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Necessary Conditions for the Existence of Group-Invariant Butson Hadamard Matrices and a New Family of Perfect Arrays

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Abstract

Let G be a finite abelian group and let $\exp(G)$ denote the least common multiple of the orders of all elements of G . A $\text{BH}(G, h)$ matrix is a G -invariant $|G| \times |G|$ matrix H whose entries are complex h th roots of unity such that $HH^* = |G|I_{|G|}$. By $\nu_p(x)$ we denote the p -adic valuation of the integer x . Using bilinear forms over abelian groups, we [10] constructed new classes of $\text{BH}(G, h)$ matrices under the following conditions.

- (i) $\nu_p(h) \geq \lceil \nu_p(\exp(G))/2 \rceil$ for any prime divisor p of $|G|$, and
- (ii) $\nu_2(h) \geq 2$ if $\nu_2(|G|)$ is odd and G has a direct factor $\mathbb{Z}_2$.

The purpose of this paper is to further study the conditions on G and h so that a $\text{BH}(G, h)$ matrix exists. We will focus on $\text{BH}(\mathbb{Z}_n, h)$ and $\text{BH}(G, 2p^b)$ matrices, where p is an odd prime. Combining our work with previously known results, there are 2687 open cases left for the existence of $\text{BH}(\mathbb{Z}_n, h)$ matrices in which $1 \leq n, h \leq 100$. Finally, we

show that $\text{BH}(\mathbb{Z}_n, h)$ matrices can be used to construct a new family of perfect polyphase arrays.

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1 Introduction

Let n and h be positive integers. An $n \times n$ matrix H whose entries are complex h th roots of unity is called a **Butson Hadamard matrix** if $HH^* = nI$, where H^* is the complex conjugate transpose of H and I is the identity matrix of order n . We also say that H is a **BH**($\mathbf{n}, \mathbf{h}$) matrix.

Let $(G, +)$ be a finite abelian group of order n . An $n \times n$ matrix $A = (a_{g,k})_{g,k \in G}$ is **G-invariant** if $a_{g+l,k+l} = a_{g,k}$ for all $g, k, l \in G$. A G -invariant $\text{BH}(n, h)$ matrix is also called a **BH**($\mathbf{G}, \mathbf{h}$) matrix. Note that in the case $G = \mathbb{Z}_n$, a cyclic group of order n , a $\text{BH}(\mathbb{Z}_n, h)$ matrix is a **circulant** matrix, i.e., a matrix each of whose rows (except the first) is obtained from the previous row by shifting one position to the right and moving the last entry to the front.

For any multiple h' of h , a $\text{BH}(G, h)$ matrix is also a $\text{BH}(G, h')$ matrix, as each h th root of unity is automatically a h' th root of unity. Therefore, it is important to find those positive integers h such that a $\text{BH}(G, h)$ matrix exists, but $\text{BH}(G, k)$ matrices do not exist for any divisor k of h . Group-invariant Butson Hadamard matrices link to many other combinatorial objects like generalized Hadamard matrices, relative difference sets, generalized Bent functions, cyclic n -roots, see [24], and perfect polyphase arrays.

A sequence $\{a_0, \dots, a_{n-1}\}$ is called a **perfect h -phase sequence** of length n if each a_i is a complex h th root of unity and

$$\sum_{i=0}^{n-1} a_i \overline{a_{i+j}} = 0 \text{ whenever } j \neq 0,$$

where the indices are taken modulo n . Such a sequence is equivalent to a $\text{BH}(\mathbb{Z}_n, h)$ matrix whose first row is $(a_0, \dots, a_{n-1})$. More generally, a multi-

dimensional array $A = (a_{i_1, \dots, i_k})$ of size $n_1 \times \dots \times n_k$ is called a **perfect h -phase array** if its entries are complex h th roots of unity and

$$\sum_{0 \leq i_j \leq n_j - 1 \ \forall \ j} a_{i_1, \dots, i_k} \bar{a}_{i_1 + s_1, \dots, i_k + s_k} = 0$$

whenever $(s_1, \dots, s_k) \neq (0, \dots, 0)$, where the indices are taken modulo n_j for $1 \leq j \leq k$. The values

$$R_{s_1, \dots, s_k} = \sum_{i_1, \dots, i_k} a_{i_1, \dots, i_k} \bar{a}_{i_1 + s_1, \dots, i_k + s_k}$$

are called **autocorrelations** of the array A . In Lemma 4.3, we show that such an array is equivalent to a $\text{BH}(\mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}, h)$ matrix. Perfect sequences and perfect arrays have a wide range of applications in communication and radar systems, see [4], [13], [15] for example. In the last section of this paper, we use (circulant) $\text{BH}(\mathbb{Z}_n, h)$ matrices to construct a new family of perfect polyphase arrays. Note that by [10, Corollary 2.5], there exist $\text{BH}(\mathbb{Z}_n, h)$ matrices whenever

- (i) $\nu_p(h) \geq \lceil \nu_p(n)/2 \rceil$ for every prime divisor p of n , and
- (ii) $\nu_2(h) \geq 2$ if $n \equiv 2 \pmod{4}$,

where $\nu_p(x)$ denotes the p -adic valuation of the integer x .

In this paper, we focus on studying necessary conditions for the existence of the following two types of Butson Hadamard matrices:

- $\text{BH}(\mathbb{Z}_n, h)$ matrices and
- $\text{BH}(G, 2p^b)$ matrices, where p is an odd prime.

The main tools used in this paper are group-ring equations combined with techniques from the field-descent method [25, 26], and upper bounds on the norm of cyclotomic integers. We sketch the approach as follows.

Let $(a_g)_{g \in G}$ be the first row of a $\text{BH}(G, h)$ matrix H . Let $D = \sum_{g \in G} a_g g$ be an element of the group ring $\mathbb{Z}[\zeta_h][G]$. By Result 2.3 of Section 2 below, the equation $HH^* = |G|I$ is equivalent to

$$DD^{(-1)} = |G|. \tag{1}$$

Put $n = |G|$ and $m = \text{lcm}(\exp(G), h)$. Let χ be a character of G and put $X = \chi(D) \in \mathbb{Z}[\zeta_m]$. The equation (1) implies

$$|X|^2 = n, \quad X \in \mathbb{Z}[\zeta_m]. \quad (2)$$

By the field-descent method, there exists an integer j such that $X\zeta_m^j$ belongs to a proper subfield $\mathbb{Q}(\zeta_k)$ of $\mathbb{Q}(\zeta_m)$. Note that $X\zeta_m^j$ is a sum of roots of unity in $\mathbb{Q}(\zeta_m)$. As $X\zeta_m^j$ belongs to the subfield $\mathbb{Q}(\zeta_k)$, usually a lot of the roots of unity in the sum cancel out. The norm of the remaining terms can be bounded using our result on the norm of cyclotomic integers. On the other hand, by (2), this norm is equal to $\sqrt{n}$. We thus obtain a restriction on n and h .

To end this section, we give a summary of known results on the existence of Butson Hadamard matrices.

Result 1.1. *The following Butson Hadamard matrices do not exist.*

1. [18, Leung-Schmidt] $\text{BH}(\mathbb{Z}_{2p^2}, 2p)$ matrices with p being an odd prime.
2. [21, Ma-Ng] $\text{BH}(\mathbb{Z}_{3pq}, 3)$ matrices with $p, q > 3$ being distinct primes.
3. [16, Hiranandani-Schlenker] $\text{BH}(\mathbb{Z}_{p+q}, pq)$ matrices with $p, q > 3$ being distinct primes.

Result 1.2. [16, Sylvester conditions]

- (i) If a $\text{BH}(n, 2)$ matrix exists, then $n = 2$ or $4 \mid n$. Furthermore, if a $\text{BH}(\mathbb{Z}_n, 2)$ matrix exists, then $4 \mid n$ and $n/4$ is a square.
- (ii) Let $p \geq 3$ be a prime. If a $\text{BH}(p+2, h)$ matrix exists, then h does not have the form $2p^b$ for some positive integer b .
- (iii) Let $q \geq 3$ be a prime. If a $\text{BH}(2q, h)$ matrix exists, then h does not have the form $2^a p^b$ for $a, b \in \mathbb{Z}^+$ and any prime $p > q$.

Result 1.3. [17, Lam-Leung] If a $\text{BH}(G, h)$ matrix exists and $p_1, \dots, p_r$ are the distinct prime divisors of h , then there exist nonnegative integers a_i such that $|G| = \sum_{i=1}^r a_i p_i$.

Result 1.4. [6, Brock] *Let n be an integer and let m be the square-free part of n . Assume that m is odd. If a $\text{BH}(n, h)$ matrix exists, then m has no prime factor p which satisfies*

- (a) p does not divide h , and
- (b) $p^j \equiv -1 \pmod{h}$ for some integer j .

Result 1.5. [10, Duc-Schmidt] *Let G be an abelian group and h be a positive integer. Then a $\text{BH}(G, h)$ matrix exists if*

- (i) $v_p(h) \geq \lceil v_p(\exp(G)/2) \rceil$ for every prime divisor p of $|G|$, and
- (ii) $v_2(h) \geq 2$ if $v_2(|G|)$ is odd and G has a direct factor $\mathbb{Z}_2$.

Moreover, if $G = \mathbb{Z}_{p^a}$ is a cyclic group of prime-power order, then (i) and (ii) are also necessary conditions for the existence of $\text{BH}(\mathbb{Z}_{p^a}, h)$ matrices.

We remark that the existence problem for $\text{BH}(\mathbb{Z}_n, h)$ matrices is in general a difficult problem. For example, the existence of $\text{BH}(\mathbb{Z}_n, 2)$ matrices is equivalent to the existence of circulant Hadamard matrices. Ryser [23, p. 134] conjectured that circulant Hadamard matrices exist only for orders $n = 1$ or $n = 4$. This conjecture remains open for more than 50 years.

2 Preliminaries

2.1 Notations and Group Rings

In this section, we fix some notation, state definitions and known results which will be used later. We start with some notations.

1. For a positive integer h , we use ζ_h to denote a primitive h th root of unity.
2. For a prime p and an integer n , let $\nu_p(n)$ denote the p -adic valuation of n , that is, the largest nonnegative integer x such that p^x divides n .

3. A positive integer is **square-free** if it is not divisible by any square of a prime. We call an integer l the **square-free part** of n if l is the product of all prime divisors p of n in which $\nu_p(n)$ is odd.
4. For coprime integers m and n , we denote the smallest positive integer j such that $m^j \equiv 1 \pmod{n}$ by $\text{ord}_n(m)$.
5. We call a complex number X a **cyclotomic integer** if it is a sum of complex roots of unity.

As it turns out, group-ring equations are pivotal in our study.

Let G be a finite abelian group of order n . Let R be a ring with identity 1 and let $R[G]$ denote the group ring of G over R . Any element $X \in R[G]$ is uniquely expressed as $X = \sum_{g \in G} a_g g$, $a_g \in R$. We call the set $\{g \in G : a_g \neq 0\}$ the **support** of X and denote it by $\text{supp}(X)$. Two elements $X = \sum_{g \in G} a_g g$ and $Y = \sum_{g \in G} b_g g$ are equal if and only if $a_g = b_g$ for all $g \in G$. We denote 1_G as the identity element of G . In the case where there is no confusion, we simply write c in the place of $c1_G$. A subgroup U of G is identified with $\sum_{u \in U} u$ in $R[G]$.

The group of complex characters of G is denoted by $\hat{G}$. It is well known that $\hat{G}$ is isomorphic to G . The trivial character of G , denoted by χ_0 , is defined by $\chi_0(g) = 1$ for all $g \in G$. For $D = \sum_{g \in G} a_g g \in R[G]$ and $\chi \in \hat{G}$, write $\chi(D) = \sum_{g \in G} a_g \chi(g)$. Let U be a subgroup of G , denote

$$U^\perp = \{\chi \in \hat{G} : \chi(g) = 1 \ \forall g \in U\}.$$

Note that $U^\perp$ is isomorphic to $\widehat{G/U}$. The following result is from [14, Lemma 2.8].

Result 2.1. *Let G be a finite abelian group, let U be a subgroup of G , and let $D = \sum_{g \in G} a_g g \in \mathbb{C}[G]$. Then, for any character $\chi \in \hat{G}$, we have*

$$\sum_{\tau \in U^\perp} \chi\tau(D) = |U^\perp| \chi \left(\sum_{g \in U} a_g g \right)$$

The next result is called Fourier inversion formula. A proof can be found in [2, Chapter VI, Lemma 3.5], for example.

Result 2.2. *Let G be a finite abelian group and let $\hat{G}$ denote the group of characters of G . Let $D = \sum_{g \in G} a_g g \in \mathbb{C}[G]$. Then*

$$a_g = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(Dg^{-1}) \quad \forall g \in G.$$

Consequently, if $D, E \in \mathbb{C}[G]$ and $\chi(D) = \chi(E)$ for all $\chi \in \hat{G}$, then $D = E$.

In our study, we focus on the ring $R = \mathbb{Z}[\zeta_h]$. Let $D = \sum_{g \in G} a_g g, a_g \in R$ for all $g \in G$, be an element of $R[G]$. Let t be an integer coprime to h and let $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_h)/\mathbb{Q})$ be defined by $\zeta_h^\sigma = \zeta_h^t$. Let $D^{(t)}$ denote

$$D^{(t)} = \sum_{g \in G} a_g^\sigma g^t.$$

As mentioned in the introduction, a $\text{BH}(G, h)$ matrix is equivalent to a solution of the group ring equation $DD^{(-1)} = |G|$ for some $D \in \mathbb{Z}[\zeta_h][G]$. We now state this result formally, see [10, Lemma 3.3] for a proof.

Result 2.3. *Let G be a finite abelian group, let h be a positive integer, and let $a_g, g \in G$, be integers. Consider the element $D = \sum_{g \in G} \zeta_h^{a_g} g$ of $\mathbb{Z}[\zeta_h][G]$ and the G -invariant matrix $H = (H_{g,k}), g, k \in G$, given by $H_{g,k} = \zeta_h^{a_{k^{-g}}}$. Then H is a $\text{BH}(G, h)$ matrix if and only if*

$$DD^{(-1)} = |G|.$$

The next result is a generalization of Ma's lemma, see [20] or [25, Lemma 1.5.1]. The proof of this result is similar to the proof of the original result. We provide it here for the convenience of the reader.

Result 2.4. *Let p be a prime and let G be a finite abelian group whose Sylow p -subgroup S is cyclic. Let P be the subgroup of S of order p . Let t, h be positive integers such that h is not divisible by p . If $D \in \mathbb{Z}[\zeta_h][G]$ satisfies*

$$\chi(D) \equiv 0 \pmod{p^t} \tag{3}$$

for all characters $\chi \in \hat{G}$ of order divisible by $|S|$, then there exist $X, Y \in \mathbb{Z}[\zeta_h][G]$ such that

$$D = p^t X + PY. \tag{4}$$

Proof. Assume $|S| = p^s$ for some $s \in \mathbb{Z}^+$. We first prove the result for $D \in \mathbb{Z}[\zeta_h][S]$. Let g be a generator of the cyclic group S . Define a ring homomorphism

$$\begin{aligned} \rho : \mathbb{Z}[\zeta_h][S] &\rightarrow \mathbb{Z}[\zeta_{hp^s}], \\ \sum_{i=0}^{p^s-1} a_i g^i &\mapsto \sum_{i=0}^{p^s-1} a_i \zeta_{p^s}^i \end{aligned}$$

for all $a_0, \dots, a_{p^s-1} \in \mathbb{Z}[\zeta_h]$. We claim that

$$\ker(\rho) = \{PY : Y \in \mathbb{Z}[\zeta_h][S]\}. \quad (5)$$

Let $\sum_{i=0}^{p^s-1} a_i g^i \in \ker(\rho)$, then $\sum_{i=0}^{p^s-1} a_i \zeta_{p^s}^i = 0$. Since p does not divide h , the minimal polynomial of ζ_{p^s} over $\mathbb{Z}[\zeta_h]$ is $\phi(x) = 1 + x^{p^{s-1}} + \dots + x^{p^{s-1}(p-1)}$. We obtain

$$\sum_{i=0}^{p^s-1} a_i x^i = \phi(x)f(x), \quad f(x) \in \mathbb{Z}[\zeta_h][x],$$

so $\sum_{i=0}^{p^s-1} a_i g^i = \phi(g)f(g) = Pf(g)$, which proves (5).

Since $\rho(D) \equiv 0 \pmod{p^t}$, there exists $X \in \mathbb{Z}[\zeta_h][S]$ so that $\rho(D) = p^t \rho(X)$. We obtain $(D - p^t X) \in \ker(\rho)$ and hence

$$D = p^t X + PY \text{ for some } Y \in \mathbb{Z}[\zeta_h][S].$$

The result is proved for $D \in \mathbb{Z}[\zeta_h][S]$. Now, write $G = S \times H$ where $|H| \not\equiv 0 \pmod{p}$ and write

$$D = \sum_{k \in H} D_k k, \quad D_k \in \mathbb{Z}[\zeta_h][S].$$

Let ψ be a character of S of order p^s . For any character γ of H , $\psi\gamma$ is a character of G defined by $\psi\gamma(xy) = \psi(x)\gamma(y)$ for all $x \in S$, $y \in H$. Note that the order of $\psi\gamma$ is divisible by p^s . By (3), we have

$$\psi\gamma(D) = \sum_{k \in H} \psi(D_k)\gamma(k) \equiv 0 \pmod{p^t}. \quad (6)$$

Define an $|H| \times |H|$ matrix by $M = (\gamma(k))_{\gamma \in \hat{H}, k \in H}$ and a column vector u by $u = (\psi(D_k))_{k \in H}^T$. As the congruence (6) holds for any character γ of H , all entries of $N := Mu$ are divisible by p^t . By the orthogonality relations

for characters, we have $M^{-1} = M^*/|H|$ where M^* is the complex conjugate transpose of M . Thus $u = M^{-1}N = M^*N/|H|$. As $\gcd(p, |H|) = 1$ and all entries of N are divisible by p^t , we obtain

$$\psi(D_k) \equiv 0 \pmod{p^t} \text{ for all } k \in H. \quad (7)$$

The equation (7) holds for every character ψ of S . By the argument at the beginning of the proof, any D_k , $k \in H$, can be written as $p^t X_k + p Y_k$ for some $X_k, Y_k \in \mathbb{Z}[\zeta_h][S]$. Therefore, the equation (4) follows, as $D = \sum_{k \in H} D_k k$. $\square$

2.2 Number Theoretic Results

We start this subsection with the following definition.

Definition 2.5. Let p be a prime, let n be a positive integer, and write $n = p^a n'$, where $\gcd(p, n') = 1$. We call p **self-conjugate modulo n** if there exists an integer j such that $p^j \equiv -1 \pmod{n'}$. A composite integer m is self-conjugate modulo n if every prime divisor of m has this property.

Result 2.6. [19, Proposition 2.11] Let $X = \sum_{i=0}^{m-1} a_i \zeta_m^i \in \mathbb{Z}[\zeta_m]$ so that $X\bar{X} = n$. Let u be the largest divisor of n which is self-conjugate modulo m . Write $u = w^2 k$, where $k = \prod_{i=1}^r p_i$ is the square-free part of u . Then

$$m \equiv 0 \pmod{k}.$$

Furthermore, for $i = 1, \dots, r$, write

$$\Theta_i = \begin{cases} 1 - \zeta_4 & \text{if } p_i = 2, \\ \sum_{j=1}^{p_i-1} \left(\frac{j}{p_i}\right) \zeta_{p_i}^j & \text{otherwise.} \end{cases}$$

where $(-)$ is the Legendre symbol. Then

$$X \equiv 0 \pmod{w \prod_{i=1}^r \Theta_i}.$$

Definition 2.7. Let m and n be positive integers and let $m = \prod_{i=1}^t p_i^{c_i}$ be the prime factorization of m . For each prime divisor q of n , define

$$\tilde{m}_q = \begin{cases} \prod_{p_i \neq q} p_i & \text{if } m \text{ is odd or } q = 2, \\ 4 \prod_{p_i \neq 2, q} p_i & \text{otherwise.} \end{cases}$$

Let $D(n)$ denote the set of all prime divisors of n . Let

$$F(m, n) = \prod_{i=1}^t p_i^{b_i}$$

be the minimum multiple of $\prod_{i=1}^t p_i$ such that for every pair (i, q) with $1 \leq i \leq t$ and $q \in D(n)$, at least one of the following conditions is satisfied.

- (a) $q = p_i$ and $(p_i, b_i) \neq (2, 1)$, or
- (b) $b_i = c_i$, or
- (c) $q \neq p_i$ and $q^{\text{ord}_{\tilde{m}_q}(q)} \not\equiv 1 \pmod{p_i^{b_i+1}}$.

Result 2.8. [26, Theorem 3.5] Let m and n be positive integers. If $X \in \mathbb{Z}[\zeta_m]$ satisfies $X\bar{X} = n$, then there exists an integer j such that $X\zeta_m^j \in \mathbb{Z}[\zeta_{F(m,n)}]$.

2.3 Bounds on Norms of Cyclotomic Integers

We start this subsection with a bound on the norm of a cyclotomic integer. This bound appeared in [11, Theorem 3.1] and will serve as the main ingredient for our results later. We provide the proof here for the convenience of the reader.

Result 2.9. Let $\alpha = \sum_{i=0}^{m-1} c_i \zeta_m^i$, $c_i \in \mathbb{Z}$, be an element of $\mathbb{Z}[\zeta_m]$. Then

$$|N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\alpha)| \leq \left(\frac{m}{\varphi(m)} \sum_{i=0}^{m-1} c_i^2 \right)^{\varphi(m)/2}. \quad (8)$$

Proof. Put $f(x) = \sum_{i=0}^{m-1} c_i x^i \in \mathbb{Z}[x]$, then $\alpha = f(\zeta_m)$ and the conjugates of α in $\mathbb{Q}(\zeta_m)$ are $f(\zeta_m^t)$, where t is coprime to m . We have

$$\sum_{t=0}^{m-1} |f(\zeta_m^t)|^2 = \sum_{i,j,t=0}^{m-1} c_i c_j \zeta_m^{(i-j)t} = m \sum_{i=0}^{m-1} c_i^2.$$

By the inequality of arithmetic and geometric means, we obtain

$$\begin{aligned}
|N(f(\zeta_m))| &= \left| \prod_{\gcd(t,m)=1} f(\zeta_m^t) \right| \\
&\leq \left(\frac{\sum_{(t,m)=1} |f(\zeta_m^t)|^2}{\varphi(m)} \right)^{\varphi(m)/2} \\
&\leq \left(\frac{m}{\varphi(m)} \sum_{i=0}^{m-1} c_i^2 \right)^{\varphi(m)/2}.
\end{aligned}$$

□

The next result can be derived from [25, Theorem 2.3.2]. We state the result below and give a simple proof which makes use of Result 2.9.

Result 2.10. *Let m and n be positive integers. Let $X = \sum_{i=0}^{m-1} c_i \zeta_m^i$ be an element of $\mathbb{Z}[\zeta_m]$ such that $X\bar{X} = n$. Let k be a divisor of m which is divisible by $F(m, n)$. Then there exists an integer j such that*

$$X\zeta_m^j \in \mathbb{Z}[\zeta_k]. \quad (9)$$

Moreover, put $d_i = c_{im/k-j}$, $0 \leq i \leq k-1$. Then

$$n \leq \min \left\{ \left(\sum_{i=0}^{k-1} |d_i| \right)^2, \left(\frac{k}{\varphi(k)} \sum_{i=0}^{k-1} d_i^2 \right) \right\}. \quad (10)$$

Proof. Since k is divisible by $F(m, n)$, we obtain $X\zeta_m^j \in \mathbb{Z}[\zeta_k]$, as $X\zeta_m^j \in \mathbb{Z}[\zeta_{F(m,n)}]$ by Result 2.8. Note that k is divisible by every prime divisor of m , as $F(m, n)$ is. Thus $1, \zeta_m, \dots, \zeta_m^{m/k-1}$ are independent over $\mathbb{Q}(\zeta_k)$. This implies $X\zeta_m^j = \sum_{i=0}^{k-1} d_i \zeta_k^i$, where $d_i = c_{im/k-j}$. Clearly, we have

$$n = |X\bar{X}| \leq \left(\sum_{i=0}^{k-1} |d_i| \right)^2.$$

The other part of (10) follows directly from (8). □

3 Necessary Conditions

We will focus only on two types of matrices: $\text{BH}(\mathbb{Z}_n, h)$ matrices and $\text{BH}(G, 2p^b)$ matrices, where p is an odd prime and n, h, b are positive integers.

3.1 Existence of $\text{BH}(\mathbb{Z}_n, h)$ Matrices

Note that by Result 1.5, there exist $\text{BH}(\mathbb{Z}_n, h)$ matrices whenever n divides $(h, n)^2$ and n and h are not both congruent to 2 modulo 4. We conjecture that these conditions are also necessary for the existence of $\text{BH}(\mathbb{Z}_n, h)$ matrices.

Conjecture 3.1. *Let n and h be positive integers. Then there exists a $\text{BH}(\mathbb{Z}_n, h)$ matrix if and only if*

- (i) $\nu_p(h) \geq \lceil \nu_p(n)/2 \rceil$ for every prime divisor p of n , and
- (ii) $\nu_2(h) \geq 2$ if $n \equiv 2 \pmod{4}$.

A special case of Conjecture 3.1 is the circulant Hadamard matrix conjecture which was mentioned in the introduction. Unfortunately, we are far from proving Conjecture 3.1. In support of it, we prove that $n \leq (h, n)^2$ under certain restrictions on n and h .

Theorem 3.2. *Let n and h be positive integers and put $m = \text{lcm}(n, h)$. Suppose that a $\text{BH}(\mathbb{Z}_n, h)$ matrix exists. Furthermore, assume that for any prime divisor p of n , we have*

- (i) p divides h , and
- (ii) $q^{\text{ord}_{\tilde{m}_q}(q)} \not\equiv 1 \pmod{p^{\nu_p(h)+1}}$ for any prime divisor $q \neq p$ of n ,

where $\tilde{m}_q$ is defined in Definition 2.7. Then

$$n \leq (h, n)^2. \quad (11)$$

Proof. Let H be a $\text{BH}(\mathbb{Z}_n, h)$ matrix and let its first row be $(\zeta_h^{a_0}, \dots, \zeta_h^{a_{n-1}})$, $a_i \in \mathbb{Z}$. Let g be a generator of the cyclic group $\mathbb{Z}_n$ and let $D = \sum_{i=0}^{n-1} \zeta_h^{a_i} g^i$ be an element of the group ring $\mathbb{Z}[\zeta_h][\mathbb{Z}_n]$. We obtain, by Result 2.3,

$$DD^{(-1)} = n. \quad (12)$$

Let χ be a character of $\mathbb{Z}_n$ such that $\chi(g) = \zeta_n$. We have $\chi(D) = \sum_{i=0}^{n-1} \zeta_h^{a_i} \zeta_n^i \in \mathbb{Z}[\zeta_m]$. By Result 2.8, there exists an integer j such that $\chi(D)\zeta_m^j \in \mathbb{Z}[\zeta_{F(m,n)}]$. Since $\gcd(m/h, m/n) = 1$, there exist integers α and β such that $\alpha m/h + \beta m/n = j$. Replacing D by $D\zeta_h^\alpha g^\beta$, if necessary, we can assume that

$$X = \chi(D\zeta_h^\alpha g^\beta) = \chi(D)\zeta_n^{\alpha n/h + \beta} = \chi(D)\zeta_m^j \in \mathbb{Z}[\zeta_{F(m,n)}].$$

Thus $X \in \mathbb{Z}[\zeta_h]$, as $h \equiv 0 \pmod{F(m,n)}$ by the definition of $F(m,n)$ and by the conditions (i) and (ii). Note that with h in the place of k , the condition (9) in Result 2.10 is satisfied. To apply the inequality (10), we need to find the exponents in

$$X = \sum_{i=0}^{n-1} \zeta_m^{a_i(m/h) + i(m/n)}$$

which are divisible by m/h . These are the ones containing i such that $n/(h,n)$ divides i . There are (h,n) such exponents, as $0 \leq i \leq n-1$. Recall that $X \in \mathbb{Z}[\zeta_h]$. When expressing X in the form $\sum_{i=0}^{h-1} d_i \zeta_h^i$, $d_i \in \mathbb{Z}$, we have $d_i \geq 0$ for all i and $\sum_{i=0}^{h-1} d_i = (h,n)$. We obtain, by (10),

$$n \leq \left(\sum_{i=0}^{h-1} d_i \right)^2 = (h,n)^2.$$

□

We remark that in the case $n = p^a$ is a prime power, the inequality (11) implies that $\nu_p(h) \geq \lceil \nu_p(n)/2 \rceil$, which is exactly the second statement in Result 1.5. Moreover in this case, the condition (ii) holds automatically and the condition (i) can be obtained using an argument from the field-descent method.

For the next result, we recall the self-conjugacy concept in Definition 2.5. We concentrate on prime divisors of n which are self-conjugate modulo $m = \text{lcm}(n, h)$.

Theorem 3.3. *Let n and h be positive integers and put $m = \text{lcm}(n, h)$. If a $\text{BH}(\mathbb{Z}_n, h)$ matrix exists, then any prime divisor of n which is self-conjugate modulo m divides h .*

Proof. Suppose that there exists a prime divisor p of n which is self-conjugate modulo m and does not divide h . Letting g be a generator of $\mathbb{Z}_n$ and defining $D = \sum_{i=0}^{n-1} \zeta_h^{a_i} g^i$ as in the proof of Theorem 3.2, we have $DD^{(-1)} = n$. For any character χ of $\mathbb{Z}_n$, we have

$$|\chi(D)|^2 = n, \quad \chi(D) \in \mathbb{Z}[\zeta_m]. \quad (13)$$

Applying χ_0 to (13) and putting $Y = \chi_0(D) \in \mathbb{Z}[\zeta_h]$, we obtain $|Y|^2 = n$. Note that p is self-conjugate modulo m and h divides m , so p is also self-conjugate modulo h . By Result 2.6 and by the condition $p \nmid h$, we obtain $\nu_p(n) = 2t$ for some $t \in \mathbb{Z}^+$. We also have $\nu_p(m) = 2t$, as $m = \text{lcm}(n, h)$ and $h \not\equiv 0 \pmod{p}$. Moreover, since $|\chi(D)|^2 = n$ and $\chi(D) \in \mathbb{Z}[\zeta_m]$, Result 2.6 implies $\chi(D) \equiv 0 \pmod{p^t}$ for any character χ of $\mathbb{Z}_n$. By Result 2.4, we obtain

$$D = p^t X + PY, \quad (14)$$

where P is the cyclic subgroup of $\mathbb{Z}_n$ of order p , and $X, Y \in \mathbb{Z}[\zeta_h][\mathbb{Z}_n]$. As $kP = P$ for any $k \in P$, we can assume that no two elements in the support of Y are contained in the same coset of P in $\mathbb{Z}_n$. Comparing the coefficients on a fixed coset of P , the equation (14) implies

$$\zeta_h^{a_i} \equiv \zeta_h^{a_{i+n/p}} \equiv \cdots \equiv \zeta_h^{a_{i+(p-1)n/p}} \pmod{p^t} \text{ for any } 0 \leq i \leq n/p - 1.$$

Hence $\zeta_h^{a_j} \equiv \zeta_h^{a_{j+n/p}} \pmod{p^t}$ for all j . Suppose that $\zeta_h^{a_j} = \zeta_h^{a_{j+n/p}}$ for all j . We have $D = PZ$, where $Z = \sum_{i=0}^{n/p-1} \zeta_h^{a_i} g^i$. Let τ be a character of $\mathbb{Z}_n$ such that $\tau(g) = \zeta_n$. We have

$$\tau(P) = \sum_{i=0}^{p-1} \tau(g^{in/p}) = \sum_{i=0}^{p-1} \zeta_p^i = 0,$$

so $\tau(D) = 0$, contradicting (13). Therefore, there exists j such that $\zeta_h^{a_j} \neq \zeta_h^{a_{j+n/p}}$.

The condition $\zeta_h^{a_j} \equiv \zeta_h^{a_{j+n/p}} \pmod{p^t}$ implies that $t = 1$ and $p = 2$, as $|\zeta_h^{a_j} - \zeta_h^{a_{j+n/p}}| \leq 2$. The congruence $\zeta_h^{a_j} \equiv \zeta_h^{a_{j+n/2}} \pmod{2}$ holds only when $2 \mid h$ and $a_{j+n/2} = a_j + h/2$, contradicting with the assumption that p does not divide h . $\square$

Corollary 3.4. *If n and h are coprime positive integers such that a $\text{BH}(\mathbb{Z}_n, h)$ matrix exists, then no prime divisor of n is self-conjugate modulo nh .*

In preparation for the next result of this section, we start with the following definition of delta function.

Definition 3.5. *Let s and t be any two complex numbers. We define the function δ_{ts} as follows.*

$$\delta_{ts} = \begin{cases} 0 & \text{if } t \neq s, \\ 1 & \text{if } t = s. \end{cases}$$

The following result is taken from [26, Lemma 2.5].

Result 3.6. *Let $m \in \mathbb{Z}^+$ and let k be a divisor of m . Let t and s be the numbers of prime divisors of m and k , respectively. Write $X = \sum_{i=0}^{m-1} a_i \zeta_m^i$, where $a_i \in \mathbb{Z}$ for all i . Then there exists an integral basis $\mathcal{B}_{m,k}$ of $\mathbb{Q}(\zeta_m)$ over $\mathbb{Q}(\zeta_k)$ which contains $\varphi(m/k)$ roots of unity. Furthermore, if $0 \leq a_i \leq C$ for all i , then we can express X as*

$$X = \sum_{x \in \mathcal{B}_{m,k}} x \left(\sum_{j=0}^{k-1} c_{xj} \zeta_k^j \right), \quad |c_{xj}| \leq 2^{t-s-1+\delta_{ts}} C \text{ for all } x, j. \quad (15)$$

Theorem 3.7. *Let n and h be positive integers and put $m = \text{lcm}(n, h)$. Let u be the largest divisor of n which is self-conjugate modulo m . Write $u = w^2 k$, where k is the square-free part of u . Let t and r be the numbers of prime divisors of h and k , respectively. Suppose that a $\text{BH}(\mathbb{Z}_n, h)$ matrix exists. We obtain the following.*

(a) *If $k \not\equiv 0 \pmod{2}$, then*

$$w \leq 2^{t-r-1+\delta_{rt}}(h, u) \sqrt{\frac{k}{\varphi(k)}}. \quad (16)$$

(b) *If $k \equiv 0 \pmod{2}$, then*

$$w \leq 2^{t-r-1}(h, u) \sqrt{\frac{2k}{\varphi(k)}}. \quad (17)$$

Proof. Similar to the proof of Theorem 3.2, we have $DD^{(-1)} = n$, where $D = \sum_{i=0}^{n-1} \zeta_h^{a_i} g^i \in \mathbb{Z}[\zeta_h][\mathbb{Z}_n]$ and g is a generator of $\mathbb{Z}_n$. Hence for any character χ of $\mathbb{Z}_n$, we have

$$|\chi(D)|^2 = n, \quad \chi(D) \in \mathbb{Z}[\zeta_m]. \quad (18)$$

Put $v = n/u$. Note that $(u, v) = 1$, as u is the largest divisor of n which is self-conjugate modulo m . Let τ be a character of $\mathbb{Z}_n$ such that $\tau(g^v) = \zeta_u$. The proof of the theorem is divided into several claims.

Claim 1. Replacing D by Dy , $y \in G$, if necessary, we can assume that

$$\tau(D \cap \mathbb{Z}_u) \neq 0. \quad (19)$$

Let $R = \{1, g, \dots, g^{v-1}\}$ be the complete set of coset representatives of $\mathbb{Z}_u$ in $\mathbb{Z}_n$. If $\tau(D \cap x\mathbb{Z}_u) = 0$ for all $x \in R$, then $\tau(D) = \sum_{x \in R} \tau(D \cap x\mathbb{Z}_u) = 0$, contradicting (18). Thus, there exists $x \in R$ such that $\tau(D \cap x\mathbb{Z}_u) \neq 0$. Note that $\tau(Dx^{-1} \cap \mathbb{Z}_u) = \tau(x^{-1})\tau(D \cap x\mathbb{Z}_u) \neq 0$. So, replacing D by Dx^{-1} if necessary, we can assume that $\tau(D \cap \mathbb{Z}_u) \neq 0$, proving (19).

Claim 2. Write $D \cap \mathbb{Z}_u = \sum_{i=0}^{u-1} \zeta_h^{a_{vi}} g^{vi} = \sum_{i=0}^{u-1} \zeta_h^{b_i} g^{vi}$ and write $k = \prod_{i=1}^r p_i$. Define Θ_i as in Result 2.6. We have

$$X = \sum_{i=0}^{u-1} \zeta_h^{b_i} \zeta_u^i \equiv 0 \pmod{w \prod_{i=1}^r \Theta_i}. \quad (20)$$

By Result 2.6, we have

$$\chi(D) \equiv 0 \pmod{w \prod_{i=1}^r \Theta_i}$$

for any character χ of $\mathbb{Z}_n$. On the other hand, by Result 2.1, we have

$$\sum_{\chi \in \mathbb{Z}_u^\perp} \chi \tau(D) = |\mathbb{Z}_u^\perp| \tau(D \cap \mathbb{Z}_u) = v \left(\sum_{i=0}^{u-1} \zeta_h^{b_i} \zeta_u^i \right), \quad (21)$$

Since $(u, v) = 1$ and each term on the left side of (21) is divisible by $w \prod_{i=1}^r \Theta_i$, we obtain the congruence (20).

Claim 3. Put $l = \text{lcm}(h, u)$, $d = \text{gcd}(h, u)$ and write $X = \sum_{i=0}^{u-1} \zeta_h^{b_i} \zeta_u^i$ in the form $X = \sum_{i=0}^{l-1} c_i \zeta_l^i$, $c_i \in \mathbb{Z}^+$. Then

$$0 \leq c_i \leq d \text{ for any } i = 0, \dots, l-1. \quad (22)$$

For a fixed $0 \leq i \leq l-1$, let a and b be fixed integers such that $(u/d)a + (h/d)b = i$. All the solutions $j \pmod{h}$ and $f \pmod{u}$ to $\zeta_l^i = \zeta_h^j \zeta_u^f$ are

$$j = a + l \frac{h}{d}, \quad f = b - l \frac{u}{d}, \quad 0 \leq l \leq d-1.$$

Thus, there are at most d solutions to $\zeta_l^i = \zeta_h^{b_z} \zeta_u^z$, $0 \leq z \leq u-1$. The claim is proved.

Claim 4. If $k \not\equiv 0 \pmod{2}$, then

$$w \leq 2^{t-r-1+\delta_{tr}}(h, u) \sqrt{\frac{k}{\varphi(k)}}.$$

First, note that $X = \tau(D \cap \mathbb{Z}_u) \neq 0$ by Claim 1. By Theorem 3.3, any prime divisor of u divides h , so the numbers of prime divisors of $l = \text{lcm}(h, u)$ and h are the same, both are equal to t . By Result 3.6 and (22), we can express X as a linear combination of the basis elements in $\mathcal{B}_{l,k} \subset \mathbb{Q}(\zeta_l)$ over $\mathbb{Q}(\zeta_k)$ as follows

$$X = \sum_{x \in \mathcal{B}_{l,k}} x \left(\sum_{j=0}^{k-1} c_{xj} \zeta_k^j \right),$$

where

$$|c_{xj}| \leq 2^{t-r-1+\delta_{tr}} d \quad \forall x \in \mathcal{B}_{l,k}, \quad 0 \leq j \leq k-1. \quad (23)$$

Note that $X \equiv 0 \pmod{w \prod_{i=1}^r \Theta_i}$ by (20). Moreover note that $w \prod_{i=1}^r \Theta_i \in \mathbb{Q}(\zeta_k)$, as $\Theta_i \in \mathbb{Q}(\zeta_{p_i})$ for all i by the definition of Θ_i . Hence $\sum_{j=0}^{k-1} c_{xj} \zeta_k^j \equiv 0 \pmod{w \prod_{i=1}^r \Theta_i}$ for all $x \in \mathcal{B}_{l,k}$. As $X \neq 0$, there exists $x \in \mathcal{B}_{l,k}$ so that

$$Y = \sum_{j=0}^{k-1} c_{xj} \zeta_k^j \neq 0 \text{ and } Y \equiv 0 \pmod{w \prod_{i=1}^r \Theta_i}. \quad (24)$$

Note that each Θ_i has absolute value $\sqrt{p_i}$. Using (23), (24) and the inequality

(8), we obtain

$$\begin{aligned} (w^2 k)^{\varphi(k)/2} = \left| N_{\mathbb{Q}(\zeta_k)/\mathbb{Q}}(w \prod_{i=1}^r \Theta_i) \right| &\leq \left| N_{\mathbb{Q}(\zeta_k)/\mathbb{Q}}(Y) \right| \leq \left(\frac{k}{\varphi(k)} \sum_{j=0}^{k-1} c_{xj}^2 \right)^{\varphi(k)/2} \\ &\leq \left(4^{t-r-1+\delta_{rt}} d^2 \frac{k^2}{\varphi(k)} \right)^{\varphi(k)/2}, \end{aligned}$$

proving Claim 4.

Claim 5. If $k \equiv 0 \pmod{2}$, then

$$w \leq 2^{t-r-1}(h, u) \sqrt{\frac{2k}{\varphi(k)}}.$$

We assume that $p_r = 2$. Similar to the proof of Claim 4, we express X as a linear combination of the basis elements in $\mathcal{B}_{l,k/2} \subset \mathbb{Q}(\zeta_l)$ over $\mathbb{Q}(\zeta_{k/2})$

$$X = \sum_{x \in \mathcal{B}_{l,k/2}} x \left(\sum_{j=0}^{k/2-1} d_{xj} \zeta_{k/2}^j \right), \quad |d_{xj}| \leq 2^{t-r} d \quad \forall x, j.$$

Note that $w \prod_{i=1}^{r-1} \Theta_i \in \mathbb{Q}(\zeta_{k/2})$ and divides X , so it divides $\sum_{j=0}^{k/2-1} d_{xj} \zeta_{k/2}^j$ for any $x \in \mathcal{B}_{l,k/2}$. As $X \neq 0$, there exists $x \in \mathcal{B}_{l,k/2}$ such that

$$Z = \sum_{j=0}^{k/2-1} d_{xj} \zeta_{k/2}^j \neq 0 \text{ and } Z \equiv 0 \pmod{w \prod_{i=1}^{r-1} \Theta_i}.$$

Using (23), (24) and the inequality (8), we obtain

$$\begin{aligned} (w^2 k/2)^{\varphi(k/2)/2} &= \left| N_{\mathbb{Q}(\zeta_{k/2})/\mathbb{Q}}(Z) \right| \leq \left(\frac{k/2}{\varphi(k/2)} \sum_{j=0}^{k/2-1} d_{xj}^2 \right)^{\varphi(k/2)/2} \\ &\leq \left(4^{t-r} d^2 \frac{k^2}{4\varphi(k)} \right)^{\varphi(k/2)/2}, \end{aligned}$$

proving Claim 5. □

Corollary 3.8. *Let n and b be positive integers and let p be a prime. Suppose that a $\text{BH}(\mathbb{Z}_n, p^b)$ matrix exists. Then $n = p^c m$ for some positive integers c and m in which $m \not\equiv 0 \pmod{p}$. Moreover, assume that p is selfconjugate modulo m . Then*

$$b \geq \lfloor c/2 \rfloor. \quad (25)$$

Proof. The claim that $n = p^c m$ follows directly from Result 1.3. We apply the inequality (11) to prove the second claim. In this case, we have $u = p^c$, $w = p^{\lfloor c/2 \rfloor}$, $t = 1$, $k \in \{1, p\}$ and $r \in \{0, 1\}$.

If $k \equiv 0 \pmod{2}$, then $k = p = 2$, $r = 1$ and the inequality (17) implies

$$p^{\lfloor c/2 \rfloor} \leq (p^b, p^c) \leq p^b,$$

proving (25).

If $k \not\equiv 0 \pmod{2}$, then $p \geq 3$ and the inequality (16) implies

$$p^{\lfloor c/2 \rfloor} \leq 2^{-r+\delta_{r1}} (p^b, p^c) \sqrt{\frac{k}{\varphi(k)}} \leq (p^b, p^c) \sqrt{\frac{p}{p-1}} < p^{b+1},$$

proving (25). □

Remark 3.9. We consider the impact of Theorem 3.2, Theorem 3.3 and Theorem 3.7 on the existence of $\text{BH}(\mathbb{Z}_n, h)$ matrices with $2 \leq n, h \leq 100$. Theorem 3.2 confirms the non-existence of 178 cases. Theorem 3.3 confirms the nonexistence of 2946 cases. Lastly, Theorem 3.7 confirms the nonexistence of 723 cases.

3.2 Existence of $\text{BH}(G, 2p^b)$ Matrices

Let G be an abelian group, let p be an odd prime and let b be a positive integer. The main result of this section relies on the following result by Leung and Schmidt, see [18, Theorem 22 and Theorem 23].

Result 3.10. *Let p be an odd prime and let a be a positive integer. Let m be a nonsquare integer and let $q_1, \dots, q_s$ be all distinct prime divisors of m . Put $f = \gcd(\text{ord}_p(q_1), \dots, \text{ord}_p(q_s))$. Suppose that $X \in \mathbb{Z}[\zeta_{p^a}]$ satisfies $|X|^2 = m$. Then the following hold.*

(i) f is an odd integer.

(ii) Either $f \leq m$ or $p \leq (f^2 - m)/(f - m)$.

(iii) $p \leq m^2 + m + 1$.

Theorem 3.11. *Let b be a positive integer, let p be an odd prime and let G be an abelian group. Write $|G| = p^c m$, where $c \geq 0$ and p does not divide m . Suppose that m is not a square. Let $q_1, \dots, q_s$ be all distinct prime divisors of m . Put $f = \gcd(\text{ord}_p(q_1), \dots, \text{ord}_p(q_s))$. If either a $\text{BH}(G, p^b)$ matrix or a $\text{BH}(G, 2p^b)$ matrix exists, then the following hold.*

(i) f is odd.

(ii) Either $f \leq m$ or $p \leq (f^2 - m)/(f - m)$.

(iii) $p \leq m^2 + m + 1$.

Proof. As $\text{BH}(G, p^b) \subset \text{BH}(G, 2p^b)$, it suffices to assume that a $\text{BH}(G, 2p^b)$ matrix H exists. Let the first row of H be $(\zeta_{2p^b}^{a_g})_{g \in G}$, $a_g \in \mathbb{Z}$ for all $g \in G$. By Result 2.3, we have

$$DD^{(-1)} = p^c m, \quad D = \sum_{g \in G} \zeta_{2p^b}^{a_g} g \in \mathbb{Z}[\zeta_{2p^b}][G].$$

Put $X = \chi_0(D) = \sum_{g \in G} \zeta_{2p^b}^{a_g}$. We obtain

$$X\bar{X} = p^c m, \quad X \in \mathbb{Z}[\zeta_{2p^b}] = \mathbb{Z}[\zeta_{p^b}].$$

In the ring $\mathbb{Z}[\zeta_{p^b}]$, we have $p\mathbb{Z}[\zeta_{p^b}] = (1 - \zeta_{p^b})^{\varphi(p^b)}$ and $(1 - \zeta_{p^b})$ is the only prime ideal above p . Thus $(1 - \zeta_{p^b})^{\varphi(p^b)c/2}$ divides (X) as ideals of $\mathbb{Z}[\zeta_{p^b}]$. Putting $Y = X(1 - \zeta_{p^b})^{-\varphi(p^b)c/2} \in \mathbb{Z}[\zeta_{p^b}]$, we obtain

$$Y\bar{Y} = m.$$

The conclusion follows directly from Result 3.10. □

Corollary 3.12. *If G is an abelian group with $|G| = 2p^c$ for some non-negative integer c and odd prime p , then there is no $\text{BH}(G, 2p^b)$ matrix.*

Proof. Using Theorem 3.11 part (iii) for $m = 2$, we obtain $p \leq 7$. The case $p = 3$ or $p = 5$ cannot satisfy the condition $f = \text{ord}_p(2)$ is odd. Therefore, the only possible value for p is $p = 7$. The case $p = 7$ was also ruled out by Leung and Schmidt [18]. $\square$

We note that Theorem 3.11 confirms the nonexistence of 2361 $\text{BH}(\mathbb{Z}_n, h)$ matrices in which $2 \leq n, h \leq 100$.

Remark 3.13. Using Results 1.1, 1.2, 1.3 and 1.4, we have 5108 open cases for the existence of a $\text{BH}(\mathbb{Z}_n, h)$ matrix in which $1 \leq n, h \leq 100$. Result 1.5 settles the existence problem of 1798 of these cases. There are 3310 cases left. Theorem 3.2, Theorem 3.3, Theorem 3.7 and Theorem 3.11 confirm the nonexistence of 623 cases in the 3310 open cases above.

In summary, there are 2687 open cases for the existence of $\text{BH}(\mathbb{Z}_n, h)$ matrices in which $1 \leq n, h \leq 100$. This list of open cases is provided in [12].

4 Application to Perfect Polyphase Arrays

We recall from the introduction that a perfect h -phase sequence of length n is equivalent to a $\text{BH}(\mathbb{Z}_n, h)$ matrix. Theorem 3.2, Theorem 3.3, Theorem 3.7 and Theorem 3.11 provide various necessary conditions for the existence of such sequences. In Lemma 4.3 below, we prove that a perfect h -phase array of size $n_1 \times \cdots \times n_k$ is equivalent to a $\text{BH}(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, h)$ matrix.

The following table gives parameters of known perfect arrays, see [3].

Author	Array size	Alphabet size (value of h)
Jedwab, Mitchell	$2^n \times 2^n$	2
Kopilovich	$(3 \times 2^{n+1}) \times (3 \times 2^{n+1})$ $(3 \times 2^n) \times (3 \times 2^{n+2})$	2
Wild	$2^n \times 2^{n+2}$	2
Blake, Hall, Tirkel	$n \times n$	n if n is odd, $2n$ if n is even
Blake, Hall, Tirkel	$n \times n^{2k+1}$	n^{k+1} if n is odd, $\text{lcm}(2n, n^{k+1})$ if n is even
Blake, Hall, Tirkel	$n \times n^2$	n if n is odd, $2n$ if n is even
Blake, Hall, Tirkel	$n^{2k+1} \times n^{2k+1}$	n^{k+1}
Blake, Hall, Tirkel	$n^2 \times n^2$	n

The following result is a direct corollary of Result 1.5, see also [22].

Result 4.1. *Let n and h be positive integers. Then a $\text{BH}(\mathbb{Z}_n, h)$ matrix exists whenever n and h satisfy the following condition:*

$$n \mid (h, n)^2 \text{ and } (\nu_2(n), \nu_2(h)) \neq (1, 1).$$

To prepare for the main result of this section, we need the following lemmas.

Lemma 4.2. *Let h_1 and h_2 be positive integers and put $h = \text{lcm}(h_1, h_2)$. Let G_1 and G_2 be finite abelian groups. Assume that H_1 is a $\text{BH}(G_1, h_1)$ matrix and H_2 is a $\text{BH}(G_2, h_2)$ matrix. Then the Kronecker product $H = H_1 \otimes H_2$ is a $\text{BH}(G_1 \times G_2, h)$ matrix.*

Proof. Assume that $H_1 = (h_{x_1, y_1})_{x_1, y_1 \in G_1}$ and $H_2 = (k_{x_2, y_2})_{x_2, y_2 \in G_2}$. The elements of the matrix $H = H_1 \otimes H_2$ are indexed by $G_1 \times G_2$ in which the $(x_1, x_2) \times (y_1, y_2)$ element is $H_{(x_1, x_2), (y_1, y_2)} = h_{x_1, y_1} k_{x_2, y_2}$. Let (a_1, a_2) be any element in $G_1 \times G_2$. We have

$$\begin{aligned}
H_{(a_1+x_1, a_2+x_2), (a_1+y_1, a_2+y_2)} &= h_{a_1+x_1, a_1+y_1} k_{a_2+x_2, a_2+y_2} \\
&= h_{x_1, y_1} k_{x_2, y_2} \\
&= H_{(x_1, x_2), (y_1, y_2)},
\end{aligned}$$

proving that H is $G_1 \times G_2$ -invariant.

On the other hand, note that each entry of H is a product of a (h_1) th root of unity and a (h_2) th root of unity, so it is a h th root of unity. It remains to verify that $HH^* = |H|I_{|H|}$, or equivalently, any two distinct rows (a_1, a_2) and (b_1, b_2) of H are orthogonal. The inner product of these two rows is

$$\begin{aligned} \sum_{x_1 \in G_1, x_2 \in G_2} H_{(a_1, a_2), (x_1, x_2)} \overline{H}_{(b_1, b_2), (x_1, x_2)} &= \sum_{x_1 \in G_1, x_2 \in G_2} h_{a_1, x_1} \overline{h}_{b_1, x_1} k_{a_2, x_2} \overline{k}_{b_2, x_2} \\ &= \left(\sum_{x_1 \in G_1} h_{a_1, x_1} \overline{h}_{b_1, x_1} \right) \left(\sum_{x_2 \in G_2} k_{a_2, x_2} \overline{k}_{b_2, x_2} \right) \\ &= 0, \end{aligned}$$

where the last equality holds as $(a_1, b_1) \neq (a_2, b_2)$ and H_1 is a $\text{BH}(G_1, h_1)$ matrix and H_2 is a $\text{BH}(G_2, h_2)$ matrix. $\square$

Lemma 4.3. *Let $k, h, n_1, \dots, n_k$ be positive integers. Then a $\text{BH}(\mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}, h)$ matrix exists if and only if a perfect h -phase array of size $n_1 \times \dots \times n_k$ exists.*

Proof. Put $G = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ and suppose that a $\text{BH}(G, h)$ matrix exists. For each $1 \leq i \leq k$, let g_i be a generator of $\mathbb{Z}_{n_i}$. By Result 2.3, there exists $D \in \mathbb{Z}[\zeta_h][G]$ such that $DD^{(-1)} = n_1 \dots n_k$. Define

$$I = \{(i_1, \dots, i_k) : 0 \leq i_j \leq n_j - 1 \text{ for } j = 1, \dots, k\}.$$

Write

$$D = \sum_{(i_1, \dots, i_k) \in I} a_{i_1, \dots, i_k} g_1^{i_1} \dots g_k^{i_k},$$

where each $a_{i_1, \dots, i_k}$ is a complex h th root of unity. Note that

$$DD^{(-1)} = \sum_{(s_1, \dots, s_k) \in I} \left(\sum_{(i_1, \dots, i_k) \in I} a_{i_1 + s_1, \dots, i_k + s_k} \overline{a}_{i_1, \dots, i_k} \right) g_1^{s_1} \dots g_k^{s_k},$$

As $DD^{(-1)} = n_1 \dots n_k$, we obtain

$$\sum_{(i_1, \dots, i_k) \in I} a_{i_1 + s_1, \dots, i_k + s_k} \overline{a}_{i_1, \dots, i_k} = \begin{cases} n_1 \dots n_k & \text{if } s_1 = \dots = s_k = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

Define the array A of size $n_1 \times \cdots \times n_k$ by

$$A = (a_{i_1, \dots, i_k}), \quad 0 \leq i_j \leq n_j - 1 \text{ for all } j.$$

The equation (26) implies that A is perfect. Conversely, it is straightforward to verify that the existence of a perfect array A implies the existence of a group ring element $D \in \mathbb{Z}[\zeta_h][G]$ whose coefficients are complex h th roots of unity such that $DD^{(-1)} = n$, which implies a $\text{BH}(G, h)$ matrix by Result 2.3. $\square$

Theorem 4.4. *Suppose that $k, h, n_1, \dots, n_k$ are positive integers such that*

$$n_i \mid (h, n_i)^2 \text{ and } (\nu_2(n_i), \nu_2(h)) \neq (1, 1) \text{ for any } 1 \leq i \leq k. \quad (\star)$$

Then a perfect h -phase array of size $n_1 \times \cdots \times n_k$ exists.

Proof. By Result 4.1, the condition $(\star)$ imply that a $\text{BH}(\mathbb{Z}_{n_i}, h)$ matrix exist for any $i = 1, \dots, k$. By Lemma 4.2, a $\text{BH}(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, h)$ matrix exists. The desired array is constructed as in Lemma 4.3. $\square$

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