Emergent commensurability from Hilbert space truncation in fractional quantum Hall fluids

Yang, Bo

2019

Yang, B. (2019). Emergent commensurability from Hilbert space truncation in fractional quantum Hall fluids. *Physical Review B*, 100(24), 241302(R)-.
doi:10.1103/PhysRevB.100.241302

<https://hdl.handle.net/10356/142494>

<https://doi.org/10.1103/PhysRevB.100.241302>

© 2019 American Physical Society. All rights reserved. This paper was published in *Physical Review B* and is made available with permission of American Physical Society.

Downloaded on 26 Apr 2023 22:45:22 SGT

Emergent commensurability from Hilbert space truncation in fractional quantum Hall fluidsBo Yang *Division of Physics and Applied Physics, Nanyang Technological University, Singapore 637371, Singapore
and Institute of High Performance Computing, A*STAR, Singapore 138632, Singapore*

(Received 19 January 2019; published 5 December 2019)

We show that model states of fractional quantum Hall fluids at all experimentally detected plateaus can be uniquely determined by imposing translational invariance with a particular scheme of Hilbert space truncation. The truncation is based on classical local exclusion conditions, motivated by constraints on physical measurements. The scheme allows us to identify filling factors, topological shifts, and clustering of topological quantum fluids universally without resorting to microscopic Hamiltonians. This prompts us to propose the notion of emergent commensurability as a fundamental property for many known fractional quantum Hall (FQH) states, which allows us to predict families of new FQH states that can be realized *in principle*. We also discuss the implications of certain missing states proposed from other phenomenological approaches, and suggest that the physics of the FQH effect could fundamentally arise from the algebraic structure of the Hilbert space in a single Landau level.

DOI: [10.1103/PhysRevB.100.241302](https://doi.org/10.1103/PhysRevB.100.241302)

Introduction. A large number of fractional quantum Hall (FQH) states with distinct topological orders have been observed experimentally and proposed theoretically, ever since the surprising discovery of the quantized Hall conductivity at $1/3$ filling factor [1,2]. The physics of the FQH effect is mainly derived from the formation of an incompressible quantum fluid with a charge excitation gap, which could be realized at specific rational filling factors when a two-dimensional electron gas system is subject to a perpendicular magnetic field. We now understand that both Abelian and non-Abelian FQH states are likely observed in the experiments [1,3]. In addition to the single-component FQH states (e.g., the Read-Rezayi series [4,5]), there can also be multicomponent or hierarchical states from the coexistence of more than one type of quantum fluid in a strongly correlated manner [6,7].

There has been much development in the microscopic theories of the FQH effect since the first proposition of the Laughlin wave functions [2] and later on, the model Hamiltonians [6]. One major approach is the phenomenological formation of “composite fermions” (CFs) with flux attachment [7], and the parton construction inspired from it [8,9]. It leads to the systematic construction of microscopic wave functions for almost all observed and proposed FQH states [10]. Another major approach is to exploit the rich algebraic structures of many-body wave functions in a single Landau level (LL) on genus 0 geometries (e.g., sphere or disk), leading to very efficient constructions of microscopic model wave functions with the Jack polynomial formalism [11–14]. The method is particularly useful for the Read-Rezayi (RR) series including the coveted non-Abelian states, revealing the particle clustering properties in an intuitive manner. The Jack polynomial formalism and the related techniques are also closely linked to the wave-function constructions from parafermion correlators in conformal field theory (CFT) [4,5], and in contrast to the CF approach, in many cases model projection Hamiltonians

can be found [15], of which the constructed wave functions are unique zero-energy ground states.

From a theoretical point of view, we can characterize the FQH states with the following expression:

$$N_\phi = \frac{q}{p}(N_e + S_e) - S_\phi, \quad (1)$$

where the system size is given by the number of orbitals N_ϕ and the number of electrons N_e . In the thermodynamic limit when both $N_\phi, N_e \rightarrow \infty$, the filling factor is given by $\nu = p/q$, and S_e, S_ϕ are integer topological shifts for the electrons and orbital [16,17], respectively. Note that p, q do not have to be coprime. While the phenomenological CF formalism is extremely useful in conjecturing about possible combinations of $[p, q, S_e, S_\phi]$, it is most successful for Abelian states, and the majority of the CF wave functions do not seem to have a local model Hamiltonian [18]. The local projection Hamiltonians (for the RR series and beyond) can serve as model Hamiltonians for many Jack polynomials or CFT-based wave functions, including the non-Abelian ones, but most of them do not have unique zero-energy ground states [11,15]. The generalization to Abelian multicomponent FQH states in this picture is also difficult. The key question we ask here is how to determine if an FQH state can form *in principle* at a particular $[p, q, S_e, S_\phi]$. This is because at the fundamental level, these topological indices should not depend on any local operators, including Hamiltonians.

In this Rapid Communication, we focus on spin-polarized states, and propose a perspective in the general understanding of the FQH effects. This perspective is based on a number of physically motivated principles and strong numerical evidence, leading to the predictions of many new FQH states with explicit model wave functions and topological indices. It also leads to very efficient ways of numerically constructing model wave functions, including previously known ones that cannot be written as Jack polynomials and thus could only

be obtained by expensive numerical diagonalization of model Hamiltonians with few-body interactions.

More specifically, we present a simple set of criteria for the constraints on physical measurements of the quantum Hall fluids, which we term as local exclusion conditions (LECs). They can unambiguously determine $[p, q, S_e, S_\phi]$ for many FQH phases, *without* resorting to microscopic Hamiltonians or other prior conjectures. Interestingly, the LECs are “classical” constraints on the reduced density matrices of the quantum Hall fluids, but here we use them to determine emergent topological properties that are purely quantum phenomena. This motivates us to propose the concept of intrinsic “commensurability” for the FQH physics, that is reminiscent of, though not equivalent to, the generalized Pauli exclusion principles [19] and clustering properties [11].

We start by examining some general ingredients for the incompressible FQH states. The ground state of a particular FQH phase needs to be the unique, highest density state that is translationally invariant. Translational invariance is needed so there are no gapless Goldstone modes, and we need the ground state to be of the highest density for it to be incompressible. If such a ground state is not unique, it generally also indicates gapless excitations [20], since the degeneracy could be split by small perturbations, unless it is protected by symmetry or topology [21]. We will also assume for any FQH phase, Eq. (1) should be valid for all allowed system sizes, and is not just valid in the asymptotic limit when N_e or N_ϕ goes to infinity.

We also conjecture there exists a minimal description as given by a minimal model ground state living in a truncated Hilbert space, for many (if not all) FQH phases. Here we define the “minimal” model state to be the state that captures all the ground state topological properties of the topological phase, but made from the smallest possible set of bases, out of all bases permissible by quantum numbers (e.g., momentum) in the Hilbert space. The RR series are explicit examples where part of the translationally invariant Hilbert space in a single LL is truncated, as manifested by their entanglement spectrum [22]. The integer quantum Hall effect also fits in this paradigm: while the permissible Hilbert space includes all Landau levels and realistic ground states have small components in higher LLs, the minimal model ground state is the Vandermonde of the lowest filled LLs with all higher LLs truncated.

We now show how “commensurability” of FQH systems arises from these principles, particularly with the important roles played by translational invariance and Hilbert space truncation. We use spherical geometry, so translational invariance is equivalent to rotational invariance, and take the model wave function of the Laughlin state at filling factor $\nu = 1/3$ as a simple illustration. We emphasize the Hilbert space truncation here is more general and less restrictive than the removal of unsqueezed bases in the Jack polynomial formalism [11]. The Laughlin state in the second quantized form is a fermionic Jack polynomial with the root configuration given by 100100100...1001, where ... represents the repeated patterns of “100.” All components of the Laughlin wave function in the occupation basis are “squeezed” from the root configuration. An important property of the wave function, valid for any system size, is that a measurement of

FIG. 1. Root configuration and squeezed bases of the Laughlin state on the sphere for four particles. The two leftmost orbitals correspond to the north pole, and no basis has those two orbitals both occupied. Other bases with $L_z = 0$ are either removed by $\mathcal{C}_{\hat{n}=\{2,1,2\}}$ or rotational invariance.

the leftmost two orbitals at the north pole will never detect more than one particle. Indeed, the *reduced density matrix* of the Laughlin state restricted to the leftmost two orbitals only contains three bases: 00,10,01; the missing basis of 11 is because of the “squeezing rule” of the Jack polynomial.

We state here the key observation based on extensive numerical evidence: if we start with the full Hilbert space containing all bases squeezed *and* unsqueezed from the root configuration (so that there is no need to select a root configuration by hand), and truncate away bases that contain two particles in the two leftmost orbitals, we are left with far more bases than the squeezed bases needed for the Laughlin state. However, when we choose N_e and N_ϕ based on $p = 1$, $q = 3$, $S_e = 0$, $S_\phi = 2$ in Eq. (1), there is a *unique* rotationally invariant $L = 0$ state, obtained from diagonalizing the total angular momentum operator L^2 on the sphere in the truncated Hilbert space: coefficients of *all remaining* unsqueezed bases are forced to be zero by rotational invariance (see Fig. 1).

A formal procedure can be set up for more general cases. Starting with a quantum Hall fluid on an infinite plane or a finite sphere, we define a set of LECs denoted as $\mathcal{S}_C$, with each element indexed by a triplet of non-negative integers $\hat{n} = \{n, n_e, n_h\}$. We use l_B to denote the magnetic length, so each magnetic flux occupies a finite area of $2\pi l_B^2$. A condition $\mathcal{C}_{\hat{n}} \in \mathcal{S}_C$ physically dictates that for a measurement over any circular droplet containing n fluxes [23–25], no more than n_e particles and n_h holes can be detected. Thus $\mathcal{C}_{\hat{n}}$ acts as a constraint on the type of bases in the quantum Hall fluid that may have nonzero coefficients. If $n = n_e = n_h$, there is no constraint if the underlying particles are fermions in a single LL. On spherical geometry, the LECs are only applied to the north or south pole, but more basis coefficients will vanish after imposing rotational invariance, effectively requiring LECs to be satisfied everywhere in the quantum fluid. The incompressibility of the FQH fluids originates from the finite energy cost of breaking such exclusion constraints, physically enforced by renormalized Coulomb interactions in various experimental conditions.

For actual implementations, we look at the Hilbert space of total z -component angular momentum $L_z = 0$ on the sphere [6], denoted by $\mathcal{H}_{N_\phi, N_e}$, indexed by the number of orbitals

TABLE I. The first row gives the triplet $\hat{n}$. With a single condition as the Hilbert space constraint, $S_e = 0$ (see [17]), and we use $[p, q, S_\phi]$ to represent a FQH state; L denotes the Laughlin state, Pf denotes the Pfaffian, while R denotes other states in the RR series. P denotes states from other projection Hamiltonians [15].

	{2, 1, 2}	{3, 2, 3}	{3, 1, 3}	{4, 3, 4}	{4, 2, 4}	{4, 1, 4}	{5, 4, 5}	{5, 3, 5}	{5, 2, 5}	{5, 1, 5}
$N_e = 2, 3, 4 \dots$	L:[1,3,2]		L:[1,5,4]			L:[1,7,6]				L:[1,9,8]
$N_e = 4, 6, 8 \dots$		Pf:[2,4,2]			H : [2, 6, 4]				P:[2,8,6]	
$N_e = 6, 9, 12 \dots$				R:[3,5,2]				P:[3,7,4]		
$N_e = 8, 12, 14 \dots$							R:[4,6,2]			

and electrons. We also define $\tilde{\mathcal{H}}_{N_\phi, N_e}^{\hat{n}}$ to be the truncated Hilbert space from $\mathcal{H}_{N_\phi, N_e}$ where all bases not satisfying the constraint of $\mathcal{C}_{\hat{n}}$ at the north pole are removed, and $\tilde{\mathcal{N}}_{N_\phi, N_e}^{\hat{n}}$ to be the number of $L = 0$ states in $\tilde{\mathcal{H}}_{N_\phi, N_e}^{\hat{n}}$. Here is one of the main statements of this work: some $\mathcal{C}_{\hat{n}}$ has a one-to-one correspondence to a combination of $[p, q, S_e, S_\phi]$ satisfying the following properties:

$$N_\phi^d = \frac{q}{p}(N_e + S_e) - S_\phi, \quad (2)$$

$$\tilde{\mathcal{N}}_{N_\phi, N_e}^{\hat{n}} = 1, \quad \tilde{\mathcal{N}}_{N_\phi < N_\phi^d, N_e}^{\hat{n}} = 0 \quad (3)$$

for all values of N_e subject to the condition that $N_e + S_e = kp, k \geq 2$. In particular, $\hat{n} = \{n, n, n\}$ corresponds to the integer quantum Hall effect. This result is computationally checked for all numerically accessible system sizes [26]. The filling factors, topological shifts, and particle clustering of the FQH states can all be unambiguously determined by specifying $\mathcal{C}_{\hat{n}}$ and the requirement of translational invariance. The basis expansion of the minimal model ground state can also be obtained as the unique $L = 0$ ground state of the L^2 operator in the truncated Hilbert space, and Eq. (3) can be interpreted as the requirement for the state to be gapped and incompressible.

Incompressibility of the FQH phases also requires gapped neutral excitations. We can show that LECs naturally forbid neutral excitations by looking at $L_z \neq 0$ sectors of different Hilbert spaces indexed by N_ϕ and N_e . Given any fixed N_e , for $N_\phi = N_\phi^d$ from Eq. (2), there are no highest weight states in any $L_z \neq 0$ sector. Thus the only highest weight state occurs in the $L_z = 0$ sector, which is the rotationally invariant ground state of the corresponding FQH phase, and all neutral excitations are excluded by the local classical constraints. Similarly for $N_\phi < N_\phi^d$, no highest weight states exist in any L_z sector.

A few FQH states and their corresponding $\hat{n}$ are listed in Table I. In particular, each $\hat{n} = \{n, 1, n\}$ gives the usual Laughlin state at $\nu = 1/(2n - 1)$, and each $\hat{n} = \{n, n - 1, n\}$ gives the Z_3 parafermion (the RR series) states [5] at $\nu = (n - 1)/(n + 1)$. States from $\hat{n} = \{n, m, n\}$ and $\hat{n} = \{kn, km, kn\}$ have the same filling factor but different shifts. For example, $\hat{n} = \{4, 2, 4\}$ gives the Haffnian state [15] at $\nu = 2/6$, $S_\phi = 4$. There is also the general relationship that $\hat{n} = \{n, m, n\}$ corresponds to the incompressible FQH state at $\nu = m/(2n - m)$ and $S_e = 0$, $S_\phi = 2(n - m)$.

All filling factors with the corresponding shifts proposed in [15] (except for the states at filling factors $\nu = 2/5, 2/9$, which we will discuss later) can be included in Table I.

Even in cases where projection Hamiltonians cannot uniquely determine the zero-energy ground states, our scheme can lead to a unique model ground state at corresponding ν and S_ϕ with transparent physical interpretations. For example, using the Hamiltonian notations in Ref. [15], the zero-energy eigenstates of P_4^5 are not unique at filling factor $\nu = 3/7$, yet a unique state can be obtained with $\hat{n} = \{5, 3, 5\}$, requiring no detection of more than three particles from a measurement in a circular droplet containing five fluxes. It is also worth mentioning that every state in Table I has its particle-hole (PH) conjugate state. They can all be uniquely determined by imposing a single condition of $\hat{n} = \{n, n, m\}$. Naturally for the PH conjugate states, the condition of “highest density” is referring to the density of the “holes.” Thus in Eq. (3) we have $\tilde{\mathcal{N}}_{N_\phi > N_\phi^d, N_e}^{\hat{n}} = 0$ instead.

The conspicuous omissions from Table I are the Gaffnian state (unique zero-energy state of P_3^3) [27], the state at $\nu = 2/9$ (unique zero-energy state of P_3^9), and some of the filling factors where the Jain or hierarchical series are expected to occur (e.g., at $\nu = 4/9$). We now proceed to show that a number of these missing states can also be realized when the $L_z = 0$ Hilbert space is truncated by more than one $\mathcal{C}_{\hat{n}}$. Explicit numerical computation shows the Gaffnian state at $\nu = 2/5$ with $S_e = 0$, $S_\phi = 3$ is a unique translationally invariant state when *either* $\mathcal{C}_{\hat{n}_1 = \{2, 1, 2\}}$ *or* $\mathcal{C}_{\hat{n}_2 = \{5, 2, 5\}}$ is satisfied by the $L_z = 0$ bases. We denote such LECs by $S_{\hat{n}_1 \hat{n}_2}^{\text{or}}$. This implies that either (a) a measurement of a circular droplet of two fluxes can at most detect *one* particle, or (b) a measurement of a circular droplet of five fluxes can at most detect *two* particles. Intuitively, this is reminiscent of the hierarchical construction [6, 28] or the CF picture for the $\nu = 2/5$ state [6, 7]. Condition (a) is the same as the Laughlin state at $\nu = 1/3$, induced effectively by a finite energy gap when $\mathcal{C}_{\{2, 1, 2\}}$ is violated. However, at $\nu = 2/5$ the violation of (a) is allowed as long as there is another effective energy gap for $\mathcal{C}_{\{5, 2, 5\}}$. It is tempting to associate the former with the creation of quasielectrons, while the latter with the incompressibility of the quantum fluid formed by the quasielectrons themselves.

It is important to note, however, the Gaffnian state is the *only* translationally invariant highest density state determined by any $S_{\hat{n}_1 \hat{n}_2}^{\text{or}}$. We discuss its relationship with the Jain state at $\nu = 2/5$ in more detail in the Supplemental Material [26]. Similarly the state at $\nu = 2/9$ can be determined by $S_{\{3, 1, 3\}\{9, 2, 9\}}^{\text{or}}$, which has very high overlap with the Jain state at the same filling and shift. We have also scanned through all possible combinations of two $\mathcal{C}_{\hat{n}}$'s, and no states at $\nu = 4/9$ can be uniquely determined with $S_{\hat{n}_1 \hat{n}_2}^{\text{or}}$. However, a unique state at the same filling factor but different shift can be

FIG. 2. The entanglement spectra of the $\nu = 3/7$ state with 15 particles from (a) $\mathcal{C}_{\hat{n}=[5,3,5]}$ and (b) $S_{\hat{n}_1\hat{n}_2}^{\text{or}}$ with $\hat{n}_1 = \{2, 1, 2\}$, $\hat{n}_2 = \{6, 3, 6\}$. (c) $\nu = 2/6$ Haffnian with 12 particles and (d) $\nu = 2/5$ Gaffnian with 16 particles. The subsystem is always chosen to be the one for which the reduced density matrix has the highest dimension.

determined by a single condition with $\{7, 7, 5\}$, which is the particle-hole conjugate of the $\nu = 5/9$ states in Table I. Another interesting finding is that there is an incompressible state at $\nu = 3/7$ and $S_\phi = 4$ corresponding to $S_{\hat{n}_1\hat{n}_2}^{\text{or}}$ with $\hat{n}_1 = \{2, 1, 2\}$, $\hat{n}_2 = \{6, 3, 6\}$, which is a competing state at the same $[p, q, S_e, S_\phi]$ to the one corresponding to a single condition with $\hat{n} = \{5, 3, 5\}$ (see Table I). The overlap of these two states quickly goes to zero as we increase the system size, suggesting they belong to two different topological phases.

To better understand the differences between the two states at $\nu = 3/7$ obtained from qualitatively different LECs, we use a topological entanglement spectrum (ES) [22] to analyze these many-body states. One should note that our model states are explicitly constructed from a truncated Hilbert space within a single LL, thus only the topological part of the ES will be present. It is worth noting from Fig. 2 that even though the two $\nu = 3/7$ states have an overlap of 0.02, their ES have the exact same counting. However, the state from $S_{\hat{n}_1\hat{n}_2}^{\text{or}}$ apparently has multiple low-lying branches, while the state from $\hat{n} = \{5, 3, 5\}$ only has a single branch. Multiple low-lying branches also appear from the Gaffnian state, which similarly requires two conditions to be determined uniquely. In contrast, the Haffnian state (or any other states) from a single condition only has one low-lying branch in its ES (see Fig. 2). This could be another interesting physical interpretation of the roles played by $\mathcal{C}_{\hat{n}}$, which warrants further studies.

Conclusions. We show that the incompressibility of the FQH states arises from translational invariance and constraints on the Hilbert space defined by a set of LECs denoted $\mathcal{C}_{\hat{n}}$. Thus topological properties of many FQH states can be determined classically, dictated by what can or cannot be measured from the quantum Hall fluids. Regarding the absence of Hamiltonians in this approach, one should note that the universal topological properties of the FQH phases should not depend on the Hamiltonian details. Moreover, the complexity of the model Hamiltonians does not *a priori* imply the corresponding FQH state is hard to realize in experiments. For example, the model Hamiltonian for the Moore-Read state is three body, but the state can be stabilized by realistic two-body interactions, and is easier to realize in experiments than many Laughlin states with two-body model Hamiltonians.

In addition to conceptual insights, we formulated an algorithm to determine an intrinsic commensurability condition between the number of particles and the number of orbitals on genus 0 geometry. The commensurability condition uniquely determines the filling factors, topological shifts, and clustering properties of potential FQH fluids that can be realized in principle. The algorithm also allows efficient construction of model ground states of these FQH fluids by diagonalizing a two-body operator in a truncated Hilbert space.

It is tempting to speculate that the minimal models arising from Hilbert space truncation could be universal for all possible FQH states. In particular, it implies different FQH

phases can be unambiguously classified by the LECs, and the truncation of permissible Hilbert space could be characteristic of the topological nature of the FQH states. Not only is this manifested by the missing of generic states from the entanglement spectrums of the model ground states, it is also a unifying description for both the *integer and fractional* quantum Hall effect. This is because the integer quantum Hall effect is uniquely defined by translational invariance and the truncation of the bases from higher LLs.

Nevertheless, our scheme cannot account for some FQH states proposed in the literature. For example, while we can

find states at $\nu = 2/5$, $2/9$ and $3/7$ that have very high overlaps with the Jain states, some Jain or hierarchical states are missing; at those filling factors, incompressible states with different topological natures are obtained with our schemes. A detailed discussion of the close relationship between the LEC and composite fermion approaches will be presented elsewhere.

Acknowledgments. I thank F. D. M. Haldane, Jainendra Jain, Ying-hai Wu, and Zlatko Papić for useful discussions. This work is supported by the NTU grant for Nanyang Assistant Professorship.

-
- [1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982); R. Prange and S. Girvin, *The Quantum Hall Effect*, Graduate Texts in Contemporary Physics (Springer-Verlag, Berlin, 1987).
 - [2] R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
 - [3] *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1996).
 - [4] G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).
 - [5] N. Read and E. Rezayi, *Phys. Rev. B* **59**, 8084 (1999).
 - [6] F. D. M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983).
 - [7] J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989).
 - [8] J. K. Jain, *Phys. Rev. B* **40**, 8079 (1989).
 - [9] X. G. Wen, *Phys. Rev. Lett.* **66**, 802 (1991).
 - [10] *Composite Fermions: A Unified View of the Quantum Hall Regime*, edited by Olle Heinonen (World Scientific, New York, 1998).
 - [11] B. A. Bernevig and F. D. M. Haldane, *Phys. Rev. Lett.* **100**, 246802 (2008).
 - [12] N. Regnault, B. A. Bernevig, and F. D. M. Haldane, *Phys. Rev. Lett.* **103**, 016801 (2009).
 - [13] Bo Yang, Z.-X. Hu, Z. Papić, and F. D. M. Haldane, *Phys. Rev. Lett.* **108**, 256807 (2012).
 - [14] Bo Yang and F. D. M. Haldane, *Phys. Rev. Lett.* **112**, 026804 (2014).
 - [15] S. H. Simon, E. H. Rezayi, and N. R. Cooper, *Phys. Rev. B* **75**, 075318 (2007).
 - [16] X. G. Wen and A. Zee, *Phys. Rev. B* **46**, 2290 (1992).
 - [17] In this Rapid Communication we only look at cases with $S_e = 0$, but there are also examples of nonzero S_e shown by Thomale *et al.* (unpublished).
 - [18] G. J. Sreejith, M. Fremling, G. S. Jeon, and J. K. Jain, *Phys. Rev. B* **98**, 235139 (2018).
 - [19] F. D. M. Haldane, *Phys. Rev. Lett.* **67**, 937 (1991).
 - [20] M. Oshikawa, *Phys. Rev. Lett.* **84**, 1535 (2000).
 - [21] In principle we can also have symmetry-protected ground state degeneracy gapped from the rest of the spectrum, or a manifold of ground state degeneracy with the dimension growing subextensively with the system size. In this work we focus on nondegenerate ground state and we will defer such FQH topological phases to future works.
 - [22] H. Li and F. D. M. Haldane, *Phys. Rev. Lett.* **101**, 010504 (2008).
 - [23] B. Estienne, N. Regnault, and B. A. Bernevig, *Phys. Rev. Lett.* **114**, 186801 (2015).
 - [24] C. Toke and J. K. Jain, *Phys. Rev. B* **80**, 205301 (2009).
 - [25] Such circular droplets are not exactly defined in real space, but rather in a single LL with an orbital cut.
 - [26] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevB.100.241302> for some specific examples of the LEC implementation, and for some further discussions on the Gaffnian state and the Jain state at $\nu = 2/5$.
 - [27] S. H. Simon, E. H. Rezayi, N. R. Cooper, and I. Berdnikov, *Phys. Rev. B* **75**, 075317 (2007).
 - [28] B. I. Halperin, *Phys. Rev. Lett.* **52**, 1583 (1984).