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Fuzzy Model Predictive Control of Discrete Systems With Time-Varying Delay and Disturbances

Long Teng, Youyi Wang, Wenjian Cai, and Hua Li

Abstract—In this work, model predictive control of discrete T-S fuzzy systems subjected to bounded time-varying delay and persistent disturbances is investigated. The Razumikhin approach is adopted for time-delay systems because it involves a Lyapunov function associated with the original non-augmented state space of system dynamics when compared to the Krasovskii approach. As such, the Razumikhin approach has a good potential to avoid the inherent complexity of the Krasovskii approach especially in the presence of large delays and disturbances. Based on which, both online and off-line model predictive control approaches for systems with time-varying delay are provided. In addition, persistent disturbances are considered that robust positive invariance and input-to-state stability under such circumstances are realized. In the off-line approach, a sequence of explicit control laws that correspond to a sequence of robust constraints sets are computed off-line. And it is proved that system states including all possibly delayed states can be steered to the terminal constraint set in finite time. Moreover, it allows the exact time delay to be unknown in the proposed two approaches. In particular, for systems with time-varying delay the special positively invariant set theory and finite-time control theory based on the Razumikhin approach are directly revealed via the proposed off-line approach.

Index Terms—Time-varying delay, model predictive control, T-S fuzzy system, Lyapunov-Razumikhin, off-line approach, input-to-state stability.

I. INTRODUCTION

Time delay can be frequently seen in many industrial systems, see [1] and references therein. Since time delay can cause degradation of the control performance or even instability of the whole systems, it has attracted extensive research interests. Generally, utilization of Lyapunov theory for time-delay systems can be categorized as two different approaches, the Lyapunov-Krasovskii functional (LKF) and Lyapunov-Razumikhin function (LRF) [2]–[5]. For discrete-time systems, the Krasovskii approach makes use of an augmentation of the state vector with all delayed states, which yields the applications of classical Lyapunov methods to an augmented system without delay. As a result, the controller synthesis and computation, especially for systems with large delays and disturbances, will become very complex. As such, time delay and persistent disturbances are treated separately

in most cases to prevent the whole control problem from becoming too intricate [6]. As an alternative, the Razumikhin approach is conservative but it involves a Lyapunov function for the original non-augmented system. Therefore, it has a good potential to get rid of the complexity induced by the Krasovskii approach especially for systems with large delays and disturbances. It is noted that extensive works can be seen on stabilization of time-delay systems adopting LKF, while LRF-based approaches have not been widely investigated until in recent years, see [6]–[10], except for a few early works [11]–[13].

Moreover, compared with time-delay systems with constant delay, a more natural aspect is to consider systems with time-varying delay. However, LKF is utilized in most existing works for this kind of systems [14]–[16]. Only a few results are based on LRF and they are only applied to continuous systems [17], [18].

On the other hand, as a kind of optimization control methods, model predictive control (MPC) has been widely investigated and utilized in industrial processes. However, there are only a few results regarding MPC of time-delay systems. Several early results can be seen in [19]–[21] where LKF is adopted. Recently, the LRF-based MPC method is proposed for linear delay difference inclusions (DDIs) [8]. For nonlinear cases, a combination of LRF and LKF conditions is adopted to decide the terminal constraint set and terminal cost function, respectively, for continuous time-delay systems [22]. MPC without terminal constraints can be seen in [23]. However, the aforementioned works only deal with systems with constant delay. In addition, time-varying input delay is investigated in [24] and [25] that it is formulated as networked control system (NCS) problems.

It can be seen that MPC for time-delay systems is not completely investigated that several problems still remain unsolved [22]. In this work, fuzzy model predictive control (FMPC) is investigated for discrete nonlinear systems where time-varying delay and persistent disturbances are involved simultaneously. Nonlinear systems with time-varying delay are modeled as T-S fuzzy systems. Then LRF is adopted that conditions of robust positive invariance and input-to-state stability (ISS) with respect to disturbances are derived.

Furthermore, one of the main drawbacks of classical MPC strategies is that it requires solving online an optimization problem at each sampling instant, which prevents its applications to some contexts due to the high computational cost. Thus several MPC approaches have been developed to reduce the online computational burden, e.g., the explicit MPC and efficient MPC algorithms. It is noted that these

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off-line MPC formulations have been extensively investigated for non-delayed systems, see explicit solutions in [26]–[29], and efficient ones in [30]–[34]. However, to exploit efficient MPC methods for time-delay systems, there exist only a few works for linear time-delay systems [25], [35], [36], where only the Krasovskii approach is investigated that the stabilization is very complex. Therefore, an efficient off-line FMPC approach for discrete nonlinear systems with time-varying delay and persistent disturbances is also provided in this work. A sequence of explicit control laws corresponding to a sequence of robust constraint sets are computed off-line that guarantee the system all (possibly delayed) states will enter the terminal constraint set in finite time.

The contributions of this work can be summarized as follows:

(1) FMPC of systems with time-varying state delay is investigated. It is noted that MPC for time-delay systems has not been extensively investigated. For systems with time-varying delay, only time-varying input delay is investigated in [24] and [25]. However, the results only apply to linear systems and disturbance is not considered. On the other hand, FMPC of non-delayed systems can be seen in [37]–[40], and in particular, in [41]–[43] for nonlinear systems modeled as hybrid systems. It has not been investigated for nonlinear time-delay systems.

(2) LRF rather than LKF is employed in the proposed MPC approach. The LKF yields the classical Lyapunov method to an augmented system without delay, thus the complexity of controller synthesis and computation increases with the number of delay that it is not tractable for large delays. However, we show that the complexity especially for systems with large delays and disturbances can be significantly alleviated by the proposed LRF-based MPC approach.

(3) Both online and off-line MPC approaches are provided for systems with time-varying delay. And the latter one has a great advantage in computational cost than the former one where online optimization is needed. It is proved that system states including all possibly delayed states can be driven to the terminal constraint set in finite time via the off-line approach. The positively invariant set theory and finite-time control theory based on LRF approach are directly revealed, and simulation results show that they are quite different from those for non-delayed systems.

(4) Persistent disturbances are considered in the proposed approach. It is noted that MPC of non-delayed systems with persistent and bounded disturbances has attracted research interests in several recent works [33], [44]–[46]. However, it has not been investigated in MPC of time-delay systems due to the complexity of existing LKF-based approaches.

The rest of this paper is structured as follows. In Section II, some preliminary knowledge about time-delay system model and T-S fuzzy systems are introduced. In Section III, online FMPC algorithm for systems with time-varying delay and disturbances is presented. An alternative off-line solution of FMPC is introduced in Section IV. In Section V, simulation examples are illustrated to verify the effectiveness of the proposed methods. Finally, some conclusions are given in Section VI.

Notations: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Z}_{[m,n]}$ denote the set of real numbers, the set of non-negative reals, the set of integers, the set of non-negative integers, and the set of integers in the interval $[m, n]$, respectively. $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. $\mathbf{x}_{[c1,c2]} := \{x(l)\}_{l \in \mathbb{Z}_{[c1,c2]}}$, with $c1, c2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c1,c2]}$. $x \in \Omega_{a1} \setminus \Omega_{a2}$ denotes that $x \in \Omega_{a1}$ and $x \notin \Omega_{a2}$. A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, i.e., $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s \rightarrow \infty} \alpha(s) = \infty$. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$, i.e., $\beta \in \mathcal{KL}$, if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is strictly decreasing and $\lim_{s \rightarrow \infty} \beta(r, s) = 0$. The symbol $*$ represents a symmetric structure in matrix inequalities.

II. PRELIMINARIES

A. System Model

Consider the following discrete-time nonlinear system with time-varying state delay

$$x(k+1) \in F(x(k), x_d(k), u(k), w(k)), \quad k \in \mathbb{Z}_+, \quad (1)$$

where $x(k), x_d(k) \in \mathbb{X}$, $u(k) \in \mathbb{U}$ and $w(k) \in \mathbb{W}$, with $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$ and $\mathbb{W} \subseteq \mathbb{R}^c$, represent system current and delayed states, inputs, and disturbances, respectively. $x_d(k) \in \mathbb{X}$ is defined as follows:

$$x_d(k) := x(k + d(k)), \quad d(k) \in \mathbb{Z}_{[-h, -1]} \quad (2)$$

where $d(k)$ denotes the number of delay, h denotes the upper bound of delay, and the minimal delay is set as 1.

Next, for time-delay systems some definitions of robust positive invariance and input-to-state stability are introduced.

Definition 1 (Robust Positively Invariant (RPI) Set, D-invariance): For LRF conditions, considering system (1), a set Ω is called an RPI set for the closed-loop system corresponding to the control law π , and for $d(k) \in \mathbb{Z}_{[-h, -1]}$, if $\forall x \in \Omega$, $\forall x_d \in \Omega$, and $\forall w \in \mathbb{W}$, it holds $x^+ \in \Omega$.

Above and in what follows, x , x_d , x^+ , and w are used to denote, at arbitrary time instant, the current state, delayed state, state in the next instant, and disturbance, respectively.

Definition 2 (Input-to-state Stability (ISS)): Consider $x^+ \in F(x(k), x_d(k), w(k))$, it is called ISS if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_+$ it holds that

$$\|x(k)\| \leq \beta(\|\mathbf{x}_{[-h,0]}\|, k) + \delta(\|\mathbf{w}_{[0,k-1]}\|), \quad (3)$$

where $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ is the initial (delayed) states, $\mathbf{w}_{[0,k-1]} \in \mathbb{W}^k$ is the disturbance sequence.

Definition 3 (ISS Lyapunov-Razumikhin Function (ISS-LRF)): For LRF conditions, suppose that there exists a positive definite function $V(x)$ such that satisfies the following conditions:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (4)$$

for all $(x, x_d, w, x_1) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times F(x_0, x_d, w)$;

(ii) For any initial condition, there exists a function $\sigma \in \mathcal{K}$ satisfying

$$V(x^+) \leq \max \{ \bar{V}(x), \sigma \|w\| \} \quad (5)$$

where $\bar{V}(x) = \max\{V(x), V(x_d)\}$, then $V(x)$ is called an ISS-LRF for $x^+ \in F(x, x_d, w)$.

It is worth to mention that detailed descriptions of ISS and ISS-LRF for discrete time-delay systems are introduced in [6].

B. T-S Fuzzy Time-delay Systems

Based on the aforementioned system model, nonlinear systems with time-varying delay can be represented by T-S fuzzy time-delay systems:

Plant Rule l :

IF: z_1 is F_1^l and $\dots$, z_v is F_v^l

THEN:

$$x^+ = A_l x + A_{dl} x_d + B_l u + E_l w \quad (6)$$

where, $z := [z_1, \dots, z_v]$ are premise variables. $F_1^l, \dots, F_v^l$ are fuzzy sets. $l \in \mathbb{Z}_{[1, L]}$, and L is the number of fuzzy rules. The T-S fuzzy system (6) can be written as,

$$x^+ = A_\mu x + A_{d\mu} x_d + B_\mu u + E_\mu w \quad (7)$$

where $A_\mu := \sum_{l=1}^L \mu_l(z) A_l$, $A_{d\mu} := \sum_{l=1}^L \mu_l(z) A_{dl}$, $B_\mu := \sum_{l=1}^L \mu_l(z) B_l$, $E_\mu := \sum_{l=1}^L \mu_l(z) E_l$, $\mu_l(z)$ is normalized membership function.

Control law for aforementioned T-S fuzzy system is given below:

Control Rule l :

IF: z_1 is F_1^l and $\dots$, z_v is F_v^l

THEN:

$$u = K_l x$$

By using the fuzzy inference method with a singleton fuzzifier, product inference and center average defuzzifiers, the final control output can be got as follows,

$$u = K_\mu x \quad (8)$$

where $K_\mu = \sum_{l=1}^L \mu_l(z) K_l$.

III. ONLINE MPC APPROACH

Online MPC methods require solving an optimization problem over a finite horizon at each sampling instant. Then in the next instant, the optimization problem is repeated based on the updated state information over a shifted horizon. Thus it is also called the receding horizon control (RHC).

In this section, the online MPC approach of systems with time-varying delay is investigated. The online control scheme about MPC for the aforementioned T-S fuzzy time-delay systems with disturbances is firstly introduced. Then the conditions of RPI set are discussed, based on which the computation of terminal constraint set is provided. In the end, the whole control algorithm is constructed, and then stability and feasibility properties are analyzed.

A. MPC Scheme

For T-S fuzzy system (7), the prediction model for online approach is given by

$$x(k+s+1|k) = A_\mu x(k+s|k) + A_{d\mu} x_d(k+s|k) + B_\mu u(k+s|k) + E_\mu w(k+s|k) \quad (9)$$

MPC optimizes a cost function under some constraints. Usually, a finite horizon cost function is adopted,

$$J(k) = \sum_{s=0}^{N-1} \ell(k+s|k) + V_T(x(k+N|k)) \quad (10)$$

where N is the prediction length, $\ell(k+s|k)$ is called the stage cost at the predicted time instant, and $V_T(x(k+N|k))$ is called the terminal cost, with $V_T(\cdot)$ being a positive definite function. The stage cost is chosen as

$$\ell(k+s|k) = x^T(k+s|k) Q x(k+s|k) + u^T(k+s|k) R u(k+s|k) - \tau v(k+s|k) \quad (11)$$

where Q and R are positive matrices, τ is a positive scalar. One can see that the disturbance is involved in the stage cost. It is noted that the choice of cost function is inspired by the H_∞ control, see Section 4.7 in [47] for the so-called " H_∞ model predictive control". And similar settings of cost functions can be found in [33], [48], [49]. However, the cost function cannot be optimised directly as it contains disturbance. Instead, a min-max approach is adopted that the worst-case cost function is minimized [50]. In addition, in the presence of persistent disturbance, asymptotic stability to the origin cannot be reached, instead, asymptotic convergence to an invariant set can be achieved [44], [45]. As such, a robust positively invariant set is employed as the terminal constraint set that, the system state is guaranteed to enter the terminal set at the end of prediction. The online optimization problem can be summarized as

$$\begin{cases} \min_{u(k+s|k)} \max_{w(k+s|k)} J(k), \\ u(k+s|k) \in \mathbb{U}, \\ w(k+s|k) \in \mathbb{W}, \\ x(k+N|k) \in \Omega_T, \end{cases}$$

where $s \in \mathbb{Z}_{[1, N-1]}$, $\Omega_T \subseteq \mathbb{X}$ represents the terminal constraint set. The disturbance and control input satisfy that

$$w \in \mathbb{W} := \{w | w^T w \leq \eta^2\}$$

$$u \in \mathbb{U} := \{u | |u_t| \leq u_{t, \max}\}$$

where η is a known constant. u_t is the t -th element of the input vector, with $t \in \mathbb{Z}_{[1, m]}$.

B. RPI Set for Online Approach

Denote $\Omega \subseteq \mathbb{X}$ as an RPI set which is described in Definition 1, thus $x, x_d \in \Omega$, and π as the corresponding control law. Define Ω as follows

$$\Omega := \{ \{x, x_d\} | \max\{x^T P_\mu x, x_d^T P_\mu x_d\} \leq \xi \} \quad (12)$$

and the corresponding control law

$$\pi(x, x_d) := \{K_\mu x\} \quad (13)$$

where $P_\mu = \sum_{l=1}^L \mu_l(x) P_l$, $K_\mu = \sum_{l=1}^L \mu_l(x) K_l$.

Lemma 1: Consider system (7), the set Ω is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{(0,1)}$ such that

$$\frac{1}{\xi} x^{+T} P_\mu^+ x^+ - \frac{1-\lambda}{\xi} \max\{x^T P_\mu x, x_d^T P_\mu x_d\} - \frac{\lambda}{\eta^2} w^T w \leq 0 \quad (14)$$

where $P_\mu^+ = \sum_{l=1}^L \mu_l(x^+) P_l$.

Proof: For (14), move the last two items from the left side to the right side. According to the definition of Ω in (12) and $w^T w \leq \eta^2$, then $\frac{1}{\xi} x^{+T} P_\mu^+ x^+ \leq (1-\lambda) + \lambda = 1$, thus $x^{+T} P_\mu^+ x^+ \leq \xi$, which indicates that $x^+ \in \Omega$. Proof is thus completed. It is noted that the RPI approach regarding disturbance is extended from the case of non-delayed systems in [33].

Conditions of RPI set and input constraint are given as solving bilinear matrix inequalities (BMIs) and linear matrix inequalities (LMIs) as follows.

Theorem 1: Consider T-S fuzzy system (7), if there exist positive definite matrix X_a (or X_b , X_l presented below), matrices Y_b , G_b , Z , and a positive scalar $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma\Phi & * & * & * \\ 0 & \gamma_d\Phi & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_a G_b + B_a Y_b & A_{da} G_b & E_a & -X_l \end{bmatrix} \leq 0 \quad (15)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & G_b + G_b^T - X_b \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (16)$$

where $\Phi = (1-\lambda)(X_a - G_b - G_b^T)$, $a, b, l \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ and γ_d are pre-set valued positive numbers that satisfy $\gamma + \gamma_d = 1$, then the set $\Omega = \{x, x_d\} | \max\{x^T P_\mu x, x_d^T P_\mu x_d\} \leq \xi\}$, with $P_\mu = \sum_{l=1}^L \mu_l(x) P_l$ and $P_l = \xi X_l^{-1}$, is an RPI set for system (7) corresponding to the feedback control law $\pi = K_\mu x$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$.

Proof: See Appendix A.

From the notation of Φ in (15), it can be seen that BMIs are introduced. Thus the solver PENBMI [51] is utilized to solve BMIs problems.

C. Terminal Constraint Set for Online Approach

Definition 4: The terminal constraint set Ω_T should satisfy two conditions. Firstly, it should be an RPI set. Secondly, there exist $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$, and a positive definite function $V(x) := x^T P_\mu x$, with $P_\mu = \sum_{l=1}^L \mu_l(x) P_l$, that $\forall x \in \Omega_T$,

$$\alpha_3(\|x\|) \leq V(x) \leq \alpha_4(\|x\|) \quad (17)$$

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (18)$$

where $\bar{V}(x) = \max\{x^T P_\mu x, x_d^T P_\mu x_d\}$. Definition 4 is concluded based on the results in [6]. In the following, the condition which assures the requirement of Definition 4 is provided.

Theorem 2: Consider T-S fuzzy system (7), if (15), (16) and the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma\Xi & * & * & * & * & * \\ 0 & \gamma_d\Xi & * & * & * & * \\ 0 & 0 & -\tau\xi I & * & * & * \\ \Psi & A_{da} G_b & E_a & -X_l & * & * \\ QG_b & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (19)$$

where $\Xi = X_a - G_b - G_b^T$, $\Psi = A_a G_b + B_a Y_b$. X_a (or X_b , X_l presented below), matrices Y_b , G_b , are the same as those defined in Theorem 1, then Ω_T is a terminal constraint set regarding the terminal cost function $\bar{V}(x)$.

Proof: See Appendix B.

D. Online MPC Algorithm

Based on the above results, the whole algorithm is discussed in this subsection to complete the online MPC controller design, and then feasibility and stability are analyzed.

The terminal constraint set Ω_T should satisfy the requirement of RPI set according to (12), which indicates,

$$\bar{V}(x) \leq \xi \quad (20)$$

thus $x(k), x_d(k) \in \Omega_T$. The following problem is considered,

$$\min \quad \xi, \text{ subjected to } \bar{V}(x) \leq \xi$$

A sufficient condition for $\bar{V}(x) \leq \xi$ is given as

$$\begin{bmatrix} 1 & x^T(k+j) \\ x(k+j) & X_l \end{bmatrix} \geq 0, j \in \mathbb{Z}_{[-h,0]}, l \in \mathbb{Z}_{[1,L]} \quad (21)$$

Remark 1: For clarity of exposition, at time instant k , it can be seen that the terminal constraint set only involves $x(k)$ and $x_d(k)$ according to (5), i.e., $x(k), x_d(k) \in \Omega_k$. However, all possibly delayed states are employed in (21), which indicates $x(k+j) \in \tilde{\Omega}_k, j \in \mathbb{Z}_{[-h,0]}$. Thus $\tilde{\Omega}_k$, rather than Ω_k , represents the terminal constraint set Ω_T computed at instant k , and it is easy to know that $\Omega_k \subseteq \tilde{\Omega}_k$. Although this choice may be less optimal because it may lead to a larger terminal constraint set, it is essential to guarantee the positive invariance of terminal constraint set as the delay is time-varying. And it will be utilized further for the feasibility analysis subsequently.

Remark 2: (21) indicates that all possibly delayed states which are available at each time instant are involved. Moreover, the upper bound of delay is assumed to be known. As such, it does not need to know the exact delay in the controller design.

Consider the condition for terminal constraint set in (19), τ can also be optimised. Taking the same idea as stated in [33], a new variable ε is defined as: $\varepsilon = \tau\xi$, thus ε is minimized instead. Then (19) is rewritten as

$$\begin{bmatrix} \gamma\Xi & * & * & * & * & * \\ 0 & \gamma_d\Xi & * & * & * & * \\ 0 & 0 & -\varepsilon I & * & * & * \\ \Psi & A_{da} G_b & E_a & -X_l & * & * \\ QG_b & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (22)$$

In summary, the MPC algorithm is given as follows. As BMIs are introduced in (15), to reduce the computational burden, the feasible value of λ got in the initial time is saved and utilized all the time. By taking this measure and the one stated in (22), the whole control problem is reduced to solving LMIs.

Algorithm 1:

Step 1. Obtain the system all (possibly delayed) states $x(k+j)$, $j \in \mathbb{Z}_{[-h,0]}$;

Step 2. If $k = 0$, solve

$$\min_{\xi, \varepsilon, X_a, G_b, Y_b, Z, \lambda} \quad \varepsilon, \quad (23)$$

subject to (15), (16), (21), (22), and $\lambda \in \mathbb{R}_{(0,1)}$. Obtain the optimal λ and denotes it as λ^* . Go to Step 3;

Else if $k \geq 1$, replace λ in (15) with λ^* , and solve

$$\min_{\xi, \varepsilon, X_a, G_b, Y_b, Z} \quad \varepsilon, \quad (24)$$

subject to (15), (16), (21), (22). Go to Step 3;

Step 3. Obtain the control input $u(k) = K_\mu x(k)$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$. Then implement the control input to system (7). Move the time instant from k to $k+1$ and go to Step 1.

Remark 3: Consider the optimization problem (24), where (21) implies that the number of LMIs will increase with the number of delay. However, the other conditions, i.e., (15), (16) and (22), will remain unchanged with the delay. Therefore, the computational burden of the online MPC algorithm will slightly increase with the number of delay.

Theorem 3: For system (7), if the constrained optimization problem has a solution at time 0, it will always be solvable. Furthermore, recursive feasibility can be achieved.

Proof: Without loss of generality, assume that (24) is satisfied at time k , according to the description in Remark 1, thus $x(k+j) \in \tilde{\Omega}_k$, $j \in \mathbb{Z}_{[-h,0]}$, with $\tilde{\Omega}_k$ being the terminal constraint set at time k . Since $x(k) \in \Omega_k$ and $x_d(k) \in \Omega_k$ as stated in Remark 1, thus $x(k+1) \in \Omega_k$. Considering $\Omega_k \subseteq \tilde{\Omega}_k$, thus $x(k+1+j) \in \tilde{\Omega}_k$, $j \in \mathbb{Z}_{[-h,0]}$. Then it can be concluded that (24) is solvable at time $k+1$. Moreover, the solution obtained at time k is feasible at time $k+1$. Applying the same process, it is observed that the optimization problem is feasible at all time.

Theorem 4: If the constrained optimization problem is feasible at the initial time 0, system (7) is ISS with respect to disturbance w .

Proof: Consider the LRF $V(x)$, at time k , suppose that it gets its maximum value at the delayed states $x(k+\theta)$, $\theta \in \{d(k), 0\}$ with $d(k)$ defined in (2), i.e., $\bar{V}^*(x(k)) = x^T(k+\theta)P_\mu^*(k)x(k+\theta)$, $P_\mu^*(k) = \sum_{l=1}^L \mu_l P_l^*(k)$, $P_l^*(k)$ is an optimal value of $P_l(k)$ at time k . In addition, as $V^*(x(k)) = x^T(k)P_\mu^*(k)x(k)$, then it can be easily got

$$\rho_{\min}^* \|x(k)\|^2 \leq V^*(x(k)) \leq \rho_{\max}^* \|x(k)\|^2 \quad (25)$$

where

$$\rho_{\min}^* := \min \{ \rho_{\min}(P_l^*(k)) | l \in \mathbb{Z}_{[1,L]}, k \in \mathbb{Z}_+ \}$$

$$\rho_{\max}^* := \max \{ \rho_{\max}(P_l^*(k)) | l \in \mathbb{Z}_{[1,L]}, k \in \mathbb{Z}_+ \}$$

where $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ represent the maximal and minimal eigenvalues, respectively.

Moreover, (18) implies

$$V_k^*(x(k+1)) - \bar{V}^*(x(k)) \leq -x(k)^T Q x(k) - u^T(k) R u(k) + \tau w^T w \quad (26)$$

where $V_k^*(x(k+1)) = x(k+1)^T P_\mu^*(k) x(k+1)$. Thus one has

$$V_k^*(x(k+1)) - \bar{V}^*(x(k)) \leq -x(k)^T Q x(k) + \tau w^T w \quad (27)$$

Owing to the optimality of (24) at time $k+1$, it holds

$$V_{k+1}^*(x(k+1)) \leq V_k^*(x(k+1)) \quad (28)$$

where $V_{k+1}^*(x(k+1))$ denotes the optimal $V(x)$ at time $k+1$, then it holds that

$$V_{k+1}^*(x(k+1)) - \bar{V}^*(x(k)) \leq -x(k)^T Q x(k) + \tau w^T w \quad (29)$$

According to Definition 3, from (25) and (29) it follows that $V(x)$ is an ISS-Lyapunov function. Hence the closed-loop system is ISS with respect to disturbance. The proof is completed.

IV. OFF-LINE MPC APPROACH

In off-line MPC schemes, usually, a sequence of control laws that correspond to a sequence of robust constraint sets are computed off-line, then the online computation only needs to look up and apply the appropriate control law. Based on which, for non-delayed systems choice of an appropriate control law only requires to look up the obtained constraint sets and find the smallest one where system current state belongs. In contrast, in this work where Razumikhin approach is adopted for time-delay systems, choice of an appropriate control law depends not only on the system current state but also on the delayed state. It should be first to find out the smallest constraint set where all (delayed) states belong, then the corresponding control law is applied.

Moreover, for non-delayed systems, it requires only one step to steer the system state from one constraint set to a neighbouring subset via the corresponding explicit control law. As a consequence, the system state can enter the terminal constraint set in finite time steps. The finite-time control performance remains effective for the provided LRF-based off-line MPC approach, however, the principle behind is more complicated and it will be proved in this work.

Remark 4: As the Krasovskii approach for time-delay systems can be regarded as an extension of the classical Lyapunov function for non-delayed systems, thus they share the same principles in Lyapunov function and set invariance theory. To better clarify the special properties of LRF-based MPC approach, above and in what follows, the contexts of off-line MPC for non-delayed systems which have been well

known are employed to compare it with proposed LRF-based approach.

In this section, computation of a terminal constraint set for off-line approach is firstly introduced, based on which a sequence of robust constraint sets corresponding to a sequence of explicit control laws are constructed. Then the whole control algorithm is summarized, and the finite-time control performance is analyzed.

A. Terminal Constraint Set for Off-line Approach

As it is discussed in the last section, the terminal constraint set should be an RPI set firstly. Define the terminal constraint set as $\Omega_t := \{x, x_d \mid \max\{x^T P_t x, x_d^T P_t x_d\} \leq \xi\}$, with P_t a positive definite matrix, then Ω_t is an RPI set if there exists $\lambda \in \mathbb{R}_{(0,1)}$ such that

$$\frac{1}{\xi} x^{+T} P_t x^+ - \frac{1-\lambda}{\xi} \max\{x^T P_t x, x_d^T P_t x_d\} - \frac{\lambda}{\eta^2} w^T w \leq 0 \quad (30)$$

Similar to (14), (30) guarantees $x^+ \in \Omega_t$, provided that $x, x_d \in \Omega_t$.

Secondly, there exist $\alpha_5, \alpha_6 \in \mathcal{K}_\infty$, and a positive definite function $V(x) := x^T P_t x$ that $\forall x \in \Omega_t$,

$$\alpha_5(\|x\|) \leq V(x) \leq \alpha_6(\|x\|) \quad (31)$$

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (32)$$

where $\bar{V}(x) = \max\{x^T P_t x, x_d^T P_t x_d\}$. In the following, the conditions which guarantee the aforementioned requirements are provided.

Theorem 5: Consider system (7), if there exist positive definite matrix X_t , matrices Y_a (or Y_b presented below), G_b , Z , positive scalars $\tau, \lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma \Pi & * & * & * \\ 0 & \gamma_d \Pi & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Gamma & A_{da} G_b & E_a & -X_t \end{bmatrix} \leq 0 \quad (33)$$

$$\begin{bmatrix} \gamma H & * & * & * & * & * \\ 0 & \gamma_d H & * & * & * & * \\ 0 & 0 & -\tau \xi I & * & * & * \\ \Gamma & A_{da} G_b & \xi E_a & -X_t & * & * \\ Q G_b & 0 & 0 & 0 & -\xi Q & * \\ R Y_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (34)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & G_b + G_b^T - X_t \end{bmatrix} \geq 0, \quad Z_{tt} \leq u_{t,\max}^2, \quad t \in \mathbb{Z}_{[0,m]} \quad (35)$$

where $\Pi = (\lambda - 1)(G_b + G_b^T - X_t)$, $H = X_t - G_b - G_b^T$, $\Gamma = A_a G_b + B_a Y_b$, $a, b \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ and γ_d are pre-set valued positive numbers that satisfy $\gamma + \gamma_d = 1$, then the terminal constraint set Ω_t is got for system (7), with $P_t = \xi X_t^{-1}$, and the corresponding feedback control law is given as $\pi_t = K_\mu x$, with $K_\mu = \sum_{a=1}^L \mu_a K_a$ and $K_a = Y_a G_a^{-1}$.

Proof: See Appendix C.

B. A Sequence of Robust Constraint Sets

Based on the obtained terminal constraint set Ω_t in the last subsection, a robust constraint set Ω_e ($\Omega_e \supseteq \Omega_t$) is employed that the system current state and delayed state $\{x, x_d\} \in \Omega_e$. A control law π_e is given that assures the system state at next instant $x(k+1) \in \Omega_t$, and it is easy to see that the positive invariance is guaranteed since $\Omega_t \subseteq \Omega_e$. When Ω_e and corresponding control law π_e are computed, then in a similar way, a sequence of constraint sets $\{\Omega_1, \Omega_2, \dots, \Omega_e, \dots, \Omega_{\tilde{N}}\}$ and corresponding control laws $\{\pi_1, \pi_2, \dots, \pi_e, \dots, \pi_{\tilde{N}}\}$ can be got, where Ω_1 and π_1 denote the terminal set Ω_t and corresponding control law π_t , respectively, $\Omega_e \subseteq \mathbb{X}$, with $e \in \mathbb{Z}_{[2,\tilde{N}]}$, $\tilde{N} \geq 2$ represents the number of robust constraint sets, where the terminal set is included.

Firstly, without loss of generality, consider two neighbouring robust constraint sets Ω_e and Ω_{e-1} , with $\Omega_{e-1} \subseteq \Omega_e$. When provided that $x, x_d \in \Omega_e$, then $x^+ \in \Omega_{e-1}$ can be satisfied by the following inequality,

$$x^{+T} X_{e-1}^{-1} x^+ - (1 - \lambda) \max\{x^T X_e^{-1} x, x_d^T X_e^{-1} x_d\} - \frac{\lambda}{\eta^2} w^T w \leq 0 \quad (36)$$

where X_e and X_{e-1} are positive definite matrices associated with the two neighbouring robust constraint sets, $\Omega_e = \{x \mid x^T X_e^{-1} x \leq 1\}$ and $\Omega_{e-1} = \{x \mid x^T X_{e-1}^{-1} x \leq 1\}$, respectively. $x, x_d \in \Omega_e$ implies that $\max\{x^T X_e^{-1} x, x_d^T X_e^{-1} x_d\} \leq 1$. Similar to (30), then $x^{+T} X_{e-1}^{-1} x^+ \leq 1$ can be got from (36), which indicates that $x^+ \in \Omega_{e-1}$.

Secondly, denote $V(x) := x^T P_e x$, thus the optimization control strategy is formulated as follows,

$$V(x^+) - \bar{V}(x) < -x^T Q x - u^T R u \quad (37)$$

where $V(x^+) = x^{+T} P_e x^+$, $\bar{V}(x) = \max\{x^T P_e x, x_d^T P_e x_d\}$.

Theorem 6: Assume the set $\Omega_{e-1} = \{x \mid x^T P_{e-1} x \leq \xi_{e-1}\}$ is known that $X_{e-1} = \xi_{e-1} P_{e-1}^{-1}$, if there exist positive definite matrix X_e , general matrices Y_a (or Y_b presented below), Z , positive scalars $\xi_e, \lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma(\lambda - 1)X_e & * & * & * \\ 0 & \gamma_d(\lambda - 1)X_e & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ A_a X_e + B_a Y_b & A_{da} X_e & E_a & -X_{e-1} \end{bmatrix} \leq 0 \quad (38)$$

$$\begin{bmatrix} -\gamma X_e & * & * & * & * \\ 0 & -\gamma_d X_e & * & * & * \\ A_a X_e + B_a Y_b & A_{da} X_e & -X_e & * & * \\ Q X_e & 0 & 0 & -\xi_e Q & * \\ R Y_b & 0 & 0 & 0 & -\xi_e R \end{bmatrix} \leq 0 \quad (39)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & X_e \end{bmatrix} \geq 0, \quad Z_{tt} \leq u_{t,\max}^2, \quad t \in \mathbb{Z}_{[0,m]} \quad (40)$$

where $a, b \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ and γ_d are pre-set valued positive numbers that satisfy $\gamma + \gamma_d = 1$, then the set $\Omega_e = \{x \mid x^T P_e x \leq \xi_e\}$, with $P_e = \xi_e X_e^{-1}$, is a robust constraint set for T-S fuzzy system (7), and

the corresponding feedback control law π_e will steer system state to Ω_{e-1} at the next instant. Moreover, π_e is given as $\pi_e = K_\mu x$, with $K_\mu = \sum_{a=1}^L \mu_a K_a$ and $K_a = Y_a X_e^{-1}$.

Proof: It is noted that (38) and (39) guarantee (36) and (37), respectively. (40) satisfies the input constraint. The proof is very similar to that of Theorem 5 for terminal constraint set. Thus is omitted for simplicity.

C. Off-line MPC Algorithm

Based on the above results, the whole control algorithm is formulated in this subsection to complete the off-line MPC design, and then the finite-time control performance is analyzed.

Algorithm 2:

1. Off-line computation

Step 1. Compute a terminal constraint set Ω_t and the corresponding control law π_t using Theorem 5;

Step 2. Base on the obtained terminal constraint set, a sequence of constraint sets $\{\Omega_1, \Omega_2, \dots, \Omega_{\tilde{N}}\}$ together with corresponding control laws $\{\pi_1, \pi_2, \dots, \pi_e, \dots, \pi_{\tilde{N}}\}$ are computed by Theorem 6;

1. Online computation

Step 1. Assume the initial conditions $x(i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_{\tilde{N}}$. Let $k = 0$ and go to Step 2;

Step 2. At time instant k , if $x(k+i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_1$ which indicates that all (possibly delayed) states are in the terminal constraint set Ω_1 , then the terminal control law π_1 is utilized all the time. Otherwise, go to Step 3;

Step 3. Implement the search to decide $\{x(k), x_d(k)\} \in \Omega_e \setminus \Omega_{e-1}$, then the corresponding control law π_e is applied, let $k = k + 1$ and return to Step 2.

Remark 5: In Step 2 of off-line computation of robust constraint set, we use $-\log \det(X_e)$ as the objective function and minimize it to get the constraint set with maximal size.

Remark 6: As it is shown in Remark 2 that, it does not need to know the exact delay for the online MPC approach. In contrast, from Step 3 of online computation of Algorithm 2, it can be seen that the delayed state $x_d(k)$ should be known to decide the constraint set Ω_e . An alternative method to avoid the dependence of the exact delay for the off-line MPC algorithm is that, we can resort to another constraint set Ω_g , which is used to replace Ω_e , decided by $x(k+i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_g \setminus \Omega_{g-1}$. Thus it does not need to know exactly the delayed state $x_d(k)$. However, this choice of constraint set also has some disadvantages. One is that it will lead to a less optimal control performance since a larger constraint set Ω_g (it is easy to know that $\Omega_g \supseteq \Omega_e$) is adopted. In addition, the computational cost of online search of constraint set Ω_g will increase because all (possibly delayed) states $x(k+i)_{i \in \mathbb{Z}[-h, 0]}$ are involved in the search.

Remark 7: Consider the computational cost of the off-line MPC approach, the off-line computation of a sequence of robust constraint sets in Algorithm 2 is not related to the size

of delay. For the online computation part of Algorithm 2, if the time-varying delay is known that Ω_e in Algorithm 2 is used as the constraint set, then the online computational cost will not increase with the number of delay. However, if the time-varying delay is unknown that the constraint set Ω_g in Remark 6 is adopted instead, then the online computational cost will increase with the number of delay.

Theorem 7: It is assumed that the initial conditions $x(i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_{\tilde{N}}$, for the aforementioned systems with time-varying delay if a sequence of robust constraint sets $\{\Omega_1, \Omega_2, \dots, \Omega_{\tilde{N}}\}$, where Ω_1 is the terminal constraint set, can be computed by Theorem 5 and Theorem 6, then the system all (possibly delayed) states will be steered to the terminal constraint set in no more than $(h+1) \cdot (\tilde{N}-1)$ steps.

Proof: Without loss of generality, suppose that system all (possibly delayed) states $x(k+i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_e$ at time instant k . Firstly, the control law π_e is applied to drive $x(k+1)$ to Ω_{e-1} . As a result, at time instant $k+1$, one can see that $x(k+1) \in \Omega_{e-1}$, while $x_d(k+1) \in \Omega_e$. Thus the control law π_e , rather than π_{e-1} , will still be adopted at instant $k+1$ to steer $x(k+2)$ to Ω_{e-1} . Therefore, considering the delay is time-varying and the upper bound of delay is h , in the worst case, it can be concluded that it requires no more than $h+1$ steps to drive system all (possibly delayed) states from Ω_e to Ω_{e-1} , i.e., $x(k+h+1+i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_{e-1}$. As a consequence, in the case of $\tilde{N}$ robust constraint sets and the initial conditions satisfying $x(i)_{i \in \mathbb{Z}[-h, 0]} \in \Omega_{\tilde{N}}$, it takes no more than $(h+1) \cdot (\tilde{N}-1)$ steps to drive system all (possibly delayed) states to the terminal constraint set. Proof is completed.

Remark 8: If the constraint set Ω_g in Remark 6 is adopted, the finite-time control theory revealed in Theorem 7 will still be effective. Actually, the result can be easily got by referring to the proof of Theorem 7.

V. ILLUSTRATIVE EXAMPLE

In this section, simulations on a numerical example and the time-delay CSTR system are provided to verify the effectiveness of the proposed two algorithms. The simulations are implemented with MATLAB and all matrix inequalities are solved via YALMIP toolbox [52].

A. A Numerical Example

Consider the following T-S fuzzy time-delay systems revised from [?], with the upper bound of delay $h = 3$,

Rule 1: If $x_1(k)$ is F_1 , then

$$x(k+1) = A_1 x(k) + A_{d1} x_d(k) + B_1 u(k) + E_1 w(k)$$

Rule 2: If $x_1(k)$ is F_2 , then

$$x(k+1) = A_2 x(k) + A_{d2} x_d(k) + B_2 u(k) + E_2 w(k)$$

where

$$A_1 = \begin{bmatrix} 1.5 & 2.5 \\ 0 & 1.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.8 & 1.5 \\ 0 & 0.8 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$E_1 = E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, A_{d1} = 0.05 A_1, A_{d2} = 0.05 A_2.$$

The membership functions are given as follows:

$$\begin{cases} F_1 = 0.5 + \sin(x_1) \cos(x_1), \\ F_2 = 1 - F_1. \end{cases}$$

The matrices Q and R in the cost function are selected as $\text{diag}\{1, 1\}$ and 0.1, respectively. The input constraint and bound of disturbance are set as: $\|u\| \leq 3$, $\|w\| \leq 0.1$. As the constraint $\gamma + \gamma_d = 1$ should be satisfied, set $\gamma = 0.8$, $\gamma_d = 0.2$. The simulation result of Algorithm 1 (online MPC approach) is shown in Fig. 1. The initial condition is given as $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [1; -1.5]$. It can be seen that with the control effort, the system states come into a very small region near the origin in the presence of persistent disturbance. And the input constraint is guaranteed. Fig. 1 also shows the number of time-varying delay at each time instant.

Fig. 1. Control performance using Algorithm 1. The two dotted lines in the third subfigure indicate the input constraint.

The simulation result of Algorithm 2 (off-line MPC approach) is shown in Fig. 2. The number of robust constraint sets including the terminal constraint set is given as 11. It can be seen that, despite the different initial conditions, system all (delayed) states enter the terminal constraint set in the end.

Fig. 3 gives a clearer view of the relevant constraint sets where all (delayed) states and the current state belong, respectively, at each time instant. It can be seen that:

- (1) In the existence of delayed states, at each instant, the constraint set for current state is always a (non-strict) subset of the constraint set for all (delayed) states, which implies that the fundamental idea of Razumikhin approach is satisfied.
- (2) Consider the constraint set for all (delayed) states at each instant, except the terminal set, the control effort will steer the system current state to a strict subset at the next instant.

Fig. 2. Control performance for three different initial conditions using Algorithm 2: $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-1.64; -0.479]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [2.6; 0.12]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-0.23; 1.08]$, respectively.

(3) Moreover, the constraint set for all (delayed) states is not monotone decreasing at all time, due to time-varying delay.

(4) The middle histogram in Fig. 3 also shows that the system current state at instant 2 enters a superset at the next instant. And the same phenomenon can also be seen in the lower histogram. Thus it verifies that the Lyapunov function is non-monotone in the Razumikhin approach.

(5) It takes no more than 4 (i.e., $h + 1$, with $h = 3$ in this example) steps to drive system all (possibly delayed) states from one constraint set to a strict subset. Based on which, it can be easily got that system all (possibly delayed) states can be steered to the terminal set in finite time. Thus the finite-time control performance introduced in Theorem 7 is verified by simulation.

Therefore, one can observe the special set invariance theory and finite-time control theory behind the Razumikhin approach that all the mentioned characteristics are different from those for non-delayed systems. As such, as shown in Remark 4, they are also different from the contexts of Krasovskii approach for time-delay systems. In addition, the number of time-varying delay for the three initial conditions are shown in Fig. 4.

B. Time-delay CSTR System

Consider the well-known CSTR system modeled as a non-linear time-delay system in [53]. In [54], the CSTR system is represented by discrete T-S fuzzy systems and the sampling time is set as $T_s = 0.2$ minutes. However, the time delay is set as a constant number in [54]. In contrast, a more natural condition is given in this paper that the time delay is allowed to be time-varying, and the upper bound of delay is set as $h = 10$.

There are three nominal states when $u = 0$: $x_{s1} = [0.1440; 0.8862]$, $x_{s2} = [0.4472; 2.7520]$, $x_{s3} = [0.7646; 4.7052]$. Then the discrete T-S fuzzy time-delay systems can be constructed as follows:

Fig. 3. Constraint sets for all (delayed) states and the current state at each instant for the three different initial conditions. The upper, middle, and lower histograms correspond to $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-1.64; -0.479]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [2.6; 0.12]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-0.23; 1.08]$, respectively. Set 1 represents the smallest constraint set (terminal constraint set) in Fig. 2, while Set 11 represents the largest constraint set.

Fig. 4. Time-varying delay for the three initial conditions.

Rule 1: If $x_2(k)$ is 0.8862, then

$$x(k+1) = A_1x(k) + A_{d1}x_d(k) + B_1u(k) + E_1w(k)$$

Rule 2: If $x_2(k)$ is 2.7520, then

$$x(k+1) = A_2x(k) + A_{d2}x_d(k) + B_2u(k) + E_2w(k)$$

Rule 3: If $x_2(k)$ is 4.7052, then

$$x(k+1) = A_3x(k) + A_{d3}x_d(k) + B_3u(k) + E_3w(k)$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.7500 & 0.0119 \\ -0.2238 & 0.8262 \end{bmatrix}, A_2 = \begin{bmatrix} 0.6203 & 0.0762 \\ -1.2337 & 1.3265 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0.3068 & 0.0442 \\ -3.6621 & 1.0765 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.0435 & 0.0003 \\ -0.0061 & 0.0455 \end{bmatrix}, \\ A_{d2} &= \begin{bmatrix} 0.0403 & 0.0019 \\ -0.0312 & 0.0581 \end{bmatrix}, A_{d3} = \begin{bmatrix} 0.0310 & 0.0013 \\ -0.1037 & 0.0528 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0.0004 \\ 0.0546 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0023 \\ 0.0698 \end{bmatrix}, B_3 = \begin{bmatrix} 0.0015 \\ 0.0634 \end{bmatrix}, \\ E_1 &= E_2 = E_3 = \begin{bmatrix} 0 \\ 0.00001 \end{bmatrix}. \end{aligned}$$

The membership functions (see, Fig. 5) are given as follows:

$$\begin{cases} F_1 = \begin{cases} 1 & \text{if } x_2 \leq 0.8862 \\ \frac{2.7520 - x_2}{2.7520 - 0.8862} & \text{if } 0.8862 < x_2 \leq 2.7520 \\ 0 & \text{if } x_2 > 2.7520 \end{cases} \\ F_2 = \begin{cases} 1 - F_1 & \text{if } 0.8862 < x_2 \leq 2.7520, \\ 1 - F_3 & \text{if } 2.7520 < x_2 \leq 4.7052. \end{cases} \\ F_3 = \begin{cases} 0 & \text{if } x_2 \leq 2.7520 \\ \frac{2.7520 - x_2}{4.7052 - 2.7520} & \text{if } 2.7520 < x_2 \leq 4.7052 \\ 1 & \text{if } x_2 > 4.7052 \end{cases} \end{cases}$$

Fig. 5. Membership functions.

The weighting matrices Q and R are selected as $\text{diag}\{1e-6, 1e-9\}$ and 0.001, respectively. The input constraint and bound of disturbance are set as: $\|u\| \leq 10$, $\|w\| \leq 1$. Since the constraint $\gamma + \gamma_d = 1$ should be satisfied, set $\gamma = 0.8$, $\gamma_d = 0.2$.

The simulation result for Algorithm 1 (online MPC approach) is shown in Figs. 6-9 where the initial (delayed) states are set as $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.6972; 3.652]$. The piecewise reference signal changes from $[0.4472; 2.7520]$, to $[0.1440; 0.8862]$,

Fig. 6. Trajectories of x_1 . Fig. 7. Trajectories of x_2 . Fig. 8. Trajectories of u .

Fig. 9. The time-varying delay.

Fig. 10. Control performance for two different initial conditions using Algorithm 2: $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.8573; 4]$ and $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.126; 3.0533]$.

to $[0.4472; 2.7520]$, and then to $[0.7646; 4.7052]$. Fig. 6 and Fig. 7 demonstrate that the closed-loop state response can track the piecewise reference well via the proposed Algorithm 1. Fig. 8 shows that the input constraint is satisfied. The number of time-varying delay is displayed in Fig. 9. It can be seen that large delays are involved that the upper bound of delay is 10. As such, if the Krasovskii approach is utilized instead, it can be easily got that the controller synthesis and computation will become very complex.

The simulation result of Algorithm 2 (off-line MPC approach) is shown in Fig. 10. The number of robust constraint sets including the terminal constraint set is given as 25. From Fig. 10, it can be seen that, despite different initial conditions, the system all (possibly delayed) states are steered to the terminal set in the end.

Fig. 11 gives a clearer view of the differences via histograms, in constraint sets where all (delayed) states and the

Fig. 11. Constraint sets for all (delayed) states and the current state at each time instant for the two different initial conditions. The upper and the lower histograms correspond to $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.8573; 4]$, $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.126; 3.0533]$, respectively. Set 1 represents the smallest constraint set (terminal constraint set) in Fig. 10, while Set 25 represents the largest constraint set.

Fig. 12. The number of time-varying delay for the two initial conditions.

current state belong to, respectively. One can see that system all (possibly delayed) states are steered to the terminal set in finite time. And the implied positively invariant set theory and finite-time control theory are the same as those demonstrated in Example 1. The number of time-varying delay for the two initial conditions are shown in Fig. 12.

VI. CONCLUSIONS

FMPC of discrete-time nonlinear systems with time-varying delay and persistent disturbances is investigated in this paper. Both online and off-line MPC approaches are provided. Based on the Lyapunov-Razumikhin approach, new conditions of robust positive invariance and input-to-state stability with respect to disturbances are realized. In the off-line approach, a sequence of explicit control laws corresponding to a sequence of robust constraint sets are computed off-line. Therefore, the online computational burden of classical MPC can be significantly reduced. Moreover, the exact time delay can be allowed unknown in the proposed two approaches. However, if the time delay is known, a less conservative result can be provided in the off-line MPC approach. Especially, The positively invariant set theory and finite-time control theory based on LRF method are directly revealed by the proposed off-line MPC approach. Simulation results show that the proposed two methods are effective.

APPENDIX A PROOF OF THEOREM 1

Proof: Resorting to the dilation lemma [55],

$$-G_b^T X_a^{-1} G_b \leq X_a - G_b^T - G_b \quad (41)$$

The following inequality is got from (15),

$$\begin{bmatrix} \gamma\Delta & * & * & * \\ 0 & \gamma_d\Delta & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_a G_b + B_a Y_b & A_{da} G_b & E_a & -X_l \end{bmatrix} \leq 0 \quad (42)$$

where $\Delta = (\lambda - 1)G_b^T X_a^{-1} G_b$. By multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I\}$ and its transpose from both sides of (42), respectively, yields that

$$\begin{bmatrix} \gamma(\lambda - 1)X_a^{-1} & * & * & * \\ 0 & \gamma_d(\lambda - 1)X_a^{-1} & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_a + B_a K_b & A_{da} & E_a & -X_l \end{bmatrix} \leq 0 \quad (43)$$

Based on which one has

$$\sum_{a=1}^L \sum_{b=1}^L \mu_a \mu_b \begin{bmatrix} \gamma\Theta & * & * & * \\ 0 & \gamma_d\Theta & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -X_l \end{bmatrix} \leq 0 \quad (44)$$

where $\Theta = (\lambda - 1)X_a^{-1}$.

Substituting X_a^{-1} with P_a/ξ , it holds that

$$\begin{bmatrix} \gamma(\lambda - 1)P_\mu/\xi & * & * & * \\ 0 & \gamma_d(\lambda - 1)P_\mu/\xi & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -X_l \end{bmatrix} \leq 0 \quad (45)$$

Applying schur complement to (45), and then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides, respectively, yields that

$$\varpi \begin{bmatrix} \gamma(\lambda - 1)P_\mu/\xi & & \\ & \gamma_d(\lambda - 1)P_\mu/\xi & \\ & & -\frac{\lambda}{\eta^2}I \end{bmatrix} \varpi^T + \varpi \Lambda X_l^{-1} \Lambda^T \varpi^T \leq 0 \quad (46)$$

where $\varpi = [x^T, x_d^T, w^T]$, $\Lambda = [A_\mu + B_\mu K_\mu, A_{d\mu}, E_\mu]^T$.

By substituting X_l^{-1} with P_l/ξ , (46) is equal to

$$\frac{1}{\xi} x^+ T P_\mu^+ x^+ - \frac{\lambda}{\eta^2} w^T w \leq \frac{1-\lambda}{\xi} (\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d) \quad (47)$$

With $\gamma + \gamma_d = 1$, it is easy to get $\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d \leq \max\{x^T P_\mu x, x_d^T P_\mu x_d\}$, put it to (47) and then (14) is got.

(16) guarantees the input constraint and the proof can be seen in [56]. The proof is thus completed.

APPENDIX B PROOF OF THEOREM 2

Proof: Firstly, (17) can be easily got by resorting to the eigenvalues of the positive definite matrices P_l , $l \in \mathbb{Z}_{[1, L]}$. Let $\rho_{\min} := \min\{\rho_{\min}(P_l) | l \in \mathbb{Z}_{[1, L]}\}$, and $\rho_{\max} := \max\{\rho_{\max}(P_l) | l \in \mathbb{Z}_{[1, L]}\}$, thus $\rho_{\min}\|x\|^2 \leq V(x) \leq \rho_{\max}\|x\|^2$. As such, the requirement of (17) is satisfied.

Then the main effort is focused on the satisfaction of optimization problem (18). Considering (41), then multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I, I, I\}$ and its transpose from both sides of (19), respectively, yields that

$$\begin{bmatrix} -\gamma X_a^{-1} & * & * & * & * & * \\ 0 & -\gamma_d X_a^{-1} & * & * & * & * \\ 0 & 0 & -\tau \xi I & * & * & * \\ \Upsilon & A_{da} & \xi E_a & -X_l & * & * \\ Q & 0 & 0 & 0 & -\xi Q & * \\ RK_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (48)$$

where $\Upsilon = A_a + B_a K_b$. (48) is equivalent to

$$\begin{bmatrix} -\gamma \xi X_a^{-1} & * & * & * & * & * \\ 0 & -\gamma_d \xi X_a^{-1} & * & * & * & * \\ 0 & 0 & -\tau I & * & * & * \\ \Upsilon & A_{da} & E_a & -\frac{X_l}{\xi} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_b & 0 & 0 & 0 & 0 & -R \end{bmatrix} \leq 0 \quad (49)$$

Substituting X_a^{-1} with P_a/ξ to (49), and rewriting (49) as

$$\begin{bmatrix} Q + K_\mu^T RK_\mu - \gamma P_\mu & * & * & * \\ 0 & -\gamma_d P_\mu & * & * \\ 0 & 0 & -\tau I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -P_l^{-1} \end{bmatrix} \leq 0 \quad (50)$$

Applying schur complement to (50), then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides of the resulting matrix inequality, respectively, noticing that $P_\mu^+ = \sum_{l=1}^L \mu_l(x^+) P_l$, then it holds that

$$x^+ T P_\mu^+ x^+ + x^T Q x + x^T K_\mu^T RK_\mu x - \tau w^T w - (\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d) \leq 0 \quad (51)$$

With $\gamma + \gamma_d = 1$, it is easy to get $\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d \leq \bar{V}(x)$, thus

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (52)$$

The proof is completed.

APPENDIX C PROOF OF THEOREM 5

Proof: It is first shown that (33) implies (30). Resorting to the dilation lemma [55],

$$-G_b^T X_t^{-1} G_b \leq X_t - G_b - G_b^T \quad (53)$$

The following inequality is got from (33),

$$\begin{bmatrix} \gamma M & * & * & * \\ 0 & \gamma_d M & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Gamma & A_{da} G_b & E_a & -X_t \end{bmatrix} \leq 0 \quad (54)$$

where $M = (\lambda - 1)G_b^T X_t^{-1} G_b$. By multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I\}$ and its transpose from both sides of (54), respectively, yields that

$$\begin{bmatrix} \gamma(\lambda - 1)X_t^{-1} & * & * & * \\ 0 & \gamma_d(\lambda - 1)X_t^{-1} & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Gamma & A_{da} & E_a & -X_t \end{bmatrix} \leq 0 \quad (55)$$

(55) can be rewritten as

$$\begin{bmatrix} \gamma(\lambda - 1)X_t^{-1} & * & * & * \\ 0 & \gamma_d(\lambda - 1)X_t^{-1} & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -X_t \end{bmatrix} \leq 0 \quad (56)$$

Applying schur complement to (56) and then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides of resulting inequality, respectively, yields that

$$\varpi \begin{bmatrix} \gamma(\lambda - 1)X_t^{-1} & & \\ & \gamma_d(\lambda - 1)X_t^{-1} & \\ & & -\frac{\lambda}{\eta^2} I \end{bmatrix} \varpi^T + \varpi \Lambda X_t^{-1} \Lambda^T \varpi^T \leq 0 \quad (57)$$

where $\varpi = [x^T, x_d^T, w^T]$, $\Lambda = [(A_\mu + B_\mu K_\mu), A_{d\mu}, E_\mu]^T$.

By substituting X_t^{-1} with P_t/ξ , (57) is equivalent to

$$\frac{1}{\xi} x^+ T P_t x^+ - \frac{\lambda}{\eta^2} w^T w \leq \frac{1-\lambda}{\xi} (\gamma x^T P_t x + \gamma_d x_d^T P_t x_d) \quad (58)$$

With $\gamma x^T P_t x + \gamma_d x_d^T P_t x_d \leq \max\{x^T P_t x, x_d^T P_t x_d\}$ due to $\gamma + \gamma_d = 1$, thus (30) is got.

Since (31) can be easily got by resorting to the eigenvalues of P_t , next we show that (32) is guaranteed by (34). Considering (53), multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I, I, I\}$ and its transpose from both sides of (34), respectively, yields that

$$\begin{bmatrix} -\gamma X_t^{-1} & * & * & * & * & * \\ 0 & -\gamma_d X_t^{-1} & * & * & * & * \\ 0 & 0 & -\tau \xi I & * & * & * \\ \bar{O} & A_{da} & \xi E_a & -X_t & * & * \\ Q & 0 & 0 & 0 & -\xi Q & * \\ RK_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (59)$$

where $\bar{O} = A_a + B_a K_b$. (59) is equivalent to

$$\begin{bmatrix} -\gamma \xi X_t^{-1} & * & * & * & * & * \\ 0 & -\gamma_d \xi X_t^{-1} & * & * & * & * \\ 0 & 0 & -\tau I & * & * & * \\ \bar{O} & A_{da} & E_a & -\frac{X_t}{\xi} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_b & 0 & 0 & 0 & 0 & -R \end{bmatrix} \leq 0 \quad (60)$$

Substituting $X_t^{-1} = \xi^{-1} P_t$ and then applying schur complement to (60), thus one has

$$\begin{bmatrix} \bar{W} & * & * & * \\ 0 & -\gamma_d P_t & * & * \\ 0 & 0 & -\tau I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -P_t^{-1} \end{bmatrix} \leq 0 \quad (61)$$

where $\bar{W} = Q + K_\mu^T R K_\mu - \gamma P_t$. Applying schur complement to (61), then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides of resulting inequality, respectively, yields that

$$x^{+T} P_t x^+ + x^T Q x + x^T K_\mu^T R K_\mu x - \tau w^T w - (\gamma x^T P_t x + \gamma_d x_d^T P_t x_d) \leq 0 \quad (62)$$

With $\gamma + \gamma_d = 1$, it is easy to get $\gamma x^T P_t x + \gamma_d x_d^T P_t x_d \leq \bar{V}(x)$, thus

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (63)$$

(35) guarantees the input constraint and the proof can be seen in [56]. The proof is thus completed.

REFERENCES

- [1] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of linear time-delay systems," *IEEE Trans. Autom. Control*, vol. 47, no. 2, pp. 253–270, Feb. 2002.
- [2] P. Pepe and Z.-P. Jiang, "A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems," *Syst. Control Lett.*, vol. 55, no. 12, pp. 1006–1014, 2006.
- [3] P. Pepe, "On Liapunov–Krasovskii functionals under carathéodory conditions," *Automatica*, vol. 43, no. 4, pp. 701–706, 2007.
- [4] M. Jankovic, "Control Lyapunov–Razumikhin functions and robust stabilization of time delay systems," *IEEE Trans. Autom. Control*, vol. 46, no. 7, pp. 1048–1060, Jul. 2001.
- [5] A. R. Teel, "Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem," *IEEE Trans. Autom. Control*, vol. 43, no. 7, pp. 960–964, Jul. 1998.
- [6] B. Liu and D. J. Hill, "Input-to-state stability for discrete time-delay systems via the Razumikhin technique," *Syst. Control Lett.*, vol. 58, no. 8, pp. 567–575, 2009.
- [7] B. Liu and H. J. Marquez, "Razumikhin-type stability theorems for discrete delay systems," *Automatica*, vol. 43, no. 7, pp. 1219–1225, 2007.
- [8] R. H. Gielen and M. Lazar, "Stabilization of polytopic delay difference inclusions via the Razumikhin approach," *Automatica*, vol. 47, no. 12, pp. 2562–2570, 2011.
- [9] R. H. Gielen, A. R. Teel, and M. Lazar, "Tractable Razumikhin-type conditions for input-to-state stability analysis of delay difference inclusions," *Automatica*, vol. 49, no. 2, pp. 619–625, 2013.
- [10] N. Stanković, S. Olaru, and S.-I. Niculescu, "Further remarks on asymptotic stability and set invariance for linear delay-difference equations," *Automatica*, vol. 50, no. 8, pp. 2191–2195, 2014.
- [11] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Springer Science & Business Media, 1999, vol. 463.
- [12] Y.-Y. Cao, Z. Lin, and T. Hu, "Stability analysis of linear time-delay systems subject to input saturation," *IEEE Trans. Circuits Syst. I, Fundam. Theory*, vol. 49, no. 2, pp. 233–240, Feb. 2002.
- [13] Z. Lin, "On asymptotic stabilizability of discrete-time linear systems with delayed input," *Communications in Information and Systems*, vol. 7, no. 3, 2007.
- [14] X. Li and H. Gao, "A new model transformation of discrete-time systems with time-varying delay and its application to stability analysis," *IEEE Trans. Autom. Control*, vol. 56, no. 9, pp. 2172–2178, Sep. 2011.
- [15] X. Su, P. Shi, L. Wu, and Y.-D. Song, "A novel control design on discrete-time Takagi–Sugeno fuzzy systems with time-varying delays," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 4, pp. 655–671, Aug. 2013.
- [16] J. Sun, Q.-L. Han, J. Chen, and G.-P. Liu, "Less conservative stability criteria for linear systems with interval time-varying delays," *Int. J. Robust Nonlinear Control*, vol. 25, no. 4, pp. 475–485, 2015.
- [17] X. Li and C. E. De Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state delay," *Automatica*, vol. 33, no. 9, pp. 1657–1662, 1997.
- [18] Y.-Y. Cao, Y.-X. Sun, and C. Cheng, "Delay-dependent robust stabilization of uncertain systems with multiple state delays," *IEEE Trans. Autom. Control*, vol. 43, no. 11, pp. 1608–1612, Nov. 1998.
- [19] M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.
- [20] W. H. Kwon, Y. Lee, and S. H. Han, "General receding horizon control for linear time-delay systems," *Automatica*, vol. 40, no. 9, pp. 1603–1611, 2004.
- [21] S. C. Jeong and P. Park, "Constrained MPC algorithm for uncertain time-varying systems with state-delay," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 257–263, Feb. 2005.
- [22] M. Reble, R. M. Esfajani, S. K. Y. Nikravesh, and F. Allgöwer, "Model predictive control of constrained non-linear time-delay systems," *IMA J. Math. Control Inf.*, vol. 28, no. 2, pp. 183–201, 2011.
- [23] M. Reble, F. D. Brunner, and F. Allgöwer, "Model predictive control for nonlinear time-delay systems without terminal constraint," in *Proceedings of the 18th IFAC World Congress*, 2011.
- [24] W. Lombardi, S. Olaru, S.-I. Niculescu, and L. Hetel, "A predictive control scheme for systems with variable time-delay," *Int. J. Control*, vol. 85, no. 7, pp. 915–932, 2012.
- [25] G. Franzè, F. Tedesco, and D. Famularo, "Model predictive control for constrained networked systems subject to data losses," *Automatica*, vol. 54, pp. 272–278, 2015.
- [26] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, no. 1, pp. 3–20, 2002.
- [27] A. Bemporad, F. Borrelli, M. Morari, and *et al.*, "Model predictive control based on linear programming: the explicit solution," *IEEE Trans. Autom. Control*, vol. 47, no. 12, pp. 1974–1985, Dec. 2002.
- [28] F. Borrelli, M. Baotic, A. Bemporad, and M. Morari, "Dynamic programming for constrained optimal control of discrete-time linear hybrid systems," *Automatica*, vol. 41, no. 10, pp. 1709–1721, 2005.
- [29] P. Tøndel, T. A. Johansen, and A. Bemporad, "An algorithm for multi-parametric quadratic programming and explicit MPC solutions," *Automatica*, vol. 39, no. 3, pp. 489–497, 2003.
- [30] Z. Wan and M. V. Kothare, "An efficient off-line formulation of robust model predictive control using linear matrix inequalities," *Automatica*, vol. 39, no. 5, pp. 837–846, 2003.
- [31] B. Ding, Y. Xi, M. T. Cychowski, and T. O'Mahony, "Improving off-line approach to robust MPC based-on nominal performance cost," *Automatica*, vol. 43, no. 1, pp. 158–163, 2007.
- [32] D. Angeli, A. Casavola, G. Franzè, and E. Mosca, "An ellipsoidal off-line MPC scheme for uncertain polytopic discrete-time systems," *Automatica*, vol. 44, no. 12, pp. 3113–3119, 2008.
- [33] W. Yang, G. Feng, and T. Zhang, "Robust model predictive control for discrete-time takagi–sugeno fuzzy systems with structured uncertainties and persistent disturbances," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 5, pp. 1213–1228, Oct. 2014.
- [34] —, "Quasi-min-max fuzzy model predictive control of direct methanol fuel cells," *Fuzzy Sets Syst.*, vol. 248, pp. 39–60, 2014.
- [35] B. Ding and B. Huang, "Constrained robust model predictive control for time-delay systems with polytopic description," *Int. J. Control*, vol. 80, no. 4, pp. 509–522, 2007.
- [36] D. Li and Y. Xi, "Constrained feedback robust model predictive control for polytopic uncertain systems with time delays," *Int. J. Syst. Sci.*, vol. 42, no. 10, pp. 1651–1660, 2011.
- [37] T. Zhang, G. Feng, and J. Lu, "Fuzzy constrained min-max model predictive control based on piecewise Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 4, pp. 686–698, Aug. 2007.
- [38] Y. Xia, H. Yang, P. Shi, and M. Fu, "Constrained infinite-horizon model predictive control for fuzzy-discrete-time systems," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 2, pp. 429–436, Apr. 2010.

- [39] Y. Zhao, H. Gao, and T. Chen, "Fuzzy constrained predictive control of non-linear systems with packet dropouts," *IET Contr. Theory Appl.*, vol. 4, no. 9, pp. 1665–1677, 2010.
- [40] X.-H. Chang and G.-H. Yang, "Fuzzy robust constrained model predictive control for nonlinear systems," *Asian J. Control*, vol. 13, no. 6, pp. 947–955, 2011.
- [41] G. Karer, G. Mušič, I. Škrjanc, and B. Zupančič, "Hybrid fuzzy model-based predictive control of temperature in a batch reactor," *Comput. Chem. Eng.*, vol. 31, no. 12, pp. 1552–1564, 2007.
- [42] J. Causa, G. Karer, A. Núñez, D. Sáez, I. Škrjanc, and B. Zupančič, "Hybrid fuzzy predictive control based on genetic algorithms for the temperature control of a batch reactor," *Comput. Chem. Eng.*, vol. 32, no. 12, pp. 3254–3263, 2008.
- [43] G. Karer, G. Mušič, I. Škrjanc, and B. Zupančič, "Model predictive control of nonlinear hybrid systems with discrete inputs employing a hybrid fuzzy model," *Nonlinear Anal.-Hybrid Syst.*, vol. 2, no. 2, pp. 491–509, 2008.
- [44] D. Q. Mayne, M. M. Seron, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.
- [45] R. Ghaemi, J. Sun, and I. V. Kolmanovsky, "Robust control of constrained linear systems with bounded disturbances," *IEEE Trans. Autom. Control*, vol. 57, no. 10, pp. 2683–2688, Oct. 2012.
- [46] B. Ding and H. Pan, "Output feedback robust MPC with one free control move for the linear polytopic uncertain system with bounded disturbance," *Automatica*, vol. 50, no. 11, pp. 2929–2935, 2014.
- [47] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.
- [48] D. M. Raimondo, D. Limon, M. Lazar, L. Magni, and E. F. Camacho, "Min-max model predictive control of nonlinear systems: A unifying overview on stability," *Eur. J. Control*, vol. 15, no. 1, pp. 5–21, 2009.
- [49] L. Magni and R. Scattolini, "Robustness and robust design of MPC for nonlinear discrete-time systems," in *Assessment and future directions of nonlinear model predictive control*. Springer, 2007.
- [50] D. Limon, T. Alamo, D. M. Raimondo, D. M. de la Peña, J. M. Bravo, A. Ferramosca, and E. F. Camacho, "Input-to-state stability: a unifying framework for robust model predictive control," in *Nonlinear model predictive control*. Springer, 2009.
- [51] M. Kocvara and M. Stingl, "PENBMI users guide," <http://www.penopt.com>, 2006, accessed: 2016-02-04.
- [52] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *IEEE International Symposium on Computer Aided Control Systems Design*, Sep. 2004.
- [53] B. Lehman, J. Bentsman, S. V. Lunel, and E. I. Verriest, "Vibrational control of nonlinear time lag systems with bounded delay: averaging theory, stabilizability, and transient behavior," *IEEE Trans. Autom. Control*, vol. 39, no. 5, pp. 898–912, May 1994.
- [54] Y.-Y. Cao and P. M. Frank, "Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 2, pp. 200–211, Apr. 2000.
- [55] J. C. Geromel, P. D. Peres, and J. Bernussou, "On a convex parameter space method for linear control design of uncertain systems," *SIAM J. Control Optim.*, vol. 29, no. 2, pp. 381–402, 1991.
- [56] N. Wada, K. Saito, and M. Saeaki, "Model predictive control for linear parameter varying systems using parameter dependent Lyapunov function," *IEEE Trans. Circuits Syst. II-Express Briefs*, vol. 53, no. 12, pp. 1446–1450, Dec. 2006.