ \sigma_{\theta z} \\ \tau_{zz} \end{Bmatrix} = \frac{h}{2^{\frac{5}{2}} r^{\frac{3}{2}} (3+\nu)} \left[1 - \left(\frac{2z}{h} \right)^2 \right] \begin{Bmatrix} -k_1 \cos\left(\frac{\theta}{2}\right) + k_2 \sin\left(\frac{\theta}{2}\right) \\ -k_1 \sin\left(\frac{\theta}{2}\right) - k_2 \cos\left(\frac{\theta}{2}\right) \\ 0 \end{Bmatrix} \quad (2.9)$$

and

$$\begin{aligned} w &= \frac{(2r)^{\frac{3}{2}}(1-\nu^2)}{2Eh(3+\nu)} \left\{ k_1 \left[\frac{1}{3} \left(\frac{7+\nu}{1-\nu} \right) \cos\left(\frac{3\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right) \right] \right. \\ &\quad \left. + k_2 \left[-\frac{1}{3} \left(\frac{5+3\nu}{1-\nu} \right) \sin\left(\frac{3\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \right] \right\}, \end{aligned} \quad (2.10)$$

where E and h denote the Young's modulus and the thickness of the plate, respectively.

According to the research by Sih *et al.* [34], k_1 is the bending SIF given by

$\lim_{r \rightarrow 0} \sqrt{2r} \sigma_{\theta\theta}(r, 0, h/2)$, and k_2 is the twisting SIF given by $\lim_{r \rightarrow 0} \frac{3+\nu}{1+\nu} \sqrt{2r} \sigma_{r\theta}(r, 0, h/2)$.

By integrating the plane stresses with the Kirchhoff–Love plate theory fields, the expression of total energy release rate G can be reformulated by [36]

$$G = \frac{1}{E} (K_I^2 + K_{II}^2) + \frac{\pi}{3E} \left(\frac{1+\nu}{3+\nu} \right) (k_1^2 + k_2^2). \quad (2.11)$$

The Kirchhoff–Love theory has been widely adopted to solve fracture problems of plates due to its simplicity [37-39]. Nonetheless, irregularities are generated by the near tip bending stress fields because the deformation is assumed to be entirely caused by the bending and in-plane stretching.

Fig. 2.4. A coordinate system located at the crack tip on the mid-plane of a plate.

(b) The Mindlin–Reissner theory-based crack tip fields

The Mindlin–Reissner plate theory is a shear deformation plate theory which takes both transverse shear and transverse normal effects into account [40, 41]. Knowles and Wang [42] determined the asymptotic stress field for a central cracked infinite plate subjected to symmetric bending. Wang [43] found that the plate thickness has no effects on the angular distribution of the stress fields under symmetric bending, whereas the bending SIF is proportional to the ratio of plate thickness to crack length. He further calculated the asymptotic stresses for a central cracked elastic plate subjected to twisting [44]. Delale and Erdogan [45] obtained SIFs for a semi-elliptic and a rectangular crack in

an infinite plate under twisting and transverse shearing. The asymptotic crack-tip stress and displacement fields associated with the coordinate system depicted in Fig. 2.4 are given by [36]

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{Bmatrix} = \frac{K_1 z}{2h\sqrt{2r}} \begin{Bmatrix} 5\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2}\right) \\ 3\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{3\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right) \end{Bmatrix} + \frac{K_2 z}{2h\sqrt{2r}} \begin{Bmatrix} -5\sin\left(\frac{\theta}{2}\right) + 3\sin\left(\frac{3\theta}{2}\right) \\ -3\sin\left(\frac{\theta}{2}\right) - 3\sin\left(\frac{3\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) - 3\cos\left(\frac{3\theta}{2}\right) \end{Bmatrix}, \quad (2.12)$$

$$\begin{Bmatrix} \sigma_{rz} \\ \sigma_{\theta z} \end{Bmatrix} = \frac{K_3}{\sqrt{2r}} \left[1 - \left(\frac{2z}{h} \right)^2 \right] \begin{Bmatrix} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{Bmatrix} \quad (2.13)$$

and

$$\begin{aligned} w = \frac{\sqrt{2}r^{\frac{3}{2}}(1-\nu)}{Eh} & \left\{ K_1 \left[\frac{1}{3} \left(\frac{7+\nu}{1-\nu} \right) \cos\left(\frac{3\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right) \right] \right. \\ & \left. + K_2 \left[-\frac{1}{3} \left(\frac{5+3\nu}{1-\nu} \right) \sin\left(\frac{3\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \right] \right\} + \frac{8\sqrt{2}(1+\nu)}{5E} \left[K_3 \sin\left(\frac{\theta}{2}\right) \right], \end{aligned} \quad (2.14)$$

where K_1 , K_2 and K_3 can be given by $\lim_{r \rightarrow 0} \sqrt{2r} \sigma_{\theta\theta}(r, 0, h/2)$, $\lim_{r \rightarrow 0} \sqrt{2r} \sigma_{r\theta}(r, 0, h/2)$ and $\lim_{r \rightarrow 0} \sqrt{2r} \sigma_{\theta z}(r, 0, 0)$, respectively. The total energy release rate G based on the Mindlin–

Reissner plate theory can thus be determined by

$$G = \frac{1}{E} (K_I^2 + K_{II}^2) + \frac{\pi}{3E} \left(\frac{1+\nu}{3+\nu} \right) \left(K_1^2 + K_2^2 + \frac{8(1+\nu)}{5} K_3^2 \right). \quad (2.15)$$

The Kirchhoff–Love theory is usually applied to the fracture problem of the thin plate involving small scale yielding, while the Mindlin–Reissner theory is a valid choice for a

thick plate. Nevertheless, neither of these two plate theories is applicable to the plastic deformation of cracked ductile materials.

(c) Experiment observations of fracture in plates

The earliest experimental study on fracture behaviour of the cracked plate can be dated back to 1962 when Erdogan *et al.* [46] validated the bending SIFs theory through tests on cracked Plexiglas sheets under pure cylindrical bending. It was found that the experimental results slightly deviated from the theoretical solutions. The discrepancy might be induced by the experimental errors, complex geometries and material heterogeneity. Paik *et al.* [47] performed both experimental and numerical validation on the maximum strength to assess the ultimate strength of a cracked plate under axial compression.

Wynn and Smith [48] further investigated fracture behaviours of Plexiglas plates subjected to both tension and bending. The experiment results were found to agree well with a fracture criterion originated from the Sih–Hartranft bending theory [34]. Smith and Smith [49] further proved that the Sih–Hartranft bending theory was close to reality through the photoelastic testing on the stress fields in a cracked plate subjected to bending. Besides cracked homogeneous plates, Chandra [50] systematically studied the fracture performance of the patched cracked composite plates. Experimental and theoretical values of SIFs were compared and found to be in good agreement with each other.

2.1.2. Plastic collapse of cracked structures

Plastic collapse is a commonly encountered failure mode as there is maximum ductility for almost every engineering structure and material. This phenomenon generally

occurs when the number and disposition of plastic zones are sufficient to convert the structure into a mechanism. For an elastic-perfectly plastic material, a further increase in the deformation of structures can be induced while the given loads remain constant. As a traditional technique, plastic analysis is widely used to assess the collapse behaviour of structures by determining its plastic limit load [51]. Many analytical solutions, including the yield line method [52] and strip method [53], have been developed for the plastic analysis. Slip-line method [54] has been further exploited to assess the limit load factors of cracked structures. The exact limit load for specific structures with simple geometries could be obtained by these methods, whereas they were mainly based on hand calculation.

Computationally effective plastic analysis methods are required to be developed to deal with complex geometries and loading conditions, especially for the cracked structures. Limit analysis has been proposed as an effective plastic analysis method that is easy to be incorporated with mathematical programming theory [55, 56]. Finding the minimum multiple of applied load distribution that triggers the plastic collapse of structures is the key to limit analysis. To estimate the exact limit load of structures, limit analysis is required to be governed by three basic relations: equilibrium, compatibility and constitutive relation. Based upon these relations, lower bound and upper bound solutions have been subsequently developed.

Fig. 2.5. Limit analysis systems: (a) a lower bound system and (b) an upper bound system.

Lower bound solutions focus on formulating the equilibrium equations for a force system subjected to applied loads and chosen yield conditions. For a body with a volume V shown in Fig. 2.5(a), its total surface area S is divided into surface S_t and S_u that are under a set of prescribed tractions $\mathbf{t}$ and unknown tractions $\mathbf{t}_u$, respectively. Meanwhile, the volume V is subjected to fixed body forces $\mathbf{f}_b$ and unknown body forces $\mathbf{f}_u$. The object of lower bound limit analysis is to maximise the total unknown energy W as

$$W = \int_{S_u} \mathbf{t}_u dS + \int_V \mathbf{f}_u dV. \quad (2.16)$$

Also, the stress distribution should satisfy the equilibrium throughout V and the traction boundary condition $\mathbf{t}_u$ over S_u . The limit load factor obtained by this approach will always be below the exact one if the critical distribution of internal actions is not identified.

Upper bound limit analysis alternatively utilises virtual work principles to achieve the minimum load at the collapse state of materials. As shown in Fig. 2.5(b), the upper bound method aims to determine a velocity field $\mathbf{u}$ that fulfils the flow rule of compatibility and

the prescribed velocities $\mathbf{w}$ acting on the surface S_w , and minimise the internal dissipated energy given by

$$W_{\text{in}} = \int_V \boldsymbol{\sigma} \dot{\boldsymbol{\varepsilon}} dV, \quad (2.17)$$

where $\boldsymbol{\sigma}$ denotes the stress field, $\boldsymbol{\varepsilon}$ denotes the strains and $\dot{\boldsymbol{\varepsilon}}$ denotes the plastic strain rates. At the collapse state, the minimum W_{in} should be equal to the dissipated work induced by the external load as

$$W_{\text{in}} = W_{\text{out}} = \int_{S_t} \mathbf{t}^T \mathbf{u} dS + \int_{S_u} \mathbf{t}_u^T \mathbf{u} dS + \int_V \mathbf{f}_b^T \mathbf{u} dV + \int_V \mathbf{f}_u^T \mathbf{u} dV. \quad (2.18)$$

The limit load factor determined by this approach will be larger than the real one unless the collapse mechanism that offers the lowest collapse load can be found among all possible ones.

Both lower and upper bound theorems aim at solving an infinite-dimensional saddle point problem. This problem becomes more complicated as nonlinearity could be introduced into the formulation under yield condition. Maier [57] addressed this issue through the linearization of convex yield condition. The resulting problem was simplified into a classical linear program that can be traditionally solved by the simplex method [58, 59] and interior-point methods [60, 61]. Andersen [62] computed the limit load by minimising a sum of norms subjected to a linear constraint. Miller [63] utilised a two-criterion failure assessment to investigate limit loads of structures with defects. Mackenzie *et al.* [64] developed an elastic compensation method which determined the lower bound limit through an iterative linear-elastic analysis. Ponter and Carter [65] extended this method to the limit analysis of cracked plates. Lei [66] adopted the J -integral to take bending into account for the plastic collapse of cracked plates. Lynch *et al.* [67]

investigated the relations between the crack length and the limit loads for cracked piping branch junctions. About the optimisation aspects, second-order cone programming (SOCP) was employed to analysis limit analysis problems [68-70]. Its application is further extended to assessing the plastic collapse of cracked structures [71].

2.2. Contact problems

The earliest study on the contact mechanics can be dated back to 1882 when Hertz [72] developed an analytical solution for the contact between two elastic bodies. As a benchmark example, the Hertzian contact problem is commonly used to validate the new numerical contact algorithms. Generally, the curvatures of the two contacting surfaces are represented by ellipsoidal shapes with local radii r_{x_α} and r_{y_α} along the x - and y -directions, respectively, where $\alpha = 1$ and 2 stands for the two contacting bodies. Such case can be further reduced to the elliptical contact model involving a half-space foundation and an ellipsoidal indenter, where the radii of curvatures r_x and r_y are respectively given by

$$\frac{1}{r_x} = \sum_{\alpha=1}^2 \frac{1}{r_{x_\alpha}} \text{ and } \frac{1}{r_y} = \sum_{\alpha=1}^2 \frac{1}{r_{y_\alpha}}. \quad (2.19)$$

Compared to the elliptical contact, circular contact is a more commonly used model in which the radii of curvatures of contacting bodies are denoted as r_α . It can be further simplified as a half-space being indented by a sphere and given by

$$\frac{1}{r} = \sum_{\alpha=1}^2 \frac{1}{r_\alpha}, \quad (2.20)$$

where R denotes the radius of the sphere.

By assuming only elastic deformation is involved in frictionless contact condition, the maximum Hertz pressure $p_{\max}$ and the Hertz radius a_h of the contact area can be obtained respectively as

$$p_{\max} = \left(\frac{6PE^2}{\pi^3 r^2} \right)^{\frac{1}{3}} \text{ and } a_h = \left(\frac{3rP}{4E} \right)^{\frac{1}{3}}, \quad (2.21)$$

where P is the normal loading and E is the effective modulus that can be further obtained by the Poisson's ratios ν_α and Young's moduli E_α as

$$\frac{1}{E} = \sum_{\alpha=1}^2 \frac{1-\nu_\alpha^2}{E_\alpha}. \quad (2.22)$$

It should be noticed that the dimensions of the contact area must remain small related to the radii of curvatures of contacting bodies r_α . Johnson [73] and Fischer–Cripps also [74] proposed analogous expressions of $p_{\max}$ and a_h .

At extreme point contact, the contact area would become linear with a constant width over the interfaces. The contacting elements are thus assumed to be infinitely long in one of the principal directions, which makes their stress σ and strain fields ε satisfy the plane strain condition as [75]

$$\sigma_y = \nu_1 \cdot (\sigma_x + \sigma_z) \text{ and } \varepsilon_y = 0, \quad (2.23)$$

where ν_1 denotes the Poisson's ratio of the half-plane. Such a model is equivalent to a half-plane indented by a cylinder with a radius r given by Eq. (2.20). The width of the linear contact area can still be referred to as the Hertz radius a_h . The expressions of a_c and maximum Hertz pressure $p_{\max}$ are reformulated by [72]

$$p_{\max} = \left(\frac{PE^2}{\pi r} \right)^{\frac{1}{2}} \text{ and } a_h = \left(\frac{4rP}{\pi E} \right)^{\frac{1}{2}}. \quad (2.24)$$

The point and line contact are the basic models for the development of dry contact treatments. Cattaneo [76] developed an analytical partial slip solution for the general elliptical contact under a tangential loading. Smith [77] took the surface friction and contact failure into consideration. McCallion and Truong [78] further investigated the influences of surface roughness on the contact stresses. Sackfield and Hills [79, 80] studied the stress field for the general elliptical contact by employing a closed-form solution. Yu and Bhushan [81] explored the plastic deformation of materials subjected to contact loading. Liu and Wang [82] investigated the thermoelastic deformations of the contact on the steady-state heat transfer condition. Besides from the analytical approaches, contact mechanics experiments have been conducted on a microscopic scale. Greenwood and Williamson [83] studied the rough contact by following the Gaussian distribution. Whitehouse and Archard [84] took the effect of the random surface on the contact stress distribution. Hu and Tonder [85] used a digital optical filter to study the contact surfaces containing both random and periodic parts.

Many numerical studies are performed on contact behaviours of homogeneous materials, whereas most engineering components contain defects such as cracks and inclusions. The presence of these defects would make the subsurface stress field and displacement more complicated to be calculated. Rosakis and Rosakis [86] proposed a stress-based method to treat the dislocation by choosing feasible strain energy functions for the total stored energy. Acharya [87] employed a continuum theory of geometrical dislocation in a neo-Hookean solid and exploited the singularity-free field solutions. Subsequently, the author [88] used the dislocation density tensor to study the thermodynamic problems of dislocations. Zubov [89] took internal degrees of freedom and couple stresses into account for dealing with the

dispersed dislocation at finite strains. Yavari and Goriely [90, 91] developed a measure of the screw and edge dislocation configurations and found that the singularity is related to the material model. Miller and Keer [92] took the sliding into account for the influences of inclusions on contact. King and O’Sullivan [93] investigated the contact of a 2D layered elastic plane and then extended their work to 3D cases. Liu and Farris [94] further explored the effects of inclusions and the fatigue on 3D sliding contact. Kuvshinov [95] developed the solutions for polyhedral inclusions through the surface integral-based Green’s function methods. By utilizing the elliptic integrals along the outward unit normal, Kim and Lee [96] studied the external field of elliptic cylindrical inclusions vector and proposed a closed-form solution. Goshima and Soda [97] took the crack into the consideration for 2D thermoelastic contact problems, and their work was further extended to problems involving two subsurface cracks. Yang and Zhou [98] investigated the elastic-plastic fracture behaviour of a cracked half-space subjected to contact loads. This contact model was further developed by Yang *et al.* [8] to study the interaction between inclusions and cracks.

2.3. Numerical solutions

Although analytical approaches for the fracture and contact problems have been developed for decades, they can only provide the exact solutions for certain cases with simple geometries and boundary conditions. For example, Zhou and Wei’s [99, 100] solutions to fracture problems are only available for an infinite half-space with defects. Analytical solutions to contact problems are also of capacity for handling large deformation, large sliding and material inhomogeneities [17]. Conversely, numerical solutions demonstrate the accuracy and efficiency in solving engineering problems that

involve complex fracture and contact phenomena. FEM, meshfree methods and IGA are three types of numerical methods that have been widely used to tackle practical fracture and contact problems.

2.3.1. Finite element method

FEM has been long available in various engineering fields, including fracture mechanics and contact mechanics. FEM can effectively solve the partial differential equations for complex geometries and boundary conditions by subdividing the whole system into simple and small finite elements. These elements are subsequently assembled into a large algorithm for the entire problem.

(a) FEM for fracture problems

For fracture analysis, Byskov *et al.* [101] employed the four-node triangular cracked elements around the crack tip to obtain accurate SIFs. Rybicki and Kanninen [102] modified the crack tip closure integral formulation to calculate SIFs for Mode I and II cracks via FEM. Barsoum [103] adopted the quadratic isoparametric element as an effective crack tip element in LEFM. Although various crack tip elements have been developed in the aforementioned research works, tedious remeshing was required to match the geometry of the discontinuities, resulting in a significant computational cost [104]. Moreover, Tong and Pian [105] found that singularity around the crack tip imposed difficulties on standard FEM solutions based upon piecewise differentiable polynomial shape functions.

To overcome the deficiencies of traditional FEM fracture modelling, Belytschko and Black [9] proposed XFEM to solve fracture problems without the conformity between meshes and boundaries of cracks or inclusions. XFEM was derived from the standard Galerkin method and local partition-of-unity property that allowed minimum remeshing at crack tips [106]. Moës *et al.* [107] and Dolbow *et al.* [108] further proposed the 2D XFEM formulation by adapting the Heaviside functions and enrichment functions which contain asymptotic near tip fields.

The overall displacement fields u_c based on the XFEM approximation are obtained by

$$u_c = \sum_{i \in I} N_i u_i + \sum_{j \in J} N_j H a_j + \sum_{o \in O} N_o \left[\sum_{m=1}^4 \chi_m b_{o_m} \right], \quad (2.25)$$

where N_i denotes the shape function corresponding to the FEM node i , I denotes the set of nodes within the problem domain, J denotes the set of nodes along the crack surface area, O stands for the number of finite element nodes around the crack tip, a_j indicates the jump in the displacement field across the crack path and b_{o_m} denotes the additional degrees of freedom (DOFs) associated with the crack-tip enrichment functions χ_m .

The Heaviside function $H(x)$ for a given point x in Eq. (2.26) is defined as

$$H(x) = \begin{cases} 1 & ((x - x') \cdot \hat{n} > 0) \\ -1 & ((x - x') \cdot \hat{n} < 0) \end{cases}, \quad (2.26)$$

where $\hat{n}$ denotes the normal vector unit of the crack path at point x , and x' denotes the projection points of x to the crack surface.

By formulating a polar coordinate system (r, θ) with its origin located at the crack tip, enrichment functions $\chi_m(r, \theta)$ can be further given by

$$\chi_m(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right); \sqrt{r} \cos\left(\frac{\theta}{2}\right); \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin \theta; \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin \theta \right\}. \quad (2.27)$$

Sukumar *et al.* [10] extended this method to the 3D modelling of static cracks. Belytschko *et al.* [109] enhanced the XFEM formulation to study arbitrary discontinuities and their derivatives in fracture problems. Dolbow *et al.* [110] explored the application of XFEM on the fracture problems of Mindlin–Reissner plates. Liu *et al.* [111] computed mixed mode SIFs for cracked structures and materials via XFEM. Bordas *et al.* [112] adopted a strain smoothing technique to preclude the shear locking. Baiz *et al.* [113] presented a unified formulation of smoothed quadrilateral finite elements and XFEM to evaluate the buckling load factors of cracked plates. Chen *et al.* [114] combined the edge-based strain smoothing technique with XFEM to predict the crack growth based upon LEFM. Nasirmanesh and Mohammadi [115] employed an 8-noded degenerated shell element to investigate the buckling behaviours of cracked composite plates. Liu *et al.* [116] introduced the discrete shear gap (DSG) technique into the 3-noded triangular plate element to analyse the buckling behaviours of cracked functionally graded plates. Tran *et al.* [117] further applied the XFEM to limit analysis of 2D cracked structures

(b) FEM for contact problems

Node-to-surface (NTS) algorithm is widely used in FEM-based simulation software. Within this method, a node on one of the contacting surfaces is set as a slave, while its edge on the other surface is denoted as a master. The contact constraints can thus be effectively enforced by applying integrals at the slave nodes. Hughes *et al.* [118, 119] firstly implemented this method on contact analysis. Wriggers and Simo [120] explored the

applications of NTS on fully non-linear contact problems. Zavarise and De Lorenzis [121] found that the NTS algorithm could not stratify the contact patch test as it cannot satisfy the balance of moments at the element level.

Surface-to-surface (STS) approach was developed to overcome the drawbacks of the NTS algorithm. The contact integrals are no longer computed over the slave nodes and enforce contact constraints integrally. Simo *et al.* [122] alternatively performed the integral over the contact segments through a perturbed Lagrangian formulation. Papadopoulos and Taylor [123] adopted a similar approach using 2D quadratic elements and later extended it to the 3D contact problems [124]. By adopting suitable intermediate surface segments, STS algorithms can fulfil the requirements of the contact patch test. However, the majority of STS methods are found to be not sufficient for the LBB stability condition [125]. Therefore, Gauss-point-to-surface (GPTS) method was developed to employ contact quadrature points over the contact surfaces where contact constraints could be enforced. Fischer and Wriggers [126] employed this method to integrate the weak form of contact contribution straightforwardly across the elements along the slave contact surface through a set of pre-determined Gauss-Legendre quadrature points. Papadopoulos *et al.* [127] developed an alternative GPTS formulation which treated each contacting surface as slave and master.

Bernardi *et al.* [128] firstly proposed the mortar methods whereby the contact constraints were non-locally enforced. Belgacem *et al.* [129] utilised the mortar FEM to solve contact problems of two elastic bodies. They subsequently studied the variational inequalities coming from the Signorini problem [130]. Yang *et al.* [131] adopted the mortar method to tackle frictional contact problems with large deformation and sliding. Hesch and

Betsch [132] used the mortar method based upon the assumption of frictionless dynamic contact condition to investigate the energy-momentum conserving schemes. Tur *et al.* [133] modified the mortar method by coupling it with a 2D contact smooth formulation. Puso *et al.* [134] proved that the mortar method could satisfy the LBB stability requirement and pass the contact patch test, whereas its mortar integrals require much higher computational cost than the previously reviewed contact algorithms.

An enhanced contact treatment technique has been proposed as the third medium method which eliminates the time-consuming contact detect algorithm [135]. The concept relies on introducing a fictitious isotropic/anisotropic medium between the contacting bodies. The penetration of contact surfaces can be prevented by selecting proper parameters to describe the constitutive behaviour of the third medium. Bog *et al.* [136] developed a similar approach using higher-order FEM shape functions. Khishvand *et al.* [137] explored the application of this method on frictionless contact problems in geomechanics.

2.3.2. Meshfree methods

Meshfree methods are developed as a potential alternative to FEM in tackling fracture and contact problems of materials and structures. Unlike conventional grid-based numerical methods, meshfree shape functions are defined from a scattered set of nodes over the problem domain, which offers more adaptivity and versatility in engineering problems involving complex discontinuities and large deformation. The meshfree method was initially proposed by Lucy [138] and Gingold and Monaghan [139] to model astrophysical phenomena but later was widely applied in fluid and solid mechanics. Especially for contact problems involving hyper-elastic materials undergoing large

deformations, FEM shows limitations on handling excessive mesh distortions that can be easily solved by meshfree methods, including element-free Galerkin (EFG) [140], meshless local Petrov-Galerkin (MLPG) [141] and reproducing kernel particle [142].

(a) Meshfree methods for fracture problems

Mesh-based FEM approaches exhibit ineffectiveness in treating material discontinuity as tedious remeshing is required around the crack tip. In response to this deficiency, meshfree methods have been developed to compute interpolation based on nodal support domains rather than elements. In this way, the local refinement can be performed straightforwardly without preserving the inter-element connectivity [143]. In 1994, Belytschko *et al.* [144] proposed the EFG method to solve 2D fracture problems involving static crack growth. This method eliminated computationally expensive remeshing but had limitations on treating meshfree nodes around the crack tip. Extrinsic and intrinsic enrichment techniques were further developed to enhance EFG-based fracture simulation [145, 146]. Sukumar *et al.* [147] extended the EFG method to solving static 3D fracture problems. Rabczuk and Belytschko [148] proposed EFG-Practical approach which could model arbitrary evolving cracks with no representation of their topologies. Zhang *et al.* [149] alternatively utilised an improved moving-least squares (MLS) approximation to improve the EFG for 2D and 3D fracture analysis. Rao and Rahman [150] applied the EFG method for the study of 2D cracked functionally graded plates. Ching and Batra [151] adopted the MLPG method to determine crack-tip fields of cracked plates in tension. Liu *et al.* [152] derived the MLPG5 formulation from the MLPG method to investigate the dynamic fracture behaviour of a cracked plate loaded in tension. Tanaka *et al.* [153]

combined the reproducing kernel meshfree approximation with the Mindlin–Reissner theory to analyse cracked plate bending problems.

(b) Meshfree methods for contact problems

Li *et al.* [154] firstly employed moment matrix of meshfree approximation to formulate a contact detection algorithm, which accurately detected contact for Taylor bar impact problem without solving complex equations. Tiago and Pimenta [155] combined EFG-based meshfree method with MLS approximation to solve the contact problems involving arbitrary large deformations for nonlinear plates. Hu *et al.* [156] studied the large deformation contact of elastomers by using the developed MLPG approach. Li and Lee [157] took sliding into account for large deformation contact problems and derived an adaptive meshless method whereby the meshfree nodes can be adaptively added into regions with large errors. Li *et al.* [158] combined complex variable theory with EFG method for 2D contact problems with large deformation. Ullah and Augarde [159] further explored the application of EFG-based adaptive meshless approach to contact problems involving non-linear materials. Although meshfree methods have shown potential in solving both fracture and contact problems, meshfree approximations are generally more computationally expensive than FEM approaches. It is because that integrals of meshfree shape functions require a large algorithm [160]. Such computational efficiency can be improved by coupling meshfree methods with finite elements and modifying the Galerkin weak form in meshfree integration, whereas the resulting formulations are complicated to be implemented [161].

2.3.3. Isogeometric analysis

IGA has been recently developed as a next-generation computational technique that bridges the gap between CAD and computer-aided engineering [15]. As two widely applied mathematical models in CAD, B-Spline and NURBS are adopted as IGA basis functions for the geometry representation and computation of displacement. A B-Spline is a one-dimensional (1D) piecewise polynomial curve with its basis functions $N_{i,p}(\xi)$ defined in terms of the knot vectors $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ as

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi_i \leq \xi < \xi_{i+1} \\ 0, & \xi < \xi_i \text{ or } \xi \geq \xi_{i+1} \end{cases} \quad (p=0); \quad (2.28)$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (p \geq 1), \quad (2.29)$$

where p denotes the polynomial order of basis functions, and n denotes the total number of IGA control points. Fig. 2.6(a) illustrates typical quadratic B-Spline basis functions associated with the knot vectors $\Xi = \{0, 0, 0, 1/3, 2/3, 1, 1, 1\}$. It should be noticed that the first and last knot vectors in Ξ should be repeated $p+1$ times. Consequently, a B-Spline curve $C_B(\xi)$ can be expressed by basis functions $N_{i,p}(\xi)$ and corresponding control points P_i as

$$C_B(\xi) = \sum_{i=1}^n N_{i,p}(\xi) P_i. \quad (2.30)$$

A B-Spline surface $S_B(\xi, \eta)$ is formed through the tensor product of basis functions $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$ along the ξ and η directions with respect to two sets of knot vectors $\Xi_1 = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ and $\Xi_2 = \{\eta_1, \eta_2, \dots, \eta_{m+q+1}\}$ as

$$S_B(\xi, \eta) = \sum_{i=1}^n \sum_{j=1}^m N_{i,p}(\xi) M_{j,q}(\eta) P_{i,j}, \quad (2.31)$$

where P_{ij} denotes control points. 2D B-Spline basis functions using the same knot vectors as those for 1D B-Spline are plotted in Fig. 2.6(b).

Fig. 2.6. (a) 1D and (b) 2D diagrams of quadratic B-Spline basis functions associated with the knot vectors Ξ given by $\{0, 0, 0, 1/3, 2/3, 1, 1, 1\}$.

NURBS is further developed from B-Spline which are more suitable to construct geometries in the CAD environment. By introducing weights ϖ into B-Spline, NURBS function $R_{i,p}(\xi)$ are subsequently given by

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)\varpi_i}{\sum_{i=1}^n N_{i,p}(\xi)\varpi_i}. \quad (2.32)$$

Subsequently, a NURBS curve $C_N(\xi, \eta)$ can be determined by

$$C_N(\xi) = \sum_{i=1}^n R_{i,p}(\xi) P_i. \quad (2.33)$$

A NURBS surface $S_N(\zeta, \eta)$ with the p^{th} and q^{th} orders in ζ and η directions and its basis functions can be expressed as

$$S_N(\zeta, \eta) = \sum_{i=1}^n \sum_{j=1}^m R_{i,j}^{p,q}(\zeta, \eta) P_{i,j}, \quad (2.34)$$

where 2D NURBS shape functions $R_{i,j}^{p,q}(\zeta, \eta)$ are given as

$$R_{i,j}^{p,q}(\zeta, \eta) = \frac{N_{i,p}(\zeta) M_{j,q}(\eta) \varpi_{i,j}}{\sum_{\hat{i}=1}^n \sum_{\hat{j}=1}^m N_{\hat{i},p}(\zeta) M_{\hat{j},q}(\eta) \varpi_{\hat{i},\hat{j}}}. \quad (2.35)$$

Compared to other numerical methods, IGA exhibits evident advantages in fracture and contact problems, including the geometry exactness and high-order approximation.

(a) Isogeometric analysis for fracture problems

Verhoosel *et al.* [162] adopted IGA to investigate the cohesive zone for material fractures through knot insertion, whereas the simulation results were affected by mesh distortion. De Luycker *et al.* [163] firstly incorporated XFEM with quartic NURBS basis functions to achieve a higher convergence rate for the LEFM problems. Based on this concept, Ghorashi *et al.* [16] further developed XIGA for modelling mixed-mode 2D stationary and propagating cracks. Subsequently, Ghorashi *et al.* [164] analysed the orthotropic cracked media by employing T-Spline as IGA basis functions. Singh *et al.* [165] incorporated the T-Spline XIGA into the existing FEM codes via the Bézier extraction method. Singh *et al.* [166] further took holes and inclusions into account for the fracture analysis of a static edge-cracked plane.

For fracture analysis of structures, Bhardwaj *et al.* [167] explored the application of XIGA on a cracked plate under different boundary and loading conditions. Ghorashi *et al.*

[168] developed a mapping technique that allowed modelling plates with curved cracks within the XIGA work-frame. Singh *et al.* [169] calculated moment intensity factors for plates with through-thickness cracks. Bhardwaj *et al.* [170] extended XIGA to simulate fatigue crack growth within the plate made from functionally graded material (FGM). Yu *et al.* [171] further investigated the internally defected FGM plates based on the Mindlin–Reissner theory for thermal buckling analysis. Alternatively, Tran *et al.* [172] conducted the vibration analysis of 2D cracked FGM plates through the high-order shear deformation plate theory. Tan *et al.* [173] further investigated the 3D crack problems of FGM plates by combining Bézier extraction method with XIGA. Bui [174, 175] utilised XIGA to study fracture behaviours of piezoelectric composite plates subjected to dynamic and static coupled electromechanical loads. Nguyen-Xuan *et al.* [176] incorporated XIGA with SOCP for the limit analysis of cracked plates.

(b) Isogeometric analysis for contact problems

IGA has also been incorporated with contact treatments that are originally available in FEM scheme. For the NTS algorithm, Matzen *et al.* [177] firstly developed its single-pass by collocating contact integrals over the physical nodes on the contacting surfaces associated to the control points. Benson *et al.* [178, 179] further derived a double-pass version through bilinear quadrilateral interpolation elements which allow direct integration of the proposed formulation with the existing FEM contact codes. Isogeometric NTS algorithm was found to be far superior to its FEM counterpart as all drawbacks related to the non-smooth discretizations are eliminated. The concept of isogeometric GPTS relies on the direct integration of Gauss-Legendre quadrature points over the slave surface where

contact constraints can be enforced. Temizer *et al.* [180] initially derived the knot-to-surface algorithm, whereas the contact constraints are not enforced over the physical points of the knot vector entries. Temizer [181] further solved thermomechanical contact via the isogeometric GPTS method. Dimitri *et al.* [182] employed a similar method for both 2D and 3D contact problems but using T-spline as the IGA basis functions. Both isogeometric GPTS and GPTS-2hp formulations are computationally efficient and sufficient for the contact patch test requirements. Nevertheless, the accuracy of simulation results is limited by the integration error.

The isogeometric mortar formulation projects contact constraints to the DOFs of the slave surface for obtaining a correct number of constraints. Kim and Youn [183] proposed this approach for 2D frictionless contact problems. Seitz *et al.* [184] further developed the isogeometric dual mortar methods. The isogeometric mortar methods deliver several advantages over its FEM counterpart, including non-negative contact pressure distribution, smooth discretisation and insensitive to the change of interpolation order. However, the mortar methods are highly computational expensive induced by the storage and computation of the mortar integrals. Kruse *et al.* [185] combined IGA with the third medium method to solve contact problems with large deformation and sliding.

Although many IGA-based approaches have been developed for both fracture and contact problems, standard NURBS basis functions suffer from the inflexibility of local refinement. Nguyen-Thanh *et al.* [186] presented an adaptive IGA mesh refinement approach for the fracture analysis based on the PHT-splines. Chen *et al.* [187] adopted truncated hierarchical NURBS to perform adaptive refinement strategy within the IGA work-frame. However, the implementation of adaptive IGA mesh refinement is

complicated owing to the tensor product structure of NURBS [188]. Alternatively, Wang and Zhang [189] coupled the IGA with the reproducing kernel particle method on the parametric domain. This approach was subsequently extended to large deformation [190], crack modelling [191] and contact problems [192]. Other research works have been performed on integrating IGA with meshfree approximations on the physical domain [193, 194]. Nevertheless, the approaches above require the whole problem domain to be represented by different basis functions, resulting in a complex algorithm. Therefore, Zhang and Wang [24] unified the meshfree and IGA basis functions across the domain by modifying meshfree nodes, support size and consistency conditions.

2.4. Summary

In this chapter, a brief review is performed on fracture problems, including fracture mechanics of cracked plates and the plastic collapse of cracked structures. Analytical approaches for these two types of fracture problems are subsequently given in details. Afterwards, basic concepts of contact problems and their corresponding analytical solutions are literally presented. Lastly, three commonly used numerical solutions for fracture and contact problems (FEM, meshfree methods and IGA) are discussed in detail.

Chapter 3 A Bézier Extraction-Based XIGA Approach for the Buckling of Cracked Plates

Buckling is one of the most common failure modes for plates under compression when such structures contain defects such as cracks or holes. Due to the presence of cracks, complex wrinkling deflection patterns could take place around such imperfections when the critical buckling load is reached. In this chapter, the effects of various crack locations, crack lengths and plate aspect ratios on the buckling load factors of cracked plates are investigated through a Bézier extraction-based XIGA approach.

3.1. Problem description and solution approach

In this chapter, governing equations for the buckling analysis are based upon the Mindlin–Reissner plate theory. As shown in Appendix A, XIGA is further coupled with the Bézier extraction method to allow direct computation of XIGA-based buckling analysis via the existing FEM codes.

3.1.1. Buckling of Mindlin–Reissner plates

Let Ω denote a domain in $\mathbb{R}^2$ of a flat isotropic plate with a thickness h under a transverse loading per unit area f . The plate is assumed to be homogenous with the Poisson's ratio ν and the Young's modulus E . The mid-surface of the plate is taken as the reference plane as illustrated in Fig. 3.1. w_0 and $\boldsymbol{\beta} = [\beta_x \ \beta_y]^T$ are introduced as parameters to present the displacement components along the x , y and z axes and the rotation in the x -

z and y - z planes, respectively. The displacements fields within the Mindlin–Reissner plate can be calculated by [195]

$$\begin{aligned} u(x, y, z) &= z\beta_x(x, y); \\ v(x, y, z) &= z\beta_y(x, y); \\ w(x, y, z) &= w_0(x, y). \end{aligned} \quad (3.1)$$

The bending strains ϵ and shear strains γ for the plate are written as

$$\epsilon = \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} \beta_x + \frac{\partial w}{\partial x} \\ \beta_y + \frac{\partial w}{\partial y} \end{bmatrix}. \quad (3.2)$$

Fig. 3.1. A Mindlin–Reissner plate.

Consequently, the governing differential equations involving ϵ and γ in the domain Ω are given by [195]

$$-div(\mathbf{D}_b \cdot \epsilon \beta) - \lambda h \gamma \beta = 0 \quad (3.3)$$

and

$$-\lambda h \cdot div(\gamma) = p, \quad (3.4)$$

where $\mathbf{D}_b$ denotes the tensor of bending moduli, and λ is given by $k_s E/2(1+\nu)$ with the shear correction factor k_s given by 5/6.

The displacement and transverse load are defined by $\mathbf{u} = [w \ \beta_x \ \beta_y]^T$ and $\mathbf{f} = [f \ 0 \ 0]^T$, respectively. The weak form of the static equilibrium formula is given by

$$\int_{\Omega} (\delta \boldsymbol{\varepsilon})^T \mathbf{D}_b \boldsymbol{\varepsilon} d\Omega + \int_{\Omega} (\delta \boldsymbol{\gamma})^T \mathbf{D}_s \boldsymbol{\gamma} d\Omega = \int_{\Omega} \delta \mathbf{u}^T \mathbf{f} d\Omega. \quad (3.5)$$

By neglecting other external forces and high powers, the weak form is rewritten by

$$\begin{aligned} & \int_{\Omega} (\delta \boldsymbol{\varepsilon})^T \mathbf{D}_b \boldsymbol{\varepsilon} d\Omega + \int_{\Omega} (\delta \boldsymbol{\gamma})^T \mathbf{D}_s \boldsymbol{\gamma} d\Omega + h \int_{\Omega} \nabla^T (\delta w) \hat{\boldsymbol{\sigma}}_0 \nabla w d\Omega \\ & + \frac{h^3}{12} \int_{\Omega} [\nabla^T (\delta \beta_x) \ \nabla^T (\delta \beta_y)] \begin{bmatrix} \hat{\boldsymbol{\sigma}}_0 & 0 \\ 0 & \hat{\boldsymbol{\sigma}}_0 \end{bmatrix} \begin{bmatrix} \nabla \beta_x \\ \nabla \beta_y \end{bmatrix} d\Omega = 0, \end{aligned} \quad (3.6)$$

where $\nabla^T = [\partial/\partial x \ \partial/\partial y]$, and the bending moduli $\mathbf{D}_b$ and shear moduli $\mathbf{D}_s$ in the above equation are respectively defined by

$$\mathbf{D}_b = \frac{Eh^3}{12(1-\nu)^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \text{ and } \mathbf{D}_s = \frac{k_s Eh}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.7)$$

$\hat{\boldsymbol{\sigma}}_0$ in the Eq. (3.6) denotes the pre-buckling stress fields that are given by

$$\hat{\boldsymbol{\sigma}}_0 = \begin{bmatrix} \sigma_x^0 & \sigma_{xy}^0 \\ \sigma_{xy}^0 & \sigma_y^0 \end{bmatrix}, \quad (3.8)$$

where σ_x^0 , σ_y^0 and σ_{xy}^0 are the pre-buckling stresses.

By introducing geometrical strains $\boldsymbol{\kappa}$ into the above equation, a revised weak form formula can be obtained as

$$\int_{\Omega} (\delta \boldsymbol{\varepsilon})^T \mathbf{D}_b \boldsymbol{\varepsilon} d\Omega + \int_{\Omega} (\delta \boldsymbol{\gamma})^T \mathbf{D}_s \boldsymbol{\gamma} d\Omega + \int_{\Omega} (\delta \boldsymbol{\kappa})^T \boldsymbol{\tau} \boldsymbol{\kappa} d\Omega = 0, \quad (3.9)$$

where the stress fields $\boldsymbol{\tau}$ can be defined in terms of $\hat{\boldsymbol{\sigma}}_0$ and h as

$$\boldsymbol{\tau} = \begin{bmatrix} h\hat{\boldsymbol{\sigma}}_0 & 0 & 0 \\ 0 & \frac{h^3\hat{\boldsymbol{\sigma}}_0}{12} & 0 \\ 0 & 0 & \frac{h^3\hat{\boldsymbol{\sigma}}_0}{12} \end{bmatrix}. \quad (3.10)$$

Next, the bounded domain Ω will be discretised into n_e finite elements Ω^e as $\Omega = \bigcup_{e=1}^{n_e} \Omega^e$. By adopting NURBS basis functions R , the solution of lateral displacement and rotations for Mindlin–Reissner plates at point $\mathbf{x}_i = [x_i \ y_i]^T$ ($i=1,2,\dots,n$) can be obtained by

$$w = \sum_{i=1}^n R_i(\mathbf{x})w_i, \beta_x = \sum_{i=1}^n R_i(\mathbf{x})\theta_{x_i} \text{ and } \beta_y = \sum_{i=1}^n R_i(\mathbf{x})\theta_{y_i}, \quad (3.11)$$

where n represents the total number of control points.

$\mathbf{q}_i = [w_i \ \theta_{x_i} \ \theta_{y_i}]$ are given for representing the DOFs for the control point i . The bending, shear and geometrical strains are rewritten by

$$\boldsymbol{\varepsilon} = \sum_{i=1}^n \mathbf{B}_{b_i} \mathbf{q}_i, \boldsymbol{\gamma} = \sum_{i=1}^n \mathbf{B}_{s_i} \mathbf{q}_i \text{ and } \boldsymbol{\kappa} = \sum_{i=1}^n \mathbf{B}_{g_i} \mathbf{q}_i. \quad (3.12)$$

The B-operators in Eq. (3.10) are defined as strain gradient matrices and given by [195]

$$\mathbf{B}_{b_i} = \begin{bmatrix} 0 & R_{i,x} & 0 \\ 0 & 0 & R_{i,y} \\ 0 & R_{i,y} & R_{i,x} \end{bmatrix}, \mathbf{B}_{s_i} = \begin{bmatrix} R_{i,x} & R_i & 0 \\ R_{i,y} & 0 & R_i \end{bmatrix} \text{ and } \mathbf{B}_{g_i} = \begin{bmatrix} R_{i,x} & 0 & 0 \\ R_{i,y} & 0 & 0 \\ 0 & R_{i,x} & 0 \\ 0 & R_{i,y} & 0 \\ 0 & 0 & R_{i,x} \\ 0 & 0 & R_{i,y} \end{bmatrix}. \quad (3.13)$$

From Eqs. (3.11) and (3.12), the buckling behaviours of a Mindlin–Reissner plate are obtained by

$$(\mathbf{K} - \lambda_{\text{cr}} \mathbf{K}_g) \mathbf{q} = 0, \quad (3.14)$$

where λ_{cr} is the critical buckling factor, and global matrix $\mathbf{K}$ and geometrical stiffness matrix $\mathbf{K}_g$ are given by

$$\mathbf{K} = \int_{\Omega} (\mathbf{B}_b)^T \mathbf{D}_b \mathbf{B}_b d\Omega + \int_{\Omega} (\mathbf{B}_s)^T \mathbf{D}_s \mathbf{B}_s d\Omega \quad (3.15)$$

and

$$\mathbf{K}_g = \int_{\Omega} (\mathbf{B}_g)^T \boldsymbol{\tau} \mathbf{B}_g d\Omega. \quad (3.16)$$

It should be noted that the buckling approximation in this chapter is based on the linear basis functions with displacement and rotations. Unlike the bending strains, the transverse shear strains contain linear interpolated functions and become spurious during the simulation, resulting in the shear locking. To address this issue, the DSG method [196-198] is further introduced into the buckling formulation, which is detailed in Appendix A.

3.1.2. XIGA formulations for cracked plates

Based on Eq. (2.25), an enriched approximation of the displacements $\mathbf{u}(\xi)$ at a point $\xi = (\xi, \eta)$ on the parametric domain is expressed in a generalized form as

$$\mathbf{u}_c(\xi) = \sum_{i \in I} R_i(\xi) \mathbf{q}_i + \sum_{j \in J} R_j(\xi) H \mathbf{a}_j + \sum_{o \in O} R_o(\xi) \left[\sum_{m=1}^4 \chi_m \mathbf{b}_{o_m} \right], \quad (3.17)$$

where I denotes all the control points on the parametric domain, J denotes control points along the crack surface area, O denotes control points around the crack tip area. $R(\xi)$ in the above equation denotes the Bézier extraction-based NURBS basis functions, the set of deflection and rotations in the displacement field $\mathbf{q}$ is given by $[_w \beta_x \beta_y]^T$ is, and $\mathbf{a}$ and $\mathbf{b}$ are the additional DOFs associated with the crack surface and tip. H denotes the Heaviside

functions and enrichment functions χ for a Mindlin–Reissner plate should be obtained by modifying Eq. (2.27) as

$$\chi(r, \theta) \begin{cases} r^{\frac{3}{2}} \sin \frac{\theta}{2}; r^{\frac{3}{2}} \cos \frac{\theta}{2}; r^{\frac{3}{2}} \sin \frac{3\theta}{2}; r^{\frac{3}{2}} \cos \frac{\theta}{2} & (\text{for } w) \\ r^{\frac{1}{2}} \sin \frac{\theta}{2}; r^{\frac{1}{2}} \cos \frac{\theta}{2}; r^{\frac{1}{2}} \cos \frac{\theta}{2} \sin \theta; r^{\frac{1}{2}} \cos \frac{\theta}{2} \sin \theta & (\text{for } \beta_x \text{ and } \beta_y) \end{cases} \quad (3.18)$$

3.2. Numerical results for the convergence test

The expressions derived in the previous content are applied to calculate the normalised buckling load factor λ_n for rectangular isotropic plates with and without the crack which is defined by

$$\lambda_n = \frac{12(1-\nu^2)l_1^2}{\pi^2 E h^3} \lambda_{cr}, \quad (3.19)$$

where h is the plate thickness, l_1 stands for the edge width of the plate, λ_{cr} represents the critical load factor, ν is the Poisson's ratio and E stands for the Young's modulus. A set of polynomial degrees $q = 2, 3$ and 4 for both sides of the plate with fine meshes are employed. Two parameters are introduced herein as the length of plate l_2 and crack length l_c .

The corresponding normalised buckling load factor λ for the convergence test of this plate with various q is presented in Fig. 3.2. It is found that the computations with the fine meshing and high polynomial order can yield accurate results. The converging rate accelerates significantly as the polynomial order increases.

Fig. 3.2. The convergence of the buckling factor λ_n for a central cracked square plate with $l_c/l_2 = 0.5$.

Three types of boundary conditions (BCs) are considered in this chapter, including free edge (F), simply supported (S) and clamped (C). The expressions for S and C are defined by

$$\begin{aligned} \text{S: } w = 0 \text{ and } \beta_t &= 0; \\ \text{C: } w = 0, \beta_n &= 0 \text{ and } \beta_t = 0, \end{aligned} \quad (3.20)$$

where t and n stand for the tangential and normal directions to the boundary, respectively.

3.3. Numerical results for buckling of plates

Material parameters of the plates are set to be constant in the following examples. The Poisson's ratio ν and the Young's modulus E are set to 0.3 and 200 GPa, respectively.

Various thickness-to-width ratios h/l_1 , length-to-width ratios l_2/l_1 and loading conditions are investigated as illustrated in Fig. 3.3.

It should be noted that different BCs will also be applied in the buckling analysis of plates to validate the efficiency and effectiveness of the Bézier extraction-based XIGA approach.

Fig. 3.3. Loading conditions of Mindlin–Reissner plates: (a) uniaxial compression, (b) biaxial compression and (c) in-plane shear.

3.3.1. Plate under uniaxial compression

Three types of BCs for the four edges of plate (SSSS, CCCC and CFCF) are first considered for uniaxial compression with two thickness-to-width $h/l_1 = 0.05$ and 0.1 as shown in Fig. 3.3(a). Three polynomial degrees $q = 2, 3$ and 4 are considered for comparison with other reference works in Table 3.1.

Next, simply supported rectangular plates with different thickness-to-width ratios $h/l_1 = 0.05, 0.1$ and 0.2 and length-to-width ratios $l_2/l_1 = 0.5, 1.0, 1.5, 2.0$ and 2.5 are also studied, and the results are presented in Table 3.2. From the two tables, it is found that the Bézier element matches well with the reference results. Besides, the rate of convergence for the present method accelerates significantly as the polynomial order q increases.

Table 3.1. Axial buckling load factors λ_n of square plates.

h/l_1	BC	Methods					
		Present method			Thai-Hoang <i>et al.</i> [198]	Liew and Chen [199]	Kitipornchai <i>et al.</i> [200]
		$q = 2$	$q = 3$	$q = 4$			
0.05	SSSS	3.96040	3.95393	3.95383	3.94460	3.94640	3.94440
	CCCC	9.56553	9.56061	9.56066	9.55350	9.58190	9.55860
	CFCF	3.81290	3.81054	3.81041	3.80170	3.81870	3.80050
0.1	SSSS	3.80005	3.79393	3.79341	3.77710	3.78530	3.78730
	CCCC	8.30043	8.29339	8.29318	8.28730	8.29310	8.29210
	CFCF	3.52148	3.51475	3.51468	3.49780	3.51380	3.50770

Table 3.2. Axial buckling load factors λ_n of rectangular plates.

b/h_1	h/h_1	Methods					
		Present method			Thai-Hoang <i>et al.</i> [198]	Liew <i>et al.</i> [201]	Kitipornchai <i>et al.</i> [200]
		$q = 2$	$q = 3$	$q = 4$			
0.5	0.05	6.07608	6.06752	6.06740	6.02370	6.04050	6.03720
	0.1	5.54962	5.50916	5.50896	5.34720	5.31160	5.47770
	0.2	4.00721	4.00125	4.00113	3.76420	3.71570	3.99630
1.0	0.05	3.98367	3.97856	3.97844	3.94490	3.92930	3.94440
	0.1	3.81773	3.80949	3.80933	3.74650	3.72700	3.78650
	0.2	3.28795	3.28201	3.28190	3.14200	3.14710	3.26370
1.5	0.05	3.98137	3.97686	3.97671	3.95940	3.86570	3.94440
	0.1	3.81433	3.80723	3.80710	3.78550	3.67970	3.78650
	0.2	3.29267	3.28664	3.28654	3.17330	3.07830	3.26370
2.0	0.05	3.98137	3.97686	3.97671	3.95940	3.86570	3.94440
	0.1	3.81433	3.80723	3.80710	3.78550	3.67970	3.78650
	0.2	3.29267	3.28664	3.28654	3.17330	3.07830	3.26370
2.5	0.05	4.11165	4.10527	4.10513	3.99380	3.96000	4.06450
	0.1	3.89216	3.88681	3.88669	3.85080	3.73110	3.86830
	0.2	3.26847	3.25903	3.25894	3.19500	3.03060	3.24210

3.3.2. Plate under biaxial compression

Next, square plates with a thickness $h = 0.01l_1$ under biaxial compression are investigated as shown in Fig. 3.3(b). Three BCs (SSSS, CCCC, SCSC) are employed in this section. The results are presented in Table 3.3 and appear to agree well with the reference solutions.

Table 3.3. Biaxial buckling load factors λ_n of a square plate with $h/l_1 = 0.01$.

BC	Methods					
	Present method			Thai-Hoang <i>et al.</i> [198]	Tham and Szeto [202]	Timoshenko and Gere [203]
	$q = 2$	$q = 3$	$q = 4$			
SSSS	2.00442	2.00221	2.00214	2.00210	2.00000	2.00000
CCCC	5.32942	5.31997	5.31982	5.31130	5.61000	5.31000
SCSC	3.84532	3.83789	3.83784	3.83390	3.83000	3.83000

3.3.3. Plate under in-plane shear

In this case, simply supported rectangular plates with a thickness $h = 0.01l_1$ loaded in-plane shear are investigated as shown in Fig. 3.3(c). Four length-to-width ratios l_2/l_1 are considered, and the simulation results are tabulated in Table 3.4. As be expected, the results obtained by the Extraction-based XIGA approach are in good accordance with the reference ones. It is also found that the length-to-width ratio l_2/l_1 of a plate is inversely proportional to its buckling load.

Table 3.4. Shear buckling load factors λ_n of square plates with $h/l_1 = 0.01$.

l_1/l_2	Methods					
	Present method			Thai-Hoang <i>et al.</i> [198]	Liew <i>et al.</i> [201]	Timoshenko and Gere [203]
	$q = 2$	$q = 3$	$q = 4$			
1.0	9.41815	9.35746	9.35537	9.33510	9.39620	9.34000
2.0	6.37083	6.36215	6.36072	6.38530	6.37410	6.34000
3.0	5.84561	5.83706	5.83586	5.71430	5.72320	5.78400
4.0	5.63211	5.62167	5.62053	5.54660	5.43670	5.59000

3.4. Numerical results for buckling of cracked plates

The enrichment functions are applied to determine the normalised buckling factor λ_n for rectangular cracked plates. Same material parameters for the plates without cracks are set in this case. As displayed in Fig. 3.4, central and edge-cracked rectangular plates are examined in this case. The crack orientation is set to align with the x -axis. Both plates are subjected to uniaxial compression along the x -axis. SSSS is further adopted as the boundary condition for the buckling analysis of cracked plates. Various crack length ratios thickness-to-length ratios h/l_2 , l_c/l_2 and length-to-width ratios l_2/l_1 are investigated. Fine meshing is applied for obtaining more accurate simulation results, and all the results are compared with available analytical and numerical solutions. The first four buckling modes of the plate are further plotted to provide us with a better understanding of the effects induced by the presence of a crack.

Fig. 3.4. Crack configurations of rectangular Mindlin–Reissner plates: (a) a central cracked plate and (b) an edge-cracked plate.

3.4.1. Central cracked plate under uniaxial compression

In the following example, the central cracked plate with a crack length ratio l_c/l_2 , as shown in Fig. 3.4(a), is investigated. The first four buckling mode shapes for the square and rectangular plates with the largest crack length ratio of 0.9 is shown in Fig. 3.5 and Fig. 3.6, respectively. It is obvious that these two plates show similar buckling patterns during the simulation. As tabulated in Table 3.5. The buckling load factors of a central cracked rectangular plate with $l_2/l_1 = 0.5$ obtained by the present method are in good match with those given by the reference research works. The buckling factor is inversely proportional to the crack length ratio l_c/l_2 due to the decrease of plate stiffness. It is also found that the decrease in the buckling load is induced by the increasing thickness.

Fig. 3.5. The first four buckling modes of a central cracked square plate with $l_c/l_2 = 0.9$.

Fig. 3.6. The first four buckling modes of a central cracked rectangular plate with $l_c/l_2 = 0.9$.

Table 3.5. Axial buckling load factors λ_n of central cracked rectangular plates.

h/l_1	l_c/l_2	Methods				
		Present method			Liu [204]	Stahl and Keer [1]
		$q = 2$	$q = 3$	$q = 4$		
0.01	0.1	3.96283	3.95627	3.95612	3.94976	3.9561
	0.2	3.83276	3.82606	3.82594	3.81427	3.8259
	0.3	3.64475	3.63752	3.63742	3.62432	3.6366
	0.4	3.42724	3.42137	3.42122	3.4137	3.4225
	0.5	3.21724	3.21144	3.21136	3.21064	3.2113
	0.6	3.03087	3.02625	3.02614	3.03341	3.0241
	0.7	2.89452	2.88696	2.88688	2.89167	2.8730
	0.8	2.76453	2.76859	2.76848	2.7899	2.7622
0.2	0.9	2.71495	2.70843	2.70831	2.73001	2.6962
	0.1	3.22627	3.22023	3.22011	3.21592	-
	0.2	3.07655	3.07154	3.0714	3.05796	-
	0.3	2.88869	2.87916	2.87905	2.86988	-
	0.4	2.70781	2.69975	2.69962	2.68535	-
	0.5	2.52991	2.52442	2.5243	2.52288	-
	0.6	2.40577	2.40084	2.40078	2.39055	-
	0.7	2.31439	2.30711	2.30703	2.29093	-
	0.8	2.23728	2.22993	2.22981	2.22383	-
	0.9	2.19937	2.19434	2.19427	2.18703	-

3.4.2. Edge-cracked plate under uniaxial compression

As depicted in Fig. 3.4(b), buckling behaviours of edge-cracked plates are investigated in this section. Fig. 3.7 shows the first four buckling modes for the edge-cracked plate with crack length ratio $l_c/l_2 = 0.5$ and length-to-width ratio l_2/l_1 of 2. The patterns are different from those for the central cracked plates. Therefore, the crack position has significant influences on the buckling modes of the plate. As expected, the results presented in Table 3.6 agree well with the solutions provided by reference research works. A thicker cracked plate is more likely to have a lower buckling load factor.

Fig. 3.7. The first four buckling modes of an edge-cracked rectangular plate with $l_c/l_2 = 0.5$.

Table 3.6. Axial buckling load factors λ_n of edge-cracked rectangular plates.

h/l_1	l_c/l_2	Methods			
		Present method			Liu [204]
		$q = 2$	$q = 3$	$q = 4$	
0.01	0.1	4.00191	3.99511	3.99498	3.98628
	0.2	3.98336	3.97775	3.97761	3.94834
	0.3	3.83607	3.82668	3.82655	3.81076
	0.4	3.33466	3.32851	3.32842	3.32145
	0.5	2.93016	2.92551	2.92542	2.90674
	0.6	2.58737	2.58368	2.58359	2.57041
	0.7	2.32718	2.32329	2.32314	2.30654
	0.8	2.13048	2.12564	2.12559	2.11441
	0.9	2.00706	1.99962	1.99953	1.99777
0.2	0.1	1.93836	1.93302	1.93288	1.92721
	0.2	1.86584	1.86035	1.86025	1.85438
	0.3	1.78853	1.78255	1.78249	1.77657
	0.4	1.72508	1.71841	1.71834	1.70716
	0.5	1.65742	1.65275	1.65267	1.64369
	0.6	1.58831	1.58447	1.58431	1.57787
	0.7	1.52023	1.51511	1.51502	1.50304
	0.8	1.43728	1.42917	1.42907	1.42818
	0.9	1.38757	1.37961	1.37956	1.3755

3.4.3. Central cracked circular plate under uniaxial compression

Fig. 3.8 shows the last example considers circular Mindlin–Reissner plates with a radius of r under radial compression to demonstrate that the present approach is able to handle more complex geometry than a rectangle.

Both central and edge-cracked circular plates are analysed to study the effects of crack position on the shapes of their buckling modes. The plate thickness-to-radius ratio h/r is set to 0.01 and the fine mesh is applied in this case. Buckling modes for the centre cracked circular plate with the largest crack length ratio $l_c/2r$ of 0.9, and the edge-cracked one with the crack size of 0.5 are presented in Fig. 3.9 and Fig. 3.10, respectively. Compared to the rectangular plates, the circular shape clearly affects the patterns of buckling modes. The similarity is proven to exist in the first buckling mode for both cases.

Fig. 3.8. Crack configurations of circular Mindlin–Reissner plates: (a) a central cracked plate and (b) an edge-cracked plate.

Fig. 3.9. The first four buckling modes of a central cracked circular plate with $l_c/l_2 = 0.9$.

Fig. 3.10. The first four buckling modes of an edge-cracked circular plate with $l_c/l_2 = 0.9$.

3.5. Summary

In this chapter, the Bézier C^0 element based XIGA is utilised for estimating the buckling load factors of Mindlin–Reissner plates. By using the Bézier extraction operator, the NURBS is transformed into the Bézier element which can be incorporated into the existing FEM work-frame. By incorporating the DSG method into the formulation, the present method can be applied to plates with different thickness straightforwardly without shear locking. The convergence and accuracy studies are established to validate the methodology. All the simulation results obtained by the present method are in good agreement with the analytical and numerical solutions. Two types of crack configurations have been investigated in this study. The increase of the crack length and thickness reduces the material stiffness, which leads to a less normalised buckling factor of a Mindlin–Reissner plate with the centre or edge crack. Furthermore, both rectangular and circular plates have been taken into consideration. The simulation results further show the effectiveness of Bézier extraction-based XIGA approach in modelling cracked plates with complex geometry.

Chapter 4 An Isogeometric-Meshfree Coupling Approach for Contact Problems

Conventional contact treatments generally require time-consuming search algorithms at the contact interface to preclude the penetration between contacting bodies. As a potential alternative, the third medium method has been proposed by Wriggers *et al.* [135] which introduces a space-filling mesh between contacting bodies as a fictive medium. In this chapter, the contact behaviours of different material modes are studied by integrating the third medium method with the isogeometric-meshfree (IMF) coupling approach.

4.1. Problem description and solution approach

In this chapter, a system with two bodies getting in contact is employed to validate the present approach. The third medium method is based on the concept of formulating a fictive medium in which contacting bodies are free to move and interact. The tangential and normal contact constraints within the medium are also formulated with a suitable set of material parameters. The material property of the medium domain gradually shifts from isotropic to anisotropic associated with the movements of contacting bodies, which makes the contact constraints discretised across a non-overlapping set of patches.

4.1.1. Third medium method

As demonstrated in Fig. 4.1, a simple contact model of the third medium method is to embed contacting bodies $\mathcal{B}_i$ ($i = 1$ and 2) in a fictive medium $\mathcal{B}_3$ where contacting bodies are allowed to move and interact. In this way, the transformation of the non-smooth contact problem into a smooth continuum mechanics problem can be achieved.

Fig. 4.1. Contacting bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ embedded in the third medium $\mathcal{B}_3$.

Based on continuum equations, the expression of stationary energy Π for bodies $\mathcal{B}_i$ ($i = 1, 2$) with volume V and surface S is given by [137]

$$\int_{\mathcal{B}_i} W \, dV - \int_{\mathcal{B}_i} \mathbf{f}_b \cdot \mathbf{x} \, dV - \int_{\Gamma_\sigma} \mathbf{t} \cdot \mathbf{x} \, dS = \Pi, \quad (4.1)$$

where W denotes the strain energy, $\mathbf{f}_b$ denotes the body forces, $\mathbf{f}_t$ denotes the traction forces acting on the boundary of body Γ_σ , and $\mathbf{x}$ denotes the deformation map which projects a body from its initial to its deformed configuration.

The weak form of balance equation is further derived from Eq. (4.1) as

$$\int_{\mathcal{B}_i} \mathbf{S} \cdot \frac{1}{2} \delta \mathbf{C}_r \, dV - \int_{\mathcal{B}_i} \mathbf{f}_b \cdot \delta \mathbf{u} \, dV - \int_{\Gamma_\sigma} \mathbf{t} \cdot \delta \mathbf{u} \, dS = 0, \quad (4.2)$$

where $\mathbf{C}_r$ denotes the right Cauchy-Green tensor, $\mathbf{S}$ denotes the Piola-Kirchhoff stress tensor given by $2\partial W/\partial \mathbf{C}$ and $\delta \mathbf{u}$ denotes the variation of the deformations.

Furthermore, the body force $\mathbf{f}_b$ and the surface traction $\mathbf{t}$ are neglected for simplicity. In this way, Eq. (4.2) can be further simplified as

$$\int_{B_i} \mathbf{S} \frac{1}{2} \delta \mathbf{C}_r \, dV = 0. \quad (4.3)$$

St. Venant-Kirchhoff and Neo-Hookean material modes are further adopted in this chapter for contact problems with large deformation and sliding. The constitutive behaviours of these hyper-elastic material modes are detailed in Appendix B.

In respect to the strain energy function W for the third medium, the expression should be divided into two parts so that the fictive medium can change from isotropic to anisotropic, which can be defined as

$$W = W_{\text{iso}} + W_{\text{aniso}}. \quad (4.4)$$

The isotropic part of the total energy W_{iso} for the third medium is obtained by

$$W_{\text{iso}} = a_1 \mathcal{I}_1 + a_2 \mathcal{I}_2 + a_3 \mathcal{I}_3 - d \ln(\sqrt{\mathcal{I}_3}), \quad (4.5)$$

where invariants of the right Cauchy-Green tensor $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ are defined by $\text{tr} \mathbf{C}_r$, $\text{tr}[(\det \mathbf{C}_r) \mathbf{C}_r^{-1}]$ and $\det \mathbf{C}$, respectively. The coefficients α_i ($i = 1, 2$ and 3) in the above equation should be above 0, and d equals $2\alpha_1 + 4\alpha_2 + 2\alpha_3$. It should be noticed that α_j should be less than the Lamé constants λ and μ of contacting bodies for the rigid body contact motion. The anisotropic part W_{aniso} can be formulated by using the surface normal $\mathbf{n}$ as

$$W_{\text{aniso}} = c \left(\frac{1}{a_4 + 1} \right) (1 - \mathcal{J}_4)^{a_4 + 1}, \quad (4.6)$$

where $\mathcal{J}_4 = \text{tr}(\mathbf{C} \mathbf{M}_n)$ is Jacobian, a_4 is another dimensionless coefficient, c is the stiffness parameter associated with the gap g_n between contacting bodies and $\mathbf{M}_n = \mathbf{n} \otimes \mathbf{n}$ is the structural tensor.

A simple expression that describes the relationship between c and g_n is given by

$$\mathbf{c} = \begin{cases} \frac{a_5}{g_n^2} & g_n \leq g_c \\ 0 & g_n > g_c \end{cases}, \quad (4.7)$$

where a_5 is a new parameter that denotes the basic stiffness along the preferred direction $\mathbf{n}$. As shown in Fig. 4.2, g_n and g_c denote the gap between contacting bodies and the threshold gap indicating the fictive medium should become anisotropic to prevent the penetration of contacting bodies into each other.

Fig. 4.2. Definition of the surface normal $\mathbf{n}$, gap g_n and threshold gap g_c .

Wriggers *et al.* [135] employ the minimum principal stretch $\lambda_{\min}$ and its direction $\mathbf{e}_{\min}$ to measure g_n and $\mathbf{n}$, respectively. It is because that the deformation state of the medium is observed to be a suitable indicator of the gap between contacting bodies. At the final stage of contact, $\lambda_{\min}$ will approach 0, and $\mathbf{e}_{\min}$ can be regarded as the surface normal $\mathbf{n}$. In this way, the switch criterion $g_n \leq g_c$ can be approximated by $\lambda_{\min} \leq \lambda_c$. According to Kruse *et al.* [185], a regularized version of the indicator function is given by

$$\kappa_b = \frac{1}{2} \left\{ \cos \left[\frac{\pi}{0.2} (\lambda_{\min} - \lambda_c + 0.1) + 1 \right] \right\}, \quad (4.8)$$

where κ_b is the bulk modulus. The above equation is valid with a range from $\lambda_c - 0.1$ to $\lambda_c + 0.1$ and will be used in the calculation of the stress tensor $\mathbf{S}_{\text{aniso}}$ corresponding to the anisotropic strain energy density.

4.1.2. Discretisation based on an IMF coupling approach

Based upon the reproducing condition shown in Appendix B, NURBS basis functions and MLS shape functions can be coupled directly on the parametric domain. As a result, the global parametric domain Ω is split into the isogeometric sub-domain Ω_{iga} , the coupling sub-domain Ω_{cp} and the meshfree sub-domain Ω_{mf} . Therefore, the displacement $u(\mathbf{x})$ can be expressed by IMF coupling basis functions $\phi(\xi)$ as

$$u_I(\mathbf{x}) = \sum_{m=1}^{n_s} \phi_m(\xi) u_I, \quad (4.9)$$

where $\mathbf{x} = [x \ y]^T$ denotes space coordinates, u_I denotes the nodal displacement and n_s is the total number of IGA, meshfree or IMF coupling basis functions, and $\phi_I(\xi)$ can be further defined by

$$\phi_I(\xi) = \begin{cases} R_I(\xi) & (\xi \in \Omega_{\text{iga}}) \\ R_I(\xi) + \psi_I(\xi) & (\xi \in \Omega_{\text{cp}}) \\ \psi_I(\xi) & (\xi \in \Omega_{\text{mf}}) \end{cases}, \quad (4.10)$$

where $R_I(\xi)$ and $\psi_I(\xi)$ stand for NURBS basis functions and MLS-based meshfree shape functions, respectively.

Reproducing conditions should be applied within the interface region between IGA and meshfree sub-domains, and they are defined by

$$\mathbf{p}(\xi) = \sum_{M=1}^{n_s} \phi_M(\xi) \mathbf{p}(\xi_M^{[I]}) = \sum_{J=1}^{n_m} \psi_J(\xi) \mathbf{p}(\xi_J^{[I]}) + \sum_{I=1}^{n_b} R_I(\xi) \mathbf{p}(\xi_I^{[I]}). \quad (4.11)$$

To make reproducing conditions satisfied across the whole domain, meshfree basis functions within the coupling region are further derived as

$$\psi_J(\xi) = \mathbf{p}^T(\xi_J^{[I]}) \mathbf{Z}^{-1}(\xi) [\mathbf{p}(\xi) - \mathbf{q}(\xi)] \widehat{W}(\xi - \xi_J^{[I]}), \quad (4.12)$$

where

$$\mathbf{Z}(\xi) = \sum_{J=1}^{n_m} \mathbf{p}^T(\xi_J^{[I]}) \mathbf{p}(\xi_J^{[I]}) \widehat{W}(\xi - \xi_J^{[I]}) \quad (4.13)$$

and

$$\mathbf{q}(\xi) = \sum_{I=1}^{n_b} R_I(\xi) \mathbf{p}(\xi_J^{[I]}). \quad (4.14)$$

The non-zero weight functions $\widehat{W}(\xi - \xi_J^{[I]})$ in Eq. (4.12) can be further obtained by

$$\widehat{W}(\xi - \xi_I) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & (r \leq 1) \\ 0 & (r > 1) \end{cases}, \quad (4.15)$$

where $r = |\xi - \xi_I|/(\gamma d_I)$, d_I denotes the supporting radius around the node I and the coefficient γ varies in a range of 2 to 4 [143].

Fig. 4.3 illustrates both 1D and 2D quadratic IMF shape functions on the reproducing condition. A smooth transition can be observed between each sub-domain, which demonstrates that reproducing conditions are satisfied for the approximation across the parametric domain. It should also be noticed that the number of coupling points within the interface should equal the polynomial order of IMF coupling shape functions.

(a)

(b)

Fig. 4.3. (a) 1D and (b) 2D diagrams of quadratic IMF coupling basis functions.

By using the Voigt notation, the variation of right Cauchy-Green tensor $\delta \mathbf{C}_r$ on the node J can be expressed by [135]

$$\delta \mathbf{C}_J = [\delta C_{11} \quad \delta C_{22} \quad 2\delta C_{12}]^T = 2 \sum_{m=1}^{n_t} \mathbf{B}_m \delta \mathbf{u}_m, \quad (4.16)$$

where n_t denotes the number of field nodes, reproducing points or control points that are located within the supporting radius around the node J . The $\mathbf{B}$ operator in the above equation can be given by IMF basis functions ϕ and deformation gradient F as

$$\mathbf{B}_J = \begin{bmatrix} F_{11}\phi_{j,1} & F_{21}\phi_{j,1} \\ F_{12}\phi_{j,2} & F_{22}\phi_{j,2} \\ F_{11}\phi_{j,2} + F_{12}\phi_{j,1} & F_{21}\phi_{j,2} + F_{22}\phi_{j,1} \end{bmatrix}. \quad (4.17)$$

According to Eq. (4.3), the Piola-Kirchoff stress tensor $\mathbf{S}$ for the node J can be calculated as

$$\mathbf{S}_J = \mathbf{S}_{\text{iso}} + \mathbf{S}_{\text{aniso}} = 2 \frac{\partial (W_{\text{iso}} + W_{\text{aniso}})}{\partial \mathbf{C}_r}. \quad (4.18)$$

The Piola-Kirchoff stress tensor $\mathbf{S}_{\text{iso}}$ associated with the isotropic part of $\mathbf{S}$ is given by

$$\mathbf{S}_{\text{iso}} = 2(a_1 + a_2 \mathcal{I}_2) \mathbf{I} - 2a_2 \mathbf{C}_r + (2a_3 \mathcal{I}_3 - d) \mathbf{C}_r^{-1}, \quad (4.19)$$

where $\mathbf{I}$ denotes the identity matrix.

The anisotropic part of the stress tensor $\mathbf{S}_{\text{aniso}}$ can be subsequently obtained by

$$\mathbf{S}_{\text{aniso}} = -\kappa \frac{2a_5}{\lambda_{\min}^2} \left[(1 - \lambda_{\min}^2)^{a_4} + \frac{(1 - \lambda_{\min}^2)^{a_4+1}}{\lambda_{\min}^2 (a_4 + 1)} \right] \mathbf{M}_n. \quad (4.20)$$

$\mathbf{M}_n$ in the above equation is defined by $\mathbf{M}_n = \mathbf{e}_{\min} \otimes \mathbf{e}_{\min}$, which is because the minimum principal stretch $\lambda_{\min}$ is applied as the measure for switch criterion.

4.2. Numerical results

In this chapter, the accuracy and effectiveness of the present approach are verified by investigating a series of benchmark problems. Since the meshfree method consists of no mesh elements but field nodes, DOFs are employed as the measure for the IMF-based simulation resolution. To achieve multi-scale generalization of contact models, f and ul are employed in this paper as the units of force and length, respectively. Moreover, the plane strain condition is applied for all the following numerical examples.

4.2.1. Material parameter test for the third medium

The effect of a_5 on the stiffness of the fictive medium is investigated in this example. Fig. 4.4(a) shows the upper edge of the indenter is under a point load of $400f$ on the middle of the upper edge. It is equivalent to a line load of $200f$ being uniformly distributed over the upper edge. The bottom edge of the lower base is subjected to the simply supported boundary condition. Both the indenter and the base are assumed to obey the St. Venant-Kirchhoff material model material properties as $E = 10000f/ul^2$ and $\nu = 0.3$. By referring to the previous research works, the material parameters are set as $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 23$. Since a_5 indicates the basic stiffness of the third medium in the preferred direction, a smaller a_5 will make the contact bodies come closer to each other at the final step, thus leading to a more realistic contact simulation. To verify this effect, different values of a_5 should be adopted for the material parameter test in this section. The overall contact system is represented by quadratic IMF coupling shape functions as plotted in Fig. 4.4(b) where the yellow points denoting meshfree nodes and reproducing points, and the IGA mesh is adopted along the boundary of each contacting bodies.

Fig. 4.4. Schematics of the material parameter test: (a) the setup diagram and (b) the discretisation configuration.

Fig. 4.5. The gap between contacting bodies verses increments of applied force with respect to different a_5 .

(a)

(b)

(c)

Fig. 4.6. Stress distribution diagrams for $a_5 = 0.0001$: (a) Horizontal stress σ_{xx} , (b) vertical stress σ_{yy} and (c) shear stress σ_{xy} distribution.

Since the medium becomes extreme stiff to prevent the penetration of contacting bodies into each other, a small residual gap will be eventually left between contacting bodies. The final size of this gap indicates the performance of a_5 shown in Fig. 4.5. Since a_5 indicates the basic stiffness of the third medium in the preferred direction, a smaller a_5 will make the contact bodies come closer to each other at the final step, thus leading to a more realistic contact simulation. To verify this effect, different values of a_5 should be adopted for the material parameter test in this section. The test begins with a very small a_5 , initialized as 0.0001, a_5 will be iteratively increased by 100 times for in subsequent rounds of simulation until a_5 hits 1. In this way, the effect of a_5 on the final gap size between contact bodies can be clearly reflected by simulation results. Therefore, a_5 is set to 0.0001 for the rest of numerical examples. Distribution diagrams of all components of stresses for $a_5 = 0.0001$ are illustrated in Fig. 4.6.

4.2.2. Hertzian contact

Hertzian contact is a benchmark problem for evaluating the robustness of a contact method. In this case, an indenter will gradually get in contact with an elastic base. As illustrated in Fig. 4.7(a), the indenter is assumed to be a semi-circle with a radius R of $1\ ul$, while the elastic base is a square. The linear material model is applied for both contact bodies with a set of the Young's moduli and the Poisson's ratios as $E_1 = 7000\ f/ul^2$, $E_2 = 70000\ f/ul^2$, $\nu_1 = 0.3$ and $\nu_2 = 0.45$. The bottom edge of the solid base is set to be simply supported on the ground. As an equivalent to the point load of $300\ f$, a linear load of $150\ f/ul^2$ is uniformly applied along the upper edge of the indenter.

The setup diagram and discretisation configuration of the overall contact domain is shown in Fig. 4.7(b). Material parameters for the third medium can be further optimized by linking them with the material property of contacting bodies. An optimised set of parameters for the fictive medium can be given as $a_1 = 0.0003\mu_{\min}$, $a_2 = 0.0003\mu_{\min}$, $a_3 = 0.0001\lambda_{\min}$, $a_4 = 1$ and $a_5 = 0.0001$, where the minimal Lamé modulus $\lambda_{\min}$ and shear modulus $\mu_{\min}$ are expressed in terms of ν and E as $\lambda_{\min} = \nu_1 E_1 / [(1 + \nu_1)(1 - 2\nu_2)]$ and $\mu_{\min} = E_1 / 2(1 + \nu_1)$, respectively.

Fig. 4.7. Schematics of the Hertzian contact simulation: (a) the setup diagram and (b) the discretisation configuration.

Fig. 4.8. Convergence test of an IMF coupling approach for Hertzian contact problem with respect to different polynomial orders of shape functions.

Both the analytical solution and IGA simulation result for the stress peak are given as $821.81 f/ul^2$ [185]. A test is thus conducted to study the relationship between the convergence rate and the polynomial order of IMF coupling basis functions. As presented in Fig. 4.8, the convergence rate of IMF becomes better as the polynomial order of the shape functions increases from 2 to 4, which demonstrates that the present approach can achieve a better convergence rate through a higher-order discretisation.

Fig. 4.9 shows that a quadratic IMF coupling approach ultimately gives a result of $821.82 f/ul^2$ while the FEM results are 824.7 [135] and $813 f/ul^2$ [137], respectively. The numerical results are in good match with both the analytical solution and reference results. Furthermore, the distribution of vertical stresses σ_{yy} over the contact domain is presented in Fig. 4.10. The stress peak is found to be located at the central contact point, and the

stress field can be captured with high resolutions. The third medium is found to be gradually pushed sideways by contacting bodies during the contact.

Fig. 4.9. Comparison of simulation results from different approaches for the Hertzian contact.

Fig. 4.10. Vertical stress σ_{yy} distribution of Hertzian contact system.

4.2.3. Contact problem with sliding

In this case, the contact behaviour of the sliding between two bodies in contact is investigated. The same setup diagram of the material parameters test is adopted for this case. Contacting bodies both obey a linear material model with a set of the Young's moduli and the Poisson's ratios as $E_1 = 7000 fl^2$, $E_2 = 70000 fl^2$, $\nu_1 = 0.3$ and $\nu_2 = 0.45$. The material parameters for the medium are still set to be as $a_1 = 0.0003\mu_{\min}$, $a_2 = 0.0003\mu_{\min}$, $a_3 = 0.0001\lambda_{\min}$, $a_4 = 1$ and $a_5 = 0.0001$. The indenter is subjected to prescribed displacements rather than an initial force. The overall loading process is divided into two stages and conducted in 70 steps. On the first stage, the upper edge of the indenter is lowered vertically by a prescribed displacement of $0.12 ul$ in 20 steps. On the second stage, the indenter is moved rightwards by $0.5 ul$ in the rest 50 steps. The bottom edge of the base is simply supported on the ground.

As presented in Fig. 4.11, the deformed configuration and the vertical stress distribution diagrams are shown in three different steps 30, 50 and 70. When the indenter gets in contact with the base and starts to slide, it is obvious that the maximum vertical stress exists around the contact point area and maintains constant during the sliding. Compared to standard FEM applied in the research by Wriggers *et al.* [135], the indenter can slide further rightwards, which demonstrates that the proposed approach is good at handling contact problems with sliding. A typical relation between the vertical and horizontal reaction force is depicted in Fig. 4.12. The vertical force reaches its maximum and remains stable after 20 steps. The isotropic stiffness of the medium will cause a small increase in the horizontal force during the sliding. This phenomenon is induced by neglecting the traction and friction forces in contact constraints.

(a)

(b)

(c)

Fig. 4.11. Vertical stress σ_{yy} distribution diagrams associated with the horizontal movement in steps (a) 30, (b) 50 and (c) 70.

Fig. 4.12. Horizontal and vertical force curves obtained by contact simulation.

4.2.4. Contact problem with large deformation

In this case, a compound semi-ring formed by a pair of inner and outer semi-rings with the same thickness of $0.5 ul$ is studied. The inner semiring has a radius $R_i = 9 ul$, while the outer one has a radius $R_o = 10 ul$. Neo-Hookean model is applied to both parts of the semi-ring. The inner and outer semirings have an identical Poisson ratio ν of 0.3, but different Young's moduli. For the inner semi-ring, the Young's modulus E is set to be $100000 f/ul^2$ which is 100 times higher than that of the outer one. The lower block is also made from the Neo-Hookean material with the Poisson ratio $\nu = 0.3$ and the Young's modulus $E = 300 f/ul^2$.

(a)

(b)

Fig. 4.13. Schematics of the contact problem with large deformation: (a) the setup diagram and (b) the discretisation configuration.

Material parameters of the third medium are set as $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 23$ and $a_5 = 0.0001$. The boundary condition of the bottom edge of the base is set to be simply supported as shown in Fig. 4.13(a), and the discretisation configuration is further depicted in Fig. 4.13 (b). A total prescribed downward displacement u of $5 ul$ is applied on the two top ends of the semi-ring in 50 steps. Fig. 4.14 shows the evolution of deformation in step 5, 20, 35 and 50. Up to 20 steps, the type of contact between the semi-ring and the base is

observed to shift from point contact to line contact. Meanwhile, a lift-off appears in the middle of the semirings and keeps increasing until the final step.

Fig. 4.14. Deformed configurations of the third medium and contact bodies in step (a) 5, (b) 20, (c) 35 and (d) 50.

It is evident that the fictive medium is gradually pushed sideways by contacting bodies, which also causes the third medium to undergo large deformation during contact. As a result, extreme mesh distortion occurs inside the contacting bodies and the fictive medium, which is a challenge to be addressed by conventional FEM contact treatment. This case further verifies the robustness of the present method in solving contact problems with large deformation.

4.3. Summary

In this chapter, the third medium method is incorporated with an IMF coupling approach to solve various contact problems. Contacting bodies and the third medium are transformed into an integrated system to establish a smooth continuum contact domain. The domain boundary is represented by IGA while the meshfree method is adopted in the interior area. The robustness of the present approach is well demonstrated by solving a series of benchmark contact problems. An IMF coupling approach is proven to inherit advantages from both IGA and meshfree approximations to exploit the application of the third medium method to contact problems.

Chapter 5 An Adaptive Isogeometric-Meshfree Coupling Approach for the Limit Analysis of Cracked Structures

The plastic collapse behaviour of cracked structures is critical to the elastic-plastic fracture mechanics and safety design of engineering structures due to the extensive usage of ductile materials in pressure vessels and reactors. Limit analysis has been proposed to assess the plastic collapse mechanism of structures by measuring its limit collapse load. In this chapter, limit analysis problems of cracked structures are solved by a numerical procedure based on an adaptive IMF coupling approach in combination with SOCP.

5.1. Problem description and solution approach

Limit analysis has been developed as an effective tool that evaluates the load-carrying capacity of structures. Enrichment functions are incorporated into the kinematic formulation to analysis the limit load factors of structures. The resulting expression is further transformed into a non-linear optimisation problem which is computed by a SOCP-based solver.

5.1.1. Kinematic formulation

Cracked elastic-perfectly plastic material is considered on a 2D domain $\Omega \in \mathbb{R}^2$ with boundary conditions of $\Gamma = \Gamma_{\mathbf{u}} \cup \Gamma_{\mathbf{t}} \cup \Gamma_{\mathbf{c}}$ and $\Gamma_{\mathbf{u}} \cap \Gamma_{\mathbf{t}} \cap \Gamma_{\mathbf{c}} = \emptyset$, where $\Gamma_{\mathbf{u}}$, $\Gamma_{\mathbf{t}}$ and $\Gamma_{\mathbf{c}}$ represent prescribed boundary, free boundary and crack path, respectively. The body is loaded in body forces $\mathbf{f}$ on $\Gamma_{\mathbf{u}}$ constrained by the displacement velocity $\dot{\mathbf{u}}$ and surface traction $\mathbf{t}$ along $\Gamma_{\mathbf{t}}$. $\dot{\mathbf{u}} = [\dot{u} \ \dot{v}]^T$ is adopted to represent the plastic flow within kinematically

admissible velocity space $\mathcal{V}$. Consequently, the equilibrium equation can be given in terms of stress $\boldsymbol{\sigma}$ and plastic velocity $\dot{\mathbf{u}}$ fields as [205]

$$W_{\text{in}}(\boldsymbol{\sigma}, \dot{\mathbf{u}}) = W_{\text{ex}}(\dot{\mathbf{u}}), \quad \forall \dot{\mathbf{u}} \in \mathcal{V}, \quad (5.1)$$

where the internal work rate W_{in} is defined in a bilinear form as

$$W_{\text{in}}(\boldsymbol{\sigma}, \dot{\mathbf{u}}) = \int_{\Omega} \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}(\dot{\mathbf{u}}) \, d\Omega, \quad (5.2)$$

the external work rate W_{ex} can be subsequently defined by

$$W_{\text{ex}}(\dot{\mathbf{u}}) = \int_{\Omega} \mathbf{f} \dot{\mathbf{u}} \, d\Omega + \int_{\Gamma_t} \mathbf{t} \dot{\mathbf{u}} \, d\Gamma \quad (5.3)$$

and space $\mathcal{V}$ can be defined by a Hilbert space $\mathcal{H}^1(\Omega)$ as

$$\mathcal{V} = \left\{ \dot{\mathbf{u}} \in (\mathcal{H}^1(\Omega))^2, \dot{\mathbf{u}} = \dot{\bar{\mathbf{u}}} \in \text{on } \Gamma_{\dot{\mathbf{u}}} \right\}. \quad (5.4)$$

A convex set $\mathcal{S}$ consisting of the statically admissible stress field is obtained by

$$\mathcal{S} = \left\{ \boldsymbol{\sigma} \in \mathcal{E}; |\mathcal{Y}(\boldsymbol{\sigma})| \leq 0 \right\}, \quad (5.5)$$

where $\mathcal{Y}$ denotes the convex yield function and $\mathcal{E}$ denotes a space of symmetric stress tensor.

The plastic strain rates $\dot{\boldsymbol{\epsilon}}$ are further determined by yield function $\mathcal{Y}$ as

$$\dot{\boldsymbol{\epsilon}} = \varphi \frac{\partial \mathcal{Y}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}, \quad (5.6)$$

where φ is a non-negative plastic multiplier.

By further defining a space $\mathcal{L} = \{ \dot{\mathbf{u}} \in \mathcal{V} | W_{\text{in}}(\dot{\mathbf{u}}) = 1 \}$, the kinematic formulation is transformed into an optimisation problem as [206]

$$\lambda_p = \max \left\{ \exists \boldsymbol{\sigma} \in \mathcal{S} | W_{\text{in}}(\boldsymbol{\sigma}, \dot{\mathbf{u}}) = \lambda W_{\text{ex}}(\dot{\mathbf{u}}), \forall \dot{\mathbf{u}} \in \mathcal{V} \right\} = \min_{\dot{\mathbf{u}} \in \mathcal{L}} D(\dot{\mathbf{u}}), \quad (5.7)$$

where λ_p is the exact limit load factor, $D(\dot{\mathbf{u}}) = \max_{\boldsymbol{\sigma} \in \mathcal{S}} W_{\text{in}}(\boldsymbol{\sigma}, \dot{\mathbf{u}})$ and $\boldsymbol{\sigma}$ denotes the admissible stresses over the convex yield surface.

By adopting the von Mises failure criterion, plastic dissipation function for the plane strain or plane stress problems can be given as [207]

$$D_p(\dot{\boldsymbol{\epsilon}}) = \sigma_0 \int_{\Omega} \sqrt{\dot{\boldsymbol{\epsilon}}^T \boldsymbol{\Theta} \dot{\boldsymbol{\epsilon}}} \, d\Omega, \quad (5.8)$$

where σ_0 denotes the yield stress of material and matrix $\boldsymbol{\Theta}$ is defined by

$$\boldsymbol{\Theta} = \frac{1}{3} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (5.9)$$

5.1.2. Discretisation based on an adaptive IMF coupling approach

Although a practical IMF coupling approach is presented in Chapter 4, its further application is limited by its formulation. To implement this method, the overall parametric domain should be divided into three subdomains described by different shape functions. Therefore, different local refinement strategies should be adopted for the representation of crack tip area, depending on its position in the parametric domain. To simplify such refinement procedure, a modified adaptive IMF coupling approach that unifies the basis functions across the whole parametric domain is developed in this chapter.

According to Appendix B, the reproducing point vector for 1D formulation with quadratic basis functions is derived as $\mathbf{p}(\xi_i^{[1]}) = \left\{ 1, \xi_i^{[1]}, \left(\xi_i^{[2]} \right)^2 \right\}^T$, in which the linear and quadratic reproducing points can be calculated as $\xi_i^{[1]} = \frac{\xi_{i+1} + \xi_{i+2}}{2}$ and $\xi_i^{[2]} = \sqrt{\xi_{i+1} \xi_{i+2}}$,

respectively. By setting the normalised support size to 1.5, the moment matrix $\mathbf{Z}(\xi)$ in Eq. (4.13) can be subsequently reformulated by

$$\mathbf{Z}(\xi) = \mathbf{p}(\xi_{i-1}^{[.]}) \mathbf{p}^T(\xi_{i-1}^{[.]}) \widehat{\mathcal{W}}_{i-1} + \mathbf{p}(\xi_i^{[.]}) \mathbf{p}^T(\xi_i^{[.]}) \widehat{\mathcal{W}}_i + \mathbf{p}(\xi_{i+1}^{[.]}) \mathbf{p}^T(\xi_{i+1}^{[.]}) \widehat{\mathcal{W}}_{i+1}. \quad (5.10)$$

The following relationships between the modified IMF coupling basis function $\Phi(\xi)$ and IGA basis functions $N(\xi)$ are determined by substituting Eq. (5.10) into Eq. (4.12)

$$\begin{aligned} \Phi_{i-1}(\xi) &= N_{i-1,p}(\xi); \\ \Phi_i(\xi) &= N_{i,p}(\xi); \\ \Phi_{i+1}(\xi) &= N_{i+1,p}(\xi). \end{aligned} \quad (5.11)$$

In this way, basis functions over the domain are unified as Φ . A typical local mesh refinement procedure is detailed in Appendix C.

Consequently, the displacement field $\mathbf{u}$ in Eq. (2.25) can be replaced by the velocity field $\dot{\mathbf{u}}$ as

$$\dot{\mathbf{u}} = \sum_{i \in I} \Phi_i \dot{\mathbf{q}}_i + \sum_{j \in J} \Phi_j H \dot{\mathbf{a}}_j + \sum_{o \in O} \Phi_o \left[\sum_{m=1}^4 \chi_m \dot{\mathbf{b}}_{o_m} \right], \quad (5.12)$$

where $\dot{\mathbf{q}}_i = [\dot{u}_i \ \dot{v}_i]^T$ denotes the velocities of the nodal displacements associated with the standard IMF nodes, H denotes the Heaviside functions and χ denotes the enrichment functions. $\dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ in the above equation are additional nodal unknowns associated with the crack surfaces and tips, respectively.

Compatible strain rates can be expressed by nodal velocities containing both standard and enriched velocity unknowns $\dot{\mathbf{d}}_i$ as

$$\dot{\boldsymbol{\epsilon}} = \sum_{i \in I} \mathbf{B}_i \dot{\mathbf{d}}_i, \quad (5.13)$$

where strain matrix $\mathbf{B}_i$ is defined as

$$\mathbf{B}_i = [\mathbf{B}_{n_i} \mathbf{B}_{h_i} \mathbf{B}_{e_i}]. \quad (5.14)$$

$\mathbf{B}_n$ in the above equation represents the standard part of matrix $\mathbf{B}$ and is defined by

$$\mathbf{B}_{n_i} = \begin{bmatrix} \Phi_{i,x} & 0 \\ 0 & \Phi_{i,y} \\ \Phi_{i,y} & \Phi_{i,x} \end{bmatrix}. \quad (5.15)$$

$\mathbf{B}_{h_i}$ and $\mathbf{B}_{e_i}$ respectively indicate the part of matrix $\mathbf{B}$ associated with Heaviside and enrichment functions as

$$\mathbf{B}_{h_i} = \begin{bmatrix} \Phi_{i,x}H_i + \Phi_i H_{i,x} & 0 \\ 0 & \Phi_{i,y}H_i + \Phi_i H_{i,y} \\ \Phi_{i,y}H_i + \Phi_i H_{i,y} & \Phi_{i,x}H_i + \Phi_i H_{i,x} \end{bmatrix} \quad (5.16)$$

and

$$\mathbf{B}_{e_i} = \begin{bmatrix} \Phi_{i,x}\chi_i + \Phi_i \chi_{i,x} & 0 \\ 0 & \Phi_{i,y}\chi_i + \Phi_i \chi_{i,y} \\ \Phi_{i,y}\chi_i + \Phi_i \chi_{i,y} & \Phi_{i,x}\chi_i + \Phi_i \chi_{i,x} \end{bmatrix}. \quad (5.17)$$

Consequently, the plastic dissipation of the body made of rigid-perfectly material can be rewritten by

$$D_p(\dot{\mathbf{u}}) = \sigma_0 \int_{\Omega} \sqrt{\dot{\boldsymbol{\varepsilon}}^T \boldsymbol{\Theta} \dot{\boldsymbol{\varepsilon}}} d\Omega = \sigma_0 \sum_{l=1}^{n_r} \int_{\Omega_l} \sqrt{\dot{\boldsymbol{\varepsilon}}^T \boldsymbol{\Theta} \dot{\boldsymbol{\varepsilon}}} d\Omega, \quad (5.18)$$

where n_r is the total number of reproducing points.

Since Gauss integral is adopted for the evaluation of integrals, the strain rate $\dot{\boldsymbol{\varepsilon}}$ is calculated through the Gauss points and weights. Eq. (5.18) can be further reformulated as

$$D_p(\dot{\mathbf{u}}) \simeq \sigma_0 \sum_{i=1}^{n_g} \varpi_i |\mathbf{J}_i| \sqrt{\dot{\boldsymbol{\varepsilon}}_i^T \boldsymbol{\Theta} \dot{\boldsymbol{\varepsilon}}_i}. \quad (5.19)$$

where ϖ denotes the weights of Gauss points, n_g is the total number of Gauss points and

$|\mathbf{J}|$ denotes the determinant of the Jacobian matrix. Consequently, the optimisation problem shown in the Eq. (5.7) can be rewritten by

$$\begin{aligned} \lambda_p = \min & \left\{ \sigma_0 \sum_{i=1}^{n_g} \varpi_i |\mathbf{J}_i| \sqrt{\dot{\mathbf{e}}_i^T \boldsymbol{\Theta} \dot{\mathbf{e}}_i} \right\}, \\ \text{s.t.} & \begin{cases} W_{\text{ex}}(\dot{\mathbf{u}}) = 1 \\ \dot{\mathbf{u}} = \dot{\bar{\mathbf{u}}} & \text{on } \Gamma_u \end{cases} \end{aligned} \quad (5.20)$$

To allow the usage of SOCP solver, new variables $\mathbf{v}_i$ and $\boldsymbol{\rho}_i$ should be further introduced into the above equation as

$$\begin{aligned} \lambda_p = \min & \left\{ \sigma_0 \sum_{i=1}^{n_g} \varpi_i |\mathbf{J}_i| \mathbf{v}_i \right\}, \\ \text{s.t.} & \begin{cases} W_{\text{ex}}(\dot{\mathbf{u}}) = 1 \\ \dot{\mathbf{u}} = \dot{\bar{\mathbf{u}}} & \text{on } \Gamma_u \\ \boldsymbol{\rho}_i = \mathbf{C}^T \dot{\mathbf{e}}_i & (i = 1, 2, \dots, n_g) \\ \sqrt{\boldsymbol{\rho}_i^T \boldsymbol{\rho}_i} \leq \mathbf{v}_i & (i = 1, 2, \dots, n_g) \end{cases} \end{aligned} \quad (5.21)$$

where the Cholesky factor $\mathbf{C}$ of matrix $\boldsymbol{\Theta}$ is defined by

$$\mathbf{C} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (5.22)$$

The last constraint in Eq. (5.21) represents quadratic cones. n_{var} is further adopted to denote the total number of variables computed by SOCP and given by $n_{\text{var}} = n_{\text{Dofs}} + 4n_g$, where n_{DOFs} stands for the total number of DOFs.

Since the plastic dissipation is concentrated within the regions of the material containing high strain rates, the local refinement of these regions would lead to an accurate result with the lowest computational cost. Therefore, the plastic dissipation $\bar{D}$ of each background element on the parametric domain is employed as an indicator to guide the

refinement [208, 209]. The total plastic dissipation $\widehat{D}_e$ in correspondence to the reproducing points within an element e can be obtained by

$$\widehat{D}_e = \sum_{i=1}^{n_e} D_i, \quad (5.23)$$

where n_e stands for the total number of reproducing points associated with the element e .

The mesh density $\bar{n}_e = n_e/S_e$ is introduced herein to indicate the number of reproducing points distributed within a unit area S_e associated with the element e . Consequently, the averaged plastic dissipation $\overline{D}_e$ can be obtained by

$$\overline{D}_e = \frac{\widehat{D}_e}{n_e}. \quad (5.24)$$

A threshold γ_t is further introduced herein to activate the refinement procedure for a set of elements when their corresponding averaged mesh intensity $\overline{D}_e$ is larger than the threshold γ_t .

The performance of an adaptive IMF coupling approach is then evaluated by solving a series of benchmark problems of rigid-perfectly plastic materials under the plane stress condition where von Mises yield criterion is adopted. Effects of various crack positions, crack lengths and structural geometries on the limit load factor are investigated.

5.2. Numerical results of structures

The limit analysis problems of structures are studied first to validate the present approach, in which different material shapes and loading conditions are considered. It should be noted that suitable threshold γ_t should be applied for different cases.

5.2.1. Square plate with a central hole

Square plates with a central hole under biaxial tensile loads σ_1 and σ_2 are firstly studied in this case as shown in Fig. 5.1(a). This is a classic benchmark problem which helps to validate the present approach. The square plate is assumed to have a length l , while the central circular hole has a radius r . Fig. 5.1(b) illustrates that this model can be further simplified into a quarter model according to the symmetric boundary condition.

Fig. 5.1. Schematics of a square plate with a central hole: (a) the full mode and (b) its quarter mode with symmetric boundary condition.

As shown in Fig. 5.2(a), the adaptive refinement procedure started with a uniform coarse mesh when $\sigma_2 = 0$ and $r = 0.2l$. Fig. 5.2(b) displays the final mesh domain after 3 steps of refinement. It is obvious that the elements associated with higher plastic deformations are automatically identified and refined.

Fig. 5.2. Mesh refinements for a square plate with a central hole ($r = 0.2l$): (a) the initial mesh and (b) the mesh after 3 steps of refinement.

By employing the von Mises yield criterion, Gaydon and McCrum [210] proposed the analytical solution of the plastic limit load factor to be $\lambda_p = 0.8\sigma_1 / \sigma_0$ for the plate subjected to uniaxial tension shown in Fig. 5.2. Therefore, λ_p will be 0.8 when $\sigma_1 = \sigma_0$. Fig. 5.3 compares the convergence rate of simulation results achieved by an adaptive IMF coupling approach with that obtained by the reference solutions. As expected, the IMF coupling approach with the uniform mesh offers a similar convergence rate to that offered by quadratic IGA.

Fig. 5.3. A companion of the convergence rates.

After the adaptive local refinement strategy is applied, the present approach significantly improves the convergence order to over 1.5. As listed in Table 5.1, the simulation results achieved by an IMF coupling approach are close to both analytical and numerical solutions. Therefore, the present approach not only achieves the highest convergence rate but also offers the most accurate results with the least optimization variables. At the collapse state, Fig. 5.4 illustrates that the plastic dissipation distribution is captured with high resolution.

Table 5.1. Limit load factors λ_p of a square plate with a central hole ($r = 0.2l$) under various loading conditions with $\sigma_1 = \sigma_0$.

Method	Loading conditions		
	$\sigma_2 = \sigma_0$	$\sigma_2 = 0.5\sigma_0$	$\sigma_2 = 0$
Gaydon and McCrum [210]	-	-	0.8000
Le <i>et al.</i> [71]	-	-	0.8006
Groß-Weege [211]	0.882	0.891	0.7820
Chen <i>et al.</i> [212]	0.8740	0.899	0.7980
Zouain <i>et al.</i> [213]	0.8940	0.9110	0.8030
Zhang <i>et al.</i> [214]	0.8890	0.8980	0.7840
Tran <i>et al.</i> [215]	0.8960	0.9120	0.8050
Nguyen-Xuan <i>et al.</i> [216]	0.8940	0.9110	0.8020
Nguyen-Xuan <i>et al.</i> [176]	0.8955	0.9112	0.8010
Present method	0.8936	0.9088	0.8001

Fig. 5.4. Plastic dissipation distribution associated to $\sigma_2 = 0$.

The geometric effect of the size of the centre on the limit load factor is investigated through various ratios of radius r to length l . The simulation results are presented in Table 5.2. Obviously, the present approach offers a great reliability to the limit analysis problems as the simulation results agree with the solutions provided by the reference research works. It is also found that the limit load factor is inversely proportional to the size of central hole. The presence of hole weakens the stiffness of the material, resulting in a significant decrease in the load-carrying capacity of the plate.

Table 5.2. Limit load factors λ_p of a square plate with various radius-to-length ratios r/l under uniaxial tension.

r/l	Methods				
	Heitzer [217]	Tran <i>et al.</i> [215]	Tran [218]	Nguyen-Xuan <i>et al.</i> [176]	Present method
0.1	0.8951	0.8932	0.9017	0.9005	0.8990
0.2	0.7879	0.7967	0.8015	0.8010	0.8001
0.3	0.6910	0.6930	0.7022	0.7016	0.7002
0.4	0.5720	0.5760	0.5914	0.5933	0.5872
0.5	0.4409	0.4011	0.4012	0.4168	0.4006
0.6	0.2556	0.2429	0.2425	0.2532	0.2410
0.7	0.1378	0.1277	0.1254	0.1315	0.1250
0.8	0.0565	0.0521	0.0523	0.0528	0.0509
0.9	0.0193	0.0133	0.0123	0.0119	0.0122

5.2.2. Grooved rectangular plate

As depicted in Fig. 5.5, a grooved rectangular plate under the uniaxial in-plane tensile load σ_0 is considered herein. The dimensions of the plate are $2l \times l$, while the groove on each side of the plate has a radius of $r = 0.25l$. A half model with the symmetry boundary condition is studied for this limit analysis problem.

Fig. 5.5. Schematics of a grooved rectangular plate: (a) the full mode and (b) its quarter mode with symmetric boundary condition.

For this limit analysis problem, Prager and Hodge [219] computed the analytical solution to be 0.5 based upon the Tresca yield criterion, while Yan [220] obtained a feasible range of 0.5 to 0.577 associated with the von Mises yield criterion. Fig. 5.6 shows that the elements across the central line of plate experience the highest strain rates when the plastic collapse occurs.

Fig. 5.6. Mesh refinements for a grooved rectangular plate: (a) the initial mesh and (b) the mesh after 3 steps of refinement.

As presented in Table 5.3, it is found that the result obtained by an adaptive IMF coupling approach not only agrees well with reference solutions but also falls within the interval of the analytical solutions presented by Yan [220]. Fig. 5.7 plots the plastic dissipation distribution across the grooved plate at the collapse state. A smooth transition of the plastic dissipation is captured. It is also found that the groove plate is most likely to be yielded along its central line under uniaxial tensile loads.

Table 5.3. Limit load factors λ_p of a grooved rectangular plate.

Methods				
Prager and Hodge [219]	Yan [220]	Casciaro and Cascini [221]	Nguyen-Xuan <i>et al.</i> [176]	Present method
0.5000	0.5000 ~ 0.5770	0.5680	0.5593	0.5649

Fig. 5.7. Plastic dissipation distribution.

5.2.3. Thin plate

The application of an adaptive IMF coupling approach is extended to the limit analysis of thin plate. Based on the Kirchhoff–Love theory, a new variable $\dot{w}$ is introduced to denote the velocity of transversal displacement along the z -direction of the plate. The curvature rates $\dot{\kappa}$ are defined by $\nabla^2 \dot{w}$, where $\nabla^2 = \begin{bmatrix} \partial^2/\partial x^2 & \partial^2/\partial y^2 & 2\partial^2/\partial xy \end{bmatrix}^T$.

By substituting the strain rates $\dot{\epsilon} = z\dot{\kappa}$ into Eq. (5.8), the plastic dissipation D_p for a thin plate can be given by

$$D_p(\dot{\kappa}) = \sigma_0 \int_{-h/2}^{h/2} \int_{\Omega} \sqrt{\dot{\epsilon}^T \Theta \dot{\epsilon}} d\Omega dz = \frac{\sigma_0 h^2}{4} \int_{\Omega} \sqrt{\dot{\kappa}^T \Theta \dot{\kappa}} d\Omega.$$

(5.25) Consequently, the optimization problem shown in Eq. (5.21) can be reformulated

by

$$\lambda_p = \min \left\{ 0.25 \sigma_p h^2 \sum_{i=1}^{n_g} \varpi_i |\mathbf{J}_i| \mathbf{v}_i \right\},$$

$$\text{s.t.} \begin{cases} W_{\text{ex}}(\dot{\kappa}) = 1 \\ \dot{\mathbf{w}} = \dot{\bar{\mathbf{w}}} & \text{on } \Gamma_w \\ \boldsymbol{\rho}_i = \mathbf{C}^T \mathbf{B}_{p_i} \dot{\mathbf{w}}_i \quad (i = 1, 2, \dots, n_g) \\ \sqrt{\boldsymbol{\rho}_i^T \boldsymbol{\rho}_i} \leq \mathbf{v}_i \quad (i = 1, 2, \dots, n_g) \end{cases} \quad (5.26) \text{ where the matrix } \mathbf{B}_{p_i}$$

for thin plate is defined as $\mathbf{B}_{p_i} = [\Phi_{i,xx} \quad \Phi_{i,yy} \quad 2\Phi_{i,xy}]^T$.

Fig. 5.8. Schematics of a thin plate subjected to the simply supported boundary condition: (a) the top view and (b) the front view.

A thin plate with each of its side subjected to the simply supported boundary condition is investigated as depicted in Fig. 5.8. The plate is assumed to have the yield stress $\sigma_p = 250$ MPa and the thickness $t = 0.1$ m. A load $\sigma_0 = 1$ MPa is uniformly applied on the plate along its z -direction. By letting m_p denote $0.25\sigma_p h^2$, the limit load factor λ_p for thin plate could be given by $\lambda_p = \sigma_0 l_1 l_2 / m_p$. Firstly, the plate is assumed to be a square one with each side to be 10 m. The adaptive refinement procedure starts with a coarse initial mesh as depicted in Fig. 5.9(a). The final refinement step depicted in Fig. 5.9(b) shows that the area associated with high plastic dissipated power exists along the diagonal lines of the plate, and the plastic dissipation is illustrated in Fig. 5.10 which further demonstrates that the plate is most likely to yield diagonally at its plastic collapse state. Although analytical solutions have not been proposed for this limit analysis problem, the simulation results still agree well with the available numerical solutions as presented in Table 5.4.

Fig. 5.9. Mesh refinements for a square thin plate: (a) the initial mesh and (b) the mesh after 4 steps of refinement.

Fig. 5.10. Plastic dissipation distribution.

Table 5.4. Limit load factors λ_p of a square thin plate.

Methods					
Le <i>et al.</i> [68]	Le <i>et al.</i> [222]	Zhou <i>et al.</i> [223]	Le <i>et al.</i> [224]	Nguyen-Xuan <i>et al.</i> [225]	Present method
24.98	25.01	25.07	24.93	25.023	24.932

Next, a rectangular thin plate with a length-width-ratio $l_1/l_2 = 2$ is investigated for its limit load factor. The adaptive refinement procedure of this plate is demonstrated in Fig. 5.11. It can be found that the pattern of the area containing high plastic dissipation within the rectangular thin plate is similar to that possessed by the square one. Fig. 5.12 further shows that the rectangular plate is also most likely to suffer plastic collapse along its diagonal lines. The limit load factor obtained by the adaptive IMF coupling approach and

reference solutions are tabulated in Table 5.5, and they agree well with each other as anticipated.

Table 5.5. Limit load factors λ_p of a rectangular thin plate.

Methods				
Le <i>et al.</i> [68]	Zhou <i>et al.</i> [223]	Melosh [226]	Nguyen-Xuan <i>et al.</i> [225]	Present method
29.88	28.90	29.88	29.93	29.88

(a)

(b)

Fig. 5.11. Mesh refinements for a rectangular thin plate: (a) the initial mesh and (b) the mesh after 4 steps of refinement.

Fig. 5.12. Plastic dissipation distribution.

5.3. Numerical results of cracked structures

After the robustness of an adaptive IMF coupling approach is verified for limit analysis of different structures and plates, its application is extended to cracked structures under a uniaxial tension load σ_0 . The limit load factor λ_p for cracked structures is defined by σ_p/σ_0 , where σ_p is the plastic collapse load of the corresponding material.

5.3.1. Central cracked plate

As shown in Fig. 5.13, a central cracked $2l \times l$ rectangular plate is considered first, and the length of crack is assumed to be l_c . A pair of uniaxial tension loads $\sigma_0 = \sigma_p$ are uniformly applied along two short ends of the plate. Firstly, the plate with a crack length ratio l_c/l of 0.5 is studied to examine the convergence of an adaptive IMF coupling approach. The initial mesh is depicted in Fig. 5.14(a), and the final step of adaptive refinement are depicted in Fig. 5.14(b) which demonstrates that the elements within the

region of material containing high plastic dissipation are automatically identified and refined along lines extended from the crack tips.

Fig. 5.13. A central cracked rectangular plate subjected to uniaxial tension.

Fig. 5.15 further demonstrates this phenomenon by plotting the plastic dissipation distribution over the problem domain at collapse state. Miller [63] developed an analytical solution of this limit analysis problem is based upon the von Mises yield criterion by $\lambda_p = 1 - l_c/l$. By setting the crack length ratio $l_c/l = 0.5$, the limit load factor can thus be calculated to be 0.5. The convergence rate achieved by the present method is compared with other available reference results in Fig. 5.16. Even if an adaptive IMF coupling approach adopts the coarse mesh with fewer variables than those employed XFEM and quadratic XIGA, the present approach is able to achieve the most accurate simulation results with the highest convergence rate.

(a)

(b)

Fig. 5.14. Mesh refinements for a central cracked rectangular plate with $l_c/l = 0.5$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.

Fig. 5.15. Plastic dissipation distribution associated to $l_c/l = 0.5$.

The effects of the crack geometry on the limit load factor is further investigated via various crack length ratios, which is tabulated in Table 5.6. Limit load factors achieved by the present approach are observed to be the closest to analytical solutions compared to other numerical methods, which demonstrates the robustness of an adaptive IMF coupling approach for the limit analysis problem of cracked structure.

Fig. 5.16. A companion of the convergence rates.

Table 5.6. Limit load factors λ_p of a central cracked rectangular plate with various crack lengths l_c .

l_c/l	Methods			
	Miller [63]	Tran <i>et al.</i> [117]	Nguyen-Xuan <i>et al.</i> [176]	Present method
0.1	0.9	0.9452	0.9183	0.8985
0.2	0.8	0.8426	0.8239	0.8105
0.3	0.7	0.7443	0.7293	0.7074
0.4	0.6	0.6398	0.6303	0.6098
0.5	0.5	0.5419	0.5299	0.5101
0.6	0.4	0.4356	0.4295	0.4099
0.7	0.3	0.3381	0.3284	0.3089
0.8	0.2	0.2292	0.2266	0.2080
0.1	0.9	0.9452	0.9183	0.8985

5.3.2. Edge-cracked plate

Fig. 5.17 shows that the $2l \times l$ rectangular plate under uniaxial tension load σ_0 now contains an edge crack with a crack length of l_c . For this benchmark limit analysis problem, Ewing and Richards [227] proposed an analytical solution by employing the von Mises yield criterion. The plastic collapse behaviours of edge-cracked plate would change once the crack length ratio l_c/l exceeds 0.146.

Fig. 5.17. An edge-cracked rectangular plate subjected to uniaxial tension.

Therefore, the adaptive procedures for the edge-cracked rectangular plates with the l_c/l of 0.06 and 0.3 are plotted in Fig. 5.18 and 5.19, respectively. Obviously, the crack length is found to affect the domains associated with large plastic dissipation. When the crack length ratio is below 0.146, the plastic collapse of the plate is most likely to take place at the crack tip and along two directions symmetric to the crack surface. On the other hand, the edge-cracked plate tends to yield along the crack surface when the crack length ratio l_c/l becomes larger than 0.146. This phenomenon could be better revealed by the plastic dissipation distribution over the problem domain as presented in Fig. 5.20. The limit load factors for various crack length ratios l_c/l are presented in Table 5.7. It is found that the limit load factor is inversely proportional to the crack length l_c . The simulation results match well with both analytical and reference solutions as anticipated.

(a)

(b)

Fig. 5.18. Mesh refinements for an edge-cracked rectangular plate with $l_c/l = 0.06$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.

(a)

(b)

Fig. 5.19. Mesh refinements for an edge-cracked rectangular plate with $l_c/l = 0.3$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.

(a)

(b)

Fig. 5.20. Plastic dissipation distribution associated to edge-cracked rectangular plates: (a) the crack length ratio $l_c/l = 0.06$ and (b) the crack length ratio $l_c/l = 0.3$.

Table 5.7. Limit load factors λ_p of an edge-cracked rectangular plate with various crack lengths.

Method	Crack length ratio l_c/l							
	0.06	0.1	0.14	0.2	0.3	0.4	0.5	0.6
Nguyen-Xuan <i>et al.</i> [176]	0.9364	0.8816	0.8327	0.7197	0.5298	0.3662	0.2339	0.1352
Present method	0.9397	0.8895	0.8369	0.7249	0.5346	0.3713	0.2366	0.1379

5.3.3. Grooved cracked plate

As depicted in Fig. 5.21, the last case investigates the plastic collapse behaviours of a grooved cracked plate under the uniaxial tension σ . The length of the plate is double of its width L , and the radius R of the groove equals $0.25l$. The crack length ratio is defined by $l_c/(1-2r)$ for this cracked structure. The adaptive refinement process is plotted in Fig. 5.22, where high strain rates are found to exist around the crack tip.

Fig. 5.21. A grooved cracked plate subjected to uniaxial tension.

(a)

(b)

Fig. 5.22. Mesh refinements for a grooved cracked rectangular plate with $l_c/l = 0.3$: (a) the initial mesh and (b) the mesh after 3 steps of refinement.

Fig. 5.23. Plastic dissipation distribution associated with $l_c/l = 0.3$.

As depicted in Fig. 5.23, the plastic dissipation distribution of the grooved cracked plate at its collapse state is found to align with the crack surface. As reported in Table 5.8, the limit load factors achieved by an adaptive IMF coupling approach agree well with the XIGA solutions. The crack length l_c are also found to have a noticeable influence on the limit load factor of cracked structures as a larger discontinues domain would further weaken the rigid-perfectly plastic materials.

Table 5.8. Limit load factors λ_p of a grooved cracked rectangular plate.

Method	Crack length ratio $l_c/(1-2r)$				
	0.1	0.2	0.3	0.4	0.5
Nguyen-Xuan <i>et al.</i> [176]	0.4787	0.3746	0.2865	0.2006	0.1395
Present method	0.4771	0.3730	0.2798	0.1980	0.1313

5.4. Summary

In this chapter, an adaptive IMF coupling approach is developed for assessing the plastic collapse behaviours of cracked structures. A plastic-dissipation-based indicator is also developed to guide the adaptive local refinement process. In this way, the computational cost induced by the excessive overhead of IGA control points due to the increase of mesh refinement can be significantly reduced for limit analysis problems. By solving benchmark limit analysis problems, the reliability and effectiveness of an adaptive IMF coupling approach can be demonstrated. It is found that the presence of crack would induce a decrease in the stiffness of material, resulting in a reduced limit load factor.

Chapter 6 Conclusions and Recommendations

In this chapter, the main conclusions based on the works conducted through this Ph.D. study are given, and corresponding future works are recommended.

6.1. Conclusions

This Ph.D. study develops IGA-based approaches for both fracture and contact problems of structures and materials. For these problems, B-Spline and NURBS exhibit evident advantages including exact geometry, tailorable inter-element connectivity and high-order approximation. However, local refinement is difficult to be performed within the IGA work-frame due to the nature of its tensor product. Therefore, it is imperative to develop feasible local refinement strategies for IGA-based numerical solutions. The detailed conclusions are summarised as follows.

In this thesis, the buckling load factors for a Mindlin–Reissner plate under different loading conditions are firstly studied. Since local refinement can be directly conducted in FEM-based computational codes, NURBS is modified into a Bézier element that can be integrated into the existing FEM codes. This method is integrated with enrichment functions for the modelling of cracks. Owing to the transverse shear locking induced by the Mindlin–Reissner theory, the DSG method is further introduced into the formulation for the plate analysis. Central cracked and edge-cracked plates with different geometries are investigated to study the effects of crack lengths, crack locations and plate thicknesses on the buckling behaviours of plates. By solving a series of bench problems, the Bézier

extraction-based XIGA is demonstrated to provide a high convergence rate as well as accurate simulation results compared with the reference solutions.

Afterwards, the contact behaviour between two bodies is investigated by formulating a fictive medium between these bodies. By adopting suitable material parameters, this fictive medium could transform from an isotropic material into an anisotropic one with respect to the gap between the contacting bodies. Consequently, an entire contact system is integrated into a smooth continuum domain without the time-consuming gap search algorithm to detect the penetration of contact surfaces. Based on the concept proposed by Wang and Zhang [189], IGA is coupled with the meshfree approximation based on the reproducing condition as meshfree methods preserve the high flexibility of local refinement. By dividing the overall domain into three sub-domains constructed by isogeometric, IMF coupling and meshfree approximations, both smooth discretisation and local refinement flexibility can be attained. Classic Hertzian contact is first studied to validate the contact treatment based on an IMF coupling approach. It is found that both the high convergence rate and accuracy for the solution can be obtained. Contact problems with large deformation and sliding are subsequently investigated. The simulation results demonstrate that extreme geometrical distortion can be well addressed by an IMF coupling approach. Nevertheless, different local refinement strategies should be adopted for each sub-domain as they are represented by different types of basis functions.

Finally, the plastic collapse of cracked structures is also analysed through the assessment of their corresponding limit load factors. Enrichment functions are employed for the modelling of cracks. The resulting equation is transformed into a SOCP optimisation problem that can be effectively tackled by interior-point solvers. An adaptive

IMF coupling approach is developed to take advantage of the fact that collapse mechanisms are generally highly localised, resulting in a significant decrease in computational cost. Compared to the method developed in Chapter 4, an adaptive IMF coupling approach is proposed to unify the basis functions across the entire domain. In this way, the local refinement of IGA can be performed in a straightforward meshfree manner. Moreover, an indicator of plastic dissipation is proposed to guide the local refinement along the area associated with high dissipated work. Structures with different geometries and boundary conditions are first examined to validate an adaptive IMF coupling approach. It is found that the adaptive refinement strategy leads to both a higher convergence rate and more accurate simulation results compared with the conventional uniform refinement procedure.

This Ph.D. study provides a solid numerical framework for analysing the fracture and contact behaviours of structures and materials. The solutions developed can provide guidance for the structural analysis and safety design of various engineering applications.

6.2. Recommendations

Both fracture and contact problems of structures and materials have been investigated through IGA-based approaches. Nonetheless, in real-life scenarios, these two problems usually merge into one for cracked materials under contact loading. Besides cracks, engineering structures and materials also contain other micro-defects such as dislocations and inclusions. Generally, the material properties of these materials with defects are depending on many factors, including surface roughness, loading types and lubrication conditions. Following works are thereby recommended below to further explore the

application of IGA-based approaches to the contact problems of materials containing defects.

Firstly, when two materials are in contact, both the contact area and contact pressure are unknown. These factors are harder to determine if there exist cracks in the material. It is because that the cracks would respond to the contact stress and traction, resulting in complex effects on the stress distribution and surface deformation. To better understand the influences of cracks on the contact behaviours of materials and structures, a future study should be conducted to assess the mechanical responses of cracked materials and structures to the contact loading.

Secondly, inclusions are another typical defects existing in various engineering components. Unlike cracks, inclusions are sub-domains with the same elastic moduli as its surrounding matrix material but possessing different eigenstrains. The interaction between inclusions and cracks would generate complex effects on the contact and tribological behaviours of the material. Since the presence of inclusions is unavoidable in the material manufacturing process, the present study should be extended to solving contact problems of materials with both inclusions and cracks.

Finally, friction and wear are undesirable phenomena that generally take place during the operation of transmission components. Therefore, lubricants are commonly employed to alleviate the material damage caused by the direct contact between surfaces. According to the research by Dong and Zhou [228] and Dong *et al.* [229], the solutions for the analysis of heterogeneous contact involving lubrication can be obtained in the elastic-plastic indentation regime via Reynolds equations and Barus law.

Appendix A

A.1. Bézier extraction formulation

For incorporating isogeometric analysis into the existing FEM codes, Bézier decomposition is utilised to calculate C^0 Bézier elements from the NURBS basis functions by constructing a Bézier extraction operation. Knot insertion is employed to reduce the continuity of the basis by one at a specified knot. In this way, the basis with a C^{p-k} continuity could be decomposed to a set of C^0 Bézier elements, where p denotes the polynomial order of the IGA basis functions and k is the multiplicity of the knot values.

To perform knot insertion, let $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ become a given knot vector. Next, the knot $\bar{\xi} \in [\xi_k, \xi_{k+1}] (k > p)$ will be inserted into the original knot vector. Consequently, a new $n + 1$ control points set $\bar{\mathbf{P}}$ is derived from the original control points set $\mathbf{P}$ as

$$\bar{\mathbf{P}}_i = \begin{cases} \mathbf{P}_1 & (i = 1) \\ \alpha_i \mathbf{P}_1 + (1 - \alpha_i) \mathbf{P}_{i-1} & (1 < i < n + 1), \\ \mathbf{P}_n & (i = n + 1) \end{cases} \quad (\text{A.1})$$

where the coefficient α_i is given by

$$\alpha_i = \begin{cases} 1 & (1 \leq i \leq k - p) \\ \frac{\bar{\xi} - \xi_i}{\xi_{i+1} - \xi_i} & (k - p + 1 \leq i \leq k). \\ 0 & (i \geq k + 1) \end{cases} \quad (\text{A.2})$$

After the expression for α_i is obtained as above, a new set of knots $\bar{\Xi} = \{\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_m\}$ is given for deriving the Bézier extraction operator $\mathbf{C}_j \in \mathbb{R}^{(n+j-1)(n+j)}$ ($j = 1, 2, \dots, m$). The operator could be defined by the following matrix as

$$\mathbf{C}_j = \begin{bmatrix} \alpha_1 & 1-\alpha_2 & 0 & \cdots & & 0 \\ 0 & \alpha_2 & 1-\alpha_3 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & 1-\alpha_4 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & & 0 & \alpha_{n+j+1} & 1-\alpha_{n+j} \end{bmatrix}. \quad (\text{A.3})$$

By constructing the Bézier extraction operator $\mathbf{C}_j$, the control points $\bar{\mathbf{P}}^{m+1}$ can be figured out and adopted to define the Bézier elements of decomposition. Let $\bar{\mathbf{P}}^{m+1}$ be replaced by $\mathbf{P}_b$ and correlated with $\mathbf{P}$ as

$$\mathbf{P}_b = \mathbf{C}^T \mathbf{P}, \quad (\text{A.4})$$

where $\mathbf{C}^T = \prod_{j=1}^m \left[(\mathbf{C}_j)^T \right]$.

For two given sets of knot vectors $\Xi_1 = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ and $\Xi_2 = \{\eta_1, \eta_2, \dots, \eta_{m+q+1}\}$, a NURBS surface is subsequently defined by Bernstein basis functions $B(\xi, \eta)$ as

$$S(\xi, \eta) = (\mathbf{P}_b)^T B(\xi, \eta) = \mathbf{P}^T N(\xi, \eta), \quad (\text{A.5})$$

where $N(\xi, \eta)$ is a new basis and linear operator set written by

$$N(\xi, \eta) = \mathbf{C} B(\xi, \eta). \quad (\text{A.6})$$

The NURBS basis functions can be given in a matrix form as

$$R(\xi, \eta) = \frac{\mathbf{W} N(\xi, \eta)}{W(\xi, \eta)} = \frac{\mathbf{W} \mathbf{C} B(\xi, \eta)}{W(\xi, \eta)}, \quad (\text{A.7})$$

where $\mathbf{W}$ is the diagonal matrix of weights ϖ , and $N(\xi, \eta)$ denotes the basis functions.

Similarly, the weight function $W(\xi, \eta)$ can further be rewritten in terms of the Bernstein basis by letting ϖ_b equal $\mathbf{C}^T \varpi$ as

$$W(\xi, \eta) = \varpi^T N(\xi, \eta) = (\varpi_b)^T B(\xi, \eta) = W_b(\xi, \eta). \quad (\text{A.8})$$

To obtain the expression for the Bézier element control points $\mathbf{P}_b$, $\mathbf{W}_b$ is defined to be the diagonal matrix of ϖ_b . $\mathbf{P}_b$ can be subsequently obtained by

$$\mathbf{P}_b = (\mathbf{W}_b)^{-1} \mathbf{C}^T \mathbf{W} \mathbf{P}. \quad (\text{A.9})$$

Finally, the original control points set $\mathbf{P}$ is mapped into the projective space to form a new set of control points $\mathbf{P}_b$. After these control points are mapped back from projective space, the NURBS surface can be expressed in terms of Bézier elements with C^0 continuity that can be incorporated into the existing FEM work-frame.

A.2. Discrete shear gap method

For Bézier C^0 plate element, the integration of the transverse shear strain γ along one direction yields:

$$v_\gamma = \int_{\alpha_0}^{\alpha_j} \gamma(\alpha) d\alpha = [w(\alpha)]_{\alpha_0}^{\alpha_j} + \int_{\alpha_0}^{\alpha_j} \beta(\alpha) d\alpha, \quad (\text{A.10})$$

where v_γ is the contribution of the transverse shear to the total deformation of material; α_j ($j=1,2$) indicate x - and y - axes, respectively and β denote the displacements and rotations of which the discretisation is given by

$$w = \sum_{i=1}^n R_i \tilde{w}_i \text{ and } \beta = \sum_{i=1}^n R_i \tilde{\beta}_i, \quad (\text{A.11})$$

where the notation $\sim$ denotes the non-interpolatory DOFs associated with the control point i , and n stands for the total number of control points.

In this way, the discretised transverse shear strains for the standard displacement are obtained by

$$\gamma = w_{,\alpha} + \beta = \sum_{i=1}^n (R_{i,\alpha} \tilde{w}_i + \tilde{\beta}_i). \quad (\text{A.12})$$

For a state of pure bending, the roots of v_γ take place on the control points on the domain for eliminating the spurious transverse shear strains. These control points are where the discrete shear gaps v_{γ_i} ($i = 1, 2, \dots, n$) are located.

Consequently, the non-interpolatory degrees of freedom $\tilde{v}_{\gamma_i}$ ($i = 1, 2, \dots, n$) could be identified in expression by discrete shear gaps v_{γ_i} ($i = 1, 2, \dots, n$) and shape function R_i as

$$\begin{bmatrix} v_{\gamma_1} \\ v_{\gamma_2} \\ \vdots \\ v_{\gamma_{np}} \end{bmatrix} = \begin{bmatrix} R_1(\alpha_1) & R_1(\alpha_2) & \cdots & R_1(\alpha_{np}) \\ R_2(\alpha_1) & R_2(\alpha_2) & \cdots & R_2(\alpha_{np}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{np}(\alpha_1) & R_{np}(\alpha_2) & \cdots & R_{np}(\alpha_{np}) \end{bmatrix} \begin{bmatrix} \tilde{v}_{\gamma_1} \\ \tilde{v}_{\gamma_2} \\ \vdots \\ \tilde{v}_{\gamma_{np}} \end{bmatrix}. \quad (\text{A.13})$$

For Bézier C^0 elements plate elements, the modified shear strain over the physical domain is the modified shear strain γ_{mod} which can be computed as

$$\gamma_{\text{mod}} = \sum_{i=1}^n (R_{i,\alpha} \tilde{w}_{\gamma_i}). \quad (\text{A.14})$$

DSGs are zero on control points with respect to any arbitrary states of pure bending, which helps to set Bézier C^0 elements plate elements free from spurious transverse shear strains. In this way, shear locking can be precluded for the analysis of plates.

Appendix B

B.1. Constitutive behaviours of hyper-elastic materials

For contacting bodies $\mathcal{B}$ made of St. Venant-Kirchhoff material, their constitutive behaviour has the energy form as

$$W = \frac{\Lambda}{8} (\mathcal{I}_1^2 - 6\mathcal{I}_1 + 9) + \frac{\mu}{4} (\mathcal{I}_1^2 - 2\mathcal{I}_2 - 2\mathcal{I}_1 + 3), \quad (\text{B. 1})$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are invariants of the right Cauchy-Green tensor $\mathbf{C}_r$ that are defined by $\text{tr}\mathbf{C}_r$ and $\text{tr}[(\det \mathbf{C}_r)\mathbf{C}_r^{-1}]$, respectively. Lamé constants Λ and μ are given in terms of The Young's modulus E and the Poisson's ratio ν as

$$\Lambda = \frac{Ev}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}. \quad (\text{B. 2})$$

For Neo-Hookean materials, the strain energy function can be defined by

$$W = \frac{\kappa}{2} (\mathcal{J} - 1) + \frac{u}{2} (\text{tr}\hat{\mathbf{C}}_r - 3). \quad (\text{B. 3})$$

where κ is the bulk modulus, $\mathcal{J} = \sqrt{\det \mathbf{C}}$ is the Jacobian of deformation and $\hat{\mathbf{C}}_r = J^{-2/3}\mathbf{C}_r$ is the isotropic part of the right Cauchy-Green deformation tensor $\mathbf{C}_r$.

B.2. Reproducing conditions for an IMF coupling approach

Reproducing conditions for univariate B-Spline basis functions $N(\xi)$ are given by

$$\sum_{i=1}^m N_i^p(\xi) \cdot \mathbf{p}(\xi_i^{[l]}) = \mathbf{p}(\xi), \quad (\text{B. 4})$$

where $\mathbf{p}(\xi_i^{[l]}) = \left\{1, \xi_i^{[1]}, (\xi_i^{[2]})^2, \dots, (\xi_i^{[p]})^p\right\}^T$ stands for reproducing point vectors on one-dimensional (1D) domain, and $\mathbf{p}(\xi) = \{1, \xi, \xi^2, \dots, \xi^p\}^T$ denotes the 1D polynomial basis.

The definition of reproducing points $\xi_i^{[l]} (l=1, 2, \dots, p)$ in Eq. (B. 4) can be given by

$$\xi_i^{[l]} = \sqrt[l]{\frac{S_p^l(G_{i+1}^{i+p})}{C_p^l}}, \quad (\text{B. 5})$$

where $C_p^l = p! / [l!(p-l)!]$, $G_{i+1}^{i+p} = \{\xi_{i+1}, \xi_{i+2}, \dots, \xi_{i+p}\}$ and I denotes the total number of elements.

The operator $S_p^l(G)$ in the above equation indicates that a set of elements are selected and multiplied from G , and all the possible terms are subsequently summed together.

Reproducing conditions for 2D basis functions can be given through the tensor product of $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$ in form as

$$\sum_{i=1}^n \sum_{j=1}^m N_{i,p}(\xi) M_{j,q}(\eta) (\xi_i^{[\alpha]})^\alpha (\eta_j^{[\beta]})^\beta = \xi^\alpha \eta^\beta, \quad (\text{B. 6})$$

where $\alpha + \beta \leq l$, l denotes the completeness order for the reproducing points. p and q in the above equation denote the polynomial order of the functions along ξ and η , respectively.

The reproducing condition can be obtained by simplifying Eq. (B. 6) as

$$\mathbf{p}(\xi) = \sum_{I=1}^{n_b} N_I^{pq}(\xi) \mathbf{p}(\xi_I^{[l]}), \quad (\text{B. 7})$$

where n_b and $N_I^{pq}(\xi)$ denote the total number and the tensor product of B-Spline, respectively.

2D reproducing point vectors $\mathbf{p}(\xi_I^{[l]})$ in Eq. (B.7) are subsequently given by

$$\mathbf{p}(\xi_I^{[l]}) = \left\{ 1, \xi_I^{[1]}, \eta_I^{[1]}, (\xi_I^{[2]})^2, \xi_I^{[1]} \cdot \eta_I^{[1]}, (\eta_I^{[2]})^2, \dots, (\xi_I^{[p]})^p, \dots, (\xi_I^{[q]})^q \right\}^T. \quad (\text{B. 8})$$

Based on the MLS method, meshfree nodes are discretely put on the domain Ω , and the 2D displacement $u(\xi)$ can be obtained by

$$u(\xi) = \mathbf{p}^T(\xi) \mathbf{z}(\xi), \quad (\text{B. 9})$$

where $\mathbf{p}(\xi)$ are meshfree polynomial basis functions and $\mathbf{z}(\xi)$ are vector coefficients. 2D meshfree polynomial basis functions with an order of q are given by

$$\mathbf{p}(\xi) = \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \dots, \xi^q, \dots, \eta^q\}^T. \quad (\text{B. 10})$$

As for $\mathbf{c}(\xi)$, it can be expressed by $\mathbf{p}(\xi)$ and the nodal displacement u_I as

$$\mathbf{L} = \sum_{I=1}^{n_m} \widehat{W}(\xi - \xi_I) [\mathbf{p}^T(\xi_I) \mathbf{z}(\xi) - u_I]^2, \quad (\text{B. 11})$$

where n_m denotes the total number of the meshfree nodes associated to ξ , $\mathbf{L}$ is the weighted L_2 norm, and $\widehat{W}(\xi - \xi_I)$ are the non-zero weight functions which are obtained by

$$\widehat{W}(\xi - \xi_I) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & (r \leq 1) \\ 0 & (r > 1) \end{cases} \text{ with } r = \frac{|\xi - \xi_I|}{\gamma d_I}, \quad (\text{B. 12})$$

where d_I denotes the supporting radius around the node I , and the coefficient γ varies in a range of 2 to 4.

Eq. (B. 11) can be further simplified through the differentiation of $\mathbf{L}$ with respect to $\mathbf{z}(\xi)$ as

$$\mathbf{A}(\xi) \mathbf{z}(\xi) = \mathbf{D}(\xi) \mathbf{U}(\xi), \quad (\text{B. 13})$$

where $\mathbf{U} = \{u_1, u_2, \dots, u_{n_m}\}$ denotes the nodal parameters; and Matrices $\mathbf{A}(\xi)$ and $\mathbf{D}(\xi)$ in the above equation can be given by

$$\mathbf{A}(\xi) = \sum_{l=1}^{n_m} \widehat{W}(\xi - \xi_l) \mathbf{p}(\xi_l) \mathbf{p}^T(\xi_l) \quad (\text{B. 14})$$

and

$$\mathbf{D}(\xi) = \begin{bmatrix} \widehat{W}(\xi - \xi_1) p(\xi_1) & \widehat{W}(\xi - \xi_2) p(\xi_2) & \cdots & \widehat{W}(\xi - \xi_{n_m}) p(\xi_{n_m}) \end{bmatrix}. \quad (\text{B. 15})$$

By substituting these equations into (B. 9), $u(\xi)$ can be reformulated as

$$u(\xi) = \sum_{l=1}^{n_m} \psi_l(\xi) u_l(\xi) = \mathbf{\Psi}(\xi) \mathbf{U}, \quad (\text{B. 16})$$

where basis functions $\psi(\xi)$ are given by

$$\psi_l(\xi) = \sum_{j=1}^{n_m} p_j(\xi) \left(A^{-1}(\xi) B(\xi) \right)_{jl} = \mathbf{p}^T(\xi) \left(\mathbf{A}^{-1} \mathbf{B} \right)_l. \quad (\text{B. 17})$$

Finally, 2D reproducing conditions for the MLS-based meshfree approach can be given by

$$\mathbf{p}(\xi) = \sum_{l=1}^{n_m} \psi_l(\xi) \mathbf{p}(\xi_l). \quad (\text{B. 18})$$

Appendix C

Fig. C.1(a) shows an initial quadric mesh on a 2D domain with a set of knot vectors $\Xi = \{0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1\}$ along each direction, which generates linear reproducing points $\xi^{[1]} = \{0, 1/8, 3/8, 5/8, 7/8, 1\}$. Its corresponding adaptive IMF basis functions are plotted in Fig. C.1(b) which matches the isogeometric B-Spline basis functions.

As shown in Fig. C.2(a), by inserting $3/8$ and $5/8$ into the initial knot vector, the new nodes set would become $\xi^{[1]} = \{0, 1/8, 5/16, 7/16, 9/16, 11/16, 7/8, 1\}$. It can be found that new reproducing nodes based on an adaptive IMF coupling approach are added into the refined elements where the nodes within the rest domain remain constant. Fig. C.2(b) further proves that a smooth transition of the basis functions between the refined and normal elements is achieved.

Fig. C.1. Schematics of (a) meshes and nodes and (b) basis functions for an adaptive IMF coupling approach at initial stage.

(a)

(b)

Fig. C.2. Schematics of (a) meshes and nodes and (b) basis functions for an adaptive IMF coupling approach after one step of refinement.

References

- [1] B. Stahl, L. M. Keer. Vibration and stability of cracked rectangular plates, *International Journal of Solids and Structures*, 1972. **8**(1): pp. 69-91.
- [2] A. A. Griffith, G. I. Taylor. Vi. The phenomena of rupture and flow in solids, *Philosophical Transactions of the Royal Society of London Series A, Containing Papers of a Mathematical or Physical Character*, 1921. **221**(582-593): pp. 163-198.
- [3] D. S. Dugdale. Yielding of steel sheets containing slits, *Journal of the Mechanics and Physics of Solids*, 1960. **8**(2): pp. 100-104.
- [4] G. R. Irwin. Linear fracture mechanics, fracture transition, and fracture control, *Engineering Fracture Mechanics*, 1968. **1**(2): pp. 241-257.
- [5] D. J. Hayes, J. G. Williams. A practical method for determining dugdale model solutions for cracked bodies of arbitrary shape, *International Journal of Fracture Mechanics*, 1972. **8**(3): pp. 239-256.
- [6] H. Riedel. A dugdale model for crack opening and crack growth under creep conditions, *Materials Science and Engineering*, 1977. **30**(3): pp. 187-196.
- [7] G. Glinka. A cumulative model of fatigue crack growth, *International Journal of Fatigue*, 1982. **4**(2): pp. 59-67.
- [8] J. Yang, X. Wang, K. Zhou. A numerical elastic-plastic contact model for a half-space with inhomogeneous inclusions and cracks, *Acta Mechanica*, 2019. **230**(6): pp. 2233-2247.
- [9] T. Belytschko, T. Black. Elastic crack growth in finite elements with minimal remeshing, *International Journal for Numerical Methods in Engineering*, 1999. **45**(5): pp. 601-620.
- [10] N. Sukumar, N. Moës, B. Moran, T. Belytschko. Extended finite element method for three-dimensional crack modelling, *International Journal for Numerical Methods in Engineering*, 2000. **48**(11): pp. 1549-1570.
- [11] D. Franke, A. Düster, V. Nübel, E. Rank. A comparison of the h-, p-, hp-, and rp-version of the fem for the solution of the 2d hertzian contact problem, *Computational Mechanics*, 2010. **45**(5): pp. 513-522.
- [12] A. Francavilla, O. C. Zienkiewicz. A note on numerical computation of elastic contact problems, *International Journal for Numerical Methods in Engineering*, 1975. **9**(4): pp. 913-924.

-
- [13] N. Kikuchi, J. T. Oden. *Contact problems in elasticity: A study of variational inequalities and finite element methods*: SIAM; 1988.
 - [14] P. Wriggers, T. Vu Van, E. Stein. Finite element formulation of large deformation impact-contact problems with friction, *Computers & Structures*, 1990. **37**(3): pp. 319-331.
 - [15] T. J. R. Hughes, J. A. Cottrell, Y. Bazilevs. Isogeometric analysis: Cad, finite elements, nurbs, exact geometry and mesh refinement, *Computer Methods in Applied Mechanics and Engineering*, 2005. **194**(39): pp. 4135-4195.
 - [16] S. S. Ghorashi, N. Valizadeh, S. Mohammadi. Extended isogeometric analysis for simulation of stationary and propagating cracks, *International Journal for Numerical Methods in Engineering*, 2012. **89**(9): pp. 1069-1101.
 - [17] L. De Lorenzis, P. Wriggers, T. J. R. Hughes. Isogeometric contact: A review, *GAMM-Mitteilungen*, 2014. **37**(1): pp. 85-123.
 - [18] L. De Lorenzis, İ. Temizer, P. Wriggers, G. Zavarise. A large deformation frictional contact formulation using nurbs-based isogeometric analysis, *International Journal for Numerical Methods in Engineering*, 2011. **87**(13): pp. 1278-1300.
 - [19] C. J. Corbett, R. A. Sauer. Nurbs-enriched contact finite elements, *Computer Methods in Applied Mechanics and Engineering*, 2014. **275**: pp. 55-75.
 - [20] M. R. Dörfel, B. Jüttler, B. Simeon. Adaptive isogeometric analysis by local h-refinement with t-splines, *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(5): pp. 264-275.
 - [21] N. Nguyen-Thanh, H. Nguyen-Xuan, S. P. A. Bordas, T. Rabczuk. Isogeometric analysis using polynomial splines over hierarchical t-meshes for two-dimensional elastic solids, *Computer Methods in Applied Mechanics and Engineering*, 2011. **200**(21): pp. 1892-1908.
 - [22] P. Hennig, S. Müller, M. Kästner. Bézier extraction and adaptive refinement of truncated hierarchical nurbs, *Computer Methods in Applied Mechanics and Engineering*, 2016. **305**: pp. 316-339.
 - [23] K. A. Johannessen, T. Kvamsdal, T. Dokken. Isogeometric analysis using lr b-splines, *Computer Methods in Applied Mechanics and Engineering*, 2014. **269**: pp. 471-514.

-
- [24] H. Zhang, D. Wang. Reproducing kernel formulation of b-spline and nurbs basis functions: A meshfree local refinement strategy for isogeometric analysis, *Computer Methods in Applied Mechanics and Engineering*, 2017. **320**: pp. 474-508.
- [25] G. R. Irwin. Analysis of stresses and strains near the end of a crack traversing a plate, *Journal of Applied Mechanics*, 1957. **24**: pp. 361-364.
- [26] F. Erdogan. Fracture mechanics, *International Journal of Solids and Structures*, 2000. **37**(1): pp. 171-183.
- [27] P. S. Theocaris, E. E. Gdoutos. The modified dugdale-barenblatt model adapted to various fracture configurations in metals, *International Journal of Fracture*, 1974. **10**(4): pp. 549-564.
- [28] L. P. Harrop. Application of a modified dugdale model to the k vs cod relation, *Engineering Fracture Mechanics*, 1978. **10**(4): pp. 807-816.
- [29] T. J. Lu, C. L. Chow. A modified dugdale model for crack tip plasticity and its related problems, *Engineering Fracture Mechanics*, 1990. **37**(3): pp. 551-568.
- [30] A. Neimitz. Modification of dugdale model to include the work hardening and in-and out-of-plane constraints, *Engineering Fracture Mechanics*, 2004. **71**(11): pp. 1585-1600.
- [31] A. T. Zehnder, M. J. Viz, Y. Potdar. Fatigue fracture in plates in tension and out-of-planeshear, *Fatigue & Fracture of Engineering Materials & Structures*, 2000. **23**(5): pp. 403-415.
- [32] A. E. H. Love, G. H. Darwin, Darwin. Xvi. The small free vibrations and deformation of a thin elastic shell, *Philosophical Transactions of the Royal Society of London (A)*, 1888. **179**: pp. 491-546.
- [33] G. Kirchho. Über das gleichgewicht und die bewegung einer elastischen scheibe. *Journal für die reine und angewandte Mathematik* 1850. p. 51.
- [34] G. C. Sih, P. C. Paris, F. Erdogan. Crack-tip, stress-intensity factors for plane extension and plate bending problems, *Journal of Applied Mechanics*, 1962. **29**(2): pp. 306-312.
- [35] M. Williams. On the stress distribution at the base of a stationary crack, *Journal of Applied Mechanics Reviews*, 1957. **24**: pp. 109-114.
- [36] C. Y. Hui, A. T. Zehnder. A theory for the fracture of thin plates subjected to bending and twisting moments, *International Journal of Fracture*, 1993. **61**(3): pp. 211-229.

-
- [37] A. M. Khludnev. Equilibrium problem of an elastic plate with an oblique crack, *Journal of Applied Mechanics and Technical Physics*, 1997. **38**(5): pp. 757-761.
- [38] L. Freddi, R. Paroni, T. Roubíček, C. Zanini. Quasistatic delamination models for Kirchhoff–Love plates, *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 2011. **91**(11): pp. 845-865.
- [39] V. V. Shcherbakov. Existence of an optimal shape of the thin rigid inclusions in the Kirchhoff–Love plate, *Journal of Applied and Industrial Mathematics*, 2014. **8**(1): pp. 97-105.
- [40] E. Reissner. On bending of elastic plates, *Quarterly of Applied Mathematics*, 1947. **5**(1): pp. 55-68.
- [41] E. Reissner. On the theory of transverse bending of elastic plates, *International Journal of Solids and Structures*, 1976. **12**(8): pp. 545-554.
- [42] J. K. Knowles, N.-M. Wang. On the bending of an elastic plate containing a crack, *Journal of Mathematics and Physics*, 1960. **39**(1-4): pp. 223-236.
- [43] N.-M. Wang. Effects of plate thickness on the bending of an elastic plate containing a crack, *Journal of Mathematics and Physics*, 1968. **47**(1-4): pp. 371-390.
- [44] N. M. Wang. Twisting of an elastic plate containing a crack, *International Journal of Fracture Mechanics*, 1970. **6**(4): pp. 367-378.
- [45] F. Delale, F. Erdogan. Line-spring model for surface cracks in a reissner plate, *International Journal of Engineering Science*, 1981. **19**(10): pp. 1331-1340.
- [46] F. Erdogan, O. Tuncel, P. C. Paris. An experimental investigation of the crack tip stress intensity factors in plates under cylindrical bending, *Journal of Basic Engineering*, 1962. **84**(4): pp. 542-546.
- [47] J. K. Paik, Y. V. Satish Kumar, J. M. Lee. Ultimate strength of cracked plate elements under axial compression or tension, *Thin-Walled Structures*, 2005. **43**(2): pp. 237-272.
- [48] R. H. Wynn, C. W. Smith. An experimental investigation of fracture criteria for combined extension and bending, *Journal of Basic Engineering*, 1969. **91**(4): pp. 841-849.
- [49] D. G. Smith, C. W. Smith. A photoelastic evaluation of the influence of closure and other effects upon the local bending stresses in cracked plates, *International Journal of Fracture Mechanics*, 1970. **6**(3): pp. 305-318.

-
- [50] R. Chandra. Experimental determination of stress intensity factors in patched cracked plates, *Engineering Fracture Mechanics*, 1989. **33**(1): pp. 65-79.
- [51] M. B. Wong. Chapter 7 - factors affecting plastic collapse. In: M. B. Wong, editor. *Plastic analysis and design of steel structures*. Boston: Butterworth-Heinemann; 2009. p. 195-217.
- [52] C. T. Morley. Yield-line theory for reinforced concrete slabs at moderately large deflexions, *Magazine of Concrete Research*, 1967. **19**(61): pp. 211-222.
- [53] A. Hillerborg, S. Commonwealth, O. Industrial Research, R. Division of Building. *Strip method for slabs on columns, l-shaped plates etc.* Melbourne: Commonwealth Scientific and Industrial Research Organization, Division of Building Research; 1964.
- [54] R. Hill. On discontinuous plastic states, with special reference to localized necking in thin sheets, *Journal of the Mechanics and Physics of Solids*, 1952. **1**(1): pp. 19-30.
- [55] D. J. F. Ewing. A series-method for constructing plastic slipline fields, *Journal of the Mechanics and Physics of Solids*, 1967. **15**(2): pp. 105-114.
- [56] C. Fleury. An efficient optimality criteria approach to the minimum weight design of elastic structures, *Computers & Structures*, 1980. **11**(3): pp. 163-173.
- [57] G. Maier. A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes, *Meccanica*, 1970. **5**(1): pp. 54-66.
- [58] M. Capurso. Limit analysis of continuous media with piecewise linear yield condition, *Meccanica*, 1971. **6**(1): pp. 53-58.
- [59] E. Anderheggen, H. Knöpfel. Finite element limit analysis using linear programming, *International Journal of Solids and Structures*, 1972. **8**(12): pp. 1413-1431.
- [60] E. Christiansen, K. O. Kortanek. Computation of the collapse state in limit analysis using the lp primal affine scaling algorithm, *Journal of Computational and Applied Mathematics*, 1991. **34**(1): pp. 47-63.
- [61] K. D. Andersen, E. Christiansen. Limit analysis with the dual affine scaling algorithm, *Journal of Computational and Applied Mathematics*, 1995. **59**(2): pp. 233-243.

-
- [62] K. D. Andersen, E. Christiansen. Minimizing a sum of norms subjected to linear equality constraints, *Computational Optimization and Applications*, 1998. **11**(1): pp. 65-79.
- [63] A. G. Miller. Review of limit loads of structures containing defects, *International Journal of Pressure Vessels and Piping*, 1988. **32**(1): pp. 197-327.
- [64] D. Mackenzie, J. Shi, J. T. Boyle. Finite element modelling for limit analysis by the elastic compensation method, *Computers & Structures*, 1994. **51**(4): pp. 403-410.
- [65] A. R. S. Ponter, K. F. Carter. Limit state solutions, based upon linear elastic solutions with a spatially varying elastic modulus, *Computer Methods in Applied Mechanics and Engineering*, 1997. **140**(3): pp. 237-258.
- [66] Y. Lei. J-integral and limit load analysis of semi-elliptical surface cracks in plates under bending, *International Journal of Pressure Vessels and Piping*, 2004. **81**(1): pp. 31-41.
- [67] M. A. Lynch, D. G. Moffat, D. N. Moreton. Limit loads for cracked piping branch junctions under pressure and branch out-of-plane bending, *International Journal of Pressure Vessels and Piping*, 2000. **77**(4): pp. 185-194.
- [68] C. V. Le, M. Gilbert, H. Askes. Limit analysis of plates and slabs using a meshless equilibrium formulation, *International Journal for Numerical Methods in Engineering*, 2010. **83**(13): pp. 1739-1758.
- [69] A. Makrodimopoulos, C. M. Martin. Lower bound limit analysis of cohesive-frictional materials using second-order cone programming, 2006. **66**(4): pp. 604-634.
- [70] A. Makrodimopoulos, C. M. Martin. Upper bound limit analysis using discontinuous quadratic displacement fields, 2008. **24**(11): pp. 911-927.
- [71] C. V. Le, H. Nguyen-Xuan, H. Askes, S. P. A. Bordas, T. Rabczuk, H. Nguyen-Vinh. A cell-based smoothed finite element method for kinematic limit analysis, 2010. **83**(12): pp. 1651-1674.
- [72] H. Hertz. Ueber die verdunstung der flüssigkeiten, insbesondere des quecksilbers, im luftleeren raume, *Annalen der Physik*, 1882. **253**(10): pp. 177-193.
- [73] K. L. Johnson, K. L. Johnson. *Contact mechanics*: Cambridge university press; 1987.
- [74] A. C. Fischer-Cripps, E. F. Gloyna, W. H. Hart. *Introduction to contact mechanics*: Springer; 2000.

-
- [75] I. S. Sokolnikoff, R. D. Specht. *Mathematical theory of elasticity*: McGraw-Hill New York; 1956.
- [76] C. Cattaneo. Sul contatto de due corpi elastici: Distribuzione locale deglisforzi, *Rendiconti dell'Accademia nazionale dei Lincei*, 1996. **6**: pp. 342-349.
- [77] J. O. Smith. Stress due to tangential and normal loads on an elastic solid with application to some contact stress problems, *J of Applid Mechanics*, 1953. pp. 157-166.
- [78] H. McCallion, N. Truong. The deformation of rough cylinders compressed between smooth flat surfaces of hard blocks, *Wear*, 1982. **79**(3): pp. 347-361.
- [79] A. Sackfield, D. A. Hills. Some useful results in the tangentially loaded hertzian contact problem, *The Journal of Strain Analysis for Engineering Design*, 1983. **18**(2): pp. 107-110.
- [80] A. Sackfield, D. A. Hills. Some useful results in the classical hertz contact problem, *The Journal of Strain Analysis for Engineering Design*, 1983. **18**(2): pp. 101-105.
- [81] M. M. H. Yu, B. Bhushan. Contact analysis of three-dimensional rough surfaces under frictionless and frictional contact, *Wear*, 1996. **200**(1-2): pp. 265-280.
- [82] G. Liu, Q. Wang. Thermoelastic asperity contacts, frictional shear, and parameter correlations, *Journal of tribology*, 2000. **122**(1): pp. 300-307.
- [83] J. A. Greenwood, J. B. P. Williamson. Contact of nominally flat surfaces, *Proceedings of the royal society of London Series A Mathematical and physical sciences*, 1966. **295**(1442): pp. 300-319.
- [84] D. J. Whitehouse, J. F. Archard. The properties of random surfaces of significance in their contact, *Proceedings of the Royal Society of London A Mathematical and Physical Sciences*, 1970. **316**(1524): pp. 97-121.
- [85] Y. Z. Hu, K. Tonder. Simulation of 3-d random rough surface by 2-d digital filter and fourier analysis, *International journal of machine tools and manufacture*, 1992. **32**(1-2): pp. 83-90.
- [86] P. Rosakis, A. J. Rosakis. The screw dislocation problem in incompressible finite elastostatics: A discussion of nonlinear effects, *Journal of elasticity*, 1988. **20**(1): pp. 3-40.

-
- [87] A. Acharya. A model of crystal plasticity based on the theory of continuously distributed dislocations, *Journal of the Mechanics and Physics of Solids*, 2001. **49**(4): pp. 761-784.
- [88] A. Acharya. Constitutive analysis of finite deformation field dislocation mechanics, *Journal of the Mechanics and Physics of Solids*, 2004. **52**(2): pp. 301-316.
- [89] L. M. Zubov. Continuously distributed dislocations and disclinations in nonlinearly elastic micropolar media, *Doklady Physics*, 2004. **49**(5): pp. 308-310.
- [90] A. Yavari, A. Goriely. Riemann-cartan geometry of nonlinear dislocation mechanics, *Archive for Rational Mechanics and Analysis*, 2012. **205**(1): pp. 59-118.
- [91] A. Yavari, A. Goriely. Weyl geometry and the nonlinear mechanics of distributed point defects, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 2012. **468**(2148): pp. 3902-3922.
- [92] G. R. Miller, L. M. Keer. Interaction between a rigid indenter and a near-surface void or inclusion, *Journal of Applied Mechanics*, 1983. **50**(3): pp. 615-620.
- [93] R. B. King, T. C. O'Sullivan. Sliding contact stresses in a two-dimensional layered elastic half-space, *International Journal of Solids and Structures*, 1987. **23**(5): pp. 581-597.
- [94] M. Liu, T. N. Farris. Effect of three-dimensional near surface defects on rolling and sliding contact fatigue, *Journal of Tribology*, 1994. **116**(4): pp. 841-848.
- [95] B. N. Kuvshinov. Elastic and piezoelectric fields due to polyhedral inclusions, *International Journal of Solids and Structures*, 2008. **45**(5): pp. 1352-1384.
- [96] B. R. Kim, H.-K. Lee. Closed form solution of the exterior-point eshelby tensor for an elliptic cylindrical inclusion, *Journal of Applied Mechanics*, 2010. **77**(2): pp. 024503.
- [97] T. Goshima, T. Soda. Stress intensity factors of a subsurface crack in a semi-infinite body due to rolling/sliding contact and heat generation, *JSME International Journal Series A Solid Mechanics and Material Engineering*, 1997. **40**(3): pp. 263-270.
- [98] J. Yang, K. Zhou. Analysis of the plastic zones of cracks in an elastic-perfectly plastic half-space under contact loading, *International Journal of Mechanical Sciences*, 2017. **121**: pp. 143-150.
- [99] K. Zhou, R. Wei. Multiple cracks in a half-space under contact loading, *Acta Mechanica*, 2014. **225**(4): pp. 1487-1502.

-
- [100] K. Zhou, R. Wei. Modeling cracks and inclusions near surfaces under contact loading, *International Journal of Mechanical Sciences*, 2014. **83**: pp. 163-171.
- [101] E. Byskov. The calculation of stress intensity factors using the finite element method with cracked elements, *International Journal of Fracture Mechanics*, 1970. **6**(2): pp. 159-167.
- [102] E. F. Rybicki, M. F. Kanninen. A finite element calculation of stress intensity factors by a modified crack closure integral, *Engineering Fracture Mechanics*, 1977. **9**(4): pp. 931-938.
- [103] R. S. Barsoum. On the use of isoparametric finite elements in linear fracture mechanics, *International journal for numerical methods in engineering*, 1976. **10**(1): pp. 25-37.
- [104] R. D. Henshell, K. G. Shaw. Crack tip finite elements are unnecessary, *International Journal for Numerical Methods in Engineering*, 1975. **9**(3): pp. 495-507.
- [105] P. Tong, T. H. H. Pian. On the convergence of the finite element method for problems with singularity, *International Journal of Solids and Structures*, 1973. **9**(3): pp. 313-321.
- [106] J. M. Melenk, I. Babuška. The partition of unity finite element method: Basic theory and applications, *Computer methods in applied mechanics and engineering*, 1996. **139**(1-4): pp. 289-314.
- [107] N. Moës, J. Dolbow, T. Belytschko. A finite element method for crack growth without remeshing, *International journal for numerical methods in engineering*, 1999. **46**(1): pp. 131-150.
- [108] J. Dolbow, N. Moës, T. Belytschko. Discontinuous enrichment in finite elements with a partition of unity method, *Finite elements in analysis and design*, 2000. **36**(3-4): pp. 235-260.
- [109] T. Belytschko, N. Moës, S. Usui, C. Parimi. Arbitrary discontinuities in finite elements, *International Journal for Numerical Methods in Engineering*, 2001. **50**(4): pp. 993-1013.
- [110] J. Dolbow, N. Moës, T. Belytschko. Modeling fracture in Mindlin–Reissner plates with the extended finite element method, *International Journal of Solids and Structures*, 2000. **37**(48): pp. 7161-7183.
- [111] X. Y. Liu, Q. Z. Xiao, B. L. Karihaloo. Xfem for direct evaluation of mixed mode sifs in homogeneous and bi-materials, *International Journal for Numerical Methods in Engineering*, 2004. **59**(8): pp. 1103-1118.

-
- [112] S. P. A. Bordas, T. Rabczuk, N.-X. Hung, V. P. Nguyen, S. Natarajan, T. Bog, et al. Strain smoothing in fem and xfem, *Computers & Structures*, 2010. **88**(23): pp. 1419-1443.
- [113] P. Baiz, S. Natarajan, S. Bordas, P. Kerfriden, T. Rabczuk. Linear buckling analysis of cracked plates by sfem and xfem, *Journal of Mechanics of Materials and Structures*, 2012. **6**(9): pp. 1213-1238.
- [114] L. Chen, T. Rabczuk, S. P. A. Bordas, G. R. Liu, K. Y. Zeng, P. Kerfriden. Extended finite element method with edge-based strain smoothing (esm-xfem) for linear elastic crack growth, *Computer Methods in Applied Mechanics and Engineering*, 2012. **209-212**: pp. 250-265.
- [115] A. Nasirmanesh, S. Mohammadi. Xfem buckling analysis of cracked composite plates, *Composite Structures*, 2015. **131**: pp. 333-343.
- [116] P. Liu, T. Q. Bui, D. Zhu, T. T. Yu, J. W. Wang, S. H. Yin, et al. Buckling failure analysis of cracked functionally graded plates by a stabilized discrete shear gap extended 3-node triangular plate element, *Composites Part B: Engineering*, 2015. **77**: pp. 179-193.
- [117] T. D. Tran, C. V. Le. Extended finite element method for plastic limit load computation of cracked structures, *International Journal for Numerical Methods in Engineering*, 2015. **104**(1): pp. 2-17.
- [118] T. J. R. Hughes, R. L. Taylor, W. Kanoknukulchai. A finite element method for large displacement contact and impact problems, *Formulations and Computational Algorithms in FE Analysis*, 1977. pp. 468-495.
- [119] T. J. R. Hughes, R. L. Taylor, J. L. Sackman, A. Curnier, W. Kanoknukulchai. A finite element method for a class of contact-impact problems, *Computer methods in applied mechanics and engineering*, 1976. **8**(3): pp. 249-276.
- [120] P. Wriggers, J. C. Simo. A note on tangent stiffness for fully nonlinear contact problems, *Communications in Applied Numerical Methods*, 1985. **1**(5): pp. 199-203.
- [121] G. Zavarise, L. De Lorenzis. The node-to-segment algorithm for 2d frictionless contact: Classical formulation and special cases, *Computer Methods in Applied Mechanics and Engineering*, 2009. **198**(41-44): pp. 3428-3451.
- [122] J. C. Simo, P. Wriggers, R. L. Taylor. A perturbed lagrangian formulation for the finite element solution of contact problems, *Computer methods in applied mechanics and engineering*, 1985. **50**(2): pp. 163-180.

-
- [123] P. Papadopoulos, R. L. Taylor. A mixed formulation for the finite element solution of contact problems, *Computer Methods in Applied Mechanics and Engineering*, 1992. **94**(3): pp. 373-389.
- [124] P. Papadopoulos, R. L. Taylor. A simple algorithm for three-dimensional finite element analysis of contact problems, *Computers & Structures*, 1993. **46**(6): pp. 1107-1118.
- [125] N. El-Abbasi, K.-J. Bathe. Stability and patch test performance of contact discretizations and a new solution algorithm, *Computers & Structures*, 2001. **79**(16): pp. 1473-1486.
- [126] K. A. Fischer, P. Wriggers. Frictionless 2d contact formulations for finite deformations based on the mortar method, *Computational Mechanics*, 2005. **36**(3): pp. 226-244.
- [127] P. Papadopoulos, R. E. Jones, J. M. Solberg. A novel finite element formulation for frictionless contact problems, *International Journal for Numerical Methods in Engineering*, 1995. **38**(15): pp. 2603-2617.
- [128] C. Bernardi. A new nonconforming approach to domain decomposition: The mortar element method, *Nonlinear partial equations and their applications*, 1989. pp.
- [129] F. B. Belgacem, P. Hild, P. Laborde. The mortar finite element method for contact problems, *Mathematical and Computer Modelling*, 1998. **28**(4-8): pp. 263-271.
- [130] F. Ben Belgacem. Numerical simulation of some variational inequalities arisen from unilateral contact problems by the finite element methods, *SIAM Journal on Numerical Analysis*, 2000. **37**(4): pp. 1198-1216.
- [131] B. Yang, T. A. Laursen, X. Meng. Two dimensional mortar contact methods for large deformation frictional sliding, *International journal for numerical methods in engineering*, 2005. **62**(9): pp. 1183-1225.
- [132] C. Hesch, P. Betsch. A mortar method for energy-momentum conserving schemes in frictionless dynamic contact problems, *International Journal for Numerical Methods in Engineering*, 2009. **77**(10): pp. 1468-1500.
- [133] M. Tur, E. Giner, F. J. Fuenmayor, P. Wriggers. 2d contact smooth formulation based on the mortar method, *Computer Methods in Applied Mechanics and Engineering*, 2012. **247**: pp. 1-14.
- [134] M. A. Puso, T. A. Laursen, J. Solberg. A segment-to-segment mortar contact method for quadratic elements and large deformations, *Computer methods in applied mechanics and engineering*, 2008. **197**(6-8): pp. 555-566.

-
- [135] P. Wriggers, J. Schröder, A. Schwarz. A finite element method for contact using a third medium, *Computational Mechanics*, 2013. **52**(4): pp. 837-847.
- [136] T. Bog, N. Zander, S. Kollmannsberger, E. Rank. Normal contact with high order finite elements and a fictitious contact material, *Computers & Mathematics with Applications*, 2015. **70**(7): pp. 1370-1390.
- [137] M. Khishvand, M. Nazem, S. W. Sloan, J. P. Carter. Application of the third medium method for frictionless contact problems in geomechanics, *Computers and Geotechnics*, 2017. **85**: pp. 117-125.
- [138] L. B. Lucy. A numerical approach to the testing of the fission hypothesis, *The astronomical journal*, 1977. **82**: pp. 1013-1024.
- [139] R. A. Gingold, J. J. Monaghan. Smoothed particle hydrodynamics: Theory and application to non-spherical stars, *Monthly notices of the royal astronomical society*, 1977. **181**(3): pp. 375-389.
- [140] T. Belytschko, Y. Y. Lu, L. Gu. Element-free galerkin methods, *International journal for numerical methods in engineering*, 1994. **37**(2): pp. 229-256.
- [141] S. N. Atluri, T. Zhu. A new meshless local petrov-galerkin (mlpg) approach in computational mechanics, *Computational mechanics*, 1998. **22**(2): pp. 117-127.
- [142] W. K. Liu, S. Jun, Y. F. Zhang. Reproducing kernel particle methods, *International journal for numerical methods in fluids*, 1995. **20**(8-9): pp. 1081-1106.
- [143] G.-R. Liu, M. B. Liu. *Smoothed particle hydrodynamics: A meshfree particle method*: World scientific; 2003.
- [144] T. Belytschko, L. Gu, Y. Y. Lu. Fracture and crack growth by element free galerkin methods, *Modelling and Simulation in Materials Science and Engineering*, 1994. **2**(3A): pp. 519.
- [145] T. Belytschko, M. Tabbara. Dynamic fracture using element-free galerkin methods, *International Journal for Numerical Methods in Engineering*, 1996. **39**(6): pp. 923-938.
- [146] T. Belytschko, M. Fleming. Smoothing, enrichment and contact in the element-free galerkin method, *Computers & Structures*, 1999. **71**(2): pp. 173-195.
- [147] N. Sukumar, B. Moran, T. Black, T. Belytschko. An element-free galerkin method for three-dimensional fracture mechanics, *Computational Mechanics*, 1997. **20**(1-2): pp. 170-175.

-
- [148] T. Rabczuk, T. Belytschko. Cracking particles: A simplified meshfree method for arbitrary evolving cracks, *International Journal for Numerical Methods in Engineering*, 2004. **61**(13): pp. 2316-2343.
- [149] Z. Zhang, P. Zhao, K. M. Liew. Analyzing three-dimensional potential problems with the improved element-free galerkin method, *Computational Mechanics*, 2009. **44**(2): pp. 273-284.
- [150] B. N. Rao, S. Rahman. Mesh-free analysis of cracks in isotropic functionally graded materials, *Engineering fracture mechanics*, 2003. **70**(1): pp. 1-27.
- [151] H. K. Ching, R. C. Batra. Determination of crack tip fields in linear elastostatics by the meshless local petrov-galerkin (mlpg) method, *CMES- Computer Modeling in Engineering and Sciences*, 2001. **2**(2): pp. 273-289.
- [152] K. Liu, S. Long, G. Li. A simple and less-costly meshless local petrov-galerkin (mlpg) method for the dynamic fracture problem, *Engineering Analysis with Boundary Elements*, 2006. **30**(1): pp. 72-76.
- [153] S. Tanaka, H. Suzuki, S. Sadamoto, M. Imachi, T. Q. Bui. Analysis of cracked shear deformable plates by an effective meshfree plate formulation, *Engineering Fracture Mechanics*, 2015. **144**: pp. 142-157.
- [154] Q. Li, S. Shen, Z. D. Han, S. N. Atluri. Application of meshless local petrov-galerkin (mlpg) to problems with singularities, and material discontinuities, in 3-d elasticity, *Computer Modeling in Engineering and Sciences*, 2003. **4**(5): pp. 571-586.
- [155] C. Tiago, P. M. Pimenta. An efg method for the nonlinear analysis of plates undergoing arbitrarily large deformations, *Engineering analysis with boundary elements*, 2008. **32**(6): pp. 494-511.
- [156] D. A. Hu, S. Y. Long, X. Han, G. Y. Li. A meshless local petrov-galerkin method for large deformation contact analysis of elastomers, *Engineering Analysis with Boundary Elements*, 2007. **31**(7): pp. 657-666.
- [157] Q. Li, K.-M. Lee. An adaptive meshless method for analyzing large mechanical deformation and contacts, *Journal of Applied Mechanics*, 2008. **75**(4): pp. 041014.
- [158] D. Li, F. Bai, Y. Cheng, K. M. Liew. A novel complex variable element-free galerkin method for two-dimensional large deformation problems, *Computer Methods in Applied Mechanics and Engineering*, 2012. **233**: pp. 1-10.
- [159] Z. Ullah, C. E. Augarde. Finite deformation elasto-plastic modelling using an adaptive meshless method, *Computers & Structures*, 2013. **118**: pp. 39-52.

-
- [160] S. D. Daxini, J. M. Prajapati. A review on recent contribution of meshfree methods to structure and fracture mechanics applications, *The Scientific World Journal*, 2014. **2014**: pp.
- [161] S. Fernández-Méndez, A. Huerta. Imposing essential boundary conditions in mesh-free methods, *Computer methods in applied mechanics and engineering*, 2004. **193**(12-14): pp. 1257-1275.
- [162] C. V. Verhoosel, M. A. Scott, R. De Borst, T. J. R. Hughes. An isogeometric approach to cohesive zone modeling, *International Journal for Numerical Methods in Engineering*, 2011. **87**(1-5): pp. 336-360.
- [163] E. De Luycker, D. J. Benson, T. Belytschko, Y. Bazilevs, M. C. Hsu. X-fem in isogeometric analysis for linear fracture mechanics, *International Journal for Numerical Methods in Engineering*, 2011. **87**(6): pp. 541-565.
- [164] S. S. Ghorashi, N. Valizadeh, S. Mohammadi, T. Rabczuk. T-spline based xiga for fracture analysis of orthotropic media, *Computers & Structures*, 2015. **147**: pp. 138-146.
- [165] S. K. Singh, I. V. Singh, B. K. Mishra, G. Bhardwaj, T. Q. Bui. A simple, efficient and accurate bézier extraction based t-spline xiga for crack simulations, *Theoretical and Applied Fracture Mechanics*, 2017. **88**: pp. 74-96.
- [166] I. V. Singh, G. Bhardwaj, B. K. Mishra. A new criterion for modeling multiple discontinuities passing through an element using xiga, *Journal of Mechanical Science and Technology*, 2015. **29**(3): pp. 1131-1143.
- [167] G. Bhardwaj, I. V. Singh, B. K. Mishra, V. Kumar. Numerical simulations of cracked plate using xiga under different loads and boundary conditions, *Mechanics of Advanced Materials and Structures*, 2016. **23**(6): pp. 704-714.
- [168] S. S. Ghorashi, N. Valizadeh, S. Mohammadi, T. Rabczuk, editors. Extended isogeometric analysis of plates with curved cracks. Proceedings of the Eighth International Conference on Engineering Computational Technology; 2012; Stirlingshire, Scotland
- [169] S. K. Singh, I. V. Singh, B. K. Mishra, G. Bhardwaj, S. K. Singh. Analysis of cracked plate using higher-order shear deformation theory: Asymptotic crack-tip fields and xiga implementation, *Computer Methods in Applied Mechanics and Engineering*, 2018. **336**: pp. 594-639.
- [170] G. Bhardwaj, I. V. Singh, B. K. Mishra. Fatigue crack growth in functionally graded material using homogenized xiga, *Composite Structures*, 2015. **134**: pp. 269-284.

-
- [171] T. Yu, T. Q. Bui, S. Yin, D. H. Doan, C. T. Wu, T. V. Do, et al. On the thermal buckling analysis of functionally graded plates with internal defects using extended isogeometric analysis, *Composite Structures*, 2016. **136**: pp. 684-695.
- [172] L. V. Tran, H. A. Ly, J. Lee, M. A. Wahab, H. Nguyen-Xuan. Vibration analysis of cracked fgm plates using higher-order shear deformation theory and extended isogeometric approach, *International Journal of Mechanical Sciences*, 2015. **96**: pp. 65-78.
- [173] P. Tan, N. Nguyen-Thanh, K. Zhou. Extended isogeometric analysis based on bézier extraction for an fgm plate by using the two-variable refined plate theory, *Theoretical and Applied Fracture Mechanics*, 2017. **89**: pp. 127-138.
- [174] T. Q. Bui. Extended isogeometric dynamic and static fracture analysis for cracks in piezoelectric materials using nurbs, *Computer Methods in Applied Mechanics and Engineering*, 2015. **295**: pp. 470-509.
- [175] T. Q. Bui, S. Hirose, C. Zhang, T. Rabczuk, C.-T. Wu, T. Saitoh, et al. Extended isogeometric analysis for dynamic fracture in multiphase piezoelectric/piezomagnetic composites, *Mechanics of Materials*, 2016. **97**: pp. 135-163.
- [176] H. Nguyen-Xuan, L. V. Tran, C. H. Thai, C. V. Le. Plastic collapse analysis of cracked structures using extended isogeometric elements and second-order cone programming, *Theoretical and Applied Fracture Mechanics*, 2014. **72**: pp. 13-27.
- [177] M. E. Matzen, T. Cichosz, M. Bischoff. A point to segment contact formulation for isogeometric, nurbs based finite elements, *Computer Methods in Applied Mechanics and Engineering*, 2013. **255**: pp. 27-39.
- [178] D. J. Benson, J. O. Hallquist. A single surface contact algorithm for the post-buckling analysis of shell structures, *Computer Methods in Applied Mechanics and Engineering*, 1990. **78**(2): pp. 141-163.
- [179] D. J. Benson, Y. Bazilevs, E. De Luycker, M. C. Hsu, M. Scott, T. J. R. Hughes, et al. A generalized finite element formulation for arbitrary basis functions: From isogeometric analysis to xfem, *International Journal for Numerical Methods in Engineering*, 2010. **83**(6): pp. 765-785.
- [180] I. Temizer, P. Wriggers, T. J. R. Hughes. Contact treatment in isogeometric analysis with nurbs, *Computer Methods in Applied Mechanics and Engineering*, 2011. **200**(9-12): pp. 1100-1112.

-
- [181] I. Temizer. Multiscale thermomechanical contact: Computational homogenization with isogeometric analysis, *International journal for numerical methods in engineering*, 2014. **97**(8): pp. 582-607.
- [182] R. Dimitri, L. De Lorenzis, M. A. Scott, P. Wriggers, R. L. Taylor, G. Zavarise. Isogeometric large deformation frictionless contact using t-splines, *Computer methods in applied mechanics and engineering*, 2014. **269**: pp. 394-414.
- [183] J. Y. Kim, S. K. Youn. Isogeometric contact analysis using mortar method, *International Journal for Numerical Methods in Engineering*, 2012. **89**(12): pp. 1559-1581.
- [184] A. Seitz, P. Farah, J. Kremheller, B. I. Wohlmuth, W. A. Wall, A. Popp. Isogeometric dual mortar methods for computational contact mechanics, *Computer Methods in Applied Mechanics and Engineering*, 2016. **301**: pp. 259-280.
- [185] R. Kruse, N. Nguyen-Thanh, P. Wriggers, L. De Lorenzis. Isogeometric frictionless contact analysis with the third medium method, *Computational Mechanics*, 2018. **62**(5): pp. 1009-1021.
- [186] N. Nguyen-Thanh, K. Zhou. Extended isogeometric analysis based on pht-splines for crack propagation near inclusions, *International Journal for Numerical Methods in Engineering*, 2017. **112**(12): pp. 1777-1800.
- [187] L. Chen, E. J. Ling, R. de Borst. Adaptive hierarchical refinement of nurbs in cohesive fracture analysis, *International Journal for Numerical Methods in Engineering*, 2017. **112**(13): pp. 2151-2173.
- [188] P. Hennig, M. Kästner, P. Morgenstern, D. Peterseim. Adaptive mesh refinement strategies in isogeometric analysis—a computational comparison, *Computer Methods in Applied Mechanics and Engineering*, 2017. **316**: pp. 424-448.
- [189] D. Wang, H. Zhang. A consistently coupled isogeometric-meshfree method, *Computer Methods in Applied Mechanics and Engineering*, 2014. **268**: pp. 843-870.
- [190] W. Li, N. Nguyen-Thanh, K. Zhou. Geometrically nonlinear analysis of thin-shell structures based on an isogeometric-meshfree coupling approach, *Computer Methods in Applied Mechanics and Engineering*, 2018. **336**: pp. 111-134.
- [191] N. Nguyen-Thanh, W. Li, K. Zhou. Static and free-vibration analyses of cracks in thin-shell structures based on an isogeometric-meshfree coupling approach, *Computational Mechanics*, 2018. **62**(6): pp. 1287-1309.

-
- [192] J. Huang, N. Nguyen-Thanh, K. Zhou. An isogeometric-meshfree coupling approach for contact problems by using the third medium method, *International Journal of Mechanical Sciences*, 2018. **148**: pp. 327-336.
- [193] A. Rosolen, M. Arroyo. Blending isogeometric analysis and local maximum entropy meshfree approximants, *Computer Methods in Applied Mechanics and Engineering*, 2013. **264**: pp. 95-107.
- [194] N. Valizadeh, Y. Bazilevs, J. S. Chen, T. Rabczuk. A coupled iga-meshfree discretization of arbitrary order of accuracy and without global geometry parameterization, *Computer Methods in Applied Mechanics and Engineering*, 2015. **293**: pp. 20-37.
- [195] L. B. Da Veiga, A. Buffa, C. Lovadina, M. Martinelli, G. Sangalli. An isogeometric method for the reissner-mindlin plate bending problem, *Computer Methods in Applied Mechanics and Engineering*, 2012. **209**: pp. 45-53.
- [196] K.-U. Bletzinger, M. Bischoff, E. Ramm. A unified approach for shear-locking-free triangular and rectangular shell finite elements, *Computers & Structures*, 2000. **75**(3): pp. 321-334.
- [197] R. Echter, M. Bischoff. Numerical efficiency, locking and unlocking of nurbs finite elements, *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(5-8): pp. 374-382.
- [198] C. Thai-Hoang, N. Nguyen-Thanh, H. Nguyen-Xuan, T. Rabczuk. An alternative alpha finite element method with discrete shear gap technique for analysis of laminated composite plates, *Applied Mathematics and Computation*, 2011. **217**(17): pp. 7324-7348.
- [199] K. M. Liew, X. L. Chen. Buckling of rectangular mindlin plates subjected to partial in-plane edge loads using the radial point interpolation method, *International Journal of Solids and Structures*, 2004. **41**(5-6): pp. 1677-1695.
- [200] S. Kitipornchai, Y. Xiang, C. M. Wang, K. M. Liew. Buckling of thick skew plates, *International Journal for Numerical Methods in Engineering*, 1993. **36**(8): pp. 1299-1310.
- [201] K. M. Liew, J. Wang, T. Y. Ng, M. J. Tan. Free vibration and buckling analyses of shear-deformable plates based on fsdt meshfree method, *Journal of Sound and Vibration*, 2004. **276**(3-5): pp. 997-1017.
- [202] L. G. Tham, H. Y. Szeto. Buckling analysis of arbitrarily shaped plates by spline finite strip method, *Computers & structures*, 1990. **36**(4): pp. 729-735.

-
- [203] S. P. Timoshenko, J. M. Gere. *Theory of elastic stability*: Courier Corporation; 2009.
- [204] F. L. Liu. Differential quadrature element method for buckling analysis of rectangular mindlin plates having discontinuities, *International Journal of Solids and Structures*, 2001. **38**(14): pp. 2305-2321.
- [205] E. Christiansen, K. D. Andersen. Computation of collapse states with von mises type yield condition, *International Journal for Numerical Methods in Engineering*, 1999. **46**(8): pp. 1185-1202.
- [206] E. Christiansen. Limit analysis of collapse states. *Handbook of numerical analysis*. 4: Elsevier; 1996. p. 193-312.
- [207] A. Capsoni, L. Corradi. A finite element formulation of the rigid-plastic limit analysis problem, *International Journal for Numerical Methods in Engineering*, 1997. **40**(11): pp. 2063-2086.
- [208] H. Nguyen-Xuan, C. T. Wu, G. R. Liu. An adaptive selective es-fem for plastic collapse analysis, *European Journal of Mechanics-A/Solids*, 2016. **58**: pp. 278-290.
- [209] H. Nguyen-Xuan, G. R. Liu. An edge-based finite element method (es-fem) with adaptive scaled-bubble functions for plane strain limit analysis, *Computer Methods in Applied Mechanics and Engineering*, 2015. **285**: pp. 877-905.
- [210] F. A. Gaydon, A. W. McCrum. A theoretical investigation of the yield point loading of a square plate with a central circular hole, *Journal of the Mechanics and Physics of Solids*, 1954. **2**(3): pp. 156-169.
- [211] J. Groß-Weege. On the numerical assessment of the safety factor of elastic-plastic structures under variable loading, *International Journal of Mechanical Sciences*, 1997. **39**(4): pp. 417-433.
- [212] S. Chen, Y. Liu, Z. Cen. Lower-bound limit analysis by using the efg method and non-linear programming, *International Journal for Numerical Methods in Engineering*, 2008. **74**(3): pp. 391-415.
- [213] N. Zouain, L. Borges, J. L. s. Silveira. An algorithm for shakedown analysis with nonlinear yield functions, *Computer Methods in Applied Mechanics and Engineering*, 2002. **191**(23-24): pp. 2463-2481.
- [214] X. Zhang, Y. Liu, Z. Cen. Boundary element methods for lower bound limit and shakedown analysis, *Engineering Analysis with Boundary Elements*, 2004. **28**(8): pp. 905-917.

- [215] T. N. Tran, G. R. Liu, H. Nguyen-Xuan, T. Nguyen-Thoi. An edge-based smoothed finite element method for primal-dual shakedown analysis of structures, *International Journal for Numerical Methods in Engineering*, 2010. **82**(7): pp. 917-938.
- [216] H. Nguyen-Xuan, T. Rabczuk, T. Nguyen-Thoi, T. N. Tran, N. Nguyen-Thanh. Computation of limit and shakedown loads using a node-based smoothed finite element method, *International Journal for Numerical Methods in Engineering*, 2012. **90**(3): pp. 287-310.
- [217] M. Heitzer. Traglast-und einspielanalyse zur bewertung der sicherheit passiver komponenten, *Berichte-forschungszentrum Julich Jul*, 1999. **1**(3704): pp. ALL-ALL.
- [218] T. N. Trần. Limit and shakedown analysis of plates and shells including uncertainties, 2008. pp.
- [219] W. Prager, P. G. Hodge. *Theory of perfectly plastic solids*: Dover Publications; 1968.
- [220] A.-M. Yan. Contributions to the direct limit state analysis of plastified and cracked structures, 1999. pp.
- [221] R. Casciaro, L. Cascini. A mixed formulation and mixed finite elements for limit analysis, *International Journal for Numerical Methods in Engineering*, 1982. **18**(2): pp. 211-243.
- [222] C. V. Le, M. Gilbert, H. Askes. Limit analysis of plates using the efg method and second-order cone programming, *International Journal for Numerical Methods in Engineering*, 2009. **78**(13): pp. 1532-1552.
- [223] S. Zhou, Y. Liu, S. Chen. Upper bound limit analysis of plates utilizing the c1 natural element method, *Computational Mechanics*, 2012. **50**(5): pp. 543-561.
- [224] C. V. Le, H. Nguyen-Xuan, H. Nguyen-Dang. Upper and lower bound limit analysis of plates using fem and second-order cone programming, *Computers & Structures*, 2010. **88**(1): pp. 65-73.
- [225] H. Nguyen-Xuan, C. H. Thai, J. Bleyer, P. V. Nguyen. Upper bound limit analysis of plates using a rotation-free isogeometric approach, *Asia Pacific Journal on Computational Engineering*, 2014. **1**(1): pp. 12.
- [226] R. J. Melosh. Basis for derivation of matrices for the direct stiffness method, *AIAA Journal*, 1963. **1**(7): pp. 1631-1637.

- [227] D. J. F. Ewing, C. E. Richards. The yield-point loads of singly-notched pin-loaded tensile strips, *Journal of the Mechanics and Physics of Solids*, 1974. **22**(1): pp. 27-36.
- [228] Q. Dong, K. Zhou. Numerical modeling of elastohydrodynamic lubrication in point or line contact for heterogeneous elasto-plastic materials, *Mechanics of Advanced Materials and Structures*, 2017. **24**(15): pp. 1300-1308.
- [229] Q. Dong, Z. Wang, D. Zhu, F. Meng, L. Xu, K. Zhou. A model of mixed lubrication based on non-normalized discretization and its application for multilayered materials, *Journal of Tribology*, 2019. **141**(4): pp. 042101-042101-042110.