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EISENSTEIN SERIES AND CONVOLUTION SUMS

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ABSTRACT. We compute Fourier series expansions of weight 2 and weight 4 Eisenstein series at various cusps. Then we use results of these computations to give formulas for the convolution sums $\sum_{a+pb=n} \sigma(a)\sigma(b)$, $\sum_{p_1a+p_2b=n} \sigma(a)\sigma(b)$ and $\sum_{a+p_1p_2b=n} \sigma(a)\sigma(b)$ where p, p_1, p_2 are primes.

Keywords: sum of divisors function, convolution sums, Eisenstein series, Dedekind eta function, eta quotients, modular forms, cusp forms, Fourier series.

Mathematics Subject Classification: 11A25, 11E20, 11F11, 11F20, 11F30, 11Y35.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{H}$ denote the sets of positive integers, non-negative integers, integers, rational numbers, complex numbers and the upper half plane, respectively. Throughout the paper we let $z \in \mathbb{H}$ and $q = e^{2\pi iz}$. Let $N \in \mathbb{N}$. Let $\Gamma_0(N)$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{N} \right\}.$$

We note that the matrices

$$(1.1) \quad S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

are the generators of $\Gamma_0(1)$ ($= SL_2(\mathbb{Z})$). An element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1)$ acts on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by

$$M(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty, \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$$

Let $k \in \mathbb{N}$. We write $M_k(\Gamma_0(N))$ to denote the space of modular forms of weight k for $\Gamma_0(N)$, and $E_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N))$, respectively. It is known (see for example [23, p. 83] and [22]) that

$$(1.2) \quad M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)).$$

Let $A_c = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$, then the Fourier series expansion of $f(z) \in M_k(\Gamma_0(N))$ at the cusp $\frac{1}{c} \in \mathbb{Q} \cup \{\infty\}$ is given by the Fourier series expansion of $f(A_c^{-1}z)$ at the cusp ∞ , see [16, pg. 35]. If the Fourier series expansion of $f(z)$ at the cusp $\frac{1}{c}$ is given by the infinite sum

$$(cz + 1)^k \sum_{n \geq 0} a_n e^{2\pi i n z c},$$

then we use the notation $[n]_c f(z) = a_n$. It is to use $[n]$, instead of $[n]_0$, at the cusp $\infty (= 1/0)$.

For $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we define the sum of divisors function by

$$\sigma_k(n) = \sum_{d|n} d^k$$

where d runs through positive divisors of n . If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$. When $k = 1$, we write $\sigma(n)$ instead of $\sigma_1(n)$. We define the weight 2 and the weight 4 Eisenstein series by

$$(1.3) \quad E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma(n) q^n,$$

$$(1.4) \quad E_4(z) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

respectively. Let $d \in \mathbb{N}$, we denote $E_2(dz)$ by $E_{(2,d)}(z)$ and $E_4(dz)$ by $E_{(4,d)}(z)$. For convenience let us define $L_d(z) = E_2(z) - dE_{(2,d)}(z)$. It is known that

$$(1.5) \quad L_d(z) \in E_2(\Gamma_0(d)),$$

$$(1.6) \quad E_4(dz) \in E_4(\Gamma_0(d)),$$

see [23, Theorem 5.8].

Let $r, s \in \mathbb{N}$. We define

$$W(r, s; n) = \sum_{\substack{a, b \in \mathbb{N}, \\ ar + bs = n}} \sigma(a) \sigma(b)$$

to be the convolution of sum of divisors function. The formula

$$W(1, 1; n) = \frac{5}{12} \sigma_3(n) + \frac{1 - 6n}{12} \sigma(n)$$

appears in the works of Besge, Glaisher and Ramanujan, see [8, 12, 20], respectively. Since then $W(r, s; n)$ has been calculated for various (r, s) . In the table below we grouped the known results according to nature of r and s , together with the references.

Table 1: Previously known formulas for convolution sums
 $W(r, s; n)$

Nature of r and s	(r, s)	References
$(r, s) = (1, 1)$	$(1, 1)$	[8, 12, 20]
$(r, s) = (1, p)$	$(1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (1, 13), (1, 23)$	[9, 10, 14, 17]
$(r, s) = (p_1, p_2)$	$(2, 3), (2, 5), (3, 5)$	[7, 11, 19]
$(r, s) = (1, p_1 p_2)$	$(1, 6), (1, 10), (1, 14), (1, 15)$	[7, 19, 21]
$(r, s) = (1, p^i)$	$(1, 4), (1, 8), (1, 9), (1, 16), (1, 25), (1, 27), (1, 32)$	[5, 6, 14, 24, 25, 26]
$(r, s) = (1, p_1^i p_2^j)$	$(1, 12), (1, 18), (1, 20), (1, 24), (1, 36)$	[2, 4, 11]
$(r, s) = (p_1^i, p_2^j)$	$(3, 4), (2, 9), (4, 5), (3, 8), (4, 9)$	[3, 4, 11]

In this paper we give formulas for $W(1, p; n)$, $W(p_1, p_2; n)$ and $W(1, p_1 p_2; n)$. In Theorem 2.2 we give the precise expression for the Eisenstein part of the formulas for $W(1, p; n)$, $W(p_1, p_2; n)$ and $W(1, p_1 p_2; n)$ for all primes p, p_1, p_2 such that $p_1 \neq p_2$. These formulas generalize the previously known formulas in rows 2–4 referenced in Table 1. Our method proves the existence of the cusp forms satisfying the formulas but fails to provide a general expression for the cusp part. Later on in the paper we will give the cusp part of the formulas for $W(r, s; n)$ with $(r, s) = (1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (2, 3), (2, 5), (2, 7), (3, 5), (1, 6), (1, 10), (1, 14), (1, 15)$ in terms of linear combinations of eta quotients.

The structure of the paper is as follows. In the next section we give the statements of main results. In the third section we derive Fourier series expansions of weight 2 and weight 4 Eisenstein series at various cusps. Then, in Section 4, we use the first terms of Fourier series expansions of Eisenstein series at certain cusps to prove the main results. In Section 5 we give a description for the cusp forms $C_{(r,s)}(z)$ with $(r, s) = (1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (2, 3), (2, 5), (2, 7), (3, 5), (1, 6), (1, 10), (1, 14), (1, 15)$ in terms of eta quotients. Finally in Section 6, for the interested reader, we describe how to extend the results of this paper to give formulas for $W(r, s; n)$ with $\text{lcm}(r, s)$ a square-free, two times a square-free number or four times a square-free number.

2. MAIN RESULTS

In this section we state the main results.

Theorem 2.1. *Let p be a prime. Then there exists a cusp form $C_{(1,p)}(z) \in S_4(\Gamma_0(p))$ such that*

$$(2.1) \quad (L_p(z))^2 = \frac{(p-1)^2}{p^2+1} E_4(z) + \frac{p^2(p-1)^2}{p^2+1} E_4(pz) + C_{(1,p)}(z).$$

Let p_1, p_2 be primes such that $p_1 \neq p_2$. Then there exists cusp forms $C_{(p_1, p_2)}(z)$, $C_{(1, p_1 p_2)}(z) \in S_4(\Gamma_0(p_1 p_2))$ such that

(2.2)

$$L_{p_1}(z)L_{p_2}(z) = \frac{(p_1^2 - p_1 + 1)(p_2^2 - p_2 + 1)}{(p_1^2 + 1)(p_2^2 + 1)}E_4(z) - \frac{p_1^3(p_2^2 - p_2 + 1)}{(p_1^2 + 1)(p_2^2 + 1)}E_4(p_1 z) \\ - \frac{p_2^3(p_1^2 - p_1 + 1)}{(p_1^2 + 1)(p_2^2 + 1)}E_4(p_2 z) + \frac{p_1^3 p_2^3}{(p_1^2 + 1)(p_2^2 + 1)}E_4(p_1 p_2 z) + C_{(p_1, p_2)}(z).$$

(2.3)

$$(L_{p_1 p_2}(z))^2 = \left(1 - \frac{2p_1 p_2}{(p_1^2 + 1)(p_2^2 + 1)}\right) E_4(z) - \frac{2p_1^3 p_2}{(p_1^2 + 1)(p_2^2 + 1)}E_4(p_1 z) \\ - \frac{2p_1 p_2^3}{(p_1^2 + 1)(p_2^2 + 1)}E_4(p_2 z) + \left(p_1^2 p_2^2 - \frac{2p_1^3 p_2^3}{(p_1^2 + 1)(p_2^2 + 1)}\right) E_4(p_1 p_2 z) \\ + C_{(1, p_1 p_2)}(z).$$

We compare the coefficients of q^n on both sides of equations (2.1)–(2.3) to get the following theorem.

Theorem 2.2. *Let p, p_1, p_2 be primes such that $p_1 \neq p_2$, then for all $n \in \mathbb{N}$ we have*

$$W(1, p; n) = \frac{5}{12(p^2 + 1)}\sigma_3(n) + \frac{5p^2}{12(p^2 + 1)}\sigma_3(n/p) + \frac{p - 6n}{24p}\sigma(n) \\ + \frac{1 - 6n}{24}\sigma(n/p) - \frac{1}{1152p}[n]C_{(1, p)}(z), \\ W(p_1, p_2; n) = \frac{5}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n) + \frac{5p_1^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_1) \\ + \frac{5p_2^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_2) + \frac{5p_1^2 p_2^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_1 p_2) \\ + \frac{p_2 - 6n}{24p_2}\sigma(n/p_1) + \frac{p_1 - 6n}{24p_1}\sigma(n/p_2) - \frac{1}{1152p_1 p_2}[n]C_{(p_1, p_2)}(z) \\ W(1, p_1 p_2; n) = \frac{5}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n) + \frac{5p_1^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_1) \\ + \frac{5p_2^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_2) + \frac{5p_1^2 p_2^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_1 p_2) \\ + \frac{p_1 p_2 - 6n}{24p_1 p_2}\sigma(n) + \frac{1 - 6n}{24}\sigma(n/p_1 p_2) - \frac{1}{1152p_1 p_2}[n]C_{(1, p_1 p_2)}(z),$$

where $C_{(1, p)}(z)$, $C_{(1, p_1 p_2)}(z)$ and $C_{(p_1, p_2)}(z)$ are the cusp forms from Theorem 2.1.

Note that Chan and Cooper in [9] gave the equation for $W(1, p; n)$, valid for $p = 3, 7, 11, 23$. The closed form they gave for Eisenstein part of the formula is

the same as Eisenstein part of the formula for $W(1, p; n)$ given in the previous theorem.

3. FOURIER SERIES EXPANSIONS OF WEIGHT 2 AND WEIGHT 4 EISENSTEIN SERIES AT VARIOUS CUSPS

In this section we find the Fourier series expansion of weight 2 and weight 4 Eisenstein series at various cusps. The results of this section will be used to prove Theorem 2.1. The transformation formula for $E_2(z)$ under the matrices T and S are given by

$$(3.1) \quad E_2(Tz) = E_2(z + 1) = E_2(z),$$

$$(3.2) \quad E_2(Sz) = E_2(-1/z) = z^2 \left(E_2(z) - \frac{1}{2\pi i z} \right),$$

see [15, Prop. 2.9].

Theorem 3.1. *Let $1 < t \in \mathbb{N}$. The Fourier series expansion of $L_t(z)$ at cusp $1 \in \mathbb{Q}$ is given by*

$$(3.3) \quad L_t(A_1^{-1}z) = (z + 1)^2 \left(E_2(z) - \frac{1}{t} E_2 \left(\frac{z + 1}{t} \right) \right).$$

Let p_1, p_2 be prime. The Fourier series expansions of $L_{p_1 p_2}(z)$ at cusps $1/p_1, 1/p_2 \in \mathbb{Q}$ are given by

$$(3.4) \quad L_{p_1 p_2}(A_{p_1}^{-1}z) = (p_1 z + 1)^2 \left(E_2(z) - \frac{p_1}{p_2} E_2 \left(\frac{p_1 z + 1}{p_2} \right) \right) \text{ and}$$

$$(3.5) \quad L_{p_1 p_2}(A_{p_2}^{-1}z) = (p_2 z + 1)^2 \left(E_2(z) - \frac{p_2}{p_1} E_2 \left(\frac{p_2 z + 1}{p_1} \right) \right),$$

respectively.

Proof. Let $t \in \mathbb{N}$ and the matrices S and T are as in (1.1). We have

$$\begin{aligned} E_{(2,t)}(A_1^{-1}z) &= E_2 \left(S^2 T^t S \left(\frac{z + 1}{t} \right) \right) \\ &= E_2 \left(T^t S \left(\frac{z + 1}{t} \right) \right) \\ &= E_2 \left(S \left(\frac{z + 1}{t} \right) \right) \\ (3.6) \quad &= \left(\frac{z + 1}{t} \right)^2 \left(E_2 \left(\frac{z + 1}{t} \right) - \frac{t}{2\pi i (z + 1)} \right). \end{aligned}$$

where in the first, second and third steps we use $E_2(S^2(z)) = E_2(z)$, $E_2(T^t(z)) = E_2(z)$ and (3.2), respectively. Let $1 < t \in \mathbb{N}$. Then we use (3.6) to get

$$L_t(A_1^{-1}z) = E_2(A_1^{-1}z) - t E_{(2,t)}(A_1^{-1}z)$$

$$\begin{aligned}
&= (z+1)^2 \left(E_2(z+1) - \frac{1}{2\pi i(z+1)} \right) \\
&\quad - t \left(\frac{z+1}{t} \right)^2 \left(E_2\left(\frac{z+1}{t}\right) - \frac{t}{2\pi i(z+1)} \right) \\
&= (z+1)^2 \left(E_2(z+1) - \frac{1}{t} E_2\left(\frac{z+1}{t}\right) \right),
\end{aligned}$$

which proves (3.3). Similarly, by using $E_2(T^t(z)) = E_2(z)$ and (3.2) we find

$$\begin{aligned}
(3.7) \quad E_2(A_{p_1}^{-1}z) &= E_2(ST^{-p_1}Sz) \\
&= \left(\frac{-p_1z-1}{z} \right)^2 \left(E_2(Sz) - \frac{z}{2\pi i(-p_1z-1)} \right) \\
&= (p_1z+1)^2 \left(E_2(z) - \frac{p_1}{2\pi i(p_1z+1)} \right),
\end{aligned}$$

and by $E_2(S^2(z)) = E_2(z)$, $E_2(T^t(z)) = E_2(z)$ and (3.2) we find

$$\begin{aligned}
(3.8) \quad E_{(2,p_1p_2)}(A_{p_1}^{-1}z) &= E_2\left(S^2T^{-p_2}S\left(\frac{p_1z+1}{p_2}\right)\right) \\
&= E_2\left(S\left(\frac{p_1z+1}{p_2}\right)\right) \\
&= \left(\frac{p_1z+1}{p_2}\right)^2 \left(E_2\left(\frac{p_1z+1}{p_2}\right) - \frac{p_2}{2\pi i(p_1z+1)} \right).
\end{aligned}$$

Combining (3.7) and (3.8) we have

$$\begin{aligned}
L_{p_1p_2}(A_{p_1}^{-1}z) &= E_2(A_{p_1}^{-1}z) - p_1p_2E_{(2,p_1p_2)}(A_{p_1}^{-1}z) \\
&= (p_1z+1)^2 \left(E_2(z) - \frac{p_1}{2\pi i(p_1z+1)} \right) \\
&\quad - p_1p_2 \left(\left(\frac{p_1z+1}{p_2}\right)^2 \left(E_2\left(\frac{p_1z+1}{p_2}\right) - \frac{p_2}{2\pi i(p_1z+1)} \right) \right) \\
&= (p_1z+1)^2 \left(E_2(z) - \frac{p_1}{p_2} E_2\left(\frac{p_1z+1}{p_2}\right) \right),
\end{aligned}$$

which proves (3.4). The proof of (3.5) is similar. □

The following theorem is a special case of Theorem 4.1 from [1].

Theorem 3.2. *Let $t, c \in \mathbb{N}$ be such that $c \mid t$. Then Fourier series expansion of $E_{(4,t)}(z)$ at the cusp $1/c$ is given by*

$$E_{(4,t)}(A_c^{-1}z) = \left(\frac{c}{t}\right)^4 (cz+1)^4 E_4\left(\frac{c^2z+c}{t}\right).$$

Proof. Let $t, c \in \mathbb{N}$ be such that $c \mid t$. Let $L = \begin{pmatrix} -t/c & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma_0(1)$ and $\gamma = \begin{pmatrix} -t & 0 \\ -c & -1 \end{pmatrix}$ be matrices. Then since $E_4(z) \in M_4(\Gamma_0(1))$, we have

$$E_{(4,t)}(A_c^{-1}z) = E_4(\gamma z) = E_4\left(L\left(\frac{c^2z+c}{t}\right)\right) = \left(\frac{c^2z+c}{t}\right)^4 E_4\left(\frac{c^2z+c}{t}\right),$$

which proves the assertion. $\square$

The following cusps in Table 2 are equivalent in the given modular subgroups.

Table 2: Equivalence of certain cusps

Modular Subgroups	Equivalent cusps
$\Gamma_0(1)$	$1 \sim 1/p_1 \sim 1/p_2 \sim \infty$
$\Gamma_0(p_1)$	$1 \sim 1/p_2$ and $1/p_1 \sim \infty$
$\Gamma_0(p_2)$	$1 \sim 1/p_1$ and $1/p_2 \sim \infty$

We construct Tables 3–5 below by using (1.3), Theorems 3.1, 3.2, the definition of cusp forms and equivalence of the cusps given in Table 2. Tables 3, 4 and 5 will be used to prove (2.1), (2.2) and (2.3), respectively.

Table 3: First terms of modular forms at certain cusps

cusp $1/c$	$[0]_c(L_p(z))^2$	$[0]_c E_4(z)$	$[0]_c E_4(pz)$	$[0]_c C_p(z)$
$c = 0$	$(1-p)^2$	1	1	0
$c = 1$	$\left(\frac{1-p}{p}\right)^2$	1	$\frac{1}{p^4}$	0

Table 4: First terms of modular forms at certain cusps

cusp $1/c$	$[0]_c L_{p_1}(z) L_{p_2}(z)$	$[0]_c E_4(z)$	$[0]_c E_4(p_1 z)$	$[0]_c E_4(p_2 z)$	$[0]_c E_4(p_1 p_2 z)$	$[0]_c C_{(p_1, p_2)}(z)$
$c = 0$	$(1-p_1)(1-p_2)$	1	1	1	1	0
$c = 1$	$\frac{(1-p_1)(1-p_2)}{p_1 p_2}$	1	$\frac{1}{p_1^4}$	$\frac{1}{p_2^4}$	$\frac{1}{p_1^4 p_2^4}$	0
$c = p_1$	$\frac{(1-p_1)(p_2-1)}{p_2}$	1	1	$\frac{1}{p_2^4}$	$\frac{1}{p_2^4}$	0
$c = p_2$	$\frac{(p_1-1)(1-p_2)}{p_1}$	1	$\frac{1}{p_1^4}$	1	$\frac{1}{p_1^4}$	0

Table 5: First terms of modular forms at certain cusps

cusp $1/c$	$[0]_c(L_{p_1 p_2}(z))^2$	$[0]_c E_4(z)$	$[0]_c E_4(p_1 z)$	$[0]_c E_4(p_2 z)$	$[0]_c E_4(p_1 p_2 z)$	$[0]_c C_{(1, p_1 p_2)}(z)$
$c = 0$	$(1 - p_1 p_2)^2$	1	1	1	1	0
$c = 1$	$\left(\frac{1 - p_1 p_2}{p_1 p_2}\right)^2$	1	$\frac{1}{p_1^4}$	$\frac{1}{p_2^4}$	$\frac{1}{p_1^4 p_2^4}$	0
$c = p_1$	$\left(1 - \frac{p_1}{p_2}\right)^2$	1	1	$\frac{1}{p_2^4}$	$\frac{1}{p_2^4}$	0
$c = p_2$	$\left(1 - \frac{p_2}{p_1}\right)^2$	1	$\frac{1}{p_1^4}$	1	$\frac{1}{p_1^4}$	0

4. PROOF OF THEOREM 2.1

Let p, p_1, p_2 be primes such that $p_1 \neq p_2$. By (1.5), we have $(L_p(z))^2 \in M_4(\Gamma_0(p))$, and $L_{p_1}(z)L_{p_2}(z), (L_{p_1 p_2}(z))^2 \in M_4(\Gamma_0(p_1 p_2))$. By [23, Theorem 5.9], the sets of Eisenstein series

$$\begin{aligned} &\{E_4(z), E_4(pz)\}, \\ &\{E_4(z), E_4(p_1 z), E_4(p_2 z), E_4(p_1 p_2 z)\} \end{aligned}$$

constitute bases for $E_4(\Gamma_0(p))$ and $E_4(\Gamma_0(p_1 p_2))$, respectively. Then by (1.2), we obtain

$$(4.1) (L_p(z))^2 = a_1 E_4(z) + a_2 E_4(pz) + C_{(1,p)}(z),$$

$$(4.2) L_{p_1}(z)L_{p_2}(z) = b_1 E_4(z) + b_2 E_4(p_1 z) + b_3 E_4(p_2 z) + b_4 E_4(p_1 p_2 z) + C_{(p_1, p_2)}(z),$$

$$(4.3) (L_{p_1 p_2}(z))^2 = b_1 E_4(z) + b_2 E_4(p_1 z) + b_3 E_4(p_2 z) + b_4 E_4(p_1 p_2 z) + C_{(1, p_1 p_2)}(z),$$

for some $a_i, b_j \in \mathbb{C}$, $C_{(1,p)}(z) \in S_4(\Gamma_0(p))$, $C_{(p_1, p_2)}(z)$, and $C_{(1, p_1 p_2)}(z) \in S_4(\Gamma_0(p_1 p_2))$.

We use the values in Table 3 to compare first terms of Fourier series expansions of the functions on both sides of equation (4.1) to obtain the following linear equations.

$$(4.4) \quad (1 - p)^2 = a_1 + a_2,$$

$$(4.5) \quad \left(\frac{p-1}{p}\right)^2 = a_1 + a_2 \frac{1}{p^4}.$$

Solving equations (4.4) and (4.5) for a_1 and a_2 we get the desired result in (2.1).

We similarly prove (2.2) and (2.3) using the values in Table 4 in (4.2) and Table 5 in (4.3), respectively.

5. THE CUSP FORMS $C_{(r,s)}(z)$ FOR $(r, s) = (1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (2, 3), (2, 5), (2, 7), (3, 5), (1, 6), (1, 10), (1, 14), (1, 15)$

In this section we express $C_{(r,s)}(z)$ ($(r, s) = (1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (2, 3), (2, 5), (2, 7), (3, 5), (1, 6), (1, 10), (1, 14), (1, 15)$) as linear combinations of

eta quotients. The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H}$ by the product formula

$$(5.1) \quad \eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in z}).$$

Let $N \in \mathbb{N}$, an eta quotient (of level N) is defined to be a finite product of the form

$$(5.2) \quad f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z),$$

where δ runs through positive divisors of N and $r_\delta \in \mathbb{Z}$, not all zeroes. For convenience we use the notation

$$\prod_{\delta|N} \eta^{r_\delta}(\delta z) = \eta[1^{r_1}, \dots, \delta^{r_\delta}, \dots, N^{r_N}](z),$$

for an eta quotient.

Theorem 5.1. *We express $C_{(r,s)}(z)$ ($(r,s) = (,)$ in terms of eta quotients as follows.*

$$\begin{aligned} C_{(1,2)}(z) &= 0, \\ C_{(1,3)}(z) &= 0, \\ C_{(1,5)}(z) &= \frac{576}{13} \eta[1^4, 5^4](z), \\ C_{(1,7)}(z) &= \frac{576}{5} \eta[1^5, 2^{-1}, 7^5, 14^{-1}](z) + \frac{2304}{5} \eta[1^2, 2^2, 7^2, 14^2](z), \\ C_{(1,11)}(z) &= \frac{17280}{61} \eta[1^6, 2^{-2}, 11^6, 22^{-2}](z) + \frac{118656}{61} \eta[1^4, 11^4](z) \\ &\quad + \frac{276480}{61} \eta[1^2, 2^2, 11^2, 22^2](z) + \frac{276480}{61} \eta[2^4, 22^4](z), \\ C_{(2,3)}(z) &= -\frac{144}{5} \eta[1^2, 2^2, 3^2, 6^2](z), \\ C_{(2,5)}(z) &= \frac{48}{13} \eta[1^{-2}, 2^8, 5^2](z) - \frac{6000}{13} \eta[1^2, 5^{-2}, 10^8](z), \\ C_{(2,7)}(z) &= \frac{288}{5} \eta[1^5, 2^{-1}, 7^5, 14^{-1}](z) - \frac{336}{25} \eta[1^{-2}, 2^6, 7^6, 14^{-2}](z), \\ &\quad + \frac{8064}{25} \eta[1^2, 2^2, 7^2, 14^2](z) - \frac{5136}{25} \eta[1^6, 2^{-2}, 7^{-2}, 14^6](z), \\ C_{(3,5)}(z) &= \frac{108}{13} \eta[1^{11}, 2^{-5}, 3^{-5}, 5^{-5}, 6^3, 10^3, 15^{11}, 30^{-5}](z) \\ &\quad - \frac{1584}{13} \eta[[1^2, 2^{-3}, 3^{-4}, 5^{-3}, 6^9, 10^4, 15^7, 30^{-4}](z) \end{aligned}$$

$$\begin{aligned}
& -\frac{2376}{13}\eta[1^4, 2^{-3}, 3^{-2}, 5^2, 6^5, 10^1, 15^0, 30^1](z) \\
& -\frac{5832}{13}\eta[1^1, 2^{-2}, 3^{-3}, 5^2, 6^8, 10^{-1}, 15^2, 30^1](z) \\
& +\frac{11448}{13}\eta[1^2, 2^{-2}, 3^0, 5^{-2}, 6^2, 10^6, 15^0, 30^2](z) \\
& +\frac{33264}{13}\eta[1^1, 2^2, 3^4, 5^3, 6^{-3}, 10^{-2}, 15^{-4}, 30^7](z) \\
& +\frac{10080}{13}\eta[1^1, 2^0, 3^{-1}, 5^{-3}, 6^0, 10^2, 15^7, 30^2](z) \\
& +\frac{83520}{13}\eta[1^1, 2^{-1}, 3^{-2}, 5^1, 6^4, 10^{-1}, 15^0, 30^6](z) \\
& +\frac{145584}{13}\eta[1^0, 2^1, 3^0, 5^2, 6^{-1}, 10^{-3}, 15^2, 30^7](z) \\
& -\frac{117720}{13}\eta[1^{-2}, 2^3, 3^5, 5^3, 6^{-4}, 10^{-4}, 15^{-4}, 30^{11}](z) \\
& -1944\eta[1^1, 2^0, 3^1, 5^3, 6^0, 10^{-4}, 15^{-5}, 30^{12}](z) \\
& +\frac{158688}{13}\eta[1^0, 2^1, 3^1, 5^2, 6^{-2}, 10^{-3}, 15^{-3}, 30^{12}](z) \\
& +\frac{216000}{13}\eta[1^{-1}, 2^3, 3^1, 5^3, 6^{-1}, 10^{-5}, 15^{-7}, 30^{15}](z) \\
& +\frac{316224}{13}\eta[1^0, 2^1, 3^2, 5^2, 6^{-3}, 10^{-3}, 15^{-8}, 30^{17}](z), \\
C_{(1,6)}(z) &= \frac{288}{5}\eta[1^2, 2^2, 3^2, 6^2](z), \\
C_{(1,10)}(z) &= \frac{2976}{13}\eta[1^{-1}, 2^5, 5^5, 10^{-1}](z) - \frac{480}{13}\eta[1^5, 2^{-1}, 5^{-1}, 10^5](z), \\
C_{(1,14)}(z) &= \frac{10272}{25}\eta[1^{-2}, 2^6, 7^6, 14^{-2}](z) - \frac{4608}{25}\eta[1^2, 2^2, 7^2, 14^2](z) \\
& +\frac{672}{25}\eta[1^6, 2^{-2}, 7^{-2}, 14^6](z), \\
C_{(1,15)}(z) &= \frac{5976}{13}\eta[1^{11}, 2^{-5}, 3^{-5}, 5^{-5}, 6^3, 10^3, 15^{11}, 30^{-5}](z) \\
& +\frac{74592}{13}\eta[[1^2, 2^{-3}, 3^{-4}, 5^{-3}, 6^9, 10^4, 15^7, 30^{-4}](z) \\
& -\frac{137808}{13}\eta[1^4, 2^{-3}, 3^{-2}, 5^2, 6^5, 10^1, 15^0, 30^1](z) \\
& -\frac{85968}{13}\eta[1^1, 2^{-2}, 3^{-3}, 5^2, 6^8, 10^{-1}, 15^2, 30^1](z) \\
& +\frac{153072}{13}\eta[1^2, 2^{-2}, 3^0, 5^{-2}, 6^2, 10^6, 15^0, 30^2](z)
\end{aligned}$$

$$\begin{aligned}
& -\frac{617184}{13}\eta[1^1, 2^2, 3^4, 5^3, 6^{-3}, 10^{-2}, 15^{-4}, 30^7](z) \\
& +\frac{1513728}{13}\eta[1^1, 2^0, 3^{-1}, 5^{-3}, 6^0, 10^2, 15^7, 30^2](z) \\
& -\frac{2807424}{13}\eta[1^1, 2^{-1}, 3^{-2}, 5^1, 6^4, 10^{-1}, 15^0, 30^6](z) \\
& -\frac{943200}{13}\eta[1^0, 2^1, 3^0, 5^2, 6^{-1}, 10^{-3}, 15^2, 30^7](z) \\
& +135504\eta[1^{-2}, 2^3, 3^5, 5^3, 6^{-4}, 10^{-4}, 15^{-4}, 30^{11}](z) \\
& +\frac{3255696}{13}\eta[1^1, 2^0, 3^1, 5^3, 6^0, 10^{-4}, 15^{-5}, 30^{12}](z) \\
& -\frac{3409344}{13}\eta[1^0, 2^1, 3^1, 5^2, 6^{-2}, 10^{-3}, 15^{-3}, 30^{12}](z) \\
& +\frac{586368}{13}\eta[1^{-1}, 2^3, 3^1, 5^3, 6^{-1}, 10^{-5}, 15^{-7}, 30^{15}](z) \\
& -\frac{10461312}{13}\eta[1^0, 2^1, 3^2, 5^2, 6^{-3}, 10^{-3}, 15^{-8}, 30^{17}](z).
\end{aligned}$$

Proof. Proofs of all these equalities are similar. We basically find combinations of eta quotients whose first couple terms in Fourier series expansion at ∞ agrees with the first couple terms in Fourier series expansion of $C_{(r,s)}(z)$ at ∞ . Then by Sturm Theorem these two modular forms are equal, see ([15, Theorem 3.13]) for Sturm Theorem (or Sturm Bound). To illustrate the proof we show

$$\begin{aligned}
(5.3) \quad C_{(1,11)}(z) &= \frac{17280}{61}\eta[1^6, 2^{-2}, 11^6, 22^{-2}](z) + \frac{118656}{61}\eta[1^4, 11^4](z) \\
&+ \frac{276480}{61}\eta[1^2, 2^2, 11^2, 22^2](z) + \frac{276480}{61}\eta[2^4, 22^4](z).
\end{aligned}$$

By Theorem 2.1 we have

$$C_{(1,11)}(z) = (L_{11}(z))^2 - \frac{50}{61}E_4(z) - \frac{6050}{61}E_4(11z) \in S_4(\Gamma_0(11)) \subset S_4(\Gamma_0(22)).$$

We use MAPLE to expand $(L_{11}(z))^2$, $E_4(z)$ and $E_4(11z)$. Then we find

$$\begin{aligned}
(5.4) \quad C_{(1,11)}(z) &= \frac{17280}{61}q + \frac{14976}{61}q^2 - \frac{8064}{61}q^3 - \frac{73728}{61}q^4 - \frac{1152}{61}q^5 - \frac{213120}{61}q^6 \\
&+ \frac{182016}{61}q^7 - \frac{80640}{61}q^8 + \frac{361728}{61}q^9 + \frac{411264}{61}q^{10} - \frac{190080}{61}q^{11} \\
&- \frac{377856}{61}q^{12} + O(q^{13}).
\end{aligned}$$

On the other hand, let

$$f(z) = \frac{17280}{61}\eta[1^6, 2^{-2}, 11^6, 22^{-2}](z) + \frac{118656}{61}\eta[1^4, 11^4](z) + \frac{276480}{61}\eta[1^2, 2^2, 11^2, 22^2](z)$$

$$+ \frac{276480}{61} \eta[2^4, 22^4](z).$$

We have $f(z) \in S_4(\Gamma_0(22))$, see [15, Theorem 5.7, p. 99], [16, Corollary 2.3, p. 37], [13, p. 174] and [18]. Again using MAPLE we also compute

$$(5.5) \quad f(z) = \frac{17280}{61}q + \frac{14976}{61}q^2 - \frac{8064}{61}q^3 - \frac{73728}{61}q^4 - \frac{1152}{61}q^5 - \frac{213120}{61}q^6 \\ + \frac{182016}{61}q^7 - \frac{80640}{61}q^8 + \frac{361728}{61}q^9 + \frac{411264}{61}q^{10} - \frac{190080}{61}q^{11} \\ - \frac{377856}{61}q^{12} + O(q^{13}).$$

That is, by (5.4) and (5.5), we have

$$f(z) - C_{(1,11)}(z) = O(q^{13}).$$

Thus by Sturm Theorem (see [15, Theorem 3.13]), we have

$$f(z) - C_{(1,11)}(z) = 0,$$

which proves (5.3). □

6. FURTHER EXTENSIONS

Let $N_1 \in \mathbb{N}$ be a square-free number and $N = N_1, 2N_1$, or $4N_1$. Let $r, s \in \mathbb{N}$ be such that $\text{lcm}(r, s) \mid N$. We can use similar arguments to give formulas for $W(r, s; n)$. We consider $L_r(z)L_s(z) \in M_4(\Gamma_0(N))$. Then by [23, Theorem 5.9] and (1.2), we have that there exists $c_d \in \mathbb{C}$ and $C_{(r,s)}(z) \in S_4(\Gamma_0(N))$, such that

$$(6.1) \quad L_r(z)L_s(z) = \sum_{d \mid N} c_d E_4(dz) + C_{(r,s)}(z).$$

Now in order to compute c_d we need to find the first terms of the modular forms on both sides of equation (6.1) at the cusps of $\Gamma_0(N)$. Note that, because of the choice of N , the set

$$\left\{ \frac{1}{c} : c \mid N \right\}$$

is a set of cusps of $\Gamma_0(N)$. Then comparing first coefficients of Fourier series expansions of modular forms in (6.1), we will find a set of linear equations. We can determine c_d by solving these equations. Also note that in many cases $C_{(r,s)}(z)$ can be given in terms of eta quotients. Finally a formula for $W(r, s; n)$ can be given by comparing coefficients of q^n on both sides of the equation (6.1).

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