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ON $\mathcal{N}_p$ -SPACES IN THE BALL

LE HAI KHOI

ABSTRACT. We present a short survey on the so-called $\mathcal{N}_p$ -spaces in the ball. Basic properties as well as the structure of $\mathcal{N}_p$ -spaces are given. Weighted composition operators and composition operators on these spaces are also studied. Some open problems are provided.

1. INTRODUCTION

For a positive integer n , if $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, then we define the inner product $\langle z, \zeta \rangle = z_1 \bar{\zeta}_1 + \dots + z_n \bar{\zeta}_n$ and norm $|z| = (z_1 \bar{z}_1 + \dots + z_n \bar{z}_n)^{1/2}$. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$ under this norm, with $\mathbb{S}$ as its boundary.

The space $\mathcal{O}(\mathbb{B})$ consists of all holomorphic functions in $\mathbb{B}$ with the compact-open topology, and $H^\infty(\mathbb{B})$ is the Banach space of all bounded functions in $\mathcal{O}(\mathbb{B})$ with the norm $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$.

Let $p > 0$, the Bergman space $A^2(\mathbb{B})$ is defined as

$$A^2(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_{A^2} = \left(\int_{\mathbb{B}} |f(z)|^2 dV(z) \right)^{1/2} < \infty \right\},$$

where dV is the normalized Lebesgue volume measure on $\mathbb{B}$ so that $V(\mathbb{B}) = 1$, while the Bergman-type space (sometimes also called the Beurling-type space) $A^{-p}(\mathbb{B})$ is defined as

$$A^{-p}(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : |f|_p = \sup_{z \in \mathbb{B}} |f(z)|(1 - |z|^2)^p < \infty \right\}.$$

The Bergman spaces and Bergman-type spaces attract a great attention of mathematicians. These spaces, on one hand play an important role in both function theoretic and operator theoretic development of function spaces, and on the other hand have a closed connection to Bloch spaces which appear as the images of the bounded functions under the Bergman projections. Bloch spaces also play the role of the dual spaces of the Bergman spaces.

Also among the Bergman and Bergman-type spaces, the so-called Q_p -spaces are of great interest. These spaces on the open unit disk $\mathbb{D}$ of the complex plane were introduced by Aulaskari, Xiao and Zhao [1]. For $p > 0$, the Q_p -space consists of functions in $\mathcal{O}(\mathbb{D})$ such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) < \infty.$$

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Here $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ is the automorphism of $\mathbb{D}$ that changes 0 and a , and dA is the Lebesgue area measure on the plane, normalized so that $A(\mathbb{D}) = 1$. It is known that Q_p -spaces coincide with the classical Bloch space $\mathcal{B}$ for $p \in (1, \infty)$; Q_1 is equal to BMOA, the space of holomorphic functions on $\mathbb{D}$ with bounded mean oscillation; and for $p \in (0, 1)$, the Q_p -spaces are all different.

If in the definition of the Q_p -space, $f'(z)$ is replaced by $f(z)$, then we have the so-called $\mathcal{N}_p$ -space. This space consists of functions in $\mathcal{O}(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) < \infty.$$

Some properties of $\mathcal{N}_p$ -spaces are: for $p > 1$, the $\mathcal{N}_p$ -space coincides with A^{-1} and for $p \in (0, 1]$, the $\mathcal{N}_p$ -spaces are all different.

We note that the Q_p -spaces have been studied intensively for over a decade. The background on these spaces can be found in the book by Xiao [18]. So due to both the similarities and the differences between $\mathcal{B}$ and A^{-1} , it is natural to study the $\mathcal{N}_p$ -spaces. For the unit disk $\mathbb{D}$ these spaces were first introduced and studied by Palmberg in [12] and then by Ueki in [17]. In these works, different properties of weighted composition operators and composition operators on $\mathcal{N}_p$ -spaces in the unit disk are also studied.

The question to ask is: how about $\mathcal{N}_p$ -spaces in higher dimensions? The aim of this paper is to present a short survey on these spaces.

One of the important topics presented in this survey concerns weighted composition operators. Let φ denote a non-constant holomorphic self-mapping of $\mathbb{B}$, while ψ will be a holomorphic function on $\mathbb{B}$, which is not identically equal to zero. These functions induce a linear operator $W_{\psi, \varphi} : \mathcal{O}(\mathbb{B}) \rightarrow \mathcal{O}(\mathbb{B})$, which is called a *weighted composition operator* with symbols ψ and φ defined by

$$W_{\psi, \varphi}(f)(z) = \psi(z) \cdot (f \circ \varphi(z)), f \in \mathcal{O}(\mathbb{B}), z \in \mathbb{B}.$$

Observe that if ψ is identically 1, then $W_{\psi, \varphi} = C_\varphi$ is the *composition operator*, and if φ is the identity, then $W_{\psi, \varphi} = M_\psi$ is the *multiplication operator*.

Composition operators and weighted composition operators acting on spaces of holomorphic functions in the unit disk $\mathbb{D}$ of the complex plane have been studied quite well. We refer the readers to the monographs [2, 15] for detailed information. Composition operators on $A^{-p}(\mathbb{D})$ have also been intensively studied (see, e.g., [4, 20] and references therein).

This survey is organized as follows. Section 2 deals with basic properties of $\mathcal{N}_p$ -spaces in connection with the Bergman-type spaces $A^{-q}(\mathbb{B})$. In particular, an embedding theorem for $\mathcal{N}_p(\mathbb{B})$ -spaces and $A^{-q}(\mathbb{B})$ is provided, together with other useful properties. Several results on the structure of $\mathcal{N}_p$ -spaces such as multipliers, $\mathcal{M}$ -invariance, the closure of all polynomials, $\mathcal{N}_p$ -norm via Carleson measures, ... are also given. The result about differences of $\mathcal{N}_p$ for small values of p is obtained in this section. In Section 3 weighted composition operators $W_{\psi, \varphi}$ between $\mathcal{N}_p$ and Bergman-type spaces A^{-q} are studied. Criteria for boundedness and compactness of these operators, the compact differences and the estimate for essential norm of weighted

composition operators $W_{u,\varphi}$ are given. Section 4 studies composition operators C_φ acting between $\mathcal{N}_p$ -spaces, which is one of the most interesting (and quite difficult) topics. The characterizations of the boundedness and compactness of these operators are obtained. In Section 5 some open problems are provided. We also note that the main results in this survey can be found in [5, 6, 7, 8, 9].

The important remark is that although the $\mathcal{N}_p$ -spaces are closely related to the Q_p -spaces, the approach and techniques used for investigating $\mathcal{N}_p$ -spaces are quite different from those of Q_p -space. Moreover, they have their own interest.

Throughout the paper, $d\sigma$ denotes the normalized surface measure on the boundary $\mathbb{S}$ of $\mathbb{B}$. For $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C , which is independent of a and b , such that $a \leq Cb$ ($a \geq Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we write $a \simeq b$.

2. BASIC PROPERTIES OF $\mathcal{N}_p$ -SPACES IN THE BALL

The $\mathcal{N}_p$ -space in $\mathbb{B}$ is introduced in [5]. It is defined as follows

$$\mathcal{N}_p(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_p = \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\},$$

where $dV(z)$ is the normalized volume measure over $\mathbb{B}$ and $\Phi_a \in \text{Aut}(\mathbb{B})$ is the involutive automorphism of the unit ball that interchanges 0 and $a \in \mathbb{B}$ (see, e.g. [13, Chapter 2]).

2.1. $\mathcal{N}_p$ -spaces and Bergman-type spaces. Several basic properties of $\mathcal{N}_p(\mathbb{B})$ -spaces can be obtained, in connection with the Bergman-type spaces $A^{-q}(\mathbb{B})$. In particular, a continuous embedding for $\mathcal{N}_p(\mathbb{B})$ -spaces and $A^{-q}(\mathbb{B})$, together with other useful properties, are given in the following theorem.

Theorem 2.1. ([5]) *The following statements hold.*

- (a) For $p > q > 0$, we have $H^\infty(\mathbb{B}) \hookrightarrow \mathcal{N}_q(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B}) \hookrightarrow A^{-\frac{n+1}{2}}(\mathbb{B})$.
- (b) For $p > 0$, if $p > 2k - 1, k \in (0, \frac{n+1}{2}]$, then $A^{-k}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$. In particular, when $p > n$, $\mathcal{N}_p(\mathbb{B}) = A^{-\frac{n+1}{2}}(\mathbb{B})$.
- (c) $\mathcal{N}_p(\mathbb{B})$ is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.
- (d) For $0 < p < \infty$, $\mathcal{B}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$, where $\mathcal{B}(\mathbb{B})$ is the Bloch space in $\mathbb{B}$.

Here if X and Y are two topological vector spaces, then the notation $X \hookrightarrow Y$ means the continuous embedding of X into Y .

The following “probe” functions in $\mathcal{N}_p$ -spaces are used very often in the sequel.

Lemma 2.2. For each $w \in \mathbb{B}$, set

$$k_w(z) = \left(\frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right)^{\frac{n+1}{2}}, \quad z \in \mathbb{B}.$$

Then $k_w \in \mathcal{N}_p(\mathbb{B})$ and $\sup_{w \in \mathbb{B}} \|k_w\|_p \leq 1$.

Such k_w is a normalized reproducing kernel function in the Bergman space $A^2(\mathbb{B})$. We have $k_w \in \mathcal{N}_p$ and $\sup_{w \in \mathbb{B}} \|k_w\|_p \leq 1$. Note that for all $w \in \mathbb{B}$, we have $k_w(w) = (1 - |w|^2)^{-(n+1)/2}$.

2.2. Multipliers and $\mathcal{M}$ -invariance of $\mathcal{N}_p$ -spaces. We first describe the space $\text{Mult}(\mathcal{N}_p)$ of multipliers of $\mathcal{N}_p$. Recall that a function $u : \mathbb{B} \rightarrow \mathbb{C}$ is a *multiplier* of $\mathcal{N}_p$ if uf belongs to $\mathcal{N}_p$ for all f in $\mathcal{N}_p$. An application of the closed graph theorem shows that for any $u \in \text{Mult}(\mathcal{N}_p)$, the multiplication operator M_u is bounded on $\mathcal{N}_p$.

Proposition 2.3. ([9]) *For any $p > 0$, we have $\text{Mult}(\mathcal{N}_p) = H^\infty(\mathbb{B})$. Furthermore, for any $u \in H^\infty(\mathbb{B})$, $\|M_u\| = \|u\|_\infty$.*

By using weighted composition operators with particular symbols, we can obtain an alternate description of the norm in $\mathcal{N}_p$. Namely, we note that for any $\Phi \in \text{Aut}(\mathbb{B})$, by [13, Theorem 2.2.5], there exists a unitary operator U such that $\Phi = U\Phi_a$, where $a = \Phi^{-1}(0)$. This shows that $|\Phi(z)| = |\Phi_a(z)|$ for all $z \in \mathbb{B}$. Let W_Φ denote the weighted composition operator $W_{k_a, \Phi}$. Then we have

$$(2.1) \quad \|f\|_p = \sup \left\{ \|W_\Phi f\|_{A_p^2} : \Phi \in \text{Aut}(\mathbb{B}) \right\}.$$

Here A_p^2 is the weighted Bergman space over $\mathbb{B}$ defined by

$$A_p^2(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_{A_p^2} = \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |z|^2)^p dV(z) \right)^{1/2} < \infty \right\}.$$

We have the following result.

Theorem 2.4. ([9]) *For any automorphism Ψ of the unit ball $\mathbb{B}$, the weighted composition operator W_Ψ is a surjective isometry on $\mathcal{N}_p$.*

Next, recall that a space $\mathcal{X}$ of functions defined on $\mathbb{B}$ is said to be *Moebius-invariant*, or simply *$\mathcal{M}$ -invariant*, if $f \circ \Phi \in \mathcal{X}$ for every $f \in \mathcal{X}$ and every $\Phi \in \text{Aut}(\mathbb{B})$ (see, e.g., [13]). As a corollary to Theorem 2.4, we obtain

Corollary 2.5. *The space $\mathcal{N}_p$ is $\mathcal{M}$ -invariant. Moreover, for any $\Phi \in \text{Aut}(\mathbb{B})$, we have*

$$(2.2) \quad \|C_\Phi\| = \|M_{1/k_a}\| = \left(\frac{1 + |a|}{1 - |a|} \right)^{\frac{n+1}{2}},$$

where $a = \Phi^{-1}(0)$.

2.3. The closure of all polynomials in $\mathcal{N}_p$. It is natural to consider what the closure of all the polynomials is in $\mathcal{N}_p$ -spaces. We introduce the little space $\mathcal{N}_p^0$ of $\mathcal{N}_p$, which is defined as follows

$$\mathcal{N}_p^0(\mathbb{B}) = \left\{ f \in \mathcal{N}_p : \lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) = 0 \right\}.$$

The main result to present is that the closure of all the polynomials on $\mathbb{B}$ coincides with the little space $\mathcal{N}_p^0$.

Proposition 2.6. ([9]) *The following assertions hold.*

- (a) $\mathcal{N}_p^0(\mathbb{B})$ is a closed subspace of $\mathcal{N}_p(\mathbb{B})$, and hence it is a Banach space.

(b) For any $p > 0$, we have $A_p^2(\mathbb{B}) \subset \mathcal{N}_p^0(\mathbb{B})$.

Theorem 2.7. ([9]) Suppose $f \in \mathcal{N}_p$. Then $f \in \mathcal{N}_p^0$ if and only if

$$\|f_r - f\|_p \rightarrow 0 \text{ as } r \rightarrow 1^-,$$

where $f_r(z) = f(rz)$ for all $z \in \mathbb{B}$.

As a corollary to Theorem 2.7, we obtain

Corollary 2.8. The set of polynomials is dense in $\mathcal{N}_p^0$.

2.4. $\mathcal{N}_p$ -norm via Carleson measures. Recall (see, e.g., [19]) that for $\xi \in \mathbb{S}$ and $r > 0$, a Carleson tube at ξ is defined as

$$Q_r(\xi) = \{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r\}.$$

A positive Borel measure μ in $\mathbb{B}$ is called a p -Carleson measure if there exists a constant $C > 0$ such that

$$\mu(Q_r(\xi)) \leq Cr^p$$

for all $\xi \in \mathbb{S}$ and $r > 0$. Moreover, if

$$\lim_{r \rightarrow 0} \frac{\mu(Q_r(\xi))}{r^p} = 0$$

uniformly for $\xi \in \mathbb{S}$, then μ is called a *vanishing p -Carleson measure*.

The following result describes a relationship between functions in $\mathcal{N}_p$ as well as $\mathcal{N}_p^0$ and Carleson measures.

Proposition 2.9. ([9]) Let $p > 0$ and $f \in \mathcal{O}(\mathbb{B})$. Define $d\mu_{f,p}(z) = |f(z)|^2(1 - |z|^2)^p dV(z)$. The following assertions hold.

- (1) $f \in \mathcal{N}_p$ if and only if $\mu_{f,p}$ is a p -Carleson measure.
- (2) $f \in \mathcal{N}_p^0$ if and only if $\mu_{f,p}$ is a vanishing p -Carleson measure.

Moreover, it holds

$$(2.3) \quad \|f\|_p^2 \simeq \sup_{r \in (0,1), \xi \in \mathbb{S}} \frac{\mu_{f,p}(Q_r(\xi))}{r^p} = \sup_{r \in (0,1), \xi \in \mathbb{S}} \frac{1}{r^p} \int_{Q_r(\xi)} |f(z)|^2(1 - |z|^2)^p dV(z).$$

Lemma 2.10. Let $p = n+1+\alpha > 0$ and μ be a finite positive Borel measure on $\mathbb{B}$. Then the following conditions are equivalent.

- (a) μ is a vanishing p -Carleson measure.
- (b) For each $s > 0$,

$$(2.4) \quad \lim_{|z| \rightarrow 1^-} \int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{p+s}} = 0.$$

- (c) For some $s > 0$, (2.4) holds.

2.5. Differences of $\mathcal{N}_p$ for small values of p . In this section, we show that for p small, that is $0 < p \leq n$, all $\mathcal{N}_p$ -spaces are different. This together with Theorem 2.1 (b) gives a complete relationship between $\mathcal{N}_p$ -spaces for all $p > 0$.

We prove this fact by a construction. In [14], the authors constructed a sequence of homogeneous polynomials $(P_k)_{k \in \mathbb{N}}$ satisfying $\deg(P_k) = k$,

$$(2.5) \quad \|P_k\|_\infty = \sup_{\xi \in \mathbb{S}} |P_k(\xi)| = 1, \text{ and } \left(\int_{\mathbb{S}} |P_k(\xi)|^2 d\sigma(\xi) \right)^{1/2} \geq \frac{\sqrt{\pi}}{2^n}.$$

Note that the homogeneity of P_k implies that $|P_k(z)| \leq |z|^k$ for all $z \in \mathbb{B}$.

Let $\{m_k\}_{k=0}^\infty$ be a sequence of positive integers such that $m_{k+1}/m_k \geq c$ for all $k \geq 0$, where $c > 1$ is a constant. Let

$$(2.6) \quad f(z) = \sum_{k=0}^{\infty} b_k P_{m_k}(z) \text{ for } z \in \mathbb{B}.$$

Such a function is said to belong to the *Hadamard gap class*. A characterization for a Hadamard gap class function to be in a weighted Bergman space was given in [16]. In the following result, we obtain an estimate for the $\mathcal{N}_p$ -norm and A^{-q} -norm of f . These results are higher dimensional versions of [12, Theorem 3.3].

Theorem 2.11. ([9]) *Let f be defined as in (2.6). Let p be a positive real number. Then the following statements hold:*

- (a) *For $0 < p \leq n$, we have $\|f\|_p^2 \simeq \sum_{k=0}^{\infty} \frac{|b_k|^2}{m_k^{p+1}}$.*
- (b) *For any $q > 0$, we have $|f|_q \simeq \sup_k \frac{|b_k|}{m_k^q}$.*

(Here, $\|f\|_p$ and $|f|_q$ denote the norm of f in the spaces $\mathcal{N}_p$ and A^{-q} , respectively).

Note that Theorem 2.11, for $n = 1$, contains the corresponding results in [12] as particular cases.

Corollary 2.12. *If $0 < p_1 < p_2 \leq n$, then we have*

$$\mathcal{N}_{p_1}(\mathbb{B}) \subsetneq \mathcal{N}_{p_2}(\mathbb{B}) \subsetneq A^{-\frac{n+1}{2}}(\mathbb{B}).$$

3. WEIGHTED COMPOSITION OPERATORS BETWEEN $\mathcal{N}_p(\mathbb{B})$ AND $A^{-q}(\mathbb{B})$

In this section, we consider the weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ to $A^{-q}(\mathbb{B})$.

3.1. Boundedness. Note that if the weighted composition operator $W_{u,\varphi}$ maps $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$, then an application of the closed graph theorem shows that $W_{u,\varphi}$ is automatically bounded from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Theorem 3.1. ([5]) *Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping, $u: \mathbb{B} \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is bounded if and only if*

$$(3.1) \quad \sup_{z \in \mathbb{B}} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty.$$

Note that when $n = 1$, we obtain Theorem 3 of [17] as a particular case.

3.2. Compactness. We have the following criteria for compactness of weighted composition operators $W_{u,\varphi}$.

Theorem 3.2. ([5]) *Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping, $u: \mathbb{B} \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is compact if and only if*

$$(3.2) \quad \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

When $n = 1$, Theorem 3.2 contains Corollary 2 in [17] as a particular case.

As it is seen from the theorems, the results in higher dimensions depend essentially on the dimension n of the space $\mathbb{C}^n$.

As a corollary of the obtained results above, we have the following criteria for composition operators.

Corollary 3.3. *Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping and $p, q > 0$. The composition operator C_φ acting from $\mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$*

(1) *is bounded if and only if*

$$\sup_{z \in \mathbb{B}} \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty.$$

(2) *is compact if and only if*

$$\lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

When $n = 1$, Corollary 3.3 contains Theorems 4.1 and 4.3 in [12] as particular cases.

3.3. Compact difference. We study the compactness of the difference of two bounded weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Let φ_1, φ_2 be two holomorphic self-mappings on $\mathbb{B}$, $u_1, u_2: \mathbb{B} \rightarrow \mathbb{C}$ two holomorphic mappings, and $p, q > 0$. Let further, W_{u_1, φ_1} and W_{u_2, φ_2} be two weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Recall that the *pseudo-hyperbolic metric* in the ball is defined as

$$\rho(z, w) = |\Phi_w(z)|, \quad z, w \in \mathbb{B}.$$

It is a true metric (see, e.g., [3]). Also it is easy to verify, in particular, that $\rho(0, w) = |w|$, $\rho(\Phi_w(z), w) = |z|$.

The following lemmas play an important role in the study of compact differences. They also have their own interest.

Lemma 3.4. *For $z, w \in \mathbb{B}$, if $\rho(z, w) \leq \frac{1}{2}$, then*

$$\frac{1}{6} \leq \frac{1 - |z|^2}{1 - |w|^2} \leq 6.$$

Lemma 3.5. *For $f \in \mathcal{N}_p(\mathbb{B})$ and $z, w \in \mathbb{B}$, we have*

$$|f(z) - f(w)| \leq M \|f\|_p \max \left\{ \frac{1}{(1 - |z|^2)^{\frac{n+1}{2}}}, \frac{1}{(1 - |w|^2)^{\frac{n+1}{2}}} \right\} \rho(z, w).$$

Here $M = \frac{6^{\frac{n+1}{2}} \cdot 2^{p+n+1} (3+2\sqrt{3}) \sqrt{n}}{3^{p/2}}$.

Inspiring by the pseudo-hyperbolic metric in the unit ball, for two holomorphic mappings $\varphi, \psi: \mathbb{B} \rightarrow \mathbb{B}$, we define

$$\rho_{\varphi, \psi}(z) = |\Phi_{\varphi(z)}(\psi(z))|, \quad z \in \mathbb{B}.$$

Evidently, $\rho_{\varphi, \psi} = \rho_{\psi, \varphi}$.

Now we are ready to formulate the following result.

Theorem 3.6. ([6]) *Let $\varphi_1, \varphi_2: \mathbb{B} \rightarrow \mathbb{B}$ be two holomorphic mappings, $u_1, u_2: \mathbb{B} \rightarrow \mathbb{C}$ two holomorphic mappings and $p, q > 0$. Let further, W_{u_1, φ_1} and W_{u_2, φ_2} be two weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$. Then $W_{u_1, \varphi_1} - W_{u_2, \varphi_2}$ is compact if and only if the following conditions are satisfied:*

(i)

$$\lim_{r \rightarrow 1^-} \sup_{|\varphi_k(z)| > r} \left\{ \frac{|u_k(z)|(1-|z|^2)^q}{(1-|\varphi_k(z)|^2)^{\frac{n+1}{2}}} \rho_{\varphi_1, \varphi_2}(z) \right\} = 0 \quad (k = 1, 2);$$

(ii)

$$\lim_{r \rightarrow 1^-} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} \left[|u_1(z) - u_2(z)| \min \left\{ \frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} \right\} \right] = 0.$$

3.4. Essential norm. Let us denote by $\mathcal{K} := \mathcal{K}(\mathcal{N}_p, A^{-q})$ the set of all compact operators acting from $\mathcal{N}_p$ into A^{-q} . Then the essential norm of $W_{u, \varphi}$ is defined as follows:

$$\|W_{u, \varphi}\|_e = \inf_{K \in \mathcal{K}} \{\|W_{u, \varphi} - K\|\}.$$

Obviously, the essential norm of a compact operator is zero.

3.4.1. Upper bound of the essential norm. We need some auxiliary results.

Lemma 3.7. *Suppose φ is a self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$ and $u \in \mathcal{N}_p$. Then the weighted composition operator $W_{u, \varphi}: \mathcal{N}_p \rightarrow \mathcal{N}_p$ is compact.*

In the case z and w are multiple of each other, Lemma 3.5 can be simplified as in the following result.

Lemma 3.8. *Let f be in $\mathcal{N}_p$. For any $a \in \mathbb{B}$ and any $\kappa \in (0, 1)$, we have*

$$(3.3) \quad |f(a) - f(\kappa a)| \leq \frac{M(1-\kappa)|a|}{1-\kappa|a|^2} \cdot \frac{\|f\|_p}{(1-|a|^2)^{\frac{n+1}{2}}} \leq \frac{M\|f\|_p}{(1-|a|^2)^{\frac{n+1}{2}}}.$$

Consequently, for any $0 < r < 1$, we have

$$(3.4) \quad \sup_{|a| \leq r} |f(a) - f(\kappa a)| \leq \frac{Mr(1-\kappa)\|f\|_p}{(1-r^2)^{\frac{n+3}{2}}}.$$

Here M is the constant from Lemma 3.5.

Now we can have an estimate for the upper bound of the essential norm of $W_{u, \varphi}$.

Theorem 3.9. ([7]) *Let p and q be two positive numbers. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic self-mapping and $u: \mathbb{B} \rightarrow \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q} . Then*

$$\|W_{u,\varphi}\|_e \leq M \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}},$$

where M is the constant from Lemma 3.5.

3.4.2. Lower bound of the essential norm. We now discuss the estimation for the lower bound of $\|W_{u,\varphi}\|_e$. We will make use of weakly convergent sequences in the Bergman space A^2 . The following lemma plays an important role.

Lemma 3.10. *Suppose $\{f_m\}_{m \geq 1} \subset A^2$ is a sequence that converges weakly to zero in A^2 . Then $\{f_m\}_{m \geq 1}$ converges weakly to zero in $\mathcal{N}_p$ as well.*

Corollary 3.11. *Let $\{w_m\}_{m \in \mathbb{N}} \subset \mathbb{B}$ and $|w_m| \rightarrow 1$ as $m \rightarrow \infty$, then $\{k_{w_m}\}$ converges weakly to zero in $\mathcal{N}_p$.*

Theorem 3.12. ([7]) *Let p and q be two positive numbers. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic self-mapping and $u: \mathbb{B} \rightarrow \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q} . Then*

$$\|W_{u,\varphi}\|_e \geq \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}}.$$

In conclusion, combining Theorems 3.9 and 3.12, we obtain a full description of the essential norm of $W_{u,\varphi}$.

Theorem 3.13. *Let p and q be two positive numbers. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic self-mapping and $u: \mathbb{B} \rightarrow \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q} . Then*

$$\|W_{u,\varphi}\|_e \simeq \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}}.$$

Theorem 3.13 provides us a characterization of compact weighted composition operators from $\mathcal{N}_p$ to A^{-q} as in Theorem 3.2.

Corollary 3.14. *Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q} as in Theorem 3.13. Then $W_{u,\varphi}$ is compact if and only if*

$$\lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

4. COMPOSITION OPERATORS BETWEEN $\mathcal{N}_p$ AND $\mathcal{N}_q$

One of the most interesting (and quite difficult) topics is to study properties of (weighted) composition operators between $\mathcal{N}_p$ -spaces. In this section, we present the results for composition operators C_φ acting from $\mathcal{N}_p$ into $\mathcal{N}_q$. Note that the one dimensional case was considered by Palmberg in [12, Theorem 4.2]. A sufficient condition and a necessary condition for C_φ to be bounded on $\mathcal{N}_p$ of the unit disk were given there. These conditions involve the generalized Nevanlinna counting function introduced by Shapiro.

4.1. Boundedness of composition operators from $\mathcal{N}_p$ to $\mathcal{N}_q$. We begin with the case φ is a univalent holomorphic self-mapping of the unit ball $\mathbb{B}$.

Theorem 4.1. ([8]) *Let φ be a univalent holomorphic self-mapping of $\mathbb{B}$ and p be any positive real number. Suppose that*

$$\delta = \inf\{|J\varphi(z)| : z \in \mathbb{B}\} > 0,$$

where $J\varphi$ the complex Jacobian of φ . Then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_p$ is a bounded operator with $\|C_\varphi\| \leq \delta^{-1}$ for all $p > 0$.

Corollary 4.2. *Suppose $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is an invertible linear operator and b is a vector in $\mathbb{C}^n$ such that $\varphi(z) = Az + b$ is a self-mapping of $\mathbb{B}$. Then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_p$ is bounded for all $p > 0$.*

For general self-mappings φ of the unit ball, we provide a necessary condition and a sufficient condition for C_φ to be bounded from $\mathcal{N}_p$ into $\mathcal{N}_q$. We make use of a sequence of homogeneous polynomials $\{P_k\}$ in (2.5). We also need the inequality

$$(4.1) \quad 1 + \sum_{k=0}^{\infty} 2^{k\gamma} x^{2^k} \gtrsim (1-x)^{-\gamma} \text{ for } 0 \leq x < 1,$$

where γ is a positive number.

We have the following necessary condition and sufficient condition for the boundedness of $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$.

Theorem 4.3. ([8]) *Let p and q be two positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$. If*

$$(4.2) \quad \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |\Phi_a(z)|^2)^q}{(1 - |\varphi(z)|^2)^{n+1}} dV(z) < \infty,$$

then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is bounded.

Conversely, if $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is bounded, then for any $0 < \varepsilon \leq p+1$,

$$(4.3) \quad \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |\Phi_a(z)|^2)^q}{(1 - |\varphi(z)|^2)^{p+1-\varepsilon}} dV(z) < \infty.$$

An application of Theorem 4.3 immediately gives the following result.

Corollary 4.4. *Let φ be a holomorphic self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is bounded for all $p, q > 0$.*

We note that in fact the operator C_φ in the corollary is compact.

4.2. Compactness of composition operators from $\mathcal{N}_p$ to $\mathcal{N}_q$. First we have the following equivalent conditions for the compactness of C_φ .

Proposition 4.5. *Let p, q be positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that the composition operator $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is bounded. Then the following statements are equivalent.*

- (a) C_φ is a compact operator from $\mathcal{N}_p$ into $\mathcal{N}_q$.
- (b) $\lim_{t \rightarrow 1^-} \left(\sup_{a \in \mathbb{B}, f \in \mathcal{N}_p} \left[\int_{|z| > t} |f(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q dV(z) \right] \right) = 0.$
- (c) $\lim_{t \rightarrow 1^-} \left(\sup_{a \in \mathbb{B}, f \in \mathcal{N}_p} \left[\int_{|\varphi(z)| > t} |f(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q dV(z) \right] \right) = 0.$

For a particular case of φ , we can also have a compactness of C_φ .

Proposition 4.6. *Let φ be a holomorphic self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is compact for all $p, q > 0$.*

Now we can formulate a necessary condition and a sufficient condition for the compactness of C_φ . These conditions are quite useful in applications.

Theorem 4.7. ([8]) *Let $p, q \in (0, n]$ be two positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is bounded. If*

$$(4.4) \quad \lim_{t \rightarrow 1^-} \sup_{a \in \mathbb{B}} \int_{|\varphi(z)| > t} \frac{(1 - |\Phi_a(z)|^2)^q}{(1 - |\varphi(z)|^2)^{n+1}} dV(z) = 0,$$

then C_φ is compact.

Conversely, if $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_q$ is compact, then for $0 < \varepsilon \leq p + 1$,

$$(4.5) \quad \lim_{t \rightarrow 1^-} \sup_{a \in \mathbb{B}} \int_{|\varphi(z)| > t} \frac{(1 - |\Phi_a(z)|^2)^q}{(1 - |\varphi(z)|^2)^{p+1-\varepsilon}} dV(z) = 0.$$

As applications of Theorems 4.3 and 4.7, we have the following result.

Corollary 4.8. *Suppose $k > 0, p, q, r \in (0, n), r \geq q, \varepsilon \in (0, q + 1)$ and φ is a holomorphic self-mapping of $\mathbb{B}$. The following statements hold*

- (1) *If $C_\varphi : A^{-k(q+1-\varepsilon)} \rightarrow A^{-k(n+1)}$ is a bounded operator, then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_r$ is a bounded operator;*
- (2) *If $C_\varphi : A^{-k(q+1-\varepsilon)} \rightarrow A^{-k(n+1)}$ is a compact operator, then $C_\varphi : \mathcal{N}_p \rightarrow \mathcal{N}_r$ is a compact operator.*

5. SOME OPEN PROBLEMS

1. In Section 2, the differences of $\mathcal{N}_p$ spaces for small values of p are obtained by using the Hadamard gap class functions. In connection with this, it is interesting to develop this topic to a more general class of functions and a more general class of spaces as well.

Problem 5.1. *Study Hadamard gap series for classes which are more general than $\mathcal{N}_p$ -spaces.*

We note that some development on $\mathcal{N}_p$ -type functions with Hadamard gaps and also Hadamard gap series in weighted-type spaces on the unit ball is presented in [10, 11].

2. In Section 3, several properties of weighted composition operators $W_{u,\varphi}$ acting from $\mathcal{N}_p(\mathbb{B})$ to $A^{-q}(\mathbb{B})$ have been investigated. It is natural to ask for about weighted composition operators acting $\mathcal{N}_p(\mathbb{B})$ to other function spaces, which are subspaces of the space $\mathcal{O}(\mathbb{B})$ of holomorphic functions in the unit ball $\mathbb{B}$.

Problem 5.2. *Let $\mathcal{H}$ be a subspace of the space $\mathcal{O}(\mathbb{B})$. Study properties of weighted composition operators $W_{u,\varphi}$ acting between the $\mathcal{N}_p$ -space and $\mathcal{H}$.*

3. In Section 4, composition operators C_φ acting between $\mathcal{N}_p$ -spaces have been studied. However, there still is a “gap” between necessary and sufficient conditions. Also weighted composition operators have not been investigated. So we have the following problems.

Problem 5.3. What are “sharp” criteria for $C_\varphi: \mathcal{N}_p \rightarrow \mathcal{N}_q$ to be bounded, compact, etc.?

Problem 5.4. Investigate weighted composition operators $W_{u,\varphi}$ acting between $\mathcal{N}_p$ -spaces.

We note that Problems 5.3–5.4 for even Q_p -spaces are still unsolved.

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