Robust support identification for direction-of-arrival estimation and EIT application

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Robust Support Identification for Direction-Of-Arrival Estimation and EIT Application

Narong Borijindargoon

School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

2020

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

22 August 2019

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Narong Borijindargoon

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Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

22 August 2019

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Assoc. Prof. Boon Poh Ng

Authorship Attribution Statement

This thesis contains material from 4 papers published in the following peer-reviewed journals in which I am listed as an author.

Chapter 4 is published as **N. Borijindargoan** and B. P. Ng “A Hybrid Sensitivity-Correlation Regularization for Electrical Impedance Tomography,” *IET Sci. Meas. Technol.*, In Press. DOI: 10.1049/iet-smt.2018.5267.

The contributions of the co-authors are as follows:

- A/Prof Boon Poh Ng provided the initial project direction and edited the manuscript drafts.
- I derived the theoretical analysis and ran simulation study.
- I co-designed the experimental study with A/Prof Boon Poh Ng.
- The experiment and data collection were conducted by me.
- Data analysis was conducted by me with guidance from A/Prof Boon Poh Ng.

Chapter 5 is published as **N. Borijindargoan**, B. P. Ng, and S. Rahardja, “MUSIC-like Algorithm for Source Localization in Electrical Impedance Tomography,” *IEEE Trans. Ind. Electron.*, vol 66, no. 6, pp. 1661-1671, June 2019. DOI: 10.1109/TIE.2018.2863196.

The contributions of the co-authors are as follows:

- A/Prof Boon Poh Ng provided the initial project direction and edited the manuscript drafts.
- A/Prof Boon Poh Ng suggested the use of DOA estimation algorithms and array processing framework for EIP applications.
- I derived the theoretical analysis and ran simulation study.
- I co-designed the experimental study with A/Prof Boon Poh Ng.
- The experiment and data collection were conducted by me.
- Data analysis was conducted by me with guidance from A/Prof Boon Poh Ng.
- Prof Susanto Rahardja assisted in revising the manuscript and provided guidance in improving the manuscript.

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The contributions of the co-authors are as follows:

- A/Prof Boon Poh Ng provided the initial project direction and edited the manuscript drafts.
- A/Prof Boon Poh Ng introduced the MUSIC-like algorithm to this research work.
- I derived additional theoretical analysis.
- I ran the simulation study.
- Data analysis was conducted by me with guidance from A/Prof Boon Poh Ng.

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Abstract

This thesis is a reconciliation between multiple bodies of work with a shared objective that is “to identify the active support” under different environment settings. The applications of focus within this thesis will be placed on electrical impedance tomography (EIT) and direction-of-arrival (DOA) estimation. The term “active support” or only “support” is commonly used in compressed sensing and sparse signal recovery communities which is essentially referring to “the most contributing parameters” in a system of linear equations. Although with a common objective shared among the works that have been done within this thesis, this thesis can be divided into two major parts based on applications’ environment.

The first part is the support identification under a process expressible by Fredholm integral equation of the first kind. For this problem, a research topic that is known as “Electrical impedance tomography (EIT),” is chosen as a representative application for a collection of problems expressible by this integral equation. The physical laws (Physics) of EIT can be explained by Maxwell’s equations under quasi-static environment. Under such condition, the energy dissipated from the source is spread throughout the region (domain) in a non-linear fashion. Together with the absent of phase or time delay information, the image reconstruction task for EIT usually faces the issue of low signal-to-noise-ratio (SNR) and support ambiguity. Multiple real-life applications typically fall under this category of problems such as medical and geophysical subsurface imaging, deconvolution, and image deblurring.

The EIT is an imaging technique that estimates a distribution of electrical admittivity γ of a “subject under test (SUT)” sometimes called as “medium” based on an inference from data captured from the medium’s boundary. Although the name would suggests the value of ‘impedance’ to be measured, in most practical scenario its reciprocal which is ‘admittance’ (admittivity value) is commonly calculated instead. The admittivity value γ encapsulates the information of medium’s conductivity (σ) and permittivity (ϵ) with the following expression $\gamma = \sigma + i\omega\epsilon$, where $\omega = 2\pi f$ is the angular frequency (rad/s) and f is frequency in Hz. The

“tomographic reconstruction” or “the process of estimating the medium’s internal admittivity distribution” can be accomplished by solving a mathematical problem that is commonly known as an “inverse problem.” Due to a quasi-static nature of this imaging modality, the image reconstruction of the EIT is known to be one of the most severely ill-posed inverse problems. The image reconstruction problem of EIT will be analyzed through two different viewpoints. Firstly, this problem will be analyzed through the lens of a discrete ill-posed inverse problem from the applied mathematics community viewpoint. Secondly, this problem will be re-analyzed through the lens of array signal processing/beamforming framework from the signal processing community viewpoint. While exploring the EIT image reconstruction techniques, it was observed that although the EIT system can be implemented economically, the implementation process is not completely straightforward. Therefore, a framework of a simple and reliable EIT prototyping system that could provide means to collect different data format (e.g. single measurement vector (SMV) and multiple measurement vectors (MMV)) were implemented in this thesis. This robust, economical, and straightforward implementation framework was developed with an aim that it could be beneficial for those who are interested in image reconstruction and inverse algorithms without the access of a full functioning commercialized EIT system.

The second part is the support identification for DOA estimation under Gaussian and α -stable distributed noises. Several improvements of a robust beamforming technique called MUSIC-like algorithm had been further investigated which helps improve the algorithm’s performance. The main focus of this part is the performance improvement of the algorithm that can be used for support identification task under different noise scenarios. Three different approaches for estimating a relaxation parameter, an essential parameter used in the MUSIC-like algorithm, are presented. By experimenting with different approaches, an adaptive framework for relaxation parameter was observed to have the potential for further improvement. The performance of the MUSIC-like algorithm was then further explored through the directional adaptive framework under both Gaussian and α -stable distributed noise. It was also compared with the conventional MUSIC algorithm coupled with a pre-conditioned covariance matrix.

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List of Abbreviations

A.C.	Alternating Current
ADC	Analogue to Digital Converter
AIC	Akaike Information Criterion
AWGN	Additive White Gaussian Noise
BEM	Boundary Element Method
CBF	Conventional Beamforming
CT	Computed Tomography
CoG	Center of Gravity
CS	Compressive Sensing/Compressed Sensing
DF	Direction Finding
DOA	Direction of Arrival
DOT	Diffuse Optical Tomography
ET	Electrical Tomography
EIT	Electrical Impedance Tomography
ECT	Electrical Capacitance Tomography
ERT	Electrical Resistance Tomography
EVD	Eigenvalue Decomposition
FEM	Finite Element Method
FER	Fidelity Embedded Regularization
FLIC	Fractional Lower Order Infinity Norm
FLOM	Fractional Lower Order Moment

GCV	Generalized Cross-Validation
GEV	Generalized Eigenvalue
GLF	Generalized Logistic Function
GSRN	Generalized Signal to Noise Ratio
I.I.D.	Independent and Identically Distributed
LCMV	Linearly Constrained Minimum Variance
MAP	Maximum A Posterior
MDL	Minimum Description Length
MUSIC	Multiple Signals Identification and Classification
MVDR	Minimum Variance Distortionless Response
MMV	Multiple Measurement Vectors
M-FLOM	Modified Fractional Lower Order Moment
MUX	Multiplexer
MRI	Magnetic Resonance Imaging
NOSER	Newton One-Step Error Reconstruction
ODE	Ordinary Differential Equation
PCB	Printed Circuit Board
PDE	Partial Differential Equation
PE	Position Error
RADAR	Radio Detection and Ranging
RES	Resolution
RIP	Restricted Isometry Property
ROI	Regions of Interest
RMSR	Root Mean Square Error
RNG	Ringling
SONAR	Sound Navigation and Ranging
SSCM	Spatial Sign Covariance Matrix
SUT	Subject under Test
SMV	Single Measurement Vector
SNR	Signal to Noise Ratio
SVD	Singular Value Decomposition

ULA	Uniform Linear Array
VCCS	Voltage Controlled Current Source

List of Symbols

Chapter 1 and 2

$\mathbb{F}(\cdot), \mathbb{F}^{-1}(\cdot)$	Forward and inverse operator
$\eta, \mathbf{n}$	noise (scalar and vector respectively)
$y, \mathbf{y}$	Measurement (scalar and vector respectively)
$m, \mathbf{x}$	Parameter of interest (scalar and vector respectively)
$\bar{\mathbf{x}}$	Estimate solution
$\mathbf{A}$	Discretized forward operator, kernel matrix, sensing matrix
$\mathbf{U}$	A matrix containing the left singular vectors
Σ	A diagonal matrix containing the singular values
σ_i	Singular value
$\mathbf{V}$	A matrix containing the right singular vectors
$\kappa(\cdot)$	Condition number of a matrix
$\mathcal{R}$	Regularization matrix
$\mathbf{E}$	Electric field
$\mathbf{B}$	Magnetic field
μ	Permeability
σ	Conductivity
ε	Permittivity
q	Electric charge
$\mathbf{J}$	Current density
$\mathbf{a}(\theta)$	Steering vector for direction θ
$\mathbf{A}(\Theta_K)$	Array manifold for Θ_K directions

$\mathbf{w}$	Weight vector of a beamforming
$\mathbf{R}$	Covariance matrix
$\mathbf{R}_{xx}$	Spatial covariance matrix from data matrix $\mathbf{X}$
$\mathbf{R}_{ss}$	Source covariance matrix
σ^2	Noise variance
$\mathbf{U}_s$	A matrix containing the eigenvectors spanning signal subspace
$\mathbf{U}_n$	A matrix containing the eigenvectors spanning noise subspace
Λ_s	Eigenvalues correspond to the signal powers
Λ_n	Eigenvalues correspond to the noise powers
β	Auxiliary parameter for MUISC-like algorithm

Chapter 3

R_i	Intracellular resisitance
R_e	Extracellular resistance
C_m	Membrane capacitance
$X(\omega C_M)$	Reactance
γ	Impedance distribution
$\bar{\Omega}$	Complete domain (Medium + Boundary)
Γ	Boundary area where electrodes are specified
Γ'	Boundary area where no electrode is specified
$\mathcal{E}$	Electrode on the boundary
Ω	Medium (Problem domain)
$\partial\Omega$	Boundary of the medium
ϕ	Potential field
$\mathbf{v}$	Voltage measurement
$\bar{E}$	Electric field
$\bar{H}$	Magnetic field
$\nabla \cdot$	Divergence of a vector field
$\nabla \times$	Curl of a vector field

c	Speed of light in vacuum
ε_0	Electrical constant or vacuum permittivity
μ_0	Magnetic constant or vacuum permeability
B	Magnetic field
E	Electric field
κ	Vector of wave number
κ	Magnitude of a wave number
f	Frequency of the wave
ω	Angular frequency of the wave
t	Time
λ	Wavelength of the wave
K	Number of sources

Chapter 4

$\mathcal{E}_k$	The k^{th} electrode
z_k	Contact impedance of electrode $\mathcal{E}_k$
L	Number of electrodes
U_k	Potential value at electrode $\mathcal{E}_k$
$v^{j,k}$	The difference potential value between $U_k - U_{k+1}$
v	Instance adjacent measurement vector
σ_0	Initial conductivity value (baseline)
$\delta\sigma$	Perturbed conductivity value
y	Measurement vector (potential variation)
$\mathcal{J}$	Jacobian matrix
$\mathbf{j}(\xi_i)$	Column i^{th} of the Jacobian matrix
n	Noise vector
M	Number of measurement obtained from one round of data collection
$\mathcal{R}$	Regularization matrix
U	Matrix with left singular vectors

$\mathbf{V}$	Matrix with right singular vectors
Σ	Diagonal matrix with singular values
λ	Regularization value
α	$\alpha = \lambda^2$
$\varphi_i(\lambda)$	Filter factor as a function of λ for i^{th} singular value σ_i
ϵ	A small positive value
$\mathbf{M}$	NOSER regularization matrix
$\mathbf{N}$	FER regularization matrix
$\mathbf{P}$	Hybrid regularization matrix

Chapter 5

$\mathbf{A}$	Sensing matrix
$\mathbf{a}_k$	A column vector extracted from matrix $\mathbf{A}$
$\mathbf{A}_k$	Sub-matrix sampled from $\mathbf{A}$
δ_k	A small positive value for sub-matrix $\mathbf{A}_k$
μ	Mutual coherence
$\mathbf{A}(\Xi)$	Array manifold for EIT
ξ_i	Steering location at i^{th} pixel within the discretized medium
$\mathbf{a}(\xi_i)$	Steering vector of the ξ_i^{th} pixel
β	Relaxation parameter of MUSIC-like algorithm
$\lambda_{\mathbf{R},min}$	Minimum eigenvalue extracted from covariance matrix $\mathbf{R}$
$\lambda_{\mathbf{R},max}$	Maximum eigenvalue extracted from covariance matrix $\mathbf{R}$
σ_n^2	Noise variance
$\mathbf{N}$	Noise matrix
$\mathbf{S}$	Source matrix
$\mathbf{S}_p$	Source matrix with p active sources
$\mathbf{A}_p(\Xi)$	Array manifold for EIT with p active support
$\mathbf{R}$	Array covariance matrix
$\mathbf{R}_{ss}$	Source covariance matrix

θ	Angle of arrival
$\mathbf{a}(\theta)$	Steering vector in the direction θ for a narrowband signal
$\mathbf{A}(\Theta)$	Array manifold matrix with steering vectors forming its columns
γ_i	Impedance value for i^{th} pixel
U_r	Matrix with the first r left singular vectors
V_r	Matrix with the first r right singular vectors

Chapter 6

$\mathcal{A}(\Xi_p)$	Array manifold of p targets
Λ	Diagonal matrix containing eigenvalues
Σ	Diagonal matrix containing singular values
X_i	Field of view within coarse-grid system
Ξ^*	Region of interest within coarse-grid system
$\tilde{\Xi}$	Field of view within fine-grid system
$\tilde{\Xi}^*$	Region of interest within fine-grid system
$p(\mathbf{a}(\tilde{\xi}_\alpha))$	Probability of $\mathbf{a}(\tilde{\xi}_\alpha)$ to be part of ROI
$d(\cdot, \cdot)$	Distance parameter in Cartesian coordinate
$\text{Corr}\{\cdot, \cdot\}$	Correlation function
$\mathcal{R}$	Regularization matrix for three-step method

Chapter 7 and 8

$y_m(t)$	Received signal by sensor m at time t
$\mathbf{y}$	Received signal vector at time t
M	Number of sensors
d	Separation between two sensors in ULA configuration
θ_k	Angle of arrival of source k
$\mathbf{a}(\theta_k)$	Steering vector for continuous direction θ_k
$\mathbf{n}(t)$	Noise vector at time t

S	Source matrix
N	Noise matrix
A (Θ_K)	Array manifold containing K steering vector (K support)
S _K	Target signal with K number of targets (K support)
p	Number of snapshots
σ_K^2	Signal variance
σ_n^2	Noise variance
$\varphi(t)$	The characteristic function of stable distribution
α	Characteristic exponential for $\varphi(t)$
β^φ	Skewness parameter for $\varphi(t)$
γ	Scaling parameter for $\varphi(t)$
μ	Location parameter for $\varphi(t)$
C _{FLOM}	Fractional lower order moment matrix
C _{FLIC}	Fractional lower infinity-norm covariance matrix
C _{SSCM}	Spatial sign covariance matrix
C _{M-FLOM}	Modified fractional lower order moment matrix
β_θ	Directional adaptive relaxation parameter
u _M	Eigenvector corresponds to the minimum eigenvalue
Θ_K	A set of directions corresponds to the K number of sources
$Y(x)$	Generalized logistic function
A, K	Lower and upper asymptotes respectively
B	Growth rate
M	Location of maximum growth rate
β_C	Directional adaptive relaxation (Capon)
β_{LC}	Directional adaptive relaxation (Capon+GLF)
L	Regularization matrix for subspace-fitting regularization

Chapter 1

Introduction

1.1 Background

An *inverse problem* is a mathematical problem that uncovers quantity of interest which is inaccessible directly, yet some other quantities connected to the quantity of interest are accessible and measurable. For example, most tomographic tasks, either it is in biomedical or geophysical subsurface imaging, are considered to be part of an inverse problem that falls under the application of non-destructive testing. In other words, we are uncovering information inside the subject without destroying it. The quantity of interest can be organs inside the human body or some valuable resources underneath the ground, which are not directly measurable. Other problems that may not seem to be an inverse problem at first glance are image deblurring and electromagnetic scattering. In the latter case, the attempt is not to preserve the subject. The parameters of interest are simply not directly accessible. For example, in direction finding we may be required to locate an object's location through a scattered electromagnetic wave. In this case, we may not have complete knowledge of the object's location, but through multiple sensors located at different locations, we are able to estimate the object's location by using phase or time delay information obtained from the sensors.

The discussion of the inverse problem will not be completed without acknowledging its counterpart, the *forward problem*. It would not be overly simplified to state that a forward problem is a simulation of a cause to its effect while an inverse problem is an estimation or an inference of the effect to find its cause. For example, if we

have a couple of electric charges, the forward problem would be a simulation or a calculation of an electric field that exerts at a particular location r . In contrast, if we have a knowledge of an electric field at location r , the inverse problem will attempt to estimate back how many electric charges there are or how far those electric charges are located. As can be seen, the inverse problem is not a trivial task when compared with the forward problem. However, since most inverse problems are based on some form of mathematical models, it is acceptable to argue that there will not be an inverse problem without a forward problem.

To solve an inverse problem, one of the most important quantity is the observation or measurement. This quantity (measurement) should be related to the quantity of interest through a mathematical model. As mentioned earlier, the forward problem is an essential part of an inverse problem. Assuming the physical laws (Physics), which relates the *quantity of interest* or the *parameter* to the *measurements* or the *data*, are known, we can specify this relationship mathematically as

$$y = \mathbb{F}(m), \quad (1.1)$$

where y is a collection of data, $\mathbb{F}$ is the forward operator, and m is the quantity of interest or the parameters. In practice, the measurement is commonly contaminated by model mismatch error, some additive noise caused by quantization error, or an instrumental noise. The relationship above can now be re-expressed under an influence of noise as

$$y = \mathbb{F}(m) + \eta, \quad (1.2)$$

where η is commonly described by an additive white Gaussian noise (AWGN) or simply Gaussian noise. The role of a forward operator $\mathbb{F}(\cdot)$ is to relate the parameters m to the data y , which is part of a forward problem.

To solve an inverse problem, it is commonly proceeded through a mathematical model which is a discretized quantity with a connection to the parameter of interest which is a continuous variable. To clarify the argument above, an example is given as follows. Let's consider a human body as a subject under test for the tomographic task. The density of a medium (organ) inside the human body is a continuous parameter. However, to estimate the density inside the human body, we may discretize the medium into pixels (2-dimensions) or voxels (3-dimensions). This

process allows us to use a mathematical tool to solve an inverse problem, which is an image reconstruction task in this case. After the discretization procedure, we are able to represent the data with a discretized model by a system of linear equations as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (1.3)$$

where $\mathbf{y} \in R^M$ is a vector of size M representing a collection of measurement (M data points), $\mathbf{A} \in R^{M \times N}$ is a discretized forward operator relating N discretized parameters of interest to M data points, $\mathbf{x} \in R^N$ is N discretized parameters (N pixels in image reconstruction or N look directions in direction finding), and $\mathbf{n} \in R^M$ is a vector of additive noise. The discretized forward operator $\mathbf{A}$ can be obtained by first identifying the underlying physical process (Physics) of our problem. In most physical phenomena, matrix $\mathbf{A}$ can be obtained by solving ordinary differential equation (ODE) or partial differential equation (PDE).

1.2 Motivation and Objectives

Consider a system of linear equations in (1.3). The majority of research interest in the field of inverse problem has been placed on the parameter estimation, which is a question of how to “best estimate $\mathbf{x}$ accurately” or how to “estimate $\mathbf{x}$ quickly within the range of acceptable error”. The concern is mostly circling around $\mathbf{x}$ and $\mathbf{n}$ in (1.3) which can be represented by the study of inverse solution under different signal-to-noise ratio (SNR) scenarios.

Our research motivation, however, lies in the scenarios when the collected data $\mathbf{y}$ were obtained under unfavorable conditions. Therefore, two different causes or problems that can lead to this “unfavorable” conditions will be the main focus of this thesis. The first problem will be focusing on the data collected under a process expressible by Fredholm integral equation of the first kind [1]. According to (1.3), a highlight is placed onto the matrix $\mathbf{A}$ or the transfer function of (1.3). The problem of image reconstruction in electrical impedance tomography (EIT) will be used as a representative case for a collection or a class of problems expressible by Fredholm integral equation of the first kind. The Physics of the EIT can be explained by Maxwell’s equations under the quasi-static environment [2]. On the other hand, the

second problem will be focusing on the data collected under α -stable distributed noise. For this case, however, a highlight is placed onto the distribution that generates vector $\mathbf{n}$, which is the additive noise. The problem of direction finding (DF) or direction-of-arrival (DOA) estimation will be used as a representative class of problem for support identification under the α -stable noise. Since both of these applications can be explained by Maxwell's equations, an insightful perspective between both environment (EIT and DOA) will be drawn.

The objective of this thesis is to explore the performance improvement of inverse algorithms under different environments from the views of conventional discrete ill-posed inverse problem and the array signal processing. A connection between the two research areas will be drawn which helps to understand the advantages and disadvantages of each approach. Furthermore, a framework for a reliable lab-based EIT prototyping system that can be implemented economically with low complexity will be provided. This can be beneficial especially in research areas where a proof-of-concept for the inverse algorithms is required without an access to a full functioning EIT system.

1.3 Contributions of the Thesis

The contribution of this thesis can be summarized as follows:

- *To design and implement a lab-based EIT prototyping system.*

To experiment with different data formats that are not available in the public domain (e.g. SMV and MMV), a simple and reliable Lab-based EIT prototyping system that can be implemented economically is required. Our contribution is to design and implement an easy-to-use and an easy-to-replicate EIT prototyping system for academic purposes. Researchers or students whose main interests are the image reconstruction or inverse algorithm (theoretically and numerically) without an access to a full EIT system will find the usefulness of the proposed implementation framework.

- *To propose a robust hybrid regularization matrix for the regularization method based on single measurement vector framework.*

The primary task of image reconstruction process in EIT is to solve a discrete ill-posed inverse problem. The estimated solution is commonly obtained under a regularization framework so that the noise amplified solution, which occurs during the matrix inversion process, can be avoided. The regularization framework aims at balancing the model-data fitness while simultaneously constraining the solution space with additional prior (commonly prescribed through the regularization matrix and solution-norm). A regularization matrix called Newton One-Step Error Reconstruction (NOSER) is one of the most robust prior that is widely used in EIT community (besides Tikhonov regularization matrix). Its popularity lies in its intuitive interpret-ability and how computationally efficient this prior provides to most image reconstruction tasks. In other word, it is mathematically and computationally robust. Recently, another regularization matrix called Fidelity-Embedded Regularization (FER) was proposed which has similar structure (diagonally structured) compared to the NOSER while the intuition and reasoning behind this proposed prior is different from the NOSER. Our contribution is to explicitly highlight the relationship between these two robust regularization matrices namely NOSER and FER in both spatial and singular value decomposition (SVD) domains. A hybrid regularization matrix which encompasses the two prior knowledge, non-uniform sensitivity distribution and array response correlation, is then proposed as an alternative prior. Experimental results along with several evaluated performance parameters validate the ability of the proposed prior to achieving a well-balanced and robust performance.

- *To highlight and analyze the connection between EIT image reconstruction and array signal processing framework and to introduce EIT image reconstruction based on the MMV framework.*

For EIT image reconstruction, the noise amplified solution caused during matrix inversion can be avoided with non-parametric spectral based estimation when the conductivity variation is bounded and spatially sparse. Among many spectral based algorithms used in direction-of-arrival (DOA) estimation, an algorithm called multiple signal classification (MUSIC) is one of the most well-known that has super resolution performance. However, its dependency on the model order estimation can lead to performance degradation especially for a quasi-static environment like EIT application and this is due

to source location changes and conductivity variation. Our contribution is to introduce the connection between EIT forward problem of elliptic PDE (Poisson equation) and the construction of array manifold for the beamforming framework. In addition, the relationship between source position, conductivity variation, ill-conditioned array manifold and eigenvalues of the covariance matrix are explored. An algorithm called MUSIC-like which has high resolution performance comparable to MUSIC is then proposed for EIT application. It is formulated under the beamforming framework, and therefore does not require an estimation of model order from the covariance matrix. Simulation results show that the proposed method is capable of obtaining high resolution performance under various noise levels. An 8-electrode EIT system prototype was built using the proposed method, and experimental results confirm the high resolution performance capability of the proposed method.

- *To propose novel β -selection methods that provide performance improvement in terms of robustness or low bias for direction finding task.*

The MUSIC-like algorithm was originally proposed as an alternative method to the MUSIC algorithm in order to circumvent a requirement on subspace segregation. The relaxation parameter β , which was introduced into the formulation of the MUSIC-like algorithm, has enabled the algorithm to achieve high resolution performance comparable to the MUSIC algorithm without requiring explicit estimation of the signal and noise subspaces. Our next contribution is to propose three different novel β -selection methods where each method contributes to an improvement over the original proposition. Additionally, a geometrical interpretation of the MUSIC-like algorithm is provided.

- *To analyze the MUSIC-like algorithm under an impulse noise environment and to propose adaptive β -selection frameworks for the MUSIC-like algorithm.*

The performance of MUSIC-like algorithm is evaluated in a new environment where impulse noise is present. To further improve the performance of the MUSIC-like algorithm. Two novel adaptive frameworks for β -selection in MUSIC-like algorithm are proposed and evaluated. Instead of using a fixed β value, the direction-dependent β_θ is proposed. Advantages and disadvantages

of each method are evaluated under both Gaussian and α -stable distributed noises.

- *To propose hybrid beamforming-regularization frameworks for support identification in DOA estimation and EIT application.*

Possible improvements including future directions are explored. The possibility of novel improvements are evaluated qualitatively. The three-step method (three-step processing: scan, map/grid-translation, and reconstruct) can be used as an alternative method to the iterative process used in l_1 convex optimization approach. We also explore source localization and joint sparse support identification under limited snapshot data scenarios. Under such condition, the covariance matrix is degenerate, and hence the subspace fitting regularization can be used.

1.4 Organization of the Thesis

This thesis contains 9 chapters in total including this chapter, where the whole thesis can be divided into two parts: Robust Support Identification for Electrical Impedance Tomography (Part I) and Robust Support Identification for Direction Of Arrival (Part II).

Part I includes **Chapter 2 - 6** which can be summarized as follows.

In **Chapter 2**, an overview of image reconstruction procedure based on discrete ill-posed inverse problem and the several beamforming techniques used in array signal processing will be reviewed.

In **Chapter 3**, an overview of the EIT system and its implementation are provided. A procedure for designing and implementing a fast, reliable, and inexpensive prototyping system will be presented in this chapter. This enables a proof-of-concept for the inverse algorithm development during the prototyping stage. The voltage controlled current source is implemented based on improved Howland architecture. A multiplexing system for current injection and voltage measurement is implemented together with the signal rectifying component and a control unit based on an Arduino board.

The SMV framework is used as a starting point for the inverse problem where several regularization methods are explored. In **Chapter 4**, a relationship between two robust regularization matrices namely Newton One-Step Error Reconstruction (NOSER) and Fidelity-Embedded Regularization (FER) is explicitly highlighted in both spatial and singular value decomposition (SVD) domains. A hybrid regularization matrix which encompasses the two prior knowledge, nonuniform sensitivity distribution and array response correlation, is then proposed as an alternative prior. Experimental results along with several evaluated performance parameters highlight the ability of the proposed prior to achieving a well-balanced and robust performance.

In **Chapter 5**, the connection between the forward problem of EIT and the array manifold construction in the array signal processing sense will be explored. With the priori knowledge of array geometry, source stimulation pattern, and measurement procedure, the connection between the solution of a partial differential equation (PDE) particularly the Poisson equation and the construction of array manifold will be drawn. A beamforming framework for source localization under quasi-static environment for EIT application is then proposed in this chapter.

Chapter 6 proposes hybrid beamforming-regularization framework for EIT application. In the three-step approach, active atoms are scanned in the coarse-grid system in the first step. Next, the regions-of-interest (ROI) are identified and mapped into the fine-grid system. Statistical information within the fine-grid system will be extracted and used as a regularization matrix. Based on local prior obtained from the coarse-to-fine grid mapping, the regularization problem is then solved in the last step.

Part II includes **Chapter 7-9** which can be summarized as follows.

Chapter 7 explores further improvement of the MUSIC-like algorithm for DOA estimation. The MUSIC-like algorithm was originally proposed as an alternative method to the MUSIC algorithm for DOA estimation without requiring explicit model order estimation. It was shown to have robust performance particularly in low signal-to-noise ratio (SNR) scenarios. The working principle of a relaxation parameter β , a parameter which was introduced into the formulation of the MUSIC-like algorithm, is further explored in this chapter. Three novel approaches for β -selection are presented. The adaptive framework for β -selection is further

developed which helps improve the algorithm performance. To illustrate its robustness, the algorithm is examined under both Gaussian and α -stable distributed noise environments.

Chapter 8 proposes the subspace fitting regularization framework. It is investigated under the condition where number of data snapshots is limited. In such scenario, a subspace searching coupling with the regularization framework can be used as an alternative method for DOA estimation and active support discovery in general.

The concluding remarks are given in **Chapter 9** where discussion of the results from previous chapters is made. Future works for both the regularization framework and the beamforming techniques are also discussed.

Chapter 2

Fundamentals of Inverse Problem & Array Processing

2.1 Introduction

This chapter provides a comprehensive review of discrete ill-posed inverse problem and array signal processing techniques. For discrete ill-posed inverse problem, the extension of the parameter from continuous domain to discrete domain will be highlighted. A brief introduction to inverse problem and its solution is given. The noise amplified solution caused by ill-condition kernel ($\mathbf{A}$) is discussed along with the regularization framework. For array signal processing framework, the physical laws (Physics) that govern the signal model will be described based on Maxwell's equations. The construction of array manifold and several other beamforming techniques will also be covered in this chapter.

2.2 Inverse Problem

Numerous problems in modern science and engineering can be categorized into one these three classes of problems based on a flowchart given below in Fig. 2.1. In Fig. 2.1, the input is defined by a set of parameters that govern the overall process (or system). The output is commonly referred to as an observation or a measurement. A kernel (or sometimes called transfer function) is a mapping function

that describes the relationship between parameters (input) and the observation (output).

FIGURE 2.1: Flowchart of modern science and engineering problems.

In most natural phenomena, the relationship above can be described by a mathematical expression in a continuous domain as

$$y(p) = \int_u^v a(p, q)x(q)dq, \quad (2.1)$$

where $y(p)$ is the observation (output), $a(p, q)$ is an input-output map (kernel), and $x(q)$ is the parameter of interest (input). When $x(q)$ is under the the integral equation that appears only inside the boundary $\int_u^v$, it is called a Fredholm integral equation of the first kind [3]. Several examples that can be explained by the Fredholm integration of the first kind include: medical imaging, geophysical prospecting, image deblurring, and deconvolution of measurement instrument's response. In contrast, the Fredholm integral equation of the second kind would also account for the condition where unknown parameter appears on both inside and outside of the integral boundary. The extension of a continuous domain in (2.1) to a discrete domain can be re-expressed by using a summation operator instead of an integral operator as

$$y_i = \sum_k a_{i,k}x_k = \mathbf{a}^T \mathbf{x}, \quad (2.2)$$

where y_i is an instance of an observation (or a measurement), x_k is the parameter of interest (while $\mathbf{x}$ is a collection of parameters of interest), $a_{i,k}$ is considered to be

a weight that maps x_k to y_i (while $\mathbf{a}$ is a collection of weights that map $\mathbf{x}$ to y_i), and $(\cdot)^T$ is a transpose operator. With measurement obtained around the boundary, the expression above can be re-expressed as

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad (2.3)$$

where $\mathbf{y} \in R^M$ is a vector of size M (M data points), $\mathbf{x} \in R^N$ is a vector of size N (N parameters of interest), and $A \in R^{M \times N}$ is a matrix of size $M \times N$ that maps vector $\mathbf{x}$ (parameter of interest) to vector $\mathbf{y}$ (observations).

2.2.1 Solution of Discrete Inverse Problem

Consider a system of linear equations in (2.3). To obtain an inverse solution, the vector $\mathbf{x}$ is estimated through the knowledge of an observation vector $\mathbf{y}$ and a system matrix (kernel) $\mathbf{A}$, which will be called the sensing matrix in this thesis. It should be noted that the so-called *naive inverse solution*, which can be obtained through $\mathbf{A}^{-1}$, does not occur in real-life applications since matrix $\mathbf{A}$ is most likely to be a non-square matrix. The solution to this inverse problem is commonly obtained by a pseudo-inverse (or Moore-Penrose inverse) which is known as a least-square solution. For an overdetermined system of linear equations ($\mathbf{A} \in R^{M \times N}$ which is sometimes called “tall/skinny matrix”, where $M > N$), the solution can be obtained by minimizing the energy of residual (error) function as

$$J(x) = \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \quad (2.4)$$

$$= (\mathbf{y} - \mathbf{A}\mathbf{x})^T (\mathbf{y} - \mathbf{A}\mathbf{x}) \quad (2.5)$$

$$= \mathbf{y}^T \mathbf{y} - \mathbf{y}^T \mathbf{A}\mathbf{x} - \mathbf{x}^T \mathbf{A}^T \mathbf{y} + \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x} \quad (2.6)$$

$$= \mathbf{y}^T \mathbf{y} - 2\mathbf{y}^T \mathbf{A}\mathbf{x} + \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x}, \quad (2.7)$$

where we use $\mathbf{y}^T \mathbf{A}\mathbf{x} = \mathbf{x}^T \mathbf{A}^T \mathbf{y}$ identity in (2.6). By setting the derivative of our residual function in (2.7) to zero we can obtain the estimate solution as

$$\mathbf{0} = \frac{\partial J(x)}{\partial \mathbf{x}} \quad (2.8)$$

$$\mathbf{0} = -2\mathbf{y}^T \mathbf{A}\mathbf{x} + \mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x}, \quad (2.9)$$

$$\bar{\mathbf{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}, \quad (2.10)$$

where $\bar{\mathbf{x}}$ is our estimate solution (least square solution). On the other hand, for an underdetermined system of linear equations ($\mathbf{A} \in R^{M \times N}$ which is sometimes called “wide/fat matrix”, where $M < N$), the solution can be obtained where we seek a minimum norm solution of the following optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_2^2 \\ \text{s.t.} \quad & \mathbf{y} = \mathbf{A}\mathbf{x}. \end{aligned} \quad (2.11)$$

The Lagrangian function of (2.11) can be expressed as

$$L(\mathbf{x}, \lambda) = \|\mathbf{x}\|_2^2 + \lambda^T(\mathbf{y} - \mathbf{A}\mathbf{x}). \quad (2.12)$$

By setting the derivative of the Lagrangian function w.r.t $\mathbf{x}$ and λ to zero, we can obtain

$$\mathbf{0} = \frac{\partial J(x)}{\partial \mathbf{x}} \quad (2.13)$$

$$= 2\mathbf{x} - \mathbf{H}^T \mathbf{y} \quad (2.14)$$

$$\mathbf{x} = \frac{1}{2} \mathbf{H}^T \lambda \quad (2.15)$$

$$\mathbf{0} = \frac{\partial J(x)}{\partial \lambda} \quad (2.16)$$

$$= \mathbf{y} - \mathbf{H}\mathbf{x} \quad (2.17)$$

$$\mathbf{y} = \mathbf{A}\mathbf{x}. \quad (2.18)$$

Combining (2.15) and (2.18), we can obtain $\mathbf{y} = (1/2)\mathbf{A}\mathbf{A}^T \lambda$. By assuming that $\mathbf{A}\mathbf{A}^T$ is invertible, we can obtain $\lambda = 2(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{y}$. Combining the two results, the minimum norm solution can be expressed under a closed form solution as

$$\bar{\mathbf{x}} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{y}. \quad (2.19)$$

When matrix $\mathbf{A}$ is well-conditioned, the above solution is commonly solved by means of matrix factorization method such as Cholesky’s decomposition, LU-decomposition, or QR-decomposition to enhance the computational speed and stability.

2.2.2 Rank Deficient and Ill-posed Inverse Problem

In this section, we provide an overview of rank deficient problem and the concept of ill-posed inverse problem. A notion of noise amplified solution will also be discussed. Before we give a definition of ill-posed problem, it would be appropriate to first define the definition of well-posed problem, or what constitutes a well-posed problem. According to Jacques Hadamard's definition of a well-posedness, a mathematical problem is considered well-posed if and only if it satisfies all three conditions given below which are

- A solution must exist.
- A solution must be unique.
- A solution must be stable.

If the problem does not satisfy any of the conditions above, the problem is regarded as an ill-posed problem. In general, most of the mathematical problems expressible by a system of linear equations focus the issue on the second condition, which is the uniqueness of a solution. It is due to the observation setting (experimental or measurement setup), which leads to an under or over-determined system matrix $\mathbf{A}$. However, a rank deficient problem is something entirely different. The main issue of rank deficient lies in the third conditions, according to Hadamard's well-posedness requirement. To further highlight this issue, consider a singular value decomposition (SVD) of an ill-conditioned matrix $\mathbf{A}$. For any matrix $\mathbf{A} \in R^{M \times N}$ (suppose $M > N$), the SVD of a matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{i=1}^N \sigma_i \mathbf{u}_i \mathbf{v}_i^T. \quad (2.20)$$

In (2.20), matrix $\mathbf{U} \in R^{M \times M}$ contains left singular vectors $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_M]$ along its column, a diagonal matrix $\mathbf{\Sigma} \in R^{N \times N}$ contains singular values $\mathbf{\Sigma} = \text{diag}[\sigma_1, \dots, \sigma_N]$ along its diagonal where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N \geq 0$, and matrix $\mathbf{V}$ contains right singular vectors $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_N]$ along its column. The sensitivity of an estimate solution corresponds to an observation can be analyzed through the perturbed linear equations as

$$\begin{aligned}\mathbf{A}(\mathbf{x} + \delta\mathbf{x}) &= \mathbf{y} + \delta\mathbf{y} \\ \mathbf{A}\mathbf{x} + \mathbf{A}\delta\mathbf{x} &= \mathbf{y} + \delta\mathbf{y},\end{aligned}\tag{2.21}$$

where $\mathbf{A}\mathbf{x} = \mathbf{y}$, and $\mathbf{A}\delta\mathbf{x} = \delta\mathbf{y}$ respectively. For ease of explanation, suppose matrix $\mathbf{A}$ is a square matrix of size $M \times M$ where $\mathbf{A}^{-1}$ exist. Since $\mathbf{A}\mathbf{x} = \mathbf{y}$, we obtain $\|\mathbf{A}\|\|\mathbf{x}\| \geq \|\mathbf{y}\|$. On the other hand, since $\mathbf{A}\delta\mathbf{x} = \delta\mathbf{y}$, we obtain $\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{y}$, and hence $\|\mathbf{A}^{-1}\|\|\delta\mathbf{x}\| \geq \|\delta\mathbf{y}\|$. Combine the two relationships above, we can now obtain the sensitivity of a system of linear equations which is bounded by

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \leq \frac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|},\tag{2.22}$$

where $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\| = \sigma_1/\sigma_M$ is called a condition number. A high condition number is usually caused by a very small singular value of σ_M (or more), which leads to a noise amplified solution issue. To highlight the noise amplified issue, let us consider a case of naive inverse problem, where $\mathbf{A}$ is an ill-conditioned square matrix of size $M \times M$. The estimate solution can be expressed as

$$\bar{\mathbf{x}} = \mathbf{V}\Sigma^{-1}\mathbf{U}^T\mathbf{y} = \sum_{i=1}^M \frac{\mathbf{u}_i^T\mathbf{y}}{\sigma_i} \mathbf{v}_i.\tag{2.23}$$

For ease of analysis, consider $M = 3$ where σ_1, σ_2 are relatively large, and σ_3 is relatively small ($0 < \sigma_3 < 1$). The expression in (2.23) can be expanded as

$$\bar{\mathbf{x}} = \frac{\mathbf{u}_1^T\mathbf{y}}{\sigma_1} \mathbf{v}_1 + \frac{\mathbf{u}_2^T\mathbf{y}}{\sigma_2} \mathbf{v}_2 + \frac{\mathbf{u}_3^T\mathbf{y}}{\sigma_3} \mathbf{v}_3.\tag{2.24}$$

As can be seen, all of the singular values σ_i become part of the denominator in the expression above. When σ_i have relatively small values, the instability in the estimate solution is unavoidable. For example, consider a measurement vector $\mathbf{y}$ to be contaminated by a very small quantization error (noise) $\mathbf{y}_n = \mathbf{y} + \delta\mathbf{y}$. We can re-express (2.24) for the contaminated observation $\mathbf{y} \rightarrow \mathbf{y}_n$ as

$$\bar{\mathbf{x}} = \left(\frac{\mathbf{u}_1^T\mathbf{y}}{\sigma_1} \mathbf{v}_1 + \frac{\mathbf{u}_2^T\mathbf{y}}{\sigma_2} \mathbf{v}_2 + \frac{\mathbf{u}_3^T\mathbf{y}}{\sigma_3} \mathbf{v}_3 \right) + \left(\frac{\mathbf{u}_1^T\delta\mathbf{y}}{\sigma_1} \mathbf{v}_1 + \frac{\mathbf{u}_2^T\delta\mathbf{y}}{\sigma_2} \mathbf{v}_2 + \frac{\mathbf{u}_3^T\delta\mathbf{y}}{\sigma_3} \mathbf{v}_3 \right).\tag{2.25}$$

Consider the last term which is $\frac{\mathbf{u}_3^T\delta\mathbf{y}}{\sigma_3} \mathbf{v}_3$, though with small amount of noise, this

term can be amplified by small value in the denominator $\frac{1}{\sigma_3}$ which is also why it is called noise amplified solution. In other words, the estimate solution is overshadowed by the amplified noise term.

2.2.3 Regularized Solution

One of the most common treatments for the ill-conditioned matrix issue is to regularize the noise amplified solution by placing additional constraint upon the solution space. In this section, several approaches and frameworks that lead to regularized solution will be presented. One robust approach that can provide a deeper insight of how the regularization method works is called a truncated singular value decomposition (TSVD).

2.2.3.1 Truncated SVD

This method first estimates the singular values of a problem at hand (the ill-conditioned matrix) by calculating the SVD of a matrix $\mathbf{A}$. Typically, a threshold of singular value will be set based on applications before estimating the solution. The singular values that are below the set threshold will be discarded while the singular values that are above the set threshold will be kept. Consider a diagonal matrix Σ with singular values along the diagonal which is extracted from an ill-conditioned matrix $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$ as

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_M \end{bmatrix}, \quad (2.26)$$

the truncated matrix Σ_λ can be defined as

$$[\Sigma_\lambda]_{ii} = \begin{cases} \sigma_{ii} & ; \quad \sigma_{ii} \geq \lambda \\ 0 & ; \quad \sigma_{ii} < \lambda, \end{cases} \quad (2.27)$$

and the ill-conditioned matrix $\mathbf{A}$ can be re-expressed as

$$\mathbf{A} = [\mathbf{U}_\lambda | \mathbf{U}_0] \left[\begin{array}{c|c} \Sigma_\lambda & 0 \\ \hline 0 & \Sigma_0 \end{array} \right] [\mathbf{V}_\lambda | \mathbf{V}_0]^T. \quad (2.28)$$

Because the diagonal values of Σ_0 are all zeros, matrix $\mathbf{A}$ can be expanded as

$$\mathbf{A} = \sum_{i=1}^j \sigma_i \mathbf{u} \mathbf{v}^T + \sum_{j+1}^M \sigma_{j+1} \mathbf{u} \mathbf{v}^T, \quad (2.29)$$

where the values of σ_1 to σ_j are above the set threshold (λ), and the values of σ_{j+1} to σ_M are set to zeros because the values are below the set threshold, the truncated $\mathbf{A}_\lambda$ can be re-expressed as $\mathbf{A}_\lambda = \mathbf{U}_\lambda \Sigma_\lambda \mathbf{V}_\lambda^T$, and the estimate solution can be expressed by

$$\bar{\mathbf{x}} = \mathbf{V}_\lambda \Sigma_\lambda^{-1} \mathbf{U}_\lambda^T \mathbf{y}. \quad (2.30)$$

Another approach that can also lead us to the estimate solution in (2.30) can be obtained from (2.28), where $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{U}_\lambda \Sigma_\lambda \mathbf{U}_\lambda^T + \mathbf{U}_\lambda \Sigma_0 \mathbf{U}_0^T, \quad (2.31)$$

and since Σ_0 are all zeros, hence the second term is vanished, and we can reach the estimate solution as given in (2.30). Consider the expression in (2.25), now when small σ_i in the denominator are being discarded, the noise amplified solution can be avoided. It should be noted that the (2.27) is valid in a theoretical sense and not in a numerical sense since any matrix with singular values equal to zero is not invertible. The truncation (setting some singular values to zero) highlighted here is only to present the notion of omitting small singular values.

2.2.3.2 l_2 -Norm Regularization

The TSVD is commonly considered to be a discrete counterpart to the l_2 -norm regularization framework. This is because in TSVD, small singular values are being discarded manually, or by an already set threshold value which is pre-defined as a discrete value. The l_2 -norm regularization on the other hand has a hyper-parameter that can be fine-tuned and has a continuous value. The l_2 -norm regularization framework proposes that the estimate solution should be obtained by

minimizing both the objective function (minimizing error or model fidelity) and the solution norm simultaneously to avoid the noise amplified solution. The optimization problem of the l_2 -norm regularization can be expressed as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{y} - \mathbf{Ax}\|_2^2 \\ \text{s.t.} \quad & \|\mathbf{x}\|_2^2 = c, \end{aligned} \quad (2.32)$$

where c is some small scalar value since the constraint in equation (2.32) attempts to enforce the notion that the solution norm should be small (to avoid noise amplified solution). A solution to the optimization in (2.32) attempts to solve this problem by minimizing the following objective function

$$J(\mathbf{x}) = \|\mathbf{y} - \mathbf{Ax}\|_2^2 + \lambda \|\mathbf{x}\|_2^2. \quad (2.33)$$

To minimize (2.33), we set the derivative of the objective function w.r.t the $\mathbf{x}$ to zero, and now we obtain

$$\mathbf{0} = \frac{\partial J(\mathbf{x})}{\partial \mathbf{x}} \quad (2.34)$$

$$\mathbf{0} = 2\mathbf{A}^T(\mathbf{Ax} - \mathbf{y}) + 2\lambda\mathbf{x} \quad (2.35)$$

$$\mathbf{A}^T\mathbf{Ax} + \lambda\mathbf{x} = \mathbf{A}^T\mathbf{y} \quad (2.36)$$

$$(\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})\mathbf{x} = \mathbf{A}^T\mathbf{y} \quad (2.37)$$

$$\bar{\mathbf{x}}_\lambda = (\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^T\mathbf{y}, \quad (2.38)$$

where the closed-form solution in (2.38) is sometimes called *Tikhonov solution* or *damped least square solution*. The hyper-parameter (regularization parameter) λ in (2.38) plays a key role in balancing the emphasize between minimizing the error and minimizing the solution norm. If λ is set too large (equivalent to enforcing c to be very small), the solution results in an under-fitted model. In image reconstruction application this leads to a blurred reconstructed image (pixels with large intensity are heavily penalized). If λ is set too small (equivalent to allowing c to be large), the solution will be resulted in an over-fitted model. In image reconstruction application this leads to a noisy reconstructed image (some pixels are the results of noise amplified solution). A general format of the l_2 -norm regularization

framework can be written as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \\ \text{s.t.} \quad & \|\mathcal{R}\mathbf{x}\|_2^2 = c, \end{aligned} \quad (2.39)$$

where $\mathcal{R}$ is called a regularization matrix. The regularization matrix typically includes certain prior knowledge of the model (or an image to be reconstructed). For example, the regularization matrix can be chosen as a difference matrix that will heavily penalize the abruptly changes of pixels intensity (image sharpness) or it can also be chosen based on local characteristic such as sensitivity of the domain. Following the steps presented earlier (to obtain Tikhonov solution), we can obtain a closed-form solution for (2.39) which can be expressed as

$$\bar{\mathbf{x}}_\lambda = (\mathbf{A}^T \mathbf{A} + \lambda \mathcal{R}^T \mathcal{R})^{-1} \mathbf{A}^T \mathbf{y} \quad (2.40)$$

2.2.3.3 l_0 -Norm Regularization

While the underlying purpose of l_2 -regularization is to prevent the noise amplified solution, the l_0 -norm regularization, on the other hand, has different underlying assumption about the solution. The l_0 -norm regularization assumes that among all of the coefficients within the solution $\mathbf{x}$, there are certain parameters that are indeed active (few x_i , $i \in \mathcal{S}$ where $\mathcal{S}$ is an set of active coefficients), and many that are inactive. Therefore, the objective of l_0 -norm regularization framework is to estimate “only” the relevant active parameters x_i in order to obtain a more accurate solution.

The l_0 -regularization attempts to solve the following optimization problem which can be expressed as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_0 \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{y}, \end{aligned} \quad (2.41)$$

where $\|\cdot\|_0$ is the l_0 -norm which provides the number of non-zero elements. At this point, it is important to highlight that l_0 is in fact not a norm by its true definition since it does not obey the property of vector norm. The optimization problem in (2.41) is also non-convex and non-deterministic polynomial-time hard (NP-hard) due to the l_0 itself, and hence there have been several approaches proposed to tackle this. Among many approaches, we can categorize them into one of the two major groups.

Greedy Methods Greedy method attempts to seek out sparsest solution in an iterative fashion where the principle of this approach can be summarized into two major steps as follows. The first step is an “element selection”. In this step the algorithm selects the best columns from an overcomplete dictionary that is best at minimizing the residual function. Subsequently, the second step is the “coefficient update”. The selected columns are alternately handled with an estimation step where the active supports (non-zero elements) are optimized through an update of $\mathbf{x}$ commonly through a least square solution. By selecting the most relevant column within the overcomplete dictionary, greedy method has an element of “thresholding” effect, where the non-relevant columns are being suppressed or disregarded. It is important to highlight that the greedy methods have been discovered independently in different domains where the underlying principles are closely related [4–6]. In statistical analysis, the greedy method is known as “forward stepwise regression” and its variants. In signal processing, the greedy method was developed and known as “Matching Pursuit (MP)” and its variants. One of the most well known greedy methods is called “Orthogonal Matching Pursuit (OMP)”. The working principle can be summarized as follows. Consider $\mathbf{a}_j$ as j^{th} column of an overcomplete matrix $\mathbf{A}$. At the initialization stage $k = 0$, $\mathbf{x}_0 = \mathbf{0}$, the residual is defined as $\mathbf{r}_0 = \mathbf{y} - \mathbf{A}\mathbf{x}_0 = \mathbf{y}$, and the support set $S_0 = \emptyset$ is an empty set. Next, we follow the steps below as

- Find $\mathbf{a}_j$ that is most correlated with $\mathbf{r}_k$
- Include j into a set of active support $S_k = j$
- Update current matrix $\mathbf{A}_S$ where the columns are corresponding to the index in set S_k
- Optimize the coefficient by solving least squares problem $\mathbf{x}_k = \mathbf{A}_S^\dagger \mathbf{y}$, where $\mathbf{A}_S^\dagger$ is the pseudo-inverse of $\mathbf{A}_S$
- Re-calculate the residual vector that will be used in the next iteration $\mathbf{r}_{k+1} = \mathbf{y} - \mathbf{A}_S \mathbf{x}_k$
- Stop the iteration process if $\mathbf{r}_{k+1}$ is less than a certain threshold value (ϵ)
- Update the iteration index $k = k + 1$

Convex Relaxation The Convex relaxation approach attempts to loosen the tight requirement on the l_0 -norm in the original problem (2.41), which is a non-convex function, to the l_1 -norm requirement. The formulation in (2.41) under l_1 -norm relaxation is sometimes called Basis Pursuit (BP) which can be expressed as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_1 \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{y}. \end{aligned} \quad (2.42)$$

In practice, $\mathbf{y}$ is commonly contaminated by noise, and the formulation expressed in (2.42) is sensitive to noise because of the tight constraint ($\mathbf{Ax} = \mathbf{y}$). In real-world applications, the formulation in (2.42) is re-formulated to a noise-tolerable form which can be expressed as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_1 \\ \text{s.t.} \quad & \|\mathbf{Ax} - \mathbf{y}\|_2^2 \leq \epsilon, \end{aligned} \quad (2.43)$$

for some choice of λ (regularization parameter). The formulation in (2.43) is called Basis Pursuit Denoising (BPDN) [7]. To solve the optimization in (2.43), we can obtain the following expression

$$\min_{\mathbf{x}} \quad \|\mathbf{Ax} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{x}\|_1, \quad (2.44)$$

which is also known in statistical regression analysis domain as Least Absolute Shrinkage and Selection Operator (LASSO) problem [8, 9]. Although the LASSO problem (BDPN) is convex, it is not differentiable at the origin (non-smooth). Based on the Method of Multiplier (MoM) and augmented Lagrangian, the LASSO problem can be solved via a method called Alternating Direction Method of Multipliers (ADMM) where the optimization problem can be expressed as

$$\begin{aligned} \min_{\mathbf{x}} \quad & (1/2) \|\mathbf{Ax} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{x}\|_1 \\ \text{s.t.} \quad & \mathbf{x} - \mathbf{z} = 0, \end{aligned} \quad (2.45)$$

and the augmented Lagrangian can be written as

$$L(\mathbf{x}, \mathbf{b}, \mathbf{z}) = \|\mathbf{Ax} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{x}\|_1 + \mathbf{b}^T(\mathbf{x} - \mathbf{z}) + (\rho/2) \|\mathbf{x} - \mathbf{z}\|_2^2. \quad (2.46)$$

The solution can be obtained through an iterative process given below

$$\mathbf{x}^{k+1} = (\mathbf{A}^T \mathbf{A} + \rho \mathbf{I})^{-1} (\mathbf{A}^T \mathbf{y} + \rho \mathbf{z}^k - \mathbf{b}^k) \quad (2.47)$$

$$\mathbf{z}^{k+1} = \mathcal{S}_{\lambda/\rho}(\mathbf{x}^{k+1} + \mathbf{b}^k/\rho) \quad (2.48)$$

$$\mathbf{b}^{k+1} = \mathbf{b}^k + \rho(\mathbf{x}^{k+1} - \mathbf{z}^{k+1}), \quad (2.49)$$

where (2.47) provides the estimate $\mathbf{x}$, and (2.48) is a thresholded $\mathbf{x}$ which is obtained through a proximal operator (soft thresholding in this case). An augmented Lagrangian parameter ρ in (2.48) is typically set to 1.0, and the the vector $\mathbf{b}$ is Lagrange multipliers.

2.2.4 Statistical Viewpoint

Given an observation of a particular data set that is contaminated by noise (with a known or assumed statistical characteristic), the task to estimate the “generative model” is known as *Maximum Likelihood Estimation* from a statistical framework.

Given an observation (measurement) vector with contaminated noise ($\mathbf{y}$), we can estimate the “Generative model” $\mathbf{x}$ by a Maximum A Posterior (MAP) estimator as

$$\bar{\mathbf{x}} = \max_{\mathbf{x}} P(\mathbf{x}|\mathbf{y}), \quad (2.50)$$

where $P(\mathbf{x}|\mathbf{y})$ is the posterior probability of the model $\mathbf{x}$ given an observation $\mathbf{y}$. This posterior probability $P(\mathbf{x}|\mathbf{y})$ can be obtained by using Bayes’ rule as

$$P(\mathbf{x}|\mathbf{y}) = \frac{P(\mathbf{y}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{y})}, \quad (2.51)$$

where $P(\mathbf{y}|\mathbf{x})$ is a likelihood function, $P(\mathbf{x})$ is the prior of the generative model, and $P(\mathbf{y})$ is the probability of the observation, which is commonly omitted because it is not a function of our unknown $\mathbf{x}$ to be maximized. The problem given in (2.50) can now be re-written as

$$\bar{\mathbf{x}} = \max_{\mathbf{x}} P(\mathbf{y}|\mathbf{x})P(\mathbf{x}). \quad (2.52)$$

In a special case where the prior $P(\mathbf{x})$ is specified as uniform distribution (or equivalently no prior information about the model), the MAP estimator is equivalent to a well-known estimator called *Maximum Likelihood Estimator* (MLE).

Under Gaussian assumption, the probabilities $P(\mathbf{y}|\mathbf{x})$ can be obtained by

$$P(\mathbf{y}|\mathbf{x}) = \frac{1}{2\pi\sigma_e^2} \exp\left(-\frac{\|\mathbf{y} - \mathbf{Ax}\|_2^2}{2\sigma_e^2}\right), \quad (2.53)$$

and the probability of prior $P(\mathbf{x})$ can be obtained by

$$P(\mathbf{x}) = \frac{1}{2\pi\sigma_x^2} \exp\left(-\frac{\|\mathbf{x}\|_2^2}{2\sigma_x^2}\right). \quad (2.54)$$

Combining $P(\mathbf{y}|\mathbf{x})$ and $P(\mathbf{x})$, we can obtain the MAP estimate in (2.52) as

$$\bar{\mathbf{x}} = \max_{\mathbf{x}} P(\mathbf{y}|\mathbf{x})P(\mathbf{x}) \quad (2.55)$$

$$\bar{\mathbf{x}} = \max_{\mathbf{x}} \exp\left(-\frac{\|\mathbf{y} - \mathbf{Ax}\|_2^2}{2\sigma_e^2} - \frac{\|\mathbf{x}\|_2^2}{2\sigma_x^2}\right). \quad (2.56)$$

The expression in (2.56) is monotonically increasing, and hence maximizing (2.56) is equivalent to

$$\bar{\mathbf{x}} = \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{Ax}\|_2^2 + \alpha^2 \|\mathbf{x}\|_2^2, \quad (2.57)$$

where $\alpha = \sigma_e/\sigma_x$. It can be seen that the solution in (2.57) is closely related to the solution of the previous l_2 -norm regularization framework.

2.3 Array Signal Processing and Beamforming Framework

In this section, several essential concepts from array signal processing will be reviewed and used as a reference in future chapters (Part II of this thesis). The mathematical basis and the physical laws are based on electromagnetic wave propagation model that is commonly used in RADAR, SONAR, and communication applications. Hence, we will start this section with an introduction to Maxwell's equations for electromagnetic wave propagation in homogeneous and lossless medium (a commonly used model in most array signal processing research). We then review some of the beamforming techniques from conventional method to high-resolution and ultimately super-resolution techniques.

2.3.1 Maxwell's Equations and Wave Propagation Model

Most wave-like physical phenomena are commonly governed by the same homogeneous solution obtained from a wave propagation model. Applications are ranging from electromagnetic (EM) wave such as RADAR and communication to applications based on acoustic waves such as microphone array, SONAR, and seismic imaging.

Since applications of our interests are only based on electric field, the effect and derivation of the magnetic field will be omitted. From Maxwell's 3rd equation which can be specified as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.58)$$

by taking curl on both sides, we can obtain

$$\nabla \times \nabla \times \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\mu \frac{\partial(\nabla \times \mathbf{H})}{\partial t}. \quad (2.59)$$

From Maxwell's 4th equation, we also know that

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \quad (2.60)$$

Under far-field assumption (source free) where $\mathbf{J} = 0$, we can obtain the following expression by combining equations (2.60) and (2.59) as

$$\nabla \times \nabla \times \mathbf{E} = -\mu\varepsilon \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{E}}{\partial t}\right). \quad (2.61)$$

By applying the following vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (2.62)$$

to the L.H.S of equation (2.61), we can obtain

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (2.63)$$

From the Maxwell's 1st equation, we know that

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon}. \quad (2.64)$$

Again, under far-field assumption (no build up charges) $q = 0$, equation (2.64) with $q = 0$ can be substituted into equation (2.63) which can be re-expressed as

$$\nabla^2 \mathbf{E} = -\mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{E} = \omega^2 \mu\varepsilon \mathbf{E}, \quad (2.65)$$

which is a source-free wave equation for time-harmonic electric field. Note that we have replaced time derivative with $j\omega$ in equation (2.65). The expression is (2.65) can be expressed as

$$\nabla^2 \mathbf{E} - k^2 \mathbf{E} = 0, \quad (2.66)$$

where $k = \omega\sqrt{\mu\varepsilon} = 2\pi/\lambda$ is called phase constant or wavenumber at angular frequency $\omega = 2\pi f$. The expression in (2.66) is a vector Helmholtz equation which is similar to the generalized scalar Helmholtz equation given by $\nabla^2 A + k^2 A = 0$. The homogeneous solution of (2.66) for single frequency can be obtained as a planar traveling wave as $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{j\mathbf{k}\cdot\mathbf{r}}$.

2.3.2 Signal Model and Steering Vector

Consider single target approaching from θ direction, the received signal from L -sensor array (assuming $L = 4$ for simplicity) without noise can be written as $\mathbf{E}(\mathbf{r}_1) = \mathbf{E}_\theta e^{j\mathbf{k}_\theta \cdot \mathbf{r}_1}$, $\mathbf{E}(\mathbf{r}_2) = \mathbf{E}_\theta e^{j\mathbf{k}_\theta \cdot \mathbf{r}_2}$, ..., $\mathbf{E}(\mathbf{r}_L) = \mathbf{E}_\theta e^{j\mathbf{k}_\theta \cdot \mathbf{r}_L}$, which can be re-written in a vector format as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\mathbf{k} \cdot \mathbf{r}_2} \\ e^{-j\mathbf{k} \cdot \mathbf{r}_3} \\ e^{-j\mathbf{k} \cdot \mathbf{r}_4} \end{bmatrix} \mathbf{s}(t) \quad \text{or} \quad \mathbf{x}(t) = \mathbf{a}(\theta) \mathbf{s}(t). \quad (2.67)$$

For K number of targets, coupling with the linearity property, the solution can be re-expressed in a form of a linear combination of uncorrelated signal sources as $\mathbf{x}(t) = \mathbf{a}(\theta_1)s_1(t) + \mathbf{a}(\theta_2)s_2(t) + \dots + \mathbf{a}(\theta_M)s_K(t)$, and the signal model under the influence of noise can be written as

$$\mathbf{x}(n) = \mathbf{A}(\Theta_K) \mathbf{s}(n) + \mathbf{n}(n), \quad (2.68)$$

where $\mathbf{A}(\Theta_K) = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_K)]$ is the array manifold spanning signal subspace, $\mathbf{s}(n)$ is the target signal, and $\mathbf{n}(n)$ is an additive white Gaussian noise.

2.3.3 Beamforming Techniques for Narrowband DOA Estimation

As discussed in the earlier part of this chapter, a discrete ill-posed inverse problem is commonly solved by incorporating additional prior knowledge or a restriction onto the solution space. For example in image reconstruction problem, through the utilization of different norm restrictions (e.g. l_2 or l_1), the regularization framework can incorporate image smoothness or sharpness prior knowledge (pixels distribution) onto the reconstructed image.

In the array signal processing domain, the techniques used in this domain aim at solving an inverse problem by imposing prior information in a different fashion. Instead of enforcing a restriction onto the solution space directly, an array signal processing framework try to exploit the knowledge of the array geometry and its response. The expected array output is then imposed by a weight vector instead. For example, when an objective of a problem at hands is to obtain and maximize a signal source $\mathbf{s}_\theta(t)|_{\theta \in \Theta_K}$ where Θ_K contains directions of all sources $\Theta_K = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_K)]$. Consider a source located at direction θ_1 , the scanning procedure can be accomplished by looking through each look direction in Θ to find $\mathbf{s}_{\theta_1}(t)$. The prior information on the array response when $\mathbf{s}_{\theta_1}(t)$ is present is $\mathbf{a}(\theta_1)$ which is essentially the forward solution when $\mathbf{s}_{\theta_1}(t)$ is present at direction θ_1 . Under beamforming framework, we seek a weight vector $\mathbf{w}_{\theta_1}$ that will allow unity gain $\mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_1) = 1$ at direction θ_1 while suppressing the signals from other directions $\mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_i) = 0$ for $i \neq 1$. The array output can be specified as

$$\begin{aligned} \mathbf{y}_{\theta_1} &= \mathbf{w}_{\theta_1}^H \mathbf{x} \\ &= \sum_{i=1}^K \mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_i) \mathbf{s}_{\theta_i}(t) \\ &= \mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_1) \mathbf{s}_{\theta_1}(t) + \sum_{i=2}^K \mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_i) \mathbf{s}_{\theta_i}(t). \end{aligned} \tag{2.69}$$

According to the relationship we enforced earlier as $\mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_1) = 1$ at direction θ_1 and $\mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_i) = 0$ for $i \neq 1$, $\mathbf{y}_{\theta_1} = \mathbf{s}_{\theta_1}(t)$ can be obtained. However in practical scenario, we can only attain $\mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_1) \approx 1$ (to get peak signal in desired direction) while minimizing $\mathbf{w}_{\theta_1}^H \mathbf{a}(\theta_i) \approx \varepsilon$ for any small ε (to suppress all signals in other

directions). This framework gives rise to a method called conventional beamformer or delay-and-sum beamformer. The direction finding function of this beamformer is given as

$$P_{CB} = \mathbf{A}(\Theta)^H \mathbf{x}, \quad (2.70)$$

where $\mathbf{A}(\Theta)$ encapsulates the steering vectors of all directions in the field-of-view (FOV). In compressed sensing terminology, $\mathbf{A}(\Theta)$ is called an ‘overcomplete dictionary’. Principally, this method is closely related with ‘match-filter’ that is widely used in signal processing. The disadvantage that comes with its simplicity is the noise rejection performance.

Before the advent of subspace-based method, many direction finding techniques suffered from the resolution limitation. The high resolution or the so called subspace-based technique exploits the eigen-structure of the spatial covariance matrix constructed from the multiple set of data (multiple snapshots) we obtained. The spatial covariance matrix along with the narrow band signal model assumption, it is possible to identify the signal subspace, which is spanned by the column vectors of $\mathbf{A}(\Theta)$ and the noise subspace, which is the orthogonal subspace to the signal subspace. The spatial covariance matrix can be constructed from multiple snapshots of $\mathbf{x}(t)$ as

$$\begin{aligned} \mathbf{R}_{xx} &= E(\mathbf{x}(t)\mathbf{x}(t)^H) \\ &= \mathbf{A}(\Theta)E(\mathbf{s}(t)\mathbf{s}(t)^H)\mathbf{A}(\Theta)^H + E(\mathbf{n}(t)\mathbf{n}(t)^H) \\ &= \mathbf{A}(\Theta)\mathbf{R}_{ss}\mathbf{A}(\Theta)^H + \sigma^2\mathbf{I}_M. \end{aligned} \quad (2.71)$$

The matrix $\mathbf{R}_{xx}$ denotes spatial covariance matrix constructed from the obtained data. Under the assumption of uncorrelated targets, the target covariance $\mathbf{R}_{ss}$ is full rank and is positive definite. The term $\sigma^2\mathbf{I}_M$ is obtained under the assumption of an uncorrelated AWGN. Assumptions other than the Gaussian distribution for noise will require appropriate consideration for that particular application (e.g. Rayleigh distribution is sometimes considered in multipath scenarios).

2.3.3.1 MVDR Algorithm

The MVDR or Capon's beamformer which aims at keeping unity gain at desired direction, while minimizing the energy originated from other signal sources and noises. The optimization problem of Capon's beamformer is defined as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{a}(\theta) = 1. \end{aligned} \quad (2.72)$$

We can obtain the Lagrangian as

$$L(\mathbf{w}, \lambda) = \mathbf{w}^H \mathbf{R} \mathbf{w} + \lambda(\mathbf{w}^H \mathbf{a}(\theta) - 1). \quad (2.73)$$

By setting the derivative of Lagrangian w.r.t $\mathbf{w}$ and λ to zero, we can obtain

$$\mathbf{0} = \frac{\partial L(\mathbf{w})}{\partial \mathbf{w}} \quad (2.74)$$

$$\mathbf{0} = 2\mathbf{R}\mathbf{w} + \lambda\mathbf{a}(\theta) \quad (2.75)$$

$$\mathbf{w} = \frac{1}{2}\lambda\mathbf{R}^{-1}\mathbf{a}(\theta) \quad (2.76)$$

$$\mathbf{0} = \frac{\partial L(\mathbf{w})}{\partial \lambda} \quad (2.77)$$

$$\mathbf{w}^H \mathbf{a}(\theta) = 1. \quad (2.78)$$

Combining (2.76) and (2.78) we can obtain $\lambda = 2/(\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))$. Substituting λ back to (2.76) we can obtain the weight vector solution which can be expressed as

$$\mathbf{w} = \frac{\mathbf{R}^{-1} \mathbf{a}(\theta)}{\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta)}. \quad (2.79)$$

Replacing the weight vector solution in (2.79) back to the objective function $\mathbf{w}^H \mathbf{R} \mathbf{w}$, we can obtain the spatial spectrum of Capon's beamformer which is defined as

$$P_{MVDR}(\theta) = 10 \log_{10} \frac{1}{|\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta)|}. \quad (2.80)$$

2.3.3.2 MUSIC Algorithm

The multiple signal classification (MUSIC) algorithm is considered to be one of the most well-known algorithms with super-resolution performance. By using eigenvalue decomposition (EVD), the spatial covariance matrix $\mathbf{R} = E(\mathbf{X}\mathbf{X}^H)$ can be decomposed into two orthogonal subspaces by the eigendecomposition $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$. The covariance matrix can now be expressed as

$$\mathbf{R} = [\mathbf{U}_s \mathbf{U}_n] \begin{bmatrix} \mathbf{\Lambda}_s & 0 \\ 0 & \mathbf{\Lambda}_n \end{bmatrix} [\mathbf{U}_s \mathbf{U}_n]^H \quad (2.81)$$

$$= \mathbf{U}_s \mathbf{\Lambda}_s \mathbf{U}_s^H + \mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n^H, \quad (2.82)$$

where $\mathbf{U}_s \in \mathbb{C}^{M \times K}$ denotes the signal subspace which comprises the eigenvectors correspond to the dominant eigenvalues denote $\mathbf{\Lambda}_s = \text{diag}\{\sigma_{s_1}^2 + \sigma_{v_1}^2, \dots, \sigma_{s_K}^2 + \sigma_{v_K}^2\}$, and $\mathbf{U}_n \in \mathbb{C}^{M \times M-K}$ denotes the noise subspace which comprises the eigenvectors correspond to the noise eigenvalues denote $\mathbf{\Lambda}_n = \text{diag}\{\sigma_{v_{K+1}}^2, \dots, \sigma_{v_M}^2\}$.

The MUSIC algorithm exploits the orthogonality between signal and noise subspaces. The subspace orthogonality property leads to the MUSIC's pseudo spatial spectrum which is defined as

$$P_{MUSIC}(\theta) = 10 \log_{10} \frac{1}{|\mathbf{a}(\theta)^H \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}(\theta)|}. \quad (2.83)$$

The performance of Capon's beamforming is known to be approaching to that of the MUSIC algorithm when the $\text{SNR} \rightarrow \infty$. From (2.81), the spatial spectrum of Capon's beamformer in (2.80) can be rewritten as

$$\begin{aligned} |\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta)| &= |\mathbf{a}(\theta)^H \mathbf{U}_s \begin{bmatrix} \frac{1}{\infty} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\infty} \end{bmatrix} \mathbf{U}_s^H \mathbf{a}(\theta)| \\ &\quad + |\mathbf{a}(\theta)^H \mathbf{U}_n \begin{bmatrix} \frac{1}{\sigma_v^2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\sigma_v^2} \end{bmatrix} \mathbf{U}_n^H \mathbf{a}(\theta)|. \end{aligned} \quad (2.84)$$

and hence the spatial spectrum of MVDR can be approximated by

$$P_{MVDR}(\theta) \approx 10 \log_{10} \frac{\sigma_v^2}{|\mathbf{a}(\theta)^H \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}(\theta)|}. \quad (2.85)$$

2.3.4 MUSIC-like Algorithm

In [10], it was shown that the MUSIC algorithm can be formulated under the beamforming framework where the optimization problem can be defined as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \|\mathbf{w} - \mathbf{a}(\theta)\|_2^2 \\ \text{s.t.} \quad & \mathbf{U}_s^H \mathbf{w} = 0, \end{aligned} \quad (2.86)$$

where the weight vector solution is $\mathbf{w} = \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}(\theta)$, and the spatial spectrum can be obtained by

$$P_{Mlike}(\theta) = 10 \log_{10} \frac{1}{|\mathbf{w}^H \mathbf{a}(\theta)|} = 10 \log_{10} \frac{1}{|\mathbf{a}(\theta)^H \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}(\theta)|}, \quad (2.87)$$

which is equivalent to (2.83). It can be seen that the most critical point in obtaining a super-resolution performance is to have a weight vector solution resided in the noise subspace.

The MUSIC algorithm was proposed with the objective to improve the resolution performance issue. However, performance degradation of the MUSIC algorithm may occur when the obtained data snapshots is limited or under low SNR scenarios. This issue can be avoided by an alternative method called MUSIC-like algorithm which does not require an estimation of number of targets. The MUSIC-like algorithm proposed in [11] [12] as an optimization problem for each look direction as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{a}(\theta) \mathbf{a}(\theta)^H \mathbf{w} + \beta \|\mathbf{w}\|_2^2 = c, \end{aligned} \quad (2.88)$$

where the weight vector $\mathbf{w}$ is the solution from an optimization problem in (2.88), the steering vector $\mathbf{a}(\theta)$ satisfies a normalized 2-norm $\|\mathbf{a}(\theta)\|_2 = 1$, a scalar value β is a control parameter which introduces certain relaxation to the constraint, and c is any constant value. The Lagrangian function of (2.88) can be expressed as

$$L(\mathbf{w}, \lambda) = \mathbf{w}^H \mathbf{R} \mathbf{w} - \lambda (\mathbf{w}^H (\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I}) \mathbf{w} - c). \quad (2.89)$$

The weight vector solution $\mathbf{w}$ should be the general eigenvector of the matrix pair $(\mathbf{R}, (\mathbf{a}(\theta)\mathbf{a}(\theta)^H + \beta\mathbf{I}))$ as

$$\mathbf{R}\mathbf{w} = \lambda(\mathbf{a}(\theta)\mathbf{a}(\theta)^H + \beta\mathbf{I})\mathbf{w}. \quad (2.90)$$

According to the minimization problem in (2.88), the weight solution $\mathbf{w}$ is the eigenvector corresponding to the minimum eigenvalue of (2.90), and therefore the spatial spectrum of the MUSIC-like algorithm is defined respectively as

$$P_{MUSIC}(\theta) = 10 \log_{10} \frac{1}{|\mathbf{w}^H \mathbf{a}(\theta)|^2}. \quad (2.91)$$

The bound of β was proposed in [12] as

$$\underbrace{\max_{p \in P} \frac{\lambda_{\mathbf{R},min}}{(\mathbf{a}(\xi)^H \mathbf{R}^{-1} \mathbf{a}(\xi))^{-1}}}_{\beta_{min}} < \beta < \underbrace{\min_{p \notin P} \frac{\lambda_{\mathbf{R},min}}{(\mathbf{a}(\xi)^H \mathbf{R}^{-1} \mathbf{a}(\xi))^{-1}}}_{\beta_{max}} \quad (2.92)$$

with the choice of β to be a value between β_{min} and β_{max} defined as

$$\beta = (1 - \alpha)\beta_{min} + \alpha\beta_{max}, \quad (2.93)$$

where $0 < \alpha < 1$ to be chosen by $\alpha = \beta_{min}/\beta_{max}$. As can be seen in (2.90), the β parameter plays an important role in regulating the eigenvalues of the matrix pair. In [12], the bound of β as well as a β -selection method was proposed.

2.4 Conclusion

This chapter provides principal concepts and backgrounds for the upcoming chapters. The first part of this chapter provides an overview of a discrete ill-posed inverse problem. The noise amplified issue caused during matrix inversion procedure while solving an inverse problem is highlighted. The significance of a regularized solution has also been pointed out. The beamforming framework is reviewed in the second part of this chapter. The conventional beamforming and its limitation is introduced. The progress towards high-resolution and super-resolution beamforming techniques are also covered.

Part I

Robust Support Identification for Electrical Impedance Tomography

Chapter 3

Fundamental of EIT & System Implementation

In this chapter, the fundamental of EIT system will be introduced which can be divided into two sections. The working principal, e.g. physical laws that govern the process, is reviewed within the first section. In the second section, an overview of the overall system is reviewed which includes the current injection and measurement procedure. The EIT prototyping system is then presented which will be used in the experimental studies in the following chapters.

3.1 Fundamental of EIT

Electrical tomography (ET) is a soft field non-invasive imaging technique that estimates the distribution of internal admittivity (γ) of a *subject under test (SUT)* or a *medium* through an inference of data collected from the medium's boundary. Several sub-categories of ET can be found which are usually referred to as Electrical Impedance Tomography (EIT), Electrical Capacitance Tomography (ECT), or sometimes also called Electrical Resistance Tomography (ERT). The variation of each ET method is a reflection of each parameter of interest (intrinsic electrical properties) for that particular application. The procedure of solving inverse problem for each type of ET is fundamentally the same. Without loss of generality, the term electrical tomography and electrical impedance tomography will be used interchangeably throughout this thesis.

The EIT system can be implemented economically with relatively low complexity in terms of implementation when compared with other hard field imaging modalities such as X-ray, Computed Tomography (CT), and Magnetic Resonance Imaging (MRI). An example of how a small system can be implemented compactly is presented in [13] where a real-time system of electrical capacitance tomography (ECT), a close relative modality to the EIT, was implemented based on a field-programmable gate array (FPGA) as an image reconstruction unit. With its high temporal resolution, the system is capable of capturing data within milliseconds.

Most of the EIT systems operate by injecting 1mA to few mA low frequency alternating current (A.C.) within the range of kHz to MHz and onto the boundary of a medium. The resulting voltage at the boundary is then measured from multiple sensors (electrodes) attached on the boundary. Due to non-uniform conductivity distribution inside the medium, the array sensors on the boundary are able to capture the perturbed field, and the image is then reconstructed subsequently by solving a discrete ill-posed inverse problem.

Although with limitation in spatial resolution capability, its high temporal resolution and low complexity implementation have been found appealing in several fields from biomedical tomography to industrial applications [14]. In biomedical application, different parts of human organs possess different bioelectrical characteristics. The EIT system were used for lung imaging where it estimates the recruitable alveolar collapse and hyper distension through images obtained during a decremental positive end-expiratory pressure (PEEP) titration [15]. The working principle is based on the perturbed admittivity distribution. For example, as used in the cross-sectional lung imaging and monitoring, when the lungs expand due to an increment of air volume, the resistivity across thorax area increases accordingly due to non-conductive property of the air. The system exploits the variation of internal electrical properties of the medium. Other possible applications in biomedical field are the study of the cerebral activity of brain function as well as tumor localization [16, 17]. For industrial application, the ECT, which is a relative imaging technique to the EIT, is used for process monitoring where transmitted gas or liquid inside a pipeline can be monitored in real-time [18–20].

3.1.1 Bio-impedance

Bio-impedance (the reciprocal of bio-admittance) is an electrical property of biological tissues which varies in accordance to the operating frequency as well as the medium's intrinsic electrical characteristic (types of tissues). This property highlights how the tissues interact with the current that flows within the medium (either electron flow or ion flow). Two fundamental electrical properties are the resistance which can be translated to a friction of how the current flow, and the capacitance which can be translated to the ability to store energy inside the tissue.

FIGURE 3.1: Low frequency current passing through extracellular and high frequency current passing through intracellular.

Bio-impedance can be represented in the complex plane. It comprises two major parts: the real part and the imaginary part. The real part corresponds to the resistance component while the imaginary part corresponds to the reactance component. The complex number can be represented by using its modulus or the magnitude of the impedance. In practice, the modulus is commonly used since it provides identical result to the amplitude of measured signal which is easier to measure and to interpret. Biological tissue comprises millions of cells as its building block. These cells can be modeled by electronic components as shown in Fig. 3.2.

The model shown in Fig. 3.2 is one of the simplest and commonly used model which comprises of three major components. The resistivity (R_e) corresponds to the extracellular space. The resistivity (R_i) and the capacitance (C_m) correspond to the intracellular space and its membrane, respectively. The cell membrane, which has lipid component, acts as an insulator to prevent the current flow (ion flow) inside and outside the cell. This is the reason why alternating current is crucial when working with bio-impedance. The frequency dependency of bio-impedance can be summarized as follows. In direct current (D.C.) case, the current flows

FIGURE 3.2: Electronic model for biological cells.

mostly through the extracellular region. The resistivity (R_e) is mainly responsible for the impedance value. In the low frequency range, the value of R_i and $X(\omega C_m)$ is relatively large, and hence R_e is mainly responsible for the impedance value (mostly). The value of $X(\omega C_m)$ drops as the frequency increases, and hence all of the components shown in Fig. 3.2 are responsible for the final impedance value. In high frequency region, the value of $X(\omega C_m)$ drops drastically, and thus the impedance is represented by only the parallel of R_e and R_i . This phenomenon is shown in Fig. 3.3 which is called a Cole-Cole plot. Instead of representing the impedance value as a point in the complex plane for single frequency, the Cole-Cole is an extension of the impedance value for a range of frequency spectrum. It should be noted that in reality, the Cole-Cole plot will not be perfectly semicircular as shown in Fig. 3.3 due to high complexity and non-linearity of real-life tissues.

FIGURE 3.3: Cole-cole plot of impedance value over multiple frequencies.

3.1.2 Forward Problem of EIT

In EIT context, the forward problem attempts to estimate the output of a known model under certain specification (boundary conditions). For example, with a known admittivity distribution $\gamma(\mathbf{r})$ and a known geometrical configuration of a medium Ω , a forward problem $\mathbb{F}(\cdot)$ aims at estimating a potential value on the boundary of the medium $\phi|_{\partial\Omega}$. In many applications, it is not possible to obtain a closed form solution, and hence an approximate solution obtained from computational method is essential. On the other hand, with known boundary data $\mathbf{v}|_{\partial\Omega}$, the inverse problem attempts to estimate the possible models that generated such boundary data $\bar{\gamma}(\mathbf{r})|_{\Omega}$.

FIGURE 3.4: (a) Forward problem and (b) Inverse problem of EIT.

As can be seen in Fig. 3.4, with a known admittivity distribution $\gamma(\mathbf{r})$ inside the medium Ω the forward problem seeks a solution which is the potential field ϕ on the medium boundary $\partial\Omega$. Note that $\phi|_{\partial\Omega}$ is a continuous variable which is the potential field along the boundary of a medium. This process can be represented by a forward operator $\mathbb{F}(\cdot)$ that maps a known model (admittivity distribution) to the solution on the boundary $\mathbb{F}(\cdot) : \gamma(\mathbf{r})|_{\Omega} \rightarrow \phi|_{\partial\Omega}$. In contrast, with a known boundary data $\mathbf{v}|_{\partial\Omega}$ the inverse problem seeks the possible (estimate) model $\bar{\gamma}(\mathbf{r})$ that can

describe such data. Note that $\mathbf{v}|_{\partial\Omega}$ is a discrete variable which is the measurement made on the boundary of a medium. In other words, an inverse operator $\mathbb{F}^{-1}(\cdot)$ works in the opposite direction to the forward operator or $\mathbb{F}^{-1}(\cdot) : \mathbb{V}|_{\partial\Omega} \rightarrow \bar{\gamma}(\mathbf{r})$.

In short, the model space (all possible combinations of $\bar{\gamma}(\mathbf{r})$) encompasses all possible electrical property distributions e.g. admittivity distribution in the medium $\gamma_1(\mathbf{r})|_{\Omega}, \dots, \gamma_k(\mathbf{r})|_{\Omega}$, where k represents the number of different models. The forward mapping maps the admittivity distribution to the corresponding data space on the boundary $\phi_1|_{\partial\Omega}, \dots, \phi_k|_{\partial\Omega}$ which represents all possible potential fields from the model space. In contrast, the inverse mapping tries to map the measurement fields on the boundary $\mathbf{v}|_{\partial\Omega}$ to the possible solution space which is an estimate of the real model space. As highlighted in the previous chapter that the relationship between the model space and the data space is not always 1-to-1, which is one of the reason that why inverse problem in EIT is ill-posed.

To gain a deeper understanding of the EIT system, the underlying physical process will be briefly reviewed. In EIT, a low frequency A.C. is typically used as an active source (stimulant). This automatically set the problem into electrical quasi-static environment (the frequency is very low and can be approximated with a D.C.), which also leads to two important points. Firstly, the time information in the sense of time delay (or phase delay) is lost (no propagation delay information at different sensor locations). In other word, the time difference between each sensor is almost a spontaneous field (no time delay between each sensor). Secondly, the electric field distribution inside the medium is considered to behave in a diffusive fashion e.g. the energy does not travel in a straight path as opposed to what happened with the X-ray or the CT scan. This causes a non-linearity behavior within the measurement field. Next, we will consider the time-harmonic field of Maxwell's equations under quasi-static approximation which can be expressed as

$$\begin{aligned}\nabla \cdot (\varepsilon \mathbf{E}) &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= i\omega \mathbf{B} \approx 0.\end{aligned}\tag{3.1}$$

where Faraday induction in the last expression of (3.1) is valid within the low frequency range (up to MHz). From $\mathbf{E} = -\nabla\phi$ and $\mathbf{J} = \sigma\mathbf{E}$, we can obtain

$$\nabla \times \mathbf{H} = \mathbf{J} + i\omega\varepsilon\mathbf{E} = (\sigma + i\omega\varepsilon)\mathbf{E} = -(\sigma + i\omega\varepsilon)\nabla\phi, \quad (3.2)$$

and by using the vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, we can obtain

$$-\nabla \cdot (\sigma + i\omega\varepsilon)\nabla\phi = 0, \quad (3.3)$$

or equivalently expressed as

$$\nabla \cdot \gamma \nabla\phi = 0, \quad (3.4)$$

where the admittivity distribution $\gamma = \sigma + i\omega\varepsilon$ is a spatial function, and for a discretized domain it can be divided up to a domain of N pixels for 2D (or N voxels in 3D), which can be expressed as $\gamma(\mathbf{r}) \in \mathbb{R}^N$. For most applications where the operating frequency is below kHz range, the admittivity value $\gamma = \sigma + i\omega\varepsilon$ can be approximated by only the real part (conductivity) $\gamma \approx \sigma$ (since $i\omega\varepsilon$ is small) [21]. This also means that in most applications when single and low frequency is used, the image reconstruction task can be considered as an estimation of a non-uniform conductivity distribution profile inside the medium. In the subsequent part of this thesis, the term conductivity will be used instead of admittivity since the operating frequency we are studying is bounded within the low frequency range.

The potential field $\phi(\mathbf{r})$ must satisfy elliptic partial differential equation (PDE) or the Laplace's equation on a complete domain denotes $\bar{\Omega} = \Omega \cup \partial\Omega$. Next, the entire problem domain will be defined with a mathematical setup. We start by denoting $\bar{\Omega}$ as a complete domain which contains the unbounded domain Ω and its boundary $\partial\Omega$ as $\bar{\Omega} = \Omega \cup \partial\Omega$ while $\bar{\Omega} \in \mathbb{R}^D$, and $D = 2$ or 3 . The unbounded domain Ω will be discretized later when the Finite Element Method is used for the forward mapping. The unbounded domain can be broken up into N sub-domain (pixels or voxels) as $\Omega = \bigcup_k \Omega_k$ for $k \in \{1, 2, \dots, N\}$. The boundary $\partial\Omega$ is divided into two regions; Γ where a set of electrodes is specified and Γ' where no electrode is specified. Finally, the region where a set of electrodes is specified can be categorized into the current injecting electrodes $\Gamma_j = \mathcal{E}_j \cup \mathcal{E}_{j+1}$ for $j \in \{1, 2, \dots, L\}$, when L is the number of the overall electrodes, and the measuring electrodes $\Gamma_l = \bigcup_l \mathcal{E}_l$ for $l \in \{1, 2, \dots, E\} \setminus \{j, j+1\}$. Note that $\{j, j+1\}$ is a pair of current injection electrodes that moves in a cyclical manner. For example, for 8 electrodes, a set of $\Gamma = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 1)\}$.

FIGURE 3.5: Problem domain: Internal medium Ω and its boundary $\partial\Omega$.

To obtain a unique solution, certain boundary conditions must be imposed or specified on the boundary of the medium. Most of the electrical tomography specification employs the Neumann boundary condition (Type II) partially on Γ_j , which generally means the current injection on specified electrodes $\mathcal{E}_j \cup \mathcal{E}_{j+1}$. Such condition is specified as

$$g = \sigma \nabla_n \phi|_{\Gamma_j}. \quad (3.5)$$

For completeness, it should be noted that the requirement for boundary condition can be Dirichlet boundary condition (Type I), Neumann boundary condition (Type II), or mixed boundary condition (Type III) depending on the application. For now, the problem that imposes Neumann boundary condition (current injection on the boundary) while measuring the Dirichlet data (potential field on the boundary) will be focused, and such mapping can be formulated as

$$\Lambda_{\sigma(\Omega_k)} : g|_{\Gamma_j} \rightarrow \phi|_{\Gamma_l}. \quad (3.6)$$

To complete the forward formulation of EIT, certain boundary conditions must be satisfied on the boundary $\partial\Omega$ depending on applications (different applications may require different treatment). Forward map in EIT can be obtained by solving a PDE or the Laplace equation as presented in (3.3) in the medium Ω with boundary conditions on the boundary $\partial\Omega$. A full formulation setup can be formulated as

$$\begin{aligned}
& \nabla \cdot (\sigma \nabla \phi) = 0 \quad \text{in } \Omega, \\
\text{BC1: } & \int_{\mathcal{E}_j} \sigma \frac{\partial \phi}{\partial \mathbf{n}} d\mathbf{s} = I = - \int_{\mathcal{E}_{j+1}} \sigma \frac{\partial \phi}{\partial \mathbf{n}} d\mathbf{s}, \\
\text{BC2: } & \sigma \frac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma' \bigcup \Gamma_l, \\
\text{BC3: } & \int_{\mathcal{E}_k} \sigma \frac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma.
\end{aligned} \tag{3.7}$$

All of the boundary conditions stated above reflect a measurement protocol called “adjacent method” which uses two electrodes for current injection input $\Gamma_j = \mathcal{E}_j \bigcup \mathcal{E}_{j+1}$ while the other electrodes are being used as measurement sensors $\Gamma_l = \bigcup_l \mathcal{E}_l$. Additional boundary condition will be added in the later section which takes the imperfection of electrodes impedance into consideration. Analytical solutions for most of the PDE problems are generally suffer from the geometrical and material complexity. Closed form solution cannot be obtained which leads to the use of numerical solution (approximate solution) typically solved by using Finite Element Method (FEM) or Boundary Element Method (BEM). Here, the FEM is chosen over BEM due to the fact that FEM has more sparse structure when compared with the BEM when multiple features appear inside the medium under arbitrary geometrical structure. The formulation above can be solved by first transforming the elliptic PDE into a weak formulation as

$$\int_{\Omega} \sigma \nabla \phi \cdot \nabla v dV = \int_{\partial \Omega} \sigma \nabla \phi \cdot \mathbf{n} v dS = \int_{\Gamma} \sigma \nabla \phi \cdot \mathbf{n} v dS. \tag{3.8}$$

The medium is then discretized into small N pixels before calculating the forward map by the FEM along with all of the boundary conditions [22]. Full derivation of FEM formulation can be found in [14]

3.1.3 Linearization of EIT Signal Model

In DOA estimation, the solution of wave equation can be obtained by solving the Helmholtz equation (2.65) which is a nonlinear problem. However, by imposing certain assumptions such as far-field and narrowband assumption, the problem is now simplified so that linear property can be attained. On the other hand, the problems in EIT do not conform to such assumptions. The potential solution of elliptic PDE has a nonlinear relationship with conductivity distribution $\sigma(\mathbf{r})$, and

hence solving $\nabla \cdot (\sigma(\mathbf{r})\nabla\phi(\mathbf{r})) = 0$ with boundary conditions will certainly yield a nonlinear potential solution (for a specified $\sigma(\mathbf{r})$). The measurement vector $\mathbf{y}(n)$ of EIT is obtained by solving Laplace equation with boundary conditions (Neumann-to-Dirichlet data). For simplicity, the expression $\sigma(\mathbf{r})$ will be simply denoted as σ while the reader should keep in mind that σ is a spatial distribution which is a function of location.

Although the potential field $\phi(\sigma)$ has a non-linear relationship with σ , we can obtain a linear approximation of $\phi(\sigma)$ through Taylor expansion. This is accomplished through a small perturbation of the conductivity variation $\delta\sigma$. The approximate solution can be written under Taylor expansion as

$$\phi(\sigma + \delta\sigma) \approx \phi(\sigma) + \phi'(\sigma)\delta\sigma + \frac{1}{2}\phi''(\sigma)\delta\sigma^2 + O(\delta\sigma^3). \quad (3.9)$$

By neglecting 2^{nd} order and higher order terms, we can obtain a linearized approximation $\delta\phi = \phi'(\sigma)\delta\sigma$ or in a vector form as $\mathbf{y} = \mathcal{J}\mathbf{x}$. The vector $\mathbf{y}$ is a vector of perturbed field $\delta\phi$ on the boundary, matrix $\mathcal{J}$ is a Jacobian matrix which is obtained by calculating $\phi'(\sigma)$, and vector $\mathbf{x}$ is a vector of small conductivity variation $\delta\sigma$.

In order to calculate the Jacobian matrix $\mathcal{J}$, additional information with regards to the perturbed field will be given. The conductivity variation is to be limited to a very small value $\Delta \rightarrow \delta$ such that we now obtain a so-called perturbed conductivity $\sigma' = \sigma + \delta\sigma$ that would contribute to a perturbed potential $\phi' = \phi + \delta\phi$. When σ' and ϕ' are substituted into the Laplace's equation $\nabla \cdot (\sigma\nabla\phi) = 0$, we have

$$\begin{aligned} \nabla \cdot (\sigma + \delta\sigma)\nabla(\phi + \delta\phi) &= 0 \\ \nabla \cdot \sigma\nabla\phi + \nabla \cdot \delta\sigma\nabla\phi + \nabla \cdot \sigma\nabla\delta\phi + \nabla \cdot \delta\sigma\nabla\delta\phi &= 0. \end{aligned} \quad (3.10)$$

We know that $\nabla \cdot (\sigma\nabla\phi) = 0$, and since $\nabla \cdot \delta\sigma\nabla\delta\phi$ is the second order approximation [14], the first order expression can be expressed as

$$\begin{aligned} \nabla \cdot \sigma\nabla\delta\phi &= -\nabla \cdot \delta\sigma\nabla\phi \\ \nabla^2\delta\phi &= -\nabla\delta\sigma \cdot \nabla\phi, \end{aligned} \quad (3.11)$$

where $\sigma = 1$ is set for simplicity in the last expression. Now, the Laplace equation has been transformed to a Poisson equation with small variation on the conductivity

value. The conductivity variation acts as a dipole source inside the medium Ω . This formulation provides another perspective on the linearized approximation. Because $\delta\sigma$ is very small, the resulted derivation is equivalent to a dipole source inside the medium. To further explain the effect of dipole source, consider a system of 16 electrodes on the circular boundary as shown in Fig. 3.6

FIGURE 3.6: (a) FEM model of 16 electrodes and (b) Current injection's position.

In Fig. 3.6, the blue and the red colors denote the location where the current is going into the medium (blue) and leaving out of the medium (red). For illustration purposes, we use opposite injection method as can be seen in Fig. 3.6 that the blue and the red colors are located at the opposite directions. Number 1 to 16 on the circle's boundary denotes the location of each sensor. In many applications where only single frequency is used, without considering the phase information this can be regarded as reconstructing an image based on the non-uniform conductivity distribution profile of the medium. Further explanation is shown in Fig. 3.7 and the following paragraph.

The current density distribution where the conductivity distribution profile of the medium is homogeneous is shown in Fig. 3.7 (a). It can be observed that when the medium has uniform conductivity profile, the current density also has a uniform distribution. In Fig. 3.7 (b), the colored inner circle denotes the change in conductivity profile. The result can be seen in Fig. 3.7 (c) where the current density distribution is perturbed. An extra small circle on the right-most shows the zoom-in view where the current density distribution is being perturbed. This

FIGURE 3.7: (a) Current distribution inside a medium without object inside (b) conductive object inside the medium (c) current distribution inside the medium with a conductive object.

perturbation (perturbed current density distribution) creates perturbed potential field (voltage change) on the boundary respectively.

The intuitive concept of how to obtain a Jacobian matrix can be summarized as follows. First, the solution of the perturbed Laplace equation $\nabla \cdot (\sigma + \delta\sigma) \nabla (\phi + \delta\phi) = 0$ is required. This can be achieved by perturbing small conductivity variation on $\delta\sigma$ at the first pixel without changing the remaining pixels. By solving with the FEM, we now obtain a vector of Neumann-to-Dirichlet data $F(\sigma_1)$ which is the solution at the electrodes due to conductivity variation at the first pixel. Since the small perturbed $\delta\sigma$ is known, the Jacobian can be identified as simply a map between the perturbed field $\delta F(\sigma)$ and a small perturbed conductivity $\delta\sigma$ or simply $\mathcal{J}_{i,j} = \frac{\partial F(\sigma)_i}{\partial \sigma_j}$. Next, we set the first pixel back to its background (reference) value and then reassign a perturbed $\delta\sigma$ to the second pixel. The same process is repeated until the N^{th} pixel of the medium is obtained. In practice, the perturbation method as explained earlier is not commonly used although it can provide a good intuitive understanding. Several methods for Jacobian calculation are available [23–26].

3.2 Design and Implementation of EIT System

3.2.1 Static and Dynamic System

In general, numerous research in EIT can be categorized into one of the two major developments. The first part is the hardware improvement, where new circuit design or system architecture is the main focus. This also includes the improvement of

high-quality sensors, fast and reliable switching architecture, and high-performance signal conditioning and demodulating scheme. The second part is the image reconstruction where a discrete ill-posed inverse problem has been the main focus from the beginning from both theoretical and computational perspectives.

On the application requirement, the EIT can be categorized into two types which is static EIT or dynamic EIT. The major difference between static and dynamic EIT is the underlying physical process of the conductivity variation. In static EIT, the underlying conductivity variation does not change with time. This is similar to the X-ray, CT-Scan, or MRI imaging (anatomical imaging). On the other hand, the conductivity variation of dynamic EIT changes with time or frequency. The image reconstruction of EIT is best suited for dynamic imaging (functional imaging). For example, a fluid or gas flow monitoring in industrial pipeline, a study of brain activities or a lung monitoring in biomedical application. The objective of the applications above is to capture differences in conductivity variation of the medium.

Until now, the most successful EIT systems are constructed under the dynamic framework. The dynamic system has the advantage over static system in a reduction of measurement errors such as electrode placement error or some small artifact reading. These errors happen at both the reference image (baseline image) as well as the consecutive image, and hence the errors can be cancelled out by calculating difference potential variation. In addition to the benefit of small errors cancellation, dynamic system has the advantage over static system due to high temporal resolution. This makes the dynamic system suitable for measuring and monitoring in on-line or real-time. One example is the brain activity monitoring. No anatomical imaging system can achieve a temporal resolution in millisecond as EIT does. Although with high temporal resolution, it should be noted that there is a limitation regarding the spatial resolution performance of EIT when compared with other anatomical imaging techniques such as X-ray, CT-Scan, or MRI imaging.

Most of the EIT system apply current of 1 mA to few mA at around 10-50 kHz by using 8 or 16 electrodes per cross-sectional plane. The electrical characteristic of low frequency is similar to that of the D.C. A commonly used method for data collection is known as the “four-electrode method” which uses two electrodes for current injection while using the other two electrodes for capturing the voltage as shown in Fig. 3.8. This method is sometimes called “adjacent method” or “neighbouring method”.

FIGURE 3.8: Four electrode method

An important point to note is that typically the current injection electrodes will not be used for data collection. This is because during current injection period, the drop in electrode impedance could cause measurement error in the collected data.

3.2.2 Flow Chart of EIT System

Next, we introduce an overview of the whole design of the EIT prototyping system. The overall system is shown in Fig. 3.9.

FIGURE 3.9: Flow chart of the EIT system.

The front-end part comprises of three major components which includes the current source, the multiplexing or switching system, and the data collection unit. Most EIT systems typically excludes phase information since it is contained with errors caused by stray capacitance and impedance mismatch between electrode impedance and the medium conductivity. Some studies suggested that the reactant part (imaginary part of the impedance) does play an important role, and it

may contain significant information of the medium. However, to use the phase information the circuit must be modified with specialized phase extraction circuit developed specifically with an ability to eliminate the errors caused by stray capacitance and impedance mismatch. Additional pre-processing part must also be implemented in order to reduce these errors.

3.2.3 Current Injection & Measurement Procedure

By using the four-electrode method, several rounds of current injection and voltage measurements can be accomplished. The whole process of current injection and measurement procedure is achieved through injecting the current into all pair of electrodes in a clockwise or counter-clockwise manner. The whole process can be summarized as shown in Fig. 3.10.

FIGURE 3.10: Current injection and measurement procedure.

During the first current injection, the a pair of electrodes that injects the current are fixed, and the rest of electrodes are used for voltage measurement from different angles. It should be noted that the current injecting electrodes are not used for voltage measurement while the current is injected. The complete measurement is accomplished through 13×16 or 208 measurements around the medium.

3.2.4 Main Components of EIT System

Next, each of the major components in the EIT prototyping system will be reviewed. The overall EIT prototyping system includes the current source, the switching system, the signal rectifier, and the Arduino board as controlling unit.

3.2.4.1 Current Source

In most applications where the medium does not contain only the resistive part, an A.C. is necessary since d.c. cannot pass through the reactive component which contains capacitive element. This can be achieved through a voltage controlled current source (VCCS) such as a “Howland current source”. It is important to note that in order to use VCCS as a current source, a signal generator that can produce sinusoidal waveform is required. The Howland current source design is commonly used in EIT application due to its high output impedance. The high output impedance is one of the most important characteristic of a good current source since it ensures that all of the current (energy) will be delivered to the load, and hence all of the energy is transferred to the medium. Due to safety concern, the Howland circuit is designed to produce an A.C. at approximately 1 mA_{pp} . The schematic of constructed VCCS is shown in Fig. 3.11.

FIGURE 3.11: Schematic of voltage controlled current source.

The 1 mA output can be achieved by setting the resistor values to $R1 = R3 = 150\text{ k}\Omega$, $R2 = R3 = 15\text{ k}\Omega$, $R_s = R_{s'} = 50\text{ }\Omega$, and 0.5 V_{pp} input. The output impedance $Z_{out} \approx \infty$, so that I_{Load} will always pass into the R_{Load} which is the water tank used in our experiment. The VCCS could only produce 1 mA for frequency up to 10 kHz of $1\text{ k}\Omega$ load. After changing with several Op-Amps, 50 kHz with $2\text{ k}\Omega$ load was later achieved. This was achieved by using TL071ACP Op-Amp.

3.2.4.2 Multiplexing or Switching System

To achieve automatic switching of the electrodes, both multiplexers and de-multiplexers are required. The chip chosen for this task is based on Maxim DG406DJ MUX/D-MUX, which is a versatile IC that can function as either a multiplexer or a de-multiplexer. It has a high current tolerance of up to 30 mA and voltage range from -15V to +15V, which is suitable for our prototyping system. Although developing for 8 channels system at the moment, the switching system is designed to support 16 channels in the future. The schematic of switching system is shown in Fig. 3.12. The electrodes are connected to the 16 bidirectional channels while the current/voltage measurement is connected to the single output channel. The selection signals are generated from Arduino digital output. For current switching system, it operates as 1:16 DMUX, selecting 1 of the 16 electrodes the current will input to the medium. The ground switching works in same way as the current switching since both of which (current and ground) moves together as source and sink pair. For voltage measurement, it operates as a 16:1 MUX by selecting 1 out of 16 electrodes to output the voltage reading (where we use the ground (sink) as a reference point). For instance, to measure voltages from electrode 1 to 16, Arduino will generate selection codes from 0000 to 1111. Thus, the entire process can be automated to collect the voltage data that allows online imaging of a moving object.

3.2.4.3 Signal Rectifier

To assist fast data collection process, an analog signal rectifier is commonly used to extract a useful information for later processing. The signal rectifier converts A.C. signal obtained from the boundary of the medium to a D.C. signal level. The process is accomplished by first passing the voltage reading on the boundary of a

FIGURE 3.12: Schematic of the switching system.

medium through the switching system. It is then passed through the signal rectifier and the analog reading pin from the Arduino board subsequently. The schematic of signal rectifier is shown in Fig. 3.13.

FIGURE 3.13: Schematic of the signal rectifier.

3.2.4.4 Controlling Unit

The controlling unit is one of the most essential component in the EIT system. It selects specific electrodes for current injection pair. It also selects specific electrodes

for voltage measurement on the other electrodes. The Arduino board is used for the following tasks:

- *Controlling the multiplexers:* This is done by the Arduino's digital output. The controlling code is generated as 4 bits command from 0000 to 1111. They are fed into the MUX select signals to command which gate will be used. For 1 full set of data, there are $16 \times 16 = 256$ readings to be taken.
- *Reading the analogue voltage:* The Arduino analogue pins can read voltage from 0 to 5V with its built-in 10-bit analogue to digital converter (ADC). Since the measured voltage at the electrodes is a sinusoidal signal, a precision rectifier is installed before passing the signal to Arduino.
- *Collecting data for image reconstruction:* The voltage values are accumulated and sent through the serial port. However, as Arduino cannot write data onto a computer directly, a secondary software named Processing is used. This allows the Arduino to write all of the accumulated data to an intermediate file before transferring to the MATLAB for image reconstruction.

FIGURE 3.14: Arduino MEGA

3.2.5 Integrated System

With all of the components presented earlier, we now combine each part to implement the integrated system where its schematic is shown in Fig. 3.15. On the left, the voltage controlled current source will take a sinusoidal waveform obtained from the signal generator. It is then passed to the switching system in the middle

which is controlled by the Arduino board. The signal rectifier on the right received signal from the switching system before passing the voltage reading to the Arduino board for data collection.

FIGURE 3.15: Schematic of the integrated system.

The implementation on PCB of Fig. 3.15 is shown in Figs. 3.16 and 3.17. In Fig. 3.16, number 1 indicates the Arduino board, number 2 indicates the VCCS, and number 3 indicates switching system with signal rectifier.

FIGURE 3.16: Implementation of the integrated system.

FIGURE 3.17: PCB of the integrated system

3.2.6 Data Handling Protocol

Data handling is a crucial step in speeding up the whole image reconstruction process. It is located between the Arduino and the image reconstruction which was done in MATLAB. On the first design, the data handling protocol is based on an intermediate software called processing 3 which was used to access data collected by the Arduino board. The collected data were put into a bracket of 64 data point (for one image frame). The difference voltage were calculated in Microsoft Excel before passing to MATLAB for image reconstruction process.

As can be seen, the data handling part is a bottleneck for the whole process since several extra steps are needed. To achieve fast image reconstruction, this data handling protocol requires a modification. This is achieved where we later redesigned in the way that the Arduino can directly communicate with the MATLAB for data acquisition. All of the difference data are now computed in MATLAB which is faster than using an intermediate software.

The EIT prototyping system in its operation is shown in Figs. 3.18-3.20 respectively. In Fig. 3.18, the water inside the container can be regarded as a homogeneous medium whereas the an additional object (metallic rod) is creating the non-uniform conductivity profiles inside the medium. The sensors on the boundary are able to detect the perturbed field, and the image is reconstructed respectively as shown on the computer screen in Fig. 3.18.

FIGURE 3.18: Image reconstruction of a conductive object located on the left.

The red color on the screen denotes the higher conductivity profile (conductive) from a metallic rod. This highlights the system's capability that it is able to detect the change in conductivity profile and reconstruct an image accordingly. After the object's location is shifted as shown in Fig. 3.19, the red color in the reconstructed image on screen is shifted accordingly.

FIGURE 3.19: Image reconstruction of a conductive object located on the right.

It can also be seen in Fig. 3.20 that the blue color on the screen denotes the lower conductivity profile (non-conductive) from a plastic rod.

FIGURE 3.20: Image reconstruction of a conductive and a non-conductive objects.

3.3 Conclusion

The fundamental working principal of EIT system is reviewed in this chapter. The concept of forward problem is highlighted with additional details on Poisson's equation with the boundary conditions specifically for EIT application. An overview of the overall EIT system is provided where the current injection and measurement procedure is explained in details. The system implementation for EIT applications is proposed and the system has been constructed with the circuit schematic diagrams provided. This system will be used to study some of the algorithms proposed in the subsequent chapters.

Chapter 4

Hybrid Regularization Matrix for Robust Image Reconstruction in EIT

4.1 Introduction

The development of EIT image reconstruction has progressed steadily from the conventional back-projection method that was used in the '90s to the post millennial regularization framework. It is important to highlight that the development of the regularization framework can be viewed as a prescription of different prior knowledge onto the estimate solution. In the image space (estimate solution) viewpoint, we may presume that a particular reconstructed image should have either a smooth or sharp characteristic. On the other hand, in the vector norm space viewpoint, the assumption that a solution obtained after solving an inverse problem should have certain norm characteristics (either l_2 or l_1) may be used instead. The problem of EIT image reconstruction was first discussed within mathematics community, particularly the inverse problem community. Most of the focus had been placed on analyzing the existence, uniqueness, and continuity of the solution in Hilbert or Sobolev spaces based on functional analysis. Additional information on this regard can be found in [14]. However, to solve the inverse problem efficiently, a modern computational method is indispensable. The back-projection method, which was popularized by the X-ray and computed tomography (CT), was used at

the beginning of EIT image reconstruction attempts. It was later realized that this approach is not efficient due to the mismatch between the underlying assumption (straight path) and the actual model which can be explained as follows. In the “hard-field” tomography such as X-ray or CT, the energy path of these techniques are known to be “local”. This means that each energy ray passes through only a limited number of pixels or voxels along a straight line. The resulting (receiving) signal is affected only by a few pixels or voxels which gives it a high precision at the receiver (sensor). This high precision is also due to a high frequency used, which also translates to high energy stimulation or a radiation. In contrast, the current distribution used in the EIT application travels in a diffusive fashion (soft-field) due to low frequency used. This means that a group of pixels or voxels (including its neighborhood) will affect the receiving signal at the receiver (sensor). This phenomenon is known as “global,” as opposed to what happened with the hard-field techniques. The diffusive nature of soft-field technique also leads to the ill-posed nature from the inverse problem perspective as will be discussed further into detail in the following section.

In the previous chapter, we have introduced the working principle of EIT, its image reconstruction process, and the mechanism of the EIT system in general. In this chapter, additional detail regarding image the reconstruction process of one snapshot or single measurement vector (SMV) will be discussed. As indicated earlier that the image reconstruction problem in EIT is considered to be one of the most severely ill-posed inverse problems. This issue is observable through a high correlation between columns of the sensitivity matrix (Jacobian matrix). From a system matrix viewpoint, the singular values decomposed from the sensitivity matrix (by the SVD) comprise several large singular values and many small singular values. With rapid decay of the singular values, the estimated solution becomes very sensitive to noise commonly occurs when the Picard’s condition is not satisfied [3], and hence the solution needs to be regularized.

In the case of on-line or real-time image reconstruction scenarios, the computational time is limited, and hence a robust and fast estimate solution is often preferred. Amongst l_2 -norm regularization matrices, Newton One-Step Error Reconstruction (NOSER) regularization matrix is one of the most well known prior with robust characteristic that is widely used within EIT community (besides Tikhonov

regularization matrix). This regularization matrix is based on an l_2 -norm regularization framework which encompasses the general knowledge (global information) of a medium (non-uniform sensitivity distribution). Its popularity lies in its intuitive interpret-ability and how computationally efficient this prior provides to most image reconstruction tasks. In short, it is mathematically intuitive and computationally robust. With its fast computation (non-iterative in nature), it can also be used as a warm start for other iterative-based methods such as the Iteratively Reweighted Least Square (IRLS) or the other l_1 -norm regularization frameworks.

A new robust regularization matrix called Fidelity-Embedded Regularization (FER) was recently proposed. It has similar structure to the NOSER (diagonal structure) while the intuition and reasoning behind both priors are different. In contrast to the NOSER, the FER regularization matrix encompasses general knowledge and global information of the medium based on orthogonality property of each pixel sensitivity [27]. Although approaching from different viewpoints, the two regularization matrices (NOSER and FER), in fact, share similarity as will be highlighted in the next section where we will address the similarities and differences of both priors in spatial domain and SVD domain.

Three major contributions of this chapter can be briefly summarized as follows. Firstly, a relationship between two regularization matrices, namely NOSER and FER, is highlighted in both spatial (pixels) and SVD domains. Secondly, a hybrid regularization matrix which is a modification of both mentioned priors is proposed as an alternative robust regularization matrix. Lastly, the experimental results along with the evaluated performance parameters are presented. In the following sections of this chapter, the main contents are organized as follows. Firstly, a forward and inverse formulation of EIT will be briefly reviewed. Then, the fundamental and working principle of NOSER and FER regularization matrices will be analyzed and compared in both spatial and SVD domains. The hybrid regularization matrix is subsequently proposed. Lastly, the experimental results together with the performance parameters from each regularization matrix will be evaluated.

4.2 Forward & Inverse Problems

FIGURE 4.1: EIT system with adjacent method for current injection and measurement procedure.

In this chapter, we follow the forward formulation with “complete electrode model” [21] which is an extension of 3.7 defined as

$$\begin{aligned}
 \nabla \cdot (\sigma \nabla \phi) &= 0 \quad \text{in } \Omega, \\
 (\phi + z_k \sigma \frac{\partial \phi}{\partial \mathbf{n}})|_{\mathcal{E}} &= U_k \quad \text{for } k = 1, \dots, L, \\
 \sigma \frac{\partial \phi}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial \Omega \setminus \bigcup_{k=1}^L \mathcal{E}_k, \\
 \int_{\mathcal{E}_k} \sigma \frac{\partial \phi}{\partial \mathbf{n}} &= 0 \quad \text{for } k \in \{1, \dots, L\} \setminus \{j, j+1\}, \\
 \int_{\mathcal{E}_j} \sigma \frac{\partial \phi}{\partial \mathbf{n}} ds &= I = - \int_{\mathcal{E}_{j+1}} \sigma \frac{\partial \phi}{\partial \mathbf{n}} ds,
 \end{aligned} \tag{4.1}$$

where $(\phi + z_k \sigma \frac{\partial \phi}{\partial \mathbf{n}})|_{\mathcal{E}} = U_k$ for $k = 1, \dots, L$ is added to this complete electrode model to simulate a realistic surface impedance of electrode sensors [28–30]. Additionally, the measurement procedure of this complete electrode model is summarized as follows. The potential U_k at electrode $\mathcal{E}_k$ on the boundary can be obtained by solving the Laplace equation with Neumann boundary condition where a pair of electrodes $\mathcal{E}_j$ and $\mathcal{E}_{j+1}$ are injecting a low frequency alternating current from different positions. In the specified formulation above, L is the number of array electrodes and z_k is the contact impedance of electrode $\mathcal{E}_k$. Since the receiving signal on the boundary are typically small, difference measurements are commonly

used $(U_k - U_{k+1})$. By using the adjacent electrodes for both current injection and potential measurements, the measurement vector is defined as

$$\mathbf{v} = [\underbrace{v^{1,1}, \dots, v^{1,L}}_{1^{st} \text{ injection}}, \dots, \underbrace{v^{L,1}, \dots, v^{L,L}}_{L^{th} \text{ injection}}], \quad (4.2)$$

where $v^{1,1} = U_1 - U_2$ from the first injection, and hence $v^{j,k} = U_k - U_{k+1}$ from the j^{th} injection. Since the current that passes through the injecting electrodes $\mathcal{E}_j$ and $\mathcal{E}_{j+1}$ can change the contact impedance z_j and z_{j+1} , the three measurements related to electrodes $\mathcal{E}_j$, and $\mathcal{E}_{j+1}$ are commonly omitted. The receiving electrodes $\mathcal{E}_k$ for $k \in \{1, \dots, L\} \setminus \{j, j+1\}$, where $\setminus$ defines exclusion, are used to obtain data. With L number of electrodes, the measurement vector is now with length of $M = (L-3)L$. Difference potential measurement either in time or frequency domain are commonly used by computing the difference between the perturbed potential on the boundary against the unperturbed or referenced potential.

The nonlinear relationship between the internal conductivity variation and the perturbed potential field on the boundary can be approximated by a linearized model as

$$\mathbf{v} = \mathbf{u}(\sigma_0) + \mathcal{J}(\sigma_0)(\delta\sigma) + \mathbf{n}, \quad (4.3)$$

where $\mathbf{v}, \mathbf{u}(\sigma_0) \in R^{M \times 1}$ are the perturbed and unperturbed potential data on the boundary respectively. The matrix $\mathcal{J}(\sigma_0) = \frac{\partial \mathbf{u}(\sigma_0)}{\partial \sigma_0} \in R^{M \times N}$ where $M \ll N$ is a sensitivity matrix (Jacobian matrix) which maps the conductivity variation from each pixel to the measurement vector, $\delta\sigma \in R^{N \times 1}$ is a vector of conductivity variation, and $\mathbf{n} \in R^{M \times 1}$ is a vector of an AWGN (unmodelled error and measurement noise). The linearized model in (4.3) can be rewritten as

$$\mathbf{y} = \mathcal{J}\mathbf{x} + \mathbf{n}, \quad (4.4)$$

where measurement vector $\mathbf{y} = \delta\mathbf{v} = \mathbf{v} - \mathbf{u}(\sigma_0) \in R^{M \times 1}$ represents the potential variation on the boundary. It should be noted that each column of the sensitivity matrix $\mathcal{J} = [\mathbf{j}(\xi_1), \dots, \mathbf{j}(\xi_N)]$ can be viewed as an array response (the expected potential variation on the boundary due to a small induced source from each pixel position). The conductivity variation vector $\mathbf{x} = \delta\sigma \in R^{N \times 1}$ is a vector of length N where each index corresponds to each pixel within the image space. With L

number of electrodes, we can obtain $M = (L - 3)L$ number of measurements based on the adjacent protocol where neighboring electrodes are used for current injection and measurement process.

The estimate solution $\mathbf{x}_\lambda$ can be obtained under the l_2 -norm regularization framework which is commonly defined as

$$\min_{\mathbf{x}} \{ \|\mathcal{J}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda^2 \|\mathcal{R}\mathbf{x}\|_2^2 \}, \quad (4.5)$$

where the first term in (4.5) attempts to minimize the discrepancy between model and data while the second term is simultaneously limiting the solution space in order to exclude possible noise amplified solutions. The estimate solution of the l_2 -norm regularization framework defined in (4.5) has a closed-form which can be expressed as

$$\mathbf{x}_\lambda = (\mathcal{J}^T \mathcal{J} + \lambda^2 \mathcal{R}^T \mathcal{R})^{-1} \mathcal{J}^T \mathbf{y}. \quad (4.6)$$

The choice of regularization matrix $\mathcal{R}$ is typically based on a priori knowledge of the solution such as the smoothness, its derivative (gradient), or other special features that should be preserved. The most fundamental case is the smoothness prior (Tikhonov regularization) where the regularization matrix is defined as $\mathcal{R} = \mathbf{I}$. With this prior, the noise amplified solution (unsmooth solutions with large norm (l_2)) will be heavily penalized or suppressed [31–33]. The estimate solution of (4.6) can be represented in the SVD domain as follows. Firstly, the Jacobian (sensitivity) matrix can be compactly decomposed by SVD as $\mathcal{J} = \mathbf{U}\Sigma\mathbf{V}^T$ where the columns of left singular matrix $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_M] \in R^{M \times M}$ are left singular vectors. A diagonal matrix $\Sigma = \text{diag}[\sigma_1, \dots, \sigma_M] \in R^{M \times M}$ is arranged in a non increasing order. The columns of right singular matrix $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_M] \in R^{N \times M}$ are the right singular vectors. For Tikhonov regularization, the estimate solution from (4.6) can be re-expressed under the SVD domain as

$$\mathbf{x}_\lambda = \sum_{i=1}^M \varphi_i(\lambda) \frac{\mathbf{u}_i^H \mathbf{y}}{\sigma_i} \mathbf{v}_i, \quad (4.7)$$

where the term $\varphi_i(\lambda)$ functions as a filter factor (weights in the SVD domain) such that

$$\varphi_i(\lambda) = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \approx \begin{cases} 1 & ; \quad \sigma_i \gg \lambda \\ \varepsilon & ; \quad \sigma_i \ll \lambda, \end{cases} \quad (4.8)$$

where the value of ε is a small positive real value. A hyper-parameter λ is used to balance between the model-data fitness (model-data discrepancy) and the prior $\mathcal{R}$ enforcing onto the estimate solution. From the results obtained by (4.7) and (4.8), we will revisit and later address how each prior (regularization matrix) from NOSER and FER affects the estimate solution from different viewpoints.

4.3 Analysis of the Regularization Matrices

The non-trivial diagonal-based regularization matrices will be explored in this section, where $\mathcal{R} \neq \mathbf{I}$. The generalized form of (4.5) can be re-written as

$$\min_{\tilde{\mathbf{x}}} \{ \|\tilde{\mathcal{J}}\tilde{\mathbf{x}} - \mathbf{y}\|_2^2 + \lambda^2 \|\tilde{\mathbf{x}}\|_2^2 \}, \quad (4.9)$$

where matrix $\tilde{\mathbf{x}} = \mathcal{R}\mathbf{x}$ and matrix $\tilde{\mathcal{J}} = \mathcal{J}\mathcal{R}^{-1}$ are the re-scaled estimate solution (pixel values) and the normalized Jacobian (sensitivity) matrix respectively. It should be noted that a direct interpretation of pixel penalization (re-scaled) is only valid when the regularization matrix has a diagonal structure. With the new formulation in (4.9), the estimated solution $\tilde{\mathbf{x}}_\lambda$ can be further analyzed in the SVD domain analogous to those in (4.7) and (4.8) where the new normalized sensitivity matrix is used which is defined as

$$\tilde{\mathcal{J}} = [\frac{1}{r_1}\mathbf{j}(\xi_1), \dots, \frac{1}{r_N}\mathbf{j}(\xi_N)], \quad (4.10)$$

where $\mathcal{R} = \text{diag}[r_1, \dots, r_N]$, and the estimated solution $\mathbf{x}_\lambda$ can be obtained subsequently by $\mathbf{x}_\lambda = \mathcal{R}^{-1}\tilde{\mathbf{x}}_\lambda$. In the following part of this section, the working principle of the regularization matrices: NOSER and FER will be briefly reviewed, the relationship of both priors will then be highlighted, and lastly the hybrid regularization matrix is proposed.

4.3.1 NOSER Regularization Matrix

Based on the Tikhonov prior perspective where the regularization matrix is defined as an identity matrix, the estimate solution obtained by this prior tends to be overly sensitive around the area closer to the boundary due to high sensitivity of electrode sensors. This non-uniform sensitivity phenomenon can be highlighted by first locating few small point sources located at different locations as shown in Fig. 4.2 where we labeled each source with number $1, 2, \dots, 10$.

FIGURE 4.2: Ten point sources located at different locations.

Here, ten point sources are situated at different locations ranging from the center area to area close to the boundary. A relationship between each point source and the corresponding column of the sensitivity matrix can be highlighted as follows

$$\mathcal{J} = \left[\mathbf{j}(\xi_1) , \dots , \mathbf{j}(\xi_k) , \dots , \mathbf{j}(\xi_N) \right]. \quad (4.11)$$

Note that the colored columns are corresponding to the source at different locations as shown in Fig. 4.2. The euclidean norm of the selected ten columns highlights the sensitivity of the array sensors and array responses as can be seen in Fig. 4.3.

It can be seen that the array responses are more sensitive to the sources situated closer the boundary and sensor areas. In order to limit the influence of the highly sensitive pixels along the areas close to the boundary, NOSER prior was utilized with a regularization matrix $\mathbf{M} = \mathcal{R}^T \mathcal{R}$ defined in (4.6) ($p \in [0, 1]$) as

FIGURE 4.3: Norm of array responses according to point sources located at different locations.

$$\mathbf{M} = \text{diag}[\{\text{diag}(\mathcal{J}^T \mathcal{J})\}^p], \quad (4.12)$$

where the value of $p = 1$ is a commonly used. Each diagonal element within this regularization matrix encompasses a global information (sensitivity information) of the each column within the sensitivity matrix. Note that in this thesis, the operator $\text{diag}(\cdot)$ is used for diagonal entries extraction while $\text{diag}[\cdot]$ is used for diagonal matrix construction. We can further analyze this by rearranging the general form in (4.5) to (4.9). It can be observed that the first term of (4.9) attempts to minimize the discrepancy between the estimated re-scaled solution $\tilde{\mathcal{J}}\tilde{\mathbf{x}}$ and the obtained data $\mathbf{y}$ where a normalized Jacobian matrix is being used. We can also observe in the second term that it simultaneously tries to minimize the norm of a re-scaled estimated solution $\tilde{\mathbf{x}} = \mathbf{M}^{1/2}\mathbf{x}$ where all values along those highly sensitive areas (pixels) are amplified by the weighting factor prescribed in matrix $\mathbf{M}^{1/2}$. This resulted in higher restriction of pixel amplitude along the sensitive areas (the amplified magnitudes/pixels are heavily suppressed to minimize the norm).

4.3.2 FER Regularization Matrix

Approaching the issue of noise amplified solution from different angle, the FER regularization matrix was proposed to solve the issue based on a priori knowledge of orthogonality property amongst the columns within the Jacobian matrix. This priori knowledge can be translated to a calculation of correlation between each column to the rest within the Jacobian matrix. In short, the more correlate between a specified column with the rest, the less orthogonal that specified column would be to the other columns. The FER regularization matrix $\mathbf{N} = \mathcal{R}^T \mathcal{R}$ can be defined as

$$\mathbf{N} = \text{diag}[\sum_{i=1, :}^N |\mathcal{J}^T \mathcal{J}|], \quad (4.13)$$

where the operation $\sum_{i, :}$ indicates a summation of all columns over i^{th} row (i is row index). In each diagonal entry of the regularization matrix, it encompasses the correlation information between a specified column to the rest. For example, in order to obtain a diagonal entry $[\mathbf{N}]_{11}$ the term $\sum_{1, :} |\mathcal{J}^T \mathcal{J}|$ can be obtained by computing a summation of the absolute value of an inner product among columns $\sum_{k=1}^N |\mathbf{j}(\xi_1)^T \mathbf{j}(\xi_k)|$. Those pixels with high correlation with the others will be resulted with a higher value, and therefore will be heavily suppressed (higher penalization).

In the image domain (pixel levels), although the prior knowledge incorporated inside the NOSER and FER regularization matrices may seemed different, they in fact share a similar trend across different locations as can be seen in Fig. 4.4.

The values depicted in Fig. 4.4 are obtained by extracting diagonal entries of both matrices $[\mathbf{M}]_{ii}$ and $[\mathbf{N}]_{ii}$ which are corresponding to different locations as shown in Fig. 4.2. The medium's shape is normalized (unit radius) with the last position of target situated at 0.9. The y-axis exhibits the intensity value of each diagonal entry obtained from the regularization matrices associated to each location within the normalized medium.

In Fig. 4.4, it can be seen that both NOSER and FER regularization matrices share similar trend across different locations within the medium. In short, they are enforcing high penalization level along the areas closer to the boundary where

FIGURE 4.4: Penalization intensity of NOSER and FER at different locations.

sensors are attached. Although with similar trend spatially, the intensity levels between the two regularization matrices are in fact different. Both NOSER and FER have comparable penalization intensity along the center area of the medium, but the NOSER regularization matrix has higher penalization levels than the FER regularization matrix over the boundary area. From the second term of equation (4.9), it is equivalent to having higher penalization levels along the pixels closer to the boundary for NOSER which also translates to a higher pixels suppression in those areas.

The fundamental principle of regularization matrices NOSER and FER would be further analyzed by re-examining (4.9) under the SVD domain (similar to the expansion in (4.7) and (4.8)). The estimate solution can be re-written as

$$\tilde{\mathbf{x}}_\lambda = \sum_{i=1}^M \varphi'_i(\lambda) \frac{\tilde{\mathbf{u}}_i^H \mathbf{y}}{\sigma'_i} \tilde{\mathbf{v}}_i, \quad (4.14)$$

where the value $\varphi'_i(\lambda)$ acts as a filter factor for the re-scaled singular value σ'_i obtained from the normalized Jacobian matrix $\tilde{\mathcal{J}}$ such that

$$\varphi'_i(\lambda) = \frac{\sigma_i'^2}{\sigma_i'^2 + \lambda^2} \approx \begin{cases} 1 & ; \quad \sigma_i' \gg \lambda \\ \varepsilon & ; \quad \sigma_i' \ll \lambda, \end{cases} \quad (4.15)$$

where the value ε is a small positive real value. The singular values obtained from the decomposition of the original Jacobian matrix are exhibited in Fig. 4.5 together with the re-scaled singular values obtained from the decomposition of normalized Jacobian matrix based on NOSER, FER, and Tikhonov regularization matrices.

FIGURE 4.5: Singular values extracted from normalized and unnormalized sensitivity matrices.

It can be seen that the levels of singular values of the normalized Jacobian matrix based on NOSER and FER regularization matrices are comparable, which are higher than the singular values of the original Jacobian matrix. The normalized Jacobian matrix based on Tikhonov prior is equivalent to the original Jacobian matrix since an identity matrix is used as a prior in the Tikhonov regularization. The normalized Jacobian matrix with Tikhonov prior is simply $\tilde{\mathcal{J}}_{Tik} = \mathcal{J}\mathbf{I}^{-1} = \mathcal{J}$.

The filter factors levels of NOSER and FER priors are depicted in Fig. 4.6 (labeled with $\square$ and $\diamond$ respectively). To further analyze the values depicted in Fig. 4.6, we shall simplify the notation in (4.14) together with the estimate solution which can be re-expressed as

 FIGURE 4.6: Filter factors of NOSER and FER for $\lambda = 0.1$.

$$\begin{aligned}
 \tilde{\mathbf{x}}_\lambda &= \tilde{\mathbf{V}} \Phi'_\lambda \tilde{\Sigma}^{-1} \tilde{\mathbf{U}}^T \mathbf{y} \\
 &= \tilde{\mathbf{V}} \Phi'_\lambda \tilde{\Sigma}^{-1} \tilde{\mathbf{U}}^T \tilde{\mathcal{J}} \tilde{\mathbf{x}}^* - \tilde{\mathbf{V}} \Phi'_\lambda \tilde{\Sigma}^{-1} \tilde{\mathbf{U}}^T \mathbf{n},
 \end{aligned} \tag{4.16}$$

where the notation $\mathbf{y} = \tilde{\mathcal{J}} \tilde{\mathbf{x}}^* + \mathbf{n}$ is used in (4.16), $\tilde{\mathbf{x}}^*$ is a true solution, the diagonal matrix $\tilde{\Sigma}$ can be defined by $\tilde{\Sigma} = \text{diag}[\sigma'_1, \dots, \sigma'_M]$, and Φ'_λ is diagonally structured matrix with filter factor terms along the diagonal entries which can be expressed as

$$\Phi'_\lambda = \begin{bmatrix} \varphi'_1(\lambda) & 0 & 0 & 0 \\ 0 & \varphi'_2(\lambda) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \varphi'_M(\lambda) \end{bmatrix}. \tag{4.17}$$

To further simplify the analysis, the matrix Φ'_λ shall now be re-defined as

$$[\tilde{\Phi}'_\lambda]_{ii} = \begin{cases} 1 & ; \quad \varphi'_i(\lambda) \geq 0.5 \\ 0 & ; \quad \varphi'_i(\lambda) < 0.5. \end{cases} \tag{4.18}$$

The value k is a cutoff or the last entry allowed in $\tilde{\Phi}'$, where $[\tilde{\Phi}'_{\lambda}]_{kk} = 1$. It is important to note that this threshold value (set to 0.5) is simply used for the analysis and comparison between the two priors in this section. In practical scenarios, the reconstructed images do not require the threshold value.

The true solution $\tilde{\mathbf{x}}^*$ can be expressed in parameter space as $\tilde{\mathbf{x}}^* = \tilde{\mathbf{V}}\tilde{\mathbf{V}}^T\tilde{\mathbf{x}}^* = \sum_{i=1}^M (\tilde{\mathbf{v}}_i^T \tilde{\mathbf{x}}^*) \tilde{\mathbf{v}}_i$ while the solution bias introduced into the estimate solution can be expressed as

$$E(\tilde{\mathbf{x}}_k) = \sum_{i=1}^k (\tilde{\mathbf{v}}_i^T \tilde{\mathbf{x}}^*) \tilde{\mathbf{v}}_i = \tilde{\mathbf{x}}^* - \sum_{i=k+1}^M (\tilde{\mathbf{v}}_i^T \tilde{\mathbf{x}}^*) \tilde{\mathbf{v}}_i. \quad (4.19)$$

It can be seen that the second term $\sum_{i=k+1}^M (\tilde{\mathbf{v}}_i^T \tilde{\mathbf{x}}^*) \tilde{\mathbf{v}}_i = \tilde{\mathbf{V}}(\mathbf{I} - \tilde{\Phi}'_{\lambda})\tilde{\mathbf{V}}^T\tilde{\mathbf{x}}^*$ is a solution bias introduced into the estimated solution. From the result depicted in equation (4.16), the error between the true solution $\tilde{\mathbf{x}}^*$ and the estimated solution $\tilde{\mathbf{x}}_{\lambda}$ or $\delta\tilde{\mathbf{x}} = \tilde{\mathbf{x}}^* - \tilde{\mathbf{x}}_{\lambda}$ can be expressed as

$$\begin{aligned} \delta\tilde{\mathbf{x}} &= \tilde{\mathbf{x}}^* - \tilde{\mathbf{V}}\tilde{\Phi}'_{\lambda}\tilde{\Sigma}^{-1}\tilde{\mathbf{U}}^T\tilde{\mathcal{J}}\tilde{\mathbf{x}}^* + \tilde{\mathbf{V}}\tilde{\Phi}'_{\lambda}\tilde{\Sigma}^{-1}\tilde{\mathbf{U}}^T\mathbf{n} \\ &= (\mathbf{I} - \tilde{\mathbf{V}}\tilde{\Phi}'_{\lambda}\tilde{\Sigma}^{-1}\tilde{\mathbf{U}}^T\tilde{\mathbf{U}}\tilde{\Sigma}\tilde{\mathbf{V}}^T)\tilde{\mathbf{x}}^* + \tilde{\mathbf{V}}\tilde{\Phi}'_{\lambda}\tilde{\Sigma}^{-1}\tilde{\mathbf{U}}^T\mathbf{n} \\ &= \tilde{\mathbf{V}}(\mathbf{I} - \tilde{\Phi}'_{\lambda})\tilde{\mathbf{V}}^T\tilde{\mathbf{x}}^* + \tilde{\mathbf{V}}\tilde{\Phi}'_{\lambda}\tilde{\Sigma}^{-1}\tilde{\mathbf{U}}^T\mathbf{n} \\ &= \underbrace{\sum_{i=k+1}^M (\tilde{\mathbf{v}}_i^T \tilde{\mathbf{x}}^*) \tilde{\mathbf{v}}_i}_{\text{solution bias}(\epsilon_b)} + \underbrace{\sum_{i=1}^k \frac{\tilde{\mathbf{u}}_i^T \mathbf{n}}{\sigma'_i} \tilde{\mathbf{v}}_i}_{\text{noise amplified perturbation}(\epsilon_p)}. \end{aligned} \quad (4.20)$$

It can be seen that the first term (ϵ_b) highlights a bias introduced into the true solution as shown earlier in (4.19) while the second term (ϵ_p) highlights the noise amplified perturbation that occurs when Picard's condition is not satisfied (when the value of σ'_i decays faster than the value of $|\tilde{\mathbf{u}}_i^T \mathbf{n}|$ and the norm of "noise amplified perturbation" increases rapidly). While not always the case, the norm of a solution bias ($\|\epsilon_b\|_2$) is smaller than the norm of a noise amplified perturbation ($\|\epsilon_p\|_2$) because the term $1/\sigma'_i$ tends to amplify the value in ϵ_p .

The hyper-parameter in Fig. 4.6 are set to $\lambda = 0.1$. As can be seen, the cutoff k for NOSER and FER are $k = 13$ and $k = 16$ respectively. This means the estimated

solution based on FER prior will be more sensitive to noise when compared with the estimated solution based on NOSER prior because

$$\begin{aligned} \|\epsilon_{p,NOSER}\|_2^2 &= \sum_{i=1}^{13} \left(\frac{\tilde{\mathbf{u}}_i^T \mathbf{n}}{\sigma'_i} \tilde{\mathbf{v}}_i \right)^2 \\ \|\epsilon_{p,FER}\|_2^2 &= \sum_{i=1}^{13} \left(\frac{\tilde{\mathbf{u}}_i^T \mathbf{n}}{\sigma'_i} \tilde{\mathbf{v}}_i \right)^2 + \sum_{j=14}^{16} \left(\frac{\tilde{\mathbf{u}}_j^T \mathbf{n}}{\sigma'_j} \tilde{\mathbf{v}}_j \right)^2, \end{aligned} \quad (4.21)$$

and hence $\|\epsilon_{p,FER}\|_2 > \|\epsilon_{p,NOSER}\|_2$. However, based on additional information being incorporated into the estimated solution in the case of FER prior ($\tilde{\mathbf{v}}_j$ for $j = 14, 15$, and 16), the resolution performance of FER prior is higher than the NOSER prior. It can be seen in Fig. 4.7 that the major difference between the performance of NOSER and FER is less pronounce when the hyper-parameter (λ) is small ($\lambda = 0.05$), and the resulting cutoff for NOSER and FER priors are $k = 16$ and $k = 17$ respectively.

FIGURE 4.7: Filter factors of NOSER and FER for $\lambda = 0.05$.

4.3.3 Hybrid Regularization Matrix

Based on the relationship depicted in Figs. 4.4 - 4.7, it can be clearly seen that both NOSER and FER priors share certain similarity. However, they have different levels of suppression intensity in certain locations especially along the area closer to the sensors. Now, a hybrid regularization matrix that tries to balance between NOSER and FER priors is proposed by enforcing a penalization intensity level as an interpolation between both priors. This can also be considered as incorporating both NOSER and FER priors knowledge into one regularization matrix. The proposed hybrid regularization matrix $\mathbf{P} = \mathcal{R}^T \mathcal{R}$ is defined as

$$\mathbf{P} = \text{diag}[\text{avg}\{\text{diag}(\mathcal{J}^T \mathcal{J})^p, (\sum_{i=1, \dots, N} |\mathcal{J}^T \mathcal{J}|)\}]. \quad (4.22)$$

Each value along the diagonal entry of the hybrid regularization matrix can be obtained by computing an average between the penalization intensity values of both NOSER and the FER priors. As will be highlighted later in the next section, the NOSER and FER priors both have their own advantages and disadvantages.

FIGURE 4.8: Penalization intensity of NOSER, FER, and Hybrid at different locations.

By combining both priors into one, we are able to alleviate the necessity to select and balance the trade-offs of two priors while at the same time being able to achieve

well a robust performance. The suppression levels of hybrid regularization matrix along with the NOSER and the FER are depicted in Fig. 4.8. In the study [27], the estimate solution was proposed by modifying (4.6) with additional weight $\sqrt{1 + \alpha^2}$ when $0 < \alpha < \infty$ as follows,

$$\mathbf{x}_\lambda = \sqrt{1 + \alpha^2}(\mathcal{J}^T \mathcal{J} + \alpha \mathcal{R}^T \mathcal{R})^{-1} \mathcal{J}^T \mathbf{y}, \quad (4.23)$$

where the α value is set by $\alpha = \lambda^2$ in (4.6). However, the image reconstruction in this chapter will only be limited to the case where $0 < \alpha < 1$ and α is limited to small positive real value, and hence the extra term can be approximated by $\sqrt{1 + \alpha^2} \approx 1$.

4.4 Experimental Results

In this section, the image reconstruction results based on real data obtained from our EIT prototyping system is presented. The experimental data were collected from an EIT system prototype as shown in Fig. 4.9. Multiple images are reconstructed based on different regularization matrices namely NOSER, FER, and Hybrid along with performance parameters of each regularization matrix.

An EIT prototyping system with 8 sensors highlighted by a red rectangular box as shown in Fig. 4.9 were used for data collection. A round-shaped polyethylene container of size 25 cm in diameter was used as a medium. Objects with different conductivity values namely polyethylene and copper which correspond to low and high conductivity profiles, were put inside the container after the water was filled. An alternating current of 1mA peak-to-peak at frequency of 1kHz was generated by

FIGURE 4.9: Experimental setup

an improved Howland voltage controlled current source (VCCS) labeled by number "2" in Fig. ???. The alternating current is injected onto the medium's boundary with the adjacent protocol. The perturbed voltage data were recorded from the non-injecting sensors with the adjacent protocol by dual 8-channel multiplexing system before passing to signal rectifier which is labeled as number "3" in Fig. ??. The rectified signal is then sampled by an analog-to-digital converter (ADC) from an Arduino board which is labeled with number "1" in Fig. ??. The data is then transferred to a personal computer for further image reconstruction process. The forward calculation is obtained by using finite element model (FEM) developed by a platform provided by [34]. Next, the images are reconstructed based on different regularization matrices for each experimental setup (P_1 to P_6) where the polyethylene and copper rods are placed at different locations inside the container as depicted in Figs. 4.11(a), 4.13(a), 4.15(a), 4.17(a), 4.19(a), and 4.21(a). The average noise levels across multiple channels for each experimental setup were recorded as $6.6\mu V$, $6.8\mu V$, $6.8\mu V$, $7.6\mu V$, $7.1\mu V$, $10.0\mu V$, and $6.8\mu V$.

In this chapter, several performance parameters namely position error (PE), resolution (RES), and ringing area (RNG) proposed in [35] will be used. After an image is successfully reconstructed (defined as $\mathbf{x}$), it is then normalized to a range

of $[0,1]$ (denoted as $\mathbf{x}_{rec}$). It is then thresholded at a value of 0.5 (now denoted as $\hat{\mathbf{x}}$). The center of gravity (CoG) of the object in the original image is denoted as $\mathbf{x}_{CoG}$ where the CoG of the reconstructed and normalized image is denoted as $\hat{\mathbf{x}}_{CoG}$. Without considering direction of the error, the position error is defined as

$$PE = |\mathbf{x}_{CoG} - \hat{\mathbf{x}}_{CoG}|. \quad (4.24)$$

Next, the resolution performance of a reconstructed image, which reflects how compact a reconstructed object in the image is, can be obtained by the following expression

$$RES = \frac{A_{\hat{\mathbf{x}}}}{A_0}, \quad (4.25)$$

where the area of a normalized and thresholded image is defined as $A_{\hat{\mathbf{x}}}$, and the total area of the overall image is defined as A_0 . Lastly, the ringing effect (RNG) which computes the pixels amplitude outside the reconstructed object (region-of-interest or ROI) area is defined as

$$RNG = \frac{\sum_{k \notin C} |\mathbf{x}|_k}{\sum_{k \in C} |\mathbf{x}|_k}, \quad (4.26)$$

where C is the area identified earlier by an object in $\hat{\mathbf{x}}$. With an absolute operator in $|\mathbf{x}|_k$, the formulation defined here can be applied to both object with higher or lower conductivity profiles. With these three performance parameters, the lower value means better performance. In certain cases where there are multiple objects inside the medium, the position errors correspond to each object will be labeled with different number (1,2, and 3) as shown in Figs. 4.15(a), 4.17(a), 4.19(a), and 4.21(a). As can be seen, each position error is denoted as PE_1 , PE_2 , and PE_3 , and the average position errors of all objects is denoted as PE_{avg} respectively.

FIGURE 4.11: Reconstructed images ($\mathbf{x}$) of the experimental setup (P_1)

In Figs. 4.11-4.22, it can be seen that NOSER prior tends to produce images with large blob (lower resolution) when compared with the FER prior (smaller blob). On the other hand, the FER prior produces underestimated images (small blob) because of high frequency information in the SVD domain are truncated. It should be noted that we can achieve a robust and reliable results which is an interpolation between NOSER and FER priors based on the proposed Hybrid method. This can be beneficial specifically under high dynamic environment where noise levels are greatly varied. In Figs. ??, 4.14, 4.16, 4.18, 4.20, and 4.22 the normalized and thresholded images $\hat{\mathbf{x}}$ are shown respectively. These images provide supplementary information of the reconstructed images specifically the resolution performance (large or small blobs). The differences between each method can be seen and compared more easily based on these supplementary images. To highlight the nuances (small variation) of the reconstructed image of each method, several dotted circles with orange color are plotted as a pointer to highlight the differences.

Now, several performance parameters for each method will be highlighted for each experimental setup. The performance parameters of one object, two objects, and three objects are exhibited in Tables 4.1-4.3. The bold numbers in Tables 4.1-4.3 highlighted the best result of all three regularization matrices. The hyper-parameter being evaluated in this thesis is limited to the case of $0 < \lambda < 1$ where $\lambda = 0.05$ and 0.1 are used. For a special case when $\lambda = \infty$, the result can be

FIGURE 4.13: Reconstructed images ($\mathbf{x}$) of the experimental setup (P_2)

FIGURE 4.14: Thresholded images ($\hat{\mathbf{x}}$) of the experimental setup (P_2)

regarded as an extension of a match filter framework which tends to have low resolution performance, and hence will not be evaluated in this thesis.

FIGURE 4.15: Reconstructed images ($\mathbf{x}$) of the experimental setup (P_3) FIGURE 4.16: Thresholded images ($\hat{\mathbf{x}}$) of the experimental setup (P_3)

TABLE 4.1: Performance Evaluation of One Object

		$\lambda = 0.05$			$\lambda = 0.1$		
		NOSER	FER	Hybrid	NOSER	FER	Hybrid
P_1	$PE(10^{-2})$	2.4	3.8	2.6	2.2	1.6	1.9
	$RES(10^{-2})$	8.1	6.6	7.7	8.1	6.0	7.4
	RNG	2.2	2.6	2.3	2.2	2.6	2.3
P_2	$PE(10^{-2})$	4.3	5.5	4.3	2.0	3.9	2.4
	$RES(10^{-2})$	6.6	4.3	5.7	7.8	5.1	0.7
	RNG	1.9	3.2	2.3	1.6	2.4	1.8

FIGURE 4.17: Reconstructed images ($\mathbf{x}$) of the experimental setup (P_4) FIGURE 4.18: Thresholded images ($\hat{\mathbf{x}}$) of the experimental setup (P_4)

TABLE 4.2: Performance Evaluation of Two Objects

		$\lambda = 0.05$			$\lambda = 0.1$		
		NOSER	FER	Hybrid	NOSER	FER	Hybrid
P_3	$PE_1(10^{-2})$	5.3	4.1	4.1	2.9	5.4	4.1
	$PE_2(10^{-2})$	6.8	7.3	6.7	4.0	6.7	4.5
	$PE_{avg}(10^{-2})$	6.1	5.7	5.4	3.5	6.1	4.3
	$RES(10^{-2})$	12.0	9.3	11.2	13.5	8.4	11.7
	RNG	1.8	2.3	1.9	1.6	2.5	1.8
P_4	$PE_1(10^{-2})$	4.4	6.4	5.2	0.5	4.0	1.9
	$PE_2(10^{-2})$	14.1	13.1	14.1	3.7	4.8	3.0
	$PE_{avg}(10^{-2})$	9.3	9.8	9.7	2.1	4.4	2.5
	$RES(10^{-2})$	14.4	7.8	12.2	21.1	10.4	17.0
	RNG	1.4	2.9	1.7	0.9	2.0	1.2

FIGURE 4.19: Reconstructed images ($\mathbf{x}$) of the experimental setup (P_5) FIGURE 4.20: Thresholded images ($\hat{\mathbf{x}}$) of the experimental setup (P_5)

TABLE 4.3: Performance Evaluation of Three Objects

		$\lambda = 0.05$			$\lambda = 0.1$		
		NOSER	FER	Hybrid	NOSER	FER	Hybrid
P_5	$PE_1(10^{-2})$	8.4	10.4	8.7	9.8	9.3	9.8
	$PE_2(10^{-2})$	14.4	16.1	15.2	15.8	14.2	14.9
	$PE_3(10^{-2})$	15.3	15.9	15.7	7.0	13.5	10.8
	$PE_{avg}(10^{-2})$	12.7	14.1	13.2	10.9	12.3	11.8
	$RES(10^{-2})$	14.9	11.7	13.8	20.0	10.2	15.9
	RNG	1.4	1.8	1.5	1.3	2.2	1.6
P_6	$PE_1(10^{-2})$	20.3	23.7	20.8	16.9	19.6	18.0
	$PE_2(10^{-2})$	5.0	14.4	8.1	1.9	5.5	3.0
	$PE_3(10^{-2})$	7.0	11.8	8.8	3.3	6.8	4.5
	$PE_{avg}(10^{-2})$	10.8	16.6	12.6	7.4	10.6	8.5
	$RES(10^{-2})$	13.2	6.3	10.6	17.3	9.7	14.5
	RNG	1.8	3.7	2.3	1.3	2.4	1.6

FIGURE 4.21: Reconstructed images ($\mathbf{x}$) of the experimental setup (P_6) FIGURE 4.22: Thresholded images ($\hat{\mathbf{x}}$) of the experimental setup (P_6)

It can be observed from Tables 4.1-4.3 that the advantage of NOSER prior lies in lower position error and less ringing area than the FER prior due to lower noise amplification as highlighted in (4.21), and hence $\|\epsilon_{p,FER}\|_2^2 > \|\epsilon_{p,NOSER}\|_2^2$. On the other hand, the advantage of FER can be noticed by having better resolution performance than the NOSER since more information are being incorporated into the estimate solution as highlighted earlier in Figs. 4.6 and 4.7 where we can obtain $\|diag(\tilde{\Phi}'_{\lambda,FER})\|_0 > \|diag(\tilde{\Phi}'_{\lambda,NOSER})\|_0$. Based on the performance parameters in Tables 4.1-4.3, the difference between NOSER and the FER is more pronounce when we set the hyper-parameter to $\lambda = 0.1$ than the setting of $\lambda = 0.05$. This effect can be explained by Figs. 4.6 and 4.7 when we set $\lambda = 0.01$, the parameter spaces of NOSER and FER are spanned by $\sum_{i=1}^k \tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^T$ ($k = 13$ for NOSER and $k = 16$ for FER). In contrast, when we set $\lambda = 0.05$, the parameter spaces for NOSER and FER are spanned by $\sum_{i=1}^k \tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^T$ ($k = 16$ for NOSER and $k = 17$ for FER). It

can also be observed that the reconstructed image based on Hybrid regularization matrix has robust characteristic (interpolate performance between NOSER and FER). It is capable of achieving better position error and less ringing effect as compared to the FER while being able to obtain higher resolution performance as compared to the NOSER. The robust characteristic is attractive especially in highly dynamic environment (a method with robust and well-balanced performance is preferred in such scenario).

It is crucial to highlight that in this chapter the hyper-parameters were selected by two values (low and high) in order to underline different penalization levels of each regularization matrix. However, in practice the hyper-parameter are commonly chosen based on one of several approaches such as the Generalized Cross Validation (GCV), the L-curve, or the Noise Figure (NF). All of the above approaches are based on model-to-noise sensitivity or a trade-off between noise amplified solution and solution bias [36–38].

4.5 A Note on Sparsity Approach

For completeness, in this section we will be highlighting the results obtained under l_1 -norm regularization framework without detailed analysis. The objective of this section is only to provide readers with a broad picture (pros and cons of different approaches). However, it is important to note that the results obtained in this section (iterative image reconstruction) are entirely different from the results obtained in previous section (direct image reconstruction). In general, the results obtained by the “direct method” can be used on its own, or it can be used as a warm start (initial solution) to the iterative method.

Two l_1 -norm regularization functions will be used in this section which includes $\|\mathbf{x}\|_1$ and $\|\nabla\mathbf{x}\|_1$. In the first case, where $\|\mathbf{x}\|_1$ is used as a constraint, the estimate solution can be obtained by solving LASSO problem: $\mathbf{x}_\lambda = \min_{\mathbf{x}} \|\mathcal{J}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|_1$. As presented in Chapter 2, although the objective function is convex (both the fidelity term and the constraint), the constraint function for l_1 -norm regularization is non-smooth and hence non-differentiable. Therefore, the LASSO problem will be solved in an iterative manner where an Alternating Direction Method of Multipliers (ADMM) is used. In the second case, where $\|\nabla\mathbf{x}\|_1$ is used as a

constraint, the estimate solution can be obtained by solving total variation regularization problem: $\mathbf{x}_\lambda = \min_{\mathbf{x}} \|\mathcal{J}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda \|\nabla \mathbf{x}\|_1$ [39–41]. The total variation regularization will be solved by the Primal Dual-Interior Point Method (PD-IPM) for EIT [42].

The results obtained from LASSO problem by ADMM method are shown in Figs. 4.23 and 4.24 subsequently, which are the reconstructed images obtained from experimental setups $P_1 - P_6$. The augmented Lagrangian parameter ρ is set to 1 in both figures. The value λ_{max} is obtained by $\lambda_{max} = \|\mathcal{J}^T \mathbf{y}\|_\infty$. To highlight the effect of the regularization parameter λ , we plot the reconstructed images where λ is set differently in two cases. The results shown in Fig. 4.25 is obtained by setting $\lambda = 0.001\lambda_{max}$. Subsequently, the results shown in Fig. 4.26 is obtained by setting $\lambda = 0.01\lambda_{max}$.

FIGURE 4.23: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_6$) based on LASSO problem, where λ is set to $0.001\lambda_{max}$

The value set by λ/ρ plays a key role during the soft thresholding step. For a fixed value ρ , we can observe that when λ is set to a higher value (as can be seen in Fig. 4.24) the estimate solution has a more sparse characteristic as compared to when λ is set to a lower value (as can be seen in Fig. 4.23).

Next, we show the result obtained from total variation regularization. The reconstructed images from experimental setups $P_1 - P_6$ are shown in Fig. 4.25 where the regularization parameter λ is set to 0.001. To highlight the effect of λ value, we

FIGURE 4.24: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_6$) based on LASSO problem, where λ is set to $0.01\lambda_{max}$

also plot the image reconstructed where λ is set to a lower value at 0.0001 which is shown in Fig. 4.26. The maximum iteration of both cases are set to 10.

FIGURE 4.25: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_6$) based on total variation prior, where λ is set to 0.001

The total variation function $\|\nabla \mathbf{x}\|_1$ is known as an “edge preserving” prior because the sum of absolute gradient is being minimized (preserving piecewise value). As

can be seen in Figs. 4.25 and 4.26, the reconstructed images tend to have “block-like” structure (the estimated solutions favor towards piecewise structure). It can also be seen in Fig. 4.26 that the lower the λ parameter is set, the more refined (less blurry) and block-like structure the reconstructed image will be.

FIGURE 4.26: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_6$) based on total variation prior, where λ is set to 0.0001

It is important to note that although the sparsity approach may produce more refined result, it is obtained through an iterative process, and hence should not be used to compare with l_2 -norm regularization approach as a 1-to-1 comparison. In most cases, the l_1 -norm regularized solution is obtained by solving least square step (or regularized least square; similarly to the l_2 -norm solution) followed by feature selection step (parameter selection) through various means e.g. soft thresholding, hard thresholding, or maximum correlation. Therefore, the l_2 -norm regularization framework can often be considered as “part of” the l_1 -norm regularization procedure, which is the l_2 -norm regularized solution can be used as a warm start.

To obtain a “good” reconstructed image through l_1 -norm regularization also requires several hyper-parameters tuning (more than regular l_2), and often times the estimate solution will be quite sensitive to these hyper-parameters setting. In highly dynamic scenarios where the SNR can be varied the robust method based on l_2 -norm regularization is still widely used due to its robustness to the environment

and application constraint e.g. computational complexity or limited time between each image frame.

4.6 Conclusion

In this chapter, the working principle of two diagonal based regularization matrices namely NOSER and FER have been explored. Similarities and differences between the two priors are examined. We have shown that although both NOSER and FER have different approaches in prescribing the prior knowledge onto the estimated solution, they share similarity in the trend of prescribing the penalization intensity over different positions in the medium. In the SVD domain, the FER method allows more information to be incorporated into the estimated solution, hence it is able to achieve better resolution performance while being more susceptible to noise than the NOSER at the same time. With such observation, a hybrid regularization matrix which encompasses both prior knowledge has been proposed. Experimental results show that the proposed hybrid regularization matrix is capable of capturing the advantages of both priors while sharing the disadvantages of the two as well.

Note that the objective of the proposed method is to provide an alternative approach with “robust performance” that can interpolate the performance of both NOSER and FER methods. From the analysis, we observed that the NOSER method has a low ringing effect and low noise sensitivity (less sensitive to over-fitting), but it also has a poor resolution which may lead to difficulty differentiating closely spaced objects. In contrast, we also observed that although the FER method has higher resolution performance than the NOSER method, it is prone to high ringing effect and noise sensitivity. In practical scenarios when the background environment has dynamic noise characteristic, the proposed method can be used as an alternative approach which can provide a robust performance instead of selecting either NOSER or FER methods for different applications. In short, under highly dynamic environment or noisy scenarios, the proposed method will not produce an over-fitting model as high as the FER method. On the other hand, the proposed method will also provide a better resolution performance than the NOSER method in low noise scenarios. Therefore, to obtain a well-balanced and robust performance, the proposed method can be used as an alternative approach to the NOSER or the FER methods.

In this chapter, we also highlight the results obtained through the l_1 -norm regularization framework. Under l_1 -norm framework, the “local prior” or the prior based on actual objects inside the medium (conductivity distributions) can lead to better results. However, most methods that rely on local prior are operated based on iterative framework such as the orthogonal matching pursuit (greedy based), iterative re-weighted least square (iterative refinement), and LASSO (through FISTA/ADMM), all of which can be categorized under the sparse signal recovery framework. Through the iterative process, the prior based on actual conductivity distribution will be incorporated into the next iteration through a refinement process, and hence better image resolution can be obtained. In this chapter the major focus is placed on direct image reconstruction (non-iterative) scheme which uses “global prior” where the prior exploits a general knowledge of the environment (sensitivity for NOSER and orthogonality property for FER) instead of extracting specific prior of the actual conductivity distribution inside the medium. We also highlighted that although the l_1 -norm approach can provide more refined solution, it requires additional hyper-parameters setting which can sometimes be sensitive especially in highly dynamic environment. To maximize the robustness of the image reconstruction system, the l_2 -norm approach is still widely used, and our proposed method provides a good alternative robust method to the EIT application.

Chapter 5

Array Signal Processing Framework for Source Localization in Electrical Impedance Tomography

5.1 Introduction

From conventional direct image reconstruction viewpoint as presented in the last chapter, we showed that the estimate solution can be obtained by solving an underdetermined system of linear equations where the Jacobian matrix is severely ill-conditioned. Detail analysis presented in the last chapter also highlights the noise amplified issue that occurs during inversion process, and hence the estimate solution should be obtained under the regularization framework. The regularization framework introduces additional a prior information that is enforcing onto the possible solution space. This can also be interpreted from a statistical viewpoint (under AWGN assumption) as a Maximum A Posterior (MAP) estimate [38].

Although the conventional inverse problem can reconstruct a whole image on pixel-by-pixel basis, in several scenarios the objective can be reformulated to only a line search for high conductivity contrast (anomalies) over a relatively homogeneous background. This concept can also be applied to solve inverse problems in many science and engineering fields such as in biomedical or industrial applications where

the problem of anomalies detection and non-destructive testing are common. For example, in biomedical problem such as tumor localization or brain EIT, the area where solid tumor is located or the highly active cerebral area can be identified based on the knowledge of higher blood flow. This also corresponds to high conductivity profile over the background conductivity in that particular area. On the other hand, the area of the brain affected by ischemic stroke can be determined based on the knowledge of blocked blood flow which is equivalent to a decrease in conductivity profile in that area [14]. Additionally, in industrial problem such as in industrial pipeline monitoring, the structure of solid contaminants can be determined by small and bounded objects (areas) with distinct electrical characteristic profile over the relatively homogeneous background. With all the aforementioned cases earlier, the profile of conductivity variations (can be considered as anomalies) are structurally (spatially) bounded and do not take up a large area in the detection space. This means that the objective of estimating a whole image can be minimized to a detection and localization problem of bounded objects that have different conductivity profile over the background conductivity profile [43]. This is analogous to a joint support identification in sparse signal recovery or compressed sensing (CS) [44–47], or a source localization problem in DOA estimation. Multiple data snapshots (samples) can be recorded and collected in time or in frequency domain. This could be achieved based on multiple excitation architecture that is specifically developed for fast signal acquisition [48]. Based on multiple measurement vector (MMV) framework, many source localization algorithms can be applied to uncover the anomalies' locations.

The problem of parameter estimation long has been one of the main interests in signal processing research community. In array signal processing research, the ability to merged collected data that are obtained from multiple sensors while exploiting temporal information (space-time information) has exhibited many great successes in sonar and radar applications. Array beamforming is one of the most well-known methods for spatial filtering used in array signal processing field. Although source localization in array signal processing and image reconstruction in EIT have certain similarities (the utilization of sensor array and the parameter estimation), limited studies have been made to consolidate and merge all of the techniques from the two research fields. In the literature, quite a number of state-of-the-art techniques for EIT image reconstruction is still around the regularization framework with different norms and priors (regularization matrix). Although several compressed

sensing techniques based on the l_1 -norm minimization have gain some tractions for research direction, the conventional l_2 -norm minimization framework is still in use in many scenarios because of its robustness and reliability.

Amongst numerous spectral-based DOA estimation methods e.g. the capon's beamforming (delay-and-sum), the minimum variance distortionless response (MVDR) beamforming, the method called multiple signal classification (MUSIC) algorithm is one of the most well-known subspace based methods that has super resolution performance capable of identifying closely situated targets (resolveability) [50, 51]. With the capability to focus the beamforming's output on the desired signal while at the same time suppressing other noises and interferences, many beamforming techniques that are originally used in DOA estimation have been adopted to use in imaging problem under target localization framework. It was shown in [52] that the MUSIC algorithm can be adopted and used as the alternative method for solving inverse problem under ill-posed condition such in diffuse optical tomography (DOT), where the MUSIC's performance outperforms the conventional compressive sensing techniques. The MUSIC algorithm was also successfully adopted and used for solving target detection and localization problem of breast tumor under quasi-static environment [17]. The LCMV algorithm, a variation of MVDR algorithm, was also proposed for determining the area of conductivity changes in human head [53], where the simulation study was later expanded to an experimental study with phantom tank [54].

It should be noted that in sparse signal recovery and compressed sensing theory, the ability to identify a set of linear combination between support set (atoms), from a pool of large dictionary (overcomplete basis) has enabled us to reconstruct (estimate) an original signal with only small and limited number of measurements. However, in certain scenarios where the sensing matrix $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_N]$ does not admit a restricted isometry property (RIP) which is typically defined as

$$(1 - \delta_k) \|\mathbf{x}\|^2 \leq \|\mathbf{A}_k \mathbf{x}\|^2 \leq (1 + \delta_k) \|\mathbf{x}\|^2, \quad (5.1)$$

where a sub-matrix $\mathbf{A}_k$ is sampled from the original dictionary $\mathbf{A}$. For EIT problem, the mutual coherence (a weaker measure than RIP) does not provide a meaningful insight, where the mutual coherence can be defined by

$$\mu = \max_{1 \leq j \leq k \leq n} \frac{\mathbf{a}_j^H \mathbf{a}_k}{\|\mathbf{a}_j\| \|\mathbf{a}_k\|}. \quad (5.2)$$

For a normalized sensing matrix that satisfies $\|\mathbf{a}_i\|_2 = 1$, it is evidently that in the scenarios with highly coherent atoms $\mathbf{a}_i^H \mathbf{a}_{i+1} \approx 1$, which leads to $\mu \approx 1$. Within this chapter, there will be no assumption regarding the mutual coherence or the RIP, and hence a high resolution performance which can be translated to high resolving power in DOA estimation sense is crucial since it addresses the capability of an algorithm to distinctively determine nearby atoms with high collinearity.

5.2 Forward Problem & Signal Model

The same forward model and measurement protocol introduced in Fig. 4.1 and (4.1) will also be applied in this chapter. A difference potential data either in time domain or in frequency domain can be used in real application by computing the discrepancy between the perturbed potential field against the referenced potential data. It was shown in [14] that when the conductivity variation is bounded (area) and spatially sparse (few bounded areas), the perturbed potential field received by the array sensors can be regarded as an electric field generated by an induced current source. The background conductivity value is set to $\sigma = 1$ for simplicity, and the Laplace equation in (4.1) can be re-written as

$$\nabla \cdot [(\sigma + \delta\sigma)\nabla(\phi + \delta\phi)] = 0 \rightarrow \nabla \cdot \nabla\delta\phi = \nabla \cdot (\delta\sigma|_p \nabla\phi), \quad (5.3)$$

where we use $\sigma \rightarrow \sigma + \delta\sigma$ and $\phi \rightarrow \phi + \delta\phi$. The whole medium Ω is then discretized into small pixels (2D) or voxels (3D) as $\Omega = \bigcup_{i=1}^N \xi_i$ where each ξ_i defines a position of each pixel/voxel and the number N is the amount of all pixel/voxel in the whole image. The perturbed potential field $\delta\phi$ is generated by small conductivity variation $\delta\sigma|_p$ situated at p^{th} pixel. Finally, the received signal snapshots can be modeled under linear approximation as

$$\mathbf{x}(n) = \mathbf{A}(\Xi)\mathbf{s}(n) + \mathbf{n}(n), \quad (5.4)$$

where vector $\mathbf{x}(n) \in R^{M \times 1}$ is single snapshot of the measurement vector $\delta \mathbf{v}$, matrix $\mathbf{A}(\Xi) \in R^{M \times N}$ is the sensing matrix, vector $\mathbf{s}(n) \in R^{N \times 1}$ is single snapshot of conductivity variation profile (also known as source vector), and vector $\mathbf{n}(n) \in R^{M \times 1}$ is a combination of additive white Gaussian noise and unmodeled error (higher order term of Taylor's expansion).

FIGURE 5.1: Sparsity structure of the conductivity variations in the medium over different time snapshots.

As can be seen on the left side of Fig. 5.1, there are two targets (sources) specified by small greyscale circles situated inside the medium which is the large circle for multiple data snapshots (t_i). The value of conductivity profile can be varied across different data snapshots (specified by different color of greyscale circles). The source matrix as shown in the right-top of Fig. 5.1 can be considered as having a joint sparse structure as shown in the right-bottom of Fig. 5.1 where each column of the matrix shares the same nonzero locations (greyscale indices). The greyscale indices of each column are essentially the conductivity variations correspond to the source locations for each data snapshots as shown in the left side of Fig. 5.1. The MMV formulation of (5.4) can now be re-written as

$$\mathbf{X} = \mathbf{A}(\Xi)\mathbf{S} + \mathbf{N} \longrightarrow \mathbf{X} = \mathbf{A}(\Xi_p)\mathbf{S}_p + \mathbf{N}, \quad (5.5)$$

The sensing matrix $\mathbf{A}(\Xi)$ and the source matrix $\mathbf{S}$ can be re-written as $\mathbf{A}(\Xi_p) \in R^{M \times p}$ and $\mathbf{S}_p \in R^{p \times K}$, when p defines a set where its members belong to the indices of non-zero rows of the matrix. The matrix $\mathbf{A}(\Xi_p) = [\mathbf{a}(\xi_1), \dots, \mathbf{a}(\xi_p)]$ can be regarded as an array manifold matrix comprising of p column vectors of length M known as steering vector or array response.

In contrast to the steering vector that is used in DOA estimation that is based on a solution of a far-field wave equation, the EIT imaging system operates at

a lower frequency band. All of the mathematical assumptions used in high frequency case such as far-field, point source, and plane wave are not valid for EIT environment. The principle behind a calculation of steering vector/array response for EIT application can be considered as an estimation of array response based on small conductivity variation at each pixel/voxel. The steering vector $\mathbf{a}(\xi_i) \in R^{M \times 1}$ used to construct the array manifold was estimated by an expected potential variation receiving on the array electrodes due to a variation at pixel ξ_i . However, because the signal strength in EIT application is a function of source's location, the steering vector/array response used in this chapter is proposed to be the equalized/normalized potential variation which can be defined as

$$\mathbf{a}(\xi_i) = [\underbrace{a_1^1(\xi_i), \dots, a_{L-3}^1(\xi_i)}_{1^{st} \text{ Induced Polarity}}, \dots, \underbrace{a_1^L(\xi_i), \dots, a_{L-3}^L(\xi_i)}_{L^{th} \text{ Induced Polarity}}]^T, \quad (5.6)$$

where $a_1^j(\xi_i), \dots, a_{L-3}^j(\xi_i)$ are the normalized (expected) potential variation due to induced polarity source situated at j^{th} pixel/voxel. Each component of the steering vector can be obtained by solving the following forward problem

$$a_k^j(\xi_i) = \frac{\int_{\xi_i} \nabla \phi^k \nabla \phi^j dr}{\sqrt{\sum_{j=1}^L \sum_{k \notin j-1, j, j+1} |\int_{\xi_i} \nabla \phi^k \nabla \phi^j dr|^2}}. \quad (5.7)$$

The steering vector/array response specified in (5.6) has an analogous meaning to the lead field vector used in active brain signal localization. However, in our case all of the information regarding the source's polarities are now being incorporated into single vector because of the rotation of current injection and measurement protocol.

It is important to highlight here that the concept of steering vector/array response for DOA estimation in RADAR/SONAR application is similar to the steering vector for EIT application. Although the process of calculating the expected array response is different, both field of applications are relying on the knowledge of array geometry (how sensors are placed). In RADAR/SONAR application, many assumptions were used a priori, and hence the solution is expressible analytically. On the other hand, the formulation that is used in EIT application requires multiple boundary conditions to be satisfied and no analytical solution is available. The steering vector for EIT application needs to be obtained through a numerical

solution based on Finite Element Method (FEM). Consider an infinite number of data snapshots with zero mean, the array covariance matrix is expressible as

$$\mathbf{R} = \mathbf{A}(\Xi_p)\mathbf{R}_{ss}\mathbf{A}(\Xi_p)^T + \sigma_n^2\mathbf{I}_M, \quad (5.8)$$

where the data covariance matrix $\mathbf{R} = E(\mathbf{X}\mathbf{X}^T)$ is expressible by the source covariance matrix $\mathbf{R}_{ss} = E(\mathbf{S}_p\mathbf{S}_p^T)$ and noise covariance matrix $\sigma_n^2\mathbf{I}_M = E(\mathbf{N}\mathbf{N}^T)$, where $E(\cdot)$ is the expectation operation. During the next section, multiple factors that affect the source signal power which also influences the source localization performance will be analyzed.

5.3 Analysis of Signal Power in EIT

Two main factors that are affecting the eigenvalues extracted from the data covariance matrix along with the analysis of ill-conditioned array manifold will be discussed in this section. Two matrix decompositions, the eigenvalue decomposition (EVD) and the singular value decomposition (SVD), are firstly defined which will later be used in for full eigenvalues analysis. An $M \times p$ array manifold can be compactly decomposed by the SVD as $\mathbf{A}(\Xi_p) = \mathbf{U}_r\mathbf{\Sigma}_r\mathbf{V}_r^T$, where the left singular vectors spanning data range space are forming the matrix $\mathbf{U}_r \in R^{M \times r}$. A diagonal matrix $\mathbf{\Sigma}_r \in R^{r \times r}$ with singular values along the diagonal entries are arranged in a non-increasing order. The right singular vectors which span the model range space are forming a matrix $\mathbf{V}_r^T \in R^{r \times p}$. Next, an $M \times M$ data covariance matrix can be decomposed by the EVD as $\mathbf{R} = [\tilde{\mathbf{U}}_r|\tilde{\mathbf{U}}_0][\mathbf{\Lambda}_r|\mathbf{\Lambda}_0][\tilde{\mathbf{U}}_r|\tilde{\mathbf{U}}_0]^T$ where the matrix $\tilde{\mathbf{U}}_r \in R^{M \times r}$ is formed by eigenvectors spanning signal subspace that also span the same range space $\mathbf{U}_r$. The matrix $\tilde{\mathbf{U}}_0 \in R^{M \times M-r}$ is formed by eigenvectors spanning noise subspace. The diagonal matrices $\mathbf{\Lambda}_r$ and $\mathbf{\Lambda}_0$ consist of eigenvalues correspond to the eigenvectors spanning signal and noise subspaces respectively. Both of the diagonal matrices are specified in a non-increasing order. The source's signal power can be varied due to multiple factors e.g. the frequency used, the level and intensity of current injection, the level of conductivity variation profile (relative change in value), the size of source/target (conductivity variation), the source/target's location, and the collinearity between steering vectors. Next, two of the most

general factors but of great importance for the analysis will first be discussed and then followed by several scenarios that could be affecting such factors.

First Factor

Location of the source/target (area of conductivity variation). The signal's energy emitted from the induced source is diffused to all across the whole domain within the medium. This leads to a complication when we localize the source that are situated further away from the boundary or the deep source due to low signal sensitivity. Let's consider two sources both of which are of equivalent conductivity variation intensity (equivalent signal power), one of which situated closer to the boundary (near sensors) while another is situated further away from the boundary (far from sensors). The data covariance matrix from (5.8) can be expanded by the SVD as the following

$$\mathbf{R} = [\mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^T] \sigma_s^2 \mathbf{I}_2 [\mathbf{V}_r \mathbf{\Sigma}_r \mathbf{U}_r^T] + \sigma_n^2 \mathbf{I}_M. \quad (5.9)$$

It can be seen that the singular values in matrix $\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2)$ are not equivalent ($\sigma_1 > \sigma_2$), and hence the data covariance matrix can be re-expressed as

$$\mathbf{R} = \tilde{\mathbf{U}} \begin{bmatrix} (\sigma_1 \sigma_s)^2 + \sigma_n^2 & 0 & 0 & 0 \\ 0 & (\sigma_2 \sigma_s)^2 + \sigma_n^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_n^2 \end{bmatrix} \tilde{\mathbf{U}}^T. \quad (5.10)$$

Although the conductivity variation intensity of both sources are at the same level (equivalent), as can be seen in (5.10) that the second eigenvalue is lower than the first $\lambda_2 = (\sigma_2 \sigma_s)^2 + \sigma_n^2$.

Second Factor

The collinearity between steering vectors of the array manifold. When the nearby pixels generate/omit similar array response profile, the column space of the array manifold becomes dependent, and hence matrix $\mathbf{A}(\Xi_p)$ becomes ill-conditioned due to collinearity between each column. Let's consider three point

sources where two of which are situated close to one another, and the last one is situated independently to the previous two sources. For the case of equivalent distance between all sources to the boundary, the signal powers would be comparable $\sigma_1^2 \approx \sigma_2^2 \approx \sigma_3^2$. A data snapshot of received signal upon the array sensors can be re-written and expanded under the SVD domain as

$$\mathbf{x} = \mathbf{U}_r \begin{bmatrix} \sigma_1 + \delta & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \mathbf{V}_r^T \mathbf{s}_p + \mathbf{n}, \quad (5.11)$$

where the value of ϵ is typically small because of high collinearity between two of the steering vectors. The dimension of data range space $\mathbf{U}_r$ is now only limited to two with an increased value on the first singular value. The data covariance matrix of the received signal in (5.11) can be expressed as

$$\mathbf{R} = \tilde{\mathbf{U}} \left[\begin{array}{ccc|c} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ \hline 0 & 0 & 0 & \mathbf{\Lambda}_0 \end{array} \right] \tilde{\mathbf{U}}^T, \quad (5.12)$$

where the matrix $\lambda_1 = (\{\sigma_1 + \delta\}\sigma_s)^2 + \sigma_n^2$, $\lambda_2 = (\sigma_2\sigma_s)^2 + \sigma_n^2$, $\lambda_3 = (\epsilon\sigma_3)^2 + \sigma_n^2$. With small positive real value ϵ , λ_3 will now become part of $\mathbf{\Lambda}_0$ (absorbed). It is important to note that the ill-conditioning effect on the array manifold does not directly cause the data covariance matrix to become ill-conditioned because $\lambda_{\mathbf{R},min}$ is always part of the noise eigenvalues in matrix $\mathbf{\Lambda}_0$, and hence it will not become singular. However, the level of eigenvalues associated with the signal subspace $\mathbf{\Lambda}_r$ can be greatly varied because of both factors mentioned earlier. Before we further explore three different scenarios (cases) affected by these important two factors, we first need to understand that the conductivity variation (intensity) produces similar signal perturbation field upon the array sensors on the boundary which is analogous to the scenario (case) of the conductivity size variation. In Fig. 5.2 (left) and (right), four snapshots of conductivity variations (intensity) along with four snapshots of conductivity size variations are shown.

FIGURE 5.2: Sources with conductivity variations (left) and size variations (right) between snapshots.

Four data snapshots were generated according to the conductivity variation ($\delta \mathbf{v}_c(t_i)$) and another four snapshots for the size variation ($\delta \mathbf{v}_s(t_i)$). For an 8-electrode system, the number of measurement obtained from one snapshot is 40 which can be considered as multichannel signal.

FIGURE 5.3: Signal perturbation due to conductivity variations and size variations.

In Fig. 5.3, the data snapshots were collected and averaged across multiple samples for each scenario. The magnitude of the averaged data are shown in Fig. 5.3 which labeled as (*) for conductivity variation (intensity) while (o) is labeled for the size variation. It can be seen in Fig. 5.3 that both scenarios lead to similar signal perturbation intensity upon the array sensor especially at channels number 1, 6, 11, 17, 24, 30, 35, and 40 where most of high signal perturbation appeared.

Since conductivity variation (intensity) and conductivity size variation can produce similar effect upon the array sensor, only the aspect of conductivity variation (intensity) will be further discussed within this chapter along with the location of source for each scenario/case.

Case I

Let's consider two scenarios with two independent sources. In the first scenario the two independent sources are situated closer to the boundary (sensors area) as shown in Fig. 5.4(a). In the second scenario the two independent sources are situated further away from the boundary as shown in Fig. 5.4(b).

As can be seen in Fig. 5.5, the signal powers of those sources that are situated closer to the boundary are fundamentally higher than signal powers of those sources that are situated further away from the boundary (as marked by $(\circ)$ and $(*)$ respectively). In addition, under the same setting but with decrease in conductivity variation intensity, the signal powers of both cases also decrease as can be seen in Fig. 5.5. (as marked by $(\diamond)$ and $(\square)$).

Case II

Let's consider p number of sources which is a combination of q non-coherent sources ($q < p$) along with $p - q$ partial coherent sources. A single data snapshot can be expressed as

$$\mathbf{x} = \sum_{i=1}^q \sigma_i(\mathbf{v}_i^T \mathbf{s}_p) \mathbf{u}_i + \sum_{j=q+1}^p \sigma_j(\mathbf{v}_j^T \mathbf{s}_p) \mathbf{u}_j + \mathbf{n}. \quad (5.13)$$

For this scenario when the terms $\forall \sigma_j \approx \epsilon$ are small positive real values, the second term and the third terms as expressed in (5.13) will be integrated and merged especially when the conductivity variation (intensity) is low (low SNR) or in the scenario of deep source.

In Fig. 5.6(a), an example of 3 independent sources is shown while Fig. 5.6(b) addresses the scenario of two non-coherent sources that are situated together with one partial coherent source inside the medium. Fig. 5.7 shows the eigenvalues of

(a) Near boundary sources I,II

(b) Deep sources I,II

FIGURE 5.4: Example of independent sources from *Case I*.

both scenarios which are marked by $(\circ)$ and $(*)$ respectively. As can be seen in Fig. 5.7, one of the eigenvalues for partial coherent case has been dropped.

It is important to highlight a distinction between partial coherent source in this case and highly coherent source in the next case. For the case of partial coherent source, the value projected in the first eigenvalue is not increased as marked in Fig. 5.7 as "Equivalent Signal Powers". However, it is not true in the case of highly coherent source which will be discussed as our next case.

FIGURE 5.5: The first ten eigenvalues of the data covariance matrix obtained from non coherent case.

Case III

Fig. 5.8(a) shows an example of three independent sources where its eigenvalues are shown in Fig. 5.9 as (o) subsequently.

In the scenario when one of the sources is highly coherent to the other as shown in Fig. 5.8(b), the signal power of the first eigenvalue is evidently increased (as λ_1 in (5.12)) due to collinearity between the two steering vectors as marked in Fig. 5.9 by a label "Increased Signal Power". The eigenvalue marked as "Highly Coherent Source" in Fig. 5.9 has also dropped (as λ_3 in (5.12)) to the level equivalent to noise power. This can also be regarded as a merging between two highly coherent sources into one distributed source.

Based on various cases discussed earlier, the occurrence of ill-conditioned array manifold can be found embedded within the data covariance matrix. With high variation in eigenvalues depending on several factors and environment, the performance of spectral based algorithms would be affected. An algorithm called MUSIC-like algorithm, however, does not depend on the subspace segregation and model order estimation. It also has the ability to incorporate certain priors into the problem formulation. Therefore, we will later show that MUSIC-like algorithm

(a) Independent sources

(b) Partial coherent sources

FIGURE 5.6: Comparison of independent sources and partial coherent sources: Example for *Case II*.

has the ability to adaptively regulate its performance under different environments and circumstances.

5.4 Joint Sparse Signal Recovery & Source Localization in EIT

Although the MUSIC algorithm seems promising due to its high resolvability performance, its dependency to subspace segregation can often lead to performance

FIGURE 5.7: The first ten eigenvalues of the data covariance matrix obtained from partial coherent case.

degradation issue. This issue can occur when a perfect and accurate subspace estimation cannot be obtained due to a lack of knowledge regarding the number of sources or because an incorrect model order estimation. These circumstances commonly arise when the obtained data sample is insufficient or when the noise level in the environment is high (low SNR). Especially for EIT environment where the power of signal source can be greatly varied because of multiple reasons. This variation can lead to the issue of incorrect model order estimation. To maintain the super resolution performance achieved by the MUSIC algorithm while eliminating certain strong requirement on the subspace segregation, a DOA estimation problem was posed under the beamforming framework and called a MUSIC-like algorithm. This algorithm does not require model order estimation [11, 12], and given such robust characteristic, the MUSIC-like algorithm will be used as an alternative method to the MUSIC algorithm specifically for EIT environment.

Next, the formulation of beamforming algorithm namely MUSIC-like algorithm adapted to EIT application will be briefly reviewed. It is important to address that the optimization problem presented here is aligned with the one presented in [11, 12] for DOA estimation used in RADAR and SONAR application. The difference lies in geometrical structure of the array sensors is being used for EIT problem which comprises the adjacent current injection, adjacent measurement

(a) Three independent sources

(b) Two independent with one highly coherent sources

FIGURE 5.8: Comparison of independent sources and highly coherent sources: Example for *Case III*.

protocol, and the conductivity variation information being incorporated within the array response $\mathbf{a}(\xi_i)$ where ξ_i is the position of conductivity variation in the discretized medium. Multiple data snapshots generated by p stationary signal sources with AWGN added are captured. From the data covariance matrix specified in (5.8), the optimization problem of MUSIC-like algorithm for EIT can be defined as

$$\begin{aligned} \min_{\mathbf{w}_{\xi_i}} \quad & \mathbf{w}^T \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T \mathbf{a}(\xi_i) \mathbf{a}(\xi_i)^T \mathbf{w} + \beta \|\mathbf{w}\|_2^2 = c, \end{aligned} \quad (5.14)$$

FIGURE 5.9: The first ten eigenvalues of the data covariance matrix obtained from highly coherent case.

where the weight vector $\mathbf{w}$ is the solution obtained from the optimization problem above, a steering vector $\mathbf{a}(\xi)$ is satisfying a normalized euclidean norm $\|\mathbf{a}(\xi_i)\|_2 = 1$, the relaxation parameter β is a control parameter that allows some relaxation onto the constraint, and c is a real positive constant value. The Lagrangian function for the optimization in (5.14) can be expressed as

$$L(\mathbf{w}, \lambda) = \mathbf{w}^T \mathbf{R} \mathbf{w} - \lambda(\mathbf{w}^T (\mathbf{a}(\xi_i) \mathbf{a}(\xi_i)^T + \beta \mathbf{I}) \mathbf{w} - c). \quad (5.15)$$

Let $\frac{\partial L(\mathbf{w}, \lambda)}{\partial \mathbf{w}^T} = \mathbf{R} \mathbf{w} - \lambda(\mathbf{a}(\xi_i) \mathbf{a}(\xi_i)^T + \beta \mathbf{I}) \mathbf{w} = 0$, the weight vector solution $\mathbf{w}$ is the general eigenvector of the matrix pair $(\mathbf{R}, (\mathbf{a}(\xi_i) \mathbf{a}(\xi_i)^T + \beta \mathbf{I}))$:

$$\mathbf{R} \mathbf{w} = \lambda(\mathbf{a}(\xi_i) \mathbf{a}(\xi_i)^T + \beta \mathbf{I}) \mathbf{w}, \quad (5.16)$$

which is equivalent to the following expression

$$(\mathbf{a}(\xi_i) \mathbf{a}(\xi_i)^T + \beta \mathbf{I})^{-1} \mathbf{R} \mathbf{w} = \lambda \mathbf{w}. \quad (5.17)$$

The weight vector solution $\mathbf{w}$ is now defined as the eigenvector corresponding to the minimum eigenvalue λ_{min} , and hence the spatial spectrum (direction finding function) of the MUSIC-like algorithm can be defined as

$$P(\xi_i) = 10 \log_{10} \frac{1}{|\mathbf{w}^T \mathbf{a}(\xi_i)|^2}. \quad (5.18)$$

The spatial spectrum (or sometimes called pseudospectrum) presented here can be obtained by scanning each pixel in the whole medium (field-of-view), and hence the result can be used as our reconstructed image. The bound of our relaxation parameter β was proposed in [12] as

$$\underbrace{\max_{p \in P} \frac{\lambda_{\mathbf{R}, \min}}{(\mathbf{a}(\xi_i)^T \mathbf{R}^{-1} \mathbf{a}(\xi_i))^{-1}}}_{\beta_{\min}} < \beta < \underbrace{\min_{p \notin P} \frac{\lambda_{\mathbf{R}, \min}}{(\mathbf{a}(\xi_i)^T \mathbf{R}^{-1} \mathbf{a}(\xi_i))^{-1}}}_{\beta_{\max}} \quad (5.19)$$

where the choice of β is set to be a value between $\beta_{\min}$ and $\beta_{\max}$ which is defined as

$$\beta = (1 - \alpha)\beta_{\min} + \alpha\beta_{\max}, \quad (5.20)$$

where $0 < \alpha < 1$ to be chosen by $\alpha = \beta_{\min}/\beta_{\max}$. It is important to highlight here that despite the objective function of MVDR and MUSIC-like algorithm are alike, the constraint used by both algorithms are very different. With a relaxation parameter β introduced into the formulation, the MUSIC-like algorithm has the ability to automatically re-adjust its performance in accordance to the signal and noise level (or SNR level within the environment) estimated by $\beta_{\min}$ and $\beta_{\max}$. When the noise level gets higher ($\beta_{\min}$ increases), the value estimated by β will be re-adjusted accordingly to be a higher value and vice versa. The end result of this influences the weight vector solution to be resided in the noise subspace. The MUSIC-like algorithm is therefore capable of producing spatial spectrum analogous to that of the MUSIC algorithm performance without requiring subspace segregation. The spatial spectrum of the MVDR algorithm, however, depends heavily on the signal power. This limits the performance of the MVDR to approach the MUSIC algorithm performance only under high SNR circumstances with infinite amount of data snapshots.

5.5 Simulation Results

Next, multiple scenarios will be simulated in order to highlight how several factors previously mentioned affect the performance of related algorithms. The forward calculation is obtained under Finite Element Method (FEM) which is provided within the platform developed and published in [34]. Simulation signal are generated from a discretized circular model of size 1,600 pixels when 8 array sensors are used with AWGN added. Let d represent a diameter of the circular medium, the medium model is normalized to a unit circle with radius $r = 1$. A matrix of measurement data $\mathbf{X}$ of size 40×100 is recorded. Four algorithms for the comparison namely One-Step Gauss Newton with Laplace prior, MVDR, MUSIC, and MUSIC-like algorithm will be used. The regularization parameter (hyper-parameter) of the One Step GN with Laplace prior is selected based on a method called Generalized Cross-Validation (GCV) which uses multiple data snapshots [55]. Since there is high variation within multiple snapshots of signal source, the median value of their regularization parameter is selected for robustness. A conductivity variation of size limited to area of 1.5% relative to the whole medium is located at $r = 0.2$ and $\theta = 0^\circ$ (polar coordinate), respectively. The intensity of conductivity variation is varied with maximum variation limited at 20% above the reference conductivity value (background). Additional white Gaussian noise with the intensity of $\sigma_n = 200\mu V$ is added to the simulated signal. The reconstructed image under the regularization framework and the beamforming techniques are shown in Fig. 5.10. It can be seen in Fig. 5.10 that the MUSIC-like algorithm is capable of achieving high resolvability performance when compared with the other methods.

FIGURE 5.10: A comparison between an image reconstructed by GN-Laplace, MVDR, MUSIC and MUSIC-like algorithm for single source.

The advantage of an algorithm with high resolution (resolvability) performance is shown in Fig. 5.11 when two (or more) sources are presented in the medium. One of the sources is situated at location $r = 0$ and $\theta = 0^\circ$ while another source is situated at the location $r = 0.4$ and $\theta = 0^\circ$. For this scenario, the conductivity size is bounded within the area of 1% over the entire area. The maximum conductivity intensity variation is limited to 10% higher relative to the background value where additional noise is added at around $\sigma_n = 10\mu V$.

FIGURE 5.11: A comparison between an image reconstructed by GN-Laplace, MVDR, MUSIC and MUSIC-like algorithm for two sources.

The regularization based approach is unable to distinctly detect both sources as can be seen in Fig. 5.11. In contrast, the MUSIC-like algorithm is able to achieve the best resolution (resolvability) performance where two distinct peaks are well separated followed by MUSIC and MVDR algorithm. The resolvability between the three DOA estimation techniques are promising where both of the sources can be distinctively identified. The resolving power of MVDR algorithm is known to be dependent to signal power (and SNR in general). In the condition where the SNR is relatively low, the discrepancy between the two peaks (resolvability) will not be well separated although both sources can be distinctively identified. The resolving power of MUSIC algorithm is commonly high, but in low SNR scenario, a performance degradation is expected. In contrast, the MUSIC-like algorithm can achieve good resolution performance even in low SNR scenario since it does not depend on subspace segregation.

In this section, two performance parameters which are the resolution and the root-mean-squared error (RMSE) of the estimated center-of-gravity (CoG) will be used

to evaluate the performance of each algorithm. After we have obtained the reconstructed image, the pixel values across the whole medium will be normalized in the range of $[0,1]$ (denoted as $\mathbf{s}_{rec}$). It is then thresholded at 50% which is now denoted as $\hat{\mathbf{s}}$. The resolution performance which reflects the compactness of a reconstructed object (defined in [35]) can be obtained by computing

$$\text{Resolution} = \frac{A_{\hat{\mathbf{s}}}}{A_0}, \quad (5.21)$$

where $A_{\hat{\mathbf{s}}}$ defines the normalized and thresholded area within an image which is related to the object (source) inside the medium, and A_0 defines the total area of the medium. Due to the fact that signal power can be varied depending on the location of each source, the performance parameter will be evaluated as follows. At each location where a source is situated from around the center ($r = 0$ and $\theta = 0^\circ$) to the area near the boundary ($r = 0.8$ and $\theta = 0^\circ$), each performance parameter will be evaluated under the Monte Carlo simulation of $N = 200$ trials. The performance parameter is then averaged across multiple locations. The calculation is reiterated based on different noise levels which includes $\sigma_n = 200\mu V, 150\mu V, 100\mu V, 50\mu V, 10\mu V, 5\mu V$, and $1\mu V$. When the signal power is fixed, having different noise levels is analogous to having different SNR levels.

FIGURE 5.12: The resolution of each algorithm based on different noise levels for the one source.

Firstly, the resolution performance is shown in Fig. 5.12, where lower value means higher specificity (high resolution). As can be seen in Figs. 5.10 and 5.11, the MUSIC-like algorithm provides the highest resolution performance followed by MUSIC, MVDR, and One Step GN-Laplace algorithm (can also be seen in Fig. 5.12). In Fig. 5.12, it can be seen that as SNR gets higher, the resolution performance of the all algorithms improve accordingly.

The center of gravity (CoG) of the source in the ground truth image is denoted as ξ_{CoG} , and the CoG of the reconstructed image (estimate solution) is denoted as $\hat{\xi}_{CoG}$, respectively. The position error metric (defined in [35]) is can be obtained by calculating $PE(\xi_{CoG}) = \hat{\xi}_{CoG} - \xi_{CoG}$. The RMSE can be calculated as follows:

$$RMSE(\xi_{CoG}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{\xi}_{CoG_i} - \xi_{CoG_i})^2}. \quad (5.22)$$

FIGURE 5.13: The average RMSE for CoG of each algorithm under different noise levels for the case of one source.

The RMSE (CoG) of the regularization framework has the lowest average error as can be seen in Fig. 5.13,. Amongst the source localization algorithms, the MUSIC-like provides the lowest average error followed by MUSIC and MVDR algorithm.

As the SNR level gets higher, the performance of source localization algorithms improves accordingly while the performance of the regularization framework exhibits certain variation across noise levels for the case of single source.

FIGURE 5.14: The resolution of each algorithm under different noise levels for the case of two sources.

FIGURE 5.15: The average RMSE for CoGs of each algorithm under different noise levels for the case of two sources.

The performance parameters for the case of two sources are further evaluated and presented in Figs. 5.14 and 5.15. Each source is bounded within the area of 4% over the entire area with maximum conductivity variation of 20% over the background. One of the sources is located at $r = 0.8$ and $\theta = 0^\circ$ while the other source is varied from $r = 0.8$ with $\theta = 180^\circ$ to $r = 0$ with $\theta = 0^\circ$. Each performance parameter is evaluated under Monte Carlo simulation of $N = 200$ trials over different positions and noise levels. It can be seen in Figs. 5.14 and 5.15 that the results follow the same trend as for the case of one source where the MUSIC-like algorithm is able to achieve the highest resolution performance followed by the MUSIC, MVDR, and regularization method. The regularization approach is however able to achieve the lowest average CoGs error followed by the MUSIC-like, MUSIC, and MVDR algorithm.

5.6 Experimental Results

In this section, experimental results are presented in order to show the possibility of applying the MUSIC-like algorithm to EIT application. Further development for real application can be implemented with more realistic 3D model design where the forward solution is solved by FEM with specific mesh implementation for each application. In this experiment, an 8-electrode EIT system prototype was constructed with a polyethylene container of size 25 cm in diameter as shown in Fig. 5.16.

After the water was filled in the tank, a current of 1 mA at 1 kHz from improved Howland current source was injected with the adjacent injection and measurement method. With 8 electrodes, the measurement vector obtained by each snapshot is a vector of length 40×1 . Several square and round shape metallic rods made by aluminum, copper, and stainless steel shown in Fig. ?? were used to create

FIGURE 5.16: An 8-electrode EIT system prototype.

different data snapshots. Each rod was used to emulate a conductivity variation found in real application where the conductivity variation occurs over different time or frequencies.

FIGURE 5.18: Three experimental setups with sources located at different positions.

In the experiment, all rods were paired up in a round-robin fashion where the pair $\{1, 2\}$ is not equivalent to the pair $\{2, 1\}$ in our case since they are located at different locations. As with the norm [56], the number of snapshots required should be more than the number of measurement M (40 in our case). With 13 rods, 156 combinations of rod pairs were generated and each pair was used to generate one data snapshot for each position setup (P_1 , P_2 , and P_3) as shown in Fig. 5.18. For example, considering a position setup P_1 . Both positions (at O_1 and O_2) will be places with a pair of 2 rods at a time (consider this as one snapshot).

A set of 156 combinations can be obtained e.g. $\{(1,2), (1,3), \dots, (1,13), (2,1), (2,3), \dots, (2,13), \dots, (13,1), (13,2), \dots, (13,12)\}$. Note that the pair (1,2) and (2,1) are considered different since each of the rod is positioned different location. These rods combination help to produce a variation of signal for each snapshot so that different snapshots of data will not be identical to each other. Although with this variation, the data captured between different snapshots still exhibit certain degree of collinearity. The collinearity between each data snapshot contributes to high condition number of the covariance matrix ($\kappa(R) \approx 1000$). In order to mitigate the ill-conditioned problem in the covariance matrix as well as to improve the robustness of the algorithms performance, we introduce diagonal loading term into the original covariance matrix as $\mathbf{R}_{DL} = \mathbf{R} + \epsilon \mathbf{I}$ where we limit the condition number to $\kappa(\mathbf{R}_{DL}) \leq 100$. For $\kappa(\mathbf{R}_{DL}) = 100$, the diagonal loading term ϵ can be calculated by $\frac{\lambda_1 + \epsilon}{\lambda_M + \epsilon} = 100$ where λ_1 and λ_M are the largest and smallest eigenvalues extracted from $\mathbf{R}$. Due to additional noise injection ($\epsilon \mathbf{I}$) introduced into the data covariance matrix which can result in higher noise floor, all of the normalized images are now thresholded at 75%. Images were reconstructed under different position setups by One Step GN with Laplace prior, MVDR, MUSIC, and MUSIC-like algorithm as shown in Figs. 5.19, 5.20, and 5.21. The performance of each algorithm was evaluated and shown in Table 4.1 where $PE(\%)$ indicates position error in percentage, RES indicates resolution performance (smaller value indicates higher resolution), and both objects are specified as O_1 and O_2 .

FIGURE 5.19: Image reconstruction based on GN-Laplace, MVDR, MUSIC, and MUSIC-like algorithm under experimental setup P_1

FIGURE 5.20: Image reconstruction based on GN-Laplace, MVDR, MUSIC, and MUSIC-like algorithm under experimental setup P_2

FIGURE 5.21: Image reconstruction based on GN-Laplace, MVDR, MUSIC, and MUSIC-like algorithm under experimental setup P_3

TABLE 5.1: Performance Evaluation of Similar Source Profiles with Diagonal Loading Term

		GN-Laplace		MVDR		MUSIC		MUSIC-like	
		O_1	O_2	O_1	O_2	O_1	O_2	O_1	O_2
P_1	$PE(\%)$	1.9	2.1	8.4	9.2	6.5	7.0	6.1	3.1
	$RES(\times 10^{-2})$	2.64	3.55	1.60	2.00	0.67	1.40	0.67	0.74
P_2	$PE(\%)$	10.2	7.1	11.3	9.5	10.6	5.0	6.9	4.1
	$RES(\times 10^{-2})$	5.03	5.55	1.90	1.90	1.30	0.80	1.00	0.93
P_3	$PE(\%)$	n/a	n/a	n/a	n/a	n/a	n/a	6.3	11.3
	$RES(\times 10^{-2})$	n/a	n/a	n/a	n/a	n/a	n/a	0.38	2.10

From Table 5.1, the regularization method is able to obtain the lowest position error for position setup P_1 . However, it also has the poorest resolution performance

which prohibits clear identification of both sources in P_2 and P_3 as marked "n/a". For algorithm with high resolution performance such as MUSIC and MUSIC-like algorithm, both sources can be distinctively identified as shown in Fig. 5.18(b). From the performance parameters evaluated in Table 5.1, the MUSIC-like algorithm is able to obtain the best overall performance for all experimental setups (P_1 , P_2 , and P_3) especially the resolution performance of the case P_3 as shown in Fig. 5.22 which exhibits the 1 dimensional slice cut of Fig. 5.18(c) along the blue dashed line. As can be seen in Fig. 5.22, the MUSIC-like algorithm produces two distinctive peaks (compared with the other algorithms) along the source positions.

FIGURE 5.22: Normalized pixel intensity along 1D slice of reconstructed images for case P_3 .

Next, the performance parameters of the spectral based algorithms (MVDR, MUSIC, and MUSIC-like) under ill-conditioned covariance matrix are evaluated and shown in Table 5.2. The performance of MVDR and MUSIC algorithm are affected since their performances are directly linked to the eigenvalues of the covariance matrix, and hence distinct identification of both sources could not be obtained for P_2 and P_3 . However, the MUSIC-like algorithm has the ability to automatically adjust its performance through β parameter. Therefore, it is able to achieve better performance compared to MVDR and MUSIC algorithm. The overall performance of the MUSIC-like algorithm without diagonal loading term is slightly better due to the absence of additional noise injection ($\epsilon \mathbf{I}$).

TABLE 5.2: Performance Evaluation of Similar Source Profiles without Diagonal Loading Term

		MVDR		MUSIC		MUSIC-like	
		O_1	O_2	O_1	O_2	O_1	O_2
P_1	$PE(\%)$	14.7	12.8	10.0	10.5	3.1	6.9
	$RES(\times 10^{-2})$	5.10	7.30	2.60	5.80	0.03	0.45
P_2	$PE(\%)$	n/a	n/a	n/a	n/a	4.8	2.9
	$RES(\times 10^{-2})$	n/a	n/a	n/a	n/a	0.09	0.29
P_3	$PE(\%)$	n/a	n/a	n/a	n/a	3.0	1.6
	$RES(\times 10^{-2})$	n/a	n/a	n/a	n/a	0.58	0.96

FIGURE 5.23: Metallic rods that are separated into two groups: group A and group B.

Lastly, we will evaluate the performance parameters in the case of two sources but with different conductivity profiles. Since we know that the conductivity value of metallic rod is based on the material it is made up (copper has higher conductivity value than aluminum which also has higher conductivity value than stainless steel) and the size of an object (large metallic rod is more conductive than small metallic rod). Now, the metallic rods that are shown in Fig. 5.23 will be divided into two groups based on the conductivity profile (relative size and conductivity value). We can obtain the first group as $A = \{1, 3, 7, 8, 10, 11\}$ and the second group as $B = \{2, 4, 5, 6, 9, 12, 13\}$. On average, the conductivity value and conductivity variation of the first group is higher than the second group. As can be seen in Fig. 5.18, source O_1 now is corresponding to the rods of the first group (large size and high conductivity value = strong source) and the source O_2 which is corresponding

to the rods from the second group (small size on average and low conductivity value = weak source). Each rod from the first group is matched up with all rods from the second group in order to simulate each data snapshot. We can then obtain 42 data snapshots for each position setup which are P_1, P_2 and P_3 . Due to the fact that the number of data snapshot is limited, only the case where diagonal loading term is used will be evaluated.

TABLE 5.3: Performance Evaluation of Different Source Profiles with Diagonal Loading Term

		GN-Laplace		MVDR		MUSIC		MUSIC-like	
		O_1	O_2	O_1	O_2	O_1	O_2	O_1	O_2
P_1	$PE(\%)$	6.5	3.6	5.0	4.7	15.5	n/a	5.5	3.7
	$RES(\times 10^{-2})$	5.30	4.33	0.80	1.42	1.64	n/a	0.30	0.80
P_2	$PE(\%)$	14.2	12.7	2.7	11.5	11.5	n/a	3.0	10.5
	$RES(\times 10^{-2})$	5.23	0.67	1.12	3.03	2.32	n/a	0.41	2.03
P_3	$PE(\%)$	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
	$RES(\times 10^{-2})$	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a

The MUSIC algorithm is unable to identify the source O_2 which corresponds to the weak source as can be seen in Table 5.3. All algorithms (both regularization and source localization) are unable to determine when the sources are closely situated as can be seen in case P_3 (weak source is situated next to strong source). Based on the results shown in Table 5.3 and Table 5.1, it is evidently clear that the performance of MUSIC-like algorithm has robust characteristic similar to that of the MVDR algorithm while having high resolution performance similar to that of the MUSIC algorithm.

5.7 A Note on Sparsity Approach

Similarly to the previous Chapter, for completeness we will be highlighting the results obtained under l_1 -norm regularization framework in this section without detailed analysis. Again, two l_1 -norm regularization functions will be used in this section which includes $\|\mathbf{x}\|_1$ and $\|\nabla \mathbf{x}\|_1$. In the first case, the estimate solution can

be obtained by solving LASSO problem: $\mathbf{x}_\lambda = \min_{\mathbf{x}} \|\mathcal{J}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{x}\|_1$ through the ADMM method. In the second case, the estimate solution can be obtained by solving total variation regularization problem: $\mathbf{x}_\lambda = \min_{\mathbf{x}} \|\mathcal{J}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda \|\nabla \mathbf{x}\|_1$ through the PD-IPM method for EIT.

The results obtained from LASSO problem by ADMM method are shown in Figs. 5.24 and 5.25 below. The reconstructed images are obtained from experimental setups $P_1 - P_3$. Since the l_1 -norm regularization we are focusing in this section is based on SMV, the inhomogeneous data is obtained by averaging over multiple data snapshots. The augmented Lagrangian parameter ρ is set to 1 in both figures. The value λ_{max} is obtained by $\lambda_{max} = \|\mathbf{A}(\Xi)^T \mathbf{x}\|_\infty$. To highlight the effect of the regularization parameter λ , we plot the reconstructed images where λ is set differently in two cases. The results shown in Fig. 5.26 is obtained by setting $\lambda = 0.001\lambda_{max}$. Subsequently, the results shown in Fig. 5.27 is obtained by setting $\lambda = 0.01\lambda_{max}$.

FIGURE 5.24: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_3$) based on LASSO problem, where λ is set to $0.001\lambda_{max}$

As mentioned earlier, the value set by λ/ρ plays a key role during the soft thresholding step. For simplicity we set ρ to a fixed value $\rho = 1$. We can now observe that when λ is set to a higher value (as can be seen in Fig. 5.25) the estimate solution has a sparse characteristic (few dominant pixels) as compared to when λ is set to a lower value which can be seen in Fig. 5.24. Intuitively, the λ/ρ acts as a threshold value. When the threshold value is set to a low number, more pixel values are allowed in the estimate solution, and hence certain noises in the reconstructed

image will be introduced. On the other hand when the threshold value is set high, only few dominant pixel values are allowed. The end result is a clipped of pixel values where only few pixels with high amplitude is left in the end solution.

FIGURE 5.25: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_3$) based on LASSO problem, where λ is set to $0.01\lambda_{max}$

We now show the result obtained from the total variation regularization. The reconstructed images from experimental setups $P_1 - P_3$ are shown in Figs. 5.26 and 5.27. The regularization parameter λ is set to 0.001 in Fig. 5.26. To highlight the effect of λ value, we show the image reconstructed where λ is set to a lower value at 0.0001 in Fig. 5.27. The maximum iteration of both cases are set to 10.

FIGURE 5.26: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_3$) based on Total Variation prior, where λ is set to 0.001

Similarly to the previous chapter, it can be seen in Figs. 5.26 and 5.27 that the reconstructed images with total variation prior tend to have “block-like” structure. It can also be seen in Fig. 5.27 that the lower the λ parameter is set, the more refined (less blurry) and block-like structure the reconstructed image will be.

FIGURE 5.27: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_3$) based on Total Variation prior, where λ is set to 0.0001

It can be seen in Fig. 5.27 that although the reconstructed images are more refined as compared to Fig. 5.26, certain artifacts may also be pronounced (especially at the top and bottom of Fig. 5.27 for P_1 and P_3). Although the l_1 -norm regularization framework may provide more refined reconstructed image, it should be noted that several hyper-parameters should also be carefully selected based on the application (sensitivity of the system) and noise within the environment.

5.8 Conclusion

In this chapter, a thorough analysis of several parameters related to conductivity variation such as intensity, size, and location has been investigated. The connection between an ill-conditioned array manifold and the eigenvalues extracted from the covariance matrix is fully explored, which helps to understand the common advantage and disadvantages of DOA estimation techniques. An alternative algorithm called MUSIC-like which has high resolution performance comparable to the MUSIC algorithm but avoids model order estimation has been proposed specifically for the EIT environment. The EIT prototype comprises 8 electrodes has been

used to test the proposed method. Simulation and experimental results using the prototype confirm the advantages of using the MUSIC-like algorithm both from the evaluated performance parameters viewpoint as well as the detectability of sources.

We also present the results obtained from the sparsity approach. The results obtained under source localization framework e.g. MVDR, MUSIC, and MUSIC-like provide similar sparse characteristic to the l_1 -norm regularization while the source localization framework is direct method (non-iterative) and has less hyper-parameters to be fine-tuned as compared with the l_1 -norm method e.g. LASSO and total variation regularization.

Chapter 6

Hybrid Framework: Beamforming-Regularization Approach for Support Identification

6.1 Introduction

This chapter introduces hybrid beamforming-regularization frameworks where a three-step method is proposed for an image reconstruction problem in EIT. In general, the computational bandwidth taken during image reconstruction process is mainly caused by the matrix inversion process while solving a system of linear equations. With a sparsity assumption, where areas occupied by the objects or the targets in the medium are bounded, the computational bandwidth can be greatly reduced through the three-step image reconstruction method. In the first step, a beamforming technique is applied as a screening step by scanning through a coarse-grid system to obtain a basic estimate location of the target. Thus, no anatomical structure of the object is assumed at this stage. In the second stage, a regions-of-interest (ROI) within the coarse-grid system is identified and then mapped to a fine-grid system. In the last step, the target is reconstructed based on the prior knowledge gained from the ROI identification of the coarse-to-fine grid mapping. This prior knowledge is prescribed into the estimate solution through the

regularization framework. Instead of inverting a full scale system matrix inversion, the size of the matrix required for the inversion is greatly minimized.

6.2 The Three-Step Processing

The MUSIC algorithm exploits the eigen-structure of data covariance matrix which is denoted as $\mathbf{R}$. Through eigendecomposition of the covariance matrix $\mathbf{R}$, a subspace segregation can be obtained by using prior information regarding the number of targets or by estimating the dominant eigenvalues after calculating eigendecomposition. The partitioned subspaces can be expressed as $\mathbf{R} = [\mathbf{U}_s, \mathbf{U}_n]\Sigma[\mathbf{U}_s, \mathbf{U}_n]^H$ where $\mathbf{U}_s$ and $\mathbf{U}_n$ denote the matrices containing the eigenvectors spanning the signal subspace and noise subspace respectively, and Σ denotes diagonal matrix containing all eigenvalues. Since $\mathbf{U}_s$ spans the same range space as the array manifold, and due to the orthogonality property between signal subspace and noise subspace, the pseudo-spectrum can be obtained.

In this section, the first hybrid framework namely the three-step processing is proposed. Due to the ability to focus on the desired signal while suppressing noises and interference, the beamforming technique is applied in the first step as a rough estimate in order to identify the ROI areas. Although the MUSIC algorithm is chosen in this thesis, it is important to note that any beamforming techniques can be used in the first step. The second step is an intermediate step that tries to map a coarse-grid system to a fine-grid system. The ROI in the fine-grid system is identified in this step. A regularization matrix $\mathcal{R}$ which incorporates statistical information regarding the possible location of targets is then calculated within the second step. In the last step, an image is reconstructed based on the prior information (regularization matrix $\mathcal{R}$) that was obtained from previous step. In summary, the hybrid beamforming regularization framework utilizes beamforming technique for support identification. The ROI areas within the coarse-grid system are identified and then translate into the fine-grid system. The regularization matrix can be obtained, and the regularized solution is calculated in the last step.

For p number of targets, under the assumption that the impedance or conductivity variation are uncorrelated between data snapshots the signal model can be expressed for each snapshot as

$$\mathbf{x}(n) = \mathbf{A}(\Xi_p)\mathbf{s}(n) + \mathbf{n}(n), \quad (6.1)$$

where $\mathbf{x} = \delta\mathbf{v} \in R^{M \times 1}$ is a measurement vector representing the potential variation on the boundary, $\mathbf{A}(\Xi_p) = [\mathbf{a}(\xi_1), \dots, \mathbf{a}(\xi_p)] \in R^{M \times p}$ is an array manifold matrix where each columns represents the expected variation (array response) on the boundary which is induced by each source ($\mathbf{a}(\xi_i)$ denotes the normalized 2-norm of the i^{th} column from the Jacobian matrix), and $\mathbf{s} = \delta\sigma \in R^{p \times 1}$ is a vector of conductivity variation.

6.2.1 First Step: Scanning for Targets

The signal model of multiple data snapshots obtained from p uncorrelated targets which is contaminated by an AWGN can be expressed as $\mathbf{X} = \mathbf{A}(\Xi_p)\mathbf{S} + \mathbf{N}$. By using eigendecomposition together with model order estimation, the covariance matrix can be expressed as $\mathbf{R} = \mathbf{U}_s\Lambda_s\mathbf{U}_s^T + \mathbf{U}_n\Lambda_n\mathbf{U}_n^T$, where $\mathbf{U}_s$ contain eigenvectors corresponding to the signal subspace, and $\mathbf{U}_n$ contain eigenvectors corresponding to the noise subspace. The pseudo spectrum of MUSIC algorithm can be obtained as $P(\xi_i) = 1/||\mathbf{U}_n^T\mathbf{a}(\xi_i)||_2^2$, where $\mathbf{a}(\xi_i)$ is a steering vector of all possible field of view (FOV) within the coarse-grid which is denoted as $\Xi = [\xi_1, \dots, \xi_N]$.

A set of steering vectors associated with the ROI in the coarse-grid system (denoted as Ξ^*) is defined as a subset of all look direction Ξ where the following condition is applied

$$\begin{aligned} \mathbf{A}(\Xi^*) &\subset \mathbf{A}(\Xi), \\ \Xi^* &:= \{\xi_i | P(\xi_i) \geq \frac{1}{2}||\mathbf{P}(\xi)||_\infty\}. \end{aligned}$$

The matrix $\mathbf{A}(\Xi^*) = [\mathbf{a}(\xi_1)^*, \dots, \mathbf{a}(\xi_q)^*] \in R^{M \times q}$ is a collection of steering vectors corresponds to the ROI in the coarse-grid system Ξ^* where the response power exceeds half power of the maximum peak.

6.2.2 Second Step: Coarse-to-Fine Mapping

The FOV in the fine-grid system is defined by $\tilde{\Xi} = [\tilde{\xi}_1, \dots, \tilde{\xi}_L]$ where L is the number of pixels or voxels in the medium ($L \gg N$). The array response for all

FOV in this fine-grid system is defined as $\mathbf{A}(\tilde{\Xi}) \in R^{M \times L}$ as shown in Fig. 6.1.

FIGURE 6.1: A map of coarse-grid system to fine-grid system (Grid Translation)

Before moving into the coarse-to-fine mapping, it is important to highlight that there are now four grid indices. The coarse grid (Ξ) and its ROI (Ξ^*) area, and the fine grid ($\tilde{\Xi}$) and its ROI ($\tilde{\Xi}^*$) which is the area to be mapped during the second step. With these four grid indices, there are also four array manifolds corresponding to the four grid indices which includes $\mathbf{A}(\Xi)$, $\mathbf{A}(\Xi^*)$, $\mathbf{A}(\tilde{\Xi})$, and $\mathbf{A}(\tilde{\Xi}^*)$. The probability that a steering vector in the fine-grid system is part of the ROI $\tilde{\Xi}^*$ can be defined as

$$p(\mathbf{a}(\tilde{\xi}_\alpha)) = \sum_{i=1}^p p(\mathbf{a}(\tilde{\xi}_\alpha) | \mathbf{a}(\xi_i^*)) p(\mathbf{a}(\xi_i^*)). \quad (6.2)$$

For simplicity, each point in an example shown in Fig. 6.2 is assumed to have Gaussian distribution with known variance σ^2 . In the fine-grid system $\mathbf{A}(\tilde{\Xi})$, the point at $\mathbf{y}_1$ has higher probability than the point at $\mathbf{y}_2$ to be part of the ROI since

$$\begin{aligned} p(\mathbf{y}_1) &= p(\mathbf{y}_1 | \mathbf{x}_1) p(\mathbf{x}_1) + \cdots + p(\mathbf{y}_1 | \mathbf{x}_4) p(\mathbf{x}_4), \\ p(\mathbf{y}_2) &= p(\mathbf{y}_2 | \mathbf{x}_1) p(\mathbf{x}_1) + \cdots + p(\mathbf{y}_2 | \mathbf{x}_4) p(\mathbf{x}_4), \end{aligned}$$

where $p(\mathbf{x}_i) = 1/4$ in this particular example, and variance σ^2 is assumed to be known. The probabilities $p(\mathbf{y}_1|\mathbf{x}_i)$ and $p(\mathbf{y}_2|\mathbf{x}_i)$ are given as

$$\begin{aligned} p(\mathbf{y}_1|\mathbf{x}_i) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mathbf{d}_i)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(1 - \mathbf{y}_1^T \mathbf{x}_i)}{\sigma^2}\right), \\ p(\mathbf{y}_2|\mathbf{x}_i) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mathbf{d}'_i)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(1 - \mathbf{y}_2^T \mathbf{x}_i)}{\sigma^2}\right), \end{aligned}$$

where $\mathbf{d}_i = \mathbf{y}_1 - \mathbf{x}_i$ and $\mathbf{d}'_i = \mathbf{y}_2 - \mathbf{x}_i$. When $\|\mathbf{y}_1\|_2^2 = 1$ and $\|\mathbf{x}_i\|_2^2 = 1$, which is true for the steering vectors from both grid systems since $\|\mathbf{a}(\xi_i)\|_2^2 = 1$ and $\|\mathbf{a}(\tilde{\xi}_j)\|_2^2 = 1$, $\mathbf{d}_i^2 = 2(1 - \mathbf{y}_1^T \mathbf{x}_i)$ and $\mathbf{d}'_i^2 = 2(1 - \mathbf{y}_2^T \mathbf{x}_i)$. Because $\sum_i (\mathbf{d}_i)^2 < \sum_i (\mathbf{d}'_i)^2$ which means $\sum_i \mathbf{y}_1^T \mathbf{x}_i > \mathbf{y}_2^T \mathbf{x}_i$, hence $p(\mathbf{y}_1) > p(\mathbf{y}_2)$.

FIGURE 6.2: The probabilities of points from fine grid system to be considered as part of the coarse grid system

As a result, the steering vectors $\mathbf{a}(\tilde{\xi}_i)$ that is coincided within the collection of Ξ^* (have the least $\sum_l d(\mathbf{a}(\xi_i), \mathbf{a}(\tilde{\xi}_j))$), or those that have the highest counts of high correlation with collection of $\mathbf{A}(\Xi^*)$ (have the most $\sum_l \text{Corr}\{\mathbf{a}(\xi_i), \mathbf{a}(\tilde{\xi}_j)\} > c$) will have higher probability to be part of the ROI. Therefore, in order to find the ROI in the fine-grid system, a correlation function ($\Phi \in R^{q \times L}$) is defined as

$$T_c[\mathbf{A}(\Xi^*)^T \mathbf{A}(\tilde{\Xi})] = \Phi = \begin{cases} [\Phi]_{ij} = 1 & ; \quad \mathbf{a}(\xi_i^*)^T \tilde{\mathbf{a}}(\xi_j) \geq c \\ [\Phi]_{ij} = 0 & ; \quad \mathbf{a}(\xi_i^*)^T \tilde{\mathbf{a}}(\xi_j) < c \end{cases}, \quad (6.3)$$

where c is specified as 0.8 in this section. The operator $T_c[\cdot]$ truncates the value $[\Phi]_{ij} < c$ to zero while the value $[\Phi]_{ij} \geq c$ will be set to 1. Without the knowledge of σ^2 from the Gaussian distribution, the probability of a steering vector in the new grid system to be chosen as part of the ROI can be substituted by calculating

$$\mathbf{p}(\mathbf{A}(\tilde{\Xi})) = \frac{\Phi^T \mathbf{1}}{\|\Phi^T \mathbf{1}\|_1}, \quad (6.4)$$

where $\mathbf{1} \in R^{q \times 1}$ is a vector of ones. The vector $\mathbf{p}(\mathbf{A}(\tilde{\Xi})) = [p_{\tilde{\xi}_1}, \dots, p_{\tilde{\xi}_L}]^T \in R^{1 \times L}$ reflects the likelihood of each steering vector in $\mathbf{A}(\tilde{\Xi})$ to be part of the ROI in the fine-grid system $\mathbf{A}(\tilde{\Xi}^*)$ where $\sum_{i=1}^L p_{\tilde{\xi}_i} = 1$. With the information from $\mathbf{p}(\mathbf{A}(\tilde{\Xi}))$, the regularization matrix is defined as

$$\mathcal{R} = \begin{bmatrix} (1/p_{\tilde{\xi}_1})^{\frac{1}{2}} & & \\ & \ddots & \\ & & (1/p_{\tilde{\xi}_L})^{\frac{1}{2}} \end{bmatrix}, \quad (6.5)$$

where the regions with high probability functions as a pass-band, and those with low probability functions as a stop-band, hence the trust regions are specified by the probability defined in the $\mathcal{R}$ matrix.

6.2.3 Third Step: Image Reconstruction

The image reconstruction for EIT can be obtained under the regularization framework which solves

$$\min_{\mathbf{s}} \{ \|\mathbf{A}(\tilde{\Xi}^*) \mathbf{s} - \mathbf{x}_m\|_2^2 + \lambda^2 \|\mathcal{R} \mathbf{s}\|_2^2 \}, \quad (6.6)$$

where the estimated solution can be expressed as

$$\mathbf{s}_\lambda = (\mathbf{A}(\tilde{\Xi}^*)^T \mathbf{A}(\tilde{\Xi}^*) + \lambda^2 \mathcal{R}^T \mathcal{R})^{-1} \mathbf{A}(\tilde{\Xi}^*)^T \mathbf{x}_m. \quad (6.7)$$

The matrix $\mathbf{A}(\tilde{\Xi}^*)$ can be decomposed by SVD as $\mathbf{A}(\tilde{\Xi}^*) = \mathbf{U} \Sigma \mathbf{V}^T$ with $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_l]$, $\Sigma = \text{diag}[\sigma_1, \dots, \sigma_l]$ where each σ_i is arranged associated to the source

location, and $\mathbf{V}^T = [\mathbf{v}_1, \dots, \mathbf{v}_l]^T$ ($l < L$). The solution from (6.7) can be re-expressed in the SVD domain as

$$\mathbf{s}_\lambda = \sum_{i=1}^L \varphi_\lambda \frac{\mathbf{u}_i^T \mathbf{x}_m}{\sigma_i} \mathbf{v}_i, \quad (6.8)$$

where φ_λ functions as a filter factor such that

$$\varphi_\lambda = \frac{\sigma_i^2}{\sigma_i^2 + \left(\frac{\lambda}{\sqrt{p_{\tilde{\xi}_i}}}\right)^2} \approx \begin{cases} 1 & ; \quad \sigma_i \gg \frac{\lambda}{\sqrt{p_{\tilde{\xi}_i}}} \\ \varepsilon & ; \quad \sigma_i \ll \frac{\lambda}{\sqrt{p_{\tilde{\xi}_i}}}, \end{cases} \quad (6.9)$$

where ε is a small positive value. The hyper-parameter λ operates as a tuning parameter between the model-to-data discrepancy and the prior obtained from $\mathcal{R}$. The regularization matrix $\mathcal{R}$ allows the information along the high probability locations $p(\mathbf{a}(\tilde{\xi}_\alpha))$ to pass while limiting or blocking the information from the low probability locations $p(\mathbf{a}(\tilde{\xi}_\beta))$. From previous example in Fig. 6.2, the information from point $\mathbf{y}_1$ will obtain higher weight than point $\mathbf{y}_2$ due to $p(\mathbf{y}_1) > p(\mathbf{y}_2)$, and the point from outside $\mathbf{A}(\tilde{\Xi})$ will obtain small weight ε or even get blocked out when $\lim_{p(\tilde{\xi}_\gamma) \rightarrow 0} \varepsilon \approx 0$. Aside from the two extreme cases, the weight value varies according to the probability along specific locations. Even though the full matrix of $\mathbf{A}(\tilde{\Xi})$ is used without reducing to $\mathbf{A}(\tilde{\Xi}^*)$, the probability in $\mathcal{R}$ matrix will filter out those $\{\tilde{\Xi} - \tilde{\Xi}^*\}$ due to their low probability. The proposed regularization matrix is separable into two parts which can be expressed as

$$\mathcal{R}^T \mathcal{R} = \left(\begin{array}{ccc|ccc} 1/p_{\tilde{\xi}_1} & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1/p_{\tilde{x}_{i_i}} & & & \\ \hline & & 0 & 1/p_{\tilde{\xi}_{i+1}} & & 0 \\ & & & & \ddots & \\ & & & 0 & & 1/p_{\tilde{\xi}_L} \end{array} \right). \quad (6.10)$$

From the zeroth-order Tikhonov prior viewpoint, the regularization matrix is specified as an identity matrix $\mathbf{I} \in R^{L \times L}$. The solution of Tikhonov prior tends to favor the reconstruction over the boundary due to higher sensitivity. The NOSER prior can be interpreted as an equalizer to reduce the over emphasize effect of the reconstructed image over the boundary area. The NOSER prior is specified as a diagonal matrix with diagonal elements $[\mathbf{D}]_{ii} = [\mathbf{J}^T \mathbf{J}]_{ii}$. Each element in the diagonal of $[\mathbf{D}]_{ii}$

represents the sensitivity of the source in that location (pixels/voxels) which is calculated by summing the values in the array response associated to the specified location. The diagonal elements in $[\mathbf{D}]_{ii}$ that are associated with the boundary area will have higher values than the area in the middle. For the proposed method, the area with low probability will be heavily penalized since $1/\epsilon \rightarrow \infty$, where ϵ is a small positive value, will put more emphasize to suppress that specific area. In contrast, the area with high probability will be less penalized since $1/c \rightarrow 0$, where c is a relatively large value, will put less emphasize to that specific area.

6.3 Simulation Results

In the first part of the simulation results, several performance parameters are discussed in order to highlight the advantage of using the MUSIC algorithm for target localization. The signal is generated by a forward model based on Finite Element Method (FEM) [34]. The forward model is constructed with 1,600 elements as a 2D slice domain. The signal's strength of the target is a function of position, hence the performance parameters will be evaluated as a function of position within a normalized unit radius medium. The simulation is computed under Monte Carlo simulation of $M = 200$ trials. The signal of 100 snapshots is captured. The measurement vector for the regularization with NOSER prior [57] is calculated by averaging over the multiple snapshots. The first performance parameter is the root-mean-squared-error (RMSE) of the estimated position compared to the ground truth. The estimated target in 2D image is first normalized as $\mathbf{s}_{rec}$ (normalized image) and then thresholded at 25% which will be denoted as $\bar{\mathbf{s}}$. The rationale is specified in the citeEval as 25% above the background is considered noticeable and should be included as part of an image. The center of gravity (CoG) of the target in the original image is denoted as $\mathbf{s}_{CoG}$ while the CoG of the estimated target is denoted as $\bar{\mathbf{s}}_{CoG}$. The position error defined in [35] is calculated as $PE = \mathbf{s}_{CoG} - \bar{\mathbf{s}}_{CoG}$. Since the direction of the position error will not be considered, the RMSE will be evaluated instead as follows

$$\text{RMSE}(CoG) = \sqrt{\frac{1}{M} \sum_{k=1}^K (\bar{\mathbf{s}}_{CoG_i} - \mathbf{s}_{CoG_i})^2}, \quad (6.11)$$

where the result is shown in Fig. 6.3. It can be seen that both the MUSIC algorithm and the regularization method can achieve good performance in high SNR scenarios while the MUSIC algorithm can achieve lower averaged RMSE while having more consistent performance along different positions.

FIGURE 6.3: The averaged RMSE of CoG from each algorithm under different SNR

The resolution which reflects the compactness of the reconstructed image defined in [35] can be calculated as follows

$$\text{Resolution} = \frac{A_{\bar{s}}}{A_0}, \quad (6.12)$$

where $A_{\bar{s}}$ is the area of the normalized and thresholded image $\bar{x}$ (related to the target), and A_0 is the total area of the whole image. The result of the resolution performance for each algorithm over different SNR is shown in Fig. 6.4.

In Fig. 6.4, both the MUSIC and regularization approaches perform better at high SNR while MUSIC algorithm can achieve better resolution as well as having more consistent performance along at different positions. The resolution of the estimated target of each algorithm are shown in Fig. 6.5. It can be seen that the estimated target from the MUSIC algorithm has a compact shape with sharp peak.

Next, the simulation result of the three-step method will be discussed. The signal is generated by a 3D model of 414,709 elements (voxels) with a target inside as shown

FIGURE 6.4: Resolution performance of each algorithm under different SNR

FIGURE 6.5: Surface plot of the localized target (a) ground truth (b) regularization with NOSER prior and (c) MUSIC algorithm

in Fig. 6.6. In order to avoid bias on the estimate solution, the number of elements in the forward model and inverse model should be different. The higher number of elements is given to the forward model in order to generate a high accuracy signal.

In the first step, the target is estimated by a coarse-grid model of 17,913 elements (Ξ) as shown in Fig. 6.7. The ROI in the coarse grid model is identified as Ξ^* .

The fine model of 58,575 elements ($\tilde{\Xi}$) is created, and the ROI of the fine-grid

FIGURE 6.6: Generative model and its ground truth in a normalized FEM model

FIGURE 6.7: Target localization by MUSIC algorithm in a normalized FEM model.

model is identified as $\tilde{\Xi}^*$ by the second step (Grid Translation). The new ROI in the fine-grid system is shown in Fig. 6.8.

The reconstructed target can be achieved in the third step through the regularization approach with prior $\mathcal{R}$ specified from the intermediate step as shown in Fig. 6.9.

In Fig. 6.10, the reconstructed target is compared with the ground truth. It can be seen that the reconstructed target while not being perfect, it is still acceptable with greatly reduced computational time.

FIGURE 6.8: ROI identification in the fine grid system in a normalized FEM model.

FIGURE 6.9: Target's image reconstruction in a normalized FEM model.

6.4 Experimental Results

Image reconstruction of the proposed method applied to the experimental data are presented within this section to highlight the performance of the algorithm with real EIT data. In the experiment, an 8-electrode EIT system prototype with the experimental setups used in Chapter 5 will also be applied to this section (refer to Figs. 6.11 and 6.12 for the experimental setups used in Chapter 5). Note that these experimental setups were conducted based on the MMV framework (multiple data snapshots), and hence the MUSIC algorithm can be used during the first step. It should also be noted that besides the MUSIC algorithm, other beamforming methods e.g. MVDR or MUSIC-like algorithm can also be used during the first step for target scanning within the coarse grid.

FIGURE 6.10: Biases of the reconstructed target in a normalized FEM model.

FIGURE 6.11: Metallic rods used in the experiment.

FIGURE 6.12: Three experimental setups with sources located at different positions.

The 2D image reconstruction can be accomplished as follows. Firstly, two different grid refinements is constructed as can be seen in Fig. 6.13. The coarse-grid has a total number of 256 pixels (FOV) while the fine-grid has a total number of 2304 pixels (FOV).

FIGURE 6.13: Depiction of coarse-grid (256 pixels) and fine-grid (2304 pixels).

During the first step, the MUSIC algorithm is used for target detection and localization within the coarse-grid system. The pseudospectrum obtained from the MUSIC algorithm is then normalized to identify the ROI as can be seen in Fig. 6.14.

With known ROI within the coarse-grid and the previously constructed fine-grid system (Fig. 6.15), ROI within the fine-grid can be identified by the second step of the proposed method (together with their probabilities), and the regularization matrix can be obtained.

FIGURE 6.14: The targets detection and localization within the coarse-grid (first step).

FIGURE 6.15: The coarse-to-fine grid mapping process (second step).

With known ROI within the fine-grid system and their corresponding probabilities, the newly constructed regularization matrix can be used for image reconstruction (as can be seen in Fig. 6.16) under $l_2 - norm$ regularization.

In Figs. 6.17 and 6.18, the reconstructed images are obtained where the λ is set to 0.01 in Fig. 6.17 and 0.001 in Fig. 6.18.

FIGURE 6.16: The reconstructed image within fine-grid (third step).

FIGURE 6.17: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_3$) based on the Three-step processing method, where λ is set to 0.01.

As can be seen in Figs. 6.17 and 6.18, the reconstructed images when λ is set to a higher value (Fig. 6.17) leads to a more relaxed solution (less sparse). In contrast, when λ is set to a lower value (Fig. 6.18) the reconstructed images are more sparse. This is due to the reciprocity property of the regularization matrix, where we specified each diagonal value to be the inverse of the probability (how likely) that certain pixels would be part of the ROI (within the fine-grid). This leads to an inverse effect where the lower value of λ contributes to more penalization on pixels with low probabilities (likelihood) to be part of the ROI and vice versa to the high value of λ .

FIGURE 6.18: Reconstructed images ($\mathbf{x}$) of the experimental setups ($P_1 - P_3$) based on the Three-step processing method, where λ is set to 0.001.

In this section, the three-step method is proposed to reconstruct the target's image. The averaged RMSE and the resolution performance of the MUSIC algorithm is shown to be superior than the regularization approach in terms of target localization task. In short, the MUSIC algorithm is able to produce more compact areas of ROI when compared with l_2 -norm regularization method (note that we are only considering the direct image reconstruction method and not the iterative method). Therefore, MUSIC algorithm is used within the first step to obtain the ROI within the coarse-grid system. The prior information prescribed in matrix $\mathcal{R}$, which can be obtained during the second step, incorporates statistical information of the target's location. After the reconstruction process within the third step based on the previously prescribed $\mathcal{R}$, the reconstructed target is shown to be acceptable with slight bias in the CoG. Lastly, the performance of the three-step method with experimental data has also been highlighted (qualitatively). It should be noted that image quality of the experimental section is not as refined as presented in the simulation section due to two reasons. Firstly, the simulation results are obtained with more electrode sensors (two rings - 8 electrodes for each ring) when compared with the actual experimental setup (one ring - 8 electrodes). Secondly, the experimental data are obtained under high noise environment because of the limitation on the hardware/prototyping system. Although with quite a number of limitation, the reconstructed image is of quality approaching to those of l_1 -norm regularization (e.g. LASSO) where the l_1 -norm methods are typically iterative based while the proposed method is direct method.

6.5 Conclusion

This chapter introduces hybrid beamforming-regularization frameworks for EIT image reconstruction. The three-step method proposed for image reconstruction in EIT has its root in sparse signal recovery where the support set is first identified before calculating (reconstruct) the estimate solution. The contribution lies in the coarse-to-fine grid mapping which reduces the time for target scanning in the first step. A regularization matrix based on local prior is proposed to further improve the estimate solution in the last step. Simulation results and some preliminary experimental results obtained from the proposed method are also presented.

Part II

Robust Support Identification for Direction Of Arrival Estimation

Chapter 7

Towards Adaptive β -parameter for MUSIC-like Algorithm

7.1 Introduction

In the previous chapter, it can be observed that the MUSIC-like algorithm can be used as an alternative method for image reconstruction, where the problem of image reconstruction has been re-framed into source localization task. The benefit of super-resolution performance that can be obtained from the MUSIC-like algorithm has also been highlighted specifically for EIT application. In addition, we have also emphasized that the algorithm can be used to uncover a set of joint support under the multiple measurement vector (MMV) framework. This means source localization task is applicable to more general applications that require an identification of active support under MMV framework.

In this chapter, we are shifting our attention towards direction-of-arrival (DOA) estimation that is used in Radio Detection and Ranging (RADAR), Sound Navigation and Ranging (SONAR), and communication applications. Although the linear models of the applications shown in this chapter are somewhat different from the previous chapter (EIT application), it should be noted that the underlying principle is applicable to both fields. Recall that when we presented the MUSIC-like algorithm, there is an important parameter denoted as β , which needs to be selected in accordance to the signal environment (signal source and noise). The organization of this chapter can be broadly separated into two parts based on the

development and improvement process of the algorithm. In the first part, three novel β -selection methods are proposed and investigated. Each novel β -selection method provides an improvement such as robustness or lower estimation bias over the original proposition such that the proposed methods can be chosen based on specific requirements of the applications. In the second part of this chapter, one of the novel β -selection method namely directional adaptive β is further explored and investigated under an impulse noise scenario. Two novel adaptive β -selection methods were derived, analyzed, and compared with the conventional methods that are commonly used for handling impulse noise scenario (fractional lower order moment (FLOM)). We also show that without relying on FLOM based methods, our proposed method (directional adaptive MUSIC-like algorithm) can provide better performance in terms of detection rate when compared to the original MUSIC-like algorithm or MUSIC with FLOM based methods.

7.2 Part A: Novel β -Parameters for MUSIC-like Algorithm

In what follows, we first present the signal model that is used for DOA estimation. The signal model is based on a simplified solution of a wave equation under narrowband signal and far-field assumption. Nonetheless, it should be emphasized that the results in the following sections are applicable to other signal models as well, where different sensing matrices $\mathbf{A}$ can be used (not limited to only to DOA estimation). Consider an linear M -element array of sensors situated far away (far-field) from the point sources (targets) such that plane wave model is valid. The terms *number of targets* and *number of active support* will be used interchangeably throughout this chapter so that the reader can link the idea and concept between the two fields. A single snapshot or a single measurement vector (SMV) of received signals is denoted as $\mathbf{x}(n) \in \mathbb{C}^{M \times 1}$, which can be expressed by a linear model as

$$\mathbf{x}(n) = \mathbf{A}(\Theta)\mathbf{s}(n) + \mathbf{n}(n). \quad (7.1)$$

The matrix $\mathbf{A}(\Theta) = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_N)] \in \mathbb{C}^{M \times N}$ is a sensing matrix which comprises all possible steering vectors (atoms) $\mathbf{a}(\theta)$ from all directions, $\mathbf{s}(n) \in \mathbb{C}^{N \times 1}$ is the

source signal vector, and $\mathbf{n}(n) \in \mathbb{C}^{M \times 1}$ denotes additive noise vector (Gaussian). Each steering vector $\mathbf{a}(\theta)$ is a function of direction θ which can be expressed as

$$\mathbf{a}(\theta) = \frac{1}{\sqrt{M}} [\exp(j\mathbf{k}_\theta^T \mathbf{r}_1), \dots, \exp(j\mathbf{k}_\theta^T \mathbf{r}_M)]^T, \quad (7.2)$$

where $\mathbf{r}_i$ incorporates the location information of the i^{th} sensor, $\mathbf{k}_\theta = 2\pi f/v \mathbf{u}_\theta$ denotes the wave number, and $\mathbf{u}_\theta$ is the unit vector along the wave propagation direction. The signal frequency and the propagation speed are denoted as f and v , respectively. For K narrowband far-field sources, the sensing matrix is now reduced to a matrix called an array manifold matrix $\mathbf{A}(\Theta_K) = [\mathbf{a}(\theta^1), \dots, \mathbf{a}(\theta^K)] \in \mathbb{C}^{M \times K}$ where $\Theta_K = \{\theta_1, \dots, \theta_K\}$ is a set that comprises active support. The MMV formulation, which is similar to (5.4), can now be expressed as

$$\mathbf{X} = \mathbf{A}(\Theta_K) \mathbf{S}_K + \mathbf{N}, \quad (7.3)$$

where $\mathbf{X} \in \mathbb{C}^{M \times p}$ denotes a matrix that contains multiple measurement vectors as its column (with p number of snapshots (samples) and $p \gg K$), the matrix $\mathbf{S}_K$ denotes the source matrix where the zero entries are being removed (only the nonzero entries are left), and $\mathbf{V} \in \mathbb{C}^{M \times p}$ is a matrix of additive noise where noises among the sensors are spatially independent.

7.2.1 The First Approach for β -Selection

For completeness on the analysis, the optimization problem of MUSIC-like algorithm is briefly repeated where the steering vector here is based on $\mathbf{a}(\theta)$ instead of $\mathbf{a}(\xi)$ as in previous chapter. The MUSIC-like algorithm proposed in [11, 12] is formulated as an optimization problem for each look direction as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{a}(\theta) \mathbf{a}(\theta)^H \mathbf{w} + \beta \|\mathbf{w}\|_2^2 = c, \end{aligned} \quad (7.4)$$

where the weight vector $\mathbf{w}$ is the solution from the optimization problem in (7.4), the steering vector $\mathbf{a}(\theta)$ satisfies a normalized 2-norm $\|\mathbf{a}(\theta)\|_2 = 1$, a scalar value β is a control parameter which introduces certain relaxation to the constraint, and c is any constant value. The Lagrangian function for (7.4) can be expressed as

$$L(\mathbf{w}, \lambda) = \mathbf{w}^H \mathbf{R} \mathbf{w} - \lambda(\mathbf{w}^H (\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I}) \mathbf{w} - c). \quad (7.5)$$

Let $\frac{\partial L(\mathbf{w}, \lambda)}{\partial \mathbf{w}^H} = \mathbf{R} \mathbf{w} - \lambda(\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I}) \mathbf{w} = 0$, the weight vector solution $\mathbf{w}$ should be the general eigenvector of the matrix pair $(\mathbf{R}, (\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I}))$ as

$$\mathbf{R} \mathbf{w} = \lambda(\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I}) \mathbf{w}. \quad (7.6)$$

According to the minimization problem in (7.4), the weight solution $\mathbf{w}$ is the eigenvector corresponds to the minimum eigenvalue of (7.6), and therefore the spatial spectrum of the MUSIC-like algorithm is defined respectively as

$$P(\theta) = 10 \log_{10} \frac{1}{|\mathbf{w}^H \mathbf{a}(\theta)|^2}. \quad (7.7)$$

As can be seen in (7.6), the β parameter plays an important role in regulating the eigenvalues of the matrix pair. In [12], the bound of β as well as a β -selection method was proposed. In this section, further investigation will be conducted. The generalized eigenvalue problem in (7.6) is equivalent to the following expression

$$(\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I})^{-1} \mathbf{R} \mathbf{w} = \lambda \mathbf{w}. \quad (7.8)$$

According to the Sherman-Morrison formula,

$$(\mathbf{A} + \mathbf{x} \mathbf{y}^H)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{x} \mathbf{y}^H \mathbf{A}^{-1}}{1 + \mathbf{y}^H \mathbf{A}^{-1} \mathbf{x}}, \quad (7.9)$$

the term $(\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I})^{-1}$ in (7.8) can be re-expressed as

$$(\mathbf{a}(\theta) \mathbf{a}(\theta)^H + \beta \mathbf{I})^{-1} = \beta^{-1} \mathbf{I} - \alpha \mathbf{a}(\theta) \mathbf{a}(\theta)^H, \quad (7.10)$$

where

$$\alpha = \frac{\beta^{-2}}{1 + \beta^{-1} \|\mathbf{a}(\theta)\|_2^2}. \quad (7.11)$$

Since $\mathbf{a}(\theta)$ defined in (7.2) satisfies $\|\mathbf{a}(\theta)\|_2 = 1$, α is equivalent to $\alpha = 1/\beta(\beta + 1)$. By using (7.10) together with the eigendecomposition of $\mathbf{R} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H$ and the unitary property $\mathbf{U} \mathbf{U}^H = \mathbf{I}$, the expression in (7.8) can now be re-expressed as

$$(\beta^{-1}\mathbf{I} - \alpha\mathbf{a}(\theta)\mathbf{a}(\theta)^H)\mathbf{U}\Lambda\mathbf{b} = \lambda\mathbf{U}\mathbf{b}, \quad (7.12)$$

where $\mathbf{b} = \mathbf{U}^H\mathbf{w}$, and hence we obtain the following expression

$$\beta^{-1}\mathbf{U}\Lambda\mathbf{b} - \alpha\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{U}\Lambda\mathbf{b} = \lambda\mathbf{U}\mathbf{b}. \quad (7.13)$$

We now denote $\mathbf{e} = \mathbf{U}^H\mathbf{a}(\theta)$, and the expression in (7.13) can be re-written as

$$\begin{aligned} \mathbf{U}(\beta^{-1}\Lambda\mathbf{b} - \alpha\mathbf{U}^H\mathbf{a}(\theta)\mathbf{a}(\theta)^H\mathbf{U}\Lambda\mathbf{b}) &= \lambda\mathbf{U}\mathbf{b}, \\ \text{or,} \quad (\beta^{-1}\Lambda - \alpha\mathbf{e}\mathbf{e}^H\Lambda)\mathbf{b} &= \lambda\mathbf{b}. \end{aligned} \quad (7.14)$$

The weight vector solution corresponding to the minimum eigenvalue of (7.6) is equivalent to the solution of (7.8), and since (7.8) can be re-expressed as (7.14), the weight vector solution should be an eigenvector corresponding to the minimum eigenvalue of matrix $\mathbf{B}$ defined as

$$\mathbf{B} = \beta^{-1}\Lambda - \alpha\mathbf{e}\mathbf{e}^H\Lambda. \quad (7.15)$$

In order to obtain high responses (peaks) along the locations of the support set while suppressing responses along the locations that does not belong to the support set, we should now further investigate the two following cases.

7.2.1.1 The Atom Belongs to the Signal Subspace

In this case, the atom $\mathbf{a}(\theta) \in \Theta_K$ is residing in the subspace spanned by $\mathbf{A}(\Theta_K)$ which implies that it is part of the signal subspace spanned by $\mathbf{U}_s$, and hence it should be orthogonal to the noise subspace ($\mathbf{a}(\theta) \perp \mathbf{U}_n$). The vector $\mathbf{e}$ is now defined as $\mathbf{e} = \begin{bmatrix} \mathbf{U}_s^H\mathbf{a}(\theta) \\ \mathbf{U}_n^H\mathbf{a}(\theta) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_s \\ \mathbf{0} \end{bmatrix}$, and the matrix $\mathbf{B}$ can be re-expressed as

$$\tilde{\mathbf{B}} = \begin{bmatrix} \beta^{-1}\Lambda_s - \alpha\mathbf{e}_s\mathbf{e}_s^H\Lambda_s & \mathbf{0} \\ \mathbf{0} & \beta^{-1}\Lambda_n \end{bmatrix}. \quad (7.16)$$

The matrix $\tilde{\mathbf{B}}$ is now a block diagonal matrix where $\tilde{\mathbf{B}}_{11} = \beta^{-1}\Lambda_s - \alpha\mathbf{e}_s\mathbf{e}_s^H\Lambda_s$ and $\tilde{\mathbf{B}}_{22} = \beta^{-1}\Lambda_n$. The pseudospectrum will produce peak along the support location

when the eigenvector corresponds to the minimum eigenvalue is residing in the noise subspace $\tilde{\mathbf{B}}_{22}$. This can be assured when the minimum eigenvalue of $\tilde{\mathbf{B}}_{22}$ is lower than the minimum eigenvalue of $\tilde{\mathbf{B}}_{11}$ ($\lambda_{\tilde{\mathbf{B}}_{22},min} < \lambda_{\tilde{\mathbf{B}}_{11},min}$). The minimum eigenvalue corresponds to the matrix $\tilde{\mathbf{B}}$ can be obtained by

$$\begin{aligned} \min_{\mathbf{b}} \quad & \mathbf{b}^H \tilde{\mathbf{B}} \mathbf{b} \\ \text{s.t.} \quad & \|\mathbf{b}\|_2^2 = 1, \end{aligned} \quad (7.17)$$

From (7.16), the term $\mathbf{b}^H \tilde{\mathbf{B}} \mathbf{b}$ can be expressed as

$$\mathbf{b}^H \tilde{\mathbf{B}} \mathbf{b} = \mathbf{b}_s^H \tilde{\mathbf{B}}_{11} \mathbf{b}_s + \mathbf{b}_n^H \tilde{\mathbf{B}}_{22} \mathbf{b}_n, \quad (7.18)$$

where $\mathbf{b} = \begin{bmatrix} \mathbf{U}_s^H \mathbf{w} \\ \mathbf{U}_n^H \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_s \\ \mathbf{b}_n \end{bmatrix}$. In order to satisfy $\lambda_{\tilde{\mathbf{B}}_{22},min} < \lambda_{\tilde{\mathbf{B}}_{11},min}$, we obtain the following relationship

$$\begin{aligned} \frac{\mathbf{b}_s^H \{\beta^{-1} \Lambda_s - \alpha \mathbf{e}_s \mathbf{e}_s^H \Lambda_s\}^* \mathbf{b}_s}{\|\mathbf{b}_s\|_2^2} &> \frac{\mathbf{b}_n^H \{\beta^{-1} \Lambda_n\}^* \mathbf{b}_n}{\|\mathbf{b}_n\|_2^2} \\ \frac{\mathbf{b}_s^H \{\beta^{-1} \Lambda_s\}^* \mathbf{b}_s}{\|\mathbf{b}_s\|_2^2} - \frac{\alpha \mathbf{b}_s^H \{\mathbf{e}_s \mathbf{e}_s^H \Lambda_s\}^* \mathbf{b}_s}{\|\mathbf{b}_s\|_2^2} &> \frac{\mathbf{b}_n^H \{\beta^{-1} \Lambda_n\}^* \mathbf{b}_n}{\|\mathbf{b}_n\|_2^2} \end{aligned} \quad (7.19)$$

where $\mathbf{x}^H \{\mathbf{Q}\}^* \mathbf{x} / \|\mathbf{x}\|_2^2$ with an $(*)$ denoting the minimum eigenvalue of the matrix $\mathbf{Q}$ (or the maximum eigenvalue of $\mathbf{Q}^{-1}$). It can be seen that the minimum eigenvalue of the term $\beta^{-1} \Lambda_s$ is equal to $\beta^{-1}(\sigma_K^2 + \sigma_v^2)$, and the minimum eigenvalue of the term $\beta^{-1} \Lambda_n$ is equal to $\beta^{-1}(\sigma_v^2)$. Since Λ_s is a diagonal matrix which is full rank and $\mathbf{e}_s \mathbf{e}_s^H$ is a rank one matrix, therefore $\text{rank}(\mathbf{e}_s \mathbf{e}_s^H \Lambda_s) = \min\{\text{rank}(\mathbf{e}_s \mathbf{e}_s^H), \text{rank}(\Lambda_s)\} = 1$ which means that the minimum eigenvalue of the term $\mathbf{e}_s \mathbf{e}_s^H \Lambda_s$ is equal to the minimum eigenvalue of $\text{tr}\{\mathbf{e}_s \mathbf{e}_s^H \Lambda_s\} = \text{tr}\{\mathbf{e}_s^H \Lambda_s \mathbf{e}_s\}$. As $\mathbf{e}_s = \mathbf{U}_s^H \mathbf{a}(\theta)$, we obtain

$$\text{tr}\{\mathbf{e}_s \mathbf{e}_s^H \Lambda_s\} = \mathbf{a}(\theta)^H \mathbf{R}_s \mathbf{a}(\theta), \quad (7.20)$$

where $\mathbf{R}_s = \mathbf{U}_s \Lambda_s \mathbf{U}_s^H$. The output of $\mathbf{a}(\theta)^H \mathbf{R}_s \mathbf{a}(\theta)$ is equivalent to the conventional beamforming (Bartlett), and hence the minimum eigenvalue of $\text{tr}\{\mathbf{e}_s \mathbf{e}_s^H \Lambda_s\}$ is corresponding to the minimum eigenvalue of $\mathbf{R}_s$ which is equal to $\sigma_K^2 + \sigma_v^2$. From (7.19), the following relationship can be obtained,

$$\frac{\beta^{-1}}{\alpha}(\sigma_K^2 + \sigma_v^2) + (\sigma_K^2 + \sigma_v^2) > \frac{\beta^{-1}}{\alpha}(\sigma_v^2), \quad (7.21)$$

and since $\beta^{-1}/\alpha = \beta + 1$, we can obtain the lower bound for the β parameter as

$$\beta > \frac{\sigma_v^2}{\sigma_K^2}. \quad (7.22)$$

7.2.1.2 The Atom Belongs to the Noise Subspace

In this case, the atom should not be part of the signal subspace. Forcing the atom to be residing in the noise subspace we obtain $\mathbf{e} = \begin{bmatrix} \mathbf{U}_s^H \mathbf{a}(\theta) \\ \mathbf{U}_n^H \mathbf{a}(\theta) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_n \end{bmatrix}$, and the matrix $\mathbf{B}$ can be expressed as

$$\bar{\mathbf{B}} = \begin{bmatrix} \beta^{-1}\Lambda_s & \mathbf{0} \\ \mathbf{0} & \beta^{-1}\Lambda_n - \alpha \mathbf{e}_n \mathbf{e}_n^H \Lambda_n \end{bmatrix}. \quad (7.23)$$

Again, the matrix $\bar{\mathbf{B}}$ is a block diagonal matrix where $\bar{\mathbf{B}}_{11} = \beta^{-1}\Lambda_s$ and $\bar{\mathbf{B}}_{22} = \beta^{-1}\Lambda_n - \alpha \mathbf{e}_n \mathbf{e}_n^H \Lambda_n$. To avoid high responses (peaks) along the locations that do not belong to the support set, the weight vector solution should not be residing in the signal subspace, and hence the minimum eigenvalue of $\bar{\mathbf{B}}$ should be part of $\bar{\mathbf{B}}_{11}$, which means that the minimum eigenvalue of $\bar{\mathbf{B}}_{11}$ should be smaller than the minimum eigenvalue of $\bar{\mathbf{B}}_{22}$ ($\lambda_{\bar{\mathbf{B}}_{22}, \min} > \lambda_{\bar{\mathbf{B}}_{11}, \min}$). Following the procedure in (7.19), we obtain

$$\begin{aligned} \frac{\mathbf{b}_s^H \{\beta^{-1}\Lambda_s\}^* \mathbf{b}_s}{\|\mathbf{b}_s\|_2^2} &< \frac{\mathbf{b}_n^H \{\beta^{-1}\Lambda_n - \alpha \mathbf{e}_n \mathbf{e}_n^H \Lambda_n\}^* \mathbf{b}_n}{\|\mathbf{b}_n\|_2^2} \\ \frac{\mathbf{b}_s^H \{\beta^{-1}\Lambda_s\}^* \mathbf{b}_s}{\|\mathbf{b}_s\|_2^2} &< \frac{\mathbf{b}_n^H \{\beta^{-1}\Lambda_n\}^* \mathbf{b}_n}{\|\mathbf{b}_n\|_2^2} - \frac{\alpha \mathbf{b}_n^H \{\mathbf{e}_n \mathbf{e}_n^H \Lambda_n\}^* \mathbf{b}_n}{\|\mathbf{b}_n\|_2^2}. \end{aligned} \quad (7.24)$$

The minimum eigenvalue of the term $\beta^{-1}\Lambda_s$ is equal to $\beta^{-1}(\sigma_K^2 + \sigma_v^2)$, and the minimum eigenvalue of the term $\beta^{-1}\Lambda_n$ is equal to $\beta^{-1}(\sigma_v^2)$. Note that $\mathbf{e}_n = \mathbf{U}_n^H \mathbf{a}(\theta)$, the minimum eigenvalue of the term $\mathbf{e}_n \mathbf{e}_n^H \Lambda_n$ can be obtained by the relationship defined as

$$\text{tr}\{\mathbf{e}_n \mathbf{e}_n^H \Lambda_n\} = \mathbf{a}(\theta)^H \mathbf{R}_n \mathbf{a}(\theta), \quad (7.25)$$

where $\mathbf{R}_n = \mathbf{U}_n \Lambda_n \mathbf{U}_n^H$. The output of the term $\mathbf{a}(\theta)^H \mathbf{R}_n \mathbf{a}(\theta)$ is equivalent to the output power of the conventional beamformer when the system is in quiescent, and therefore the minimum eigenvalue of $\text{tr}\{\mathbf{e}_n \mathbf{e}_n^H \Lambda_n\}$ is corresponding to the minimum noise power σ_v^2 . The result from the expression (7.24) can be expressed as

$$\frac{\beta^{-1}}{\alpha}(\sigma_K^2 + \sigma_v^2) < \frac{\beta^{-1}}{\alpha}(\sigma_v^2) - \alpha(\sigma_v^2), \quad (7.26)$$

and since $\beta^{-1}/\alpha = \beta + 1$, we can obtain the upper bound for the β parameter as

$$\beta < \frac{\sigma_K^2 - \sigma_v^2}{\sigma_K^2}. \quad (7.27)$$

Since the K^{th} eigenvalue is not known a priori, and due to the fact that $\sigma_1^2 > \sigma_K^2$ and $\lambda_{\mathbf{R},\min} \approx \sigma_v^2$ are true, we propose the loose-bounds with σ_K^2 being replaced by $\lambda_{\mathbf{R},\max}$ and $\lambda_{\mathbf{R},\min}$ by σ_v^2 . The new bounds of β are now given by

$$\underbrace{\frac{\lambda_{\mathbf{R},\min}}{\lambda_{\mathbf{R},\max}}}_{\beta_{\min}} < \beta < \underbrace{\frac{\lambda_{\mathbf{R},\max} - \lambda_{\mathbf{R},\min}}{\lambda_{\mathbf{R},\max}}}_{\beta_{\mathbf{R},\max}}, \quad (7.28)$$

with the condition that $\lambda_{\mathbf{R},\max} > 2\lambda_{\mathbf{R},\min}$. The β value can be chosen by

$$\beta = (1 - \alpha)\beta_{\min} + \alpha\beta_{\max}, \quad (7.29)$$

where $\alpha = \beta_{\min}/\beta_{\max}$.

The performance of the new β is denoted as $\tilde{\beta}$ is shown in Fig. 7.1 along with the performance of the old beta denotes β and the performance of the MUSIC algorithm. Here, two targets (members of the support set Θ_K where $K = 2$) are located at $\theta = 80^\circ$ and 90° . With 10 sensor elements ($M = 10$), the pseudospectrums from the received signals at the SNR of 5dB with 100 snapshots data ($K = 100$) are shown in Fig. 7.1. In [12], it was mentioned that the pseudospectrum of the original β leads to higher bias in the estimated angle as reflected in the root-mean-square error (RMSE). However, it can be seen in Fig. 7.1 that $\tilde{\beta}$ provides lower bias than that produced by the original β .

FIGURE 7.1: The spatial spectrum of the MUSIC and MUSIC-like algorithm with β and $\tilde{\beta}$.

7.2.2 The Second Approach for β -Selection

In this part, a geometrical interpretation of the MUSIC-like algorithm is presented followed by a proposition for the second method for β -selection. For ease of visualization, the 2 dimensional contour plot is used for illustration purposes in stead of M -dimensional ellipsoid. For the case of higher dimensions, it can be easily extended where a line becomes a hyper-plane and a contour plot of the ellipse becomes a hyper-ellipsoid. The optimization problem of the Capon's beamforming attempts to minimize the output power as $\min_{\mathbf{w}} \{\mathbf{w}^H \mathbf{R} \mathbf{w}\}$ while the weight vector solution is required to satisfy the constraint $\mathbf{w}^H \mathbf{a}(\theta) = c$ (where $c = 1$ is commonly used). The constraint of the Capon's beamforming provide feasible region where the weight vector solution should belong to which is on a line (hyper-plane in higher dimensional spaces) as shown in Fig 7.2(a). In contrast, it can be seen in Fig. 7.2(b) that the feasible region of the MUSIC-like algorithm provides a wider

space for the weight vector to be resided. This relaxation in fact allows the weight vector solution to be resided in the noise subspace, and hence the pseudo spectrum of the MUSIC-like is able to obtain high response along the support locations.

FIGURE 7.2: Optimization problem of the (a) Capon beamforming and the (b) MUSIC-like algorithm

From Fig. 7.2(b), it can be seen that the β parameter plays a key role in regulating the feasible region of the MUSIC-like algorithm. It is now further presented in Fig. 7.3 on how the β parameter is able to adaptively regulate itself depending on the ellipsoidal shape of the covariance matrix. In Fig. 7.3(a) when the SNR is low, both axes (P_1) and (P_2) are equivalent, and hence the β inflates the feasible region ellipsoid accordingly. On the other hand as shown in Fig. 7.3(b) when the SNR is high, the principle axis (P_1) is no longer equivalent to the minor axis (P_2) as can be seen that now the contour is skewed, and hence the β parameter deflates the feasible region ellipsoid respectively.

FIGURE 7.3: Adaptability of the MUSIC-like algorithm under (a) low SNR and (b) high SNR scenarios.

It should be noted that the β cannot be set unconditionally large. The result where the β is set too large is shown in Fig. 7.4, and the weight vector solution now approaches the same solution as Pisarenko's method where the eigenvector

corresponds to the minimum eigenvalue is being used. The result is known to be sensitive to noise which creates multiple spurious peaks along the pseudospectrum. As can be seen in Fig. 7.4 that when the β is set too large, the feasible region does not provide valuable information since the weight vector solution is not constrained to be residing in any specific region. With the geometrical insight into the

FIGURE 7.4: Geometrical interpretation of the over-extended β

algorithm, we further investigate whether there is a relationship between the ratio of principle-to-minor axis of the covariance matrix and the principle-to-minor axis of the feasible region ellipsoid.

We denote τ as a ratio between the principle-to-minor axis of the covariance matrix to the principle-to-minor axis of the feasible region ellipsoid which can be expressed as

$$\tau = \frac{\lambda_{\mathbf{R},max}/\lambda_{\mathbf{R},min}}{(\beta + 1)/\beta}, \quad (7.30)$$

and hence β can be expressed as a function of τ as

$$\beta = \frac{\tau}{\lambda_{\mathbf{R},max}/\lambda_{\mathbf{R},min} - \tau}. \quad (7.31)$$

Next, we investigate the appropriate range for τ that would provide a robust and high performance β . Since τ is a ratio between two condition numbers which means that $\tau > 0$, and therefore it can be expressed as $\tau = a/b$ where $a, b > 0$. This leads to two possible cases which are

$$\tau = \begin{cases} 0 < \tau \leq 1 & \text{if } a < b \\ \tau > 1 & \text{if } a > b. \end{cases} \quad (7.32)$$

In the first case where $a < b$, the following relationship can be obtained

$$\begin{aligned}\lambda_{\mathbf{R},max}\beta &< \lambda_{\mathbf{R},min}\beta + \lambda_{\mathbf{R},min} \\ \beta(\sigma_s^2 + \sigma_v^2) &< \sigma_v^2(\beta + 1) \\ \beta &< \frac{\sigma_v^2}{\sigma_s^2}.\end{aligned}\tag{7.33}$$

Since $\beta < \sigma_v^2/\sigma_s^2$ does not satisfy the condition shown in (7.22), we will be focusing on the other case where $a > b$. Following the same procedure done in (7.33), we obtain the relationship $\beta > \sigma_v^2/\sigma_s^2$ which agree with the result in (7.22). Therefore, we propose that the value where $\tau > 1$ is to be further investigated. Since $\beta > \sigma_v^2/\sigma_s^2$ which means that β is always positive. From the relationship in (7.31) along with the knowledge that τ is always positive, it is important that $\lambda_{\mathbf{R},max}/\lambda_{\mathbf{R},min} - \tau > 0$. We now obtain that $\tau < \lambda_{\mathbf{R},max}/\lambda_{\mathbf{R},min}$, and therefore the appropriate range for τ is

$$1 < \tau < \frac{\lambda_{\mathbf{R},max}}{\lambda_{\mathbf{R},min}}.\tag{7.34}$$

We now denote the new β -selection method as β_τ since it can be obtained as a function of τ . In Fig. 7.5, the performance of the MUSIC-like algorithm based on the original β and the new β_τ where τ is ranging from $\tau = 1.5, 2.0, 2.5, 3.0$ are compared. With 10 sensor elements ($M = 10$) and 100 snapshots ($p = 100$) obtained, the pseudospectrums of the two targets (members of support set Θ_K where $K = 2$) located at $\theta = 60^\circ$ and 120° under the SNR of 0 dB are shown in Fig. 7.5. It can be seen that although the larger value of τ contributes to higher response of the pseudospectrum, the bias also increases accordingly.

The advantage of the new β_τ is shown in Fig. 7.6. Under the same setting as shown in Fig. 7.5, now the two targets are located at $\theta = 50^\circ$ and 120° with the SNR of -10 dB. We can see that the higher response of the β_τ will contribute to higher probability of detection which provides more robustness to the support recovery stage in the lower SNR condition.

FIGURE 7.5: The spatial spectrum of the MUSIC-like algorithm with β and β_τ .

7.2.3 The Third Approach for β –Selection

In this part, we propose an adaptive framework for the β –selection method where the value of β is automatically adjusted based on the distance between the principle axis of feasible region hyper-ellipsoid and the subspace spanned by the eigenvector $\mathbf{u}_M$ corresponds to $\lambda_{\mathbf{R},min}$. The basis of this approach is based on the notion that high value of β leads to the scenarios as shown in Fig. 7.4 where the result is approaching to the Pisarenko’s method which is prone to spurious peaks. Since the principle axis of each feasible region hyper-ellipsoid is tilted with different angles to $\mathbf{u}_M$ as shown in Fig. 7.7, the inflating factor (β value) should be adjusted according to this information.

FIGURE 7.6: The spatial spectrum of the MUSIC and MUSIC-like algorithm with β and β_τ under low SNR condition.

FIGURE 7.7: Automatic adjustment of the β value corresponding to the ellipsoidal surface adjustment for (a) source direction and (b) non-source direction.

We now propose the distance parameter d_θ which is a function of θ as

$$d_\theta = \left\{ \frac{1 - |\mathbf{u}_M^H \mathbf{a}(\theta)|}{\|\mathbf{a}(\theta)\|_2^2} \right\}, \quad (7.35)$$

where $\mathbf{a}(\theta)$ satisfies $\|\mathbf{a}(\theta)\|_2^2 = 1$. In other applications the θ can be replaced by the discretized parameter of the continuous domain in that specific application, or it can directly be a function of the atoms itself. Note that when $\mathbf{a}(\theta) \in \Theta_K$, the atom is part of the signal subspace, and hence the distance between $\mathbf{a}(\theta)$ and $\mathbf{u}_M$ will be the largest as shown in Fig. 7.7(a). This allows us to apply larger value of β . In contrast, when $\mathbf{a}(\theta) \notin \Theta_K$, the atom is not part of the signal subspace, and hence the distance between $\mathbf{a}(\theta)$ and $\mathbf{u}_M$ will be small as shown in Fig. 7.7(b). In this case the value of β should be small respectively. The variability of the parameter

d_θ is shown in Fig. 7.8 where three targets (members of the support set Θ_K) are located at $\theta = 30^\circ$, $\theta = 70^\circ$ and 130° .

FIGURE 7.8: Variability of d_θ regarding the support locations

Another important parameter is ρ which is used as a gain factor for the β value. We now define the new approach for β -selection as $\beta_\theta = \rho d_\theta$ where ρ sets the gain of the β_θ , and d_θ functions as a weight value.

In Figs. 7.9 and 7.10, the pseudospectrums of the MUSIC-like algorithm with the original β , β_τ where $\tau = 1.75$, and β_θ where $\rho = 0.7$ are shown. Three targets are located at $\theta = 30^\circ$, $\theta = 70^\circ$ and $\theta = 140^\circ$, and 10 sensor elements are being used. As can be seen in Fig. 7.9, with the SNR of -10 dB and 100 snapshots of data, the pseudospectrum of β_θ is able to increase the responses along the targets (support) directions without increasing the noise floor.

Following the same setting as shown in Fig. 7.9, in Fig. 7.10 only 50 snapshots of data is used. It can be seen that β_θ provides more robust performance when compared with β and β_τ .

7.2.4 Simulation Results

In this section, simulation results are presented. Comparison between the performance parameters namely the probability of detection and the root-mean-square

FIGURE 7.9: The spatial spectrum of the MUSIC-like algorithm with β , β_τ , and β_θ .

error (RMSE) of the MUSIC algorithm, the MUSIC-like algorithm with the original β , and the MUSIC-like algorithm with the new β -selection methods are discussed. In what follows, the performance parameters are calculated based on 10 sensor elements, and three targets (support) are located at $\theta = 50^\circ$, $\theta = 65^\circ$ and 110° respectively. Each performance parameter is conducted under the Monte Carlo simulation of 1000 trials.

In the first study, we analyze the performance of the β_τ where the value of τ is varied from $\tau = 1.25$ to $\tau = 3.00$ as shown in Figs. 7.11 and 7.12. It can be seen in Fig. 7.12 that higher value of τ leads to the higher RMSE. However, this does not contribute to higher probability of detection. It can be seen in Fig. 7.11 that the value of τ between $\tau = 1.75$ to $\tau = 2.00$ contributes to the highest probability of detection without heavily compromising on the RMSE performance.

Next, we analyze the performance of β_θ where the value of ρ is varied between $\rho = 0.9$ to $\rho = 0.5$. From Figs 7.13 and 7.14, we can see that higher value of ρ

FIGURE 7.10: The spatial spectrum of the MUSIC-like algorithm with β , β_τ , and β_θ in low snapshots scenario.

FIGURE 7.11: The probability of detecting all of the support plotted against different SNR values with different τ values.

contributes to higher probability of detection while leading to higher RMSE at the same time as can be seen in Fig. 7.14.

We now compare the performance of the MUSIC and the MUSIC-like algorithms with the original β along with the new β which are $\tilde{\beta}$, β_τ where $\tau = 1.75$, and β_θ where $\rho = 0.7$. In the first scenario as shown in Figs. 7.15 and 7.16, 100 snapshots

FIGURE 7.12: The RMSE of the estimated support set plotted against different SNR values with different τ values.

FIGURE 7.13: The probability of detecting all of the support plotted against different SNR values with different ρ values.

of data are obtained. It can be seen in Fig. 7.15 that the β_τ is able to achieve the highest probability of detection. However as can be seen in Fig. 7.16, the $\tilde{\beta}$ is able to achieve the lowest RMSE among different variations of MUSIC-like algorithm in both high and low SNR.

Next, we further investigate the scenario where only 50 snapshots of data can be obtained. It can be seen in Figs. 7.17 and 7.18 that although β_θ leads to the highest

FIGURE 7.14: The RMSE of the estimated support set plotted against different SNR values with different ρ values.

FIGURE 7.15: The probability of detecting all of the support plotted against different SNR values for MUSIC and MUSIC-like algorithms with β , $\tilde{\beta}$, β_τ , and β_θ .

RMSE, it is able to achieve the highest probability of detection as well. This could be beneficial in the case where full recovery is more important than recovering the exact location. On the other hand, the β_τ provides a robust performance where it is able to achieve fair probability of detection while still maintaining low RMSE. However, the probability of detection is slightly lower than β_θ .

FIGURE 7.16: The RMSE of the estimated support set plotted against different SNR values for MUSIC and MUSIC-like algorithms with β , $\tilde{\beta}$, β_τ , and β_θ .

FIGURE 7.17: The probability of detecting all of the support plotted against different SNR values for MUSIC and MUSIC-like algorithms with β , $\tilde{\beta}$, β_τ , and β_θ in low snapshots scenario.

7.3 Part B: Support Identification under Impulse Noise Based on Beamforming Techniques

Signal models in array and statistical signal processing are commonly based on Gaussian distribution assumption, which helps simplifying the design and analysis

FIGURE 7.18: The RMSE of the estimated support set plotted against different SNR values for MUSIC and MUSIC-like algorithms with β , $\tilde{\beta}$, β_τ , and β_θ in low snapshots scenario.

of related algorithms. However, in practical scenarios the signal and particularly the noise model may not admit Gaussian distribution. As mentioned in [58], one particular type of non-Gaussian distributed noise frequently arises in array and statistical signal processing applications is impulse noise, which can be categorized under stable distributions.

It is known that MUSIC algorithm is one of the most well-known super resolution spectral and DOA estimation techniques [50]. This subspace-based method requires either a priori knowledge of the targets or a model order estimated from the second-order moment of the obtained data to segregation two orthogonal subspaces namely the signal and the noise subspaces. When an impulse noise is present ($\alpha < 2$), the second-order moment is infinite. The model order estimators based on second order moment information such as the Akaike Information Criterion (AIC) and the Minimum Description Length (MDL) [59, 60] are likely to produce erroneous estimations, and hence performance degradation of the MUSIC algorithm is expected [61].

In this section, several classes of fractional lower order moment (FLOM) based matrices for spectral and direction-of-arrival (DOA) estimation under isotropic symmetric α -stable (S α S) distributed noise are reviewed. These non-iterative

methods which can be regarded as 1-step M-estimator (a class of weighted covariance matrix) [62] were proposed in the literature to extract valuable information from the data matrix in the case where a complete knowledge of accurate second-order moment is unavailable [63–65]. The techniques namely FLOM, SSCM, and FLIC will be briefly reviewed. Inspired by a symmetric structure of the SSCM method, a modified FLOM (M-FLOM) will be investigated. In this section, we show that the high resolution performance of the MUSIC algorithm can be further improved based on the M-FLOM method. The improvement of this method are two folds. Firstly, we show that the root-mean-squared error (RMSE) of the original FLOM can be reduced by using the modified FLOM (M-FLOM). Secondly, we show that the proposed method can further improve the probability of resolution performance over the original FLOM, SSCM, and FLIC techniques.

7.3.1 The α -Stable Distribution

The α -stable distribution, a sub-class of stable distributions, has been used as a statistical model to encompass the behavior of both Gaussian and Non-Gaussian models [58]. The α -stable distribution can be parameterized by several parameters as $S(\alpha, \beta, \gamma, \mu)$, where $0 < \alpha \leq 2$ is the characteristic exponential, β is the skewness parameter, γ is the scaling parameter, and μ is the location parameter. In this letter, we assume no skewness in the distribution ($\beta = 0$), and hence the characteristic function of the symmetric α -stable distribution can be expressed as

$$\varphi(t) = \exp(j\mu t - \gamma|t|^\alpha). \quad (7.36)$$

Consider signals with zero mean ($\mu = 0$), we can now obtain an isotropic symmetric α -stable distribution (isotropic S α S) which is parameterised mainly by α and γ parameters. It is known that for a stable distribution with characteristic exponent α , only the moments of order less than or equal to α is finite. When $\alpha = 2$, the stable distribution reduces to Gaussian process, and hence the second-order moment is finite. However, when an impulse noise is present, the characteristic exponential is less than 2 ($\alpha < 2$), and hence only moments of order less than or equal to α is finite.

Several techniques were proposed to extract valuable information from the data matrix although the information based on second-order moment is not available. To obtain the MUSIC pseudospectrum in the presence of impulse noise, the fractional lower order moment (FLOM) technique was proposed and reported to have comparable performance with the robust covariation technique [63,64]. An intuitive approach, the spatial sign covariance matrix (SSCM), can also produce satisfactory results [65]. To further improve the performance of FLOM method, the fractional lower order infinity-norm based covariance (FLIC) matrix was proposed, where it was reported to have better performance than the original FLOM technique [66].

7.3.2 Fractional Lower Order Moment Matrices

Several classes of fractional lower order moment (FLOM) based matrices for spectral and direction-of-arrival (DOA) estimation under isotropic symmetric α -stable (S α S) distributed noise are briefly reviewed here. Inspired by the symmetric structure of the spatial sign covariance matrix (SSCM), a modified fractional lower order moment (M-FLOM) matrix will also be investigated. The performances of FLOM, SSCM, and fractional lower infinity-norm covariance (FLIC) matrix will then be compared.

7.3.2.1 Different FLOM-based Matrices:

- Fractional Lower Order Moment ($\mathbf{C}_{FLOM}$): Each element of an $M \times M$ FLOM matrix is defined as

$$C_{ik} = E[x_i(t)|x_k(t)|^{p-2}x_k^*(t)], \quad (7.37)$$

where $1 < p < \alpha \leq 2$. Note that the term $|x_k(t)|^{p-2}$ plays a key role in reducing the effect of the outlier produced by an impulse noise by mean of re-scaling each data point in $x_k^*(t)$ with the re-scaling factor $|x_k(t)|^{p-2}$.

- Fractional Lower Infinity-Norm Covariance Matrix ($\mathbf{C}_{FLIC}$): Inspired by the FLOM matrix, the FLIC matrix was proposed where each element of $\mathbf{C}_{FLIC}$ is defined as

$$C_{ik} = E[(\max\{|x_i(t)|\}_{i=1}^M)^{p-2} x_i(t) x_k^*(t)]. \quad (7.38)$$

It was reported in [66] that the best performance can be achieved when $p = 0$ is set. In the presence of strong impulse noise, it can be seen in (7.38) that all of the data are heavily penalized by the re-scaling factor $(\max\{|x_i(t)|\}_{i=1}^M)^{p-2}$.

- **Spatial Sign Covariance Matrix ($\mathbf{C}_{SSCM}$):** The SSCM approach simply normalizes each data point by the l_2 -norm within its data snapshot. Each element of the $\mathbf{C}_{SSCM}$ can be defined as

$$C_{ik} = E[(\sum_{i=1}^M |x_i(t)|^2)^{-1/2} x_i(t) (\sum_{k=1}^M |x_k(t)|^2)^{-1/2} x_k^*(t)]. \quad (7.39)$$

Comparing the re-scaling factors of FLOM, FLIC, and SSCM, it can be seen that each element of $\mathbf{C}_{SSCM}$ has a symmetric structure. Both $x_i(t)$ and $x_k(t)$ were being re-scaled in the case of SSCM while either $x_i(t)$ or $x_k(t)$ is being re-scaled in the case of FLOM and FLIC.

- **Modified Fractional Lower Order Moment ($\mathbf{C}_{M-FLOM}$):** To obtain a symmetric structure similar to the $\mathbf{C}_{SSCM}$, the FLOM matrix is now modified where each element of $\mathbf{C}_{M-FLOM}$ can be obtained by

$$C_{ik} = E[|x_i(t)|^{p-2} x_i(t) |x_k(t)|^{p-2} x_k^*(t)], \quad (7.40)$$

when $1 < p < \alpha \leq 2$. By having symmetric re-scaling structure in both $x_i(t)$ and $x_k(t)$, the effect of the outlier can be further reduced.

7.3.3 Simulation Results

Performance of the MUSIC algorithm based on the second order moment and the fractional lower order moments are presented in this section. The original FLOM and the modified FLOM methods are obtained by setting $p = 1.1$. Consider a uniform linear array (ULA) of $M = 10$ element-sensors with 3 targets situated at $\Theta = [50^\circ, 65^\circ, 110^\circ]$ and 100 data snapshots obtained. The generalized signal-to-noise ratio (GSNR) according to [63] is used which is defined as $GSNR(dB) = 10 \log_{10}(E[|s(t)|^2]/\gamma^\alpha)$. The impulse noise under the consideration

is complex isotropic $S\alpha S$ distributed. The characteristic function in (7.36) is parameterized by $\alpha = 1.1$ which is equivalent to a strong impulse noise. According to Figs. 7.19 and 7.20, it can be seen that the M-FLOM method is able to resolve all targets in both low and high GSNR scenarios.

FIGURE 7.19: The spatial spectrum of the MUSIC algorithm based on the second order moment and the fractional lower order moments with GSNR = 2 dB.

FIGURE 7.20: The spatial spectrum of the MUSIC algorithm based on the second order moment and the fractional lower order moments with GSNR = 8 dB.

To obtain different GSNR levels, the parameter γ , which is associated with the standard deviation (σ) in Gaussian distribution case ($\alpha = 2$), is varied. The performance parameters (probability of resolution and RMSE) of the MUSIC algorithm based on original FLOM, FLIC, SSCM, and modified FLOM, which were obtained from Monte Carlo simulation of 1000 trials, are shown in Figs. 7.21 and 7.22.

FIGURE 7.21: The probability of resolution for all targets plotted against different levels of GSNR.

FIGURE 7.22: The RMSE for all targets plotted against different levels of GSNR.

It can be seen in Fig. 7.21 that the probability of resolution under low GSNR can be further improved based on the M-FLOM method. Additionally, the RMSE of the original FLOM has been greatly reduced in the M-FLOM method as shown in Fig. 7.22. With the obtained results, the M-FLOM method is considered to be more robust than the other methods since there is a noticeable improvement on targets resolvability while the RMSE has been kept low in a level comparable to FLIC and SSCM methods.

7.3.4 Adaptive MUSIC-like Algorithm under Impulse Noise

To avoid subspace estimation altogether, the MUSIC-like algorithm can be used as an alternative method. The algorithm was originally proposed as a mean to circumvent model order estimation of the MUSIC algorithm, a step which is required when a priori knowledge on the number of targets is unavailable. A high resolution performance comparable to the MUSIC algorithm was shown to be attainable under a beamforming framework formulated in [11]. Detailed analysis of the algorithm was further carried out in [12] where the authors highlighted the role of a relaxation parameter β in regulating the weight vector solution. Aside from the theoretical investigation, additional experiments based on real data under controlled environment were also explored [67, 68].

In this section, the working principle of a relaxation parameter β , a parameter which was explored in the last section, will be further investigated under different noise environment (impulse noise). To illustrate its robustness, the algorithm will be examined under symmetric α -stable distributed noise. The extended adaptive framework will then be developed and proposed in the next section to further optimize the algorithm. The proposed adaptive framework in this section is compared with the original MUSIC-like, MUSIC, FLOM-MUSIC, and SSCM-MUSIC algorithms. A notable improvement in terms of targets resolvability of the proposed method is observed under different impulse noise scenarios as well as different SNR levels.

In an earlier section, we highlighted that the β parameter plays a key role in regulating the feasible region and the ellipsoidal surface for weight vector solution of the MUSIC-like algorithm. The bound for β was originally proposed in [12] as

$$\underbrace{\max_{\theta \in \Theta} \frac{\lambda_{\mathbf{R}, \min}}{(\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}}}_{\beta_{\min}} < \beta < \underbrace{\min_{\theta \notin \Theta} \frac{\lambda_{\mathbf{R}, \min}}{(\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}}}_{\beta_{\max}}, \quad (7.41)$$

with the choice of β to be a value between $\beta_{\min}$ and $\beta_{\max}$ defined as

$$\beta = (1 - \xi)\beta_{\min} + \xi\beta_{\max}, \quad (7.42)$$

where $0 < \xi < 1$ can be chosen by $\xi = \beta_{\min}/\beta_{\max}$.

Detailed analysis regarding the upper bound $\beta_{\max}$ and the lower bound $\beta_{\min}$ can be found in [12] Section IV which can be summarized as follows. According to (26) in [12] section IV, β value should be much larger than $\beta_{\min}$ along the source directions, and according to (29) in [12] section IV, β value should be much smaller than $\beta_{\max}$ along the non-source directions. Note that the β value proposed in [12] is a fixed value throughout all look directions.

The variation of β parameter over different SNR levels is shown in Fig. 7.23. Note that Fig. 7.23 was obtained from a ULA of $M = 10$ sensors. Two targets were situated at $\Theta_K = [50^\circ, 110^\circ]$, and 200 snapshots of data were obtained.

FIGURE 7.23: Variation of relaxation parameter under different SNR levels.

Until recently [67, 68], the value of β has been set according to the SNR of the obtained data (data-dependent), where it is fixed throughout all look directions

(β in Fig. 7.8). In the next section, β will be adaptively readjusted not only according to the SNR (data-dependent) but also throughout each look direction (direction-dependent). This adaptive beta is denoted as β_θ .

7.3.5 The Working Principle of β_θ Parameter

The constraint of the MUSIC-like algorithm can be regarded as an inflated hyper-ellipsoidal surface where the principle axis of the hyper-ellipsoid is spanned by a rank 1 matrix $\mathbf{a}(\theta)\mathbf{a}(\theta)^H$, and the minor axes are inflated by the β value.

For each look direction, the distance between the noise subspace and the hyper-plane as well as the orientation of the hyper-plane itself are varied relative to each steering vector $\mathbf{a}(\theta)$. Since the noise subspace $\mathbf{U}_n$ is not known a priori, the eigenvector $\mathbf{u}_M$ corresponds to the minimum eigenvalue of the covariance matrix $\lambda_{\mathbf{R},min}$ is used as an anchor point. We now propose a distance parameter ξ_θ which approximates the distance between each steering vector and the anchor point. The distance parameter is defined as

$$\xi_\theta = 1 - |\mathbf{u}_M^H \mathbf{a}(\theta)|, \quad (7.43)$$

where $\|\mathbf{a}(\theta)\|_2^2 = 1$, and the value of ξ_θ is varied within the range of $0 \leq \xi_\theta \leq 1$. The new adaptive β_θ can be obtained by substituting (7.43) into (7.42) which can be re-expressed for each look direction as

$$\beta_\theta = \beta_{max} - \delta\beta |\mathbf{u}_M^H \mathbf{a}(\theta)|, \quad (7.44)$$

where $\delta\beta = \beta_{max} - \beta_{min}$. The working principle of β_θ can be summarized as follows. Consider a ULA with $M = 10$ sensors. Two targets are situated at $\Theta_K = [50^\circ, 110^\circ]$ where 200 snapshots of data were obtained with SNR = -5dB ($\alpha = 2$). When $\theta \in \Theta_K$, ξ_θ admits a relatively large value since $|\mathbf{u}_M^H \mathbf{a}(\theta)|$ is small. Hence, the corresponding β_θ is assigned to a large value as shown in Fig. 7.24. With large β_θ , the ellipsoidal surface is then inflated. In contrast, when $\theta \notin \Theta_K$, ξ_θ admits a relatively small value since $|\mathbf{u}_M^H \mathbf{a}(\theta)|$ is large. The corresponding β_θ is now assigned to a relatively small value. With small β_θ , the ellipsoidal surface is now deflated accordingly.

FIGURE 7.24: Variation of relaxation parameters corresponding to each look direction.

The variation of ξ_θ allows β_θ to be adaptively readjusted itself for each look direction, and hence the hyperellipsoid is automatically inflated or deflated corresponding to each look direction. On the other hand, the old β (marked as β in Fig. 7.24) remains constant for each look direction.

7.3.5.1 Simulation Results

Consider a ULA with $M = 10$ sensors. Three targets are situated at $\Theta_K = [50^\circ, 60^\circ, 110^\circ]$ where 100 data snapshots were obtained. When $\alpha < 2$, a generalized signal-to-noise ratio (GSNR), which is defined as $GSNR(dB) = 10\log_{10}(E[|s(t)|^2]/\gamma^\alpha)$, will be used. A complex $S\alpha S$ distributed noise is considered in all simulation studies. When $\alpha = 2$, the GSNR reduces to the traditional SNR where γ plays the same role as the standard deviation σ in a Gaussian process. To obtain the FLOM matrix, $p = 1.1$ is set according to the best result reported in [63]. In Fig. 7.25, α is set to 1.8 where the GSNR is set to -2dB. It can be seen that the MUSIC-like algorithm with adaptive β_θ has the ability to differentiate closely spaced sources better than the MUSIC, FLOM-MUSIC, SSCM-MUSIC, and the original MUSIC-like algorithm with fixed β .

Next, performance parameters (probability of resolution and average root-mean-squared error (RMSE)) of related algorithms are studied under the Monte Carlo simulation of 1,000 trials based on different values of α . To illustrate the effect of each selected α , the real part of complex isotropic $S\alpha S$ distributed noise corresponds to $\alpha = 2, 1.9, 1.8$, and 1.7 are shown in Fig. 7.26. To obtain performance parameters, three targets are situated at $\Theta_K = [50^\circ, 65^\circ, 110^\circ]$ where 100 snapshots of data were obtained. The probability of resolution and the average RMSE with different levels of α are shown in Figs. 7.27 and 7.28. When $\alpha = 2$, the $S\alpha S$ distribution reduces to a Gaussian process, and as α decreases ($\alpha < 2$) the likelihood of outlier occurrence increases accordingly. It can be seen in Fig. 7.27 that the proposed method is able to achieve very good performance for the probability of resolution (best targets resolvability) especially in low GSNR conditions. Slight improvement from MUSIC algorithm can be obtained from the SSCM-MUSIC especially when the value of α decreases (e.g. $\alpha = 1.7$). Performance of the original MUSIC-like algorithm with fixed β is comparable to the FLOM-MUSIC.

FIGURE 7.25: The spatial spectrum of related algorithms with 100 snapshots for $M = 10$, $K = 3$, and $\text{GSNR} = -2\text{dB}$.

FIGURE 7.26: Real part of the complex isotropic $S\alpha S$ distributed noises for $\alpha = 2, 1.9, 1.8$, and 1.7 , respectively.

Although with significant improvement on targets resolvability, we also note a crucial trade-off between such improvement and the estimation RMSE. It can be seen in Fig. 7.28 that the proposed method tends to have higher bias than the other methods on estimated directions. Such trade-off would be tolerable in certain applications such as the source localization under quasi-static environment, where detection and resolvability are of highest priority and certain biasness is acceptable [17, 52, 68].

FIGURE 7.27: The probability of resolution plotted against different levels of GSNR with $\alpha = 2, 1.9, 1.8$, and 1.7 , respectively.

FIGURE 7.28: The average RMSE plotted against different levels of GSNR with $\alpha = 2, 1.9, 1.8$, and 1.7 , respectively.

7.3.6 Improved Directional Adaptive MUSIC-like Algorithm

In the last section, an adaptive framework for β -selection method was discussed. The proposed method provided additional improvement on target's resolvability performance over the fixed β . However, it is noted that a trade-off between such improvement and estimation bias is inherent. It should be noted that the proposed adaptive framework could be beneficial in applications where target's resolvability is of highest priority, and certain estimation bias is tolerable.

Although with great improvement on target's resolvability performance, a trade-off between such improvement and the estimation bias is inherent. In this section, two novel directional adaptive β -selection methods for MUSIC-like algorithm under α -stable distributed noise are proposed. The proposed methods aim at reducing estimation bias and noise sensitivity which are inherent in prior adaptive β framework. Simulation results highlight noticeable reduction in the estimation bias as well as the noise sensitivity of the proposed methods without excessive compromise

on target's resolvability performance when compared with the original adaptive β framework.

To summarize, the underlying notion was based on a distance parameter ξ_θ which approximates the distance between each steering vector and the anchor point $\mathbf{u}_M$, which is an eigenvector corresponds to the minimum eigenvalue $\lambda_{\mathbf{R},min}$ of the data covariance matrix $\mathbf{R}$. Each ξ_θ can be obtained by $\xi_\theta = 1 - |\mathbf{u}_M^H \mathbf{a}(\theta)|$, where $\|\mathbf{a}(\theta)\|_2^2 = 1$, and the value of ξ_θ is varied within the range of $\xi_\theta \in [0, 1]$. The adaptive $\beta_P(\theta)$ can be obtained by substituting ξ_θ into (7.42), and the adaptive $\beta_P(\theta)$ can now be re-expressed for each look direction as

$$\beta_P(\theta) = \beta_{max} - \delta\beta |\mathbf{u}_M^H \mathbf{a}(\theta)|, \quad (7.45)$$

where $\delta\beta = \beta_{max} - \beta_{min}$. The working principle of $\beta_P(\theta)$ can be briefly summarized as follows. The value of $\beta_P(\theta)$ along the source directions ($\theta \in \Theta$) are allowed to take up a relatively large value without exceeding the value of β_{max} . On the other hand, the value of $\beta_P(\theta)$ along the non-source directions ($\theta \notin \Theta$) should be kept low without extending lower than the value of β_{min} . It was noted that though with great improvement on resolvability performance, the estimated direction of $\beta_P(\theta)$ is inherently bias. In the next section, two novel adaptive β -selection methods are proposed in order to reduce the estimation bias and noise sensitivity of $\beta_P(\theta)$.

7.3.6.1 The First Method: Adaptive $\beta_C(\theta)$

With result obtained from (7.45), it can be observed that the choice of $\beta_P(\theta)$ is associated with an inverse of Pisarenko's output [69], which is an inner product between an eigenvector $\mathbf{u}_M$ and the steering vector corresponds to each look direction $\mathbf{a}(\theta)$. Due to noise sensitivity characteristic of Pisarenko's method, we propose an alternative method with an objective to reduce the estimation bias of $\beta_P(\theta)$ while maintaining a satisfactory targets resolvability performance. A new adaptive $\beta_C(\theta)$ parameter based on a notion of re-scaled Capon's output is proposed, where $\beta_C(\theta)$ can be varied within the range of $[\beta_{min}, \beta_{max}]$. The new adaptive $\beta_C(\theta)$ is defined as

$$\beta_C(\theta) = \frac{(\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1} - \min_{\theta \notin \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}}{\max_{\theta \in \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1} - \min_{\theta \notin \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}} \delta\beta + \beta_{min}, \quad (7.46)$$

The formulation behind (7.46) can be summarized as follows. To obtain a re-scaled Capon's output within the range of $[\beta_{min}, \beta_{max}]$, the Capon's output is first obtained by computing $(\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}$ for each look direction [70]. Next, the minimum noise power ($\min_{\theta \notin \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}$) and the maximum targets power ($\max_{\theta \in \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}$) are determined to obtain the range of the spatial spectrum. The $\beta_C(\theta)$ can now be obtained by the procedure denoted in (6) which comprises of 4 steps as follows. Firstly, the Capon's output for each look direction is shifted down by the amount reflected in the minimum power of the noise floor. Secondly, it is normalized by the range of the spatial spectrum. Thirdly, it is re-scaled up to the range of $[\beta_{min}, \beta_{max}]$ by $\delta\beta$. Lastly, it is shifted up so that the $\beta_C(\theta) \geq \beta_{min}$. Consider a uniform linear array (ULA) of $M = 10$ sensors with 3 targets situated at $\Theta = [50^\circ, 65^\circ, 110^\circ]$ where 100 data snapshots are obtained and SNR = -6 dB. The variation of $\beta_P(\theta)$ and $\beta_C(\theta)$ can be seen in Fig. 7.29 (a) and (b), respectively.

(a) $\beta_P(\theta)$

(b) $\beta_C(\theta)$

FIGURE 7.29: Variation of (a) $\beta_P(\theta)$ and (b) $\beta_C(\theta)$ correspond to each look direction.

In Fig. 7.29 (a), the variation of $\beta_P(\theta)$ along each look direction reflects noise sensitivity characteristic of this method. On the other hand, as can be seen in Fig. 7.29 (b), the larger values of $\beta_C(\theta)$ are allowed along source directions while they are kept low along the non-source directions. As will be shown in the next

section, this enhancement of $\beta_C(\theta)$ along the source directions together with a suppression along the non-source directions is able to lower the estimation bias without compromising the resolvability performance of the algorithm.

7.3.6.2 The Second Method: Adaptive $\beta_{LC}(\theta)$

With an objective to further optimize the values of $\beta_C(\theta)$ along source and non-source directions, it is observed that the values of $\beta_C(\theta)$ along the non-source directions as shown in Fig. 7.29 (b) can be further reduced. We now propose a new adaptive $\beta_{LC}(\theta)$ where the working principle is as follows. Along the source directions, $\beta_{LC}(\theta)$ is allowed to take up maximum value (β_{max}) while it should be kept minimum (β_{min}) along the non-source directions. This procedure can be accomplished through a generalized logistic function (GLF) which is defined as $Y(x) = A + (K - A)/(1 + \exp(-B(x - M)))$, where $Y(x)$ is the GLF output, A and K are the lower and upper asymptotes, B is the growth rate, and M is the location with maximum growth rate. To illustrate the GLF output, a plot of an arbitrary GLF is shown in Fig. 7.30 (a) where A and K are set as 1 and 3, respectively, B is set to 2, and M is set to 4.

(a)

(b)

FIGURE 7.30: (a) Generalized logistic function and (b) Variation of $\beta_{LC}(\theta)$ corresponds to each look direction.

The new $\beta_{LC}(\theta)$ can be obtained by passing Capon's output to the GLF where lower and upper asymptotes to be set as β_{min} and β_{max} , respectively. By doing so, the value of $\beta_{LC}(\theta)$ will be limited within the range of $[\beta_{min}, \beta_{max}]$. The growth rate is now defined by β_{max}/β_{min} which corresponds to the signal-to-noise ratio

(SNR) of the obtained data. When SNR is high, the obtained data are with high fidelity, and hence the growth rate ($\beta_{max}/\beta_{min} = 1/\xi$) is automatically set to a high value, which is equivalent to a sharp slope in Fig. 7.30 (a). This can be regarded as a sharp cut-off when considering if a particular direction has a high probability of being source direction. In contrast, when SNR is low, the obtained data are with low fidelity, and hence the growth rate will be automatically set to a lower value. The location for maximum growth rate works in a similar fashion as a location of the cutoff point, where any value that is greater than the cutoff point will be pushed toward the upper asymptote, and any value that is less than the cutoff point will be pushed toward the lower asymptote. The cutoff point is specified by $\chi \max_{\theta \in \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}$, where $\chi = 0.8$ is used. Although not identical, this can be regarded in a similar fashion as a way to specify the angles of half-power-beamwidth in a spectral plot. The new adaptive $\beta_{LC}(\theta)$ can now be defined as

$$\beta_{LC}(\theta) = \beta_{min} + \delta\beta / (1 + \exp(-\frac{(\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1} - \chi \max_{\theta \in \Theta} (\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta))^{-1}}{\xi})). \quad (7.47)$$

As can be seen in Fig. 7.30 (b), $\beta_{LC}(\theta)$ along the source directions were assigned to a relatively large value while it is kept to the minimum value along the non-source directions. The discriminating power of $\beta_{LC}(\theta)$ can be observed by comparing Fig. 7.30 (b) with Fig. 7.29 (b).

7.3.6.3 Simulation Results

In this section, performance of the MUSIC-like algorithm based on the proposed $\beta_C(\theta)$ and $\beta_{LC}(\theta)$ are compared with the MUSIC, FLOM-MUSIC, and the MUSIC-like algorithm based on fixed β and $\beta_P(\theta)$. For signal source simulation, we follow the same configuration as presented in the previous section (ULA of $M = 10$ with 100 snapshots of data). The generalized signal-to-noise ratio (GSNR) will be used when $\alpha < 2$, which is defined as $GSNR(dB) = 10 \log_{10}(E[|s(t)|^2]/\gamma^\alpha)$. The impulse noise under the consideration is complex isotropic symmetric α -stable ($S\alpha S$) distributed. The α parameter specifies the likelihood of outlier occurrence as $\alpha < 2$ gets smaller. To obtain different GSNR levels, the γ parameter, which can be associated with the standard deviation under Gaussian distribution, is varied. To obtain the spatial spectrum of FLOM-MUSIC algorithm (when $\alpha < 2$),

each element of an $M \times M$ FLOM matrix is first estimated prior to the subspace decomposition. This can be accomplished through the weighted covariance matrix $C_{ik} = E[x_i(t)|x_k(t)|^{p-2}x_k^*(t)]$, where $p = 1.1$ is set.

FIGURE 7.31: The spatial spectra of (a) MUSIC, FLOM-MUSIC, and MUSIC-like algorithm with $\beta_P(\theta)$ and (b) MUSIC-like algorithm with β and $\beta_P(\theta)$, $\beta_C(\theta)$, and $\beta_{LC}(\theta)$

The spatial spectra of related algorithms are shown in Fig. 7.31 where two targets are situated at $\Theta = [70^\circ, 85^\circ]$ and $\text{GSNR} = 0$ dB. As can be seen, the MUSIC algorithm fails under an impulse noise condition ($\alpha = 1.7$). However, the FLOM-MUSIC and the MUSIC-like with either fixed or adaptive β are able to identify all targets in non-Gaussian scenarios. Among the MUSIC-like algorithms, note that $\beta_{LC}(\theta)$ has the lowest noise floor followed by $\beta_C(\theta)$, $\beta_P(\theta)$, and β .

A comparison between spatial spectra produced by the MUSIC-like algorithm with $\beta_C(\theta)$ and $\beta_{LC}(\theta)$ is shown in Fig. 7.32. As can be seen in Figs. 7.32(a) and 7.32(b), the spectra plots obtained from $\beta_{LC}(\theta)$ are less sensitive along the non-source directions when compared with the spectra plots obtained from $\beta_C(\theta)$. Additional arrows are provided in both figures to highlight the sharp response characteristic of $\beta_{LC}(\theta)$ over $\beta_C(\theta)$. Although with sharp response characteristic of $\beta_{LC}(\theta)$, it should be noted that this is accomplished through an extra step (GLF) taken while computing $\beta_{LC}(\theta)$.

FIGURE 7.32: The spatial spectra of the MUSIC-like algorithm with $\beta_C(\theta)$ and $\beta_{LC}(\theta)$ where (a) $\alpha = 2$, $\text{GSNR} = -5$ dB and (b) $\alpha = 1.7$, $\text{GSNR} = 5$ dB.

Next, the performance parameters, the probability of resolution and the average root-mean-square error (RMSE), of related algorithms under Gaussian and non-Gaussian noise scenarios ($\alpha=2$ and 1.7) at different levels of GSNR based on Monte Carlo simulation of 1000 trials with three targets situated at $\Theta = [50^\circ, 65^\circ, 110^\circ]$ are shown in Figs. 7.33 and 7.34. Note that in the case of $\alpha = 2$, the complex isotropic $S\alpha S$ reduces to Gaussian distribution, and hence FLOM-MUSIC will not be used

FIGURE 7.33: The probability of resolution of the related algorithms where (a) $\alpha = 2$ and (b) $\alpha = 1.7$ were set.

FIGURE 7.34: The RMSE of the related algorithms where (a) $\alpha = 2$ and (b) $\alpha = 1.7$ were set.

for comparison. It can be seen in Figs. 7.33 and 7.34, the proposed $\beta_C(\theta)$ is able to reduce estimation bias of $\beta_P(\theta)$ without compromising the targets' resolvability performance. Additional reduction on estimation bias can be obtained by $\beta_{LC}(\theta)$.

It can be seen in Fig. 7.34 that the estimation bias of $\beta_{LC}(\theta)$ is approaching to the fixed β . However, the rate of targets resolvability of $\beta_{LC}(\theta)$ is slightly slower than the other methods in a certain range of GSNR levels as can be seen in Fig. 7.33. Note that both fixed and adaptive β -selection methods exhibit a robust characteristic which work in both Gaussian and non-Gaussian noise without requiring a preconditioned covariance matrix as the FLOM-MUSIC algorithm does.

7.4 Conclusions

In this chapter, we have further investigated the MUSIC-like algorithm with different approaches for β -selection methods. The first β -selection method ($\tilde{\beta}$) provides lower RMSE when compared with the original β . The analysis of $\tilde{\beta}$ also serves as a foundation for further analysis on the second and the third β -selection methods (β_τ and β_θ). The β_τ provides the most robust performance in term of probability of detection under different SNR scenarios without introducing high RMSE while the β_θ provides a robust performance in term of probability of detection under different SNR scenarios with varying number of data snapshots. The applicability of the proposed method to the joint sparse signal recovery under the MMV framework has been covered throughout the chapter. With the robust performance characteristic of the proposed method under different SNR and data snapshots scenarios, the MUSIC-like algorithm is able to be applied as an alternative method to the MUSIC algorithm for support recovery stage in joint sparse signal reconstruction.

Investigations of the original MUSIC-like algorithm and the proposed method under $S\alpha S$ distributed noise were conducted under impulse noise environment. The original MUSIC-like and the adaptive MUSIC-like algorithms were compared with the MUSIC, FLOM-MUSIC, and the SSCM-MUSIC algorithms. Simulation results highlight a notable improvement of the proposed method over other methods in terms of targets resolvability. We also note a crucial trade-off between such improvement and the estimation bias which is inherent in the proposed method. The proposed method is therefore suitable for applications where detection and targets resolvability are of highest priority and slight bias is acceptable.

Two novel adaptive β -selection methods ($\beta_C(\theta)$ and $\beta_{LC}(\theta)$) were proposed. Simulation results highlight a reduction in estimation bias and noise sensitivity characteristic, which is inherent in the original adaptive framework obtained from $\beta_P(\theta)$. In short, both $\beta_C(\theta)$ and $\beta_{LC}(\theta)$ are able to obtain lower estimation bias than $\beta_P(\theta)$. The $\beta_C(\theta)$ is best at balancing both performance parameters. Within the adaptive framework, $\beta_{LC}(\theta)$ has the lowest estimation bias with slight trade-off on the rate of targets resolvability performance.

Chapter 8

Hybrid Framework: Subspace Fitting for Support Identification

8.1 Introduction

In this chapter, a set of support (targets) in DOA estimation environment is identified under the MMV framework where the number of data snapshots is limited. Under such scenario, performance degradation of the subspace based method such as the MUSIC algorithm is expected. Thus, the subspace fitting regularization is proposed. Without estimating the model order of the covariance matrix, the noise subspace is estimated through the iterative process. A Rayleigh quotient is used to estimate one signal subspace at a time. Subsequently, the noise subspace is then estimated through the orthogonality property between the signal subspace and the noise subspace. The process is repeated through several iterations. The prior knowledge for the regularization framework can be obtained by projecting measurement data onto the estimated noise subspace. The solution is then estimated through an iterative process until two consecutive estimate solutions are bounded within a small threshold value.

This chapter introduces hybrid beamforming-regularization frameworks for direction-of-arrival (DOA) estimation application. A set of support (targets) in DOA estimation is identified under the MMV framework where the number of data snapshots is limited. Under such scenario, performance degradation of the subspace based method such as the MUSIC algorithm is expected. Thus, the subspace fitting

regularization is proposed. Without estimating the model order of the covariance matrix, the noise subspace is estimated through the iterative process. A Rayleigh quotient is used to estimate one signal subspace at a time. Subsequently, the noise subspace is then estimated through the orthogonality property between the signal subspace and the noise subspace. The process is repeated through several iterations. The prior knowledge for the regularization framework can be obtained by projecting measurement data onto the estimated noise subspace. The solution is then estimated through an iterative process until two consecutive estimate solutions are bounded within a small threshold value.

8.2 Subspace Fitting Regularization for Support Identification

In this section, we propose the second hybrid beamforming-regularization framework for DOA estimation under the scenario where only limited data snapshots are available. As mentioned in the previous chapter, it should be noted that the proposed method is applicable to other support identification task and not limited to only DOA estimation. In DOA estimation, the data snapshot (sample) obtained is commonly assumed to be sufficiently large ($K \gg M$ where K is the number of snapshots and M is the number of measurements or sometimes sensors). In certain scenario especially in the context of highly dynamic targets, the received data snapshots could be limited. The estimated data covariance matrix is thus inaccurate and unreliable, and that could lead to performance degradation especially for subspace based technique such as MUSIC algorithm. In this section, a subspace fitting regularization method is proposed where statistical information from data covariance matrix are extracted iteratively without estimating the model order directly. The information are then combined with the regularization framework to uncover a set of active support. The procedure of the proposed method is summarized and shown in Fig. 8.1.

It is important to note that the regularization framework attempts to minimize the model-to-data fitness which is the term $\min_{\mathbf{S} \in \mathbb{R}^{N \times K}} \{\|\mathbf{A}\mathbf{S} - \mathbf{X}\|_2^2\}$ and the penalty term $\min_{\mathbf{S} \in \mathbb{R}^{N \times K}} \{\|\mathbf{L}\mathbf{S}\|_2^2\}$ simultaneously through a tuning parameter λ . As highlighted in the previous chapters (especially in Chapter 4), the regularization matrix

$$\begin{aligned}
& \underset{\mathbf{S} \in \mathbb{R}^{N \times K}}{\text{minimize}} \|\mathbf{A}\mathbf{S} - \mathbf{X}\|_2^2 + \lambda^2 \|\mathbf{L}\mathbf{S}\|_2^2 & \mathbf{R} = \frac{\mathbf{X}\mathbf{X}^H}{K} \\
& \hat{\mathbf{S}} = (\mathbf{A}^H \mathbf{A} + \lambda^2 \mathbf{L}^H \mathbf{L})^{-1} \mathbf{A}^H \mathbf{X} & \mathbf{R} = [\mathbf{U}_s | \mathbf{U}_n][\Sigma_s | \Sigma_n][\mathbf{U}_s | \mathbf{U}_n]^H
\end{aligned}$$

$$\begin{aligned}
& \mathbf{L}^H \mathbf{L} = \mathbf{A}^H (\mathbf{I} - \mathbf{U}_s \mathbf{U}_s^H) \mathbf{A} \\
& \|\mathbf{L}\mathbf{S}\|_2^2 = \|(\mathbf{I} - \mathbf{U}_s \mathbf{U}_s^H) \mathbf{A}\mathbf{S}\|_2^2 \\
& \|\mathbf{L}\mathbf{S}\|_2^2 = \|(\mathbf{U}_n \mathbf{U}_n^H) \mathbf{X}\|_2^2 \\
& \downarrow \\
& \hat{\mathbf{S}} = (\mathbf{A}^H \mathbf{A} + \lambda^2 \mathbf{A}^H (\mathbf{U}_n \mathbf{U}_n^H)^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{X}
\end{aligned}$$

FIGURE 8.1: The procedure of subspace fitting regularization.

is commonly chosen or constructed based on the information of the image pixels e.g. smoothness, sharpness, or sparsity. In this section, the objective is to uncover useful information from data covariance matrix with limited sample and to use such information as a prior for the regularization framework.

As shown in Fig. 8.1, the basis of this approach can be summarized as follows. It is more comprehensible to first focus on the penalty term $\|\mathbf{L}\mathbf{S}\|_2^2 = \|(\mathbf{I} - \mathbf{U}_s \mathbf{U}_s^H) \mathbf{A}\mathbf{S}\|_2^2$ in Fig. 8.1. This suggests that on the penalty term, we require that the obtained data $\mathbf{X}$ does not reside in the noise subspace since we minimize the term $\|\mathbf{L}\mathbf{S}\|_2^2 = \|(\mathbf{U}_n \mathbf{U}_n^H) \mathbf{X}\|_2^2$. This is an appropriate prior on the penalty term since it is known that $\mathbf{X} = \mathbf{A}_{Supp} \mathbf{S}$, where $\mathbf{A}_{Supp}$ are the atoms which correspond to the signal in $\mathbf{X}$. We also know that $\mathbf{A}_{Supp}$ is orthogonal to the noise subspace. Hence, the projection of $\mathbf{X}$ into the noise subspace spanned by $\mathbf{U}_n \mathbf{U}_n^H$ should be minimized. We can extend that $\mathbf{U}_n \mathbf{U}_n^H = \mathbf{I} - \mathbf{U}_s \mathbf{U}_s^H$ as well as $\mathbf{X} = \mathbf{A}\mathbf{S}$, which leads to the expression $\|\mathbf{L}\mathbf{S}\|_2^2 = \|(\mathbf{I} - \mathbf{U}_s \mathbf{U}_s^H) \mathbf{A}\mathbf{S}\|_2^2$. Using the fact that $(\mathbf{U}_n \mathbf{U}_n^H)^2 = \mathbf{U}_n \mathbf{U}_n^H$, we obtain $\mathbf{L}^H \mathbf{L} = \mathbf{A}^H (\mathbf{I} - \mathbf{U}_n \mathbf{U}_n^H) \mathbf{A}$.

In scenario where only limited data snapshot is available, the complete knowledge of $\mathbf{U}_n$ is unavailable. We now propose an iterative process to uncover the noise subspace which is summarized in Figs. 8.2 and 8.3.

The procedure starts by estimating the covariance matrix in the first iteration which is labeled as $\mathbf{R}^0$. By using the power method, the first eigenvector and eigenvalue

FIGURE 8.2: First iteration of subspace fitting regularization.

can be obtained which are labeled as $\mathbf{u}_1$ and λ_1 respectively. Next, the estimated noise subspace in the first iteration can be obtained by $\mathbf{U}_n^1 = \mathbf{I} - \mathbf{u}_1 \mathbf{u}_1^H$. This will be used to construct the regularization matrix as $\mathbf{L}^{1H} \mathbf{L}^1 = \text{diag}\{\mathbf{A}^H \mathbf{U}_n^1 \mathbf{U}_n^{1H} \mathbf{A}\}$, and the inverse problem can be solved subsequently.

FIGURE 8.3: Second iteration of subspace fitting regularization.

In the second iteration, the contribution from the previous eigenvector and eigenvalue will be excluded. The new covariance matrix can be obtained by calculating $\mathbf{R}^1 = \mathbf{R}^0 - \lambda_1 \mathbf{u}_1 \mathbf{u}_1^H$. The new eigenvector and eigenvalue can be obtained through the power method again, which leads to the second eigenvector ($\mathbf{u}_2$) and eigenvalue λ_2 respectively. We follow the same procedure as the last iteration which is to construct the estimated noise subspace for this iteration as $\mathbf{U}_n^2 = \mathbf{I} - \mathbf{u}_1 \mathbf{u}_1^H - \mathbf{u}_2 \mathbf{u}_2^H$. The regularization matrix can then be obtained by calculating $\mathbf{L}^{2H} \mathbf{L}^2 = \text{diag}\{\mathbf{A}^H \mathbf{U}_n^2 \mathbf{U}_n^{2H} \mathbf{A}\}$, and the inverse problem can be solved subsequently. The procedure continues until the stopping criterion is met. The

stopping criterion will be presented in the later section after some preliminary results are shown.

8.2.1 Working Principle of Subspace Fitting Regularization

The working principle of subspace fitting regularization is highlighted in Figs. 8.4. One of the most important aspect for this method is the calculation of regularization matrix $\mathbf{L}^{iH}\mathbf{L}$ at each iteration. Consider three underlying supports at $\mathbf{A}_{Supp} = \mathbf{a}_{30}, \mathbf{a}_{80}, \mathbf{a}_{150}$, where $\mathbf{a}_i$ indicates i^{th} column as an active parameter. The data is obtained through $M = 16$ measurements and $K = 30$ data snapshots at $\text{SNR} = 0$ dB. For infinite number of data snapshots, the covariance matrix will contain three dominant eigenvalues. However, in the scenario where the available data snapshot is limited, we estimate the noise subspace iteratively through $\mathbf{U}_n^i \mathbf{U}_n^{iH}$. By computing $\text{diag}\{\mathbf{A}^H \mathbf{U}_n^i \mathbf{U}_n^{iH} \mathbf{A}\}$, the indices that are corresponding to the support will yield smaller values since those locations are not part of the noise subspace.

FIGURE 8.4: The working principle of subspace fitting regularization.

The small values contribute to small weight values in the penalization part of the regularization framework which means less penalization in the inverse solving process. The result is refined through several iterations where at each iteration the noise subspace is re-evaluated. Typically, the result of the first few iterations will

yield a good estimate. However, the result is improved (refined) through several iterations.

8.2.2 Simulation Results

Preliminary simulation results are presented in this section. In the first simulation experiment, we consider $M = 16$ measurements with a sensing matrix (array manifold) of size $\mathbf{A} \in \mathbb{R}^{16 \times 180}$. The support location are located at 30, 40, 120, and 150 atom (deg). In the later part, we will be experimenting with a random sensing matrix due to its RIP property where each atom (especially the nearby atoms) are uncorrelated.

FIGURE 8.5: Spatial spectrum of four supports located at 30, 40, 120, and 150 with 100 data snapshots and $\text{SNR} = 0$ dB.

The spatial spectrum as shown in Fig. 8.5 represents the result of DOA estimation when the data snapshot is sufficient ($K = 100$ in this case). It can be seen that all targets (supports) can be detected. With non-degraded covariance matrix, the subspace fitting regularization is also able to identify all targets as shown in Fig. 8.6. It can be seen that the resolution of the estimated amplitude is refined over the iterations.

FIGURE 8.6: Four iterations of estimate solution with four supports located at 30, 40, 120, and 150 with 100 data snapshots and $\text{SNR} = 0$ dB.

The DOA estimation techniques fail at identifying most targets when the obtained data snapshot is insufficient (in this case $K = 15$). It can be seen in Fig. 8.7 that DOA estimation techniques cannot identify all targets correctly. As mentioned in the previous section for the three-step processing, the method will fail from the first step (target scanning).

FIGURE 8.7: Spatial spectrum of four supports located at 30, 40, 120, and 150 with 15 data snapshots and $\text{SNR} = 0$ dB.

However, through the subspace fitting regularization, all of the supports can be identified through several round of iterations as can be seen in Fig. 8.8. The resolution is refined through each step of iteration. In the context of three-stage processing, this subspace fitting regularization can be used in the case where number of data snapshots is small.

FIGURE 8.8: Four iterations of estimate solution with four supports located at 30, 40, 120, and 150 with 15 data snapshots and $\text{SNR} = 0$ dB.

8.2.2.1 A Test for Random Sensing Matrix

Next, we are experimenting with random sensing matrix that has good RIP property when compared with the array manifold matrix. The random matrix is shown in Fig. 8.9. In what follows, we specify three active supports which are situated at the 30^{th} , 80^{th} , and 150^{th} atoms $\{\mathbf{a}_{30}, \mathbf{a}_{80}, \mathbf{a}_{150}\}$ as can be seen in Fig. 8.9.

FIGURE 8.9: Signal model for random sensing matrix.

The result of utilizing DOA estimation techniques for support identification is highlighted in Fig. 8.10. It can be seen that with sufficient number of data snapshots (consider $K = 30$ in this case), although not a very high number, the DOA estimation techniques are able to identify all of the support locations (30^{th} , 80^{th} , and 150^{th} in the search space (atom)).

FIGURE 8.10: Spatial spectrum of MUSIC and MUSIC-like algorithm based on 30 data snapshots.

As shown in Fig. 8.11, it can be seen that the subspace fitting regularization can confidently identify all of support locations. It can be seen that the result is refined through several round of iterations.

FIGURE 8.11: Four iterations of estimate solution of subspace fitting regularization with three supports based on 30 data snapshots.

The more challenging scenario is shown in Fig. 8.12 where only 14 data snapshots were obtained. It can be seen that the DOA estimation techniques which rely on degraded covariance matrix are unable to identify the support locations. In contrast, it can be seen in Fig. 8.13 that the subspace fitting regularization can confidently identify all of the support locations.

FIGURE 8.12: Spatial spectrum of MUSIC and MUSIC-like algorithm based on 14 data snapshots.

FIGURE 8.13: Four iterations of estimate solution of subspace fitting regularization with three supports based on 14 data snapshots.

8.2.3 Stopping Criterion

In this section, we will be discussing one of the methods that can be used as a stopping criterion. This method is briefly summarized and shown in Fig. 8.14. In Fig. 8.14, there are two images which are located on the left side and the right side. On the left side, a matrix of reconstructed images are stacked up where each column represents one reconstructed image. The $\hat{\mathbf{S}}^1$ represents the images reconstructed from the 1st iteration, and $\hat{\mathbf{S}}^j$ represents the images reconstructed from the j^{th} iteration. The $\hat{\mathbf{S}}^1(:, 1)$ indicates the 1st column of $\hat{\mathbf{S}}^1$ which represents the first image reconstructed from $\hat{\mathbf{S}}^1$ (Frame 1), and $\hat{\mathbf{S}}^1(:, k)$ means the k^{th} column of $\hat{\mathbf{S}}^1$ which represents the first image reconstructed from $\hat{\mathbf{S}}^1$ (frame k). We then calculate the l_1 -norm of each column of the reconstructed image, hence we can obtain a vector $\mathbf{a}^j$ for j^{th} iteration. The stopping criterion can be defined to terminate the iteration process once the difference between variance of current iteration ($\mathbf{a}^{j+1}$) when compared with the variance of previous iteration ($\mathbf{a}^j$) is less than a small positive value ϵ or $\|\text{var}(\mathbf{a}^{j+1}) - \text{var}(\mathbf{a}^j)\|_p < \epsilon$. The ϵ value can be chosen based on

the environment and requirement of each application.

FIGURE 8.14: Stopping criterion based on amplitude variation.

As can be seen from previous simulation results, the reconstructed image in the first few iterations are with high variation. The result is refined through several round of iteration process, and the result become relatively steady (not much variation when compared with the first few iteration).

8.3 Conclusion

This chapter introduces hybrid beamforming-regularization frameworks for DOA estimation that is applicable for other classes of support identification tasks. The proposed method is called subspace fitting regularization which is investigated to handle the problem of limited data snapshot. Based on an iterative process, the estimate solution attempts to uncover (extract) meaningful information from the data matrix. It is important to note that this method can be used in the first step of the three-step method (presented earlier) when the available data snapshots are limited. Preliminary results highlight its ability to uncover support set while simple beamforming methods alone might fail to do so.

Chapter 9

Conclusions and Future Directions

9.1 Conclusions

The objective of the thesis is to propose several algorithms and frameworks for support identification tasks where we put our focus on EIT application and DOA estimation. In EIT application, to experiment with different data format (e.g. SMV and MMV), a robust and reliable EIT system that can be implemented economically with simple procedure was proposed in Chapter 3. The fundamental of EIT system and its working principal is also briefly reviewed within this chapter. The proposed framework for implementing EIT prototype system should be beneficial for those whose interest lies in image reconstruction and solving discrete ill-posed inverse problem. Two frameworks namely SMV and MMV are investigated in Chapter 4 and Chapter 5.

A robust hybrid regularization matrix is proposed in Chapter 4 which is based on the SMV framework. With an insight of the recently proposed FER regularization matrix coupled with the understanding of the conventional NOSER regularization matrix, a robust hybrid regularization matrix is proposed. One of the essential contributions of this chapter is an insight which connects the FER and NOSER regularization matrices together. The connection is highlighted in both spatial domain and SVD domain. The proposed hybrid regularization matrix, which has robust performance (trade-off between FER and NOSER), is also investigated with

real experimental data. It is shown in the experimental result that the hybrid regularization matrix inherit the characteristic of both NOSER and FER. Although NOSER regularization matrix provides higher resolution performance than that of the FER regularization matrix, the FER provides a more robust estimation (more resistant to noised amplified solution). In highly dynamic environment, the proposed hybrid method is a good alternative approach which can be used as a warm start for other iterative methods such as those used in l_1 -norm optimization solutions.

Several beamforming techniques used in DOA estimation have been applied to uncover a set of active support in EIT application under MMV framework. The connection between steering vector used in DOA estimation and the array response for EIT application has been drawn in Chapter 5. Several beamforming techniques were explored under the Quasi-static environment of EIT application. Although the beamforming techniques are able to produce high resolution for support identification task in EIT, certain issues were explored based on the analysis of signal powers in EIT. An alternative method named MUSIC-like algorithm that has super-resolution performance comparable to the MUSIC algorithm is proposed specifically for the EIT application. The array response which incorporate medium geometry, current injection profile, and measurement procedure can be obtained in a similar fashion of the Jacobian calculation. Simulation study and real experiment were conducted to investigate several algorithms. The signal power analysis for EIT application has shown that the a robust DOA estimation technique is essential for the quasi-static environment. We showed that MUSIC-like is a good alternative method to the MUSIC algorithm especially in EIT application. The connection between the MMV framework and the beamforming techniques are also drawn which highlights the applicability of DOA estimation to support identification tasks.

Besides support identification for EIT application, we also investigated performance improvement in terms of support detection rate and the estimation bias of the MUSIC-like algorithm for DOA estimation. It should be noted that the results are applicable to other support identification tasks and are not limited to only DOA estimation. In MUSIC-like algorithm, an auxiliary parameter β is crucial and should be treated with care. In Chapter 6, three directions for β -selection were investigated. It was found that the directional adaptive framework has the most potential for further investigation. Hence, we further explored the directional

adaptive β_θ -selection method under α -stable distributed noise. Several weighted covariance matrix based methods (weighted covariance matrix coupled with MUSIC algorithm) were explored and compared with the directional adaptive MUSIC-like algorithm. The simulation results highlight that the proposed method provides a good performance especially for applications where high recovery rate (detection rate) is the core objective and some estimation biases are tolerable.

Lastly, two hybrid beamforming-regularization frameworks are explored in Chapter 7. The first framework is dedicated to image reconstruction in EIT where we proposed the three-step method. Without reconstructing the image pixel-by-pixel with fine resolution at once, we proposed a two grid systems namely the coarse-grid and the fine-grid system. The coarse-grid system is used in the first step where beamforming techniques can be applied to obtain a rough estimate. The ROI is identified within the coarse-grid and then mapped to the fine-grid system. The image is reconstructed in the last step where we proposed a local regularization matrix which incorporates statistical information of the target (possible areas that targets should be located). The use of two grid systems has also reduced the requirement of the scanning process in the first step. Simulation results have shown that this method is able to reconstruct image with high resolution without using iterative process.

In the second framework, a subspace fitting regularization is proposed specifically for the scenario when the obtained data snapshot is limited. The example of the second framework is based on DOA estimation, but it is also applicable to the first step (scanning in coarse-grid system) of the first framework that we proposed previously. The core objective of this method is to minimize the discrepancy between model-data fitness while suppressing the projection of measurement data onto the noise subspace. As proposed that this framework is dedicated to the scenario where the data snapshot is limited, the noise subspace is iteratively estimated without model order estimation or prior knowledge regarding the number of active supports. Preliminary study showed that the proposed method is able to uncover the active support under limited data snapshot scenario while the beamforming techniques failed.

9.2 Future Works

Further improvement on the EIT system prototype can be done by simultaneously capturing the boundary data. This would require additional signal conditioning and a new design of multiplexing unit and data handling protocol. To apply the system for a wide-range of applications, high performance instrument amplifier is needed. The VCCS that is capable of handling wide range of frequencies will provide multi-frequency system. Adjustable band-pass filter will also be essential for the multi-frequency system.

For hybrid regularization, an adaptive weight that could automatically adjust the intensity between FER and NOSER will help improve the overall performance especially under different levels of SNR. In high SNR environment, the hybrid regularization should place more emphasize on the resolution performance and allow more noise sensitivity into the model. On the other hand when the SNR is low, the regularization matrix should be able to put more weight on the robustness (lower noise sensitivity). An important point where further investigation has to be made is the method to obtain the weight value. In most scenarios, the SNR of the environment is not known a priori, hence certain framework to estimate this information has to be carefully designed.

The MMV framework for EIT application can be further investigated in the direction of multi-frequency data snapshots. Different types of tissue structure will also affect the frequency dependency and correlation between data snapshots taken by subsequent frequency bands. Therefore, scenarios of high dependency between data snapshot is also an interesting and challenging direction to be explored. To explore such scenario, an improvement on the hardware system prototype is also necessary.

The MUSIC-like algorithm can be improved towards fully adaptive framework where β parameter is adjustable according to direction and SNR levels. During the analysis, we recognized that β is more sensitive to β_{max} than β_{min} , hence a fast estimation of β_{max} based on Gershgorin circle theorem can be further explored. The effect of different SNR levels and number of data snapshots to the fast estimation of λ_{max} based on Gershgorin circle theorem should be thoroughly analyzed.

List of Author's Publications

Journal Articles

- **N. Borijindargoan**, B. P. Ng, and S. Rahardja*, "MUSIC-like Algorithm for Source Localization in Electrical Impedance Tomography," *IEEE Trans. Ind. Electron.*, vol. 66, no. 6, pp. 1661-1671, June 2019.
DOI: 10.1109/TIE.2018.2863196.
- **N. Borijindargoan** and B. P. Ng, "A Hybrid Sensitivity-Correlation Regularization for Electrical Impedance Tomography," *IET Sci. Meas. Technol.*, vol. 13, no. 8, pp. 1092-1101, Oct. 2019.
DOI: 10.1049/iet-smt.2018.5267.
- **N. Borijindargoan** and B. P. Ng, "Novel Directional Adaptive Relaxation Parameter for MUSIC-like Algorithm," *Prog. Electromagn. Res. L.*, vol 81, pp. 21-28, Jan. 2019.
DOI: 10.2528/PIERL18112002.
- **Narong Borijindargoan** and B. P. Ng, "Directional Adaptive MUSIC-like Algorithm under α -Stable Distributed Noise," *Prog. Electromagn. Res. L.*, vol. 86, pp. 29-37, Aug. 2019.
DOI: 10.2528/PIERL19062605
- D. A. Dharmawan, B. P. Ng, and **N. Borijindargoan**, "Design of Optimal Adaptive Filters for Two-Dimensional Filamentary Structures Segmentation," *IEEE Signal Process. Lett.*, vol. 26, no. 10, pp. 1511-1515, Oct. 2019.
DOI: 10.1109/LSP.2019.2938631
- **N. Borijindargoan** and B. P. Ng, "Fully Adaptive MUSIC-like Algorithm with Tapered Auxiliary Parameter," (In preparation)

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- **N. Borijindargoan** and B. P. Ng, “Novel β –Selection Method for MUSIC-like Algorithm,” (In preparation)
 - **N. Borijindargoan** and B. P. Ng, “Three-Step Method for EIT Image Reconstruction,” (In preparation)
 - **N. Borijindargoan** and B. P. Ng, “Subspace Fitting Regularization for Support Identification in Low Snapshots Scenario,” (In preparation)

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