Multi-period inventory sharing between independent retailers : models for different degrees of information sharing

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**Multi-Period Inventory Sharing
Between Independent Retailers:
Models for Different Degrees of
Information Sharing**

**ZHANG ZHIBIN
SCHOOL OF CIVIL AND ENVIRONMENTAL
ENGINEERING**

2019

**Multi-Period Inventory Sharing
Between Independent Retailers:
Models for Different Degrees of
Information Sharing**

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School of Civil and Environmental Engineering

A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2019

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Nomenclature

CSCMP	Council of Supply Chain Management Professionals
VMIP	Vendor Managed Inventory Partnership
MMFE	Martingale Model of Forecast Evolution
N	number of retailers
$\mathcal{N}$	set of retailers
$\emptyset$	empty set
t	time
t_0	current time
L	lead time
D	demand
D_i^t	demand of retailer i in time t
$\mathbb{P}_{D_i^t}$	distribution of demand D_i^t
O	order quantity
O_i^t	ordered quantity of retailer i that arrives in time t
$O_i^{t,L}$	ordered quantity of retailer i is made in time t and replenished in time $t + L$
I_i^t	inventory level of retailer i at the end of time t

s_i^t	sharing quantity of retailer i in time t
x_+	$\max(x, 0)$
x_-	$-\min(x, 0)$
$\mathcal{G}$	class of all cooperative games
$\mathcal{T}_1$	class of all transshipment games
$\mathcal{T}_2$	class of all centralized transshipment games
Φ	allocation rule
n_b^k	reduction of backorders in time k
n_t^k	total number of lateral transshipments in time k
n_s^k	increment of loaned out quantity in time k
p_{ij}	is the profit of transshipping a unit from i to j
p, p_b	unit backorder penalty in one time unit
h	the unit holding cost in each time unit
q	unit ordering price
r	unit revenue
r_{ij}^j	profit of j for sharing one unit of product from i to j
r_{ji}^j	profit of j for sharing one unit of product from j to i
T	replenishment cycle
$I_{i,i_1}^+(O_i, D)$	non-negative amounts of inventory residuals that retailer i willing to share with retailer i_1
$I_{i,i_1}^-(O_i, D)$	non-negative amounts of unfulfilled demands that retailer i willing to share with retailer i_1

$I_{i_k}^+(\mathbf{O}, \mathbf{D})$	non-negative amount of residual inventories that retailer i_k is willing to share with retailer i
$I_{i_k}^-(\mathbf{O}, \mathbf{D})$	non-negative amount of unfulfilled demands that retailer i_k is willing to share with retailer i
$I_{i,i_k}^+(\mathbf{O}, \mathbf{D})$	non-negative amount of residual inventories that retailer i is willing to share with retailer i_k
$I_{i,i_k}^-(\mathbf{O}, \mathbf{D})$	non-negative amount of unfulfilled demands that retailer i is willing to share with retailer i_k

Abstract

The problem of inventory sharing between independent retailers is studied. Current literature focuses on the single-period problem, of which a large majority is restricted to two retailers. In contrast, we consider the multi-period, multi-ordering problem that permits more than two retailers. In this research, we consider two related variants of the problem that are differed by the extent of inter-retailer information sharing: (1) “limited information sharing” and (2) “complete information sharing”.

In “limited information sharing”, the retailers only share information on their up-to-date inventory levels. A sharing policy that is based on the concept of stock lending to control the sharing process for all possibilities is formulated. When designing the sharing policy, the key difficulty is to quantify the risk into a form of compensation to these retailers. This lending policy overcomes this difficulty. We first formulate the allocation rule for distributing the profit that are generated by reduction in backorder and holding cost. The rule that attains the essential optimality conditions of this problem, i.e., complete sharing and full inventory pooling is proved. Next, the lateral transshipment cost is included in the analysis and a transshipment rule for matching “surplus” retailers with those in “deficit” is developed. An upper bound of transshipment frequency for this rule is established, and it shows that under specific cost structure, the rule leads to complete sharing and guarantees additional profit for every retailer.

In the “complete information sharing” problem, retailers share information on demands and order quantities (in addition to inventory information in “limited information sharing”). Information is characterized. It explains that how these information can be utilized to enhance sharing efficiency

in terms of expectation and a modeling framework that determines each retailer's optimal order quantity is developed.

An analytical model is first developed, and then the complexity in obtaining the optimal solution and the difficulty in attaining the joint demand distribution (a key model input) in practice are demonstrated. A simulation approach that computes the optimal solution using the analytical model is then developed. The simulation avoids the need for the joint demand distribution by allowing demands to be characterized through demand sampling; this is enabled by an extension of the Leibniz Rule embedded in the analytical model, which allows for general demand distributions. The sharing problem without system-wide cooperation is first studied, then the concavity of the best response function is investigated. Unlike typical single-period problems in which the retailers have only one opportunity to order, retailers can choose their own order timing in the multi-period setting. Spreading the replenishment evenly over time (known as balanced ordering) can increase system-wide stock availability and consequently improve system efficiency. In addition, as a side benefit, the sequential ordering by the retailers considerably improves tractability by eliminating the possibility of game-based interactions.

Chapter 1

Introduction

1.1 Background and Motivations

According to Council of Supply Chain Management Professionals (CSCMP), Supply Chain Management is the management of supply chain activities, which encompass product development, procurement, manufacturing, transportation, as well as the transmission of information to coordinate these actions (www.cscmp.org). In a profit-maximizing supply chain, organizations are connected via flows of goods and information, each attempting to improve efficiency in these flows and to enhance customer experience.

Physical flow of goods in a supply chain typically involves transportation between storage points, which are usually facilities such as manufacturing plants, warehouses, distribution centers and retail stores. These goods, which are stored while waiting to serve customer demand or simply to be transported to the next forward facility, are known as inventories. Inventory Management is an essential and fundamental managerial aspect of Supply Chain Management. With the trend of accelerating industrial clockspeed and raising customer expectations, inventory management has become even more important today. In most businesses, a stock-out

situation often results in dissatisfied customers, which often leads to detrimental consequences affecting both short-term and long-term revenue prospects. However, on the other hand, it is also costly for businesses to hold high levels of inventories to achieve the desired level of customer service. Holding of inventories incurs storage cost and capital cost. Furthermore, holding inventories of short life-cycle products or perishables result in obsolescence cost as their market values depreciate at a rapid rate. Hence the fundamental trade-off in inventory management is between the incurrence of holding cost versus shortage cost.

Managing inventories can be a daunting task as the inventory decisions, such as how much to hold, where to hold and when to hold, typically involves many sources of uncertainties. Common uncertainties include customer demand (quantity and timing), lead-time of replenishing inventories (subject to supplier's stock availability and reliability of in-transit transport modes) and variation in product quality. Over the decades, there are many research areas related to inventory management. Examples include supply network planning, demand forecasting, supply chain coordination, choice of supply sources, impact of supply disruptions, inventory sharing between independent entities, and many others.

The area of production planning has been researched for decades. Van Hoesel and Wagelmans (1993) study sensitivity analysis of the uncapacitated single level economic lot-sizing problem. Chen (2004) shows the optimal hedging point policy for stochastic two-item, periodic review, single facility, flexible manufacturing systems. Later Feng (2005) extends the control policy in Chen (2004) to multiple products system and the results in Van Hoesel and Wagelmans (1993) to allow for backlogging. These outcomes were also published in Chen et al. (2008) and in Feng et al. (2011).

The vendor managed inventory partnership (VMIP) problem that in-

volves managing order fulfillment to improve collaboration between suppliers and their distribution partners also has been studied. E.g., Yang et al. (2012) obtain an optimal replenishment policy for the single-warehouse multi-retailer problem. Later, Sadeghi et al. (2013) investigate a constrained multi-vendor, multi-retailer supply chain problem and find the optimal order quantities and shipment frequency received by retailers and vendors to minimize the total inventory cost. However in VMIP the problem of inventory sharing is usually not considered.

In this research, we consider the problem of inventory sharing between independent retailers. This research aligns with the growing trend among businesses, which is the sharing of resources that aims to benefit every party's self-interest. Such resource sharing is often supported by the use of information technology and is widely regarded as part of sharing economy. One such example is inventory sharing (also known as inventory pooling) between independent firms. Inventory sharing helps businesses reduce stock-outs and/or inventory investment by deriving the benefit of risk pooling. Many practices of inventory sharing are reported. Applications include spare parts in semi-conductor industry (Kranenburg, 2006), auto dealers of General Motors, Ford and Honda (Anupindi and Bassok, 1999), distributors of machine tool builder Okuma (Anupindi et al., 2001) and major power tool manufacturer Bosch (Rudi et al., 2001). Online platforms also exist to support such sharing, such as www.warehousetwo.com, which are created for inventory sharing between durable goods manufacturers and their authorized distributors. Registered members form "communities" for sharing specific products and are then allowed to access real-time inventory availability of fellow members. It is worthwhile to mention that we consider the demand is unchanged by retailer's ordering and sharing. This assumption is valid, for example, if retailers form non-competitive alliance among themselves or establish an agreement with

customers for a fixed purchase channel. It is possible to establish such setups that target on specific locations, shopping preferences, membership, loyalty, privileged information for customers, etc. As a result, the demand of a retailer is unchanged for the inventory management. Therefore it is reasonable for retailers to consider sharing their inventories.

We consider the multi-period, multi-ordering version of the n ($n \geq 2$) independent retailers inventory sharing problem. In contrast to this work, existing literature primarily focuses on single-period (or repeated single-period) problem. The problem is characterized by one-time ordering and demand realization, with no demand backlog. The typical setting can be described as follows. There are n independent retailers whose common objective is to maximize (minimize) her own profit (cost). Each retailer first selects her own inventory level of an identical product. After the demand is realized, the retailers fulfill their own demands with their on-hand inventories. Thereafter, some may be left with surplus stocks (residual inventories) while others experience unfulfilled demands (residual demands). The retailers with surpluses can trade with those with deficits. Such same-echelon stock transfers are also known as lateral transshipments. Any excess inventories are typically salvaged at an amount less than the unit purchase cost. In the repeated single-period problem, the above problem recurs over time without permitting for demand backlogs in successive periods.

Paterson et al. (2011) provide a comprehensive survey on the literature on lateral transshipment where the related literature are categorized according to the timing of transshipments (reactive versus proactive), and by characteristics related to the ordering policy, inventory network and the underlying modeling approach. Notably, they provide the scope of what “lateral transshipment” encompasses. First, the transshipment must take place between inventory locations of the same echelon. Second, such

transshipment include both the transfers of the physical products and the demand (where demand can be directly satisfied from a different location). On the other hand, lateral transshipments exclude reallocation of in-transit items before their arrival at their original destination in the context of the multi-echelon distribution networks; it also exclude emergency shipments from the upstream supplier or any location of a different echelon. Given the above scope, the inventory sharing problem between independent retailers can be regarded as part of lateral transshipment, but with an additional specification that the stocks or demands are transferred between independent entities who focus on their own self-interest.

In this research, inventory sharing policies and models are developed. Current literature on inventory sharing focuses on single-period or repeated single-period (newsvendor-like) problems. E.g., Anupindi et al. (2001), Rudi et al. (2001) and Granot and Sošić (2003). Furthermore, a large majority is restricted to two retailers. E.g., Tagaras (1989), Herer and Rashit (1999) and Yang and Qin (2007). From the aspect of information sharing in VMIP problems. Examples include Yu et al. (2001) and Angulo et al. (2004). Yan and Zhao (2011) first consider asymmetric information in a two-retailer model. They assume each retailer holds private demand distribution parameters. Yan and Zhao (2015) extend their results to multi-retailer model. However they still assume the type of demand distribution is known. Thus the optimality is achieved in terms of “averages” (expectation). This research attempts to bridge these gaps in the inventory sharing literature by considering the multi-period, multi-ordering problem that permits more than two retailers. In addition, we formulate a range of policies and models that are tailored to the different extent of information sharing between the retailers. We demonstrate that the extent of information sharing can lead to considerably different optimality. From the modeling point of view, they lead to significantly different analysis and tractability. Specifically, we

consider the following two levels of information sharing:

1. Limited information sharing (sharing of information on only current inventory levels) (Chapter 2)
2. Complete information sharing (sharing of information on current and future inventory levels, demands, ordering quantities and ordering timing) (Chapter 3)

In the first part (limited information sharing), there is lack of information (ordering and demands) for each individual retailer to support their optimal ordering decisions. Thus there is no consideration of optimal ordering in this part; with a given ordering policy, we formulate a sharing policy that based on stock lending. In the second part (complete information sharing), given the available information, the objective is how to rely on these information independently and collectively as a system to improve individual and system performance in expectation. Besides coordination through sharing of information, the multi-period problem (in contrast to single-period problems) presents an opportunity to coordinate the timing of their individual order cycles to enhance system-wide stock availability. In this aspect, we consider the “even spreading” of order timing (known as balanced ordering) that increases probabilities of stock surpluses among retailers when some face shortages.

1.2 Objectives

As investigated in the literature review the multi-period, multi-retailer vendor managed inventory partnership problem (VMIP) has been studied. Moreover the single period and repeated period inventory sharing problems and the information asymmetry in VMIP problems have also been well studied separately. Furthermore most existing studies focus on

the optimality in terms of optimizing expected profits and optimization are often carried out periodically. Hence the allocation problem has also been identified and studied. However few of the existing literature discuss about information sharing in the inventory sharing problem. In summary, information in the inventory sharing problem has not been characterized and the different degrees of information sharing in inventory sharing problem have not been studied to control the risk or to enhance the sharing efficiency.

The objective of this research is to develop inventory sharing policies and corresponding analytical models with different degrees of information for the multi-period n -retailer ($n \geq 2$) inventory sharing problem where the demand is unchanged by retailer's ordering and sharing. With limited information, we aim to analytically optimize centralized systems and the controllable bound for transshipment frequency in decentralized system for all possible demands. With complete information, we perform analysis to demonstrate the benefits and desirable properties of the policies and validate their respective optimality in terms of expected values (average) numerically. We aim to address the following challenges in the model development and analysis:

1. Optimality in inventory sharing

With demand distribution, the inventory sharing problem is usually translated into the optimization problem for the expected value. However, with limited information there is no way to estimate the expectations. Thus the challenge is how do we define the optimization problem for inventory sharing with limited information. Particularly, what is the optimal for multiple periods?

2. Models should capture stock-out risks of retailers who share out their stocks

A key feature of multi-period sharing is the consequential risk of stock-outs/left-overs in subsequent periods if a retailer shares out her stock/unfulfilled demand. In other words, the result holds for all possibilities. Thus, when formulating and modeling a sharing policy, the key challenge is to quantify the risk into a form of compensation to these retailers. How do we overcome this challenge?

3. Incorporate the different extent of information sharing and their implications

How should the sharing policies and models be developed depends on the types of information that is shared between the retailers. What are the information that can be shared (or not shared) and how can the policies and the corresponding models be developed that can tailor to the different types of information?

4. Sharing policies that satisfy the essential optimality conditions

In many inventory sharing problems, additional profits from sharing need to be distributed among the participating retailers. In such problems, the central feature of most policies is the profit allocation rule, which allocates the resulting profits among the retailers according to a set of predetermined rules. To maximize the profits from sharing, it is sufficient that the rule achieves two conditions. First, retailers must be willing to share all their excess demands or inventories. Second, retailers must be willing to share their inventory with other participating retailers. Huang and Sošić (2010a) refer to the conditions as complete sharing condition and full participation condition respectively. In the existing literature (as will be discussed in this report), none of the allocation rules can achieve both conditions concurrently without resorting to third-party intervention (e.g. the retailers' supplier provides subsidy to induce both conditions,

in Yan and Zhao 2015). Furthermore, Granot and Sošić (2003) show that no complete sharing core allocation rule exists for $(n \geq 4)$ in the conventional game framework. The aforementioned difficulties on the allocation rule are based on the single-period literature. In the multi-period setting, inter-retailers' interaction is more complicated since decisions have to rely on the entire history of previous ordering and demands. In such a context, how do we formulate a policy and an allocation rule that lead to optimality?

5. Transshipment frequency

In decentralized systems transshipment cost is increased if there is a high transshipment frequency. The problem is: how could retailers control the transshipment frequency in the multiple periods when faced with limited information?

6. Sharing without system-wide cooperation

A realistic problem for independent n retailers sharing is that the system-wide alliance is difficult to form. Without system-wide alliance, how do we model the sharing process?

7. Nash equilibria of the sharing system

In sharing inventories, inventory choices of the retailers involve strategic interaction. In particular, a retailer who carries a large inventory would make it easier for others to utilize this inventory if stock-outs occur. Alternatively, small inventories held by one retailer present opportunities for others to share out their excess stocks. Given this interaction of strategies in the ordering and inventory decisions, how does the Nash equilibria of the resulting game be captured and addressed? Or even, can a sharing policy be formulated so as to avoid the complexity of ensuring the equilibria?

8. Incorporate coordinated ordering timing (balanced ordering) and their implications

In the single-period problem, all retailers only order once before the selling season. In the multi-period problem, retailers can choose to order at different timings, which implies their respective inventory levels can be at “peaks” and “troughs” at different points of time. Clearly, the effectiveness of sharing depends on system-wide stock availability. Can the ordering and replenishment between retailers be coordinated to enhance system-wide stock availability? If so, how best can this be carried out?

1.3 Research Methodology

In developing the analytical models and in proving their properties and optimality, the approaches are predominantly based on methodologies in the discipline of operations research. In particular, a few sub-fields are involved. Game theory is applied in modeling the sharing mechanism and profit allocation (Chapter 2), as they encompass strategic interactions between the retailers. In addition, inventory theories (Chapter 3) are involved in modeling the inventory control, specifically in deciding the order quantity and in characterizing the forecasting process. Additionally, computer-based simulations (Monte Carlo and Markov Chain Monte Carlo) are also employed to support the optimization using the developed analytical models (Chapter 3). Numerical experiments were also carried out using the simulation to illustrate the insights and benefits of the policies and models (Chapter 3).

1.4 Research Scope

As this research problem could involve a wide range of planning variables and could also implicate other supply chain entities, we clarify the high-level aspects of the research scope as follows.

- The inventory sharing occurs among the same-echelon supply chain players, without considering the direct involvement of supplier(s) or immediate customer(s).
- Even though the term “retailers” are used extensively in this report, most models and analyses are applicable to other forms of inventories, e.g. raw materials of manufacturing firms and stocks stored in distribution centers.
- In this research, we consider “reactive transshipments” for the inventory sharing, which are transshipments that occur in response to impending or actual occurrences of stock-outs. This is opposed to “proactive transshipments” in which stocks are re-distributed amongst retailers in advance, at predetermined points in time; refer to recent works such as Bouma and Teunter (2014), and Bouma and Teunter (2016). See Paterson et al. (2011) for a review of the research on both types of transshipments.
- We restrict the scope of information sharing to those directly related to inventory decisions, including inventory status, demands, and ordering quantities and timing. Furthermore, we do not consider or discuss the related technical feasibility such as information transmission and visibility between the retailers. Nevertheless, as the requirements on information technology for the proposed policies are not high by today’s technological standard, we believe that all the information sharing considered in this research are technically

feasible.

1.5 Organization of the Thesis

The subsequent chapters are organized as follows:

In Chapter 2, we consider the sharing of limited information in that only inventory information is shared between the retailers. The policy developed can be regarded as stock lending and borrowing, where the profit brought about by the sharing are allocated to the retailers according to their net inventory quantity that are loaned out. We prove that the allocation rule attains the key conditions of complete sharing and full inventory pooling for centralized systems. For the decentralized system, we develop a transshipment rule for support decision on “from who to whom” and establish an upper bound of transshipment frequency for this rule; in addition, we show that it leads to complete sharing and guarantees additional profit for every retailer under certain cost structure.

In Chapter 3, we consider the “complete information sharing” problem where retailers share information on demands and order quantities (in addition to inventory information in Chapter 2). We develop a multi-period newsvendor-based model that optimizes each retailer’s order quantity. Due to the complexity in computing the optimal solution (as will be demonstrated) and the difficulty in attaining or approximating the joint demand distribution of the retailers (which a key model input), we develop a simulation approach that to solve the analytical model. We extend the Leibniz Rule to allow for general demand distributions, which allows demands to be characterized through demand sampling and thus avoiding the need for the joint demand distribution. We then show that balanced ordering can increase system-wide stock availability, enhance system efficiency and improves model tractability. Furthermore, we formulate a sharing rule

that pairs the sharing retailers, which significantly reduces the otherwise intractably large solution space.

In Chapter 4, which is the concluding chapter, first a summary of the thesis and the contributions to literature are presented. Then, the opportunities for future research are discussed.

Chapter 2

Models with Sharing of Inventory Information

2.1 Introduction

In this chapter, we consider the multi-period, n -retailer inventory sharing problem with limited information sharing, i.e., the sharing of only inventory information. We formulate a sharing policy that is based on the concept of stock lending. The lending concept avoids the difficulty of quantifying the risk of subsequent stock-outs if one shares out their stocks. Under the policy, the stock-out retailers are compensated according to their net inventory quantity that are loaned out. We first formulate the allocation rule for distributing the increase in profit caused by decrease in backorder and holding cost. We prove that the rule attains complete sharing and full participation, which are essential conditions for optimality. Next, we include lateral transshipment cost in our analysis and develop a transshipment rule for matching "surplus" retailers with those in "deficit". We establish an upper bound of transshipment frequency for this rule, and show that under specific cost structure, it leads to complete sharing and guarantees additional profit for every retailer. Our policy possesses some

desirable properties. For example, it does not require approximation of demand distributions as model input and is not restricted to any specific ordering policy; it is also more stable than most other policies as it deters deviation of any subcoalition of retailers (rather than deviation of just a single player).

In the next section, we review related papers and highlights this chapter's contribution to the literature. In section 2.3, we describe the problem setting, state the underlying assumptions and present the fundamental workings of the lending system. In Section 2.4, we derive the basic properties of the lending system. Then in Section 2.5, we discuss the development of the allocation rule for centralized systems. In Section 2.6, we develop the transshipment rule that controls the transshipment frequency. Finally, we offer our concluding remarks in Section 2.7.

2.2 Literature Review and Contributions

In the following, we review related papers and highlight this chapter's contributions to the literature. The system-optimality of inventory sharing has been studied by as early as Krishnan and Rao (1965), who build a model for a single-period system with two locations. Other early papers include Karmarkar and Patel (1977) and Karmarkar (1979), who examine general models for multi-location inventory systems. In the independent-retailer problem, retailers only participate in such transshipments if it can benefit their self-interest. Hence much of the literature emphasizes on strategic interactions between rational retailers, and the problem is predominantly modeled as newsvendor game. In a typical setting, the retailers first unilaterally determine their own order quantities and the analysis attempts to induce subsequent sharing. The sharing comprises two main phases. The retailers first cooperatively determine a transshipment pattern for

distributing the residual inventories and in the second phase, divide the residual profits according to an allocation rule specified at the beginning of the game. Most of the literature focuses on two-retailer problems, such as Tagaras (1989); Herer and Rashit (1999); Rudi et al. (2001); Zhao et al. (2006); Yang and Qin (2007); Hu et al. (2007); Hezarkhani and Kubiak (2010). One example of the two-retailer model is the use of transshipment pricing as an allocation rule to coordinate the retailers (Rudi et al. 2001 and Hu et al. 2007). Specifically, the residual profit is allocated among the retailers according to a priori determined price for each unit transshipped. The analyses are limited to two retailers, as the modeling of more retailers significantly complicates the setting of the transshipment prices as well as how residual inventories can be optimally matched with residual demands (as detailed in Huang and Sošić 2010a). In addition, Nash equilibrium of the transshipment price does not always exist even in a two-retailer system (see Anupindi and Bassok 1999 and Hu et al. 2007) and is difficult to be established in an n -retailer ($n > 2$) system (see Huang and Sošić 2010a).

Far fewer papers have considered the more realistic n -retailer ($n > 2$) sharing problem due to its complexities. In this stream of literature, Anupindi et al. (2001) propose an allocation rule based on the dual prices for the transshipment problem. Other models that are related to the same rule include Granot and Sošić (2003), Huang and Sošić (2010b), and Yan and Zhao (2015). Granot and Sošić (2003) call it the *dual allocation* rule. For the $n > 2$ problem, it is sufficient that the allocation rule satisfy two conditions in order for the residual profits to be maximized. First, retailers are motivated to share all their residual demands or residual supplies. Second, retailers share inventory with everyone else in the system; that is, from the game theory viewpoint, they form only grand coalition (and do not form subcoalitions). We follow Huang and Sošić (2010b) in referring the above conditions as *full inventory pooling* and *full participation* respec-

tively. However, these conditions cannot be easily achieved simultaneously. Anupindi et al. (2001) automatically attain the full inventory pooling condition by implicitly assume it in their analysis. In addition, they show that the dual allocation rule satisfies the full participation condition. Huang and Sošić (2010a) prove that the dual allocation only results in a first-best solution (i.e., system-optimality) under certain conditions and it requires the implicit assumption of full inventory pooling. Granot and Sošić (2003) relax the full inventory pooling assumption in which retailers are allowed to hold back some of their residual stocks. By relying on the results of Owen (1975) on linear production game, they point out that the dual allocation rule is a core allocation rule; however, if retailers are allowed to withhold residual stocks, the dual allocation may not be able to induce complete sharing as retailers may hold back some residuals to increase their allocations. Furthermore, they prove that coalitionally monotonic allocation rules (including the fractional rule and Shapley value) satisfy the complete sharing or value preserving condition, but these rules are not core allocation rules in general (i.e., they do not achieve full participation). Huang and Sošić (2010b) show that in a repeated game setting with dual allocation, retailers may completely share their residuals if their emphasis on future payoffs is sufficiently high.

The above review shows that none of the existing allocation rules can achieve both conditions concurrently. Furthermore, Granot and Sošić (2003) show that in the conventional game framework, no completely sharing core allocation rule exists for an n -retailer inventory sharing game for $n \geq 4$. Yan and Zhao (2015) design a coordination policy that achieves both conditions of full participation and complete sharing (full inventory pooling). Their paper includes a mechanism to ensure system stability in terms of full participation, albeit in different forms – they induce it by providing third-part subsidy to deter exit of grand coalition. This

chapter differs from theirs in that they study the single-period problem and therefore, many of the fundamental considerations and assumptions, such as allowing for demand backlogs in succeeding periods, are considerably different.

In the multi-period setting, a major challenge from the modeling viewpoint lies with the need to incorporate each retailer's risk of stock-out if she shares out her stock. To compensate for such risk, we formulate a policy that constitutes a system of borrowing and lending. The fundamental parameter that needs to be constantly tracked in this policy is the *sharing quantity* of each retailer; this is the stock quantity that is loaned out *net* of the quantity borrowed. For example, a retailer who faces stock-out and concurrently holds a positive sharing quantity would imply that her stock-out (or part of it) is caused by her prior lending. Under the allocation rule, the retailer will be compensated based on her sharing quantity so that she will not be worse off from the loan.

In this chapter, we assume the demands are exogenous; in particular, they are not affected by retailers' ordering and sharing. We start our analysis by first describing how the ordering in the absence of sharing (referred to as the *non-sharing* case) can be conveniently reconstructed from the lending system model. We then consider only the backorder cost and holding cost in the lending system, while deferring the consideration of lateral transshipment cost. Note that we can also regard this part of the analysis as the case of zero (or negligible) transshipment cost; this case can be applied directly in a centralized inventory system, where all inventories are stored at a central location (such as in an e-commerce fulfillment center serving different e-tailers). We prove that complete sharing leads to minimum total backorders and total inventory level in the lending system. We then formulate the allocation rule for distributing the profit among the retailers where we focus on achieving system optimality and full inventory

pooling. Next we include lateral transshipment in our analysis, which is incurred in decentralized inventory system where inventories are held in locations that are of significant distance apart. Based on our result in the centralized case where complete sharing leads to minimum total backorders and inventories, our objective is to achieve complete sharing while efficiently limiting the frequency of lateral transshipments. With the aim to control transshipment frequency, we develop a transshipment rule for matching the “surplus” retailers with “deficit” ones. We then establish an upper bound in that the transshipment frequency is bounded by a function of the reduction of total backorders. More significantly, under a condition based on the unit transshipment cost relative to the unit backorder and holding costs, we show that the transshipment rule guarantees complete sharing and non-negative additional profit for every retailer.

We summarize this chapter’s contributions to the literature as follows.

1. As discussed earlier, the existing literature on inventory sharing is predominantly based on single-period problems, of which most consider the two-retailer problem. To our knowledge, this is the first that develops a sharing rule for the multi-period, multi-ordering ($n \geq 2$) independent-retailer problem. The policy takes into account the key challenge in such setting, which is to address the risk of future stock-outs when sharing out inventories.
2. We show that the policy leads to complete sharing and full inventory pooling. In addition, the policy possesses some nice properties that facilitate real-life applications, which we state in the following:
 - a. The results in this chapter are independent of the demand distributions and their implicit demand uncertainties. From the application standpoint, this implies that the policy does not require any demand forecasts or approximations as model inputs.

- b. The sharing rule and its allocation rule are applicable to different ordering policies because we do not assume retailers adopt any specific ordering policy.
 - c. For centralized systems (or zero transshipment cost), the payoff received by each retailer under the policy's allocation rule (through reduction in backorders and inventories), is independent of how the quantities of surplus inventories are allocated to each stock-out retailer. This considerably reduces the complexity in determining the "transshipment pattern" among the retailers. Furthermore, each retailer's payoff will not be improved even if a subcoalition of retailers change their strategies. This is related to strong Nash equilibrium and it is more stable than most other allocation rules that are usually in Nash equilibrium, which is only valid for strategy changes of a single retailer.
3. Building upon the results of the centralized case, we develop a simple transshipment rule for the decentralized case and derive an upper bound of its lateral transshipment frequency. More significantly, the bound implies non-negative residual profits in the existence of positive transshipment costs if the policy is adopted under specific condition of the transshipment cost.

2.3 Problem

Consider a set of N retailers that sell a common product. Each retailer faces her own uncertain demand in every discrete time unit. The demands are exogenous; in particular, they are not affected by sharing and ordering. We assume unfulfilled demands are backlogged and all demands are eventually fulfilled. In every time unit, each retailer suffers a backorder penalty cost for each unit of backorder. Additionally, each unit of inventory

held incurs a unit holding cost in every time unit. *Inventory* herein refers to on-hand inventory that are ready to be transshipped when needed and agreed upon.

The retailers unilaterally place their orders, i.e., orders are placed at their own selected timing and quantity. To the retailers in the system, the objective is to minimize their own total cost, comprising the inventory holding cost and backorder cost. From a system perspective, our aim is to minimize the system-wide backorders and inventory holding. To achieve the optimality, our goal is to achieve complete sharing and to control the frequency of lateral transshipments.

In addition, we assume the retailers share all information of on-hand inventories and stock-outs. A retailer with positive on-hand inventory can share with other retailers who suffer from stock-outs. The sharing scheme is equivalent to lending of stocks. The essence of the scheme is the sharing quantity, which is a value assigned to every retailer. It is the net quantity of stocks of a retailer that is loaned out to other retailers. At the start of the scheme, all retailers start with zero sharing quantity. As an example, suppose retailer i encounters stock-out of 1 unit and meanwhile retailer j has surplus on-hand inventory. Retailer j transships the 1 unit of stock to retailer i , which results in retailer i 's inventory to increase by 1 unit and sharing quantity to correspondingly decrease by 1 unit. On the other hand, retailer j 's on-hand inventory drops by 1 unit while sharing quantity increases by 1 unit. We further assume instantaneous transshipment (or that it takes negligible duration relative to the length of the time unit).

As defined earlier, complete sharing means that retailers are willing to share their inventories at all available opportunities (till no sharing opportunity left). In the next section, we will show that complete sharing leads to minimum total backorders plus inventory holding in the lending system. Given the significance of complete sharing, our key goal in this

paper is to achieve complete sharing in lending system; subsequently, we aim to develop a transshipment rule that limits the lateral transshipment cost (in the decentralized case).

2.4 Basic Properties of Lending System

In this section, we characterize the lending system and derive its basic properties. We consider only the backorder and inventory holding costs. (We will include lateral transshipment cost in our analysis in Section 2.6.) As stated earlier, the analysis that excludes the transshipment cost can be regarded as the case of centralized inventory system where transshipment cost is zero.

We index the discrete time by $t = 1, 2, 3, \dots$ and let t_0 be the current time. In addition, we let $\mathcal{N} = \{1, 2, 3, \dots, N\}$ be the set of N retailers. Let D_i^t be the realized demand of retailer i in time t , O_i^t be the ordered quantity of retailer i that arrives in time t , and I_i^t be the inventory level of retailer i at the end of time t (after the replenishments have arrived and the demands are realized). The sharing quantity of retailer i in time t is denoted by s_i^t which represents the net amount of loaned out or borrowed inventory by time t . For convenience of discussion, we assume that the sharing quantity of retailer i at the beginning of $t = 1$ $s_i^0 = 0$, and that I_i^0 denotes the corresponding initial inventory level.

Property 1. *For any $t \geq 1$, D_i^t , O_i^t , I_i^t and s_i^t as defined above, satisfy the following conditions.*

1. $\sum_{i=1}^N s_i^t = 0$;
2. For each i , $I_i^t + s_i^t = I_i^0 + \sum_{j=1}^t O_i^j - \sum_{j=1}^t D_i^j$;
3. For each i , $I_i^t + s_i^t - s_i^{t-1} \geq I_i^t \geq 0$ or $I_i^t + s_i^t - s_i^{t-1} \leq I_i^t \leq 0$.

Proof. (1) Initially $s_i^0 = 0$, and thus $\sum_{i=1}^N s_i^0 = 0$. Sharing between any pair of retailers causes sharing quantity of one to increase and another to decrease by the same value. That is, the sharing changes the value of s_i^t , but $\sum_{i=1}^N s_i^t$ remains unchanged. Hence it means $\sum_{i=1}^N s_i^t = 0$. (2) Initially $I_i^0 + s_i^0 = I_i^0$. Sharing of any quantity and in any time t does not change the value of $I_i^t + s_i^t$, because any change in I_i^t results in an equal and opposite change in s_i^t . Its value depends only on the accumulated replenishments and demands since time $t \geq 1$. Therefore $I_i^t + s_i^t = I_i^0 + \sum_{j=1}^t O_i^j - \sum_{j=1}^t D_i^j$. (3) Sharing can only decrease the absolute value of $I_i^{t-1} + O_i^t - D_i^t = I_i^t + s_i^t - s_i^{t-1}$ and cannot change its sign. $\square$

Conversely, if conditions (1),(2) and (3) in Property 1 are satisfied for $t \leq t'$, then the inventory balance equation for time $t \leq t'$ can be re-expressed as $I_i^{t-1} + O_i^t - D_i^t = I_i^t + s_i^t - s_i^{t-1}$. Note that the difference $s_i^t - s_i^{t-1}$ uniquely determines the net quantity of stocks loaned out by retailer i in time t . If $I_i^t + s_i^t - s_i^{t-1} \geq 0$, retailer i lends $s_i^t - s_i^{t-1} \geq 0$ units such that her inventory level becomes I_i^t ; if $I_i^t + s_i^t - s_i^{t-1} \leq 0$, retailer i borrows $s_i^{t-1} - s_i^t \geq 0$ such that her inventory level becomes I_i^t . The non-sharing case, which we define as the *base-case* in which no sharing is performed, can be regarded as a special case, where $s_i^t = 0$ for all i and t . In addition, note that conditions (1) and (3) are satisfied for the non-sharing case, and its inventory balanced equation can be expressed by condition (2).

As illustrated above, the sharing quantity plays a key role in the lending system, as it can represent an *ownership* of stocks. Specifically, when retailers share their inventories, it can be viewed as an exchange between the physical stocks and the sharing quantity. For example, retailer i has stocks I_i^t and sharing quantity s_i^t . From the ownership perspective, the retailer i owns a total of $I_i^t + s_i^t$ of goods. By Property 1 $I_i^t + s_i^t$ is unchanged by sharing. This means that when retailers lend their stocks, their own-

erships will not be altered. With this property, as will be shown, $I_i^t + s_i^t$ allows the inventory level and ordering in the non-sharing case (base case) to be conveniently characterized (i.e., reconstructed) in the lending system model. The reconstruction serves two purposes. First, it enables us to prove the superiority of the lending system over the non-sharing case. Second, it is an essential function in the formulation of the allocation rule for determining the amount of savings (resulted from the sharing) which are then allocated to every retailer.

We let $\mathbb{P}_{D_i^t}$ be the distribution of demand D_i^t that retailer i may utilize for her forecast when deciding on her order quantity. As an extension of our prior assumption of exogenous demand, the distribution $\mathbb{P}_{D_i^t}$ is not affected by the sharing and ordering i.e., values of $s_j^{t'}$ and $O_j^{t'}$ ($j \in \mathcal{N}$ and $t' \geq 0$). (Note that D_i^t and $\mathbb{P}_{D_i^t}$ are fixed but unknown before realization.) Indeed the results we obtain in this paper are independent of D_i^t and $\mathbb{P}_{D_i^t}$. Finally, assume order $O_i^{t,L}$ is made in time t and is replenished in time $t + L$, where $L > 0$ is the replenishment lead time. Retailers determine $O_i^{t,L}$, s_i^t and I_i^t in time t . Note that $t = 0$ means the initial value.

Now we consider the non-sharing case, where $s_i^t = 0$ for every i and t . Assume the initial orders $O_i^{0,L}$ are fixed. The ordering function for retailer i in the non-sharing case $O_i^{t,L}(\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}, I_i^t)$ is a function of $\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}$ and I_i^t . By the inventory balanced equation, I_i^t is in turn determined by $D_i^{t'}$ and $O_i^{t'}$ ($t' \leq t$). It means that $O_i^{t,L}(\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}, I_i^t)$ uniquely determines the orders in the non-sharing case. Next we consider reconstructing the orders of the non-sharing case into the corresponding orders of the sharing case by using the ordering function $O_i^{t,L}(\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}, I_i^t + s_i^t)$.

Property 2. *For any non-sharing ordering policy $O_i^{t,L}(\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}, I_i^t)$, there is an ordering policy $O_i^{t,L}(\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}, I_i^t + s_i^t)$ such that the orders $O_i^{t,L}(\mathbb{P}_{D_i^{t+1}}, \mathbb{P}_{D_i^{t+2}}, \dots, \mathbb{P}_{D_i^{t+L}}, I_i^t)$ for retailer i in the non-sharing case*

can be reconstructed.

Proof. First initial orders $O_i^{0,L}$ are fixed, there is no difference for all cases before time $t = 1$. Suppose at time m , for $t \leq m$, the order $O_i^{t-1,L}$ are same (equal to the order in non-sharing case). Thus it implies O_i^t are same. By Property 1, condition (2): $I_i^t + s_i^t = I_i^0 + \sum_{j=1}^t O_i^j - \sum_{j=1}^t D_i^j$ always hold. By the inventory balance equation $I_i^t + s_i^t$ is the value of I_i^t in the non-sharing case for $t \leq m$. Therefore, $O_i^{m,L}(\mathbb{P}_{D_i^{m+1}}, \mathbb{P}_{D_i^{m+2}}, \dots, \mathbb{P}_{D_i^{m+L}}, I_i^m + s_i^m)$ is the order $O_i^{m,L}(\mathbb{P}_{D_i^{m+1}}, \mathbb{P}_{D_i^{m+2}}, \dots, \mathbb{P}_{D_i^{m+L}}, I_i^m)$ in the non-sharing case. By mathematical induction, the orders can be reconstructed for all time t . $\square$

In the rest of this paper, we characterize the lending system such that the non-sharing case can be conveniently reconstructed, with both sharing and non-sharing cases subject to the same D_i^t and O_i^t . Therefore in the lending system, $\sum_{i=1}^N s_i^t = 0$; $I_i^t + s_i^t$, $\sum_{i=1}^N I_i^t$, $(\sum_{i=1}^N I_i^t)_-$ and $(\sum_{i=1}^N I_i^t)_+$ are fixed for time t (where we define $x_+ = \max(x, 0)$ and $x_- = -\min(x, 0)$). Our aim is to determine the sharing actions in every time t , i.e., the appropriate changes of s_i^t (and I_i^t) that satisfy the conditions (1), (2) and (3) in Property 1.

Property 3. *In the lending system, complete sharing leads to minimum total backorders and total inventory level across all retailers in time t .*

Proof. By the definition of complete sharing, sharing continues to take place till there is no sharing opportunity left in any time t . In other words, in complete sharing, there is no $I_i^t I_j^t < 0$ for any $i \neq j$ and any t . First, we show that complete sharing can exist in lending system by construction. In order to construct the complete sharing, we only need to add some restrictions of sharing into the construction of lending system. Suppose for current time t_0 , $s_i^{t_0} = s_i^{t_0-1}$ (which implies no sharing in time t_0). If there is no $I_i^{t_0} I_j^{t_0} < 0$, then it has done. Otherwise $I_i^{t_0} > 0$ and $I_j^{t_0} < 0$, we can let $s_i^{t_0} = s_i^{t_0-1} + 1$ and $s_j^{t_0} = s_j^{t_0-1} - 1$, while $I_i^{t_0} = I_i^{t_0} - 1$ and $I_j^{t_0} = I_j^{t_0} + 1$. If

this process is repeated and since all inventory levels are finite, eventually there is no $I_i^{t_0} I_j^{t_0} < 0$.

In general, the total backorders is bounded by $\sum_{i=1}^N (I_i^t)_- \geq (\sum_{i=1}^N I_i^t)_-$, where $(\sum_{i=1}^N I_i^t)_-$ is the system's fixed lower bound. Under complete sharing, there is no $I_i^t I_j^t < 0$, i.e., no different signs for I_i^t . Therefore, the backorders in complete sharing equals the lower bound, i.e., $\sum_{i=1}^N (I_i^t)_- = (\sum_{i=1}^N I_i^t)_-$. Similarly, the total inventory under complete sharing equals the corresponding lower bound, i.e., $\sum_{i=1}^N (I_i^t)_+ = (\sum_{i=1}^N I_i^t)_+$. $\square$

Since the ownership $I_i^t + s_i^t$ is fixed in the lending system, its value can be employed to represent the inventory level in the non-sharing case. By Property 3, the optimal total backorders and inventories under complete sharing is no worse than all other cases. Particularly they are no worse than the base case. Thus our next step is to formulate an allocation rule to distribute the resulting savings (from sharing). By reconstructing the inventory level of the non-sharing case, we can then derive the total savings generated by the sharing, and then determine how to allocate them among the retailers, which we discuss in the next section.

2.5 Allocation Rules

In this section we consider the sharing and allocation problem in current time t_0 as a cooperative game. We focus on the allocation rules that leads to full inventory pooling, i.e., retailers are willing to share their on-hand inventories. (Note that in this context, complete sharing can be considered as the subsequent stage of decision after full inventory pooling where the retailers share all their residuals with others who are in deficit.)

First, we introduce the cooperative game and the underlying assumptions. We let $\mathcal{N}$ be the set of players called grand coalition, $|\mathcal{N}| = N$. We assume $\mathcal{N}$ is fixed. In addition, we let $f : 2^{\mathcal{N}} - \{\emptyset\} \rightarrow \mathbb{R}$ be a characteristic

function and $\mathcal{G}$ be the class of all cooperative games. Without ambiguity and for brevity, we use f to denote game $(\mathcal{N}, f)$. Next, we introduce the games involved in the model. In the current time t_0 , the difference between the initial on-hand inventory $I_i^{t_0}$ (before any sharing in time t_0 , i.e., $s_i^{t_0} = s_i^{t_0-1}$) and ownership of stocks $I_i^{t_0} + s_i^{t_0}$ is fixed. The sharing in time t_0 is based on the initial on-hand inventory $I_i^{t_0}$ (when $s_i^{t_0} = s_i^{t_0-1}$). In the subsequent formulations, we follow the three-stage sharing scheme in Granot and Sošić (2003). We let I'_i be the amount of on-hand inventory (or excess demand) that retailer i is willing to share. Hence $I'_i \in [0, I_i^{t_0}]$ if $I_i^{t_0} \geq 0$; otherwise $I'_i \in [I_i^{t_0}, 0]$. Thus the retailer can be considered as the player and I'_i can be viewed as a sharing strategy of retailer i . Further, we let $I' = (I'_1, \dots, I'_N)$ be the strategy profile and define the third stage of the sharing process in time t_0 as a cooperative game with parameter I' . In other words, the third stage is a function that maps strategy profile to a game. As defined in Granot and Sošić (2003) the characteristic function of transshipment game $f_1(S; I')$ is the solution of the following linear program $L(S, I')$:

$$\begin{aligned} & \max \sum_{i \in S} \sum_{j \neq i \text{ and } j \in S} p_{ij} x_{ij} \\ & \left\{ \begin{array}{ll} \sum_{i \in S, i \neq j} x_{ij} \leq (I'_j)_- & \text{for all } j \in S \\ \sum_{j \in S, i \neq j} x_{ij} \leq (I'_i)_+ & \text{for all } i \in S \\ x_{ij} \geq 0 & \text{for all } i, j \in S \text{ and } i \neq j \end{array} \right. \end{aligned}$$

where p_{ij} is the profit of transshipping a unit from i to j . By the definition of transshipment game $f_1(\cdot; I')$, it implies $f_1(S; I') \geq 0$ and $f_1(\mathcal{N}; I') \geq f_1(S; I')$. Let the class of all transshipment games $f_1(\cdot; I')$ be $\mathcal{T}_1$. By Property 3, it is not difficult to see if all p_{ij} are the same and positive ($p_{ij} = c > 0$), then $f_1(S; I')$ is equivalent to $\sum_{i \in S} (cI'_i)_- - (\sum_{i \in S} cI'_i)_-$. Since I' is the

parameter, the transshipment games with equally positive p_{ij} , e.g., the games without transshipment cost or in centralized systems, are all games with form $f_2(S; I') = \sum_{i \in S} (I'_i)_- - (\sum_{i \in S} I'_i)_-$. Let $\mathcal{T}_2$ be the class of all games with form $f_2(S; I') = \sum_{i \in S} (I'_i)_- - (\sum_{i \in S} I'_i)_-$. The payoff $f_2(S; I')$ can be considered as the benefit of sharing in time t_0 in terms of backorders (or on-hand inventory levels) in subcoalition S with I' . Due to limited information and uncertainties, the payoff function of the game focuses on retailers' myopic benefits in current time t_0 .

Next we introduce the allocation rules. We define an allocation rule of game $\Phi(f)$ to be a function of cooperative games that maps game f to a vector $\Phi(f) = (\Phi_1(f), \Phi_2(f), \dots, \Phi_N(f))$ with $\sum_i^N \Phi_i(f) \leq f(\mathcal{N})$; herein, the domain of Φ is a certain class of cooperative games, e.g., the domain is $\mathcal{G}$, we say Φ is an allocation rule on $\mathcal{G}$; $\Phi_i(f)$ is the allocation for each player i . An allocation rule is efficient if and only if $\sum_i^N \Phi_i(f) = f(\mathcal{N})$. In this paper, we only consider allocation rules that are efficient. Moreover, we define an allocation rule of strategy I' as a function of strategy profile. For a given allocation rule Φ on games and $f(; I')$, we can construct the allocation of retailer i as the composition function $\Phi_i(f(; I'))$, where the sharing strategy profile I' is controlled by the retailers. The retailers are able to change their strategies in order to gain better allocations. An allocation rule leads to full inventory pooling entails that when $I' = I^{t_0}$, all retailers do not intend to change their strategies (by reducing I').

For game $f(; I')$, the core shows the full participation of the players. Granot and Sošić (2003) show that there is no core complete sharing allocation rule for four or more retailers, and no value preserving allocation rule for six or more retailers in transshipment games. It can be interpreted as follows: for any allocation rule of all transshipment games, either full participation or full inventory pooling is not stable. The existing literature focuses on providing a core allocation rule with the full inventory pooling

inherently assumed. Rather, we focus on the allocation rules that lead to full inventory pooling with the full participation imposed. By Property 3, in the centralized systems, complete sharing (no $I_i^{t_0} I_j^{t_0} < 0$) is not just myopic but also optimal for the multi-period setting. Since full inventory pooling always leads to complete sharing when the lateral transshipment costs is less than the savings resulted from sharing, our intent is to find an allocation rule that results in full inventory pooling. In the following, we explain how certain allocation rules lead to full inventory pooling and we aim to derive such allocation rules.

A few properties are known to have enabled allocation rules to attain the full inventory pooling. Granot and Sošić (2003) define an allocation rule that leads to full inventory pooling as follows. Consider an allocation rule Φ on transshipment games $f_1(\cdot; I') \in \mathcal{T}_1$. When every retailer chooses full inventory pooling $I'_i = I_i^{t_0}$, $\Phi_i(f)$ is allocated, where $f \in \mathcal{T}_1$ is the transshipment game decided by full inventory pooling strategy. Suppose one retailer i deviates her strategy from $I'_i = I_i^{t_0}$ while other retailers keep the same strategies. The deviated I' decides a new transshipment game $g \in \mathcal{T}_1$, such that $\Phi_i(f) \geq \Phi_i(g)$ or $\Phi_i(f) > \Phi_i(g)$. Granot and Sošić (2003) offer two properties of allocation rule on $\mathcal{G}$ that consider a single retailer's deviation and show that they lead to full inventory pooling. These properties, as stated below, motivates full inventory pooling for the allocation rules on $\mathcal{T}_1$:

Property 4 (Coalitionally Monotonic). (Granot and Sošić, 2003) For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.1) holds for $\Phi(f)$ and $\Phi(g)$: for any $i \in \mathcal{N}$,

$$\left. \begin{array}{ll} f(S) \geq g(S), & \text{if } i \in S \\ f(S) = g(S), & \text{if } i \notin S \end{array} \right\} \implies \Phi_i(f) \geq \Phi_i(g) \quad (2.1)$$

Property 5 (Value Preserving). (Granot and Sošić, 2003) For any fixed $f, g \in \mathcal{G}$

with fixed $\mathcal{N}$, the following (2.2) holds for $\Phi(f)$ and $\Phi(g)$: for any $i \in \mathcal{N}$,

$$\left. \begin{aligned} f(\mathcal{N}) &> g(\mathcal{N}), \\ f(S) &\geq g(S), \quad \forall S \neq \mathcal{N}, i \in S \\ f(S) &= g(S), \quad \forall S \neq \mathcal{N}, i \notin S \end{aligned} \right\} \implies \Phi_i(f) > \Phi_i(g) \quad (2.2)$$

The only way for $f_1(; I') \in \mathcal{T}_1$ to deviate from full inventory pooling is by reducing $|I'_i|$ (no change of sign of I'_i); therefore by definition of $f_1(; I')$, it results in some specific relation between games in terms of $f(S) \geq g(S)$, where f is $f_1(; I')$ with full inventory pooling and g is $f_1(; I')$ with reduced I'_i . Property 4 and 5 are based on the specific relation between games derived from single retailer's deviation. Hence:

Property 6. *If an allocation rule Φ on $\mathcal{T}_1$ has (2.1) for any transshipment games $f, g \in \mathcal{T}_1$, then under Φ , full inventory pooling is a Nash equilibrium for the transshipment games.*

Proof. In Appendix A.1.1. □

Property 7. *If an allocation rule Φ on $\mathcal{T}_1$ has (2.2) for any transshipment games $f, g \in \mathcal{T}_1$, then under Φ the following holds. For any retailer i , suppose $f(\mathcal{N})$ of a sharing strategy $(I_i(f), -I_i(f))$ is larger than $g(\mathcal{N})$ of sharing strategy $(I_i(g), -I_i(g))$ where $-I_i(f) = -I_i(g)$. This implies $\Phi_i(f) > \Phi_i(g)$. (Here $I_i(f)$ denotes the value of I'_i in f , $I_i(g)$ denotes the value of I'_i in g and $-I_i(f)$ represents all the values of I'_j with $j \neq i$ in f .)*

Under this property, every retailer does not intend to decrease $f(\mathcal{N})$ by only changing her own strategy.

Proof. In Appendix A.1.2. □

As shown in the following example, there are allocation rules that satisfy the properties of coalitional monotonicity and value preserving.

Example 1. *The Shapley Value allocation rule satisfies coalitional monotonicity and value preserving (Granot and Sošić, 2003); therefore Property 6 and 7 hold for Shapley Value.*

However, Property 6 and 7 only deter a single retailer's deviation. Coalition deviation, referred to as deviation of a subgroup of retailers, may also cause a failure to attain full inventory pooling. Therefore an allocation rule, even if Property 6 and 7 hold, may not refrain from coalition deviation. In the following example, we show that under the Shapley Value allocation rule, full inventory pooling is not a strong Nash equilibrium.

Example 2. *Suppose there is no transshipment cost in a transshipment game with three retailers. Under Shapley Value, $I_1^t = I_2^t = 3$ and $I_3^t = -3$, full inventory pooling is not a strong Nash equilibrium.*

Proof. In Appendix A.1.3. □

In Example 2, if Retailer 1 and 2 both reduce their $|I'_i|$ from 3 to 2, then they both get better payoffs. It shows that under Shapley Value, coalition deviation from full inventory pooling can occur. Hence we extend how an allocation rule can lead to full inventory pooling by generalizing Property 4 and 5. Consider an allocation rule Φ of transshipment game $f_1(; I') \in \mathcal{T}_1$. Let f be the game $f_1(; I')$ where all the retailers choose strategy of full inventory pooling $I'_i = I_i^{t_0}$, then $\Phi_i(f)$ is allocated to each retailer. For any subcoalition $\mathcal{M} \subset \mathcal{N}$, some retailers $i \in \mathcal{M}$ deviate their strategies from $I'_i = I_i^{t_0}$. The deviated strategy profile decides a new transshipment game $g \in \mathcal{T}_1$, such that there are $\Phi_i(f) \geq \Phi_i(g)$ for all $i \in \mathcal{M}$ or there exists $i \in \mathcal{M}$ with $\Phi_i(f) > \Phi_i(g)$. Since the only way for $f_1(; I') \in \mathcal{T}_1$ to deviate from full inventory pooling is to reduce $|I'_i|$ while preserving the sign of $|I'_i|$, it results in some specific relation in terms of $f(S) \geq g(S)$ (where f is $f_1(; I')$ with full inventory pooling and g is $f_1(; I')$ with reduced I'_i). In the following, we provide properties of allocation rules on $\mathcal{G}$ similar to

Property 4 and 5 that deter coalition deviation for $f, g \in \mathcal{G}$. In addition, note that these properties are based on the specific relation between games that derived from coalition deviation.

Property 8. *For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.3) holds for $\Phi(f)$ and $\Phi(g)$: for any $\mathcal{M} \subset \mathcal{N}$,*

$$\left. \begin{aligned} f(S) &\geq g(S), & S \cap \mathcal{M} &\neq \emptyset \\ f(S) &= g(S), & S \cap \mathcal{M} &= \emptyset \end{aligned} \right\} \implies \Phi_i(f) \geq \Phi_i(g) \text{ for } i \in \mathcal{M} \text{ or } \exists j \in \mathcal{M}, \Phi_j(f) > \Phi_j(g). \quad (2.3)$$

The above (2.3) implies that a coalition deviation leads to no one's payoff be better or one retailer's payoff be worse (Pareto efficient) in the subcoalition. Property 8 is a sufficient condition for an allocation rule of transshipment games to deter any coalition deviation from full inventory pooling.

Property 9. *If an allocation rule on $\mathcal{T}_1$ has (2.3) for any transshipment games $f, g \in \mathcal{T}_1$, then under this allocation rule, full inventory pooling is a strong Nash equilibrium.*

Proof. In Appendix A.1.4. □

Suppose the payoffs in the subcoalition is transferable. It implies coalition deviation can improve every retailer's payoff in the subcoalition if and only if the subcoalition's total payoff improves. We provide the property for allocation rules on $\mathcal{G}$ as follows.

Property 10. *For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$ the following (2.4) holds for $\Phi(f)$ and $\Phi(g)$: for any $\mathcal{M} \subset \mathcal{N}$,*

$$\left. \begin{aligned} f(S) &\geq g(S), & S \cap \mathcal{M} &\neq \emptyset \\ f(S) &= g(S), & S \cap \mathcal{M} &= \emptyset \end{aligned} \right\} \implies \sum_{i \in \mathcal{M}} \Phi_i(f) \geq \sum_{i \in \mathcal{M}} \Phi_i(g). \quad (2.4)$$

It is easy to see that (2.4) $\implies$ (2.3). Property 10 shows that coalition deviation cannot improve every retailer's payoff in the subcoalition, even if their payoffs can be transferred. Hence coalition deviation from full inventory pooling strategy will not be agreed by everyone in the subcoalition. We provide a stronger sufficient condition of Property 10 in the next property.

Property 11. *For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.5) holds for $\Phi(f)$ and $\Phi(g)$: for any $\mathcal{M} \subset \mathcal{N}$,*

$$\left. \begin{array}{l} f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies \Phi_i(f) \geq \Phi_i(g) \quad \text{for } i \in \mathcal{M}. \quad (2.5)$$

The above (2.5) implies that coalition deviation does not improve everyone's payoff in the subcoalition. Hence Property 11 implies no retailer in the subcoalition intend to change her strategy. It is not difficult to see that (2.5) $\implies$ (2.4) $\implies$ (2.3) $\implies$ (2.1).

Next we consider the allocation rules on $\mathcal{G}$ that satisfy (2.5) for all f and g in $\mathcal{G}$. Before we consider the allocation rules, we first show an equivalent representation of (2.5).

Property 12. *For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.6) holds for $\Phi(f)$ and $\Phi(g)$:*

$$f(S) \geq g(S), \quad \text{for } S \subset \mathcal{N} \implies \Phi_i(f) \geq \Phi_i(g), \quad \text{for all } i \in \mathcal{N}. \quad (2.6)$$

The above (2.6) is equivalent to (2.5).

The assumption for (2.6), $f(S) \geq g(S)$, for $S \subset \mathcal{N}$, is the general relation between games that deviate from full inventory pooling, i.e., the decreasing of payoff for every subcoalition. The above (2.6) is the case $\mathcal{M} = \mathcal{N}$ in (2.5). It means that decreases in payoffs of all subcoalitions result in a decrease of allocation for every player.

Proof. In Appendix A.1.5. □

For fixed $f, g \in \mathcal{G}$, $\Phi(f)$ and $\Phi(g)$, (2.6) $\iff$ (2.5). It means that the allocation rules on $\mathcal{G}$ with (2.5) for any $f, g \in \mathcal{G}$ are the same as those on $\mathcal{G}$ with (2.6) for any $f, g \in \mathcal{G}$. Therefore we aim to characterize the allocation rules on $\mathcal{G}$ that satisfy (2.6) for all f and g in $\mathcal{G}$.

Theorem 1. *The allocation rule Φ on $\mathcal{G}$ has (2.6) for all $f, g \in \mathcal{G}$ if and only if the allocation rule $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$, where Φ_i is an increasing function of $f(\mathcal{N})$.*

Proof. In Appendix A.1.6. □

Since the only way to deviate from full inventory pooling is to reduce $|I'_i|$ with the sign of I'_i unchanged, the deviation generally results in $f(S) \geq g(S)$, for $S \subset \mathcal{N}$, where f is the transshipment game with full inventory pooling and g is the transshipment game determined by the deviation. Hence we have the following Property 13.

Property 13. *If an allocation rule Φ on $\mathcal{T}_1$ has (2.6) that holds in $\mathcal{T}_1$, then under Φ any deviation from full inventory pooling implies $\Phi_i(f) \geq \Phi_i(g)$, where f is the transshipment game with full inventory pooling and g is the transshipment game determined by the deviation.*

Similarly, if any subcoalition of retailers change their strategies from full inventory pooling to strictly decrease $f(\mathcal{N})$, it should lead to strictly decreasing payoff for at least one retailer in the subcoalition in order to deter this deviation. The properties on $\mathcal{G}$ based on the specific relation between games of coalition deviation are stated as follows.

Property 14. *For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.7) holds for*

$\Phi(f)$ and $\Phi(g)$: for any $\mathcal{M} \subset \mathcal{N}$,

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies \Phi_i(f) > \Phi_i(g) \exists i \in \mathcal{M}. \quad (2.7)$$

Suppose the payoff is transferable, to restrain coalition deviation, the total payoff should be reduced. Hence the property for allocation rule on $\mathcal{G}$ is given in the following:

Property 15. For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.8) holds for $\Phi(f)$ and $\Phi(g)$: for any $\mathcal{M} \subset \mathcal{N}$,

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies \sum_{i \in \mathcal{M}} \Phi_i(f) > \sum_{i \in \mathcal{M}} \Phi_i(g). \quad (2.8)$$

A stronger property would be that the coalition deviation reduces every retailer's payoff. That is:

Property 16. For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.9) holds for $\Phi(f)$ and $\Phi(g)$: for any $\mathcal{M} \subset \mathcal{N}$,

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies \Phi_i(f) > \Phi_i(g) \text{ for all } i \in \mathcal{M}. \quad (2.9)$$

It is obvious that $(2.9) \implies (2.8) \implies (2.7) \implies (2.2)$. We now consider the allocation rules on $\mathcal{G}$ that satisfy (2.9) for all f and g in $\mathcal{G}$. We first present an equivalence of (2.9) as follows.

Property 17. For any fixed $f, g \in \mathcal{G}$ with fixed $\mathcal{N}$, the following (2.10) holds for

$\Phi(f)$ and $\Phi(g)$:

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \subsetneq \mathcal{N} \end{array} \right\} \implies \Phi_i(f) > \Phi_i(g) \text{ for all } i \in \mathcal{N}. \quad (2.10)$$

The above (2.10) is equivalent to (2.9).

The assumption of (2.10) is the general relation between games that deviate from full inventory pooling with strictly decreasing total payoff. The above (2.10) is the case $\mathcal{M} = \mathcal{N}$ in (2.9). It shows that decreases in the payoffs of all subcoalitions and a strictly decreasing payoff of the grand coalition result in a strictly decreasing allocation for every player.

Proof. In Appendix A.1.7. □

Since for fixed $f, g \in \mathcal{G}$, $\Phi(f)$ and $\Phi(g)$, (2.10) $\iff$ (2.9), it means that the allocation rules on $\mathcal{G}$ with (2.9) for any $f, g \in \mathcal{G}$ are equivalent to those on $\mathcal{G}$ with (2.10) for any $f, g \in \mathcal{G}$. Hence our aim is to characterize the allocation rules on $\mathcal{G}$ that satisfy (2.10) for all f and g in $\mathcal{G}$.

Theorem 2. *The allocation rule Φ on $\mathcal{G}$ has (2.10) for all $f, g \in \mathcal{G}$ if and only if the allocation rule $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$, where Φ_i is a strictly increasing function of $f(\mathcal{N})$.*

Proof. In Appendix A.1.8. □

Since the only way to deviate from full inventory pooling is to reduce $|I'_i|$ without changing the sign of I'_i , it implies that the deviation of full inventory pooling with $f(\mathcal{N}) \geq g(\mathcal{N})$, where f is the transshipment game with full inventory pooling and g is the transshipment game determined by the deviation. Thus we have the following Property 18.

Property 18. *If an allocation rule Φ on $\mathcal{T}_1$ has (2.10) in $\mathcal{T}_1$, then under Φ , any deviation from full inventory pooling with $f(\mathcal{N}) > g(\mathcal{N})$ implies $\Phi_i(f) > \Phi_i(g)$,*

where f is the transshipment game with full inventory pooling and g is the transshipment game determined by the deviation.

Note that the assumptions in Property 13 and 18 only need that the allocation rules have (2.6) (or (2.10)) for all $f, g \in \mathcal{T}_1$. The allocation rules $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$, where Φ_i is a (strictly) increasing function of $f(\mathcal{N})$ restricted on $\mathcal{T}_1$, satisfies (2.6) (or (2.10)) for all $f, g \in \mathcal{T}_1$. It is not clear for the converse. However we show that if (2.6) (or (2.10)) holds for all $f, g \in \mathcal{T}_2$ (e.g., in centralized cases), then Φ_i is a (strict) increasing function of $f(\mathcal{N})$ for $f \in \mathcal{T}_2$.

Theorem 3. *The allocation rule Φ on $\mathcal{T}_2$ has (2.6) for any $f, g \in \mathcal{T}_2$ if and only if the allocation rule $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$, where Φ_i is an increasing function of $f(\mathcal{N})$.*

Proof. In Appendix A.1.9. □

Theorem 4. *The allocation rule Φ on $\mathcal{T}_2$ has (2.10) for any $f, g \in \mathcal{T}_2$ if and only if the allocation rule $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$, where Φ_i is a strictly increasing function of $f(\mathcal{N})$.*

Proof. In Appendix A.1.9. □

An allocation rule Φ , where $\Phi_i(f(\mathcal{N}))$ is a (strictly) increasing function, implies optimality of the games with maximum $f(\mathcal{N})$. The maximum $f(\mathcal{N})$ can be achieved by full inventory pooling. There exist such allocation rules. One example is the fractional allocation rule $\Phi_i(f) = \theta_i f(\mathcal{N})$, where $\theta_i > 0$ and $\sum_{i=1}^N \theta_i = 1$. Note that since $I_i^{t_0} + s_i^{t_0}$ is fixed and realized in time t_0 , it is possible to define the fractional parameter θ_i in the form $\theta_i(I_1^{t_0} + s_1^{t_0}, \dots, I_N^{t_0} + s_N^{t_0})$.

In the lending system, there are different ways to consider the allocation rule (on different game classes). One is to allocate the benefits of sharing actions in time t_0 . For example, consider the fractional allocation rule of

transshipment games; it leads to strategy of full inventory pooling for every retailer. Hence the allocation for retailer i is $\theta_i f_1(\mathcal{N}, I^{t_0})$, where $I^{t_0} = (I_1^{t_0}, \dots, I_N^{t_0})$. In addition, compared to the non-sharing case, retailer i already experienced $(I_i^{t_0})_- - (I_i^{t_0} + s_i^{t_0})_-$ change in backorders and $(I_i^{t_0})_+ - (I_i^{t_0} + s_i^{t_0})_+$ change in holding costs before sharing in time t_0 . Alternatively, one can allocate the benefits from all the sharing actions to the retailers directly. Let p be the unit backorder penalty in one time unit and h be the unit holding cost in each time unit. We define $f(\cdot; I')$ as:

$$\begin{aligned} f(S, I') \\ = f_1(S, I') - p \sum_{i \in S} ((I_i^{t_0})_- - (I_i^{t_0} + s_i^{t_0})_-) - h \sum_{i \in S} ((I_i^{t_0})_+ - (I_i^{t_0} + s_i^{t_0})_+). \end{aligned}$$

Under an allocation rule Φ , where $\Phi_i(f(\mathcal{N}))$ is a (strictly) increasing function, it implies optimality of the game with maximum $f(\mathcal{N})$; therefore it leads to full inventory pooling. Since $(I_i^{t_0})_- - (I_i^{t_0} + s_i^{t_0})_-$ and $(I_i^{t_0})_+ - (I_i^{t_0} + s_i^{t_0})_+$ are fixed in time t_0 , there is a one-to-one correspondence in terms of games (and allocations) between two game classes – one with sharing actions in time t_0 and the other based upon all sharing actions.

As stated earlier, if there is no lateral transshipment cost (e.g., in centralized systems), then it results in complete sharing. Comparing the complete sharing case with the non-sharing case in time t_0 , the total savings is $(p + h)(\sum_{i \in \mathcal{N}} (I_i^{t_0} + s_i^{t_0})_- - (\sum_{i \in \mathcal{N}} I_i^{t_0} + s_i^{t_0})_-) \geq 0$. By Property 3, it is optimal in every time t . However if lateral transshipment cost is incurred, complete sharing may not coincide with the maximum payoff. In the next section, we discuss how to control the number of lateral transshipments while achieving complete sharing in decentralized systems.

2.6 Transshipment Rule

In this section, we consider the lateral transshipments when performing sharing in decentralized systems. In particular, we look into how to establish the *transshipment pattern* for sharing. Under a simple transshipment rule for sharing, we establish an upper bound for the transshipment frequency by a function of the reduction of backorders and the loaned out quantity. More notably, with additional assumption on the relationship between the cost types, the bound entails non-negative total profit under the transshipment rule.

We first present an example that illustrates the excessive transshipments that could occur for complete sharing, if there is no deliberate control on the transshipment pattern.

Example 3. *Initially, at time 1, suppose the on-hand inventories of the N retailers are $\{-1, 1, 0, \dots, 0\}$. Now, suppose 1 unit of stock is transshipped from retailer 2 to 1; the on-hand inventories then become $\{0, 0, 0, \dots, 0\}$. At time 2, all retailers replenish 1 unit of stock except retailer 2, who receives 1 unit of demand; the on-hand inventories are now: $\{1, -1, 1, \dots, 1\}$. Suppose 1 unit of stock is then transshipped from retailer 3 to 2, then the on-hand inventories now become $\{1, 0, 0, 1, \dots, 1\}$. Next, at time 3, retailer 3 faces 1 unit of demand and the on-hand inventories become $\{1, 0, -1, 1, \dots, 1\}$. To counter the stock-out at retailer 3, 1 unit is transshipped from retailer 4 to 3, then the on-hand inventories are: $\{1, 0, 0, 0, 1, \dots, 1\}$. Suppose this process repeats and happens to all N retailers, then there will be N lateral transshipments.*

In the above example, the non-sharing case would have resulted in only one unit of backorder and an extra unit of holding, without incurring any lateral transshipment; however, as demonstrated above, the complete sharing leads to N lateral transshipments. This example shows that establishing the transshipment pattern in an arbitrary way could lead

to excessive number of lateral transshipments over time. It is therefore important to develop a transshipment rule for sharing that can control the transshipment frequency. Furthermore, it is clear from the example that determining the “optimal” transshipment pattern is nontrivial, as the eventual number of transshipments is subject to uncertainty of future demands.

Transshipment Rule. *The rule attempts to keep all $s_i^{t_0} = 0$ or as near as possible to 0 when establishing the transshipment pattern. Formally that is: if a retailer faces stock-out, then she first looks for sharing with retailers with negative $s_i^{t_0}$; if a retailer has surplus inventory, then she first looks for sharing with retailers with positive $s_i^{t_0}$.*

Suppose the current time is t_0 . Let the initial value of $s_i^{t_0}$ be $s_i^{t_0-1}$, which refers to no sharing action in time t_0 . We develop the transshipment rule to determine the value of $s_i^{t_0}$ (i.e., the sharing actions in time t_0). The transshipment rule decides the sharing based on current values of $s_i^{t_0}$ and $I_i^{t_0}$. Define A_+ as the set of all retailers with $s_i^{t_0} > 0$ and A_- as the set of all retailers with $s_i^{t_0} < 0$, where $s_i^{t_0}$ is the current value of sharing quantity. One transshipment from i to j means one sharing action from i to j . It results in $I_i^{t_0} = I_i^{t_0} - 1$ and $s_i^{t_0} = s_i^{t_0} + 1$ for retailer i ; $I_j^{t_0} = I_j^{t_0} + 1$ and $s_j^{t_0} = s_j^{t_0} - 1$ for retailer j . The transshipment rule comprises two steps:

1. If there is retailer $i \in A_-$ with $I_i^{t_0} > 0$ and retailer $j \in A_+$ with $I_j^{t_0} < 0$, then transship between i and j .
2. Share until there is no $I_i^{t_0} I_j^{t_0} < 0$.

In the above, $I_i^{t_0}$ represents the current value of inventory level, which can be changed by sharing. However, a sharing action decreases the absolute value of $I_i^{t_0}$ with its sign unchanged, while $I_i^{t_0} + s_i^{t_0}$ is fixed. Thus it is not possible to find new retailers in A_+ with $I_i^{t_0} < 0$ or in A_- with $I_i^{t_0} > 0$

during the sharing. Moreover since each sharing action decreases the value of $\sum_{i=1}^N (I_i^{t_0})_-$ by 1, after finite number of sharing actions, $\sum_{i=1}^N (I_i^{t_0})_-$ eventually decreases to $(\sum_{i=1}^N I_i^{t_0})_-$. Therefore the transshipment rule terminates in current time t_0 .

If one applies the above transshipment rule to Example 3, at time 2, 1 unit of stock would be transshipped from retailer 1 to retailer 2. The total number of lateral transshipments would only be 2.

By the definition of complete sharing, we know that complete sharing is achieved by following the transshipment rule all the time. Furthermore, the following lemma shows that non-negative reduction of backorders $(\sum_{i=1}^N (I_i^{t_0} + s_i^{t_0})_- - \sum_{i=1}^N (I_i^{t_0})_- \geq 0)$ is achieved at the end of Step 1. It only requires that the current values of $s_i^{t_0}$ and $I_i^{t_0}$ to satisfy the terminating condition of Step 1.

Lemma 2.6.1. Let $a_i, b_i \in \mathbb{R}$, for $i = 1, 2, \dots, m$. If there is no i , s.t. $a_i < 0$ and $b_i > 0$, then $\sum_{i=1}^m (a_i + b_i)_- - \sum_{i=1}^m (a_i)_- \geq 0$. Furthermore assume $\sum_{i=1}^m b_i = 0$. If there is no i , s.t. $a_i > 0$ and $b_i < 0$, then $\sum_{i=1}^m (a_i + b_i)_- - \sum_{i=1}^m (a_i)_- \geq 0$

Proof. In Appendix A.2.1. □

At the end of Step 1, $I_i^{t_0}$ and $s_i^{t_0}$ satisfy the assumption of Lemma 2.6.1; hence $\sum_{i=1}^N (I_i^{t_0} + s_i^{t_0})_- - \sum_{i=1}^N (I_i^{t_0})_- \geq 0$. It means that non-negative reduction of backorders in time t_0 is achieved. In the following, we establish the relationship between the reduction of backorders, the number of lateral transshipments and the loaned out quantity. For convenience of discussion, let $n_b^k = \sum_{i=1}^N (I_i^k + s_i^k)_- - \sum_{i=1}^N (I_i^k)_-$ be the reduction of backorders in time $k \leq t_0$, n_t^k be the total number of lateral transshipments in time $k \leq t_0$ and $n_s^k = \sum_{i=1}^N (s_i^k)_- - \sum_{i=1}^N (s_i^{k-1})_-$ be the increment of loaned out quantity in time $k \leq t_0$. Moreover let $n_b^{[1, t_0]} = \sum_{j=1}^{t_0} n_b^j$ be the reduction of backorders during $[1, t_0]$, $n_t^{[1, t_0]} = \sum_{j=1}^{t_0} n_t^j$ be the total number of lateral

transshipments during $[1, t_0]$ and $n_s^{[1, t_0]} = \sum_{j=1}^{t_0} n_s^j = \sum_{i=1}^N (s_i^{t_0})_-$ be the total number of loaned out quantity during $[1, t_0]$. Note that in current time t_0 , n_b^k , n_t^k and n_s^k are fixed for $k < t_0$. The sharing actions in t_0 can only change $n_b^{t_0}$, $n_t^{t_0}$ and $n_s^{t_0}$. In other words, n_b^k , n_t^k and n_s^k for $k < t_0$ are decided in previous time $k < t_0$.

Theorem 5. *Following the transshipment rule in time t_0 leads to $2n_b^{t_0} \geq n_t^{t_0} + n_s^{t_0}$. To construct sharing in lending system by following the transshipment rule all the time, there is $2n_b^{[1, t_0]} \geq n_t^{[1, t_0]} + n_s^{[1, t_0]}$.*

By following the transshipment rule, Example 3 shows that the equality in Theorem 5 can be achieved.

Proof. When $s_i^{t_0} = s_i^{t_0-1}$ for all $i \in \mathcal{N}$, $n_t^{t_0} = 0$ and $n_s^{t_0} = 0$. Any sharing action in Step 1 increases $n_t^{t_0}$ by 1 while decrease $n_s^{t_0}$ by 1, thus during Step 1 of the transshipment rule, $n_t^{t_0} + n_s^{t_0} = 0$ is unchanged. Hence at the end of Step 1, $n_t^{t_0} + n_s^{t_0} = 0$. Meanwhile $n_b^{t_0} \geq 0$ at the end of Step 1 by Lemma 2.6.1. Hence at the end of Step 1, $2n_b^{t_0} \geq n_t^{t_0} + n_s^{t_0}$. Because any sharing action increases $n_b^{t_0}$ and $n_t^{t_0}$ by 1, and $n_s^{t_0}$ by at most 1, it implies in Step 2, $n_b^{t_0} \geq n_t^{t_0} + n_s^{t_0}$ always holds.

Because n_b^k , n_t^k and n_s^k for $k < t_0$ are decided in previous time $k < t_0$, by following the transshipment rule in time k , after all sharing actions in time k , $n_b^k \geq n_t^k + n_s^k$. It will be unchanged in the future (time $t > k$). It means $n_b^k \geq n_t^k + n_s^k$ in time t_0 . Therefore $2n_b^{[1, t_0]} \geq n_t^{[1, t_0]} + n_s^{[1, t_0]}$. $\square$

The significance of Theorem 5 is to construct a complete sharing in lending system that the total number of lateral transshipments $n_t^{[1, t_0]}$ (cost) is kept under control by the total reduction in the number of backorders $n_b^{[1, t_0]}$ (savings). Next, we show that Theorem 5 guarantees non-negative profit under the condition that unit backorder cost plus unit holding cost (denote as Saving) is larger than or equal to two times of the supremum of unit lateral transshipment costs (denoted as Cost); that is: $\text{Saving} \geq 2\text{Cost}$.

Since $2n_b^{[1,t_0]} \geq n_t^{[1,t_0]} + n_s^{[1,t_0]}$ and $n_s^{[1,t_0]} = \sum_i (s_i^{t_0})_- \geq 0$, $2n_b^{[1,t_0]} \geq n_t^{[1,t_0]} + n_s^{[1,t_0]} \geq n_t^{[1,t_0]}$. Therefore:

$$\begin{aligned} \text{Saving} \times n_b^{[1,t_0]} &= \frac{\text{Saving}}{2} \times 2n_b^{[1,t_0]} \\ &\geq \frac{\text{Saving}}{2} \times n_t^{[1,t_0]} \geq \text{Cost} \times n_t^{[1,t_0]} \\ &\geq \text{Sum of all the lateral transshipment costs during } [1, t_0]. \end{aligned}$$

The following non-negative profit can then be allocated to the retailers in any time t :

$$\text{Saving} \times n_b^t - \text{Cost} \times (n_t^t + n_s^t) \geq \text{Cost}(2n_b^t - (n_t^t + n_s^t)) \geq 0.$$

Since $\sum_{i=1}^t (\text{Saving} \times n_b^i - \text{Cost} \times (n_t^i + n_s^i)) = \text{Saving} \times n_b^{[1,t]} - \text{Cost} \times n_i^{[1,t]} - \text{Cost} \times n_s^{[1,t]} \leq \text{Saving} \times n_b^{[1,t]} - \text{Cost} \times n_i^{[1,t]}$, there are always enough savings to attain non-negative allocations.

Together with the transshipment rule, the above allocation rule leads to full inventory pooling. As described in Section 2.5, each retailer can choose $I'_i \in [0, I_i^{t_0}]$ (or $I'_i \in [I_i^{t_0}, 0]$). For any fixed I' (where $I'_i \in [0, I_i^{t_0}]$ or $I'_i \in [I_i^{t_0}, 0]$) can be considered as a part of the sharing when $I' = I^{t_0}$. With I'_i increasing, n_b^t may increase. If n_b^t increases by 1, then n_t^t increases by 1 and the number of loaned out n_s^t increase by at most 1. By $\text{Saving} \geq 2\text{Cost}$, the profits allocated to retailers increase when there is more sharing. By following the transshipment rule, n_s^t is minimum which implies maximum profits. It leads to in full inventory pooling which entails highest frequency of sharing.

Even if no assumption on savings and costs is imposed (where complete sharing does not guarantee non-negative additional profit), the transshipment rule guarantees minimum backorders and with a controlled number of lateral transshipments. As this transshipment rule aims to keep all

$s_i^{t_0}$ equal to or as near as possible to zero, it helps to reduce the sharing quantity and thus prevents any retailer from borrowing excessive amount of stocks.

2.7 Conclusion

We present a sharing policy for $n \geq 2$ independent retailers in a multi-period setting. Under the policy, the system functions as a lending system, where the sharing quantity is central to its operation. We characterize the lending system such that the non-sharing case can be conveniently reconstructed. In the system, the ownership $I_i^t + s_i^t$ is unchanged for any time t and equals to the inventory level in the non-sharing case where $s_i^t = 0$ for all i and t . We show that complete sharing leads to minimum total backorders and inventory level in the lending system. In current time t_0 , at the stage in which the retailers make decisions on sharing, the problem is equivalent to a cooperative game between all retailers. However the game is decided by I' , which is the amount of inventories or demands each retailer is willing to share. The allocation rule affects retailers' decision I' . When formulating the allocation rule, we focus on the rule that leads to full inventory pooling (which results in complete sharing in centralized systems). We show that allocation rules that are (strictly) monotonic increasing with total payoff strongly prevent coalition deviation from full inventory pooling. Furthermore these rules are exactly the allocation rules that satisfy (2.6) (or (2.10)) on $\mathcal{G}$ and $\mathcal{T}_2$. For centralized systems, it results in complete sharing in time t_0 .

Next, we incorporate lateral transshipment cost in our analysis, where complete sharing may not coincide with optimal total profit. We develop a transshipment rule for determining the transshipment pattern of complete sharing. We derive an upper bound such that the number of lateral

transshipment is bounded by two times the reduction of backorders minus the loaned out quantity. Finally, under the condition that two times of the unit transshipment cost is less than or equal to the savings (sum of unit backorder cost and unit holding cost), we show that the transshipment rule guarantees complete sharing and non-negative profit.

Chapter 3

Models with Sharing of Complete Information

3.1 Introduction

The results in Chapter 2 show that under the same ordering policy, the sharing rule and complete sharing provide better performance in terms of fewer backorders and lateral transshipments for any given demand. However with more information, it is possible for retailers to optimize their expected profits through their ordering decisions. The scope of this chapter is to develop analytical results and simulation methods based on sharing of information between the retailers. Specifically, the information comprises demands, order quantities and timing, in addition to sharing of inventory information in Chapter 2. We define this form of information sharing as *complete information sharing*. (Note that we retain the prior definition of “complete sharing” in Chapter 2 as the condition where “retailers are motivated to share all their residual supplies and demands”.)

The rest of this chapter is organized as follows. First the related literature is reviewed in section 3.2. In section 3.3, we extend the Leibniz Rule in order to obtain the differentiation of the objective function with

general demand distributions, and we find a sufficient condition for the continuity of the best response function. Next in section 3.4, we discuss the information that is used to optimize the expectation of profits and how the information can be used to update the forecasting in MMFE-like models(MMFE: martingale model of forecast evolution). The single retailer models, namely the basic newsvendor model and the multi-period newsvendor model, are introduced in section 3.5. In that section the analytical solution and a simulation method for general demand distributions are derived based on the extended Leibniz Rule(of section 3.3). Then in section 3.6, a two-retailer model is introduced. With complete information sharing, for the general demand distributions, some analytical results of order quantities are derived for the two-retailer model under the complete sharing policy. We consider the balanced system where retailers coordinate their order timings to “spread” their orders and replenishments over time so as to increase system-wide stock availability; we build a two-retailer model to determine the order quantity under a fixed sharing policy, and show that the balanced system improves sharing performance significantly. For the multi-retailer problem, which are considerably more complicated, we first provide the principles and a special case for the multi-retailer sharing policy in section 3.7. We then extend the two-retailer model to the multi-retailer setting. We assume the multi-retailer system is regulated by a deterministic sharing rule in which each retailer pre-prioritize which of the other retailers that they would conduct sharing (e.g., based on proximity); this rule significantly facilitates tractability by reducing the solution space. Subsequently we integrate the balanced system into the multi-retailer model and perform numerical experiments by simulation. Finally we offer our concluding remarks in section 3.8.

3.2 Literature Review and Contributions

In Chapter 2, we have reviewed some the literature related to the inventory sharing problem. Here, we build on that and focus on those more related to this chapter. In particular, we review the literature streams related to the newsvendor model, information sharing, forecasting models (MMFE) and balanced ordering.

In this chapter, we employ the classic newsvendor model which is the basic inventory optimization problem in terms of expectation. In particular, we extend the Leibniz rule to allow for general demand distributions, which is then incorporated into the basic newsvendor model. Furthermore, we also consider the two-retailer sharing problem based on the extended model. We then extend this model into a corresponding a multi-period newsvendor model. For a survey of the newsvendor problem and extensions, see Porteus (1990) for a comprehensive review of earlier works. Petruzzi and Dada (1999) provide a review that synthesizes the results for the problem that include the market factors of demand and selling price and how they interact with them to influence optimal decisions. For a more recent survey, Qin et al. (2011) review the extensions of the newsvendor model in incorporating the various aspects of customer demand, supplier costs, and the buyer risk profile. Other extensions that consider game-based interaction have previously been considered, such as Parlar (1988), Karjalainen (1992), and Lippman and McCardle (1997) . However, these papers consider interaction in a competitive environment, rather than in a collaborative sharing context. For example, in Lippman and McCardle (1997), if a newsvendor runs out of stock, the customer moves to a newsvendor competitor such that demand is transferred competitively. Other inventory sharing papers rely much on the extensions of the newsvendor model. For example, Rudi et al. (2001) extend the model by incorporating transshipment prices for two retailers; Anupindi et al.

(2001) applies the newsvendor model in developing an allocation rule based on the dual prices. Clearly, the main difference between our work and theirs is that we employ the multi-period newsvendor (and extended it for general distribution) instead of the basic single-period model.

To our knowledge, very few papers explicitly consider information sharing in the independent-retailer inventory sharing problem. For example, Anupindi et al. (2001) requires the sharing of realized demands, demand distributions and order quantities to determine the total profit allocation. However, information is not explicitly considered for coordination purpose. To our knowledge, Yan and Zhao (2011) are the first to consider decentralized inventory sharing with asymmetric information. The model a two-retailer system and focus on the system coordination when both retailers uphold their own demand information. Yan and Zhao (2015) extend the model to the multi-retailer ($n > 2$) problem, and as such, shares the same focus as this work. They develop a coordination mechanism that they called the nRCM, which has a few important properties, such as complete sharing and the formation of grand coalition. In this mechanism, a third-party (such as a manufacturer) is needed to coordinate the retailers by subsidizing the transshipment profit allocation. In addition, a premium fee collected by the third-party before each retailer's order placement finances the subsidy. Our work in this chapter differs in that we do not consider a third-party coordinator in our analysis and that our interest is on the multi-period setting without system-wide cooperative, which leads to considerable differences in assumptions and modeling considerations.

Demand forecasting is considered as an important information. In this work, we assume for a rolling forecast over the multi-period ordering. Based on the rolling forecast, Heath and Jackson (1994) develop a model that characterize the process of demand forecasting, known as the MMFE (Martingale model of forecast evolution) where the forecasting is

modeled as a conditional expectation on the information collection, and that the retailer's information collection is constantly updated by demand observations. Since then a number of MMFE-like model has been studied for inventory problems. Güllü (1996) study the value of information in dynamic inventory problem. Graves et al. (1998) build a MMFE model for a network of production/inventory stages. Aviv (2001) uses the MMFE-like forecasting approach to model local forecasting(LF) and collaborative forecasting(CF). The author demonstrates that both LF and CF benefits the supply chain system. Iida and Zipkin (2006) develop a computational approach to approximate solutions of dynamic MMFE model. Lu et al. (2006) develop some bounds for the optimal base-stock level in the MMFE model, and employ these bounds in a heuristic solution. Schoenmeyr and Graves (2009) show that the forecast evolution process can significant benefit placement of safety stocks. Iida (2015) studies the benefits of lead-time information and demand forecast information for inventory control operations. The author shows that the state-dependent base-stock policy is optimal under certain conditions, and develops an effective heuristic policy based on the conditions. MMFE models were also applied in many production planning models such as Beril Toktay and Wein (2001), Iida and Zipkin (2006) and Lu et al. (2006).

In this research, we incorporate balanced ordering which assumes retailers share the information on their order timings such that retailers spread their orderings and replenishments evenly over time. In this stream of literature, the papers primarily focus on a supply chain comprising of a supplier and a group of retailers. In a typical setting, the supplier receives replenishment orders at fixed intervals. Within any two successive replenishments, the retailers place their orders in a balanced manner, i.e., at each period, only one retailer (or one group of retailers) is scheduled to order in a fixed sequence. Lee et al. (1997) study and compare the impact

of three different ordering policies on the bullwhip effect and show that balanced ordering outperforms the other two, namely random ordering and synchronized ordering. Cachon (1999) and Chen and Samroengraja (2000) develop the procedural framework for evaluating the supply chain performance that practise balanced ordering. Chen and Samroengraja (2004) show that periodic order-up-to policy with staggered ordering (like balanced ordering) is superior in terms of reducing the bullwhip effect than the conventional continuous review policy but it results in a higher total cost. Zhang and Cheung (2008) model dissimilar retailer orders under balanced ordering and derive the supplier's optimal sequence. Zhang and Cheung (2011) extend their work by including the information sharing and advance order information and characterize how the value of demand information affects the supplier's optimal sequence. Obviously, our work differs from the aforementioned papers in that we consider balanced ordering system in a single-echelon setting for inventory sharing with the aim of improving system-wide inventory availability, instead of focusing on the impact on supplier's cost and demand variability.

We summarize this chapter's contributions to the literature as follows.

- Like the work reported in Chapter 2, this chapter contributes to the inventory sharing literature by considering the multi-period, multi-ordering ($n \geq 2$) independent-retailer problem, albeit with different extent of information sharing.
- We characterize how sharing are carried out with the sharing of information on demands, order quantity and up-to-date inventory levels.
- We extend the Leibniz rule to incorporate general demand distribution in the inventory models, which we employ to enable characterization of demands through sampling in the simulation; this is

opposed to the use of joint demand distribution in typical inventory sharing problems. We believe this will widen the applicability of the simulation approach.

- Without system-wide cooperation, we develop the multi-retailer sharing model under a deterministic sharing policy and translate it into a convex programming problem by a simulation-based approach. We believe it will greatly facilitate real-life applications of the model. Furthermore we provide a sufficient condition to find a pure Nash equilibrium.
- This is the first research to our knowledge that applies balanced ordering in the inventory sharing problem. We demonstrate its benefits in his sharing system in terms of system-wide stock availability, besides the side benefit of contributing to the model tractability.

3.3 Development of Properties

In this chapter we employ the future information such as demand distribution (samples) in our research. We will consider the expectations of objectives, which is integration of objective functions. In this subsection we introduce and extend some existing mathematical results, which will be used to prove some of our analytical results related with expectation in the subsequent subsections. In particular, the properties are developed for the general demand distributions, and are used to prove the differentiation of objective function, the continuity of the best response function, and the partial derivatives in basic inventory problems and inventory sharing problems. Therefore any method based on the analytical result will be available for general demands.

We emphasize that the purpose of generalizing the assumptions is to

widen the applicability of the methods. For example, in practice the general demand distribution is usually unknown. The simulation methods rely on the samples (e.g. historical data) may be from an unknown general distribution. Hence the results are useful. As most literature assumes the existence of probability density function for the demand or that the function is known, and then apply the Leibniz Rule or the Fundamental Theorem of Calculus to interchange differentiation and integration. However for general demand distribution, this assumption may not hold. By the general demand distribution, it means that, for example, it can be of any probability measures, could be in either discrete or continuous or a probability set without density function. Hence it may be problematic if one encounters demand distribution that is non-differentiable or non-continuity. For example, when there exists a discrete random variable of demand, then the objective function is not differentiable.

In order to get similar result for the optimal order quantities without the assumption of density functions, the following extended Leibniz Rule is required.

Theorem 6 (Extension of Leibniz Rule). *Let X be an open subset of $(\mathbb{R}, m)$, (Ω, μ) be a measure space. Suppose $f: X \times \Omega \rightarrow \mathbb{R}$ satisfies the following conditions:*

1. *$f(x, \omega)$ is an integrable function of ω for each $x \in X$;*
2. *There exists set E , $\mu(E) = \mu(\Omega)$, such that fix $\omega \in E$, $f(x, \omega)$ is absolutely continuous on X ; and there is an integrable function $g: \Omega \rightarrow \mathbb{R}$ such that $|f_x(x, \omega)| \leq g(\omega)$ a.e. in Lebesgue measure m on X ; furthermore for $x_0 \in X$, $f_x(x_0, \omega)$ (or $f_x(x_0 \pm, \omega)$, the left and right partial derivatives) exists in E .*

Then

$$\left. \frac{d}{dx} \int_{\Omega} f(x, \omega) d\mu(\omega) \right|_{x=x_0} = \int_E f_x(x_0, \omega) d\mu(\omega)$$

or

$$\left(\left. \frac{d}{dx} \int_{\Omega} f(x, \omega) d\mu(\omega) \right|_{x=x_0 \pm} = \int_E f_x(x_0 \pm, \omega) d\mu(\omega) \right).$$

Proof. The proof is in Appendix B.1.1 □

Theorem 6 will be employed to show the differentiability of expected objectives (such as, the expected profit in newsvendor problem). So it is possible to study the optimality by first order differentiation. However it is not a problem in most literature in which the differentiability is assumed or the explicit demand distribution is given, but it is a problem for simulation based method which employed the analytical results derived from differentiation. Fortunately, the Lipschitz continuity which is usually satisfied in the inventory model is stronger than the assumption in Theorem 6.

Furthermore in order to find a pure Nash equilibrium for two-retailer ordering strategy, a common way is to show that their order quantities are best response to each other. (e.g. Rudi et al. 2001)

Definition 1. Let $f(x, y)$ be a real valued function of $(x, y) \in A \times B$. Suppose for fixed $y \in B$, there is a unique maximum $f(x, y)$ for $x \in A$. Denote this x as $BR_{f(x,y)}(y)$, which we call the best response function.

It means a pure Nash equilibrium is a fixed point of the compose function of the best response functions. In order to find the fixed point, normally the existence and continuity of the best response function are assumed. The next theorem shows that under suitable conditions the best response function is continuous which is useful to prove the existence of the fixed point.

Theorem 7. Suppose $f(x, y)$ is uniformly continuous of y for all $x \in [a, b]$, $f(x, y)$ is continuous of x for fixed y and $BR_{f(x,y)}(y)$ exists (i.e. there is a unique maximum point $BR_{f(x,y)}(y)$ in $[a, b]$ to maximize $f(x, y)$), $BR_{f(x,y)}(y)$ is continuous.

Proof. The proof is in Appendix B.1.2 □

3.4 Forecasting Demand Distribution

Since the future demand is a random variable, a feature of it is that the future information can be updated along the time-marching. In this subsection, we briefly introduce the future information updating problem in our research and others' related works. (We consider the information collection of demand as an σ -algebra. The mathematical explanation is in Appendix B.2.)

Forecasts for demands are often based on information collections of sales and market conditions, which are then used to update the demand forecasts over time. The more information is collected, the more accurate the forecasts tend to become. For MMFE-like model, it considers the forecasts in $\mathcal{L}^2$ space; and judge how good the forecast is by the $\mathcal{L}^2$ distance. Based on the concept of MMFE and forecasting in general, the forecasts that are performed nearer in time to demand realizations tend to be closer to the actual demand in the $\mathcal{L}^2$ sense.

Let $L \geq 1$ be the lead-time of replenishment for the ordered product. It means that the order quantities are based on demand forecasting for the next L periods. The forecasting of demand D^t is usually based on the observations X^t , where X^t includes all demand observations before period t . The key concept of MMFE-like models is that forecasting is performed based on information collection that is updated by demand observations. Hence the forecasting is updated with new observations. Let $\mathcal{F}^t := \sigma(X^t)$

be the information collection at period t . The demand forecasting of period $t + k$, where $k = 0, 1, 2, \dots$, is $\mathbb{E}[D^{t+k}|\mathcal{F}^t]$ and it is denoted as $M^t = g(X^t)$ which is a martingale process for t . Since observations $X^t = x$ are realized at period t , it implies M^t is a constant $g(x)$. Meanwhile the demand distribution is updated as conditional distribution $\mathbb{P}(\cdot | X^t = x)$.

However it is difficult to forecast the joint distribution of all demands. As a consequence the updated demand distribution at period t is usually a new forecast of D^{t+k} denoted as $D^{t,(t+k)}$, which is rolling forecasting. This also explains that the forecasting errors are inevitable. Therefore in practice, for better solutions for MMFE-like models, they model should be solved with updated demand distribution in every period.

3.5 Inventory Model for a Single Retailer

This section is about how a single retailer determines her order quantity. For a single retailer, the most well-known model is probably the newsvendor model which determines the optimal order quantity that maximizes the expected profit based on demand forecast. We show some general results by applying the extended Leibniz Rule derived earlier Theorem 6 and the solution for MMFE-like multi-period model. In addition, we give an example of the simulation method.

3.5.1 Newsvendor Problem for General Demand Distribution

Problem 3.5.1. (Basic Newsvendor Problem) A retailer sells a single product in a single period (or a single selling season) with only one opportunity to order (i.e. no reordering). Suppose the retailer has information about the future demand D of this product. How does the retailer determine the

optimal order quantity O to maximize her expected profit?

The order quantity O is a non-negative integer that is less than a fixed big integer number N^O , which can be viewed as the capacity of retailer's storage or the supplier's ordering limit. Assume the future demand D is a random variable. If its distribution is known, then there is an analytical result of the optimal order quantity.

Theorem 8 (Optimality of Newsvendor Problem). *Let q be the unit ordering price, $r > q$ be the unit revenue and h be the unit holding cost. For any random demand D ($\mathbb{E}[D] < +\infty$), the optimal solutions form an interval $[a, b]$ such that*

$$\mathbb{P}(D < O) \leq \frac{r - q}{r + h} \leq \mathbb{P}(D \leq O) \quad (3.1)$$

for $O \in [a, b]$. The optimal feasible order quantities are the feasible non-negative integers in $[a, b]$ or the closest feasible non-negative integers to $[a, b]$.

Proof. The proof is in Appendix B.3.1 □

The profit for the single retailer is

$$\begin{aligned} \text{Profit} &= \min(D, O)r - (O - D)_+h - Oq \\ &= rD - r(O - D)_- - h(O - D)_+ - Oq. \end{aligned}$$

The expectation of profit for the single retailer is

$$\begin{aligned} \mathbb{E}[\text{Profit}] &= \mathbb{E}[\min(D, O)r - (O - D)_+h - Oq] \\ &= r\mathbb{E}[D] - r\mathbb{E}[(O - D)_-] - h\mathbb{E}[(O - D)_+] - Oq. \end{aligned}$$

The optimal solutions may not be unique, which may happen when there are gaps in the demand distribution. That is, the probabilities of some “in-between” values of demand are zero. One such example is the demand distribution as shown in Figure 3.1.

Figure 3.1: Frequency distribution of demand

One can first assume that the demand distribution is uniformly distributed on $[0, 10]$ and that there is an additional customer who may buy 30 units of the product with one-half probability, leading to the right hand side uniform distribution $[0+30, 10+30]$ in Figure 3.1. It results in a percentile of demand that is not unique. So the problem may be difficult to solve, particularly when a simulation method is applied. Because of the gap in distribution, it is difficult to determine when the simulations can be terminated due to possible slow convergence in simulation output.

3.5.2 Multi-Period Newsvendor Problem

Problem 3.5.2. (Newsvendor Problem with Lead time) Now consider the multi-period version of the same problem. Let D^t be the demand of period t and I^t be the inventory level at the end of period t . Assume a replenishment lead-time L . How does the retailer determine the order quantity O^t ?

Suppose retailer knows the demand distribution. Let q be the unit ordering price, $r > q$ be the unit revenue and h be the unit holding cost. Besides ordering, the inventory level is also decided by the uncertain demand; but the retailer can only attempt to control the inventory level through the order quantity.

Now we consider the number of periods that should be included in the objective function. Given the replenishment lead-time L , O^t only affects I^{t+L+k} for $k = 0, 1, 2, \dots$ and thus O^t is the last chance for the retailer to control I^{t+L} . Meanwhile the forecasts for demands in periods $t + L + 1$, $t + L + 2, \dots$ which will be updated after current period t are still unknown at current period t . (As discussed in subsection 3.4, it is similar to the concept of MMFE.) Hence only one period is considered in the objective function. In other words, the solution to the multi-period problem is equivalent to considering the effect of O^t on inventory level in period $t + L$ in every period t repeatedly.

Positive I^t indicates the amount of left-over stock in period t ; negative I^t shows the amount of stock-outs in period t . For the multi-period problem, the unfulfilled demand results in either *backorder* or *lost sale*. The assumption of either backorder or lost sale can significantly affect the inventory level and the resulting analysis.

If backorder assumption is imposed, the problem is more analytically tractable. The inventory balance equation is,

$$I^{t+k} = I^{t-1+k} + O^{t-L+k} - D^{t+k}; \quad (3.2)$$

Therefore for period $t + L$,

$$I^{t+L} = I^{t-1+L} + O^t - D^{t+L}$$

$$\begin{aligned}
&= I^{t-2+L} + O^{t-1} + O^t - D^{t+L-1} - D^{t+L} \\
&= I^t + \sum_{k=1}^L O^{t-L+k} - \sum_{k=1}^L D^{t+k}.
\end{aligned}$$

Because in the case of backorder, all demands are eventually fulfilled (i.e., all revenues are realized), the optimal order quantity is thus decided only by cost. Let p_b be the penalty of each unit of backorder per period. The cost in period t is

$$\text{Cost}^t = hI_+^t + p_b I_-^t.$$

Hence to find the optimal O^t , one needs to perform the following minimization:

$$\begin{aligned}
&\min_{s^t} \mathbb{E}[\text{Cost}^{t+L}] \\
&= \min_{s^t} (\mathbb{E}[hI_+^{t+L}] + \mathbb{E}[p_b I_-^{t+L}]) \\
&= \min_{s^t} (h\mathbb{E}[(I^t + \sum_{k=1}^L O^{t-L+k} - \sum_{k=1}^L D^{t+k})_+]) \\
&\quad + p_b \mathbb{E}[(I^t + \sum_{k=1}^L O^{t-L+k} - \sum_{k=1}^L D^{t+k})_-].
\end{aligned}$$

Let $\hat{D}^{t+L} := I^t + \sum_{k=1}^{L-1} O^{t-L+k} - \sum_{k=1}^L D^{t+k}$. The problem become

$$\begin{aligned}
&\min_{O^t} \mathbb{E}[\text{Cost}^{t+L}] \\
&= \min_{O^t} (h\mathbb{E}[(O^t - \hat{D}^{t+L})_+] + p_b \mathbb{E}[(O^t - \hat{D}^{t+L})_-]).
\end{aligned}$$

By Theorem 6, for $\mathbb{P}(\hat{D}^{t+L} = O^t) = 0$, there is

$$\begin{aligned}
&\frac{d}{dO^t} \mathbb{E}[\text{Cost}^{t+L}] \\
&= h\mathbb{P}(\hat{D}^{t+L} < O^t) - p_b \mathbb{P}(\hat{D}^{t+L} > O^t)
\end{aligned}$$

$$= -p_b + (h + p_b)\mathbb{P}(\hat{D}^{t+L} < O^t).$$

Hence O^t is optimal, if and only if

$$\mathbb{P}(\hat{D}^{t+L} < O^t) \leq \frac{p_b}{h + p_b} \leq \mathbb{P}(\hat{D}^{t+L} \leq O^t). \quad (3.3)$$

On the other hand, in the case of *lost sale*, since retailer loses the unfulfilled demand I_-^t in current period t , the stocks that are carried over to next period $t + 1$ should be I_+^t . Hence the inventory balance equation is,

$$I^{t+k} = I_+^{t-1+k} + O^{t-L+k} - D^{t+k}. \quad (3.4)$$

The fulfilled demand gains revenue $\min(D^{t+L}, I_+^{t-1+L} + O^t)r$, while the unfulfilled demand $I_-^{t+L} = (I_+^{t-1+L} + O^t - D^{t+L})_-$ results in *lost sale* (i.e., no profits). Additionally the left-over stock $I_+^{t+L} = (I_+^{t-1+L} + O^t - D^{t+L})_+$ incurs a holding cost $I_+^{t+L}h$. Given that the cost of ordering in period t is O^tq , the profit of a single retailer at period $t + L$ is:

$$\begin{aligned} & \text{Profit}^{t+L} \\ &= \min(D^{t+L}, I_+^{t-1+L} + O^t)r - I_+^{t+L}h - O^tq \\ &= D^{t+L}r - I_-^{t+L}r - I_+^{t+L}h - O^tq \\ &= D^{t+L}r - (O^t - D^{t+L} + I_+^{t-1+L})_-r \\ &\quad - (O^t - D^{t+L} + I_+^{t-1+L})_+h - O^tq. \end{aligned}$$

Let $\hat{D}^{t+L} := D^{t+L} - I_+^{t-1+L}$. Thus, finding the optimal $O(t)$ can be viewed as the basic newsvendor problem with demand $D^{t+L} - I_+^{t-1+L}$.

The problem for both the backorder and lost sale cases is that what is the value of O^t such that (3.1) or (3.3) holds? It is not an easy task, for example, to compute distribution $\mathbb{F}_{\hat{D}^{t+L}}$. Normally, because of the lead-time and the difficulty in expressing $\hat{D}^{t+L}$, computing the distribution

$\mathbb{F}_{\hat{D}^{t+L}}$ which usually involves a multiple integral is very difficult. With suitable conditional probabilities, to compute $\mathbb{F}_{D^{t+L}-I_+^{t-1+L}}$ for any fixed O ,

$$\begin{aligned}
& \mathbb{E}[1_{D^{t+L}-I_+^{t-1+L}>O}] \\
&= \mathbb{E}[\mathbb{E}[1_{D^{t+L}-I_+^{t-1+L}>O} | I_+^{t-1+L}]] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{E}[1_{D^{t+L}-I_+^{t-1+L}>O} | I_+^{t-1+L}] | I_+^{t-2+L}]] \\
&\quad \dots \\
&= \underbrace{\mathbb{E}[\mathbb{E}[\dots \mathbb{E}[\mathbb{E}[1_{D^{t+L}-I_+^{t-1+L}>O} | I_+^{t-1+L}] | I_+^{t-2+L}]]}_{L} \dots | I_+^{t+1}]].
\end{aligned} \tag{3.5}$$

The above (3.5) is based on the knowledge of the conditional probabilities and it needs implement of Theorem 12 (as shown in Appendix B.2) to compute the conditional expectations. However even if (3.5) is computable, inversely it is inefficient. That is to find the value O such that

$$\mathbb{P}(D^{t+L} - I_+^{t-1+L} < O) \leq \frac{r-q}{r+h} \leq \mathbb{P}(D^{t+L} - I_+^{t-1+L} \leq O).$$

Besides, in reality, the forecasted distributions of demands are also difficult to attain. Therefore the more convenient way is to sample from demands' distribution to approximate the distribution of either $\hat{D}$, (3.1), (3.3) or the value of objective function (which is the profit or cost).

3.5.3 Simulation Method

For the basic or multi-period newsvendor problem, the aim is to find an order quantity such that (3.1) or (3.3) holds. As discussed earlier, it is difficult in practice to get this analytical result (or the distribution of demand). However the simulation method, which is based on the law of large numbers and the central limit theorem, is efficient in approximating

the optimal solution. The biggest advantage of the simulation method is that it does not require for the expression of demands' distribution; instead only samples from demands' distribution are required. That is also the reason why we extend Leibniz Rule to generalize demand condition in newsvendor problem, because the demand distribution is unknown in practice.

Method (Simulation Method). Assume $X_1, X_2, \dots, X_n$ are samples from demand D . The samples can be obtained from historical data or forecasting. Because X_i are samples from random variable D , X_i are i.i.d.. By the law of large numbers, for O there is

$$\frac{\sum_{i=1}^n 1_{X_i < O}}{n} \xrightarrow{n \rightarrow \infty} \mathbb{E}[1_{D < O}] = \mathbb{P}(D < O). \quad (3.6)$$

The purpose of this simulation is to approximate the value of objective function at O , where $\mathbb{P}(D < O) \leq \frac{r-q}{r+h} \leq \mathbb{P}(D \leq O)$. It is to sort the samples and find an O^* such that the proportion of samples that are less than O^* is less than $\frac{r-q}{r+h}$ and the proportion of samples that are less than or equal to O^* is more than $\frac{r-q}{r+h}$. The value of objective function at O^* will approximate the optimal value when n is big enough.

Note that it is possible that O^* and optimal O are not close. As shown in Figure 3.1 the demand distribution has a gap between 10 and 30. Suppose $\mathbb{P}(D < 10) = \mathbb{P}(D < 30) = \frac{r-q}{r+h} = \frac{1}{2}$, the optimal order quantity O can be any number in $[10, 30]$ and obviously O^* cannot be any number in $(10, 30)$. However it does not affect the above results, since the derivatives between $(10, 30)$ are 0.

It is worthwhile to mention that the simulation convergence rate is $\sqrt{n}$ and its theoretic bases are Law of Large Number and Central Limit Theorem (Borovkov, 2013). Levi et al. (2015) show the probabilistic bound of sample average approximation simulation. The Metropolized Gibbs

Sampler/Metropolis-within-Gibbs developed by Müller (1991) is widely used to approximate the independent samples.

Simulation

We give an example of solving the multi-period newsvendor problem by the simulation method.

Example 4. Suppose the ordering replenishment lead-time L is 7 days, current time is $t = 1$, the inventory level $I^0 = 10$, the order quantity $O^t = 100$, for $t = 1, 2, 3, 4, 5, 6$ and $\frac{r-q}{r+h} = 60\%$. There are 100000 samples from $U[60, 140]$ demands as shown in Figure 3.2.

Figure 3.2: Multi-period newsvendor problem

We solve this problem by simulation method with R language MonteCarlo package. The optimal order quantity is around 77. As shown in Figure 3.2, the first 7 histograms are the samples for demands of day $1, 2, \dots, 7$. The 8th histogram is for D , and the last one is the approximated density function of D with 60 percentile. The related codes are in Appendix C.3.

For multiple periods, it is simply repeating the process of Example 4. After demands are updated and orders arrived in each period, the retailer faces exactly the same situation in Example 4. Hence in each period, the retailer can make ordering decision for the current period based on this simulation method.

3.6 Inventory Model for Two Retailers

In this section we consider the ordering problem for two retailers. Differing from the single retailer problem in the last section, the retailers share their inventories if one encounters stock-out while the other holds surplus stocks. Thus they have to take into account such sharing in their ordering decisions. In subsection 3.6.1, we assume complete information sharing and complete sharing and show the general analytical results by using the properties developed in subsection 3.3. In subsection 3.6.2, we give simulation methods to deal with the problem that no analytical demand distribution. In subsection 3.6.3, we integrate the balance ordering system with the model, and demonstrate its benefits to the sharing system and how it simplifies the analysis.

3.6.1 Complete Information Sharing under Complete Sharing

In this subsection, we consider two retailers who completely share their information under complete sharing policy. We show the existence of the best response function as well as its continuity and differentiation.

As shown in subsection 3.5.2, the problem with ordering lead time $L(L > 1)$ can be translated into the problem with one period ordering lead time, because their objective functions are of the same form. The only difference is to determine or approximate the demands' distributions for a given lead time L . Hence from this subsection, in the model development and in deriving the analytical results, we do not explicitly express the model as a function of the lead time. In particular, in the objective function, the demands (D_i, D_j) is just considered as a random variable.

Problem 3.6.1. (Two retailers) A common product is sold by two retailers $i = 1, 2$. Suppose each retailer has certain information for forecasting the future demand D_i of this product. How do the retailers determine their order quantities O_i ?

It is possible for one retailer to maximize the expectation of her profit with known order quantities of the other retailer.

Differentiation of Expected Profit

For retailers i and j (here $i, j = 1, 2$ and $i \neq j$), suppose they are willing to share all the residuals whenever the other encounters stock-out. Let $r_{ij}^j(r_{ji}^j)$ be the profit of j for sharing one unit of product from $i(j)$ to $j(i)$. Because selling one unit of product only gain r , it is necessary to assume that $r \geq r_{ij}^i + r_{ij}^j$. We assume retailers have equal bargaining powers, then

it is reasonable to assume $r_{ij}^j + r_{ji}^j \leq r$.

$$\begin{aligned}
& \text{Profit}_j(O_j) \\
&= \text{Basic Profit}_j(O_j) + \text{Additional Profit}_j(O_j) \\
&= r \min(O_j, D_j) - h(O_j - D_j)_+ - O_j q \\
&\quad + (r_{ji}^j + h) \min((O_j - D_j)_+, (D_i - O_i)_+) \\
&\quad + r_{ij}^j \min((O_j - D_j)_-, (D_i - O_i)_-) \\
&= r D_j - r(O_j - D_j)_- - h(O_j - D_j)_+ - O_j q \\
&\quad + (r_{ji}^j + h)(O_j - D_j)_+ - (r_{ji}^j + h)((D_i - O_i)_+ - (O_j - D_j)_+)_- \\
&\quad + r_{ij}^j(O_j - D_j)_- - r_{ij}^j((D_i - O_i)_- - (O_j - D_j)_-)_-. \tag{3.7}
\end{aligned}$$

Take the expectation,

$$\begin{aligned}
& \mathbb{E}[\text{Profit}_j](O_j) \\
&= \mathbb{E}[r D_j - r(O_j - D_j)_- - h(O_j - D_j)_+ - O_j q] \\
&\quad + (r_{ji}^j + h)(O_j - D_j)_+ - (r_{ji}^j + h)((D_i - O_i)_+ - (O_j - D_j)_+)_- \\
&\quad + r_{ij}^j(O_j - D_j)_- - r_{ij}^j((D_i - O_i)_- - (O_j - D_j)_-)_-] \\
&= \mathbb{E}[r D_j] - r \mathbb{E}[(O_j - D_j)_-] - h \mathbb{E}[(O_j - D_j)_+] - O_j q \\
&\quad + (r_{ji}^j + h) \mathbb{E}[(O_j - D_j)_+] - (r_{ji}^j + h) \mathbb{E}[(D_i - O_i)_+ - (O_j - D_j)_+)_-] \\
&\quad + r_{ij}^j \mathbb{E}[(O_j - D_j)_-] - r_{ij}^j \mathbb{E}[(D_i - O_i)_- - (O_j - D_j)_-)_-]. \tag{3.8}
\end{aligned}$$

Similar as single retailer model, by Theorem 6, when $\mathbb{P}(D_j = O_j) = 0$, $\mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) = 0$ and $\mathbb{P}((O_j - D_j)_- = (D_i - O_i)_-, O_j < D_j) = 0$,

$$\begin{aligned}
& \frac{d}{dO_j} \mathbb{E}[\text{Profit}_j](O_j) \\
&= -r \int_{O_j < D_j} -1 d\omega - h \int_{O_j > D_j} d\omega - q + (r_{ji}^j + h) \int_{O_j > D_j} d\omega
\end{aligned}$$

$$\begin{aligned}
& - (r_{ji}^j + h) \int_{O_j > D_j, (O_j - D_j) > (D_i - O_i)_+} d\omega + r_{ij}^j \int_{O_j < D_j} -1 d\omega \\
& - r_{ij}^j \int_{O_j < D_j, (D_i - O_i)_- < (O_j - D_j)_-} -1 d\omega \\
& = -q + r_{ji}^j + (r - r_{ji}^j - r_{ij}^j) \int_{O_j < D_j} d\omega \\
& - (r_{ji}^j + h) \int_{O_j > D_j, O_j > D_j + (D_i - O_i)_+} d\omega \\
& + r_{ij}^j \int_{O_j < D_j, O_j < D_j - (D_i - O_i)_-} d\omega. \tag{3.9}
\end{aligned}$$

Proof. The proof is in Appendix B.4, Property 34. $\square$

Furthermore $\mathbb{E}[\text{Profit}_j](O_j)$ is absolutely continuous, so it is differentiable a.e. and

$$\mathbb{E}[\text{Profit}_j] \big|_{O_j=O_1} - \mathbb{E}[\text{Profit}_j] \big|_{O_j=O_2} = \int_{O_2}^{O_1} \frac{d}{dO_j} \mathbb{E}[\text{Profit}_j] dO_j.$$

Since by (3.9), $\frac{d}{dO_j} \mathbb{E}[\text{Profit}_j](O_j)$ is monotonic decreasing function of O_j (a.e.), then similar to the proof of Theorem 8, $\mathbb{E}[\text{Profit}_j](O_j)$ is concave function of O_j , there is an interval $[a, b]$, such that $\mathbb{E}[\text{Profit}_j](O_j)$ strictly increases to point a , reaches maximum in $[a, b]$, and then strictly decreases from point b . The optimal feasible solution should be in $[0, N_j^O]$.

Continuity of Best Response

Consider the case where there is a unique optimal feasible solution for a fixed O_i . For example when $\mathbb{P}(D_j > O_j)$ is strictly decreasing by O_j .

Property 19. *If $\mathbb{P}(D_j > O_j)$ is strictly decreasing by O_j in $[0, N_j^O]$, then it leads to a unique optimal feasible solution of $\mathbb{E}[\text{Profit}_j](O_i, O_j)$ for $O_j \in [0, N_j^O]$ with a fixed O_i .*

Proof. For a fixed O_i , $\mathbb{P}(D_j > O_j)$ is strictly decreasing by O_j in $[0, N_j^O]$ and

it implies that (3.9) is strictly decreasing by O_j in $[0, N_j^O]$. By

$$\mathbb{E}[\text{Profit}_j] |_{O_j=O_1} - \mathbb{E}[\text{Profit}_j] |_{O_j=O_2} = \int_{O_2}^{O_1} \frac{d}{dO_j} \mathbb{E}[\text{Profit}_j] dO_j,$$

there is a unique optimal feasible solution of O_j for a fixed O_i . $\square$

Hence $\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)$ exists. Furthermore we have the following property.

Property 20. $\mathbb{E}[\text{Profit}_j](O_i, O_j)$ is uniformly continuous of O_i for all O_j .

Proof. The proof is in Appendix B.4.1 $\square$

By Theorem 7, the following corollary holds.

Corollary 1. $\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)(O_i)$ is continuous in $[0, N_i^O]$.

Proof. Because by Property 20, $\mathbb{E}[\text{Profit}_j](O_i, O_j)$ is uniformly continuous of O_i for all $O_j \in [0, N_j^O]$; $\mathbb{E}[\text{Profit}_j](O_i, O_j)$ is continuous of O_j in $[0, N_j^O]$ for fixed O_i ; by Property 19, $\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)(O_i)$ exists; by Theorem 7, $\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)(O_i)$ is continuous in $[0, N_i^O]$. $\square$

When both $\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)(O_i)$ and $\text{BR}_{\mathbb{E}[\text{Profit}_i]}(O_i, O_j)(O_j)$ are continuous, then compose function $\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)(\text{BR}_{\mathbb{E}[\text{Profit}_i]}(O_i, O_j)(O_j))$ is continuous too. With appropriate conditions, the following fixed point exists

$$\text{BR}_{\mathbb{E}[\text{Profit}_j]}(O_j, O_i)(\text{BR}_{\mathbb{E}[\text{Profit}_i]}(O_i, O_j)(O_j)) = O_j.$$

It represents a pure Nash equilibrium.

Continuity of Partial Derivatives of Profit

We already give some conditions about first differentiation existence and the continuity of expected profit. We further discuss the continuity of partial derivatives of expected profit.

The total profit is the sum of all individuals' profits.

$$\mathbb{E}[\text{Total Profit}](O_1, O_2) = \mathbb{E}[\text{Profit}_1](O_1, O_2) + \mathbb{E}[\text{Profit}_2](O_1, O_2).$$

By (3.7) and (3.8),

$$\begin{aligned} & \mathbb{E}[\text{Total Profit}](O_1, O_2) \\ &= \sum_{\substack{j=1,2 \\ i \neq j; i=1,2}} (\mathbb{E}[rD_j] - r\mathbb{E}[(O_j - D_j)_-] - h\mathbb{E}[(O_j - D_j)_+] - O_jq \\ & \quad + (r_{ji}^j + h)\mathbb{E}[(O_j - D_j)_+] - (r_{ji}^j + h)\mathbb{E}[(D_i - O_i)_+ - (O_j - D_j)_+]) \\ & \quad + r_{ij}^j(O_j - D_j)_- - r_{ij}^j\mathbb{E}[(D_i - O_i)_- - (O_j - D_j)_-]). \end{aligned}$$

For fixed a O_1 (or O_2), except countable many points where $\mathbb{P}(D_j = O_j) > 0$ or $\mathbb{P}(D_j - O_j = O_i - D_i, O_j < D_j) > 0$ or $\mathbb{P}(D_j - O_j = O_i - D_i, O_j > D_j) > 0$, $\mathbb{E}[\text{Profit}_i](O_i, O_j)$ is differentiable for O_2 (or O_1). Therefore for fixed O_1 (or O_2), except countable many points, $\mathbb{E}[\text{Total Profit}](O_1, O_2)$ is differentiable.

Property 21. Suppose for any fixed O_i , and for all O_j , $\mathbb{P}(D_j = O_j) = 0$, $\mathbb{P}(O_j - D_j = D_i - O_i, O_j > D_j) = 0$ and $\mathbb{P}(O_j - D_j = D_i - O_i, O_j < D_j) = 0$. It results in the partial derivatives $\frac{\partial}{\partial O_1}\mathbb{E}[\text{Profit}_1](O_1, O_2)$, $\frac{\partial}{\partial O_1}\mathbb{E}[\text{Profit}_2](O_1, O_2)$, $\frac{\partial}{\partial O_1}\mathbb{E}[\text{Total Profit}](O_1, O_2)$, $\frac{\partial}{\partial O_2}\mathbb{E}[\text{Profit}_1](O_1, O_2)$, $\frac{\partial}{\partial O_2}\mathbb{E}[\text{Profit}_2](O_1, O_2)$ and $\frac{\partial}{\partial O_2}\mathbb{E}[\text{Total Profit}](O_1, O_2)$ to exist everywhere. The partial derivatives $\frac{\partial}{\partial O_1}\mathbb{E}[\text{Profit}_1](O_1, O_2)$, $\frac{\partial}{\partial O_1}\mathbb{E}[\text{Profit}_2](O_1, O_2)$, $\frac{\partial}{\partial O_1}\mathbb{E}[\text{Total Profit}](O_1, O_2)$, $\frac{\partial}{\partial O_2}\mathbb{E}[\text{Profit}_1](O_1, O_2)$, $\frac{\partial}{\partial O_2}\mathbb{E}[\text{Profit}_2](O_1, O_2)$ and $\frac{\partial}{\partial O_2}\mathbb{E}[\text{Total Profit}](O_1, O_2)$ are all continuous at (O_1, O_2) .

Proof. The proof is in Appendix B.4.2 □

By the well-known result that a function is differentiable, if the partial derivatives are continuous, it means that $\mathbb{E}[\text{Profit}_1](O_1, O_2)$, $\mathbb{E}[\text{Profit}_2](O_1, O_2)$ and $\mathbb{E}[\text{Total Profit}](O_1, O_2)$ are differentiable. With suitable additional conditions, like the condition of twice differentiable, it is possible to find

the first-best solution(optimal expected total profit) or Nash equilibrium for the sharing problem.

3.6.2 Sharing Model with Limited Knowledge of Demand Distribution

However in practice, demand distribution in analytical may be difficult to obtain or approximate accurately. As in (3.7) and (3.8), the joint demand distribution and the order quantities of all retailers are required to compute the expected profit of each retailer. Even if information is completely shared, it is often difficult to determine their trustworthiness, as it is usually not clear whether the related data are well validated. Another problem about shared information is that other retailers usually do not take responsibility for any loss caused by unreliable information. Inaccurate information can lead to large forecast errors, which results in excessive inventories or serious stock-outs. Hence, rather than relying on other retailers' demand related information which can be uncertain, retailers may choose to use more reliable information such as inventory levels, ordered quantities and historical data. Hence in practice the retailers usually determine their order quantities with limited knowledge of demand (such as historical data).

Simulation Methods

First, in practice it is possible to draw samples from historical demands. Hence it makes simulation-based methods available.

Method 1:

In subsection 3.6.1, we consider the two-retailer model under complete sharing policy with the following conditions: 1. $\mathbb{P}(D_j = O_j) = 0$, $\mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) = 0$ and $\mathbb{P}((O_j - D_j)_- = (D_i - O_i)_-, O_j < D_j) = 0$.

$D_j) = 0$; 2. $\mathbb{P}(D_j > O_j)$ is strictly decreasing. By Theorem 13, there is a unique zero in (3.9) which is the optimal solution.

$$\begin{aligned}
& \frac{d}{dO_j} \mathbb{E}[\text{Profit}_j](O_j) \\
&= -q + r_{ji}^j + (r - r_{ji}^j - r_{ij}^j) \mathbb{P}(D_j > O_j) \\
&\quad - (r_{ji}^j + h) \mathbb{P}(D_j < O_j, O_j > D_j + (D_i - O_i)_+) \\
&\quad + r_{ij}^j \mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-) \\
&= 0.
\end{aligned} \tag{3.10}$$

Similar to the newsvendor problem in section 3.5.3, it can be approximated by the simulation method with samples of (D_i, D_j) . That is when number of simulations is big enough, the value at O^* such that (3.10) holds approximates the optimal solution. As shown in subsection 3.5.2, the optimization problem in multiple periods is similar. The only difference is in the approximation for the distribution of $(\hat{D}_i, \hat{D}_j)$.

Method 2:

An alternative method based on simulation is to approximate the expected profit by the average of profits from samples.

$$\begin{aligned}
& \max_{O_j} \mathbb{E}[\text{Profit}_j(O_j)] \\
& \approx \max_{O_j} \frac{1}{N} \sum_{n=1}^N (r \min(O_j, x_j^{(n)}) - h(O_j - x_j^{(n)})_+ - O_j q + (r_{ji}^j + h) \min \\
& \quad ((O_j - x_j^{(n)})_+, (x_i^{(n)} - O_i)_+) + r_{ij}^j \min((O_j - x_j^{(n)})_-, (x_i^{(n)} - O_i)_-)) \tag{3.11}
\end{aligned}$$

For each sample $(x_i^{(n)}, x_j^{(n)})$ and a fixed O_i , $r \min(O_j, x_j^{(n)}) - h(O_j - x_j^{(n)})_+ - O_j q + (r_{ji}^j + h) \min((O_j - x_j^{(n)})_+, (x_i^{(n)} - O_i)_+) + r_{ij}^j \min((O_j - x_j^{(n)})_-, (x_i^{(n)} - O_i)_-)$ represents the profits which is a concave piecewise linear function of O_j . It results in the summation (3.11) to be concave too. It means that maximizing $\mathbb{E}[\text{Profit}_j(O_j)]$ is equivalent to solving a convex programming.

3.6.3 Balanced System

The order quantities of the other retailer has to be known in the above methods, otherwise it becomes a cooperative game between the two retailers which is much more complicated. Complexity also depends on the retailer interaction when deciding their order quantities. For example, it could happen in synchronized ordering system where retailers order at same time in every replenishment cycle; this can lead to game-based interaction between retailers on their ordering decisions. This is in contrast balanced ordering system where retailers spread their order timing evenly over time. In this ordering policy, each retailer decides on her order quantity based on past known order quantities of other retailers, thus avoiding game-based interactions. We consider with the same length of replenishment cycle in synchronized system retailers order and replenishment at the same time, and the balanced system spread both ordering times and replenishment times evenly. The advantage of the balanced system is not only about enhancing modeling tractability but also in improving the performance of the sharing system.

For a two-retailer sharing system, there exists backorders if system-wide stock-out occurs, i.e., if $I_1^t + I_2^t < 0$. In a synchronized system, both retailers experience their lowest inventory levels concurrently, i.e., at the end of every replenishment cycles. This implies that the lowest system-wide inventory level happens at the same time. In contrast, the balanced ordering system avoids this scenario since the arrival of replenishments are timed such that one retailer has relatively high inventory level when the other's inventory level is low. Clearly, the probability of stock-outs in a balanced ordering system tends to be smaller than in a synchronized system. The next example illustrates and contrasts their differences.

Example 5. Assume the replenishment cycle $T = 2$. Thus in the synchronized system (Figure 3.3), the two retailers replenish at the same time, at time 0, 2,

4, 6, 8, In the balanced system (Figure 3.4), retailer 1 and 2 replenish at different times, kT and $1 + kT$ respectively. Hence in the balanced case, the lowest inventory levels of both retailers do not happen at same time. The probability of stock-outs in the balanced ordering system is smaller.

Figure 3.3: Synchronized ordering system

Figure 3.4: Balanced ordering system

In fact the balanced ordering system indeed decreases the variance of $I_1^t + I_2^t < 0$. It is helpful to avoid the lost sales.

Simulation

In order to show the impact of the balanced system, we set all assumptions and parameters the same, except the time of replenishment. The

simulations are programmed in R language. All the codes and results are available in Appendix C.4

Problem 3.6.2. Consider a two-retailer inventory sharing system. How does the replenishment timing affect the number of sharings and the number of lost sales?

We simulate the balanced system and the synchronized system simultaneously with the same demand samples D_i^t and the same order quantities O_i^t . We then count the number of sharings and the number of lost sales resulted from for both systems. The process is described in the following:

1. Randomly generate the integer demand samples D_i^t for each retailer in every two periods. Set each retailer's replenishment order quantities O_i^t which are the same for both balanced and synchronized cases. The only difference for the two cases is the replenishment timing. Assume both retailers replenish at odd periods for the synchronized case. In the balanced case one retailer replenishes at odd periods while the other retailer at even periods.
2. Calculate the residual inventories and unfulfilled demands in current period by the inventory balanced equation

$$I_i^t = (I_i^{t-1})_+ + O_i^{t-1} - D_i^t.$$

3. Retailers share the residual inventories and unfulfilled demands in one period. Count the number of sharings and the number of lost sales in current period.
4. If current period is less than M , go to next period and repeat Step 2 and 3.

Scenario 1:

First we generate D_i^t from $[60, 140]$, set ordering quantities O_i^t as fixed numbers in $[90, 110]$ and let $M = 21$. For each setting of order quantities, we simulate 10000 times. The average number of sharings for Scenario 1 is stated in Table C.1. It shows that the balanced case has more sharings than the synchronized case. Meanwhile the balanced case results in lower average number of lost sales than in the synchronized case. That is shown in Table C.2.

Scenario 2:

In practice the order quantities in different periods may fluctuate by multiple factors. To simulate this, we add a random error ϵ_1 for each order quantity setting. We generate D_i^t from $[60, 140]$ and set order quantities as O_i^t in $[90, 110]$ and include a random number ϵ_1 . Again we let $M = 21$, and simulate 10000 times for each setting of order quantity. The average numbers of sharings for Scenario 2 are stated in Table C.3. It shows that the balanced case has more sharings than the synchronized case. Meanwhile the average number of lost sales in Table C.4 shows that the balanced case has fewer lost sales than the synchronized case.

Scenario 3:

Finally, in this scenario, we generate the demands with different demand means. We adjust the order quantities by adding the means' difference every two periods. We generate D_i^t from $[60, 140]$ and add a fluctuation term ϵ_2 to it. Since each retailer orders once in every two periods, and adjust the mean fluctuation every two periods. In particular, we set the order quantity as

$$O_i^t + \epsilon_{1i}^t + \epsilon_{2i}^{t+1} + \epsilon_{2i}^{t+2}.$$

We let $M = 21$, and simulate 10000 times for each setting of order quantity. We show the average number of sharings in Table C.5 and the average number of lost sales in Table C.6. It indicates that the balanced case has more sharings and fewer lost sales than the synchronized case.

As shown in Appendix C.1, smaller order quantities lead to more stock-outs and results in more sharings. Under the same ordering setting the balanced system provides more opportunities than the synchronized system to share the residuals. Therefore it results in fewer lost sales.

For example, Figure 3.5 shows the histograms of the number of lost sales for both the balanced and synchronized cases, while order settings are: 1. less than demand mean; 2. equal to demand mean; 3. more than demand mean.

Figure 3.5: Histograms for lost sale

Therefore we conclude that for two retailers under the same ordering setting, the number of lost sales in balanced system is significantly fewer than in the synchronized system.

3.7 Inventory Model for Multi-Retailers

In this section, we extend the two-retailer model to multi-retailer case. First in subsection 3.7.1 we define three principles for sharing policy and demonstrate a sharing policy based on distances. In the next subsection 3.7.2, we focus on modeling the multi-retailer problem based on the principles we defined. We derived some properties of the model. Eventually we show the concavity of this model and translate the model into a convex programming problem by simulation-based approach. The last subsection 3.7.3 we integrate the balance ordering system with the multi-retailer model, and show its benefits to the sharing system by the simulation study.

3.7.1 Sharing Policy

We already have the two-retailer model under the complete sharing and complete information sharing. We rewrite it as

$$\begin{aligned}
& \text{Profit}_j(O_j) \\
&= r \min(O_j, D_j) - h(O_j - D_j)_+ - O_j q \\
&\quad + (r_{ji}^j + h) \min((O_j - D_j)_+, (D_i - O_i)_+) \\
&\quad + r_{ij}^j \min((D_j - O_j)_+, (O_i - D_i)_+) \\
&= r D_j - r(D_j - O_j)_+ - h(O_j - D_j)_+ - O_j q \\
&\quad + (r_{ji}^j + h)(D_i - O_i)_+ - (r_{ji}^j + h)((D_i - O_i)_+ - (O_j - D_j)_+)_+ \\
&\quad + r_{ij}^j(O_i - D_i)_+ - r_{ij}^j((O_i - D_i)_+ - (D_j - O_j)_+)_+. \tag{3.12}
\end{aligned}$$

However, in generally for multiple retailers it is very difficult to extend this model. The main difficulty lies with the numerous ways of pairing the retailers for sharing. The sharing process of n retailers, which we now define as the process of determining “who share with whom”, is not deterministic. In addition, it is questionable to complete share residuals.

For multi-retailer problem, we state the steps of the three-stage sharing scheme defined in Granot and Sošić (2003).

1. After demands are realized, net inventories are determined;
2. Each retailer determines the amount of inventory residuals or the unfulfilled demands to share;
3. Share the inventory residuals and the unfulfilled demands within a sharing rule.

Some previous works assume retailers form a cooperative alliance, where they maximize the system-wide profit in step 3, and then allocate the profit to each retailer. It requires high degree of system-wide cooperative between retailers. However Granot and Sošić (2003) show that there is no core complete sharing allocation rule for four or more retailers, and no value preserving allocation rule for six or more retailers. It means either the cooperative alliance or the cooperative game is not stable. Hence, we assume there is no system-wide cooperative alliance for individual retailers. That is, the retailers only focus on their own self-interest and share freely.

First we provide an example of sharing based on inter-retailer distances. The sharing policy is simulated in the following example.

Example 6 (A Sharing Policy based on Distances). *Suppose retailers are willing to share all their residual inventories. Assume that they prefer to share with retailer who locates as near as possible, so they determine the sharing orders by distances. If one retailer i has stock-out, she will ask for sharing with the nearest retailer. If the demand is still unfulfilled, she will looking for sharing with the second nearest retailer. Meanwhile if retailer j has residual inventories to share, she will consider the sharing requests from the nearest retailer first.*

Among all sharing opportunities, if the distance between retailer i and j is the shortest, then there is no reason for them to reject this sharing. Hence the sharing process can be determined as follows.

- 1 Sort all distances between retailers.
- 2 Share between two retailers, only if among all pairs of retailers who have unfulfilled demands and inventory residuals, the distance between them is the shortest.
- 3 Determine all sharings one by one in the order of distance.

We simulate the sharing process with complete sharing by the R language. First we randomly generate 10 locations for 10 individual retailers in a 50×50 map. Then we randomly generate the residual inventories and unfulfilled demands. Finally we determine the sharing order by the distances between retailers. As shown in Figure 3.6, 10 randomly generated locations are marked in blue for 10 retailers. The number of residual inventories and unfulfilled demands are in black. The red lines indicate the pairing of retailers. The first sharing of 12 units is between location 1 and 3 which has the shortest distance for all possible sharing opportunity. The second sharing of 17 units is between 2 and 8. The third sharing of 23 units is between 3 and 4. Finally the fourth sharing of 2 units is between 2 and 3. In the end all the retailers have non-negative inventory levels. The related R codes are in Appendix

We note that under this distance-based sharing rule, once the inventory residuals and unfulfilled demands are realized the sharing process are decided. C.5

Figure 3.6: Sharing Process

Next we present some principles for inventory sharing between individual retailers which significantly reducing the complexity of the model.

Assume retailer i has inventory level $I_i^t < 0$ and retailer j has $I_j^t > 0$. It means there is a sharing opportunity between them. Without consider other sharings in the future, this sharing between i, j does not affect retailer i . Because her leftover stock is still $(I_i^t)_+ = 0$, by inventory balanced equation, $I_i^{t+1} = (I_i^t)_+ + O_i^{t+1} - D_i^t$ is unchanged. Meanwhile retailer j needs to consider the risk of suffering future stock-out if she shares out her stocks. Because each unit shared out will decrease $(I_j^t)_+$ by 1, it is thus more likely for stock-out to occur given $(I_j^{t+1})_- = ((I_j^t)_+ + O_j^{t+1} - D_j^t)_-$.

Principle 1. Retailer i with $I_i^t < 0$ always like to share.

Principle 2. Due to the risk of future stock-out, retailer j with $I_j^t > 0$ may only share part of her inventory. We assume it is a rate θ .

Assume any two sharings between i, j with a certain amount are identical. Hence it is comparable for the pairs of $\{i, j\}$.

Principle 3. *There is a total priority for all the pairs of $\{i, j\}$. Therefore according to the total priority, each retailer has own priority. Retailers attempt sharing by the total priority.*

For example, the total priority can be compared by the distances (Example 6) or transshipment fees.

3.7.2 Modeling

With the three principles in previous section, we show that the sharing process is deterministic. Furthermore we build the model of one individual retailer's profit.

In period t , after demands D_i are realized, retailer i has the net inventory I_i^t . By Principle 1 and 2, she is willing to share $-I_i^t$ unfulfilled demand if $I_i^t < 0$ (or θI_i^t inventory residuals if $I_i^t > 0$). Since the sharing process completes in one period t , for simplicity we denote I_i^t (or θI_i^t) as I_i , D_i^t as D_i and O_i^t as O_i . Moreover we define $\mathbf{I} = (I_1, I_2, \dots, I_n)$, $\mathbf{D} = (D_1, D_2, \dots, D_n)$ and $\mathbf{O} = (O_1, O_2, \dots, O_n)$. By Principle 3, there is a total priority $>$ is for all $\{i, j\}$.

Under the given priority $>$, and an initial I , the sharing process can be decided by sharing attempt in the order of priority $>$.

1. Follow the total priority $>$, to pick a (i, j) .
2. Then the sharing amount from i to j is $\min((I_i)_+, (I_j)_-)$ and the sharing amount from j to i is $\min((I_i)_-, (I_j)_+)$. Meanwhile compute

the following:

$$\begin{cases} (I_i)_+ = (I_i)_+ - \min((I_i)_+, (I_j)_-) \\ (I_j)_- = (I_j)_- - \min((I_i)_+, (I_j)_-) \\ (I_i)_- = (I_i)_- - \min((I_i)_-, (I_j)_+) \\ (I_j)_+ = (I_j)_+ - \min((I_i)_-, (I_j)_+) \end{cases} \quad (3.13)$$

3. If (i, j) is the last, then stop; otherwise go back to 1.

All terms (as the terms $(I_i)_+$, $\min((I_i)_-, (I_j)_+)$, $\dots$) at any step (i, j) above are determined by initial $I = (I_1, I_2, \dots, I_n)$. Therefore they are functions of I . Furthermore their expressions are decided by the priority $>$. Therefore the sharing process is deterministic. That is for a fixed I , all terms in step 2 are fixed. Because $>$ is fixed, then the priority of retailer i is also fixed, then assume this priority of retailer i is $\{i_1 > i_2 > \dots i_{n-1}\}$.

By the inventory balance equation and the assumption that each retailer decides the initial amount she is willing to share with others by her own net inventory, the initial I_i is determined by (O_i, D) . Hence we assume $I_{i,i_1}^+(O_i, D)$ and $I_{i,i_1}^-(O_i, D)$ are the non-negative amounts of inventory residuals and the unfulfilled demands respectively that she willing to share with retailer i_1 . Other functions of the initial I also induce the functions of (O, D) . It means all sharing amounts or inventory levels are functions of (O, D) . Hence we assume $I_{i_k}^+(O, D)$ is the non-negative amount of residual inventories that retailer i_k is willing to share with retailer i ; $I_{i_k}^-(O, D)$ is the non-negative amount of stock-outs that retailer i_k is willing to share with retailer i ; $I_{i,i_k}^+(O, D)$ is the non-negative amount of residual inventories that retailer i is willing to share with retailer i_k ; $I_{i,i_k}^-(O, D)$ is the non-negative amount of stock-outs that retailer i is willing to share with retailer i_k . For consistency, we express the inventory residuals after demands realized as $I_i(O_i, D)_+ = \max(O_i + (I_i^{t-1})_+ - D_i, 0)$ and the unfulfilled demands

after demands realized as $I_i(O_i, D)_- = -\min(O_i + (I_i^{t-1})_+ - D_i, 0)$. Let $I_{i,i_1}^+(O, D) = I_{i,i_1}^+(O_i, D)$ and $I_{i,i_1}^-(O, D) = I_{i,i_1}^-(O_i, D)$.

Hence retailer i 's profit is

$$\begin{aligned}
& \text{Profit}_i(O, D) \\
&= r(D_i - I_i(O_i, D)_-) - hI_i(O_i, D)_+ - O_i q \\
&\quad + (r_{i,i_1}^i + h) \min(I_{i,i_1}^+(O_i, D), I_{i,i_1}^-(O, D)) \\
&\quad + r_{i_1,i}^i \min(I_{i,i_1}^-(O_i, D), I_{i_1}^+(O, D)) + \dots \\
&\quad + (r_{i,i_{n-1}}^i + h) \min(I_{i,i_{n-1}}^+(O, D), I_{i_{n-1}}^-(O, D)) \\
&\quad + r_{i_{n-1},i}^i \min(I_{i,i_{n-1}}^-(O, D), I_{i_{n-1}}^+(O, D)), \tag{3.14}
\end{aligned}$$

where $\min(I_{i,i_1}^+(O_i, D), I_{i,i_1}^-(O, D))$ is the amount of inventory sharing from retailer i to retailer i_1 . After sharing attempt with i_1 and before sharing with i_2 ,

$$\begin{aligned}
I_{i,i_2}^+(O, D) &= I_{i,i_1}^+(O_i, D) - \min(I_{i,i_1}^+(O_i, D), I_{i,i_1}^-(O, D)) \\
&= (I_{i,i_1}^+(O_i, D) - I_{i,i_1}^-(O, D))_+;
\end{aligned}$$

similarly $I_{i,i_2}^-(O, D) = (I_{i,i_1}^-(O_i, D) - I_{i,i_1}^+(O, D))_+$; moreover there are

$$\begin{cases} I_{i,i_{k+1}}^+(O, D) = (I_{i,i_k}^+(O, D) - I_{i,i_k}^-(O, D))_+ \\ I_{i,i_{k+1}}^-(O, D) = (I_{i,i_k}^-(O, D) - I_{i,i_k}^+(O, D))_+ \end{cases}, \quad \text{for } k = 1, 2, \dots, n-2. \tag{3.15}$$

Intuitively, suppose at a certain step of the sharing process, retailer i has positive inventory residual. If retailer i were increased her initial I_i by 1 unit, then sharing process should not be affected before this unit be shared-out. Next we mathematically prove this. In fact it is to rewrite the expression of (3.14). In order to show more clearly, we denote $O_i- :=$

$(O_1, O_2, \dots, O_{i-1}, O_{i+1}, \dots, O_n)$, so $(O, D) = (O_i, O_{i-}, D)$. Note that O_{i-} is fixed means the order quantities of other retailers are fixed.

Property 22. Let $S_+ := \{I_{i,i_k}^+(O, D) > 0\}$. If $x_1, y_1 \in S_+$, $x_1 = (O_i^{x_1}, O_{i-}, D)$ and $y_1 = (O_i^{y_1}, O_{i-}, D)$, then $I_{i_k}^-(x_1) = I_{i_k}^-(y_1)$. Let $S_- := \{I_{i,i_k}^-(O, D) > 0\}$. If $x_2, y_2 \in S_-$, $x_2 = (O_i^{x_2}, O_{i-}, D)$ and $y_2 = (O_i^{y_2}, O_{i-}, D)$, then $I_{i_k}^+(x_2) = I_{i_k}^+(y_2)$.

Intuitively, Property 22 shows that the amount that retailer i_k is willing to share with retailer i is fixed when the sharing between i and i_k exists. It is because these: 1. under the sharing rule the sharing order is fixed. 2. The sharing between i and i_k exists means that before this sharing attempt, there is enough inventory residuals or unfulfilled demands for retailer i to share. Hence when the other inventory residuals and unfulfilled demands are fixed. The sharing process is unique before this sharing attempt.

Proof. By assumption, initially functions $I_{i,i_1}^+(O_i, D)$ and $I_{i,i_1}^-(O_i, D)$ are determined by O_i and D_i . For x_1 and y_1 the only difference is O_i , hence initially for $j \neq i$, $I_{j,j_1}^+(x_1) = I_{j,j_1}^+(y_1)$ and $I_{j,j_1}^-(x_1) = I_{j,j_1}^-(y_1)$. By assumption $I_{i,i_k}^+ > 0$ at x_1 and y_1 , it means $I_{i,i_l}^+ > 0$ and $I_{i,i_l}^- = 0$ for $l \leq k$.

We only need show in the recursion formula (3.13), $\min(I_{i,i_l}^+, I_{i_l}^-)(x_1) = \min(I_{i,i_l}^+, I_{i_l}^-)(y_1)$ and $\min(I_{i,i_l}^-, I_{i_l}^+)(x_1) = \min(I_{i,i_l}^-, I_{i_l}^+)(y_1)$ for all $l < k$. It is obviously, $\min(I_{i,i_l}^-, I_{i_l}^+)(x_1) = \min(I_{i,i_l}^-, I_{i_l}^+)(y_1) = 0$. For $l \leq k$, $I_{i,i_l}^+ > 0$, it implies $I_{i,i_{l-1}}^+ - \min(I_{i,i_{l-1}}^+, I_{i_{l-1}}^-) > 0$, i.e. $\min(I_{i,i_{l-1}}^+, I_{i_{l-1}}^-) = I_{i_{l-1}}^-$. By mathematical induction, since initially for $j \neq i$, $I_{j,j_1}^+(x_1) = I_{j,j_1}^+(y_1)$ and $I_{j,j_1}^-(x_1) = I_{j,j_1}^-(y_1)$, it implies that in the recursion formula (3.13), function $\min((I_i)_+, (I_j)_-)$ and $\min((I_i)_-, (I_j)_+)$ are same at x_1 and y_1 before computing the pair $\{i, i_k\}$. Hence I_j are always same at x_1 and y_1 (where $j \neq i$) before computing the pair $\{i, i_k\}$. It means $I_{i_k}^-(x_1) = I_{i_k}^-(y_1)$.

Similarly, we have $I_{i_k}^+(x_2) = I_{i_k}^+(y_2)$. □

We note that in the proof of Property 22 there are no requirements for

the expressions of I_{j,j_1}^+ and I_{j,j_1}^- . In fact all the functions are the functions of initial I composing with function $I(O, D)$ which determines initial I . The Property 22 generally holds for initial I , where $I_j(j \neq i)$ is fixed and $I_{i,i_k}^+(I) > 0$. Hence the only requirement for I_{j,j_1}^+ and I_{j,j_1}^- is that they are determined by (O_i-, D) which implies $I_j(j \neq i)$ is fixed. In other words, Property 22 holds for the sharing policies in which the amount of inventory to share is determined by retailer herself.

Now we rewrite the terms $\min(I_{i,i_k}^+(O, D), I_{i,i_k}^-(O, D))$ and $\min(I_{i,i_k}^-(O, D), I_{i,i_k}^+(O, D))$ in (3.14).

Property 23. $\min(I_{i,i_k}^+(O, D), I_{i,i_k}^-(O, D))$ and $\min(I_{i,i_k}^-(O, D), I_{i,i_k}^+(O, D))$ can be expressed as $\min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D))$ and $\min(I_{i,i_k}^-(O, D), C_{i_k}^+(O_i-, D))$, where $C_{i_k}^-(O_i-, D)$ and $C_{i_k}^+(O_i-, D)$ are non-negative constants when (O_i-, D) is fixed.

Proof. By property 22, a function is well-defined as

$$C_{i_k}^-(O_i, O_i-, D) = \begin{cases} I_{i_k}^-(x) & \text{if there is a } x = (O'_i, O_i-, D) \in S_+ \\ 0 & \text{if there is no } x = (O'_i, O_i-, D) \in S_+ \end{cases} \quad (3.16)$$

Obviously $C_{i_k}^-(O_i, O_i-, D)$ is a non-negative function determined by (O_i-, D) . Denote it as $C_{i_k}^-(O_i-, D)$. Then we show that

$$\min(I_{i,i_k}^+(O, D), I_{i,i_k}^-(O, D)) = \min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D))$$

For any $x = (O, D) \in S_+$, by definition of $C_{i_k}^-(O_i-, D)$,

$$\min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D)) = \min(I_{i,i_k}^+(O, D), I_{i,i_k}^-(O, D)).$$

For any $x \notin S_+$

$$\min(I_{i,i_k}^+(O, D), I_{i,i_k}^-(O, D)) = 0 = \min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D)).$$

Similarly $\min(I_{i,i_k}^-(O, D), I_{i_k}^+(O, D)) = \min(I_{i,i_k}^-(O, D), C_{i_k}^+(O_i-, D))$. $\square$

Property 23 states that in (3.14), when (O_i-, D) is fixed, the term represents the amount of inventory transfer from retailer i to retailer i_k

$$\min(I_{i,i_k}^+(O, D), I_{i_k}^-(O, D)) = \min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D))$$

is determined by $I_{i,i_k}^+(O, D)$ which is the inventory of retailer i left from previous sharing attempts. In other words, when (O_i-, D) is fixed, the sharing amount $\min(I_{i,i_k}^+(O, D), I_{i_k}^-(O, D))$ between i and i_k is a piecewise function of O_i , where it values either 0 or $C_{i_k}^-(O_i-, D)$ out of the region $\{I_{i,i_k}^+(O, D) \in (0, C_{i_k}^-(O_i-, D))\}$.

The important feature of this model is that when the priority is ranked by the profit of sharings, then it is concave. By Property 23, we are able to express the model as follows.

$$\begin{aligned} & \text{Profit}_i(O_i, D; O_i-) \\ &= -O_i q + rD_i - rI_i(O_i, D)_- - hI_i(O_i, D)_+ \\ & \quad + (r_{i,i_1}^i + h) \min(I_{i,i_1}^+(O_i, D), C_{i_1}^-(D; O_i-)) \\ & \quad + r_{i_1,i}^i \min(I_{i,i_1}^-(O_i, D), C_{i_1}^+(D; O_i-)) + \dots \\ & \quad + (r_{i,i_{n-1}}^i + h) \min(I_{i,i_{n-1}}^+(O_i, D; O_i-), C_{i_{n-1}}^-(D; O_i-)) \\ & \quad + r_{i_{n-1},i}^i \min(I_{i,i_{n-1}}^-(O_i, D; O_i-), C_{i_{n-1}}^+(D; O_i-)), \end{aligned} \quad (3.17)$$

where

$$\begin{cases} I_{i,i_{k+1}}^+(O_i, D; O_i-) = (I_{i,i_k}^+(O_i, D; O_i-) - C_{i_k}^-(D; O_i-))_+ \\ I_{i,i_{k+1}}^-(O_i, D; O_i-) = (I_{i,i_k}^-(O_i, D; O_i-) - C_{i_k}^+(D; O_i-))_+ \end{cases},$$

for $k = 1, 2, \dots, n-1$. (3.18)

Here we consider O_i- is fixed.

Property 24. *If the deterministic sharing rule satisfies following conditions:*

1. *For a fixed D , $I_{i,i_1}^-(O_i, D) = \theta^- I_i(O_i, D)_-$ and $I_{i,i_1}^+(O_i, D) = \theta^+ I_i(O_i, D)_+$, where $0 \leq \theta^-, \theta^+ \leq 1$;*
2. *$r_{i,i_k}^i \geq r_{i,i_{k+1}}^i \geq 0$, $r_{i_k,i}^i \geq r_{i_{k+1},i}^i \geq 0$ and $r_{i_1,i}^i + r_{i,i_1}^i < r$.*

Then the $\text{Profit}_i(O_i, D; O_i-)$ is concave for fixed D .

The condition 1 in Property 24 is the stage of determining the amount of inventory to share. As assuming in Principle 1 and 2, the $\theta^- = 1$ and $\theta^+ = \theta$. As noted Property 22 holds for the sharing policies in which the amount of inventory to share is determined by retailer herself, the expression in (3.17) has no change when $0 \leq \theta^-, \theta^+ \leq 1$. The condition 2 in Property 24 means retailer i 's profit priority is consisted with total priority, since the order of $i_1 > i_2 > \dots > i_k$ is following the total priority. Under the equal bargaining power, we can suppose priority is the fewer cost of sharing (e.g. transshipment fee).

Proof. Obviously in (3.17), $\text{Profit}_i(O_i, D; O_i-)$ is a piecewise function of O_i and

$$\begin{aligned} & \frac{\partial}{\partial O_i} \text{Profit}_i(O_i, D; O_i-) \\ &= \begin{cases} -q + r & \text{for } O_i \in \{I_{i,i_{n-1}}^-(O_i, D; O_i-) > C_{i_{n-1}}^+(D; O_i-)\} \\ -q + r - \theta^- r_{i_k,i}^i & \text{for } O_i \in \{0 < I_{i,i_k}^-(O_i, D; O_i-) < C_{i_k}^+(D; O_i-)\} \\ -q - h + \theta^+ (r_{i,i_k}^i + h) & \text{for } O_i \in \{0 < I_{i,i_k}^+(O_i, D; O_i-) < C_{i_k}^-(D; O_i-)\} \\ -q - h & \text{for } O_i \in \{I_{i,i_{n-1}}^+(O_i, D; O_i-) > C_{i_{n-1}}^-(D; O_i-)\} \end{cases} \end{aligned} \quad (3.19)$$

Since $0 \leq \theta^-, \theta^+ \leq 1$, $r_{i,i_k}^i \geq r_{i,i_{k+1}}^i$, $r_{i_k,i}^i \geq r_{i_{k+1},i}^i$ and $r_{i_1,i}^i + r_{i,i_1}^i < r$, the slopes are decreasing. Hence the $\text{Profit}_i(O_i, D; O_i-)$ is concave. $\square$

By Property 24, it immediately implies that $\mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$ is concave.

Simulation based method:

The concavity is considered as a very important feature in optimization. There are a lot of existence methods and solvers to find or approximate the optimality of concave problem. However the practical problems for this model under the deterministic sharing rule are: 1. it is costly to obtain the analytical expressions of the amounts that retailers are willing to share; 2. in practice it is difficult to get the exact expression of the joint demand distribution for multiple retailers; 3. Taking the expectation of the joint demand distribution is extremely difficult.

Alternatively we are able to approximate the expected profit by the average of profits of samples $\mathbf{x}^{(n)}$ from D. That is

$$\begin{aligned}
& \mathbb{E}[\text{Profit}_i(\mathbf{O}, D)] \\
& \approx \frac{1}{N} \sum_{n=1}^N (r(x_i^{(n)} - I_i(O_i, \mathbf{x}^{(n)})_-) - hI_i(O_i, \mathbf{x}^{(n)})_+ - O_i q \\
& \quad + (r_{i,i_1}^i + h) \min(I_{i,i_1}^+(O_i, \mathbf{x}^{(n)}), C_{i_1}^-(O_i-, \mathbf{x}^{(n)})) \\
& \quad + r_{i_1,i}^i \min(I_{i,i_1}^-(O_i, \mathbf{x}^{(n)}), C_{i_1}^+(O_i-, \mathbf{x}^{(n)})) + \dots \\
& \quad + (r_{i,i_{n-1}}^i + h) \min(I_{i,i_{n-1}}^+(\mathbf{O}, \mathbf{x}^{(n)}), C_{i_{n-1}}^-(O_i-, \mathbf{x}^{(n)})) \\
& \quad + r_{i_{n-1},i}^i \min(I_{i,i_{n-1}}^-(\mathbf{O}, \mathbf{x}^{(n)}), C_{i_{n-1}}^+(O_i-, \mathbf{x}^{(n)}))), \tag{3.20}
\end{aligned}$$

Since for fixed D, (3.17) is concave and the summation of concave functions is still concave, the function (3.20) is concave of O_i . It means that maximizing $\mathbb{E}[\text{Profit}_i(\mathbf{O}, D)]$ is transferred to solving a convex programming (3.20). It avoids above three practical problems by solve (3.20) numerically.

We then show the partial differentiation of one individual retailer's profit. In order to implement the Extended Leibniz Rule (Theorem 6)

on the expectation of profit, the following condition for measurability is necessary.

Property 25. For any fixed O , suppose $I_{i,i_1}^-(O_i, D)$ and $I_{i,i_1}^+(O_i, D)$, are measurable functions of D . There are $I_{i,i_{k+1}}^-(O, D) = (I_{i,i_k}^-(O, D) - C_{i_k}^+(O_i-, D))_+$, $I_{i,i_{k+1}}^+(O, D) = (I_{i,i_k}^+(O, D) - C_{i_k}^-(O_i-, D))_+$, $\min(I_{i,i_k}^-(O, D), I_{i_k}^+(O, D)) = \min(I_{i,i_k}^-(O, D), C_{i_k}^+(O_i-, D))$, $\min(I_{i,i_k}^+(O, D), I_{i_k}^-(O, D)) = \min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D))$ and all the terms are measurable.

Proof. The proof is in Appendix B.5.1. □

Therefore we rewrite the profit of retailer i in (3.14) and (3.15) as follows,

$$\begin{aligned}
& \text{Profit}_i(O_i, O_i-, D) \\
&= r(D_i - I_i(O_i, D)_-) - hI_i(O_i, D)_+ - O_i q \\
& \quad + (r_{i,i_1}^i + h)(I_{i,i_1}^+(O_i, D) - I_{i,i_2}^+(O, D)) \\
& \quad + r_{i_1,i}^i(I_{i,i_1}^-(O_i, D) - I_{i,i_2}^-(O, D)) + \dots \\
& \quad + (r_{i,i_{n-1}}^i + h)(I_{i,i_{n-1}}^+(O, D) - I_{i,i_n}^+(O, D)) \\
& \quad + r_{i_{n-1},i}^i(I_{i,i_{n-1}}^-(O, D) - I_{i,i_n}^-(O, D)), \tag{3.21}
\end{aligned}$$

where

$$\begin{cases} I_{i,i_{k+1}}^+(O, D) = (I_{i,i_k}^+(O, D) - C_{i_k}^-(O_i-, D))_+ \\ I_{i,i_{k+1}}^-(O, D) = (I_{i,i_k}^-(O, D) - C_{i_k}^+(O_i-, D))_+ \end{cases}, \quad \text{for } k = 1, 2, \dots, n-1. \tag{3.22}$$

Note that for $k = 1, 2, \dots, n-2$, $I_{i,i_{k+1}}^+(O, D)$ and $I_{i,i_{k+1}}^-(O, D)$ represent the amount of inventory residuals and stock-outs that retailer i willing to share with retailer i_{k+1} ; for $k = n-1$, we just define $I_{i,i_{k+1}}^+(O, D)$ and $I_{i,i_{k+1}}^-(O, D)$ as the same way as in (3.22).

Let the objective function be expected profit for demand D , $\mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$, where O_i- is fixed. We show the differentiation of $\mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$ for O_i .

Property 26. Suppose for any fixed O , $I_{i,i_1}^-(O_i, D)$ and $I_{i,i_1}^+(O_i, D)$ are measurable and all functions are bounded. $|I_{i,i_1}^+(O_i, D) - I_{i,i_1}^+(O'_i, D)| \leq \alpha|O_i - O'_i|$, where α is a constant. There is

1. $\mathbb{E}[I_{i,i_1}^+(O_i, D)]$ is absolutely continuous.
2. Suppose for any fixed D , $I_{i,i_1}^+(O_i, D)$ is differentiable for O_i , in $\{I_{i,i_1}^+(O_i, D) > 0\}$. For a given x , if $\mathbb{P}(I_{i,i_1}^+(x, D) = 0 \text{ and } I_{i,i_1}^-(x, D) = 0) = 0$ and $I_{i,i_1}^-(O_i, D)$ is continuous, then $\mathbb{E}[I_{i,i_1}^+(O_i, D)]$ is differentiable at x ,

$$\left. \frac{d}{dO_i} \mathbb{E}[I_{i,i_1}^+(O_i, D)] \right|_{O_i=x} = \int_{\{I_{i,i_1}^+(x, D) > 0\}} \left. \frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) \right|_{O_i=x} dD. \quad (3.23)$$

Furthermore if $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) = \theta^+$ in $\{I_{i,i_1}^+(x, D) > 0\}$, then

$$\left. \frac{d}{dO_i} \mathbb{E}[I_{i,i_1}^+(O_i, D)] \right|_{O_i=x} = \theta^+ \mathbb{P}(I_{i,i_1}^+(x, D) > 0). \quad (3.24)$$

The θ^+ in assumption can be explained the rate of the residual inventories that retailer willing to share.

Proof. The proof is in Appendix B.5.2. □

Property 26 states the differentiability of the initial amount of inventory residuals or unfulfilled demands that retailer are willing to share, when Lipschitz continuity exists. Similarly we have same result for $\mathbb{E}[I_{i,i_1}^-(O_i, D)]$. Particularly, if $\frac{\partial}{\partial O_i} I_{i,i_1}^-(O_i, D) = -\theta^-$, then $\left. \frac{d}{dO_i} \mathbb{E}[I_{i,i_1}^-(O_i, D)] \right|_{O_i=x} = -\theta^- \mathbb{P}(I_{i,i_1}^-(x, D) > 0)$. Next we show the differentiation for $\mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)]$ and $\mathbb{E}[I_{i,i_{k+1}}^-(O_i, D; O_i-)]$.

Property 27. Suppose for any fixed O , $I_{i,i_1}^-(O_i, D)$ and $I_{i,i_1}^+(O_i, D)$ are measurable and all functions are bounded. $|I_{i,i_1}^+(O_i, D) - I_{i,i_1}^+(O'_i, D)| \leq \alpha|O_i - O'_i|$, where α is a constant.

1. $\mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)]$ is absolutely continuous.
2. Suppose for any fixed D , $I_{i,i_1}^+(O_i, D)$ is differentiable for O_i in $\{I_{i,i_1}^+(O_i, D) > 0\}$. For a given x , if

$$\mathbb{P}(I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) = 0 \text{ and } I_{i,i_k}^+(x, D; O_i-) > 0) = 0$$

and

$$\mathbb{P}(I_{i,i_1}^+(x, D) = 0 \text{ and } I_{i,i_1}^-(x, D) = 0) = 0,$$

then $\mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)] = \mathbb{E}[(I_{i,i_k}^+(O_i, D; O_i-) - C_{i_k}^-(D; O_i-))_+]$ is differentiable at x ,

$$\begin{aligned} & \left. \frac{d}{dO_i} \mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)] \right|_{O_i=x} \\ &= \int_{\{I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) > 0\}} \left(\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) \right)_{O_i=x} dD. \end{aligned} \quad (3.25)$$

Furthermore if $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) = \theta^+$ in $\{I_{i,i_1}^+(O_i, D) > 0\}$, then

$$\begin{aligned} & \left. \frac{d}{dO_i} \mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)] \right|_{O_i=x} \\ &= \theta^+ \mathbb{P}(I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) > 0). \end{aligned} \quad (3.26)$$

Note that by Property 25, the set in (3.25), $\{I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) > 0\} = \{(I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-))_+ > 0\}$ is measurable.

Proof. The proof is in Appendix B.5.3 □

Similar as Property 26, Property 27 states the differentiability of the amount of inventory residuals or unfulfilled demands that retailer are

willing to share with another retailer, when Lipschitz continuity exists. It is because by the expression of (3.22) $I_{i,i_1}^+(O_i, \mathbf{D})$ transfers its Lipschitz continuity to $I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)$. Similarly we have the same result for $\mathbb{E}[I_{i,i_{k+1}}^-(O_i, \mathbf{D}; O_i-)]$. Particularly, if $\frac{\partial}{\partial O_i} I_{i,i_1}^-(O_i, \mathbf{D}) = -\theta^-$, then

$$\left. \frac{d}{dO_i} \mathbb{E}[I_{i,i_{k+1}}^-(O_i, \mathbf{D}; O_i-)] \right|_{O_i=x} = -\theta^- \mathbb{P}(I_{i,i_k}^-(x, \mathbf{D}; O_i-) - C_{i_k}^+(\mathbf{D}; O_i-) > 0).$$

Therefore by (3.21), under all assumptions, $\mathbb{E}[\text{Profit}_i(O_i, \mathbf{D}; O_i-)]$ is differentiable.

$$\begin{aligned} & \frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, \mathbf{D}; O_i-)] \\ = & \frac{d}{dO_i} (r(\mathbb{E}[D_i] - \mathbb{E}[I_i(O_i, \mathbf{D})_-]) - h\mathbb{E}[I_i(O_i, \mathbf{D})_+] - O_i q \\ & + (r_{i,i_1}^i + h)(\mathbb{E}[I_{i,i_1}^+(O_i, \mathbf{D})] - \mathbb{E}[I_{i,i_2}^+(\mathbf{O}, \mathbf{D})]) \\ & + r_{i_1,i}^i(\mathbb{E}[I_{i,i_1}^-(O_i, \mathbf{D})] - \mathbb{E}[I_{i,i_2}^-(\mathbf{O}, \mathbf{D})]) \\ & + \dots \\ & + (r_{i,i_{n-1}}^i + h)(\mathbb{E}[I_{i,i_{n-1}}^+(\mathbf{O}, \mathbf{D})] - \mathbb{E}[I_{i,i_n}^+(\mathbf{O}, \mathbf{D})]) \\ & + r_{i_{n-1},i}^i(\mathbb{E}[I_{i,i_{n-1}}^-(\mathbf{O}, \mathbf{D})] - \mathbb{E}[I_{i,i_n}^-(\mathbf{O}, \mathbf{D})])) \\ = & r\mathbb{P}(I_i(O_i, \mathbf{D})_- > 0) - h\mathbb{P}(I_i(O_i, \mathbf{D})_+ > 0) - q \\ & + (r_{i,i_1}^i + h)(\theta^+ \mathbb{P}(I_{i,i_1}^+(O_i, \mathbf{D}) > 0) \\ & - \theta^+ \mathbb{P}(I_{i,i_1}^+(O_i, \mathbf{D}; O_i-) - C_{i_1}^-(\mathbf{D}; O_i-) > 0)) \\ & + r_{i_1,i}^i(-\theta^- \mathbb{P}(I_{i,i_1}^-(O_i, \mathbf{D}) > 0) + \theta^- \mathbb{P}(I_{i,i_1}^-(O_i, \mathbf{D}; O_i-) - C_{i_1}^+(\mathbf{D}; O_i-) > 0)) \\ & + \dots \\ & + (r_{i,i_{n-1}}^i + h)(\theta^+ \mathbb{P}(I_{i,i_{n-2}}^+(O_i, \mathbf{D}; O_i-) - C_{i_{n-2}}^-(\mathbf{D}; O_i-) > 0) \\ & - \theta^+ \mathbb{P}(I_{i,i_{n-1}}^+(O_i, \mathbf{D}; O_i-) - C_{i_{n-1}}^-(\mathbf{D}; O_i-) > 0)) \\ & + r_{i_{n-1},i}^i(-\theta^- \mathbb{P}(I_{i,i_{n-2}}^-(O_i, \mathbf{D}; O_i-) - C_{i_{n-2}}^+(\mathbf{D}; O_i-) > 0) \\ & + \theta^- \mathbb{P}(I_{i,i_{n-1}}^-(O_i, \mathbf{D}; O_i-) - C_{i_{n-1}}^+(\mathbf{D}; O_i-) > 0)) \end{aligned} \tag{3.27}$$

Because $I_{i,i_1}^+(O_i, D)$ and $I_{i,i_1}^-(O_i, D)$ are the non-negative amounts of inventory residuals and the unfulfilled demands that retailer i is willing to share, and by assuming $I_{i,i_1}^-(O_i, D) = \theta^- I_i(O_i, D)_-$ and $I_{i,i_1}^+(O_i, D) = \theta^+ I_i(O_i, D)_+$, then $\mathbb{P}(I_{i,i_1}^-(O_i, D) > 0) = \mathbb{P}(I_i(O_i, D)_- > 0)$ and $\mathbb{P}(I_{i,i_1}^+(O_i, D) > 0) = \mathbb{P}(I_i(O_i, D)_+ > 0)$.

Theorem 9. Suppose for all k , $r_{i_k,i}^i$ are the same and r_{i,i_k}^i are the same. In addition, $0 < \theta^+, \theta^- \leq 1$. $\frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$ is monotone decreasing. If $\mathbb{P}(I_i(O_i, D)_+ > 0)$ is strictly increasing in the feasible region of O_i , then $\frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$ is also decreasing strictly.

Proof. The proof is in Appendix B.5.4. □

If $\frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$ is strictly monotone decreasing, then there is a unique optimal solution. Furthermore if the probabilities are continuous of O_i , then the optimal solution occurs when $\frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_i-)] = 0$. By Property 24, $\mathbb{E}[\text{Profit}_i(O_i, D; O_i-)]$ is concave, thus a solution of equations,

$$\frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_i-)] = 0 \quad \text{for all } i,$$

is a sufficient condition of a pure Nash equilibrium.

3.7.3 Balanced System

Similar to the two-retailer case discussed in subsection 3.6.2, retailer's replenishment times are evenly distributed in the balanced system. Hence it is possible for each retailer to order optimal quantities as others' prior order quantities are fixed. Thus it avoids the consideration of more complicated cooperative game between multiple retailers.

Additionally assume the length of replenishment cycles are the same in both balanced system and synchronized system. In balanced ordering

system, since retailers replenish at different times, it prevents the situation where all retailers experience their respective lowest inventory levels at same time. Usually the system-wide lowest inventory in balanced system is larger than it in synchronized system. Meanwhile the variance of total inventory level of the multiple retailers is smaller than in the synchronized system. Hence the probability of stock-out is smaller than in the synchronized system.

We show that balanced system results in fewer lost sales than the synchronized system by simulations.

Simulation Study

In order to show the affection of different replenishment times, we show that balanced system is better than synchronized system under same ordering setting, rather than showing the advantage of sharing model integrated balanced system. Hence we need set all assumptions are same for both balanced and synchronized system except the replenishing times.

Problem 3.7.1. Consider the multiple retailers at random locations in the inventory sharing problem. How does the replenishment timing affect the number of sharings and the number of lost sales?

We simulate the balanced system and the synchronized system with the same demand samples D_i^t and the same order quantities O_i^t . We then count the number of sharings and the number of lost sale resulted from both ordering system. The simulations are programmed in R language. All the codes and results are available in Appendix C.4. We describe the simulation process in the following:

1. Randomly generate the 10 locations for 10 individual retailers in a 50×50 map.

2. For each retailer in every period, randomly generate demand samples D_i^t from integers in $[60, 140]$ while adding a random mean fluctuation term ϵ_1 . Meanwhile set each retailer's replenishment order quantities O_i^t at every 10 periods. The value of O_i^t is a integer from $[90, 110]$ with an added fluctuation term ϵ_2 and adjusting terms of ϵ_1 in the next 10 periods. That is $O_i^t = x + \epsilon_2^t + \epsilon_1^t + \dots + \epsilon_1^{t+9}$. Assume all retailers replenish in first periods for the synchronized case. In the balanced case the replenishment times are evenly distributed. That is in each period there is exactly one replenishment for each one retailer. The values of replenishments in corresponding replenishment cycles for both the balanced and the synchronized cases are the same. The only difference between two cases is the replenishment timing.
3. Calculate the residual inventories and unfulfilled demands at current period by the inventory balanced equation

$$I_i^t = (I_i^{t-1})_+ + O_i^{t-1} - D_i^t.$$

4. Under the distance-based sharing rule (in example 6), retailers share their residual inventories and unfulfilled demands. Count the number of sharings and the number of lost sales in current period t .
5. If current period t is less than M , go to next period and repeat Step 3 and 4.

First, as shown in 3.7 we generate the locations for 10 retailers.

Figure 3.7: Locations of the 10 retailers

Scenario 1: First we simulate over a short term. Set $M = 20 + 5$. Since the lost sales are usually occur at the end of replenishment cycles, it may affect the number of lost sales in the short term. Hence we set an extra 5 periods (a half of the replenish cycle) to balance the effect. For both balanced and synchronized cases, we simulate 21 ordering settings with mean values $[90 : 110] + \epsilon_1^t + \dots + \epsilon_1^{t+9}$. We simulate 10000 times for each ordering setting. We show the 10 retailers' average number of sharings in the balanced case in Table C.7 and C.8, the average number of sharings in the synchronized case in Table C.9 and C.10, the average number of lost sales in the balanced case in Table C.11 and C.12, and the average number of lost sales in the synchronized case in Table C.13 and C.14. State the differences of lost sales between two cases in Table C.15 and C.16.

Hence as shown in Appendix C.2, for most retailers under the same ordering setting the numbers of sharings in the balanced system are more

than that in the synchronized system. Meanwhile for all retailers, the numbers of lost sales in the balanced system are fewer than that in the synchronized system. It is because the balanced system provides more opportunities for sharing and thus results in fewer lost sales.

Scenario 2: We perform the simulations over a long term. We set $M = 105$. For both balanced and synchronized cases, we simulate 21 ordering settings with mean values $[90 : 110] + \epsilon_1^t + \dots + \epsilon_1^{t+9}$. We simulate 10000 times for each ordering setting. We state the 10 retailers' average number of sharings in the balanced case in Table C.17 and C.18, the average number of sharings in the synchronized case in Table C.19 and C.20, the average number of lost sales in the balanced case in Table C.21 and C.22, and the average number of lost sales in the synchronized case in Table C.23 and C.24. State the differences of lost sales between two cases in Table C.25 and C.26.

We can observe the following under the same ordering settings: 1. for most retailers the numbers of sharings in the balanced system are more than that in the synchronized system; 2. for all retailers the numbers of lost sales in the balanced system are less than that in the synchronized system, except retailer 4 under ordering setting 1 and retailer 10 under ordering settings 1-6.

Because in ordering settings 1-6, the replenishments are fewer than the mean of the total demands, thus the chances of stock-outs are higher. As shown in Figure 3.7, retailer 4 and 10 are located in remote areas. Hence they are at the lowest end of the sharing ranking. It results in them having more chances to encounter lost sales. However when replenishments are more than the mean of the total demands, then the chances of stock-outs become lower. left-over stocks are large enough, eventually all the demands are fulfilled and the lost sales become 0.

Also based on the distances, these 10 retailers can be divided into 4 groups (A, B, C and D): A: retailer 1,2,3,7,8; B: retailer 4,5; C: retailer 9; D: retailer 6,10. A retailer in a bigger group has more opportunity to share and thus experiences fewer lost sales. We average the differences of lost sales for all ordering settings for each retailer. They are 258.55, 248.25, 265.47, 98.92, 176.73, 141.51, 238.15, 265.20, 117.34, -211.39. We can see that retailers in group A enjoy most benefits from the balanced ordering system. On the other hand, the retailers 4,5,6,9,10 who are in the other groups B,C,D experience the least benefits.

We summarize the lost sales in Table 3.1 for the 10 retailer system. We compare the short term and long term scenarios. The balanced system in over the short term improves the performance of sharing in terms of significantly reducing the lost sales. Even when the replenishment is less than the demand mean, the balanced system still prevents retailers from encountering lost sales. Over the long term, the balanced system also significantly reduces the system-wide lost sales. However when replenishment is less than the demand mean, it eventually leads to in loss sales.

Table 3.1: System-wide lost sales

Ordering setting	Short term		Long term	
	Balanced	Synchronized	Balanced	Synchronized
1	0.21	2000.84	6455.48	9995.44
2	0.03	1798.08	5354.01	8981.61
3	0.00	1602.30	4295.61	8014.29
4	0.00	1392.38	3202.89	7010.44
5	0.00	1201.86	2107.57	5994.30
6	0.00	1005.30	1131.22	5026.76
7	0.00	817.41	377.70	4010.64
8	0.00	639.60	72.96	3048.73
9	0.00	468.42	5.75	2105.94
10	0.00	327.90	0.25	1286.30
11	0.00	215.61	0.00	634.32
12	0.00	133.10	0.00	273.12
13	0.00	74.64	0.00	110.83
14	0.00	37.76	0.00	46.89
15	0.00	19.48	0.00	21.14
16	0.00	9.73	0.00	9.04
17	0.00	4.39	0.00	3.96
18	0.00	1.46	0.00	1.65
19	0.00	0.51	0.00	0.73
20	0.00	0.25	0.00	0.18
21	0.00	0.11	0.00	0.08

3.8 Conclusion

In this chapter we focus on the inventory sharing problem between individual retailers with “complete information sharing” which includes the future information. Hence, given that the future parameters are uncertain, most results in this chapter are given in terms of expectations which are integrations on the probability spaces. We first consider the problem that without explicit assumption on the demand distribution in section 3.3. We modify the assumptions in Leibniz Rule (Theorem 6) such that it provides conditions to interchange the differentiation with integration. We also point out that the feature on Lipschitz continuity is stronger than the assumption in Theorem 6. The inventory sharing problem can also be considered as a cooperative game between individual retailers. Hence in Theorem 7, we show the conditions for the existence and continuity of best response function which is essential in finding the pure Nash equilibrium.

In section 3.5, Theorem 8, based on Lipschitz continuity of single retailer inventory model, we employ Theorem 6 to show the optimality condition (3.1) for this inventory model with general demand distribution. As discussed in section 3.4, we translate the multi-period problem into a sequence of single period problem and discuss the difficulty of obtain the analytical form of demand distribution during lead time L (or its analytical form does not exist). We overcome this difficulty by using simulation methods. Also we provide an example on the use of the simulation method which is based on the analytical result from inventory model with no explicit specification on the demand distribution; this approach contributes in widening the real-life applicability of the model.

Next in section 3.6, we discuss the two-retailer model. Based on Lipschitz continuity, we can also employ Theorem 6 to show the differentiation of a single retailer’s expected profit when others’ orders are fixed. It provides the condition for the existence of the best response function. We then

employ Theorem 7 to show the continuity of the best response function. Furthermore, we show the condition that when the partial differentiations of profits exist. Due to the difficulty in obtaining the analytical demand distribution (as discussed in the single-retailer model), we introduce the simulation based methods for the two-retailer setting. However to compute the optimal solution for a retailer, the method requires that the other retailer's order quantity to be fixed. To avoid the difficulty of game-based interaction, the balanced system is implemented into two-retailer model. We also study the system performance of balanced system by simulation. Under same ordering setting, balanced system provides more sharing opportunities and thus results in fewer lost sales.

In section 3.7, we define the deterministic sharing rule, which fixes the sharing process once the demands are realized and give a total priority for all the pairs of retailers. Hence it determines the priority for every retailer. Under a deterministic sharing rule, we build the sharing model (3.14) which generalized the two-retailer model (3.7). We prove that under the deterministic sharing rule, the amount that retailer i_k willing to share with retailer i is fixed when sharing exists (Property 22). We show

$$\min(I_{i,i_k}^+(\mathbf{O}, \mathbf{D}), I_{i_k}^-(\mathbf{O}, \mathbf{D})) = \min(I_{i,i_k}^+(\mathbf{O}, \mathbf{D}), C_{i_k}^-(O_i-, \mathbf{D}))$$

in Property 23. It means for a fixed $(O_i-, \mathbf{D})$, $\min(I_{i,i_k}^+(\mathbf{O}, \mathbf{D}), I_{i_k}^-(\mathbf{O}, \mathbf{D}))$ is a piecewise function of O_i . It implies that for a fixed $(O_i-, \mathbf{D})$, the multi-retailer model under a deterministic sharing rule (3.17) is a piecewise function. We also provides the conditions for the differentiability of all the terms in the multi-retailer model. Furthermore we prove that (3.17) is concave (Theorem 24) and present a simulation based method to solve the optimal order quantity for the multi-retailer problem. Finally, similar to the two retailer model, we carry out the simulation study of the balanced

system in multi-retailer model shows that the balanced system significantly improve the sharing performance and reduce the lost sales.

Chapter 4

Conclusion

4.1 Conclusions

In this research, we have developed policies and models for the multi-period, multi-retailer inventory sharing models based on two different extent of information sharing - “limited information sharing” (Chapter 2) and “complete information sharing” (Chapter 3). This problem is analytically challenging due to the large number of dimensions involved, across time and the interactions between the retailers. We have found ways to overcome the research challenges outlined in Chapter 1 by integrating policies (e.g. stock lending, balanced ordering and deterministic sharing rule) that overcome some of the major complexities and yet also contribute to system optimality. In summary, the proposed models and analysis have achieved the objectives defined in Chapter 1. The solutions of the objectives in section 1.2 are listed below. (For a summary of the specific contributions of this research, refer to the Conclusion sections of Chapter 2 and 3.)

1. Optimality in inventory sharing

With limited information, the optimality in multiple periods is achieved

in the lending system for all possibilities. The analytical results of optimality and bound are provided for both centralized and decentralized systems respectively.

2. Models should capture stock-out risks of retailers who share out their stocks

The sharing quantity is implemented in the lending system to track the sharing. The lending system is built upon reconstructed orders in the non-sharing case. The optimal solution in lending system overcomes the risk challenge.

3. Incorporate the different extent of information sharing and their implications

With basic information, inventory level, the results in Chapter 2 can be achieved. With larger degree of information sharing, such as ordering quantities, demand distributions or forecasts, demand historical data, sharing pattern, order timings, etc., the sharing efficiency can be enhanced in terms of expected value.

4. Sharing policies that satisfy the essential optimality conditions

The special class of allocation rules, which are (strictly) increasing with $f(\mathcal{N})$, are introduced to strongly lead to full inventory pooling for all independent retailers. It overcomes the subcoalition deviation problem.

5. Transshipment frequency

The transshipment rule is developed to control the transshipment frequency in multiple periods by the reduction of backorder penalty.

6. Sharing without system-wide cooperation

The sharing priority is defined. Under the sharing priority, the sharing process is deterministic. Thus it simplifies modeling of the sharing process for multiple retailers without system-wide cooperation.

7. Nash equilibria of the sharing system

Under proper condition, a solution of equations,

$$\frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_{-i})] = 0 \quad \text{for all } i,$$

is a sufficient condition of a pure Nash equilibrium. The balanced system is considered to avoid the game based interaction between the retailers.

8. Incorporate coordinated ordering timing (balanced ordering) and their implications

Based on the simulation results, it shows that there are more sharing opportunities and fewer lost sales in the balanced system than in the synchronized system.

As proposed in section 1.2, the models with different degrees of information in this report are for the multi-period n -retailer ($n \geq 2$) inventory sharing problem where the demand is unchanged by retailer's ordering and sharing. It is worthwhile to mention that the assumption fails, if customer can change her purchase channel and there are competitions between retailers. If the assumption is not valid, so are the results in this report. Hence in practical applications, retailers may make an agreement/-contract with their customers to contract-bound them and the sale/purchase channels; alternatively all retailers form a non-competitive alliance.

4.2 Future extension

This research could be extended in several directions as follows:

- The changeable demands: If a customer can change her purchase channel and there are competitions between retailers, the demand may be affected. In this case, a competitive model should be more appropriate. Furthermore, the result of stock-out can be considered as a combination of backorders and lost sales (Silver et al., 2016).
- Consideration of transshipment lead-times: We did not explicitly consider lateral transshipment lead-times and inherently, it implies that all transshipments are assumed to arrive in time to prevent (or partially reduce) stock-outs. It would be worthwhile to relax this assumption by considering positive transshipment lead-times in the models. This could necessitate a delivery time window at each retailer where only stocks transshipped within the window can meet the demands. In this context, it would also be beneficial to include the consideration of arriving in-transit inventory that could be transshipped within the windows of stock-out retailers. This could also include re-routing of in-transit stocks to stock-out retailers, similar in spirit to re-routing of component supplies among Dell's facilities that is reported in Foreman et al. (2010).
- Extension to dual echelons: The model and policies developed in this research assume lateral transshipments, where stocks are transshipped within the same echelon between the retailers. A worthwhile future research is to extend the analysis to include the additional option of expediting stocks from the supplier's location or from the pipeline inventories. This added flexibility would likely enhance the inventory performance.

- Transshipment Pricing: A worthwhile extension of the work is to consider pricing, which we did not explicitly consider in this research. There are two aspects of pricing that future research could incorporate - transshipment pricing and dynamic pricing.
 - Transshipment pricing. The models can include transshipment prices for the transfers of stocks between the retailers. The transshipment rule in Chapter 2, the distance-based sharing rule and the deterministic sharing rule take into account the differences in transshipment efforts, but did not explicitly include the consideration of transshipment pricing. Such inclusion will offer an added incentive parameter for the sharing system to create a fairer compensation to the respective retailers for the different transshipment costs incurred.
 - Dynamic pricing. In many businesses, firms adjust their selling price based on the amount of available stocks in order to better balance supply and demand; some perform this form of price adjustment dynamically in real-time, which is known as dynamic pricing. Dynamic pricing has become more important in today's context with the growth of e-commerce, where dynamic pricing is important for sales competitiveness and can be more easily implemented than in typical brick-and-mortar. As the opportunity to share obviously depends on stock availability, such pricing tactics can greatly affect the sharing policy and its efficiency. Thus it would be worthwhile to study how such pricing and inventory sharing can interact, and how optimal pricing decisions can be incorporated in the inventory sharing models.
 - Management of Collaboration Contract: We developed the poli-

cies and models based on assumptions that we believe are realistic from the operational standpoint. However, we have not considered and validated the associated legal contractual agreements that are usually needed in such collaborative systems. A meaningful research is to study the “contract management” for the sharing policy, which includes the process of formulating contractual terms, executing and operationalize contracts, and analyzing the resulting operational and financial performance. This will further enhance the validation of the policies and models for real-life applications.

Appendices

Appendix A

A.1 The Proofs in Section 2.5

A.1.1 Proof of Property 6

Proof. Any retailer i changes her strategy: $I_i^{t_0}$ to $I'_i \in [0, I_i^{t_0})$ if $I_i^{t_0} > 0$, or $I'_i \in (I_i^{t_0}, 0]$ if $I_i^{t_0} < 0$. Let the new transshipment game be g . The solution $g(S)$ of linear programming $L(S, I')$ will not be better than the original $f(S)$. It means $f(S) \geq g(S)$ if $i \in S$, and $f(S) = g(S)$ if $i \notin S$. Since $f, g \in \mathcal{T}_1$, by assumption, (2.1) holds. Therefore $\Phi_i(f) \geq \Phi_i(g)$, i.e., full inventory pooling is a Nash equilibrium. $\square$

A.1.2 Proof of Property 7

Proof. Any retailer i changes her strategy: I'_i leading to $g(\mathcal{N}) < f(\mathcal{N})$ with $-I'_i$ unchanged, only if $|I'_i|$ decreases and the sign is unchanged, i.e., the feasible range of $L(S, I')$ becomes smaller. Therefore $g(S) \leq f(S)$ if $i \in S$ and $g(S) = f(S)$ if $i \notin S$. By (2.2), $\Phi_i(f) > \Phi_i(g)$. $\square$

A.1.3 Proof of Example 2

Proof. If Retailer 1 and 2 choose strategies $I'_1 = I'_2 = 2$, then they both get $\Phi_1(g) = \Phi_2(g) = \frac{2}{3}$, which is better than $\Phi_1(f) = \Phi_2(f) = \frac{1}{2}$ with full inventory pooling. $\square$

A.1.4 Proof of Property 9

Proof. The only way to change full inventory pooling strategy is by reducing $|I'_i|$ with the sign of I'_i unchanged. For any subcoalition $\mathcal{M} \subset \mathcal{N}$, any retailers in $\mathcal{M}$ change their strategies by reducing the $|I'_i|$ with the sign of I'_i unchanged, then by definition of $L(S, I')$, we have if $S \cap \mathcal{M} \neq \emptyset$, then $g(S) \leq f(S)$; if $S \cap \mathcal{M} = \emptyset$, then $g(S) = f(S)$. Since (2.3) holds for all $f, g \in \mathcal{T}_1$, it shows $\Phi_i(g) \leq \Phi_i(f)$ for all $i \in \mathcal{M}$ or $\exists j \in \mathcal{M}, \Phi_j(f) > \Phi_j(g)$. It means full inventory pooling is a strong Nash equilibrium. $\square$

A.1.5 Equivalence of Property 12

Proof. First, we prove that for $f, g \in \mathcal{G}$, $\Phi(f)$ and $\Phi(g)$ satisfy (2.5) $\implies$ $\Phi(f)$ and $\Phi(g)$ satisfy (2.6). $\Phi(f)$ and $\Phi(g)$ satisfy (2.5), i.e., for any $\mathcal{M} \subset \mathcal{N}$,

$$\left. \begin{array}{l} f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies \Phi_i(f) \geq \Phi_i(g) \text{ for } i \in \mathcal{M}$$

holds. Assume $f(S) \geq g(S)$, for $S \subset \mathcal{N}$. Let $\mathcal{M} = \mathcal{N}$,

$$\begin{aligned} f(S) \geq g(S) &\implies \left. \begin{array}{l} f(S) \geq g(S), \quad S \cap \mathcal{N} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{N} = \emptyset \end{array} \right\} \\ &\implies \Phi_i(f) \geq \Phi_i(g) \text{ for all } i \in \mathcal{N}. \end{aligned}$$

Conversely we show that for $f, g \in \mathcal{G}$, $\Phi(f)$ and $\Phi(g)$ satisfy (2.6) $\implies$ $\Phi(f)$ and $\Phi(g)$ satisfy (2.5). $\Phi(f)$ and $\Phi(g)$ satisfy (2.6), i.e.,

$$f(S) \geq g(S), \quad \text{for } S \subset \mathcal{N} \implies \Phi_i(f) \geq \Phi_i(g), \quad \text{for all } i \in \mathcal{N}$$

holds. Then for any $\mathcal{M} \subset \mathcal{N}$,

$$\left. \begin{array}{l} f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies f(S) \geq g(S), \text{ for } S \subset \mathcal{N}$$

$$\implies \Phi_i(f) \geq \Phi_i(g), \text{ for all } i \in \mathcal{N}$$

$$\implies \Phi_i(f) \geq \Phi_i(g), \text{ for } i \in \mathcal{M}.$$

Therefore (2.5) and (2.6) are equivalent. $\square$

A.1.6 Proof of Theorem 1

Proof. Suppose $f, g \in \mathcal{G}$ with $g(\mathcal{N}) = f(\mathcal{N})$. Let $B = \sup_{S \subset \mathcal{N}} \{|f(S)|, |g(S)|\}$. Define $h(S) = B$ for all $S \subsetneq \mathcal{N}$; $h(\mathcal{N}) = f(\mathcal{N})$. Hence by (2.6)

$$h(S) \geq f(S), \quad \text{for } S \subset \mathcal{N} \implies \Phi_i(h) \geq \Phi_i(f), \quad \text{for all } i \in \mathcal{N},$$

and

$$h(S) \geq g(S), \quad \text{for } S \subset \mathcal{N} \implies \Phi_i(h) \geq \Phi_i(g), \quad \text{for all } i \in \mathcal{N}.$$

Since Φ is efficient, $\sum_{i=1}^N \Phi_i(h) = h(\mathcal{N}) = f(\mathcal{N}) = \sum_{i=1}^N \Phi_i(f)$. Hence $\Phi_i(h) = \Phi_i(f)$ for all i . Similarly $\Phi_i(h) = \Phi_i(g)$ for all i . It implies $\Phi_i(f) = \Phi_i(g)$ for all i . Therefore Φ_i is a function of $f(\mathcal{N})$, i.e., $\Phi_i(f(\mathcal{N}))$. Furthermore, let $f(\mathcal{N}) > g(\mathcal{N})$ with $f(S) \geq g(S)$ for $S \subset \mathcal{N}$, then by (2.6), $\Phi_i(f) = \Phi_i(f(\mathcal{N})) \geq \Phi_i(g(\mathcal{N})) = \Phi_i(g)$. It means Φ_i is an increasing function of $f(\mathcal{N})$.

Conversely $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$ on $\mathcal{G}$, where Φ_i is an increasing function of $f(\mathcal{N})$, satisfies (2.6) for all $f, g \in \mathcal{G}$. $\square$

A.1.7 Equivalence of Property 17

Proof. First, we prove that for $f, g \in \mathcal{G}$, $\Phi(f)$ and $\Phi(g)$ satisfy (2.9) $\implies \Phi(f)$ and $\Phi(g)$ satisfy (2.10). $\Phi(f)$ and $\Phi(g)$ satisfy (2.9), i.e., for any $\mathcal{M} \subset \mathcal{N}$,

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} \implies \Phi_i(f) > \Phi_i(g) \text{ for all } i \in \mathcal{M}$$

holds. Assume $f(\mathcal{N}) > g(\mathcal{N})$; $f(S) \geq g(S)$, $S \subsetneq \mathcal{N}$. Let $\mathcal{M} = \mathcal{N}$,

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \subsetneq \mathcal{N} \end{array} \right\} \implies \left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \cap \mathcal{N} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{N} = \emptyset \end{array} \right\} \implies \Phi_i(f) > \Phi_i(g) \text{ for all } i \in \mathcal{N}$$

Conversely we show that for $f, g \in \mathcal{G}$, $\Phi(f)$ and $\Phi(g)$ satisfy (2.10) $\implies \Phi(f)$ and $\Phi(g)$ satisfy (2.9). $\Phi(f)$ and $\Phi(g)$ satisfy (2.10), i.e.,

$$\left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \subsetneq \mathcal{N} \end{array} \right\} \implies \Phi_i(f) > \Phi_i(g) \text{ for all } i \in \mathcal{N}$$

holds. Then for any $\mathcal{M} \subset \mathcal{N}$,

$$\begin{aligned} \left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \cap \mathcal{M} \neq \emptyset \\ f(S) = g(S), \quad S \cap \mathcal{M} = \emptyset \end{array} \right\} &\implies \left. \begin{array}{l} f(\mathcal{N}) > g(\mathcal{N}) \\ f(S) \geq g(S), \quad S \subsetneq \mathcal{N} \end{array} \right\} \\ &\implies \Phi_i(f) > \Phi_i(g) \text{ for all } i \in \mathcal{N}. \\ &\implies \Phi_i(f) \geq \Phi_i(g), \text{ for } i \in \mathcal{M}. \end{aligned}$$

Therefore (2.9) and (2.10) are equivalent. □

A.1.8 Proofs of Theorem 2

Proof. Suppose $f, g \in \mathcal{G}$ with $g(\mathcal{N}) = f(\mathcal{N})$. Let $B = \sup_{S \subset \mathcal{N}} \{|f(S)|, |g(S)|\}$. Define $h(S) = B$ for all $S \subsetneq \mathcal{N}$; $h(\mathcal{N}) = f(\mathcal{N}) + \epsilon$, where $\epsilon > 0$. Hence by (2.10)

$$\left. \begin{array}{l} h(\mathcal{N}) > f(\mathcal{N}) \\ h(S) \geq f(S), \quad S \subsetneq \mathcal{N} \end{array} \right\} \implies \Phi_i(h) > \Phi_i(f), \quad \text{for all } i \in \mathcal{N},$$

and

$$\left. \begin{array}{l} h(\mathcal{N}) > g(\mathcal{N}) \\ h(S) \geq g(S), \quad S \subsetneq \mathcal{N} \end{array} \right\} \implies \Phi_i(h) > \Phi_i(g), \quad \text{for all } i \in \mathcal{N}.$$

Since Φ is efficient, $\sum_{i=1}^N \Phi_i(h) = h(\mathcal{N}) = f(\mathcal{N}) + \epsilon = \sum_{i=1}^N \Phi_i(f) + \epsilon$. Hence $\Phi_i(f) < \Phi_i(h) < \Phi_i(f) + \epsilon$ for all i . Similarly $\Phi_i(g) < \Phi_i(h) < \Phi_i(g) + \epsilon$ for all i . It implies $|\Phi_i(f) - \Phi_i(g)| \leq 2\epsilon$ for all i . By the arbitrariness of ϵ , $\Phi_i(f) = \Phi_i(g)$. Therefore Φ_i is a function of $f(\mathcal{N})$, i.e., $\Phi_i(f(\mathcal{N}))$. Furthermore, let $f(\mathcal{N}) > g(\mathcal{N})$ with $f(S) \geq g(S)$ for $S \subsetneq \mathcal{N}$, then by (2.10), $\Phi_i(f) = \Phi_i(f(\mathcal{N})) > \Phi_i(g(\mathcal{N})) = \Phi_i(g)$. It means Φ_i is a strictly increasing function of $f(\mathcal{N})$.

Conversely $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$ on $\mathcal{G}$, where Φ_i is a strictly increasing function of $f(\mathcal{N})$, satisfies (2.10) for all $f, g \in \mathcal{G}$. $\square$

A.1.9 Proofs of Theorem 3 and 4

Obviously the allocation rule $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$ on $\mathcal{T}_2$, where Φ_i is an increasing function or a strictly increasing function of $f(\mathcal{N})$, satisfies (2.6) or (2.10) for any $f, g \in \mathcal{T}_2$. Therefore we only need to show that all allocation rules on $\mathcal{T}_2$ satisfying (2.6) or (2.10) have the form $\Phi(f) = (\Phi_1(f(\mathcal{N})), \Phi_2(f(\mathcal{N})), \dots, \Phi_N(f(\mathcal{N})))$, where Φ_i is an increasing

function or a strictly increasing function of $f(\mathcal{N})$. The proof is to show the following: (i). Φ_i is a function of $f(\mathcal{N})$; (ii). $\Phi_i(f(\mathcal{N}))$ is (strictly) increasing. If (i) is true, then by (2.6) or (2.10) it is not difficult to show (ii) is also true.

$\mathcal{T}_2$ is the class of all games with form $f_2(S; I') = \sum_{i \in S} (I'_i)_- - (\sum_{i \in S} I'_i)_-$. Hence with given I' and the form $f_2(S; I')$, a game $f = f_2(\cdot; I') \in \mathcal{T}_2$ is determined. It also means that for any $f \in \mathcal{T}_2$, there exists at least one I' , s.t. $f = f_2(\cdot; I')$. Hence for any $f \in \mathcal{T}_2$, let $I(f) = (I_1(f), \dots, I_N(f))$ be a I' , s.t. $f = f_2(\cdot; I')$. Hence $I(f)$ determines f , but it may not be unique. Since $I(f)$ determines f , it is not difficult to see that for any $f, g \in \mathcal{T}_2$:

$$\begin{aligned} & \exists I(f), I(g) \text{ s.t. for } i \in S, I_i(f) \leq I_i(g) \leq 0 \text{ or } I_i(f) \geq I_i(g) \geq 0 \\ \implies & f(S) \geq g(S). \end{aligned}$$

By Property 12 or Property 17 with $f(\mathcal{N}) > g(\mathcal{N})$,

$$\begin{aligned} \Phi_i(f) &= \Phi_i(f_2(\cdot; I(f))) \geq \Phi_i(f_2(\cdot; I(g))) = \Phi_i(g) \\ \text{or} \\ \Phi_i(f) &= \Phi_i(f_2(\cdot; I(f))) > \Phi_i(f_2(\cdot; I(g))) = \Phi_i(g) \end{aligned} \quad .$$

Since for $f \in \mathcal{T}_2$, $f(S) \geq 0$ for $S \subset \mathcal{N}$ and $f(\mathcal{N}) \geq f(S)$ for $S \subset \mathcal{N}$, there is only one game $f \in \mathcal{T}_2$ with $f(\mathcal{N}) = 0$. Hence there is a unique allocation $\Phi_i(f)$, for $f \in \mathcal{T}_2$ with $f(\mathcal{N}) = 0$. Next we need show that $\Phi_i(f)$ is a function $\Phi_i(f(\mathcal{N}))$, for $f(\mathcal{N}) > 0$.

For $f \in \mathcal{T}_2$, $I(f)$ exists. Let $S_+(f)$ be a set of players with non-negative $I_i(f)$ and $S_-(f)$ be a set of players with non-positive $I_i(f)$ s.t. $S_+(f) \cap S_-(f) = \emptyset$ and $S_+(f) \cup S_-(f) = \mathcal{N}$. Note that $S_+(f)$ and $S_-(f)$ may not be unique even $I(f)$ is fixed, and there is no problem for the existence when $f(\mathcal{N}) > 0$. For any possible $I(f)$, $S_+(f)$ and $S_-(f)$, $f(\mathcal{N}) = \min(\sum_{i \in S_+(f)} I_i(f), \sum_{i \in S_-(f)} -I_i(f))$.

Lemma A.1.1. Suppose allocation rule Φ satisfies (2.6) in $\mathcal{T}_2$. For any $f, g \in \mathcal{T}_2$ with $f(\mathcal{N}) = g(\mathcal{N}) > 0$, if $\exists I(f), I(g), S_+ = S_+(f) = S_+(g)$ and $S_- = S_-(f) = S_-(g)$, s.t.

$$\begin{cases} I_i(f) = I_i(g) \text{ for } i \in S_+ \text{ and } \sum_{i \in S_+} I_i(f) = f(\mathcal{N}) \\ \text{or} \\ I_i(f) = I_i(g) \text{ for } i \in S_- \text{ and } |\sum_{i \in S_-} I_i(f)| = f(\mathcal{N}), \end{cases}$$

then $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$.

Proof. Without loss of generality, assume $I_i(f) = I_i(g)$ for $i \in S_+$ and $\sum_{i \in S_+} I_i(f) = f(\mathcal{N})$. For fixed $I(f)$ and $I(g)$, there exists an upper boundary B for all $|I_i(f)|$ and $|I_i(g)|$. For $\forall \epsilon > 0$, let game f' be: $I_i(f') = I_i(f) + \epsilon$ for $i \in S_+$ and $I_i(f') = -B - \epsilon$ for $i \in S_-$. Hence for $\forall S \subset \mathcal{N}$, there exist $I(f')$ and $I(f)$ s.t. for $i \in S$, $I_i(f') \leq I_i(f) \leq 0$ or $I_i(f') \geq I_i(f) \geq 0$. It implies $f'(S) \geq f(S)$ for $S \subset \mathcal{N}$. By (2.6), $\Phi_i(f') \geq \Phi_i(f)$ for all $i \in \mathcal{N}$. Since $f(\mathcal{N}) \leq f'(\mathcal{N}) = \min(\sum_{i \in S_+} I_i(f'), \sum_{i \in S_-} -I_i(f')) = \min(\sum_{i \in S_+} (I_i(f) + \epsilon), \sum_{i \in S_-} (B + \epsilon)) \leq f(\mathcal{N}) + N\epsilon$, it implies $\Phi_i(f') \leq \Phi_i(f) + N\epsilon$ for all $i \in \mathcal{N}$. Similarly $\Phi_i(g) \leq \Phi_i(f') \leq \Phi_i(g) + N\epsilon$ for all $i \in \mathcal{N}$. Hence $|\Phi_i(f) - \Phi_i(g)| \leq 2N\epsilon$ for all $i \in \mathcal{N}$. By the arbitrariness of ϵ , $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$. $\square$

Lemma A.1.2. Suppose allocation rule Φ satisfies (2.6) in $\mathcal{T}_2$. For any $f, g \in \mathcal{T}_2$ with $f(\mathcal{N}) = g(\mathcal{N}) > 0$, if $\exists I(f), I(g), S_+ = S_+(f) = S_+(g), S_- = S_-(f) = S_-(g)$, then $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$.

Proof. There are f' and g' with $f'(\mathcal{N}) = f(\mathcal{N}) = g(\mathcal{N}) = g'(\mathcal{N})$ defined by $I(f')$ and $I(g')$, where $I(f)$ and $I(f')$ as well as $I(g)$ and $I(g')$ satisfy the assumption of Lemma A.1.1, s.t. $\sum_{i \in S_+} I_i(f') = |\sum_{i \in S_-} I_i(f')| = f'(\mathcal{N})$ and $\sum_{i \in S_+} I_i(g') = |\sum_{i \in S_-} I_i(g')| = g'(\mathcal{N})$. Hence by Lemma A.1.1, $\Phi(f) =$

$\Phi(f')$ and $\Phi(g) = \Phi(g')$. Let game f'' be:

$$\begin{cases} I_i(f'') = I_i(f') & \text{for } i \in S_+ \\ I_i(f'') = I_i(g') & \text{for } i \in S_- \end{cases}.$$

Again by Lemma A.1.1, $\Phi_i(f'') = \Phi_i(f')$ and $\Phi_i(f'') = \Phi_i(g')$ for all $i \in \mathcal{N}$. Hence $\Phi(f') = \Phi(g')$, i.e., $\Phi(f) = \Phi(g)$. $\square$

Lemma A.1.3. Suppose allocation rule Φ satisfies (2.6) in $\mathcal{T}_2$. For any $f, g \in \mathcal{T}_2$ with $f(\mathcal{N}) = g(\mathcal{N}) > 0$, if $\exists I(f), I(g), S_+(f) \cap S_+(g) \neq \emptyset$ or $S_-(f) \cap S_-(g) \neq \emptyset$, then $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$.

Proof. Without loss of generality, assume $1 \in S_+(f) \cap S_+(g)$. Let game f' be:

$$\begin{cases} I_1(f') = f(\mathcal{N}) \\ I_i(f') = 0 & \text{for } i \neq 1 \in S_+(f) , \\ I_i(f') = I_i(f) & \text{for } i \in S_-(f) \end{cases}$$

$\exists S_+(f) = S_+(f'), S_-(f) = S_-(f')$ and $f(\mathcal{N}) = f'(\mathcal{N})$, by Lemma A.1.2, $\Phi(f) = \Phi(f')$. Let game g' be:

$$\begin{cases} I_1(g') = g(\mathcal{N}) \\ I_i(g') = 0 & \text{for } i \neq 1 \in S_+(g) , \\ I_i(g') = I_i(g) & \text{for } i \in S_-(g) \end{cases}$$

$\exists S_+(g) = S_+(g'), S_-(g) = S_-(g')$ and $g(\mathcal{N}) = g'(\mathcal{N})$, by Lemma A.1.2, $\Phi(g) = \Phi(g')$. Since $f'(\mathcal{N}) = f(\mathcal{N}) = g(\mathcal{N}) = g'(\mathcal{N})$, and $\exists S_+(f') = \{1\} = S_+(g'), S_-(f') = \mathcal{N} - \{1\} = S_-(g')$, by Lemma A.1.2, $\Phi(f') = \Phi(g')$. Hence $\Phi(f) = \Phi(g)$. $\square$

Theorem 10. Suppose allocation rule Φ satisfies (2.6) in $\mathcal{T}_2$. Φ_i is an increasing function of $f(\mathcal{N})$.

Proof. There is a unique allocation $\Phi_i(f)$, for $f \in \mathcal{T}_2$ with $f(\mathcal{N}) = 0$. For any $f \in \mathcal{T}_2$ with $f(\mathcal{N}) = c > 0$, when $N \geq 3$, let game $f' \in \mathcal{T}_2$ be:

$$\begin{cases} I_1(f') = c \\ I_2(f') = -c \\ I_i(f') = 0 \quad \text{for } i \neq 1, 2. \end{cases}$$

It is easy to see $f'(\mathcal{N}) = c = f(\mathcal{N}) > 0$. Since $N \geq 3$, there is $I_3(f') = 0$. By the definitions of S_+ and S_- , $\exists S_+(f')$, s.t. $3 \in S_+(f')$ and also $\exists S_-(f')$, s.t. $3 \in S_-(f')$. It means for any f and fixed f' with $f(\mathcal{N}) = f'(\mathcal{N}) = c > 0$, $\exists I(f), I(f'), S_+(f) \cap S_+(f') \neq \emptyset$ or $S_-(f) \cap S_-(f') \neq \emptyset$. By Lemma A.1.3, $\Phi_i(f) = \Phi_i(f')$. Hence Φ_i is a function of $f(\mathcal{N})$, when $N \geq 3$. By (2.6), Φ_i is an increasing function of $f(\mathcal{N})$. When $N = 2$, since $f(S) = 0$ except $S = \mathcal{N}$, there is a unique $f \in \mathcal{T}_2$ s.t. $f(\mathcal{N}) = c$. Therefore Φ_i is a function of $f(\mathcal{N})$. It also implies that $f, g \in \mathcal{T}_2$ with (2.6), if and only if

$$f(\mathcal{N}) \geq g(\mathcal{N}) \implies \Phi_i(f) \geq \Phi_i(g).$$

Therefore Φ_i is an increasing function of $f(\mathcal{N})$. □

Lemma A.1.4. Suppose allocation rule Φ satisfies (2.10) in $\mathcal{T}_2$. For any $f, g \in \mathcal{T}_2$ with $f(\mathcal{N}) = g(\mathcal{N}) > 0$, if $\exists I(f), I(g), S_+ = S_+(f) = S_+(g), S_- = S_-(f) = S_-(g)$, s.t.

$$\begin{cases} I_i(f) = I_i(g) \text{ for } i \in S_+ \text{ and } \sum_{i \in S_+} I_i(f) = f(\mathcal{N}) \\ \text{or} \\ I_i(f) = I_i(g) \text{ for } i \in S_- \text{ and } |\sum_{i \in S_-} I_i(f)| = f(\mathcal{N}), \end{cases}$$

then $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$.

Proof. Without loss of generality, assume $I_i(f) = I_i(g)$ for $i \in S_+$ and

$\sum_{i \in S_+} I_i(f) = f(\mathcal{N})$. For fixed $I(f)$ and $I(g)$, there exists an upper bound B for all $|I_i(f)|$ and $|I_i(g)|$. For $\forall \epsilon > 0$, let game f' be: $I_i(f') = I_i(f) + \epsilon$ for $i \in S_+$ and $I_i(f') = -B - \epsilon$ for $i \in S_-$. Hence $f(\mathcal{N}) < f'(\mathcal{N}) = \min(\sum_{i \in S_+} I_i(f'), \sum_{i \in S_-} -I_i(f')) = \min(\sum_{i \in S_+} (I_i(f) + \epsilon), \sum_{i \in S_-} (B + \epsilon)) \leq f(\mathcal{N}) + N\epsilon$. For $\forall S \subset \mathcal{N}$, there exist $I(f')$ and $I(f)$ s.t. for $i \in S$, $I_i(f') \leq I_i(f) \leq 0$ or $I_i(f') \geq I_i(f) \geq 0$. It implies $f'(S) \geq f(S)$ for $S \subset \mathcal{N}$. By (2.10), $\Phi_i(f) < \Phi_i(f') < \Phi_i(f) + N\epsilon$ for all $i \in \mathcal{N}$. Similarly $\Phi_i(g) < \Phi_i(f') < \Phi_i(g) + N\epsilon$ for all $i \in \mathcal{N}$. Hence $|\Phi_i(f) - \Phi_i(g)| < 2N\epsilon$ for all $i \in \mathcal{N}$. By the arbitrariness of ϵ , $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$. $\square$

Lemma A.1.5. Suppose allocation rule Φ satisfies (2.10) in $\mathcal{T}_2$. For any $f, g \in \mathcal{T}_2$ with $f(\mathcal{N}) = g(\mathcal{N}) > 0$, if $\exists I(f), I(g), S_+ = S_+(f) = S_+(g), S_- = S_-(f) = S_-(g)$, then $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$.

Proof. There are f' and g' with $f'(\mathcal{N}) = f(\mathcal{N}) = g(\mathcal{N}) = g'(\mathcal{N})$ defined by $I'(f)$ and $I'(g)$, s.t. (1). $I(f), I(f')$ and $I(g), I(g')$ satisfy the assumption of Lemma A.1.4; (2). $\sum_{i \in S_+} I_i(f') = |\sum_{i \in S_-} I_i(f')| = f'(\mathcal{N})$ and $\sum_{i \in S_+} I_i(g') = |\sum_{i \in S_-} I_i(g')| = g'(\mathcal{N})$. Hence by Lemma A.1.4, $\Phi(f) = \Phi(f')$ and $\Phi(g) = \Phi(g')$. Let game f'' be:

$$\begin{cases} I_i(f'') = I_i(f') & \text{for } i \in S_+ \\ I_i(f'') = I_i(g') & \text{for } i \in S_- \end{cases}.$$

Again by Lemma A.1.4, $\Phi_i(f'') = \Phi_i(f')$ and $\Phi_i(f'') = \Phi_i(g')$ for all $i \in \mathcal{N}$. Hence $\Phi(f') = \Phi(g')$, i.e., $\Phi(f) = \Phi(g)$. $\square$

Lemma A.1.6. Suppose allocation rule Φ satisfies (2.10) in $\mathcal{T}_2$. For any $f, g \in \mathcal{T}_2$ with $f(\mathcal{N}) = g(\mathcal{N}) > 0$, if $\exists I(f), I(g), S_+(f) \cap S_+(g) \neq \emptyset$ or $S_-(f) \cap S_-(g) \neq \emptyset$, then $\Phi_i(f) = \Phi_i(g)$ for all $i \in \mathcal{N}$.

Proof. Without loss of generality, assume $1 \in S_+(f) \cap S_+(g)$. Let game f'

be:

$$\begin{cases} I_1(f') = f(\mathcal{N}) \\ I_i(f') = 0 & \text{for } i \neq 1 \in S_+(f) , \\ I_i(f') = I_i(f) & \text{for } i \in S_-(f) \end{cases}$$

$\exists S_+(f) = S_+(f')$, $S_-(f) = S_-(f')$ and $f(\mathcal{N}) = f'(\mathcal{N})$, by Lemma A.1.5, $\Phi(f) = \Phi(f')$. Let game g' be:

$$\begin{cases} I_1(g') = g(\mathcal{N}) \\ I_i(g') = 0 & \text{for } i \neq 1 \in S_+(g) , \\ I_i(g') = I_i(g) & \text{for } i \in S_-(g) \end{cases}$$

$\exists S_+(g) = S_+(g')$, $S_-(g) = S_-(g')$ and $g(\mathcal{N}) = g'(\mathcal{N})$, by Lemma A.1.5, $\Phi(g) = \Phi(g')$. Since $f'(\mathcal{N}) = f(\mathcal{N}) = g(\mathcal{N}) = g'(\mathcal{N})$, and $\exists S_+(f') = \{1\} = S_+(g')$, $S_-(f') = \mathcal{N} - \{1\} = S_-(g')$, by Lemma A.1.5, $\Phi(f') = \Phi(g')$. Hence $\Phi(f) = \Phi(g)$. $\square$

Theorem 11. Suppose allocation rule Φ satisfies (2.10) in $\mathcal{T}_2$. Φ_i is a strictly increasing function of $f(\mathcal{N})$.

Proof. There is a unique allocation $\Phi_i(f)$, for $f \in \mathcal{T}_2$ with $f(\mathcal{N}) = 0$. For any $f \in \mathcal{T}_2$ with $f(\mathcal{N}) = c > 0$, when $N \geq 3$, let game $f' \in \mathcal{T}_2$ be:

$$\begin{cases} I_1(f') = c \\ I_2(f') = -c \\ I_i(f') = 0 & \text{for } i \neq 1, 2. \end{cases}$$

It is easy to see $f'(\mathcal{N}) = c = f(\mathcal{N}) > 0$. Since $N \geq 3$, there is $I_3(f') = 0$. By the definitions of S_+ and S_- , $\exists S_+(f')$, s.t. $3 \in S_+(f')$ and also $\exists S_-(f')$, s.t. $3 \in S_-(f')$. It means for any f and fixed f' with $f(\mathcal{N}) = f'(\mathcal{N}) = c > 0$, $\exists I(f), I(f'), S_+(f) \cap S_+(f') \neq \emptyset$ or $S_-(f) \cap S_-(f') \neq \emptyset$. By Lemma A.1.6,

$\Phi_i(f) = \Phi_i(f')$. Hence Φ_i is a function of $f(\mathcal{N})$, when $N \geq 3$. By (2.10), Φ_i is a strictly increasing function of $f(\mathcal{N})$. When $N = 2$, since $f(S) = 0$ except $S = \mathcal{N}$, there is a unique $f \in \mathcal{T}_2$ s.t. $f(\mathcal{N}) = c$. Therefore Φ_i is a function of $f(\mathcal{N})$. It also implies that $f, g \in \mathcal{T}_2$ with (2.10), if and only if

$$f(\mathcal{N}) > g(\mathcal{N}) \implies \Phi_i(f) > \Phi_i(g).$$

Therefore Φ_i is a strictly increasing function of $f(\mathcal{N})$. $\square$

A.2 The Proof in Section 2.6

A.2.1 Proofs of Lemma 2.6.1

Proof. Suppose there is no i , s.t. $a_i < 0$ and $b_i > 0$, thus $a_i \geq 0$ or $b_i \leq 0$. Since $x_- (\geq 0)$ is a decreasing function of x , $(a_i + b_i)_- \geq (a_i)_-$ for $a_i \geq 0$ or $b_i \leq 0$. it follows that $\sum_{i=1}^m (a_i + b_i)_- - \sum_{i=1}^m (a_i)_- \geq 0$.

Suppose there is no i , s.t. $a_i > 0$ and $b_i < 0$. Thus if $b_i < 0$, then $a_i \leq 0$. It implies $(a_i + b_i)_- - (a_i)_- = -a_i - b_i + a_i = -b_i$. If $b_i \geq 0$, since x_+ is an increasing function of x , then $(a_i + b_i)_+ - (a_i)_+ \geq 0$. Because $b_i = (a_i + b_i) - a_i = (a_i + b_i)_+ - (a_i)_+ - ((a_i + b_i)_- - (a_i)_-)$, it implies $(a_i + b_i)_- - (a_i)_- = (a_i + b_i)_+ - (a_i)_+ - b_i \geq -b_i$. Hence $\sum_{i=1}^m ((a_i + b_i)_- - (a_i)_-) \geq \sum_{i=1}^m (-b_i) = 0$. $\square$

Appendix B

B.1 The Proofs in Section 3.3

B.1.1 The proof of Theorem 6

Proof. Assume $\{x_n\}$ is a sequence converging to x_0 in X , where $x_n \neq x_0$ (or $x_n < x_0$ or $x_n > x_0$). The problem is equivalent to proving that for any sequence $\{x_n\}$, there is

$$\lim_{n \rightarrow \infty} \frac{\int_{\Omega} f(x_0, \omega) d\mu(\omega) - \int_{\Omega} f(x_n, \omega) d\mu(\omega)}{x_0 - x_n} = \int_E f_x(x_0, \omega) d\mu(\omega) \quad (\text{B.1})$$

and $\int_E f_x(x_0, \omega) d\mu(\omega)$ is finite. For left and right derivative cases, the proofs are the same. The only difference is the restrictions of $\{x_n\}$. Therefore we only prove (B.1) and that $\int_E f_x(x_0, \omega) d\mu(\omega)$ is finite.

First we define $f_E(\omega) =: 1_E(\omega)f(\omega)$. For any fixed sequence $\{x_n\}$, let $h_n(\omega) := \frac{f(x_0, \omega) - f(x_n, \omega)}{x_0 - x_n}$. By Assumption 1, $h_n(\omega)$ is measurable. For $\omega \in E$, by the definition of $f_x(x_0, \omega)$, there is

$$\lim_{n \rightarrow \infty} h_n(\omega) = f_x(x_0, \omega).$$

It means

$$\lim_{n \rightarrow \infty} (h_n(\omega))_E = (f_x(x_0, \omega))_E,$$

and implies $(f_x(x_0, \omega))_E$ is measurable. By Assumption 2, for $\omega \in E$, $f(x, \omega)$ is absolutely continuous. It implies

$$f(x, \omega) - f(x_0, \omega) = \int_{x_0}^x f_x(s, \omega) ds.$$

Without loss of generality, we assume $x_0 < x$. For $x < x_0$ the following (B.2) still holds. Furthermore by Assumption 2, $f_x(x, \omega)$ is bounded by $g(\omega)$ a.e. on X ,

$$\begin{aligned} & |f(x, \omega) - f(x_0, \omega)| \\ &= \left| \int_{x_0}^x f_x(s, \omega) ds \right| \leq \int_{x_0}^x |f_x(s, \omega)| ds \leq \int_{x_0}^x g(\omega) ds = g(\omega) |x - x_0|. \end{aligned} \quad (\text{B.2})$$

It means $\frac{|f(x, \omega) - f(x_0, \omega)|}{|x - x_0|} \leq g(\omega)$. Therefore $|h_n(\omega)| \leq g(\omega)$ holds for every n . Since $g(\omega)$ is integrable, by dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\Omega} h_n(\omega) d\mu(\omega) = \int_{\Omega} (f_x(x_0, \omega))_E d\mu(\omega) = \int_E f_x(x_0, \omega) d\mu(\omega)$$

is finite. Hence (B.1) holds. □

B.1.2 The proof of Theorem 7

Proof. For any fixed y and any $\epsilon > 0$, by the assumption $\text{BR}_{f(x,y)}(y)$ exists, there is a unique maximum point $\text{BR}_{f(x,y)}(y)$ in $[a, b]$ to maximize $f(x, y)$. Suppose the set $A := [a, b] - (\text{BR}_{f(x,y)}(y) - \epsilon, \text{BR}_{f(x,y)}(y) + \epsilon)$ is compact (assume ϵ is small enough, then A is non-empty). Therefore $\max_{x \in A} f(x, y)$ exists. Furthermore by the uniqueness in the definition of $\text{BR}_{f(x,y)}(y)$, let $d := f(\text{BR}_{f(x,y)}(y), y) - \max_{x \in A} f(x, y)$, then $d > 0$.

By the uniform continuity of $f(x, y)$, there is a $\delta > 0$, when $|y - y'| < \delta$ s.t. $|f(x, y') - f(x, y)| < d/2$ for all x . Because for $x_0 \in A$, we can then

obtain

$$\begin{aligned} f(\text{BR}_{f(x,y)}(y), y') &> f(\text{BR}_{f(x,y)}(y), y) - d/2 = \max_{x \in A} f(x, y) + d/2 \\ &> f(x, y'). \end{aligned}$$

By the assumption that $\text{BR}_{f(x,y)}(y')$ exists, it means $\text{BR}_{f(x,y)}(y') \notin A$, i.e. $\text{BR}_{f(x,y)}(y') \in [a, b] - A \subset (\text{BR}_{f(x,y)}(y) - \epsilon, \text{BR}_{f(x,y)}(y) + \epsilon)$. $\square$

B.2 The Results in Probability Theory

Definition 2. A random variable Y is called a conditional expectation of X given $\mathcal{F}$, $\mathbb{E}[X \mid \mathcal{F}] := Y$, if: Y is $\mathcal{F}$ -measurable and for any $A \in \mathcal{F}$, we have $\mathbb{E}[X1_A] = \mathbb{E}[Y1_A]$. For $B \in \mathcal{A}$, $P[B \mid \mathcal{F}] := \mathbb{E}[1_B \mid \mathcal{F}]$ is called a conditional probability of B given the σ -algebra $\mathcal{F}$.

Property 28. $\mathcal{L}^2(\Omega)$ is a Hilbert space.

Proof. This is a well-known result. A proof is given by Rudin (1987), Theorem 3.11. $\square$

Property 29. Let $\mathcal{G}, \mathcal{F}$ are information collections(σ -algebra). Suppose $\mathcal{G} \subset \mathcal{F}$, $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ is isometric to a close subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$.

Proof. Because $\mathcal{G}$ -measurable implies the $\mathcal{F}$ -measurable. So it is not difficult to see this $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ can be embedded into $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$. We need show $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ is closed in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$.

By Property 28, $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ are two Hilbert spaces. Suppose $\{x_n\}$ is a sequence of points in $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ and $x_n \xrightarrow{\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})} x$ ($n \rightarrow \infty$) in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$. Since distances of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ and its subspaces are coincide, the $\{x_n\}$ is Cauchy sequence in subspace $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$. Because $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ is Hilbert space, there exists a limit point $\tilde{x}$ of $\{x_n\}$ in subspace $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$. Again since distances are coincide, $\tilde{x}$ is a limit point of $\{x_n\}$ in

$\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ too. By the uniqueness of limit point, $x = \tilde{x}$. Hence $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ is a close subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ $\square$

Definition 3. Assume H is a inner product space, x is a point in H and M is a subspace of H . Define $x^\perp = \{y \mid \langle x, y \rangle = 0\}$. And $M^\perp = \{y \mid \langle x, y \rangle = 0 \text{ for all } x \in M\}$. Obviously, $M^\perp = \bigcap_{x \in M} x^\perp$.

Property 30. Assume H is a Hilbert Space and M is a closed subspace of H . Then each $x \in H$ has a unique representation $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in M^\perp$. The map $p : x \rightarrow x_1$ is called the projection of H onto M .

Proof. A proof is given by Rudin (1987), Theorem 4.11(a). $\square$

Property 31 (Properties of Conditional Expectation). Given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. X and Y are the random variables. The σ -algebra $\mathcal{G}_1 \subset \mathcal{G} \subset \mathcal{F}$.

1. $\mathbb{E}[\lambda_1 X + \lambda_2 Y \mid \mathcal{G}] = \lambda_1 \mathbb{E}[X \mid \mathcal{G}] + \lambda_2 \mathbb{E}[Y \mid \mathcal{G}]$;
2. If $X \geq Y$, then $\mathbb{E}[X \mid \mathcal{G}] \geq \mathbb{E}[Y \mid \mathcal{G}]$;
3. If $\mathbb{E}[|XY|] < \infty$ and Y is measurable with respect to $\mathcal{G}$, then $\mathbb{E}[XY \mid \mathcal{G}] = Y \mathbb{E}[X \mid \mathcal{G}]$;
4. $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{G}_1] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}_1] \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{G}_1]$;
5. If $\sigma(X)$ is independent with $\mathcal{G}$, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$

Proof. A proof given by Klenke (2008), Theorem 8.14. $\square$

Property 32. For probability space Ω , $\mathcal{L}^2(\Omega) \subset \mathcal{L}^1(\Omega)$.

Proof. For $X \in \mathcal{L}^2(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} |X| dP &= \int_{\{|X|>1\}} |X| dP + \int_{\{|X|\leq 1\}} |X| dP \\ &\leq \int_{\{|X|>1\}} |X|^2 dP + \int_{\{|X|\leq 1\}} 1 dP \end{aligned}$$

$$\begin{aligned} &\leq \int_{\Omega} |X|^2 dP + \int_{\Omega} 1 dP \\ &< +\infty \end{aligned}$$

So $X \in \mathcal{L}^1(\Omega)$. □

Property 33. *Conditional expectation is a best approach in the sense of $\mathcal{L}^2$ norm. Hence the expectation is a best approach as a constant in the sense of $\mathcal{L}^2$ norm. (The conditional expectation is isometric to the projection)*

Proof. Given probability space $(\Omega, \mathcal{F}, P)$, then denote the $\mathcal{L}^2$ space as $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. Suppose there is a information collection (sub- σ -algebra) $\mathcal{G}$ in $\mathcal{F}$. By property 29, $\mathcal{L}^2(\Omega, \mathcal{G}, P) \subset \mathcal{L}^2(\Omega, \mathcal{F}, P)$ is a close subspace. It is to show that for any random variable $X \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$, the conditional expectation $\mathbb{E}[X | \mathcal{F}]$ is the closest point in subspace $\mathcal{L}^2(\Omega, \mathcal{G}, P)$.

First is to show the projection is closest. By property 30, for any $X \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ there is a unique decomposition

$$X = X_1 + X_2$$

where $X_1 \in \mathcal{L}^2(\Omega, \mathcal{G}, P)$, $X_2 \in \mathcal{L}^2(\Omega, \mathcal{G}, P)^\perp$ and the inner product $\langle X_1, X_2 \rangle = \mathbb{E}[X_1 X_2] = 0$. Since $X_2 \in \mathcal{L}^2(\Omega, \mathcal{G}, P)^\perp$, for any Y in $\mathcal{L}^2(\Omega, \mathcal{G}, P)$, there is $X - Y = (X_1 - Y) + X_2$. By the definition of $\mathcal{L}^2(\Omega, \mathcal{G}, P)^\perp$, the square of distance is

$$\begin{aligned} \|X - Y\|^2 &= \|X_1 - Y + X_2\|^2 \\ &= \|X_1 - Y\|^2 + \|X_2\|^2 \\ &\geq \|X_2\|^2. \end{aligned}$$

The equality holds only when $Y = X_1$.

Second is to show a random variable $X_1 \in \mathcal{L}^2(\Omega, \mathcal{G}, P)$ is the conditional expectation. Because $\mathbb{E}[X_2 Y] = \mathbb{E}[(X - X_1) Y] = 0$, let $Y = 1_A$ for any $A \in \mathcal{G}$,

$\mathbb{E}[X_1 1_A] < +\infty$. Therefore $\mathbb{E}[X 1_A] = \mathbb{E}[X_1 1_A]$. Consider a random variable $X_1 \in \mathcal{L}^2(\Omega, \mathcal{G}, P)$ is $\mathcal{G}$ -measurable. By definition 2, $X_1 = \mathbb{E}[X | \mathcal{G}]$. It means $\mathbb{E}[X | \mathcal{G}]$ is closest to X . In other words, $\mathbb{E}[X | \mathcal{G}]$ is isometric to the projection of X . $\square$

Theorem 12. Let Ω be the probability space. $X(\omega)$ valued in E_1 and $Y(\omega)$ valued in E_2 are random variables on Ω . f is measurable function from $E_1 \times E_2$ to $\mathbb{R}$, such that $f(X(\omega), Y(\omega)) \in \mathcal{L}^1(\Omega)$. Suppose the conditional probability $\mathbb{P}_X(\omega, y)$ exists. There is

$$\mathbb{E}[f(X(\omega), Y(\omega)) | X] = \int_{E_2} f(X(\omega), y) \mathbb{P}_X(\omega, y). \quad (\text{B.3})$$

Proof. For any measurable sets $S_1 \subset E_1$ and $S_2 \subset E_2$. The formula (B.3) holds for indicate function $1_{S_1 \times S_2}$. Since

$$\begin{aligned} \int_{E_2} 1_{S_1 \times S_2}(X(\omega), y) \mathbb{P}_X(\omega, y) &= 1_{S_1}(X(\omega)) \int_{E_2} 1_{S_2}(y) \mathbb{P}_X(\omega, y) \\ &= 1_{S_1}(X(\omega)) \mathbb{P}_X(\omega, S_2) \\ &= \mathbb{E}[1_{S_1 \times S_2}(X(\omega), Y(\omega)) | X]. \end{aligned}$$

Let $\mathcal{C}$ be the collection of all the sets such that their indicate function satisfies (B.3). Because

1. $\Omega \in \mathcal{C}$.
2. If $A \in \mathcal{C}$, then

$$\begin{aligned} \int_{E_2} 1_{A^c}(X(\omega), y) \mathbb{P}_X(\omega, y) &= \int_{E_2} 1 - 1_A(X(\omega), y) \mathbb{P}_X(\omega, y) \\ &= 1 - \int_{E_2} 1_A(X(\omega), y) \mathbb{P}_X(\omega, y) \\ &= 1 - \mathbb{E}[1_A(X(\omega), Y(\omega)) | X] \\ &= \mathbb{E}[1_{A^c}(X(\omega), Y(\omega)) | X]. \end{aligned}$$

3. If $A_1, A_2, A_3, \dots$, is a sequence in $\mathcal{C}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then for any $i \neq j$

$$\begin{aligned}
& \int_{E_2} 1_{A_i \cup A_j}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \int_{E_2} 1_{A_i}(X(\omega), y) + 1_{A_j}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \int_{E_2} 1_{A_i}(X(\omega), y) \mathbb{P}_X(\omega, y) + \int_{E_2} 1_{A_j}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \mathbb{E}[1_{A_i}(X(\omega), Y(\omega)) | X] + \mathbb{E}[1_{A_j}(X(\omega), Y(\omega)) | X] \\
&= \mathbb{E}[1_{A_i}(X(\omega), Y(\omega)) + 1_{A_j}(X(\omega), Y(\omega)) | X].
\end{aligned}$$

Hence for any finite n , there is $\cup_{i=1}^n A_i$ is in $\mathcal{C}$. Furthermore, by dominated convergence theorem,

$$\begin{aligned}
\int_{E_2} 1_{\cup_{i=1}^\infty A_i}(X(\omega), y) \mathbb{P}_X(\omega, y) &= \int_{E_2} \lim_{n \rightarrow \infty} 1_{\cup_{i=1}^n A_i}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \lim_{n \rightarrow \infty} \int_{E_2} 1_{\cup_{i=1}^n A_i}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \lim_{n \rightarrow \infty} \mathbb{E}[1_{\cup_{i=1}^n A_i}(X(\omega), Y(\omega)) | X] \\
&= \mathbb{E}[\lim_{n \rightarrow \infty} 1_{\cup_{i=1}^n A_i}(X(\omega), Y(\omega)) | X] \\
&= \mathbb{E}[1_{\cup_{i=1}^\infty A_i}(X(\omega), Y(\omega)) | X].
\end{aligned}$$

Because all the sets with $S_1 \times S_2$ form are in $\mathcal{C}$, and these sets form a π system, by Dynkin's π - λ theorem, the σ -algebra generated by $S_1 \times S_2$ is in $\mathcal{C}$.

For any positive measurable simple function g from $E_1 \times E_2$ to $\mathbb{R}$. It can be expressed as a linear combination of indicate functions

$$g = \sum_{i=1}^n a_i 1_{A_i}(x, y).$$

So by linearity,

$$\begin{aligned}
\int_{E_2} g(X(\omega), y) \mathbb{P}_X(\omega, y) &= \int_{E_2} \sum_{i=1}^n a_i 1_{A_i}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \sum_{i=1}^n a_i \int_{E_2} 1_{A_i}(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \sum_{i=1}^n a_i \mathbb{E}[1_{A_i}(X(\omega), Y(\omega)) | X] \\
&= \mathbb{E}\left[\sum_{i=1}^n a_i 1_{A_i}(X(\omega), Y(\omega)) | X\right] \\
&= \mathbb{E}[g(X(\omega), Y(\omega)) | X]
\end{aligned}$$

For any positive measurable function f_+ such that $f_+(X(\omega), Y(\omega)) \in \mathcal{L}^1(\Omega)$, it can be expressed as a limitation of a sequence of monotonic increasing positive measurable simple functions,

$$f_+(x, y) = \lim_{i \rightarrow \infty} g_i(x, y).$$

So by monotone convergence theorem and dominated convergence theorem,

$$\begin{aligned}
\int_{E_2} f_+(X(\omega), y) \mathbb{P}_X(\omega, y) &= \int_{E_2} \lim_{i \rightarrow \infty} g_i(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \lim_{i \rightarrow \infty} \int_{E_2} g_i(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \lim_{i \rightarrow \infty} \mathbb{E}[g_i(X(\omega), Y(\omega)) | X] \\
&= \mathbb{E}[f_+(X(\omega), Y(\omega)) | X].
\end{aligned}$$

Finally, for any $f(x, y) = f_+(x, y) - f_-(x, y)$, such that $f(X(\omega), Y(\omega)) \in \mathcal{L}^1(\Omega)$, by linearity

$$\int_{E_2} f(X(\omega), y) \mathbb{P}_X(\omega, y)$$

$$\begin{aligned}
&= \int_{E_2} f_+(X(\omega), y) - f_-(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&\stackrel{a.e.X}{=} \int_{E_2} f_+(X(\omega), y) \mathbb{P}_X(\omega, y) - \int_{E_2} f_-(X(\omega), y) \mathbb{P}_X(\omega, y) \\
&= \mathbb{E}[f_+(X(\omega), Y(\omega)) | X] - \mathbb{E}[f_-(X(\omega), Y(\omega)) | X] \\
&= \mathbb{E}[f(X(\omega), Y(\omega)) | X].
\end{aligned}$$

So for any f , $\int_{E_2} f(X(\omega), y) \mathbb{P}_X(\omega, y)$ is a conditional expectation of $f(X(\omega), Y(\omega))$ given X . $\square$

B.3 Single Retailer Model

B.3.1 The proof of Theorem 8

Notice that there is no non-negative restriction of D in Theorem 8. Although it is impossible for negative D to occur in single period problems, it is a possible case for multi-period problems (where D is modeled as the net demand over lead time, as will be discussed). Furthermore as shown in Example 7, the solutions of the non-negative case and the general case are the same. That is, they are interconvertible. It would be helpful, if either the non-negative case or the general case is relatively easy to model or compute. One such example is the normally distributed demand.

Proof. The expectation of profit for the single retailer is

$$\begin{aligned}
\mathbb{E}[\text{Profit}] &= \mathbb{E}[\min(D, O)r - (O - D)_+h - Oq] \\
&= r\mathbb{E}[D] - r\mathbb{E}[(O - D)_-] - h\mathbb{E}[(O - D)_+] - Oq.
\end{aligned}$$

It is easy to see $|(O_1 - D)_- - (O_2 - D)_-| \leq |O_1 - O_2|$ and $|(O_1 - D)_+ - (O_2 - D)_+| \leq |O_1 - O_2|$. Hence $\mathbb{E}[|(O_1 - D)_- - (O_2 - D)_-|] \leq |O_1 - O_2|$ and $\mathbb{E}[|(O_1 - D)_+ - (O_2 - D)_+|] \leq |O_1 - O_2|$. It means both $(O_1 - D)_-$ and

$(O_1 - D)_+$ are Lipschitz continuous, so they are both absolutely continuous. Therefore $\mathbb{E}[\text{Profit}]$ is absolutely continuous, and $\mathbb{E}[\text{Profit}]$ is differentiable a.e..

$(O - D)_-$ and $(O - D)_+$ are integrable for every O . Since for any fixed D , both $(O - D)_-$ and $(O - D)_+$ are Lipschitz continuous, so they are absolutely continuous. For fixed D , $|\frac{\partial}{\partial O}(O - D)_-| \leq 1$ a.e. and $|\frac{\partial}{\partial O}(O - D)_+| \leq 1$ a.e.

Because for $O = O_0$ and fixed D , both $(O - D)_\pm$ are not differentiable at O_0 only when $D = O_0$. By Theorem 6, if $\mathbb{P}(D = O_0) = 0$, then $\mathbb{E}[(O - D)_\pm]$ is differentiable at $O = O_0$ and

$$\begin{aligned}
& \frac{d}{dO} \mathbb{E}[\text{Profit}] \Big|_{O=O_0} \\
&= -r \frac{d}{dO} \mathbb{E}[(O - D)_-] \Big|_{O=O_0} - h \frac{d}{dO} \mathbb{E}[(O - D)_+] \Big|_{O=O_0} - q \\
&= -r \mathbb{E}\left[\frac{\partial}{\partial O}(O_0 - D)_-\right] - h \mathbb{E}\left[\frac{\partial}{\partial O}(O_0 - D)_+\right] - q \\
&= -r \mathbb{E}[-1_{\{D > O_0\}}(D)] - h \mathbb{E}[1_{\{D < O_0\}}(D)] - q \\
&= r\mathbb{P}(D > O_0) - h\mathbb{P}(D < O_0) - q \\
&= r - (r + h)\mathbb{P}(D < O_0) - q.
\end{aligned}$$

However if $\mathbb{P}(D = O_0) \neq 0$, then $\mathbb{E}[(O - D)_\pm]$ may not be differentiable at $O = O_0$. Note that there is only countable many value O_0 such that $\mathbb{P}(D = O_0) > 0$, since $\{O_0 \mid \mathbb{P}(D = O_0) > \frac{1}{n}\}$ is finite. Apparently $\frac{d}{dO} \mathbb{E}[\text{Profit}] \Big|_{O=O_0}$ is monotonic decreasing of O . Furthermore for $O = O_0$, both $(O - D)_\pm$ are left (right) differentiable at every D . Therefore by Theorem 6, $\mathbb{E}[(O - D)_\pm]$ is left(right) differentiable at every $O = O_0$,

$$\begin{aligned}
& \frac{d}{dO} \mathbb{E}[\text{Profit}] \Big|_{O=O_0-} \\
&= -r \frac{d}{dO} \mathbb{E}[(O - D)_-] \Big|_{O=O_0-} - h \frac{d}{dO} \mathbb{E}[(O - D)_+] \Big|_{O=O_0-} - q \\
&= -r \mathbb{E}\left[\frac{\partial}{\partial O}((O_0-) - D)_-\right] - h \mathbb{E}\left[\frac{\partial}{\partial O}((O_0-) - D)_+\right] - q
\end{aligned}$$

$$\begin{aligned}
&= -r\mathbb{E}[-1_{\{D \geq O_0\}}(D)] - h\mathbb{E}[1_{\{D < O_0\}}(D)] - q \\
&= r\mathbb{P}(D \geq O_0) - h\mathbb{P}(D < O_0) - q \\
&= r - (r + h)\mathbb{P}(D < O_0) - q
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dO}\mathbb{E}[\text{Profit}] \Big|_{O=O_0+} \\
&= -r \frac{d}{dO}\mathbb{E}[(O - D)_-] \Big|_{O=O_0+} - h \frac{d}{dO}\mathbb{E}[(O - D)_+] \Big|_{O=O_0+} - q \\
&= -r\mathbb{E}\left[\frac{\partial}{\partial O}((O_0+) - D)_-\right] - h\mathbb{E}\left[\frac{\partial}{\partial O}((O_0+) - D)_+\right] - q \\
&= -r\mathbb{E}[-1_{\{D > O_0\}}(D)] - h\mathbb{E}[1_{\{D \leq O_0\}}(D)] - q \\
&= r\mathbb{P}(D > O_0) - h\mathbb{P}(D \leq O_0) - q \\
&= r - (r + h)\mathbb{P}(D \leq O_0) - q
\end{aligned}$$

It means $\frac{d}{dO}\mathbb{E}[\text{Profit}] \Big|_{O=O_0}$ is decreasing from $r - q$ to $-h - q$ and exists a.e. except the points O_0 where $\mathbb{P}(D = O_0) \neq 0$. By the property of absolutely continuous

$$\mathbb{E}[\text{Profit}] \Big|_{O=O_1} - \mathbb{E}[\text{Profit}] \Big|_{O=O_2} = \int_{O_2}^{O_1} \frac{d}{dO}\mathbb{E}[\text{Profit}] dO.$$

It is easy to show the maximum of $\mathbb{E}[\text{Profit}]$ exists, and it achieves maximum at point O_0 if and only if

$$\begin{cases} r - (r + h)\mathbb{P}(D < O_0) - q \geq 0 \\ r - (r + h)\mathbb{P}(D \leq O_0) - q \leq 0 \end{cases}.$$

It then follows that

$$\mathbb{P}(D < O_0) \leq \frac{r - q}{r + h} \leq \mathbb{P}(D \leq O_0). \quad (\text{B.4})$$

Notice that the maximum point O_0 may not be unique and all such maximum points form an interval $[a, b]$. Furthermore $\mathbb{E}[\text{Profit}]$ is concave. It first strictly increases to its maximum value at points in $[a, b]$ and then strictly decreases.

However the feasible order quantity has to be non-negative integer less than N^O . If $b < 0$, then the feasible optimal order quantity is 0; if $0 \in [a, b]$, then the feasible optimal order quantity is 0 or any feasible non-negative integer in $[a, b]$; if $a > 0$, then the optimal order quantity is any feasible non-negative integer in $[a, b]$ or $\min(\lfloor a \rfloor, N^O)$ or $\min(\lceil b \rceil, N^O)$. $\square$

The proof of Theorem 8 holds for any distribution of D . However the actual demand is usually non-negative. The next example shows that to decide the optimal order quantity for demand distribution D_+ one only needs to compute the case for demand D .

Example 7. *The optimal solution of $\mathbb{E}[\text{Profit}]$ for demands D_+ and D are the same.*

Proof. Since $\mathbb{E}[\text{Profit}]$ is absolutely continuous,

$$\mathbb{E}[\text{Profit}]|_{O=O_1} - \mathbb{E}[\text{Profit}]|_{O=O_2} = \int_{O_2}^{O_1} \frac{d}{dO} \mathbb{E}[\text{Profit}] dO$$

and by the proof of Theorem 8 the value of $\frac{d}{dO} \mathbb{E}[\text{Profit}]$ at feasible order quantities are the same for both D and D_+ . $\square$

B.4 Two-Retailer Model

Property 34. *Show the partial differentiation of (3.8).*

Fix O_i . For any O_j all terms are integrable. For fixed $D_i(\omega)$ and $D_j(\omega)$,

$$\left\{ \begin{array}{l} |(O_j - D_j)_- - (O'_j - D_j)_-| \leq |O_j - O'_j| \\ |(O_j - D_j)_+ - (O'_j - D_j)_+| \leq |O_j - O'_j| \\ |O_j - O'_j| \leq |O_j - O'_j| \\ |((D_i - O_i)_+ - (O_j - D_j)_+)_- - ((D_i - O_i)_+ - (O'_j - D_j)_+)_-| < |O_j - O'_j| \\ |((D_i - O_i)_- - (O_j - D_j)_-)_- - ((D_i - O_i)_- - (O'_j - D_j)_-)_-| < |O_j - O'_j| \end{array} \right.$$

It implies that all terms are Lipschitz continuous, so they are absolutely continuous. It also implies that the derivatives are bounded by a fixed constant a.e.. Hence by Theorem 6, when $\mathbb{P}(D_j = O_j) = 0$, $\mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) = 0$ and $\mathbb{P}((O_j - D_j)_- = (D_i - O_i)_-, O_j < D_j) = 0$,

$$\begin{aligned} & \frac{d}{dO_j} \mathbb{E}[\text{Profit}_j](O_j) \\ &= -r \int_{O_j < D_j} -1 d\omega - h \int_{O_j > D_j} d\omega - q + (r_{ji}^j + h) \int_{O_j > D_j} d\omega \\ & \quad - (r_{ji}^j + h) \int_{O_j > D_j, (O_j - D_j) > (D_i - O_i)_+} d\omega + r_{ij}^j \int_{O_j < D_j} -1 d\omega \\ & \quad - r_{ij}^j \int_{O_j < D_j, (D_i - O_i)_- < (O_j - D_j)_-} -1 d\omega \\ &= -q + r_{ji}^j + (r - r_{ji}^j - r_{ij}^j) \int_{O_j < D_j} d\omega - (r_{ji}^j + h) \int_{O_j > D_j, O_j > D_j + (D_i - O_i)_+} d\omega \\ & \quad + r_{ij}^j \int_{O_j < D_j, O_j < D_j - (D_i - O_i)_-} d\omega. \end{aligned} \tag{B.5}$$

We claim that only countable many O_j are not differentiable for a fixed O_i . We need to show that there are only countable many O_j such that $\mathbb{P}(D_j = O_j) > 0$ or $\mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) > 0$ or $\mathbb{P}((O_j - D_j)_- = (D_i - O_i)_-, O_j < D_j) > 0$ for fixed O_i . Since by the proof of Theorem 8, there are countable many O_j such that $\mathbb{P}(D_j = O_j) > 0$, we only need to show that there are countable many O_j such that $\mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j >$

$D_j) > 0$. Suppose $O'_j \neq O_j$, for any ω , s.t. $(O_j - D_j(\omega))_+ = (D_i(\omega) - O_i)_+$ and $O_j > D_j(\omega)$, there is $O_j - D_j(\omega) = (D_i(\omega) - O_i)_+ > 0$. Hence $O_j - D_j(\omega) = D_i(\omega) - O_i$. It means $O'_j - D_j(\omega) \neq D_i(\omega) - O_i$. In other words, it does not hold that $(O'_j - D_j(\omega))_+ = (D_i(\omega) - O_i)_+$ and $O'_j > D_j(\omega)$. Since $\{O_j \mid \mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) > 0\} = \cup_{n=1}^{\infty} \{O_j \mid \mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) > \frac{1}{n}\}$, and $|\{O_j \mid \mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) > \frac{1}{n}\}|$ is finite, it implies there are only countable many O_j such that the condition: $\mathbb{P}(D_j = O_j) = 0$, $\mathbb{P}((O_j - D_j)_+ = (D_i - O_i)_+, O_j > D_j) = 0$ and $\mathbb{P}((O_j - D_j)_- = (D_i - O_i)_-, O_j < D_j) = 0$, is not satisfied.

B.4.1 The Proof of Property 20

Proof.

$$\begin{aligned} & \mathbb{E}[\text{Profit}_j](O_i, O_j) \\ = & \mathbb{E}[rD_j] - r\mathbb{E}[(O_j - D_j)_-] - h\mathbb{E}[(O_j - D_j)_+] - O_j q + (r_{ji}^j + h)\mathbb{E}[(O_j - D_j)_+] \\ & - (r_{ji}^j + h)\mathbb{E}[((D_i - O_i)_+ - (O_j - D_j)_+)_-] + r_{ij}^j(O_j - D_j)_- \\ & - r_{ij}^j\mathbb{E}[((D_i - O_i)_- - (O_j - D_j)_-)_-]. \end{aligned}$$

There are only two terms $\mathbb{E}[((D_i - O_i)_+ - (O_j - D_j)_+)_-]$ and $\mathbb{E}[((D_i - O_i)_- - (O_j - D_j)_-)_-]$ that are related with O_i . Because for all O_j ,

$$\begin{aligned} & |\mathbb{E}[((D_i - O_1)_+ - (O_j - D_j)_+)_-] - \mathbb{E}[((D_i - O_2)_+ - (O_j - D_j)_+)_-]| \\ = & |\mathbb{E}[((D_i - O_1)_+ - (O_j - D_j)_+)_- - ((D_i - O_2)_+ - (O_j - D_j)_+)_-]| \\ \leq & \mathbb{E}[|((D_i - O_1)_+ - (O_j - D_j)_+)_- - ((D_i - O_2)_+ - (O_j - D_j)_+)_-|] \\ \leq & \mathbb{E}[|O_1 - O_2|] \\ = & |O_1 - O_2| \end{aligned}$$

and

$$\begin{aligned} & |\mathbb{E}[(D_i - O_1)_- - (O_j - D_j)_-] - \mathbb{E}[(D_i - O_2)_- - (O_j - D_j)_-]| \\ & \leq |O_1 - O_2|. \end{aligned}$$

By definition of uniformly continuous, $\mathbb{E}[\text{Profit}_i](O_i, O_j)$ is uniformly continuous of O_i for all O_j . $\square$

B.4.2 The proof of Property 21

By (3.9) and

$$\begin{aligned} & \frac{\partial}{\partial O_j} \mathbb{E}[\text{Profit}_i](O_i, O_j) \\ & = - (r_{ij}^i + h) \int_{O_j < D_j, D_i - O_i < O_j - D_j} d\omega + r_{ji}^i \int_{O_j > D_j, (D_j - O_j) > O_i - D_i} d\omega, \end{aligned}$$

it is sufficient to prove that the probability as $(\mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j))$ or $\mathbb{P}(O_j < D_j, O_j < D_j - (D_i - O_i)_-)$ is continuous at (O_i, O_j) .

Theorem 13. *If $\mathbb{P}(D_j = O_j) = 0$, $\mathbb{P}(D_i + D_j = O_i + O_j, D_j < O_j) = 0$ and $\mathbb{P}(D_i + D_j = O_i + O_j, D_j > O_j) = 0$, then $\mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j)$ or $\mathbb{P}(O_j < D_j, O_j < D_j - (D_i - O_i)_-)$ is continuous at (O_i, O_j) .*

Proof. Since

$$\begin{aligned} & \mathbb{P}(D_j \leq O_j, D_i + D_j \geq O_i + O_j) \\ & \leq \mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j) + \mathbb{P}(D_j = O_j) \\ & \quad + \mathbb{P}(D_i + D_j = O_i + O_j, D_j < O_j) \\ & = \mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j) + 0 + 0 \\ & = \mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j), \end{aligned}$$

$$\begin{aligned}
& \mathbb{P}(D_j \geq O_j, O_j \leq D_j - (D_i - O_i)_-) \\
& \leq \mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-) + \mathbb{P}(D_j = O_j) \\
& \quad + \mathbb{P}(D_i + D_j = O_i + O_j, D_j > O_j) \\
& = \mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-) + 0 + 0 \\
& = \mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-),
\end{aligned}$$

and $\mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j) \leq \mathbb{P}(D_j \leq O_j, D_i + D_j \geq O_i + O_j)$,
 $\mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-) \leq \mathbb{P}(D_j \geq O_j, O_j \leq D_j - (D_i - O_i)_-)$
there are

$$\begin{aligned}
& \mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j) \\
& = \mathbb{P}(D_j \leq O_j, D_i + D_j \geq O_i + O_j) := C,
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-) \\
& = \mathbb{P}(D_j \geq O_j, O_j \leq D_j - (D_i - O_i)_-) := C'.
\end{aligned}$$

Let $\{a_k\}$, $\{c_k\}$ and $\{b_k\}$ be strictly monotonic increasing sequences to O_j , O_i and $O_i + O_j$. Let $\{a'_k\}$, $\{c'_k\}$ and $\{b'_k\}$ be strictly monotonic decreasing sequences to O_j , O_i and $O_i + O_j$.

First we show that $\mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j)$ is continuous at (O_i, O_j) . Let

$$A_k = \{D_j < a_k, D_i + D_j > b'_k\} \quad \text{and} \quad B_k = \{D_j < a'_k, D_i + D_j > b_k\}.$$

Hence $A_k \nearrow$ and $B_k \searrow$. By continuity of probability,

$$\lim_{k \rightarrow \infty} \mathbb{P}(A_k) = C \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathbb{P}(B_k) = C.$$

For $\forall \epsilon > 0, \exists K$, such that when $k > K$,

$$|\mathbb{P}(A_k) - C| < \epsilon \quad \text{and} \quad |\mathbb{P}(B_k) - C| < \epsilon.$$

Since for all $a_k < O'_j < a'_k$ and $b_k < O'_i + O'_j < b'_k$,

$$\begin{aligned} A_k &= \{D_j < a_k, D_i + D_j > b'_k\} \subset \{D_j < O'_j, D_i + D_j > O'_i + O'_j\} \subset B_k \\ &= \{D_j < a'_k, D_i + D_j > b_k\}, \end{aligned}$$

it implies

$$|\mathbb{P}(D_j < O'_j, D_i + D_j > O'_i + O'_j) - C| < \epsilon.$$

Because $a_k < O_j < a'_k$ and $b_k < O_i + O_j < b'_k$, there is a neighbor U of (O_i, O_j) such that for $(O'_i, O'_j) \in U$

$$|\mathbb{P}(D_j < O'_j, D_i + D_j > O'_i + O'_j) - C| < \epsilon.$$

Therefore

$$\begin{aligned} &\lim_{(O'_i, O'_j) \rightarrow (O_i, O_j)} \mathbb{P}(D_j < O'_j, D_i + D_j > O'_i + O'_j) \\ &= \mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j) \\ &= C. \end{aligned}$$

This means that $\mathbb{P}(D_j < O_j, D_i + D_j > O_i + O_j)$ is continuous at (O_i, O_j) .

Next we show that $\mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-)$ is continuous at (O_i, O_j) . Let

$$\begin{aligned} \tilde{A}_k &= \{D_j > a'_k, a'_k < D_j - (D_i - c'_k)_-\} \\ \tilde{B}_k &= \{D_j > a_k, a_k < D_j - (D_i - c_k)_-\}. \end{aligned}$$

Hence $\tilde{A}_k \nearrow$ and $\tilde{B}_k \searrow$. It is not difficult to see

$$\cup_{k=1}^{\infty} \tilde{A}_k \subset \{D_j > O_j, O_j < D_j - (D_i - O_i)_-\}$$

and

$$\cap_{k=1}^{\infty} \tilde{B}_k \supset \{D_j \geq O_j, O_j \leq D_j - (D_i - O_i)_-\}.$$

Furthermore for $\forall$ fixed $\omega \in \{D_j > O_j, O_j < D_j - (D_i - O_i)_-\}$, $\exists a'_k$ and c'_k such that

$$D_j(\omega) > a'_k, a'_k < D_j(\omega) - (D_i(\omega) - c'_k)_- \implies \omega \in \tilde{A}_k.$$

Hence

$$\cup_{k=1}^{\infty} \tilde{A}_k = \{D_j > O_j, O_j < D_j - (D_i - O_i)_-\}.$$

Also for $\forall \omega \in \cap_{k=1}^{\infty} \tilde{B}_k$, there are

$$D_j(\omega) > a_k, a_k < D_j(\omega) - (D_i(\omega) - c_k)_- \text{ for all } k.$$

Let $k \rightarrow \infty$, then the limitations are

$$D_j(\omega) \geq O_j, O_j \leq D_j(\omega) - (D_i(\omega) - O_i)_-.$$

Hence

$$\cap_{k=1}^{\infty} \tilde{B}_k = \{D_j \geq O_j, O_j \leq D_j - (D_i - O_i)_-\}.$$

Therefore by continuity of probability $\lim_{k \rightarrow \infty} \mathbb{P}(\tilde{A}_k) = C'$ and

$\lim_{k \rightarrow \infty} \mathbb{P}(\tilde{B}_k) = C'$. For $\forall \epsilon > 0, \exists K$, such that when $k > K$,

$$|\mathbb{P}(\tilde{A}_k) - C'| < \epsilon \quad \text{and} \quad |\mathbb{P}(\tilde{B}_k) - C'| < \epsilon.$$

Since for all $a_k < O'_j < a'_k$ and $c_k < O'_i < c'_k$,

$$\begin{aligned} \tilde{A}_k &= \{D_j > a'_k, a'_k < D_j - (D_i - c'_k)_-\} \subset \{D_j > O'_j, O'_j < D_j - (D_i - O'_i)_-\} \\ &\subset B_k = \{D_j > a_k, a_k < D_j - (D_i - c_k)_-\}, \end{aligned}$$

it implies

$$|\mathbb{P}(D_j > O'_j, O'_j < D_j - (D_i - O'_i)_-) - C'| < \epsilon.$$

Because $a_k < O_j < a'_k$ and $c_k < O_i < c'_k$, there is a neighbor U of (O_i, O_j) such that for $(O'_i, O'_j) \in U$

$$|\mathbb{P}(D_j > O'_j, O'_j < D_j - (D_i - O'_i)_-) - C'| < \epsilon.$$

Therefore $\mathbb{P}(D_j > O_j, O_j < D_j - (D_i - O_i)_-)$ is continuous at (O_i, O_j) . $\square$

If for all (O_1, O_2) , there are $\mathbb{P}(D_1 = O_1) = 0, \mathbb{P}(D_2 = O_2) = 0, \mathbb{P}(D_1 + D_2 = O_1 + O_2, D_1 < O_1) = 0, \mathbb{P}(D_1 + D_2 = O_1 + O_2, D_2 < O_2) = 0, \mathbb{P}(D_1 + D_2 = O_1 + O_2, D_1 > O_1) = 0$ and $\mathbb{P}(D_1 + D_2 = O_1 + O_2, D_2 > O_2) = 0$; by Theorem 13, $\frac{\partial}{\partial O_1} \mathbb{E}[\text{Profit}_1](O_1, O_2), \frac{\partial}{\partial O_2} \mathbb{E}[\text{Profit}_1](O_1, O_2), \frac{\partial}{\partial O_1} \mathbb{E}[\text{Profit}_2](O_1, O_2), \frac{\partial}{\partial O_2} \mathbb{E}[\text{Profit}_2](O_1, O_2), \frac{\partial}{\partial O_1} \mathbb{E}[\text{Total Profit}](O_1, O_2)$ and $\frac{\partial}{\partial O_2} \mathbb{E}[\text{Total Profit}](O_1, O_2)$ are all continuous. Hence $\mathbb{E}[\text{Profit}_1](O_1, O_2), \mathbb{E}[\text{Profit}_2](O_1, O_2)$ and $\mathbb{E}[\text{Total Profit}](O_1, O_2)$ are differentiable.

B.5 Multi-Retailer Model

B.5.1 The proof of Property 25

Proof. Since $I_{i,i_1}^-(O_i, D)$, $I_{i,i_1}^+(O_i, D)$, are measurable, by (3.13), all $I_{i,i_k}^-(O, D)$, $I_{i,i_k}^+(O, D)$, $I_{i_k}^-(O, D)$, $\min(I_{i,i_k}^-(O_i, D), I_{i_k}^+(O, D))$ and $\min(I_{i,i_1}^+(O_i, D), I_{i_1}^-(O, D))$ are measurable.

By Property 23, $\min(I_{i,i_k}^+(O, D), I_{i_k}^-(O, D)) = \min(I_{i,i_k}^+(O, D), C_{i_k}^-(O_i-, D))$ and $\min(I_{i,i_k}^-(O, D), I_{i_k}^+(O, D)) = \min(I_{i,i_k}^-(O, D), C_{i_k}^+(O_i-, D))$. Hence

$$\begin{aligned} I_{i,i_{k+1}}^-(O, D) &= (I_{i,i_k}^-(O, D) - I_{i_k}^+(O, D))_+ \\ &= I_{i,i_k}^-(O, D) - \min(I_{i,i_k}^-(O, D), I_{i_k}^+(O, D)) \\ &= I_{i,i_k}^-(O, D) - \min(I_{i,i_k}^-(O, D), C_{i_k}^+(O_i-, D)), \\ &= (I_{i,i_k}^-(O, D) - C_{i_k}^+(O_i-, D))_+. \end{aligned}$$

Similarly $I_{i,i_{k+1}}^+(O, D) = (I_{i,i_k}^+(O, D) - C_{i_k}^-(O_i-, D))_+$. □

B.5.2 The proof of Property 26

Proof. Since for a fixed O , $I_{i,i_1}^-(O_i, D)$ and $I_{i,i_1}^+(O_i, D)$ are measurable and all functions are bounded, then $I_{i,i_1}^-(O_i, D)$ and $I_{i,i_1}^+(O_i, D)$ are integrable. By assumption $|I_{i,i_1}^+(O_i, D) - I_{i,i_1}^+(O'_i, D)| \leq \alpha|O_i - O'_i|$, for fixed D , $I_{i,i_1}^+(O_i, D)$ is Lipschitz continuous. It implies $I_{i,i_1}^+(O_i, D)$ is absolutely continuous and $|\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D)| \leq \alpha$ a.e.. Since

$$\begin{aligned} &|\mathbb{E}[I_{i,i_1}^+(O_i, D)] - \mathbb{E}[I_{i,i_1}^+(O'_i, D)]| \\ &= |\mathbb{E}[I_{i,i_1}^+(O_i, D) - I_{i,i_1}^+(O'_i, D)]| \\ &\leq \mathbb{E}[|I_{i,i_1}^+(O_i, D) - I_{i,i_1}^+(O'_i, D)|] \\ &\leq \mathbb{E}[\alpha|O_i - O'_i|] = \alpha|O_i - O'_i|, \end{aligned}$$

$\mathbb{E}[I_{i,i_1}^+(O_i, D)]$ is Lipschitz continuous. It implies $\mathbb{E}[I_{i,i_1}^+(O_i, D)]$ is absolutely continuous.

Let $A := \{I_{i,i_1}^-(O_i, D) > 0\}$ and $B := \{I_{i,i_1}^+(O_i, D) > 0\}$. By definition of $I_{i,i_1}^-(O_i, D)$ and $I_{i,i_1}^+(O_i, D)$, if $I_{i,i_1}^-(O_i, D) > 0$, then $I_{i,i_1}^+(O_i, D) = 0$. Hence $A \cap B = \emptyset$, $\{I_{i,i_1}^+(O_i, D) = 0 \text{ and } I_{i,i_1}^-(O_i, D) = 0\} = (A \cup B)^c$. Because for fixed D , $I_{i,i_1}^-(O_i, D)$ is continuous, A is open. Therefore for fixed D , $I_{i,i_1}^+(O_i, D)$ is differentiable in A and the derivative $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) = 0$ in A . By assumption for fixed D , $I_{i,i_1}^+(O_i, D)$ is also differentiable in B . Hence for a given x , $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) \Big|_{O_i=x} = 0$ in $\{D | I_{i,i_1}^-(x, D) > 0\}$; and $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) \Big|_{O_i=x}$ exists in $\{D | I_{i,i_1}^+(x, D) > 0\}$. Because $\mathbb{P}(I_{i,i_1}^+(x, D) = 0 \text{ and } I_{i,i_1}^-(x, D) = 0) = 0$, hence by the Extended Leibniz Rule (Theorem 6),

$$\frac{d}{dO_i} \mathbb{E}[I_{i,i_1}^+(O_i, D)] \Big|_{O_i=x} = \int_{\{I_{i,i_1}^+(x, D) > 0\}} \frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) \Big|_{O_i=x} dD.$$

Furthermore if $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) = \theta$ in $\{I_{i,i_1}^+(x, D) > 0\}$, then

$$\begin{aligned} \frac{d}{dO_i} \mathbb{E}[I_{i,i_1}^+(O_i, D)] \Big|_{O_i=x} &= \int_{\{I_{i,i_1}^+(x, D) > 0\}} \theta dD \\ &= \theta \mathbb{P}(I_{i,i_1}^+(x, D) > 0). \end{aligned}$$

□

B.5.3 The proof of Property 27

Proof. By Property 25 and the assumption that all functions are bounded, $I_{i,i_{k+1}}^+(O, D)$ is integrable for fixed O_i . For any fixed D , any O_i and O'_i ,

$$\begin{aligned} &|I_{i,i_{k+1}}^+(O_i, D; O_i-) - I_{i,i_{k+1}}^+(O'_i, D; O_i-)| \\ &= |(I_{i,i_k}^+(O_i, D; O_i-) - C_{i_k}^-(D; O_i-))_+ - (I_{i,i_k}^+(O'_i, D; O_i-) - C_{i_k}^-(D; O_i-))_+| \end{aligned}$$

$$\begin{aligned}
&\leq |(I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-)) - (I_{i,i_k}^+(O'_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-))| \\
&= |I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - I_{i,i_k}^+(O'_i, \mathbf{D}; O_i-)| \\
&\leq \dots \dots \\
&= |I_{i,i_1}^+(O_i, \mathbf{D}) - I_{i,i_1}^+(O'_i, \mathbf{D})| \leq \alpha |O_i - O'_i|.
\end{aligned}$$

It implies that $I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)$ for any fixed $\mathbf{D}$ is Lipschitz continuous of O_i , and $\mathbb{E}[I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)]$ is also Lipschitz continuous of O_i . Hence they are absolutely continuous. Furthermore $|\frac{\partial}{\partial O_i} I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)| \leq \alpha$ a.e..

Fix $\mathbf{D}$, let $A := \{O_i | I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-) < 0\}$, $B := \{O_i | I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-) > 0\}$. Since $I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-)$ is continuous for fixed $\mathbf{D}$, A and B are open sets. Because $I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-) = (I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-))_+ = 0$ in A . It implies $\frac{\partial}{\partial O_i} I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)$ is 0 in A . For any $O_i \in B$, $I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-) > 0$ implies $I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) > 0$. Therefore

$$\begin{aligned}
I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-) &= I_{i,i_k}^+(O_i, \mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-) \\
&= I_{i,i_{k-1}}^+(O_i, \mathbf{D}; O_i-) - C_{i_{k-1}}^-(\mathbf{D}; O_i-) - C_{i_k}^-(\mathbf{D}; O_i-) \\
&= \dots \dots \\
&= I_{i,i_1}^+(O_i, \mathbf{D}) - C_{i_1}^-(\mathbf{D}; O_i-) - \dots - C_{i_k}^-(\mathbf{D}; O_i-),
\end{aligned}$$

By assumption for fixed $\mathbf{D}$, $I_{i,i_1}^+(O_i, \mathbf{D})$ is differentiable in $\{I_{i,i_1}^+(O_i, \mathbf{D}) > 0\}$, hence $I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)$ is differentiable in B . The differentiation $\frac{\partial}{\partial O_i} I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-) = \frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, \mathbf{D})$ in B . So for fixed $\mathbf{D}$, if $I_{i,i_{k+1}}^+(O_i, \mathbf{D}; O_i-)$ is not differentiable at x , then

$$\begin{aligned}
I_{i,i_l}^+(x, \mathbf{D}; O_i-) - C_{i_l}^-(\mathbf{D}; O_i-) &= 0, & l \leq k. \\
I_{i,i_{l-1}}^+(x, \mathbf{D}; O_i-) - C_{i_{l-1}}^-(\mathbf{D}; O_i-) &> 0, & l \leq k.
\end{aligned}$$

or

$$I_{i,i_1}^+(x, D) = 0 \text{ and } I_{i,i_1}^-(x, D) = 0$$

By the assumption and by applying the Extended Leibniz Rule (Theorem 6), $\mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)]$ is differentiable at x and

$$\begin{aligned} & \frac{d}{dO_i} \mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)] \Big|_{O_i=x} \\ &= \mathbb{E} \left[\frac{\partial}{\partial O_i} I_{i,i_{k+1}}^+(O_i, D; O_i-) \Big|_{O_i=x} \right] \\ &= \int_{\{I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) > 0\}} \left(\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) \Big|_{O_i=x} \right) dD \end{aligned}$$

Furthermore suppose $\frac{\partial}{\partial O_i} I_{i,i_1}^+(O_i, D) = \theta$ in $\{I_{i,i_1}^+(O_i, D) > 0\}$, then

$$\begin{aligned} \frac{d}{dO_i} \mathbb{E}[I_{i,i_{k+1}}^+(O_i, D; O_i-)] \Big|_{O_i=x} &= \int_{\{I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) > 0\}} \theta dD \\ &= \theta \mathbb{P}(I_{i,i_k}^+(x, D; O_i-) - C_{i_k}^-(D; O_i-) > 0). \end{aligned}$$

□

B.5.4 The proof of Theorem 9

Proof. Since $r_{i_k, i}^i$ are the same and r_{i, i_k}^i are the same,

$$\begin{aligned} & \frac{d}{dO_i} \mathbb{E}[\text{Profit}_i(O_i, D; O_i-)] \\ &= -q + r\mathbb{P}(I_i(O_i, D)_- > 0) - h\mathbb{P}(I_i(O_i, D)_+ > 0) \\ & \quad + (r_{i,i_1}^i + h)\theta^+\mathbb{P}(I_{i,i_1}^+(O_i, D) > 0) \\ & \quad - r_{i_1, i}^i\theta^-\mathbb{P}(I_{i,i_1}^-(O_i, D) > 0) \\ & \quad - (r_{i, i_{n-1}}^i + h)\theta^+\mathbb{P}(I_{i, i_{n-1}}^+(O_i, D; O_i-) - C_{i_{n-1}}^-(D; O_i-) > 0) \end{aligned}$$

$$+ r_{i_{n-1},i}^i \theta^- \mathbb{P}(I_{i,i_{n-1}}^-(O_i, \mathbf{D}; O_i-) - C_{i_{n-1}}^+(\mathbf{D}; O_i-) > 0).$$

Since $r > r_{i,i_1}^i + r_{i_1,i}^i$, the term

$$\begin{aligned} & r \mathbb{P}(I_i(O_i, \mathbf{D})_- > 0) - h \mathbb{P}(I_i(O_i, \mathbf{D})_+ > 0) \\ & + (r_{i,i_1}^i + h) \theta \mathbb{P}(I_{i,i_1}^+(O_i, \mathbf{D}) > 0) - r_{i_1,i}^i \theta \mathbb{P}(I_{i,i_1}^-(O_i, \mathbf{D}) > 0) \end{aligned}$$

and

$$\begin{aligned} & -(r_{i,i_{n-1}}^i + h) \theta \mathbb{P}(I_{i,i_{n-1}}^+(O_i, \mathbf{D}; O_i-) - C_{i_{n-1}}^-(\mathbf{D}; O_i-) > 0) \\ & + r_{i_{n-1},i}^i \theta \mathbb{P}(I_{i,i_{n-1}}^-(O_i, \mathbf{D}; O_i-) - C_{i_{n-1}}^+(\mathbf{D}; O_i-) > 0) \end{aligned}$$

are monotone decreasing. If $\mathbb{P}(I_i(O_i, \mathbf{D})_+ > 0)$ is strictly increasing in the feasible region of O_i , then

$$\begin{aligned} & r \mathbb{P}(I_i(O_i, \mathbf{D})_- > 0) - h \mathbb{P}(I_i(O_i, \mathbf{D})_+ > 0) \\ & + (r_{i,i_1}^i + h) \theta \mathbb{P}(I_{i,i_1}^+(O_i, \mathbf{D}) > 0) - r_{i_1,i}^i \theta \mathbb{P}(I_{i,i_1}^-(O_i, \mathbf{D}) > 0) \end{aligned}$$

is strictly decreasing in the feasible region of O_i . □

Appendix C

Data and Code

C.1 Two-Retailer Model

Table C.1: Average number of sharings in Scenario 1

Order quantity	Balanced	Synchronized
90	1259.67	68.95
91	1233.17	72.76
92	1204.27	76.14
93	1157.23	79.35
94	1091.23	80.49
95	1032.33	81.70
96	950.83	81.73
97	859.11	80.04
98	759.19	78.24
99	650.20	74.24
100	547.04	70.65
101	461.89	64.32
102	372.15	59.01
103	301.27	53.45
104	228.50	46.11
105	180.31	40.39
106	138.46	35.42
107	106.34	30.16
108	79.89	25.97
109	64.55	22.41
110	49.29	18.25

Table C.2: Average number of lost sales in Scenario 1

Order quantity	Balanced	Synchronized
90	342.82	408.68
91	301.96	369.80
92	265.74	335.57
93	224.87	296.95
94	185.96	258.49
95	157.07	229.96
96	124.23	196.55
97	98.66	168.80
98	74.34	141.09
99	53.83	114.68
100	37.16	92.37
101	26.08	75.68
102	16.83	59.72
103	10.65	47.39
104	6.45	36.28
105	3.99	28.84
106	2.01	22.06
107	1.25	17.50
108	0.54	13.34
109	0.35	10.97
110	0.24	8.68

Table C.3: Average number of sharings in Scenario 2

Order quantity	Balanced	Synchronized
90+ ϵ_1	1069.96	60.54
91+ ϵ_1	1060.96	63.28
92+ ϵ_1	1036.73	65.29
93+ ϵ_1	1005.72	67.13
94+ ϵ_1	971.49	68.43
95+ ϵ_1	925.31	70.99
96+ ϵ_1	868.34	71.06
97+ ϵ_1	796.39	71.53
98+ ϵ_1	717.50	69.81
99+ ϵ_1	641.55	68.54
100+ ϵ_1	556.08	65.60
101+ ϵ_1	473.07	61.62
102+ ϵ_1	390.72	57.22
103+ ϵ_1	310.46	51.91
104+ ϵ_1	245.33	45.70
105+ ϵ_1	185.55	40.30
106+ ϵ_1	144.78	34.97
107+ ϵ_1	109.47	30.61
108+ ϵ_1	82.40	25.80
109+ ϵ_1	63.90	21.81
110+ ϵ_1	49.08	18.56

Table C.4: Average number of lost sales in Scenario 2

Order quantity	Balanced	Synchronized
90+ ϵ_1	359.67	425.82
91+ ϵ_1	321.46	389.22
92+ ϵ_1	281.04	350.91
93+ ϵ_1	244.46	315.61
94+ ϵ_1	209.29	281.19
95+ ϵ_1	174.59	246.89
96+ ϵ_1	142.50	214.55
97+ ϵ_1	112.15	182.17
98+ ϵ_1	86.50	152.89
99+ ϵ_1	66.27	129.53
100+ ϵ_1	46.79	104.75
101+ ϵ_1	32.93	85.37
102+ ϵ_1	22.27	67.77
103+ ϵ_1	13.09	51.88
104+ ϵ_1	7.58	40.63
105+ ϵ_1	4.30	30.56
106+ ϵ_1	2.64	23.94
107+ ϵ_1	1.31	18.12
108+ ϵ_1	0.70	13.91
109+ ϵ_1	0.35	10.80
110+ ϵ_1	0.15	8.45

Table C.5: Average number of sharings in Scenario 3

Ordering setting	Balanced	Synchronized
1	1046.47	61.36
2	1031.21	62.97
3	1007.32	65.36
4	987.60	66.95
5	944.08	68.72
6	894.25	70.96
7	846.89	70.91
8	768.93	70.45
9	704.70	69.93
10	627.86	67.79
11	549.12	65.47
12	465.91	61.62
13	378.98	56.73
14	303.73	51.80
15	242.32	45.80
16	194.24	41.57
17	147.19	35.44
18	114.32	30.76
19	85.54	26.08
20	66.22	21.58
21	52.02	18.64

Table C.6: Average number of lost sales in Scenario 3

Order setting	Balanced	Synchronized
1	361.46	424.95
2	323.31	388.78
3	282.85	350.40
4	249.95	318.60
5	210.38	279.97
6	176.49	245.82
7	145.09	213.81
8	114.31	181.27
9	90.43	154.83
10	69.36	130.00
11	49.67	106.12
12	34.75	85.58
13	23.17	66.58
14	14.55	51.95
15	9.01	40.23
16	5.51	31.73
17	3.01	23.96
18	1.70	18.74
19	0.82	14.07
20	0.46	11.10
21	0.21	8.47

C.2 Multi-Retailer Model

Table C.7: Average of sharings in short term balanced system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	1788.14	1369.70	1826.88	860.76	836.56
2	1592.99	1198.00	1694.54	743.66	709.05
3	1382.41	1029.25	1544.14	653.62	614.65
4	1168.62	870.21	1363.35	580.30	537.10
5	982.91	742.62	1178.56	518.03	477.21
6	812.16	634.50	983.90	465.39	426.49
7	674.29	550.47	795.14	419.34	382.55
8	563.44	478.43	621.86	373.39	340.15
9	457.72	411.45	458.27	328.62	300.30
10	383.39	355.40	333.89	288.45	264.47
11	318.11	305.12	240.63	251.90	232.31
12	263.93	260.67	168.58	213.31	198.65
13	216.37	220.63	124.01	181.22	169.30
14	174.83	182.56	86.75	152.97	144.33
15	144.72	154.59	66.25	125.12	118.99
16	109.70	122.00	49.06	100.75	96.39
17	88.27	100.12	37.31	82.27	79.56
18	69.88	81.25	29.66	67.03	65.11
19	54.90	64.20	23.43	53.52	52.54
20	44.83	53.22	19.14	42.14	41.49
21	34.19	41.22	15.85	33.64	33.21

Table C.8: Average of sharings in short term balanced system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	1171.43	1220.38	2040.73	371.49	1118.33
2	1065.01	1102.03	1922.53	279.10	1020.25
3	969.85	983.78	1788.45	217.13	932.52
4	875.28	863.01	1626.87	169.59	845.56
5	779.20	752.11	1455.04	141.20	756.01
6	676.36	647.30	1274.45	120.71	659.69
7	592.30	553.75	1094.38	105.21	580.68
8	504.27	466.55	913.95	91.41	495.89
9	431.98	381.37	732.38	80.52	426.28
10	353.65	313.72	586.03	69.09	350.32
11	295.29	260.27	464.31	60.14	293.36
12	244.56	211.85	366.99	51.04	243.44
13	196.76	171.78	288.39	43.25	196.22
14	151.03	138.59	222.24	35.72	150.81
15	119.74	113.07	179.63	30.29	119.65
16	92.49	89.77	137.66	25.16	92.44
17	72.11	69.29	106.84	20.04	72.09
18	53.74	54.45	84.45	16.95	53.74
19	40.95	43.74	67.52	12.81	40.95
20	31.19	34.41	54.05	10.64	31.19
21	23.57	26.18	42.07	8.17	23.57

Table C.9: Average of sharings in short term synchronized system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	158.15	167.42	125.74	120.60	127.29
2	157.57	168.02	124.46	121.14	127.60
3	156.80	166.96	121.58	118.91	126.21
4	158.83	170.01	120.09	118.14	125.73
5	159.27	169.63	121.07	117.04	124.95
6	160.35	170.43	121.05	117.65	124.76
7	162.48	172.41	122.27	117.84	125.51
8	163.86	172.73	122.81	117.72	125.09
9	164.81	169.69	125.23	120.26	127.38
10	159.68	167.02	123.12	118.29	125.27
11	153.04	158.25	121.11	114.68	119.35
12	144.17	147.70	113.10	106.84	111.88
13	131.47	136.67	103.88	96.86	99.35
14	117.15	120.59	89.18	84.29	87.74
15	103.71	107.65	76.24	71.87	75.61
16	85.26	90.38	61.24	59.10	63.13
17	71.08	75.51	48.49	48.81	52.31
18	58.79	63.59	39.16	39.59	42.63
19	47.18	51.40	30.74	32.26	35.08
20	37.47	42.27	23.40	26.41	28.21
21	29.11	33.82	18.22	20.81	22.60

Table C.10: Average of sharings in short term synchronized system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	120.20	128.08	159.11	102.74	111.79
2	121.57	126.96	157.87	102.20	112.47
3	117.99	126.20	158.06	101.12	111.87
4	117.90	124.34	158.29	99.42	111.19
5	116.21	124.48	156.47	97.09	110.21
6	115.60	123.28	158.51	97.51	109.28
7	116.08	127.51	159.83	97.54	110.82
8	117.34	127.67	160.38	100.58	112.75
9	119.37	127.10	161.47	101.63	111.43
10	116.54	128.11	159.93	100.20	109.65
11	112.69	124.24	153.78	97.06	105.21
12	103.81	117.39	145.43	89.05	96.69
13	94.88	105.16	132.68	78.94	89.09
14	80.32	92.35	119.36	65.32	75.16
15	67.89	80.39	106.36	54.45	64.83
16	57.10	66.61	88.76	42.72	55.20
17	47.13	53.93	73.96	34.03	46.05
18	38.06	42.93	61.96	27.39	37.10
19	31.30	35.55	51.67	20.53	31.18
20	25.41	28.36	42.08	16.47	25.28
21	20.48	22.28	33.65	12.94	20.45

Table C.11: Average of lost sales in short term balanced system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00

Table C.12: Average of lost sales in short term balanced system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	0.06	0.01	0.00	0.00	0.14
2	0.01	0.00	0.00	0.00	0.02
3	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00

Table C.13: Average of lost sales in short term synchronized system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	198.04	193.70	201.79	202.20	200.90
2	178.70	173.99	183.50	181.11	179.07
3	161.24	155.61	163.96	160.94	158.67
4	139.90	133.50	142.95	140.70	137.54
5	118.32	112.64	124.20	121.21	120.25
6	96.82	90.53	104.72	102.86	101.11
7	75.55	70.67	85.43	84.73	81.99
8	55.79	51.85	65.89	67.87	64.94
9	37.13	35.57	48.44	50.06	47.64
10	24.90	22.83	33.12	37.15	34.58
11	15.11	14.10	21.36	24.48	22.88
12	8.66	8.08	13.00	15.29	14.11
13	4.32	4.17	6.89	9.22	7.89
14	1.94	1.85	3.61	4.76	4.31
15	1.02	1.04	1.76	2.41	2.18
16	0.41	0.39	0.88	1.10	0.99
17	0.17	0.20	0.35	0.48	0.46
18	0.04	0.06	0.09	0.24	0.22
19	0.02	0.01	0.03	0.08	0.06
20	0.00	0.00	0.03	0.03	0.03
21	0.01	0.03	0.00	0.01	0.01

Table C.14: Average of lost sales in short term synchronized system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	199.77	202.67	199.16	201.40	201.21
2	178.82	183.55	179.54	179.58	180.22
3	158.96	163.17	161.10	159.01	159.64
4	137.36	143.18	140.29	139.01	137.94
5	118.72	125.03	119.98	121.19	120.32
6	100.79	104.53	99.21	102.82	101.92
7	83.14	84.75	79.31	85.12	86.73
8	66.84	67.63	60.58	67.64	70.57
9	51.00	49.70	42.56	51.92	54.41
10	36.07	34.44	27.59	36.98	40.24
11	24.39	23.38	17.85	24.94	27.12
12	15.21	13.86	10.35	15.97	18.58
13	8.73	7.70	5.28	9.71	10.73
14	4.34	3.76	2.53	4.75	5.91
15	2.40	2.00	1.24	2.53	2.91
16	1.23	1.03	0.70	1.44	1.56
17	0.59	0.42	0.27	0.59	0.85
18	0.17	0.12	0.03	0.18	0.30
19	0.06	0.06	0.01	0.07	0.10
20	0.03	0.04	0.01	0.03	0.04
21	0.02	0.01	0.00	0.02	0.00

Table C.15: The differences of average lost sales in short term (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	198.04	193.70	201.79	202.20	200.90
2	178.70	173.99	183.50	181.11	179.07
3	161.24	155.61	163.96	160.94	158.67
4	139.90	133.50	142.95	140.70	137.54
5	118.32	112.64	124.20	121.21	120.25
6	96.82	90.53	104.72	102.86	101.11
7	75.55	70.67	85.43	84.73	81.99
8	55.79	51.85	65.89	67.87	64.94
9	37.13	35.57	48.44	50.06	47.64
10	24.90	22.83	33.12	37.15	34.58
11	15.11	14.10	21.36	24.48	22.88
12	8.66	8.08	13.00	15.29	14.11
13	4.32	4.17	6.89	9.22	7.89
14	1.94	1.85	3.61	4.76	4.31
15	1.02	1.04	1.76	2.41	2.18
16	0.41	0.39	0.88	1.10	0.99
17	0.17	0.20	0.35	0.48	0.46
18	0.04	0.06	0.09	0.24	0.22
19	0.02	0.01	0.03	0.08	0.06
20	0.00	0.00	0.03	0.03	0.03
21	0.01	0.03	0.00	0.01	0.01

Table C.16: The differences of average lost sales in short term (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	199.70	202.66	199.16	201.40	201.07
2	178.81	183.55	179.54	179.58	180.20
3	158.96	163.17	161.10	159.01	159.64
4	137.36	143.18	140.29	139.01	137.94
5	118.72	125.03	119.98	121.19	120.32
6	100.79	104.53	99.21	102.82	101.92
7	83.14	84.75	79.31	85.12	86.73
8	66.84	67.63	60.58	67.64	70.57
9	51.00	49.70	42.56	51.92	54.41
10	36.07	34.44	27.59	36.98	40.24
11	24.39	23.38	17.85	24.94	27.12
12	15.21	13.86	10.35	15.97	18.58
13	8.73	7.70	5.28	9.71	10.73
14	4.34	3.76	2.53	4.75	5.91
15	2.40	2.00	1.24	2.53	2.91
16	1.23	1.03	0.70	1.44	1.56
17	0.59	0.42	0.27	0.59	0.85
18	0.17	0.12	0.03	0.18	0.30
19	0.06	0.06	0.01	0.07	0.10
20	0.03	0.04	0.01	0.03	0.04
21	0.02	0.01	0.00	0.02	0.00

Table C.17: Average of sharings in long term balanced system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	14968.57	14374.69	14838.71	12848.12	13206.98
2	14710.65	14015.04	14604.99	12565.14	12852.24
3	14370.61	13548.86	14280.24	12153.18	12366.43
4	13889.97	12891.65	13818.26	11514.84	11620.57
5	13260.63	12027.51	13222.87	10624.07	10588.18
6	12401.01	10860.11	12418.97	9311.78	9115.61
7	11165.50	9237.36	11353.92	7363.53	6970.16
8	9423.42	7319.34	10072.35	5099.65	4611.69
9	7118.16	5291.89	8513.67	3083.36	2649.71
10	4655.69	3483.29	6446.51	1760.22	1500.79
11	2641.34	2095.65	4052.72	1041.68	918.47
12	1464.87	1273.45	2220.52	697.54	637.40
13	852.71	814.70	1000.33	493.83	459.65
14	521.16	531.34	435.46	350.73	330.21
15	332.18	353.24	214.50	247.67	235.14
16	229.22	251.63	106.41	175.18	167.81
17	152.56	174.46	64.23	120.66	116.77
18	105.36	123.97	41.01	87.50	85.24
19	75.66	89.61	30.05	65.39	64.14
20	52.62	64.11	22.21	50.15	49.37
21	40.80	49.78	18.01	37.85	37.41

Table C.18: Average of sharings in long term balanced system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	12858.75	13492.54	15068.39	11931.10	10829.15
2	12584.05	13178.70	14854.14	11526.21	10760.73
3	12228.92	12738.42	14568.26	10980.76	10611.47
4	11615.10	12104.38	14177.22	10167.14	10229.53
5	10753.48	11263.28	13668.82	9043.12	9600.89
6	9591.28	10156.26	12985.63	7462.10	8642.80
7	7968.55	8739.45	12097.18	5210.69	7214.91
8	6191.65	7247.44	10968.74	2868.30	5679.46
9	4484.69	5711.65	9478.81	1175.35	4241.27
10	3169.16	4136.34	7414.18	445.35	3077.82
11	2158.35	2674.27	4956.91	222.68	2123.73
12	1473.35	1664.09	2998.68	155.63	1455.85
13	928.37	971.87	1619.80	118.34	920.74
14	574.91	579.65	883.69	87.65	571.63
15	351.19	362.61	519.95	62.85	350.09
16	203.02	222.44	310.20	46.93	202.77
17	134.93	146.67	203.98	35.18	134.86
18	87.65	95.53	134.57	25.88	87.59
19	58.67	66.51	96.13	18.31	58.67
20	40.87	47.98	69.47	13.74	40.87
21	28.58	32.76	50.70	10.34	28.58

Table C.19: Average of sharings in long term synchronized system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	791.89	837.23	617.62	602.19	638.28
2	785.85	835.83	619.57	601.44	636.73
3	786.64	833.69	614.47	596.62	632.15
4	786.43	844.43	607.88	590.32	628.64
5	795.89	849.27	602.07	586.56	625.22
6	802.24	855.53	602.22	585.03	626.21
7	818.14	863.24	613.99	590.80	632.44
8	820.34	864.37	624.50	602.97	641.59
9	813.83	851.47	631.63	605.88	640.48
10	789.52	818.32	621.49	585.67	615.24
11	715.90	733.22	561.47	526.37	545.22
12	600.23	611.68	461.02	421.47	434.42
13	464.31	472.55	335.10	309.55	321.68
14	336.67	347.93	229.24	217.63	229.23
15	234.91	249.41	159.10	156.46	165.11
16	171.89	184.38	108.05	113.07	118.50
17	121.09	132.78	75.37	80.61	85.43
18	85.44	96.25	51.12	58.05	62.00
19	62.66	71.52	37.97	44.47	47.76
20	46.03	52.87	27.49	32.56	35.09
21	35.05	40.89	20.60	25.31	27.31

Table C.20: Average of sharings in long term synchronized system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	598.68	636.84	790.22	513.72	562.74
2	603.13	632.92	790.13	513.85	563.85
3	594.35	631.53	792.41	506.04	558.62
4	586.59	627.50	791.55	496.42	556.16
5	583.33	620.95	791.77	492.43	552.67
6	578.48	622.99	796.43	487.16	554.83
7	585.14	634.53	809.28	497.65	556.69
8	594.86	642.92	812.85	512.01	561.09
9	601.29	653.97	807.92	517.07	565.16
10	574.27	639.86	790.70	499.19	538.16
11	501.52	580.85	721.29	431.19	470.25
12	394.32	486.77	613.45	322.62	373.61
13	284.39	362.31	477.97	219.11	273.72
14	204.14	257.97	350.39	146.80	197.82
15	148.70	184.90	257.24	97.73	145.84
16	102.43	128.69	183.36	68.31	100.55
17	78.66	92.11	132.90	49.42	77.17
18	57.99	66.65	96.10	36.12	56.85
19	42.67	49.00	71.94	25.95	42.13
20	32.00	36.85	53.10	19.03	31.84
21	24.25	26.52	39.81	14.66	24.06

Table C.21: Average of lost sales in long term balanced system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	0.00	0.00	79.85	1033.19	608.23
2	0.00	0.00	56.91	867.58	478.86
3	0.00	0.00	39.59	700.61	358.09
4	0.00	0.00	25.41	523.20	248.25
5	0.00	0.00	14.38	343.86	149.87
6	0.00	0.00	6.31	184.18	73.52
7	0.00	0.00	1.57	60.27	21.24
8	0.00	0.00	0.27	11.16	3.64
9	0.00	0.00	0.01	0.87	0.33
10	0.00	0.00	0.00	0.03	0.02
11	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00

Table C.22: Average of lost sales in long term balanced system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	805.14	264.83	3.40	919.50	2741.33
2	647.10	203.50	2.12	766.68	2331.26
3	502.62	151.75	1.07	623.84	1918.04
4	362.42	103.63	0.58	463.91	1475.50
5	230.93	62.37	0.24	306.44	999.49
6	121.45	30.27	0.08	158.30	557.12
7	39.33	8.67	0.02	51.84	194.76
8	7.66	1.47	0.01	9.44	39.31
9	0.51	0.16	0.01	0.71	3.16
10	0.02	0.00	0.00	0.05	0.12
11	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00

Table C.23: Average of lost sales in long term synchronized system (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	987.86	966.46	1012.95	1009.23	996.27
2	897.75	872.63	912.65	904.74	891.20
3	804.68	776.79	818.49	807.90	795.29
4	703.39	671.75	719.11	706.30	694.20
5	592.22	562.32	620.33	606.61	594.32
6	481.54	457.79	521.75	515.20	500.54
7	366.60	345.69	415.10	417.83	403.35
8	262.51	246.99	315.65	326.57	311.10
9	168.47	156.63	219.31	231.32	215.31
10	94.57	89.55	131.98	146.28	134.27
11	42.38	40.57	64.93	73.75	66.73
12	17.15	15.93	27.92	32.10	29.22
13	6.22	6.10	10.85	13.89	12.24
14	2.43	2.21	4.71	5.75	5.41
15	1.08	1.10	1.94	2.74	2.24
16	0.52	0.34	0.78	1.30	0.99
17	0.16	0.18	0.37	0.49	0.40
18	0.05	0.09	0.19	0.18	0.15
19	0.04	0.03	0.06	0.06	0.07
20	0.00	0.01	0.02	0.01	0.02
21	0.00	0.01	0.01	0.00	0.00

Table C.24: Average of lost sales in long term synchronized system (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	999.83	1012.25	993.73	1007.37	1009.50
2	890.65	915.14	901.51	898.45	896.89
3	791.42	818.31	805.85	798.53	797.03
4	693.33	720.90	708.44	697.99	695.01
5	593.05	621.39	603.50	601.32	599.26
6	504.09	521.80	499.05	511.94	513.06
7	410.91	419.73	389.70	414.99	426.74
8	318.25	321.83	286.04	324.61	335.18
9	224.31	222.92	192.29	231.80	243.58
10	139.74	136.69	110.70	145.17	157.35
11	70.73	67.05	51.53	74.57	82.09
12	30.33	28.98	20.97	33.28	37.25
13	12.77	11.60	7.70	13.77	15.70
14	5.23	5.02	3.32	6.13	6.66
15	2.37	2.39	1.35	2.76	3.16
16	1.04	0.98	0.55	1.14	1.39
17	0.46	0.40	0.26	0.58	0.66
18	0.20	0.21	0.08	0.26	0.23
19	0.08	0.10	0.05	0.11	0.13
20	0.01	0.03	0.02	0.02	0.04
21	0.02	0.00	0.01	0.01	0.02

Table C.25: The differences of average lost sales in long term (a)

Ordering setting	Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
1	987.86	966.46	933.10	-23.96	388.03
2	897.75	872.63	855.75	37.16	412.34
3	804.68	776.79	778.90	107.29	437.20
4	703.39	671.75	693.70	183.11	445.95
5	592.22	562.32	605.95	262.74	444.45
6	481.54	457.79	515.45	331.02	427.02
7	366.60	345.69	413.52	357.56	382.11
8	262.51	246.99	315.38	315.41	307.46
9	168.47	156.63	219.30	230.46	214.99
10	94.57	89.55	131.98	146.25	134.24
11	42.38	40.57	64.93	73.75	66.73
12	17.15	15.93	27.92	32.10	29.22
13	6.22	6.10	10.85	13.89	12.24
14	2.43	2.21	4.71	5.75	5.41
15	1.08	1.10	1.94	2.74	2.24
16	0.52	0.34	0.78	1.30	0.99
17	0.16	0.18	0.37	0.49	0.40
18	0.05	0.09	0.19	0.18	0.15
19	0.04	0.03	0.06	0.06	0.07
20	0.00	0.01	0.02	0.01	0.02
21	0.00	0.01	0.01	0.00	0.00

Table C.26: The differences of average lost sales in long term (b)

Ordering setting	Retailer 6	Retailer 7	Retailer 8	Retailer 9	Retailer 10
1	194.68	747.41	990.32	87.87	-1731.84
2	243.55	711.64	899.39	131.77	-1434.37
3	288.80	666.56	804.78	174.70	-1121.00
4	330.91	617.27	707.86	234.08	-780.48
5	362.12	559.02	603.26	294.88	-400.23
6	382.65	491.54	498.98	353.63	-44.06
7	371.58	411.06	389.68	363.15	231.98
8	310.59	320.36	286.04	315.17	295.87
9	223.80	222.76	192.28	231.09	240.42
10	139.72	136.69	110.70	145.12	157.23
11	70.73	67.05	51.53	74.57	82.09
12	30.33	28.98	20.97	33.28	37.25
13	12.77	11.60	7.70	13.77	15.70
14	5.23	5.02	3.32	6.13	6.66
15	2.37	2.39	1.35	2.76	3.16
16	1.04	0.98	0.55	1.14	1.39
17	0.46	0.40	0.26	0.58	0.66
18	0.20	0.21	0.08	0.26	0.23
19	0.08	0.10	0.05	0.11	0.13
20	0.01	0.03	0.02	0.02	0.04
21	0.02	0.00	0.01	0.01	0.02

C.3 Example 4

```
newsvendor_demand<-function(n,mean,sd,orders,
inventory){
  # normal distribution
  # sample<-rnorm(n,mean,sd)
  # uniform distribution
  sample<-runif(n,min = 60, max =140)
  sample<-sample*(sample>0)
  # print(sample)
  for (m in 1:(n-1)) {
    # print(inventory)
    inventory<-inventory+orders-sample[m]
    # print(inventory)
    inventory<-inventory*(inventory>0)
    # print(typeof(inventory))
  }
  Demand<-sample[n]-inventory
  Demands<-c(d=sample, list)
  Demands<-Demands[-(n+1)]
  return(c(list("Demand"=Demand),Demands))
}
```

```
read_chunk("Newsvendor.R")
library(MonteCarlo)
source('~/Documents/paper2/Rcode/Newsvendor.R')
param_list = list("n"=7,"mean"=100,"sd"=20,
"orders"=100, "inventory"=10)
result<-MonteCarlo(func = newsvendor_demand,
nrep = 100000, param_list = param_list)

df<-MakeFrame(result)
D<-df[6:13]
library(xtable)
ta1<-xtable(D[1:80,])
```

```

par(mfrow=c(3,3))
hist(D$d1,breaks = 30)
hist(D$d2,breaks = 30)
hist(D$d3,breaks = 30)
hist(D$d4,breaks = 30)
hist(D$d5,breaks = 30)
hist(D$d6,breaks = 30)
hist(D$d7,breaks = 30)
hist(D$Demand,breaks = 30)
plot(density(D$Demand))
orderquantity<-quantile(D$Demand, probs = 0.6)
print(orderquantity)

##          60%
## 77.01336

abline(v=orderquantity)

```

C.4 Two-Retailer Balanced System

Simulation 3.6.3

```
BShare2Order<-function(Ostart,Oend){  
  source('~/Thesis/Rcode/SharingSimulation/BShare2.R')  
  for (i in Ostart:Oend) {  
    a<-replicate(10,BShare2(10,2,i))  
    if (i == Ostart){  
      result<-a  
    }else{  
      result<-rbind(result,a)  
    }  
  }  
  return(result)  
}
```

```
BShare2<-function(m,n,O){  
  source('~/Thesis/Rcode/SharingSimulation/Dem2.R')  
  library(data.table)  
  Two_retailers_demand<-Demands2(2*m+1,n)  
  #Two_retailers_demand<-matrix(c(117,136,99,75,135,  
  #137,99,129,100,100),nrow = 2,ncol = 2*m+1)  
  #print(Two_retailers_demand)  
  #O<-110  
  
  D1mean<-sample(-20:20,2*m+1,replace = TRUE)  
  D2mean<-sample(-20:20,2*m+1,replace = TRUE)  
  Two_retailers_demand<-Two_retailers_demand+  
  rbind(D1mean,D2mean)  
  Demandss<-data.table(Two_retailers_demand)  
  fwrite(Demandss,file = 'GeneratedDemands.txt',  
  append = TRUE)  
  error<-sample(-10:10,m+1,replace = TRUE)  
  error2<-sample(-10:10,m+1,replace = TRUE)
```

```

orders1<-error+2*0
orders2<-error2+2*0
Retailer1_order<-as.vector(rbind(orders1,0))
Retailer2_order<-as.vector(rbind(0,orders2))
I<-c(0,0)
#Retailer1_order<-rep(c(2*0,0),m+1)
#Retailer2_order<-rep(c(0,2*0),m+1)
for (i in 1:m){
Retailer1_order[2*i-1] = Retailer1_order[2*i-1] +
Dlmean[2*i-1]+Dlmean[2*i]
}
Retailer1_order[2*m+1] = Retailer1_order[2*m+1]
+Dlmean[2*m-1]
for (i in 1:m){
Retailer2_order[2*i] = Retailer2_order[2*i] +
D2mean[2*i]+D2mean[2*i+1]
}
Two_retailers_order <-
rbind(Retailer1_order,Retailer2_order)
Total_share_amout<-0
Total_lost_sale<-0
for (i in 1:(2*m+1)){
I<-I+Two_retailers_order[,i]-Two_retailers_demand[,i]
if(I[[1]]*I[[2]]<0){
share_amout<-min(abs(I))
I<-(I<0)*(I+share_amout)+(I>0)*(I-share_amout)
Total_share_amout<-Total_share_amout+share_amout
}
lost_sale<--1*(I<0)*I
Total_lost_sale<-Total_lost_sale+sum(lost_sale)
I<-(I>0)*I*1
}
I_S<-rep(0,n)
Retailer1_order_S<-as.vector(rbind(orders1,0))
Retailer2_order_S<-as.vector(rbind(orders2,0))

```

```

#Retailer1_order_S<-rep(c(2*O, 0), m+1)
#Retailer2_order_S<-rep(c(2*O, 0), m+1)
for (i in 1:m){
  Retailer1_order_S[2*i-1] = Retailer1_order_S[2*i-1]
+ D1mean[2*i-1]+D1mean[2*i]
}
Retailer1_order_S[2*m+1] = Retailer1_order_S[2*m+1]
+D1mean[2*m-1]
for (i in 1:m){
  Retailer2_order_S[2*i-1] = Retailer2_order_S[2*i-1]
+ D2mean[2*i-1]+D2mean[2*i]
}
Retailer2_order_S[2*m+1] = Retailer2_order_S[2*m+1]
+D1mean[2*m-1]

Two_retailers_order_S <-
rbind(Retailer1_order_S, Retailer2_order_S)
Total_share_amout_S<-0
Total_lost_sale_S<-0
for (i in 1:(2*m+1)) {
  I_S<-I_S+Two_retailers_order_S[,i]
-Two_retailers_demand[,i]
  if(I_S[[1]]*I_S[[2]]<0){
    share_amout_S<-min(abs(I_S))
    I_S<-(I_S<0)*(I_S+share_amout_S)+(I_S>0)*(I_S
-share_amout_S)
    Total_share_amout_S<-Total_share_amout_S+share_amout_S
  }
  lost_sale_S<--1*(I_S<0)*I_S
  Total_lost_sale_S<-Total_lost_sale_S+sum(lost_sale_S)
  I_S<-(I_S>0)*I_S*1
}
return(c(Total_share_amout, Total_lost_sale,
Total_share_amout_S, Total_lost_sale_S))
}

```

C.5 Example 6

Step 1: Randomly generate 10 locations for 10 individual retailers. List the x - y coordinations and the distances.

```
Location<-function(n){
  library(ggplot2)
  library(data.table)
  #generate locations:
  loc<-data.table(x = sample(1:50,n), y =sample(1:50,n))
  #plot map:
  #plot(loc$x,loc$y)
  # label with names:
  #retailernames<-as.character(1:n)
  # label names
  #text(loc$x,loc$y, labels = retailernames, pos = 4)
  #dev.off()

  #compute distances:
  dis<-array(,c(n,n))
  i<-1
  for (x in loc$x){
    x2<-(loc$x-x)^2
    dis[i,]<-x2
    i<-i+1
  }
  i<-1
  for (y in loc$y){
    y2<-(loc$y-y)^2
    dis[i,]<-dis[i,]+y2
    i<-i+1
  }
  # compute distances
  LateraltransshipmentCost<-sqrt(dis)
  # record
  #fwrite(loc,file = "Loc.csv")
}
```

```

#fwrite(LateraltransshipmentCost,file = "Lateral.csv")
# return locations and distances:
return(list(loc, LateraltransshipmentCost))
}

```

```

#opts_chunk$set(fig.width=10)
opts_chunk$set(fig.width=9, fig.height=6)
options(width = 50)
read_chunk("LOC1.R")
source('~/Thesis/Rcode/SharingSimulation/LOC1.R')
Location(10)

## Loading required package: snow
## x being coerced from class: matrix to data.table

## [[1]]
##      x  y
## 1: 23 22
## 2: 46 17
## 3: 26  5
## 4: 35 26
## 5:  2 44
## 6: 31 39
## 7: 44 33
## 8: 29  4
## 9:  3 10
## 10: 48 45
##
## [[2]]
##      [,1]      [,2]      [,3]      [,4]      [,5]
## [1,]  0.00000 23.53720 17.262677 12.64911 30.41381
## [2,] 23.53720  0.00000 23.323808 14.21267 51.62364
## [3,] 17.26268 23.32381  0.000000 22.84732 45.79301
## [4,] 12.64911 14.21267 22.847319  0.00000 37.58989
## [5,] 30.41381 51.62364 45.793013 37.58989  0.00000

```



```
## [6,] 18.78829 26.62705 34.365681 13.60147 29.42788
## [7,] 23.70654 16.12452 33.286634 11.40175 43.41659
## [8,] 18.97367 21.40093 3.162278 22.80351 48.25971
## [9,] 23.32381 43.56604 23.537205 35.77709 34.01470
## [10,] 33.97058 28.07134 45.650849 23.02173 46.01087
##           [,6]      [,7]      [,8]      [,9]     [,10]
## [1,] 18.78829 23.70654 18.973666 23.32381 33.97058
## [2,] 26.62705 16.12452 21.400935 43.56604 28.07134
## [3,] 34.36568 33.28663 3.162278 23.53720 45.65085
## [4,] 13.60147 11.40175 22.803509 35.77709 23.02173
## [5,] 29.42788 43.41659 48.259714 34.01470 46.01087
## [6,] 0.00000 14.31782 35.057096 40.31129 18.02776
## [7,] 14.31782 0.00000 32.649655 47.01064 12.64911
## [8,] 35.05710 32.64966 0.000000 26.68333 45.18849
## [9,] 40.31129 47.01064 26.683328 0.00000 57.00877
## [10,] 18.02776 12.64911 45.188494 57.00877 0.00000
```

Step 2: Generate demand samples and compute inventory levels

```
Demands_Next<-function(m,n) {
  library(data.table)
  Demands <- data.table()
  for (l in 1:m+1) {
    Demands <- Demands[,as.character(l)]:=sample(60:140
  ,n,replace= TRUE)]}
  #fwrite(Demands,file = "Demand.csv")
  return(Demands)
}
```

```
read_chunk("Dem.R")
#library(MonteCarlo)
source
('~/Documents/paper2/Rcode/SharingSimulation/Dem.R')
D<-Demands_Next(1,10)
#print(D)
```

```

Inventories_Curr_input<-100-D
print(Inventories_Curr_input)

##          2
## 1:  12
## 2:  32
## 3: -37
## 4:  23
## 5:  39
## 6:  29
## 7:  34
## 8: -17
## 9:  17
## 10:  37

```

Step 3: Sharing process

```

Share<-function(n,lt,Inventories_Curr_input){
  library(data.table)
  library(ggplot2)
  library(gridExtra)
  # loading locations:
  locations<-as.data.table(lt[1])
  # loading residuals:
  Inventories_Curr<-as.data.table(Inventories_Curr_input)
  # loading distances costs:
  fees<-lt[[2]]
  # setting initial profits of sharing:
  add_profit<-rep(0, n)
  # setting the profits of sellers and buyers:
  sell<-5
  buy<-5

  # generate map:
  Retailers_Map<-ggplot(locations, aes(x= x, y= y,
label = as.character(1:n)))+geom_point()

```

```

#plot(locations$x, locations$y)
Inventories<-as.character(Inventories_Curr$'2')

Retailers_Map<-Retailers_Map+geom_text(aes(label=
as.character(1:n)),hjust=-1,colour="blue")

#print list
prt<-list()
lpoint<-list()
linepix<-list()
linepiy<-list()
linepjx<-list()
linep jy<-list()
pt<-1

#print map with original residuals:
lpoint[[pt]]<-Inventories
pt<-pt+1

# sharing: find the shortest distance to share
while ((sum(Inventories_Curr>0)*sum(Inventories_Curr<0))>0){
fee<-10000
for (i in 1:n){
for (j in 1:n){
I_res<-(Inventories_Curr>0)[i]
D_res<-(Inventories_Curr<0)[j]

if (I_res*D_res){
#toj<-as.character(j)
if (fees[i,j]<fee){
fee<-fees[i,j]
Min_i<-i
Min_j<-j
}
}
}
}
}

```

```

}

#fee

# residuals:
Residual_Inv<-Inventories_Curr*(Inventories_Curr>0)
Residual_Dem<--Inventories_Curr*(Inventories_Curr<0)

# sharing amount:
share_amout<-min(Residual_Inv[Min_i],Residual_Dem[Min_j])

# update residuals:
Inventories_Curr[Min_i]<-Inventories_Curr[Min_i]-share_amout
Inventories_Curr[Min_j]<-Inventories_Curr[Min_j]+share_amout

# additional profits:
add_profit[Min_i]<-add_profit[Min_i]+share_amout*sell
add_profit[Min_j]<-add_profit[Min_j]+share_amout*buy


#dev.off()

Retailers_Map<-ggplot(locations, aes(x= x, y= y,
label = as.character(1:n)))+geom_point()
Retailers_Map<-Retailers_Map+geom_text(aes(label=
as.character(1:n)),hjust=-1,colour="blue")
Inventories<-as.character(Inventories_Curr$'2')
lpoint[[pt]]<-Inventories
linepix[[pt]]<-locations$x[Min_i]
linepiy[[pt]]<-locations$y[Min_i]
linepjx[[pt]]<-locations$x[Min_j]
linepjy[[pt]]<-locations$y[Min_j]

pt<-pt+1
}

# update the final residual inventories:
Residual_Inv<-Inventories_Curr*(Inventories_Curr>0)
#do.call(grid.arrange, prt)
print(lpoint)
l<-length(lpoint)

```

```

Retailers_Map<-ggplot(locations, aes(x= x, y= y,
label = as.character(1:n)))+geom_point()
Retailers_Map<-Retailers_Map+geom_text(aes(label=
as.character(1:n)),hjust=-1,colour="blue")
plots1 = list(Retailers_Map+
geom_text(aes(label=lpoint[[1]],vjust=-1))
+theme(panel.background = element_rect(fill='white',
colour = 'black'))
+geom_segment(aes(x = 0, y = 0,
xend = 0, yend =0,color="share")))
plots2 = lapply(2:l, function(.x) Retailers_Map+
geom_text(aes(label=lpoint[[.x]],vjust=-1))
+geom_segment(aes(x = linepix[[.x]], y = linepiy[[.x]],
xend = linepjax[[.x]], yend =linepjy[[.x]],color="share"))
+theme(panel.background = element_rect(fill='white',
colour = 'black'))))
plots <- c(plots1,plots2)
do.call(grid.arrange, plots)
return(list(Inventories_Curr,add_profit,Residual_Inv))
}

```

```

## [[1]]
## [1] "12" "32" "-37" "23" "39" "29" "34" "-17" "17"
## [10] "37"
##
## [[2]]
## [1] "0" "32" "-25" "23" "39" "29" "34" "-17" "17"
## [10] "37"
##
## [[3]]
## [1] "0" "15" "-25" "23" "39" "29" "34" "0" "17"
## [10] "37"
##
## [[4]]
## [1] "0" "15" "-2" "0" "39" "29" "34" "0" "17" "37"

```

```
##  
## [[5]]  
## [1] "0" "13" "0" "0" "39" "29" "34" "0" "17" "37"
```

```
## [[1]]  
##      2  
## 1:  0  
## 2: 13  
## 3:  0  
## 4:  0  
## 5: 39  
## 6: 29  
## 7: 34  
## 8:  0  
## 9: 17  
## 10: 37  
##  
## [[2]]  
## [1] 60 95 185 115 0 0 0 85 0 0  
##  
## [[3]]  
##      2  
## 1:  0  
## 2: 13  
## 3:  0  
## 4:  0  
## 5: 39  
## 6: 29  
## 7: 34  
## 8:  0  
## 9: 17  
## 10: 37
```

C.6 Multi-Retailer Balanced System Simulation

3.7.3

```
BShareMOrder<-function(Ostart,Oend){  
  source('~/Thesis/Rcode/SharingSimulation/BShareM.R')  
  library("data.table")  
  DD<-as.matrix(fread('DD.txt'))  
  for (i in Ostart:Oend) {  
    a<-replicate(10000,BShareM(DD,10,10,i))  
    if (i == Ostart){  
      result<-a  
    }else{  
      result<-rbind(result,a)  
    }  
  }  
  return(result)  
}
```

```
BShareM<-function(DD,m,n,O){  
  source('~/Thesis/Rcode/SharingSimulation/Dem2.R')  
  library("data.table")  
  Multi_retailers_demand<-Demands2(3*m,n)  
  Dmean<-sample(-20:20,3*m,replace = TRUE)  
  for (k in 2:10){  
    Dmean<-rbind(Dmean,sample(-20:20,3*m,replace = TRUE))  
  }  
  Multi_retailers_demand<-Multi_retailers_demand+Dmean  
  Demandss<-data.table(Multi_retailers_demand)  
  fwrite(Demandss,file = 'Multidemand17.txt',append = TRUE)  
  #test  
  errors<-sample(-100:100,n+2,replace = TRUE)  
  for (k in 2:10){  
    errors<-rbind(errors,sample(-100:100,n+2,replace = TRUE))  
  }  
}
```

```

#orders<-error+2*0
#Retailer1_order<-as.vector(rbind(orders,0))
#Retailer2_order<-as.vector(rbind(0,orders))
I<-c(0,0,2*0,3*0,4*0,5*0,6*0,7*0,8*0,9*0)
Retailer1_order<-c(n*0,rep(0,n-1),n*0,rep(0,n-1))
for (k in 1:10){
Retailer1_order<-c(Retailer1_order,n*0,rep(0,n-1))
}
Multi_retailers_order<-Retailer1_order
Retailer2_order<-Retailer1_order
for (i in 2:n)
{
Retailer2_order<-c(0,Retailer2_order[1:(k+2)*n-1])
Multi_retailers_order <-
rbind(Multi_retailers_order,Retailer2_order)
}
for (k in 1:10){
for (l in 1:2){
Multi_retailers_order[k,(l-1)*10+k]<-
Multi_retailers_order[k,(l-1)*10+k]+errors[k,l]
for (kk in 1:10) {
Multi_retailers_order[k,(l-1)*10+k]<-
Multi_retailers_order[k,(l-1)*10+k]
+Dmean[k,(l-1)*10+k+kk-1]
}
}
}
#Total_share_amout<-0
Multi_share_amout<-rep(0,n)
#Total_lost_sale<-0
Multi_lost_sale<-rep(0,n)
for (i in 1:(2*m+5)){
I<-I+Multi_retailers_order[,i]-Multi_retailers_demand[,i]
for (j in 1:45) {
x<-DD[,j][1]

```



```

y<-DD[,j][2]
if (I[[x]]*I[[y]]<0){
share_amout<-min(abs(I[[x]]),abs(I[[y]]))
I[[x]]<-(I[[x]]<0)*(I[[x]]+share_amout)
+(I[[x]]>0)*(I[[x]]-share_amout)
I[[y]]<-(I[[y]]<0)*(I[[y]]+share_amout)
+(I[[y]]>0)*(I[[y]]-share_amout)
#I<-(I<0)*(I+share_amout)+(I>0)*(I-share_amout)
Multi_share_amout[x]<-Multi_share_amout[x]+share_amout
Multi_share_amout[y]<-Multi_share_amout[y]+share_amout
#Total_share_amout<-Total_share_amout+share_amout
}
}
lost_sale<--1*(I<0)*I
Multi_lost_sale<-Multi_lost_sale+lost_sale
#Total_lost_sale<-Total_lost_sale+sum(lost_sale)
I<-(I>0)*I*1
}

Retailer1_order_S<-c(n*0,rep(0,m-1),n*0,rep(0,m-1))
for (k in 1:10){
Retailer1_order_S<-c(Retailer1_order_S,n*0,rep(0,m-1))
}
Multi_retailers_order_S<-Retailer1_order_S
for (i in 2:n)
{
Multi_retailers_order_S <-
rbind(Multi_retailers_order_S,Retailer1_order_S)
}
for (k in 1:10){
for (l in 1:2){
Multi_retailers_order_S[k,(l-1)*10+1]<-
Multi_retailers_order_S[k,(l-1)*10+1]+errors[k,l]

```

```

for (kk in 1:10){
Multi_retailers_order_S[k, (l-1)*10+1]<-
Multi_retailers_order_S[k, (l-1)*10+1]
+Dmean[k, (l-1)*10+kk]
}
}
}

I_S<-rep(0,n)
Multi_share_amout_S<-rep(0,n)
Multi_lost_sale_S<-rep(0,n)

for (i in 1:(2*m+5)){
I_S<-I_S+Multi_retailers_order_S[,i]
-Multi_retailers_demand[,i]
for (j in 1:45) {
x<-DD[,j][1]
y<-DD[,j][2]
if (I_S[[x]]*I_S[[y]]<0){
share_amout_S<-min(abs(I_S[[x]]),abs(I_S[[y]]))
I_S[[x]]<-(I_S[[x]]<0)*(I_S[[x]]+share_amout_S)
+(I_S[[x]]>0)*(I_S[[x]]-share_amout_S)
I_S[[y]]<-(I_S[[y]]<0)*(I_S[[y]]+share_amout_S)
+(I_S[[y]]>0)*(I_S[[y]]-share_amout_S)
#I<-(I<0)*(I+share_amout)+(I>0)*(I-share_amout)
Multi_share_amout_S[x]<-Multi_share_amout_S[x]
+share_amout_S
Multi_share_amout_S[y]<-Multi_share_amout_S[y]
+share_amout_S
#Total_share_amout<-Total_share_amout+share_amout
}
}
lost_sale_S<--1*(I_S<0)*I_S
Multi_lost_sale_S<-Multi_lost_sale_S+lost_sale_S
#Total_lost_sale<-Total_lost_sale+sum(lost_sale)

```

```
I_S<- (I_S>0) *I_S*1
}
#return(c(Multi_lost_sale,Multi_lost_sale_S))
return(c(Multi_share_amout,Multi_lost_sale,
Multi_share_amout_S,Multi_lost_sale_S))
}
```

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