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Persistently-Excited Adaptive Relative Localization and Time-Varying Formation of Robot Swarms

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Abstract—In this paper we investigate the problem of controlling a multi-robot team to follow a leader in formation, supported by relative position estimate derived from distance and self-displacement measurements, thus waiving the need of external localization infrastructure. The main challenge of the problem, which is to simultaneously fulfill both relative localization and control tasks, is efficiently and novelly resolved by embedding a distance-displacement-based *persistently excited adaptive relative localization* technique into a *time-varying formation* with bounded control input (PEARL-TVF). By assuming that the leader is globally reachable and selecting proper parameters, it is shown that PEARL-TVF ensures exponentially convergent localization, which leads to exponentially convergent formation when the leader's behavior is deterministic, and bounded formation error for a non-deterministic leader. Numerical simulations and experiments on quadcopters are provided to verify the theoretical findings.

I. INTRODUCTION

Distributed formation control for multi-robot swarms has been a highly active topic in recent years, and in order to achieve independence from external localization system, many control schemes based on relative position [1]–[8] or relative bearing measurements [9]–[11] have been proposed over the years. To realize these schemes, many researchers have attempted to develop new relative localization (RL) methods based on different types of sensors, e.g. Lidar [12], array of beacons [13], or visual marker [14]. However, these methods are still quite restrictive for most applications. For example, the laser scan in [12] may fail to distinguish the robots from one another, or from other objects with similar shape, while the array of beacons used in [13], besides having limited range, is also quite difficult to be installed on most robot platforms. On the other hand, while the vision-based method in [14] can provide good RL accuracy, the limited camera field of view and detectable range of markers would also restrict the flexibility of the system. For these limitations, it can be seen that the capability to directly measure the relative position or bearing remains a strong assumption from a technological

point of view, especially in the absence of external localization systems.

To address this issue, some researchers have investigated alternative strategies by employing relative distance measurements [15]–[17]. Specifically, it requires an integration of RL with formation control. To this end, in [15] a stop-and-go coordination strategy was proposed for a robot team with the leader moving at a constant velocity. Similarly, in [16], to facilitate RL, Jiang et. al. prescribed that neighboring agents perform ideal linear-circular motions with commensurate angular frequencies over each interval. In a recent work [17], the authors have attempted to integrate RL with *similar formation* under an adaptive control framework. However, there remain several limitations in these works. Clearly, the stop-and-go strategy in [15] would reduce the formation efficiency, and the ideal circular trajectories required in [16] would be too restrictive in practice. Finally, the proposed method in [17] can only achieve formation with bounded error when the initial localization is not perfect. To overcome the aforementioned issues, we believe an appropriate approach must undertake a more holistic and efficient strategy to simultaneously solve the RL and formation control problems.

To this end, an integrated estimation-control scheme, so-called PEARL-TVF (*Persistently-Excited Adaptive Relative Localization and Time-Varying Formation*), is proposed to efficiently tackle both RL and formation control problems with a globally reachable leader. Specifically, by embedding the so-called *persistent excitation* (PE) characteristics into TVF reference trajectories, we are able to achieve RL with exponential convergence through the use of a recursive least-square optimization based estimator, which in turn induces the exponentially convergent formation if the leader's input is deterministic. Furthermore, when the leader has non-deterministic but bounded input, under the simplification of the topology to a directed acyclic graph (DAG), the TVF can still be achieved with an ultimate error whose bound can be tightly stipulated.

The contributions of this work are threefold. First, we propose an effective and efficient integration scheme for RL and TVF, where the TVF objective and the RL objective are inherently reinforcing each other: by design, the predefined TVF reference trajectories satisfy a PE condition, which facilitates the exponential convergence of RL; reciprocally, the TVF can be achieved with exponential convergence for a globally reachable and deterministic leader, and ultimately bounded error for a non-deterministic leader. Second, we consider discrete-time dynamics with bounded inputs, which maximizes the compatibility of the proposed scheme with

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A video summarizing the main concepts and experiment results of this paper can be downloaded as supplementary media materials on <http://ieeexplore.ieee.org>, or viewed directly at <https://youtu.be/0MUBIM8vNGY>.

practical implementation. Specifically, the use of discrete-time formulation can help waive the requirement of the derivative of distance measurements in previous work [17], which is difficult to obtain in practice. Finally, we also study the ultimate formation error bound for a leader with non-deterministic input (unmodeled dynamics) in a DAG, and provide an explicit error bound which is dependent on the network topology, the input bound, and the control gain. Note that the non-deterministic input is unpredictable and can be seen as bounded unknown disturbance, and the coordination problem under input saturation and external disturbance is rarely investigated in the literature, except a few papers [18], [19] which studied output regulation with a known disturbance dynamic model. Different from the above works which employed small gain approach to avoid saturation and can only achieve semi-global stability, we address the saturation in the analysis which saves the trouble of gain tuning and achieves global stability. Also, it is worth mentioning that the predicted error bound matches closely with the simulation result.

The remainder of this paper is organized as follows: after providing the basic definitions and the problem formulation in Section II, we present the localization and control laws in Section III. The convergence results and analyses are then provided in Section IV. Numerical simulations and physical implementations are conducted in Section V to validate the theoretical findings. Finally, Section VI concludes our work.

Notations: In this paper, the sets of natural numbers and real numbers are respectively denoted by $\mathbb{N}$ and $\mathbb{R}$. For $k \in \mathbb{N}$, we denote $k^+ \triangleq k + 1$ and $k^- \triangleq k - 1$. $\mathbb{R}^m$ denotes the m -dimension real vector space with $m = 2$ or 3 . For a vector $v \in \mathbb{R}^m$, v' denotes its transpose and $\|v\|$ stands for the Euclidean norm. For a matrix $M \in \mathbb{R}^{p \times m}$, $\|M\| = \sup\{\|Mx\| : x \in \mathbb{R}^m, \|x\| = 1\}$. For a set N , $|N|$ denotes the cardinality of N . We abbreviate *exponential convergence* and *exponentially convergent* respectively as EC and e.c. Finally, we use $\pi_U(u)$ to denote the projection of a vector u as follows:

$$\pi_U(u) \triangleq s_U(u)u, \quad s_U(u) \triangleq U / \max\{U, \|u\|\}. \quad (1)$$

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section we shall put forth some basic definitions before formulating the problem under consideration.

A. Topology

For a leader-follower network of $n + 1$ agents, let us describe the sensing and communication within the network by a directed graph $G = \{\bar{V}, \bar{E}\}$. Specifically, $\bar{V} = \{0, 1, 2, \dots, n\}$ is the node set, $\bar{E} = \{(i, j) : i, j \in \bar{V}, i \neq j\}$ is the edge set where $(i, j) \in \bar{E}$ if agent i can obtain measurements from agent j . $N_i = \{j : (i, j) \in \bar{E}\}$ is the neighbor set of agent i . We assign agent 0 as a leader which has no neighbors, and denote the set of followers by $V = \bar{V} \setminus \{0\}$. A path from node i_1 to node i_l is defined as $(i_1, i_2, \dots, i_l)$ where $(i_k, i_{k+1}) \in \bar{E}$ for $k = 1, \dots, l - 1$, and it is a cycle if $i_1 = i_l$. G is called a DAG if it has no cycle. The leader 0 is said to be globally reachable if there exists a path from any other node to 0. We use $a_{ij} \geq 0$ to model the connection weight between two

agents i and j , and $a_{ij} > 0$ if and only if $(i, j) \in \bar{E}$. In this work, we adopt the following standard assumption:

Assumption II.1. *The leader 0 is globally reachable, and $\sum_j a_{ij} = 1$ for any $i \in V$.*

B. Dynamics and Measurements

Assume that the agents' positions in some common coordinate frame $\mathcal{F}$ are modeled by the following discrete-time dynamics with bounded velocity:

$$p_0(k^+) = p_0(k) + T\bar{u}_0(k) + Tv_0(k); \quad (2a)$$

$$p_i(k^+) = p_i(k) + T\bar{u}_i(k), \quad \forall i \in V; \quad (2b)$$

$$\|\bar{u}_i(k)\| \leq \bar{U}_i, \quad \forall i \in \bar{V}, \|v_0(k)\| \leq U_0, \quad (2c)$$

where $p_i(k) \in \mathbb{R}^m$ denotes the position of agent i at time kT , T is the sampling period, and $U_0, \bar{U}_i > 0$ are some user-defined maximum velocities. Note that the leader's input consists of two terms: $\bar{u}_0(k)$ denotes a predetermined input to track a reference trajectory; $v_0(k)$ denotes an input generated in real-time to address unpredictable situations during operation (e.g. obstacles, user's input), which is not available to followers. Accordingly, the leader is said to be *deterministic* if $v_0(k) \equiv 0$, and *non-deterministic* if $v_0(k) \neq 0$.

Remark II.1. *In this paper, we forgo a detailed discussion on the dynamics of specific types of robots. In the experiments of Section V-B, the proposed algorithms are implemented on some quadrotor platforms, which employ an open-sourced flight control system called px4 control flight stack¹ that offers velocity control mode and has been popularly adopted by the research community. Moreover, since we have imposed that the control input is subject to a pre-determined bound in (2c), we can treat $p_i(k)$ and $p_i(k^+)$ as the starting point and final point of a trajectory from time kT to $(k+1)T$, and use some control technique to drive the agent from $p_i(k)$ to $p_i(k) + T\bar{u}_i(k)$ (or $p_0(k)$ to $p_0(k) + T\bar{u}_0(k) + Tv_0$ in the leader's case). Depending on the specific type of robot, there are several control techniques such as model predictive control, optimal control that can take the robots dynamics into account and realize such a trajectory from $p_i(k)$ and $p_i(k^+)$ with high accuracy.*

We assume that at every time step k agent i can measure the self-displacement $\phi_i(k^-)$, and the distance $d_{ij}(k)$ to its neighbor j , and receive the displacement $\phi_j(k^-)$ sent from the neighbor j to calculate relative displacement $\phi_{ij}(k^-)$. These quantities are formally defined as follows:

$$d_{ij}(k) = \|p_i(k) - p_j(k)\|, \quad (3a)$$

$$\phi_i(k^-) = p_i(k) - p_i(k^-), \quad (3b)$$

$$\phi_{ij}(k^-) = \phi_i(k^-) - \phi_j(k^-), \quad j \in N_i. \quad (3c)$$

An illustration of the measurements over a simple network of three agents can be seen in Fig. 1.

Remark II.2. *In practice, communication and ranging capabilities can be fulfilled by Ultra-Wideband (UWB) sensors*

¹<https://px4.io/>

Fig. 1: An illustration of measurements over a network of three agents. A pointed arc from agent i to agent j is used to represent an edge (i, j) in the topology. The displacement measurements of the agents between time steps $k-1$ and k are illustrated by dashed arrows, and the distance measurements by solid straight lines. In this topology, at the time step k , agent 1 range to and receive data from agent 0; while agent 2 can range to and communicate with both agents 0 and 1. Hence, the distance $d_{10}(k)$ and displacement $\phi_0(k-1)$ will be available to agent 1, and agent 2 will have access to the measurements $d_{20}(k)$, $d_{21}(k)$, $\phi_0(k-1)$, $\phi_1(k-1)$.

[20]–[26]. On the other hand, displacement measurements can be directly obtained from some visual-inertial-odometry [27] or optical flow [28] combined with IMU, which essentially determine the motion of the robot between two consecutive camera frames. Moreover, displacement measurement can also be obtained from wheel odometry on a ground robot platform. Also, we would note that the formulation of $\phi_{ij}(k^-)$ in (3c) implies that the agents' local coordinate frames share a common orientation. In practice this can be achieved by aligning all of these frames with the earth's magnetic frame.

C. Problems

Relative Localization (RL): At time kT , denote the relative position between agents i and j as

$$q_{ij}(k) = p_i(k) - p_j(k). \quad (4)$$

Moreover, denote $\hat{q}_{ij}(k)$ as an estimate of $q_{ij}(k)$, and $\tilde{q}_{ij} \triangleq \hat{q}_{ij}(k) - q_{ij}(k)$ as the *relative localization error*. Based on the measurements (3), we aim to design an estimator for $\hat{q}_{ij}$ to achieve the following convergence:

$$\lim_{k \rightarrow \infty} \|\tilde{q}_{ij}(k)\| = 0, \quad \forall (i, j) \in \bar{E}. \quad (5)$$

Time-Varying Formation (TVF): Let us assign each agent $i \in \bar{V}$ a reference trajectory $\Sigma_i = \{\sigma_i(k)\}_{k=0}^{\infty}$ and define the formation error as follows:

$$\bar{q}_i(k) \triangleq \bar{p}_i(k) - \bar{p}_0(k), \quad \bar{p}_i(k) \triangleq p_i(k) - \sigma_i(k). \quad (6)$$

From this definition, the overall TVF is achieved if the following convergence is achieved for any $i \in V$:

$$\limsup_{k \rightarrow \infty} \|\bar{q}_i(k)\| = \begin{cases} 0, & v_0(k) \equiv 0; \\ Q_i, & v_0(k) \neq 0, \end{cases} \quad (7a)$$

$$(7b)$$

where Q_i is a positive constant depending on the network topology and the upper bound U_0 of $\|v_0(k)\|$. Note that in traditional consensus-based fixed points formation [1],

the reference trajectories would be defined as constants, or $\sigma_i(k) \equiv \sigma_i^* \in \mathbb{R}^m, \forall i \in \bar{V}$.

To efficiently solve the RL and TVF problems, we consider a case when the formation setpoints $\sigma_i(k)$ between neighboring agents satisfy the following PE condition:

Assumption II.2. For any $(i, j) \in \bar{E}$, there exist $\gamma_{ij,2} \geq \gamma_{ij,1} > 0$ and $K_{ij}, n_{ij} \in \mathbb{N}$ such that $\forall l \geq n_{ij}$, there exists $\kappa_{ij}(l) \subseteq \{l, l+1, \dots, l+K_{ij}-1\}$ satisfying

$$\gamma_{ij,1}I \leq \sum_{k \in \kappa_{ij}(l)} \Delta\sigma_{ij}(k) \Delta\sigma'_{ij}(k) \leq \gamma_{ij,2}I, \quad (8)$$

where $\Delta\sigma_{ij}(k) = [\sigma_i(k^+) - \sigma_i(k)] - [\sigma_j(k^+) - \sigma_j(k)]$.

Remark II.3. Simply speaking, Assumption II.2 requires that for every K_{ij} consecutive steps, there exist some steps where the relative reference velocity $\Delta\sigma_{ij}(k)$ satisfies condition (8), i.e. “persistently spans $\mathbb{R}^m$ ”. This condition can be satisfied by selecting appropriate periodic relative time-varying trajectories $\{\sigma_{ij}(k)\}_{k=0}^{\infty}$ that are non-collinear (in 2D) or non-coplanar (in 3D case), and K_{ij} is the period of $\Delta\sigma_{ij}(k)$. Such periodic motions occur in several scenarios where the agents move in a loop for surveillance and patrol, as well as target capture [29], [30]. Furthermore, other than pre-defined periodic motions, the p.e. steps can be negotiated ahead online by neighbouring agents.

III. INTEGRATED ESTIMATION-CONTROL LAWS

In this part, we propose the distributed adaptive estimation-control laws to solve the RL and TVF problems (5) and (7).

A. Relative Localization Estimator

For each agent $i \in V$ and $j \in N_i$, it can be checked that $\zeta_{ij}(k) \triangleq \frac{1}{2}[d_{ij}^2(k^+) - d_{ij}^2(k) - \|\phi_{ij}(k)\|^2] = \phi'_{ij}(k)q_{ij}(k)$. Accordingly, we can define the innovation $\epsilon_{ij}(k)$ as follows:

$$\epsilon_{ij}(k) \triangleq \zeta_{ij}(k) - \phi'_{ij}(k)\hat{q}_{ij}(k) = -\phi'_{ij}(k)\tilde{q}_{ij}(k). \quad (9)$$

Now we can design a least-square estimator to update the estimate $\hat{q}_{ij}(k)$ for the relative position $q_{ij}(k)$ as follows:

$$\begin{cases} \Gamma_{ij}(k^+) = \frac{1}{\beta_f} \left[\Gamma_{ij}(k) - \frac{\Gamma_{ij}(k)\phi_{ij}(k)\phi'_{ij}(k)\Gamma_{ij}(k)}{\beta_f + \phi'_{ij}(k)\Gamma_{ij}(k)\phi_{ij}(k)} \right], \\ \hat{q}_{ij}(k^+) = \hat{q}_{ij}(k) + \phi_{ij}(k) + \Gamma_{ij}(k^+)\phi_{ij}(k)\epsilon_{ij}(k), \end{cases} \quad (10a)$$

$$(10b)$$

where $\Gamma_{ij}(k) \in \mathbb{R}^{m \times m}$ with $\Gamma_{ij}(0) > 0$, and $\beta_f \in (0, 1)$ is a forgetting factor. Note that the equations (9) and (10) represent the updates at time $(k+1)T$, and employ the measurements $d_{ij}(k+1)$, $d_{ij}(k)$, $\phi_{ij}(k)$ in its calculations, which are available as the agent reaches the position $p_i(k^+)$ after executing the control input $\bar{u}_i(k)$ (or $\bar{u}_0(k) + v_0(k)$) generated at time kT .

Remark III.1. Compared with [17] which is based on continuous-time formulation and needs the derivative of distance measurements in the relative localization, we only need distance measurements in the estimator which is suitable for implementation. On the other hand, we employ a recursive least-square estimator to improve the localization robustness.

B. Formation Controller

Based on the relative position estimate $\hat{q}_{ij}(k)$ and $\sigma_j(k)$ sent from $j \in N_i$, we can design the distributed control law $\bar{u}_i(k)$ as follows:

$$\begin{cases} u_i(k) = -\beta \sum_j a_{ij} (\hat{q}_{ij}(k) - \sigma_{ij}(k)), \quad \beta < 1/T, \\ \Delta\sigma_i(k) \triangleq \sigma_i(k^+) - \sigma_i(k), \quad \sigma_{ij}(k) = \sigma_i(k) - \sigma_j(k), \\ \bar{u}_i(k) = \pi_{U_i}(u_i(k)) + T^{-1}\Delta\sigma_i(k), \quad U_i \in (0, \bar{U}_i), \end{cases} \quad (11a)$$

(11b)

(11c)

for all $i, j \in \bar{V}$. Note that $u_0(k) \equiv 0$ as $N_0 = \emptyset$.

Remark III.2. Note that in (11b), the term $\Delta\sigma_i(k)$ is defined by a future formation setpoint $\sigma_i(k^+)$ (11c). Thus, we can consider $T^{-1}\Delta\sigma_i(k)$ as a feedforward term in (11c), where the remaining term $\pi_{U_i}(u_i(k))$ represents a feedback term based on the relative localization estimate. Moreover, it should be noted that the notion of the global frame $\mathcal{F}$ is only introduced for convenience in formulation. From (11c) and (11a), we see that the control $\bar{u}_i(k)$ only consists of $\Delta\sigma_i(k)$, which is a differentiation, and $u_i(k)$, which only involves relative terms. Thus if one changes the global frame $\mathcal{F}$ to another arbitrary frame $\mathcal{F}'$, all relative position estimates and control inputs will remain the same, i.e the system is invariant to the choice of reference frame. Moreover, we can see from (11c) that the control input of each agent is bounded by $\bar{U}_i = U_i + U_{\sigma_i}$, where U_i is the user-defined upper bound of the consensus tracking term $\pi_{U_i}(u_i(k))$ (or the non-deterministic leader's velocity), and U_{σ_i} is the upper bound of the user-defined TVF velocity term $T^{-1}\Delta\sigma_i(k)$.

IV. CONVERGENCE ANALYSIS

A. Convergence of RL error

Under Assumption II.2, select small T to satisfy that

$$T\sqrt{\mathcal{K}_{ij}}(U_i + U_j) < \gamma_{ij,1}, \quad \forall (i, j) \in \bar{E}, \quad (12)$$

where $\mathcal{K}_{ij} = \max_{l \geq n_{ij}} \{\kappa_{ij}(l)\} \leq K_{ij}$. We can show the EC of the relative position estimate on each edge in the following theorem.

Theorem IV.1. Under Assumption II.2 and the condition (12), given $(i, j) \in \bar{E}$, the relative displacement $\phi_{ij}(k)$ satisfies the following PE condition for any $l \geq n_{ij}$:

$$\rho_{ij,1}I \leq \sum_{k \in \kappa_{ij}} \phi_{ij}(k) \phi'_{ij}(k) \leq \rho_{ij,2}I, \quad (13)$$

where $\rho_{ij,1} = [\gamma_{ij,1} - T\sqrt{\mathcal{K}_{ij}}(U_i + U_j)]^2$ and $\rho_{ij,2} = [\gamma_{ij,2} + T\sqrt{\mathcal{K}_{ij}}(U_i + U_j)]^2$. As a result, the estimation error $\tilde{q}_{ij}(k)$ converges to 0 exponentially fast under the estimation law (10).

Proof. Given $(i, j) \in \bar{E}$, denote $v_{ij}(k) = \pi_{U_i}(u_i(k)) - \pi_{U_j}(u_j(k)) - v_j(k)$, where $v_j(k) = 0$ if $j \neq 0$. We can see from (3b), (3c) and (11a) that

$$\phi_{ij}(k) = \Delta\sigma_{ij}(k) + Tv_{ij}(k). \quad (14)$$

To show (13), it is equivalent to show that for any unit vector x , $x'S_{ij}x \in [\rho_{ij,1}, \rho_{ij,2}]$ with $S_{ij} = \sum_{k \in \kappa_{ij}} \phi_{ij}(k) \phi'_{ij}(k)$. Denote

$$W_{ij} = [\Delta\sigma_{ij}(k_1), \Delta\sigma_{ij}(k_2), \dots, \Delta\sigma_{ij}(k_{|\kappa_{ij}|})],$$

$$V_{ij} = [v_{ij}(k_1), v_{ij}(k_2), \dots, v_{ij}(k_{|\kappa_{ij}|})]; \quad k_1, k_2, \dots \in \kappa_{ij}.$$

Direct computation shows that

$$x'S_{ij}x = \|W'_{ij}x\|^2 + 2T\langle W'_{ij}x, V'_{ij}x \rangle + T^2\|V'_{ij}x\|^2.$$

Let $w = \|W'_{ij}x\|$ and $v = \|V'_{ij}x\|$. Since $-wv \leq \langle W'_{ij}x, V'_{ij}x \rangle \leq wv$ by Cauchy-Schwartz inequality, we have $x'S_{ij}x \in [(w - Tv)^2, (w + Tv)^2]$. Note that $v \leq \sqrt{|\kappa_{ij}|}(U_i + U_j)$ as a result of $\|v_{ij}(k)\| \leq U_i + U_j$, and the condition (8) implies that $w \in [\gamma_{ij,1}, \gamma_{ij,2}]$. Therefore, it holds under the condition (12) that $x'S_{ij}x \in [\gamma_{ij,1}^2, \gamma_{ij,2}^2]$, which is equivalent to (13). As the relative position evolves by $q_{ij}(k^+) = q_{ij}(k) + \phi_{ij}(k)$. Combining this with (10) and (9), the dynamics of $\tilde{q}_{ij}(k) \triangleq \hat{q}_{ij}(k) - q_{ij}(k)$ is dictated by $\tilde{q}_{ij}(k^+) = [I - \Gamma_{ij}(k)\phi_{ij}(k)\phi'_{ij}(k)]\tilde{q}_{ij}(k)$. With the PE condition on $\phi_{ij}(k)$, the EC of this system can be proved along the same line as in Theorem 1 of [31]. ■

B. Convergence of TVF error

Based on the convergence of $\hat{q}_{i,j}(k)$ in the last section, here we will solve the TVF problem respectively for a deterministic leader and a nondeterministic leader, by examining the formation error for each follower i . To this end, we first rewrite $u_i(k)$ in (11a) as

$$u_i(k) = -\beta \sum_j a_{ij} (\bar{q}_i(k) - \bar{q}_j(k) + \tilde{q}_{ij}(k)), \quad (15)$$

where we used the equations $\hat{q}_{ij} = q_{ij} + \tilde{q}_{ij}$ and $q_{ij} - \sigma_{ij} = \bar{p}_i - \bar{p}_j = \bar{q}_i - \bar{q}_j$ from (4) and (6). Then the dynamics of the formation error can be found as

$$\begin{aligned} \bar{q}_i(k^+) &= \bar{q}_i(k) + \pi_{U_i}(u_i(k)) - Tv_0(k) \\ &= \bar{q}_i(k) - T\beta s_i(k) \sum_j a_{ij} (\bar{q}_i(k) - \bar{q}_j(k)) \\ &\quad - \varepsilon_i(k) - Tv_0(k), \end{aligned} \quad (16)$$

where $s_i(k) = s_{U_i}(u_i(k))$ and $\varepsilon_i(k) = T\beta s_i(k) \sum_j a_{ij} \tilde{q}_{ij}(k)$.

Let us put forth the following condition and some extra definitions to be used in the next analyses on convergence of the formation error:

$$0 < T(U_i + U_0)^2 / [2(U_i - U_0)] < U_i / \beta, \quad \forall i \in V. \quad (17)$$

Moreover, define $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ and $\hat{C} = C^{-1} = [\hat{c}_{ij}]$ by

$$c_{ij} = \begin{cases} 1, & i = j \in V; \\ -a_{ij}, & i \neq j \in V. \end{cases} \quad (18)$$

Note that C is invertible under Assumption II.1 (Lemma 2.1, [32]). Now the convergence of formation error can be stated as follows:

Theorem IV.2. Let Assumptions II.1 and II.2 hold, and select T by the condition (12). If $v_0(k) \equiv 0$, then the TVF problem (7a) is solved with EC under the estimator (10) and the controller (11a). ■

Proof. See Appendix A.

Theorem IV.3. Under Assumptions II.1 and II.2, let the conditions (12) and (17) hold. Also, assume that G has no cycles, i.e. G is DAG. For a nondeterministic leader, the TVF problem (7b) can be solved by the estimator (10) and the controller (11) with the ultimate bound Q_i given by $Q_i = \sum_j |\hat{c}_{ij}| U_0 / \beta$.

Proof. See Appendix B. ■

Remark IV.1. Given $U_i > U_0$, (17) can be easily satisfied by small T or β . The requirement $U_i > U_0$ is also intuitively meaningful: if we see the upper bound U_0 of the input $v_0(k)$ as the maximum uncertainty, then the maximum control effort U_i must be larger than U_0 to overcome the uncertainty for global stability, otherwise $v_0(k)$ may cancel out $\bar{u}_i(k)$ and destabilize the system in the worse case.

Remark IV.2. As can be seen in the proof, for a follower i connected to the leader, the term $T(U_i + U_0)^2 / [2(U_i - U_0)]$ in (17) is indeed its ultimate bound of formation error, while U_i / β is the scaled bound of its control input. Consequently, inequality (17) ensures that the saturation will be waived in finite time. In a DAG where followers can be categorized into different layers according to the longest path to the leader, the condition (17) ensures that each agent is able to overcome the maximum uncertainty induced by previous layers and achieve the global ultimate error bound. Note that here we address input saturation directly, instead of using the small gain approach which can only guarantee semi-global stability [18], [19].

Remark IV.3. Given any digraph satisfying Assumption II.1, a pre-operation hierarchical decomposition [33] can be used to identify a DAG G^* of the original graph, and the algorithm can be applied on G^* without any compromise on the convergence and boundedness of the localization and formation errors. In other words, we can pragmatically ignore some edges to avoid the effect of cycles in the network. Indeed, in a networked system, such simplification can offer multiple advantages. First, reducing the topology to DAG would help reduce the amount of communication and sensing channels needed, thus easing the implementation. Second, for the leader-following consensus, it has been shown in [34], [35] that the existence of cycles will degrade the convergence performance. In this paper we gloss over the discussion on hierarchical decomposition and study the stability of the system under DAG directly. For this special topology, we can see in the subsequent simulation that the global bound is almost as exactly as predicted by Theorem IV.3.

Remark IV.4. In practice, measurement noises and disturbances are unavoidable. However, thanks to the EC of RL and formation errors in the aforementioned analysis, we can expect that the RL and TVF errors would remain bounded under measurement error. Indeed, it can be easily seen that under sufficiently small measurement errors, the PE feature (13) can still be preserved. Hence, a semi-global stability result can be established in the case that measurement error is bounded with sufficiently small bounds. However, a comprehensive theoretical analysis would be too convoluted and is beyond

the scope of this paper, thus we would like to defer this issue to a future work. Nevertheless, the simulation of Section V and the experiment in Section V-B are used to demonstrate the robustness of the system under measurement noise.

V. SIMULATION AND EXPERIMENTS

A. Numerical simulations

Two simulations with 196 agents under a DAG topology (see Fig. 2) are presented to validate the theoretical findings of Theorems IV.1, IV.2 and IV.3 in this part. The specific choices of parameters and reference trajectories $\sigma_i(k)$ are overlaid in Fig. 3. Note that we choose $\sigma_i(k) = R(\psi(k))(q_i^* + [-20, 0]')$ where $R(\psi(k))$ is a periodic time-varying rotation matrix and q_i^* is the coordinate of agent i in Fig. 2. We can verify (12) by direct computations with K_{ij} being the period of $R(\psi(k))$, and hence the localization errors converge to 0 exponentially as seen in Fig. 3c for the deterministic leader case and Fig. 3d for the non-deterministic leader case.

Fig. 2: A swarm of 196 agents with DAG topology in a desired formation. The leader is indicated by the red dot.

As predicted by Theorem IV.2, the formation error is also observed to be e.c. in Fig. 3c. For the non-deterministic leader case, we first find U_0 from the choice of $v_0(k)$ in Fig. 3b, and then verify the condition (17). In Fig. 3d we compare the maximum ultimate formation error bound with $\max_i Q_i$, and it can be seen that the predicted $\max_i Q_i$ is almost identical with the simulation result. Animated plots of these simulations as well as others with different topologies and formation shapes can be viewed at <https://youtu.be/0MUBIM8vNGY>.

To inspect the robustness of the system, we rerun the simulation in Fig. 3b with all measurements corrupted by some random noise. Specifically, the distance measurement is corrupted by random noise of uniform distribution on the interval $[-0.05, 0.05]$, and the displacement measurement is corrupted by random noise of uniform distribution on the interval $[-0.01, 0.01]$ on both x and y direction. The RL and formation errors are shown in Fig. 4. It can be seen that the relative localization error remains bounded below 0.2m, and the formation error stays between 0.4m to 0.6m, thus demonstrating the robustness of the system under the presence of measurement noise.

(a) Snapshots of the swarm system at three different time instances with a deterministic leader.

(c) Convergence of RL and TVF error confirm predictions of Theorems IV.1 and IV.2

(b) Snapshots of the swarm system at three different time instances with a non-deterministic leader.

(d) Convergence of RL and TVF error confirm predictions of Theorems IV.1 and IV.3.

Fig. 3: Simulation of the swarm system in the deterministic and non-deterministic cases. The leader's trajectory through time is plotted by the red path in Fig. 3a and Fig. 3b. The positions of the followers at an intermediate time step and the final step are respectively marked with green dots and blue dots. The leader's positions at these steps are always marked with the red dot.

Fig. 4: Simulation results for the nondeterministic leader case with noisy distance and displacement measurement. It can be seen that both the RL and formation error remain bounded.

B. Experiment on quadcopters

The proposed PEARL-TVF scheme is further verified by implementation on three quadrotors in an indoor testing area. The quadrotors are enumerated by 0, 1, 2 and the edge set is chosen as $\bar{E} = \{(1, 0), (2, 0), (2, 1)\}$. A set of high accuracy UWB ranging sensors, which have been popularly adopted by several research groups ([21], [24], [25]), is employed for both distance measurements and communication between the quadrotors (an accuracy of about 2cm is reported

by the manufacturer of the UWB sensor). The quadrotors are tracked by the Vicon motion capture system² and the tracking information is fused with onboard IMU and optical flow measurements to obtain position estimates with 1cm accuracy. This is then used to calculate the displacement and communicated over the UWB network. The estimation and control signals are updated locally by each quadrotor every 100 ms in synchronization with the UWB network.

The user-defined parameters and other values are overlaid on Fig. 5a, which shows the time evolution of the RL and formation errors. From the choice of the TVF trajectories, it can be seen that when TVF is achieved the two followers will circle around the leader at opposite sides on a circle (see Fig. 6). It can be seen that both errors reduce to some small values, which can be expected due to the existence of measurement errors. Moreover, it can be noticed that the bounds of RL and formation errors are quite close to that of the simulation in Fig. 4, as we have chosen the noise levels to be close to that of the actual measurements in the simulation. Fig. 5b shows the time evolution of the relative position estimates $\hat{q}_{21}(k)$,

²<https://vicon.com/>

(a) Relative localization and formation errors in the experiment. Note that the errors are quite close to the simulation result in Fig. 4.

(b) Relative position estimate, ground truth and relative reference trajectory on the edge (2,1).

Fig. 5: Time evolution of the errors and relative states under the PEARL-TVF algorithm.

(a) When the leader stays static, the followers' trajectory will converge to a circular reference trajectory centered at the leader.

(b) When the leader is shifted away, the followers' trajectory will deviate from the circle but will converge back to it later.

Fig. 6: Recorded 3D trajectories of the robots over some short periods.

in comparison with the ground truth $q_{21}(k)$ from the motion capture system and the relative reference trajectory $\sigma_{21}(k)$. We can observe that $\hat{q}_{21}(k)$ and $q_{21}(k)$ quickly converge to the reference trajectory $\sigma_{ij}(k)$ under the effect of the PEARL-TVF algorithm. Similar convergence can also be observed on other edges.

Also, it is worth mentioning that the formation error slightly increases and then decreases when the leader is shifted away. Fig. 6 shows some snapshots of the formation when the leader is static and then moves to a new position, temporarily increasing the formation error. A video recording of the experiment overlaid with other real-time plots can be viewed at <https://youtu.be/0MUBIM8vNGY>.

VI. CONCLUSION

In this paper, we have introduced a distributed distance-displacement-based RL and TVF control scheme for multi-robot system with bounded input. By embedding the PE condition into the TVF reference trajectories, we achieve EC for RL, which facilitates e.c. TVF in the deterministic leader case, and a globally stable TVF in the non-deterministic leader under DAG. Numerical and empirical examples were used to demonstrate the theoretical findings. Besides, we believe this study on a discrete-time multi-robot system with input saturation and external disturbance can also provide a useful reference for future studies on distributed multi-robot systems.

APPENDIX

A. Proof of Theorem IV.2

With $v_0(k) \equiv 0$, we can simplify (16) as

$$\bar{q}_i(k^+) = \alpha_i(k)\bar{q}_i(k) + \sum_{j \neq 0} \alpha_{ij}(k)\bar{q}_j(k) - \varepsilon_i(k), \quad (19)$$

where $\alpha_{ij}(k) = T\beta s_i(k)a_{ij}$, and $\alpha_i(k) = 1 - \sum_j \alpha_{ij}(k) = 1 - T\beta s_i(k)$. Note that the summation in (19) excludes the leader as $\bar{q}_0(k) \equiv 0$ by definition, and hence

$$c_i(k) \triangleq \alpha_i(k) + \sum_{j \neq 0} \alpha_{ij}(k) = 1 - \alpha_{i0}(k) \leq 1. \quad (20)$$

Accordingly, we can define a matrix $A(k) \in \mathbb{R}^{n \times n}$ whose (i, i) -th entry is given by $\alpha_i(k)$, and the non-diagonal (i, j) -th entry is given by $\alpha_{ij}(k)$. Clearly, $A(k)$ is a nonnegative matrix with each row sum $c_i(k) \leq 1$. Now we can establish the EC of $\bar{q}_i$ after showing its boundedness.

1) Boundedness of $\bar{q}_i$: $\varepsilon_i(k)$ is e.c. as a result of Theorem IV.1 and $s_i(k) \in (0, 1]$. Thus, there exist $e > 0$ and $\lambda \in (0, 1)$ such that $\|\varepsilon_i(k)\| \leq e\lambda^k$ for all $i \in V$. Denote $b(k) = [\|\bar{q}_1(k)\|, \dots, \|\bar{q}_n(k)\|]^T$. By applying the triangle inequality to (19) for each i we get the following componentwise inequality

$$b(k^+) \leq A(k)b(k) + \mathbf{1}e\lambda^k, \quad (21)$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector of ones. Noticing that $A(k)\mathbf{1} \leq \mathbf{1}$, we get:

$$b(k) \leq b(0) + \mathbf{1}e \sum_{l=0}^{k-1} \lambda^l \leq b(0) + \mathbf{1} \frac{e}{1-\lambda},$$

which is the required boundedness.

2) EC of $\bar{q}_i$: By the boundedness we can immediately get from (15) that u_i is bounded, and there exists $s_* \in (0, 1)$ so that $s_i(k) \geq s_*$. Consequently, we have $T\beta s_* a_{ij} \leq \alpha_{ij}(k) \leq T\beta a_{ij}$. Below we will establish the convergence based on (21).

For each follower i , let the shortest path from i to the leader be of length l_i . Accordingly, we can classify the followers into different groups S_l , so that $l_i \equiv l$ for $i \in S_l$. Assume that $l \leq L$ and denote $Q(k) = \max_i \|\bar{q}_i(k)\|$. We shall show that for any $i \in S_l, l = 1, \dots, L$, there exists $\eta_{i,l} \in (0, 1)$ such that

$$\|\bar{q}_i(k+d)\| \leq \eta_{i,l} Q(k) + e \sum_{t=k}^{k+d-1} \lambda^t, \quad d \in [l, L]. \quad (22)$$

Thus, by letting $\eta = \min_{i,l} \eta_{i,l} \in (0, 1)$ we can obtain

$$Q(k+L) \leq \eta Q(k) + e \sum_{t=k}^{k+L-1} \lambda^t, \quad (23)$$

which implies the EC of $\bar{q}_i$.

To show (22), we first note that (21) implies that

$$b(k+d) \leq A(k+d-1:k)b(k) + \mathbf{1} e \sum_{t=k}^{k+d-1} \lambda^t, \quad (24)$$

where $A(k+d-1:k)$ is recursively defined as $A(k_2:k_1) = A(k_2)A(k_2-1:k_1)$ if $k_2 \geq k_1$, otherwise $A(k_2:k_1) = I$. Denote $A_i(k+d-1:k)$ as the i -th row of $A(k+d-1:k)$. Comparing (24) with (22), it suffices to show that $A_i(k+d-1:k)\mathbf{1} \leq \eta_{i,l}$, where $A_i(k+d-1:k)\mathbf{1}$ is the i -th row sum of $A(k+d-1:k)$.

For $i \in S_1$, it is clear that $A_i(k)\mathbf{1} = c_i(k) = 1 - \alpha_{i0}(k) \triangleq \eta_{i,1}(k) < 1$. To compute $A_i(k+1:k)\mathbf{1}$, note that $A(k)\mathbf{1} \leq \gamma_i(k)$, where $\gamma_i(k) \in \mathbb{R}^n$ is obtained by replacing the i -th entry of $\mathbf{1}$ with $\eta_{i,1}(k)$. Let us first consider the following two cases:

1) $\alpha_{i0}(k) \leq \alpha_{i0}(k+1)$, which gives:

$$\begin{aligned} A_i(k+1:k)\mathbf{1} &\leq A_i(k+1)\gamma_i(k) \\ &= \alpha_i(k+1)\eta_{i,1}(k) + \sum_{j \neq 0} \alpha_{ij}(k+1) \dots \\ &= [1 - \sum_j \alpha_{ij}(k+1)]\eta_{i,1}(k) + \sum_{j \neq 0} \alpha_{ij}(k+1) \\ &= \eta_{i,1}(k) - [1 - \alpha_{i0}(k)] \sum_j \alpha_{ij}(k+1) + \sum_{j \neq 0} \alpha_{ij}(k+1) \\ &= \eta_{i,1}(k) - \alpha_{i0}(k+1)\alpha_i(k+1) \leq \eta_{i,1}(k). \end{aligned}$$

2) $\alpha_{i0}(k) > \alpha_{i0}(k+1)$, in this case since $\alpha_{i0}(k) = 1 - \eta_{i,1}(k)$ we have $\eta_{i,1}(k) < 1 - \alpha_{i0}(k+1)$. Hence:

$$\begin{aligned} A_i(k+1:k)\mathbf{1} &\leq A_i(k+1)\gamma_i(k) \\ &< \alpha_i(k+1)[1 - \alpha_{i0}(k+1)] + \sum_{j \neq 0} \alpha_{ij}(k+1) \\ &= 1 - \alpha_{i0}(k+1) - \alpha_i(k+1)\alpha_{i0}(k+1) \\ &= \eta_{i,1}(k+1) - \alpha_{i0}(k+1)\alpha_i(k+1) \leq \eta_{i,1}(k+1). \end{aligned}$$

By inference and induction we can show (22) for $l = 1$ and $\eta_{i,1} = 1 - T\beta s_* a_{i0}$.

For $j \in S_2$ which is connected to the above $i \in S_1$ but not to the leader, it follows from $A(k)\mathbf{1} \leq \gamma_i(k)$ that

$$\begin{aligned} A_j(k+1:k)\mathbf{1} &\leq \alpha_{ji}(k+1)\eta_{i,1}(k) + 1 - \alpha_{ji}(k+1) \\ &= 1 - \alpha_{ji}(k+1)(1 - \eta_{i,1}(k)) \triangleq \eta_{j,2}(k+1) < 1. \end{aligned}$$

Consequently we get that $A_j(k+1:k)\mathbf{1} \leq \gamma_{j,i}(k+1)$ where $\gamma_{j,i}(k+1) \in \mathbb{R}^n$ is obtained by replacing the j -th entry of $\gamma_i(k)$ with $\eta_{j,2}(k+1)$. Moving on, we find that

$$\begin{aligned} A_j(k+2:k)\mathbf{1} &\leq \alpha_j(k+2)\eta_{j,2}(k+1) + \alpha_{ji}(k+2)\eta_{i,1}(k) \dots \\ &\quad + 1 - \alpha_{ji}(k+2) - \alpha_j(k+2) \\ &= 1 - \alpha_{ji}(k+2)\alpha_{i0}(k) - \alpha_j(k+2)\alpha_{ji}(k+1)\alpha_{i0}(k) \\ &\leq 1 - (T\beta s_*)^2 a_{ji}a_{i0} \triangleq \eta_{j,2}. \end{aligned}$$

Switching j to i , we can show (22) for $l = 2$ and $\eta_{i,2} = 1 - (T\beta s_*)^2 a_{ij}a_{j0}$.

The above analysis can be extended to general cases with $\eta_{i,l} = 1 - (T\beta s_*)^l a_{i,i_1} \dots a_{i_{l-1},i_l}$, where $i_l = 0$ and $(i, i_1), \dots, (i_{l-1}, i_l)$ is the shortest path from i to 0. If we define $a_* = \min\{a_{ij} : (i, j) \in \bar{E}\}$, then we can take $\eta = 1 - (T\beta s_* a_*)^L$.

B. Proof of Theorem IV.3

For each follower i , let the longest path from i to the leader be of length l_i . Accordingly, we can classify the followers into different groups H_l , so that $l_i \equiv l$ for $i \in H_l$. Also, denote $H_0 = \{0\}$. Since there is no loop, it holds that $N_i \subseteq \bigcup_{l=0}^{i-1} H_l$ for any $i \in H_l, l \geq 1$. Below we analyze the ultimate bound in sequence of H_l .

1) For the agent $i \in H_1$ which has the only neighbor of the leader, it follows by replacing $a_{i0} = 1$ in (16) that

$$\bar{q}_i(k^+) = [1 - T\beta s_i(k)]\bar{q}_i(k) - T[\beta s_i(k)\bar{q}_{i0}(k) + v_0(k)]. \quad (25)$$

Under the condition (17), we will show that $s_i(k) = 1$ for sufficiently large k , and (25) becomes

$$\bar{q}_i(k^+) = (1 - T\beta)\bar{q}_i(k) - T[\beta\bar{q}_{i0}(k) + v_0(k)]. \quad (26)$$

To begin, let us define $D_i(k) = \|\bar{q}_i(k)\|^2$ and examine $\Delta D_i(k) = D_i(k^+) - D_i(k)$ (henceforth, we may omit the subscripts $i, ij, i0$ and the time step k when they can be easily inferred). From (25) one can obtain

$$\begin{aligned} \Delta D &= (\bar{q} + T s u - T v)'(\bar{q} + T s u - T v) - \bar{q}'\bar{q} \\ &= T [T(s^2 \|u\|^2 - 2 s u'v + \|v\|^2) + 2 \bar{q}'(s u - v)] \\ &= T [\hat{s}^2 \|\hat{u}\|^2 - 2 \hat{s} \hat{u}'v + \|v\|^2] + 2 \bar{q}'(\hat{s} \hat{u} - v) + \varepsilon, \end{aligned}$$

where $\hat{s} = U_i / \max\{U_i, \|\hat{u}\|\}$ with $\hat{u} = -\beta \bar{q}$, and $\varepsilon = (s u - \hat{s} \hat{u})'[T s u + T \hat{s} \hat{u} - 2 T v + 2 \bar{q}]$. In the sequel we shall find the ultimate bound of $\bar{q}$ by examining when $\Delta D \leq 0$.

We firstly show that ε is e.c. Actually, as $\|T s u + T \hat{s} \hat{u} - 2 T v + 2 \bar{q}\| \leq 2[T(U_i + U_0) + \|\bar{q}\|]$ only grows at most linearly fast by (25), it suffices to show the EC of $s u - \hat{s} \hat{u} = s(u - \hat{u}) - (\hat{s} - s)\hat{u}$. For the first term, we have $\|s(u - \hat{u})\| \leq \beta \|\bar{q}\|$ which is e.c. For the second term, since $\|\bar{q}\|$ is e.c., we can take $\beta \|\bar{q}\| < U_i/2$ and find that $\max\{U_i, \|u\|\} \geq \max\{U_i, \beta \|\bar{q}\|\} - \beta \|\bar{q}\| > 0$ for sufficiently large k . Then we can see that

$$\begin{aligned} s - \hat{s} &= U_i / \max\{U_i, \|u\|\} - U_i / \max\{U_i, \|\hat{u}\|\} \\ &\leq \frac{U_i \beta \|\bar{q}\|}{(\max\{U_i, \beta \|\bar{q}\|\} - \beta \|\bar{q}\|) \max\{U_i, \beta \|\bar{q}\|\}} \\ &\leq [U_i \beta \|\bar{q}\|] / [(U_i/2) U_i] = 2 \beta \|\bar{q}\| / U_i, \end{aligned}$$

which implies that $|s - \hat{s}|$ is e.c. Since $\|\hat{u}\| = \beta\|\bar{q}\|$ grows at most linearly fast, we can achieve the EC of $|su - \hat{s}\hat{u}|$. In summary, for any $\varepsilon^* > 0$, there exists $K^* = K^*(\|\bar{q}(0)\|, \|\bar{q}(0)\|, \varepsilon^*)$ such that $|\varepsilon| \leq \varepsilon^*$ for $k \geq K^*$.

Below we shall show that $\hat{s} = 1$ in finite time, and we firstly consider $\hat{s} < 1$.

If $\hat{s} < 1$, then $\|\bar{q}\| > U_i/\beta$, $\hat{s} = U_i/\|\hat{u}\|$ and

$$\begin{aligned} \Delta D &\leq T \left[T(U_i + 2U_i U_0 + U_0^2) - 2\bar{q}' \left(\frac{U_i}{\|\bar{q}\|} \bar{q} + v \right) + |\varepsilon| \right] \\ &\leq T \left[T(U_i + U_0)^2 - 2(U_i - U_0)U_i/\beta + \varepsilon^* \right] \triangleq \delta(\varepsilon^*). \end{aligned}$$

Based on the condition (17), we can find a small ε^* so that $\delta(\varepsilon^*) < 0$. Therefore, $\Delta D \leq \delta(\varepsilon^*) < 0$ after some time steps. Hence $\|\bar{q}\|$ will reduce to U_i/β in finite time, and we can proceed to $\hat{s} = 1$ in the next case.

If $\hat{s} = 1$, then $\|\bar{q}\| \leq U_i/\beta$ and it holds that

$$\begin{aligned} \Delta D &\leq T \left[(T\beta - 2)\beta\|\bar{q}\|^2 + 2(1 - T\beta)U_0\|\bar{q}\| + TU_0^2 + |\varepsilon| \right] \\ &= -T \left[((2 - T\beta)\|\bar{q}\| + TU_0)(\beta\|\bar{q}\| - U_0) - |\varepsilon| \right]. \end{aligned}$$

For $\beta\|\bar{q}\| > (U_i + U_0)/2 > U_0$, the above becomes

$$\Delta D \leq -\frac{T}{2} \left[(1/\beta - T/2)(U_i + U_0) + TU_0 \right] (U_i - U_0) + T|\varepsilon|.$$

Similar to the previous case, we can show that there exists $K = K(\|\bar{q}(0)\|, \|\bar{q}(0)\|)$ such that $\beta\|\bar{q}(K)\| \leq (U_i + U_0)/2$.

At the same time, K can also be taken sufficiently large so that $\beta\|\bar{q}(k)\| \leq (U_i - U_0)/4$ for $k \geq K$, which follows that $\|u(K)\| = \beta\|\bar{q}(K) + \bar{q}(K)\| \leq (3U_i + U_0)/4 < U$, and (26) holds for $k = K$. Furthermore, we have $\beta\|\bar{q}(K+1)\| \leq (U_i + U_0)/2 - T\beta(U_i - U_0)/4$, and hence $\|u(K+1)\| = \beta\|\bar{q}(K+1) + \bar{q}(K+1)\| < (3U_i + U_0)/4 < U_i$. By induction we conclude that (26) holds for $k \geq K$, and the ultimate bound U_0/β can be established by noticing that $\bar{q}(k)$ converges to 0. Thus, the following is achieved:

$$\limsup_{k \rightarrow \infty} \|\bar{q}_i(k)\| \leq U_0/\beta, \quad i \in H_1. \quad (27)$$

2) Moving on to $i \in H_2$ where $N_i \subseteq H_0 \cup H_1$. Define $r_i(k) = \sum_j a_{ij}[\bar{q}_j(k) - \bar{q}_j(k)]$,

then we have $u_i(k) = -\beta[r_i(k) + \sum_j a_{ij}\bar{q}_{ij}(k)]$, and the dynamics of $r_i(k)$ can be derived from the above and (16) as

$$r_i(k^+) = (1 - T\beta s_i(k))r_i(k) - \varepsilon_i(k) - f_i(k), \quad (28)$$

where $\varepsilon_i(k)$ converges to 0 exponentially fast, and $f_i(k) = T\sum_j a_{ij}\bar{u}_j(k)$. Since $\limsup_{k \rightarrow \infty} \|\bar{u}_j(k)\| \leq U_0$ follows as a result of (27), we have $\limsup_{k \rightarrow \infty} \|f_i(k)\| \leq TU_0$. Following a similar analysis in 1), we can also maintain that $\limsup_{k \rightarrow \infty} \|r_i(k)\| \leq U_0/\beta$ for $i \in H_2$.

3) From H_3 onwards, we can repeat the same steps and conclude that $\limsup_{k \rightarrow \infty} \|r_i(k)\| \leq U_0/\beta$. Note that $r(k) = (C \otimes I)\bar{q}(k)$, where $\otimes$ denotes the Kronecker product, $\bar{q}(k) \triangleq [\bar{q}_{10}(k), \dots, \bar{q}_{n0}(k)]'$, $r(k) \triangleq [r_1'(k), \dots, r_n'(k)]'$, and C is defined in (18). Since $\bar{q}(k) = (C^{-1} \otimes I)r(k)$, we immediately get that $\limsup_{k \rightarrow \infty} \|\bar{q}_i(k)\| \leq \sum_j |\hat{c}_{ij}|U_0/\beta$.

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