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Two-stage Bayesian system identification with Gaussian discrepancy model

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Abstract:

System identification aims at updating the model parameters (e.g., mass, stiffness) associated with the mathematical model of a structure based on measured structural response. In this process, a two-stage approach is commonly adopted. In Stage I, modal parameters including natural frequencies and mode shapes are identified. In Stage II, the modal parameters are used to update structural parameters such as those related to stiffness, mass and boundary conditions. A recent Bayesian formulation allows the identification results in the first stage to be incorporated in the second stage directly via Bayes' rule without using a heuristic model (often based on classical statistics) that transfers the information from Stage I to II. This opens up opportunities for explicitly accounting for modeling error in the structural model (Stage II) through the conditional distribution of modal parameters given structural model parameters. Following this approach, this paper investigates a methodology where the modeling error between the two stages is incorporated with Gaussian distributions whose statistical parameters are also updated with available data. Leveraging on special mathematical structure induced by the model, computational issues are resolved and an analytical investigation is performed that yields insights on the role of modeling error and whether its statistics can be distinguished from those of identification uncertainty (defined for given structural model). The proposed methodology is verified using synthetic data and applied to a laboratory-scale structure.

Key words: system identification, two-stage formulation, Bayesian, modal identification, modeling error

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1. Introduction

Structural system identification is concerned with the determination of structural model parameters based on measured response collected from the subject structure in its constructed state. The resulting information provides a basis for understanding the current state of the structure, damage detection, or structural health monitoring in general [1]-[8]. As a general feature of inverse problems, the difficulty of obtaining a satisfactory solution (if exists) depends critically on the complexity of the structural model used and the adequacy of data. Uncertainty always exists in this process. To better address the uncertainty issues, some advanced methods based on hierarchical Bayesian framework [9][10] and sparse Bayesian learning [11][12] have been developed recently.

For structures responding in the linear-elastic range, modal properties are under frequent demand for practical reasons as they are the interface between the structural properties and dynamic response. Knowing the modal properties of a few contributing modes is already sufficient for predicting reliably the dynamic response in the corresponding frequency regime by virtue of modal superposition. With dynamic response data, one common strategy identifies the structural model parameters using a ‘two-stage’ approach [13][14][15]. In Stage I, the modal parameters using the measured response data are identified. In Stage II, model updating is performed using the information in Stage I. The motivation for adopting a two-stage approach instead of directly identifying the structural model parameters from measured data is multi-folded. Identifying the properties of well-excited modes is mathematically a globally identifiable problem [16][17]. The existence of mode(s) in a frequency band can be detected based on the Fourier spectrum of data or stabilization diagram [18][19]. Except for challenging situations which do exist (e.g., [20][21]), reported laboratory experiments and field tests revealed that modal properties can be identified in a robust manner from data of reasonable quality (e.g., [17][22]-[39]). In contrast, it is not always possible to determine the structural model parameters from data, e.g., when there are too many structural parameters or too few data, which can cause non-uniqueness in the solution and rendering the problem ‘locally identifiable’ or ‘unidentifiable’ [40]. Whether the structural identification problem has a unique solution can be non-trivial in many cases. Resolving the problem into two therefore has the advantage of isolating the most difficult problem (Stage II) from the relatively solvable one (Stage I).

Formulating the two-stage identification problem in strict accordance with probability and structural dynamics modeling assumptions is not trivial, however. Examples include [13][42][45]-[48] and applications can be found in [49][51]-[57]. In order to establish the posterior distribution for the structural model parameters in the second stage, conventionally a ‘discrepancy model’ is introduced to connect the most probable modal properties (instead of the original uncertain counterparts) determined in Stage I with the model (e.g., finite element) counterparts that depend on the structural model parameters. In this sense the procedure is a ‘fusion’ of Bayesian and frequentist inferences [61], i.e., a combination of concepts/definitions from the two approaches to come up with a mathematically reasonable approach, although the introduction of such discrepancy model appeared unavoidable. How to properly account for the modeling error [58][59] in Bayesian system identification is an important issue.

A general theory for the Bayesian two-stage problem has been recently developed [60] and applied to structural system identification based on ambient vibration data [61]. The posterior probability distribution function (PDF) of structural parameters in Stage II was expressed in a fundamental manner in terms of the posterior PDF of modal parameters in Stage I and the conditional prior PDF of modal properties for given structural parameters. Let $\boldsymbol{\theta}$ denote the set of ‘structural parameters’ and $\boldsymbol{\varpi}$ denote the set of modal parameters related to $\boldsymbol{\theta}$. In the studies so far, the modeling error between the two stages has been ignored, i.e., it has been assumed that the knowledge of $\boldsymbol{\theta}$ completely determines the value of $\boldsymbol{\varpi}$ so that the conditional distribution $p(\boldsymbol{\varpi} | \boldsymbol{\theta})$ can be taken as a Dirac-Delta function of $\boldsymbol{\varpi}$. This method has also been extended to deal with the multiple setups model updating problem [62].

Modeling error in the general context refers to the discrepancy between observations and model prediction, which may be related to test-to-test variability, model falsification, etc. This work focuses on the discrepancy in the modal properties identified in Stage I (based on linear superposition in the frequency domain) and those predicted by structural model in Stage II. For example, the natural frequencies of a pure shear building may follow the ratio 1, 3, ..., but this need not be the case for those identified from data. Similarly, mode shapes corresponding to computed models bear spatial continuity but the same is generally not true for mode shape values identified from experimental data. One can speculate that ignoring modeling error can lead to under-estimation of identification uncertainty in structural model parameters or bias of the

posterior distribution. This motivates the current work to incorporate modeling error and investigates its effect on identification results. Ideally, modeling error better be ‘model-free’ but in reality assumption on the class of models is inevitable to allow a mathematical treatment. As a trade-off, in this paper a Gaussian model class is adopted, i.e., $p(\varpi | \theta)$ is a Gaussian distribution whose parameters are updated from data. Theoretical and computational issues are investigated, leading to an efficient algorithm for calculating the posterior statistics of structural parameters and insights on whether the statistics of modeling error can be distinguished from those of identification error.

This paper is organized as follows. In Section 2, the general formulation will be reviewed first, followed by the development of the formulation considering modeling error in Section 3. Section 4 presents the computational issues encountered in the model updating based on ambient vibration data. In Section 5, the proposed method is illustrated by synthetic and experimental data. Section 6 summarizes the conclusions.

2. General formulation in Bayesian two-stage formulation

Let D denote the measured data for modal identification in Stage I. Assume that a ‘data prediction model’ exists whose set of parameters α can be identified from D . Let θ denote a set of model parameters to be updated from measured data D . Let $\alpha = [\varpi, \mathbf{v}]$, where ϖ is related to θ but $\mathbf{v}$ is not. In the two-stage formulation for model updating, the posterior distribution of θ given the data D and prior information is given by [60]:

$$p(\theta | D) \propto p(\theta) \int p_0(\varpi | D) p(\varpi | \theta) d\varpi \quad (1)$$

Here, ϖ comprises the natural frequencies and partial mode shapes (i.e., confined to measured degrees of freedom only), which bridge the information between Stage I and Stage II; $p(\varpi | \theta)$ is the prior PDF of modal parameters ϖ given structural model parameters θ ; the conditional PDF $p_0(\varpi | D)$ denotes the marginal posterior PDF of ϖ given data D using (hypothetically) a uniform (i.e., constant) prior distribution for ϖ in Stage I. The latter assumption on prior distribution, reminded in the subscript ‘0’, is necessary to avoid ‘double-counting’ prior information at Stage I

and II; see Section 4 of [60]. If one ignores modeling errors of the finite element model (FEM) in predicting ϖ , the conditional PDF $p(\varpi | \boldsymbol{\theta})$ is given by a Dirac-Delta function [61]:

$$p(\varpi | \boldsymbol{\theta}) = \delta(\varpi - \tilde{\varpi}(\boldsymbol{\theta})) \quad (2)$$

with

$$\tilde{\varpi}(\boldsymbol{\theta}) = [\tilde{\mathbf{f}}(\boldsymbol{\theta}), \tilde{\boldsymbol{\Phi}}(\boldsymbol{\theta})] \quad (3)$$

where $\tilde{\mathbf{f}}(\boldsymbol{\theta})$ and $\tilde{\boldsymbol{\Phi}}(\boldsymbol{\theta})$ are respectively the theoretical natural frequencies and mode shapes determined from FEM for given $\boldsymbol{\theta}$. In reality, modeling error may exist in many structural model updating problems. In the next section, this issue will be addressed.

3. Gaussian structural discrepancy model

Equation (1) is applicable to the general situation as long as $p(\varpi, \mathbf{v} | \boldsymbol{\theta}) = p(\varpi | \boldsymbol{\theta})p(\mathbf{v})$ holds and the quantities $p(\boldsymbol{\theta})$, $p(\varpi | \boldsymbol{\theta})$, $p(D | \varpi, \mathbf{v})$, $p(\mathbf{v})$ are available. When determining the posterior statistics (e.g., the most probable value (MPV), covariance) of $\boldsymbol{\theta}$ in Stage II, the posterior PDF in (1) will often need to be evaluated for different values of $\boldsymbol{\theta}$, e.g., during numerical optimization. Efficient computation of its value is therefore important for feasible implementation. The integral in (1) presents an obvious difficulty. Advancing over (2) in [61] that ignored modeling error, in this work we assume that the structural modeling error is Gaussian. It turns out that the integral in this case can be evaluated analytically, resulting in a simple expression for $p(\boldsymbol{\theta} | D)$. Based on that we will investigate the role of the structural modeling error in the two-stage problem, which turn out to yield insights on when modeling error can be distinguished from data.

3.1. Gaussian approximation of $p_0(\varpi | D)$

With sufficient amount of data, the modal identification problem in Stage I is ‘globally identifiable’ in the sense that the posterior PDF $p_0(\varpi, \mathbf{v} | D)$ has a unique maximum, at the MPV $(\hat{\varpi}, \hat{\mathbf{v}})$. In this case the posterior PDF can be approximated by a Gaussian PDF [61], i.e.,

$$p_0(\boldsymbol{\alpha} | D) = (2\pi)^{-(n_1+n_2)/2} (\det \mathbf{C}_0)^{-1/2} \exp[-\frac{1}{2}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})^T \mathbf{C}_0^{-1}(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}})] \quad (4)$$

where n_1 and n_2 are the number of parameters in ϖ and $\mathbf{v}$, respectively; $\hat{\mathbf{a}} = [\hat{\varpi}, \hat{\mathbf{v}}]$ is the posterior MPV of $\mathbf{a}$ which maximizes $p_0(\mathbf{a} | D)$, or equivalently minimizes the negative log-likelihood function (NLLF) $L(\mathbf{a}) = -\ln p_0(D | \mathbf{a})$; $\mathbf{C}_0 \in R^{(n_1+n_2) \times (n_1+n_2)}$ is the posterior covariance matrix of $\mathbf{a}$, equal to the inverse of the Hessian of the NLLF evaluated at the MPV. Since $\mathbf{a} = [\varpi, \mathbf{v}]$ is Gaussian according to the posterior PDF in (4), the marginal PDF of ϖ is also Gaussian:

$$p_0(\varpi | D) = (2\pi)^{-n_1/2} (\det \mathbf{C}_D)^{-1/2} \exp\left[-\frac{1}{2}(\varpi - \hat{\varpi})^T \mathbf{C}_D^{-1}(\varpi - \hat{\varpi})\right] \quad (5)$$

where $\mathbf{C}_D \in R^{n_1 \times n_1}$ is the posterior covariance matrix of ϖ , equal to the corresponding $n_1 \times n_1$ partition in $\mathbf{C}_0$.

3.2. Gaussian structural discrepancy model

Assume that the set of modal parameters ϖ is related to the structural model prediction $\tilde{\varpi}(\boldsymbol{\theta})$ by

$$\varpi = \tilde{\varpi}(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}_{\varpi} \quad (6)$$

where $\boldsymbol{\varepsilon}_{\varpi}$ denotes the discrepancy between ϖ and $\tilde{\varpi}(\boldsymbol{\theta})$. In this work, $\boldsymbol{\varepsilon}_{\varpi}$ is assumed to follow a Gaussian distribution with zero mean and covariance matrix $\mathbf{C}_p(\boldsymbol{\theta}) \in R^{n_1 \times n_1}$, with $\mathbf{C}_p(\boldsymbol{\theta})$ being the covariance matrix of modeling errors of natural frequencies and mode shapes. The latter generally depends on $\boldsymbol{\theta}$ because it is unknown and so its defining parameters should also be identified in Stage II. It accounts for the potential difference between the modal properties identified from data and those predicted within the class of structural model (e.g., finite element model) parameterized by $\boldsymbol{\theta}$.

According to this model, for given $\boldsymbol{\theta}$, ϖ is Gaussian with mean $\tilde{\varpi}(\boldsymbol{\theta})$ and covariance matrix $\mathbf{C}_p(\boldsymbol{\theta})$, i.e.,

$$p(\varpi | \boldsymbol{\theta}) = (2\pi)^{-n_1/2} [\det \mathbf{C}_p(\boldsymbol{\theta})]^{-1/2} \exp\left\{-\frac{1}{2}[\varpi - \tilde{\varpi}(\boldsymbol{\theta})]^T \mathbf{C}_p(\boldsymbol{\theta})^{-1}[\varpi - \tilde{\varpi}(\boldsymbol{\theta})]\right\} \quad (7)$$

3.3. Analytical expression for posterior PDF

It can be difficult to evaluate the integral in the general case as shown in equation (1). However, when $p_0(\varpi | D)$ is a Gaussian PDF and a Gaussian model is chosen for $p(\varpi | \theta)$, the integral in (1) can be evaluated analytically. This can be done by expressing the integrand as a Gaussian PDF multiplied by a constant. The value of the integral is then simply equal to that constant because the Gaussian PDF integrates to 1. Substituting (5) and (7) into the integrand of (1) and integrating analytically (see Appendix for details):

$$\int p_0(\varpi | D) p(\varpi | \theta) d\varpi = \phi(\tilde{\varpi}(\theta); \hat{\varpi}, \bar{\mathbf{C}}(\theta)) \quad (8)$$

where $\phi(\tilde{\varpi}(\theta); \hat{\varpi}, \bar{\mathbf{C}}(\theta))$ denotes the value of a Gaussian PDF with mean $\hat{\varpi}$, covariance matrix $\bar{\mathbf{C}}(\theta)$ and evaluated at $\tilde{\varpi}(\theta)$; and

$$\bar{\mathbf{C}}(\theta) = \mathbf{C}_D + \mathbf{C}_p(\theta) \quad (9)$$

Substituting (8) into (1), the posterior PDF of θ is given by

$$p(\theta | D) \propto p(\theta) \phi(\tilde{\varpi}(\theta); \hat{\varpi}, \bar{\mathbf{C}}(\theta)) \quad (10)$$

As a quick check, when the structural modeling error is zero, i.e., $\mathbf{C}_p = \mathbf{0}$, $\bar{\mathbf{C}} = \mathbf{C}_D$ and so $p(\theta | D) \propto p(\theta) \phi(\tilde{\varpi}(\theta); \hat{\varpi}, \mathbf{C}_D)$. This gives the same result as (5) because $p_0(\tilde{\varpi}(\theta) | D) = \phi(\tilde{\varpi}(\theta); \hat{\varpi}, \mathbf{C}_D)$.

Similar to [60], the set of model parameters is partitioned as $\theta = [\theta_s, \theta_p]$, where θ_s is related to the structural model ϖ and θ_p is related to the modeling error variance of ϖ . Equation (10) can then be written as:

$$p(\theta_s, \theta_p | D) \propto p(\theta_s, \theta_p) \phi(\tilde{\varpi}(\theta_s); \hat{\varpi}, \bar{\mathbf{C}}(\theta_p)) \quad (11)$$

where $\bar{\mathbf{C}}(\theta_p) = \mathbf{C}_D + \mathbf{C}_p(\theta_p)$.

3.4 Characteristic resolution of structural modeling error

At this point it is worth mentioning an implication resulting from introduction of the structural modeling error, which is intuitive and may hold in more general settings. In [60], it has been shown that when a perfect match is possible then $\boldsymbol{\theta}_p$ always has a MPV at $\mathbf{0}$, which is at the boundary of the permissible domain. That is $\mathbf{C}_p = \mathbf{0}$, $\bar{\mathbf{C}} = \mathbf{C}_D$. In the general case when a perfect match is not possible, i.e., $\mathbf{C}_p \neq \mathbf{0}$ due to the presence of modeling error, the posterior PDF of $\boldsymbol{\theta}_p$ depends on the discrepancy between $\tilde{\omega}(\hat{\boldsymbol{\theta}}_s)$ and $\hat{\omega}$. Intuitively, if the discrepancy is large then it should be possible to distinguish $\boldsymbol{\theta}_p$ from zero, in which case a non-zero MPV can be expected. On the other hand, the MPV of $\boldsymbol{\theta}_p$ may still remain at the boundary zero value if the discrepancy is not large enough. We shall illustrate these intuitive statements using a simple example. Consider the simple case when

$$\mathbf{C}_D = \text{diag}([\hat{v}_1, \dots, \hat{v}_{n_\alpha}]) \quad (12)$$

$$\mathbf{C}_p = \text{diag}([\boldsymbol{\theta}_p]) = \text{diag}([\theta_{p1}, \dots, \theta_{pn_\alpha}]) \quad (13)$$

where $\hat{v}_i$ and θ_{pi} are the posterior variance in Stage I and structural modeling error variance of the i -th parameter in $\boldsymbol{\omega}$. Then

$$\bar{\mathbf{C}} = \mathbf{C}_D + \mathbf{C}_p = \text{diag}([\hat{v}_1 + \theta_{p1}, \dots, \hat{v}_{n_\alpha} + \theta_{pn_\alpha}]) \quad (14)$$

and so

$$\det \bar{\mathbf{C}} = \prod_{i=1}^{n_\alpha} (\hat{v}_i + \theta_{pi}) \quad (15)$$

$$\bar{\mathbf{C}}^{-1} = \text{diag}([\hat{v}_1 + \theta_{p1}]^{-1}, \dots, [\hat{v}_{n_\alpha} + \theta_{pn_\alpha}]^{-1}) \quad (16)$$

According to (10) with a non-informative prior PDF, the NLLF $L(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p)$ such that $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p | D) = \exp[-L(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p)]$ is given by (omitting constants)

$$L(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p) = \frac{1}{2} \sum_{i=1}^{n_a} \ln(\hat{v}_i + \theta_{p_i}) + \frac{1}{2} \sum_{i=1}^{n_a} (\hat{v}_i + \theta_{p_i})^{-1} [\tilde{\omega}_i(\boldsymbol{\theta}_s) - \hat{\omega}_i]^2 \quad (17)$$

where $\tilde{\omega}_i(\boldsymbol{\theta}_s)$ and $\hat{\omega}_i$ are the i -th elements of $\tilde{\omega}(\boldsymbol{\theta}_s)$ and $\hat{\omega}$, respectively.

Differentiating (17) with respect to θ_{p_i} gives

$$\frac{\partial L}{\partial \theta_{p_i}} = \frac{1}{2} (\hat{v}_i + \theta_{p_i})^{-2} \{ \hat{v}_i + \theta_{p_i} - [\tilde{\omega}_i(\boldsymbol{\theta}_s) - \hat{\omega}_i]^2 \} \quad (18)$$

Solving $\partial L / \partial \theta_{p_i} = 0$ and noting that $\hat{v}_i + \theta_{p_i} \neq 0$ gives the MPV

$$\hat{\theta}_{p_i} = [\tilde{\omega}_i(\boldsymbol{\theta}_s) - \hat{\omega}_i]^2 - \hat{v}_i \quad (19)$$

This solution is valid only if it is positive, i.e., $[\tilde{\omega}_i(\boldsymbol{\theta}_s) - \hat{\omega}_i]^2 > \hat{v}_i$. Otherwise, according to (18)

$\partial L / \partial \theta_{p_i}$ is always positive, implying that the minimum of L occurs at the zero boundary, i.e.,

$\hat{\theta}_{p_i} = 0$. We thus see that the MPV of the modeling error variance has a non-trivial value only when

the squared difference between the model and posterior MPV is bigger than the posterior variance in Stage I. The quality of information about the structural modeling error thus depends in a characteristic manner on the discrepancy between the most probable structural model prediction and the posterior MPV determined in Stage I. If the discrepancy is smaller than the identification uncertainty $\hat{v}_i$ in Stage I then it is theoretically impossible to distinguish the modeling error from zero in good resolution (as reflected by a zero MPV). Intuitively, when the discrepancy is small compared to the identification uncertainty it tells little about the modeling error because there is a good chance that the discrepancy merely results from identification error in Stage I.

4. Structural model identification using ambient vibration data

In the present context, we define $\boldsymbol{\theta}_s$ to be the set of structural stiffness parameters that may possibly be involved in the characterization of the stiffness and/or mass matrix, which may be symbolically written as $\mathbf{K}(\boldsymbol{\theta}_s)$ and $\mathbf{M}(\boldsymbol{\theta}_s)$, respectively. The damping matrix $\mathbf{C}$ is not included in this parameterization because practically the interest regarding damping is on the damping ratios

$\{\zeta_i\}$, which have already been identified in Stage I. For civil engineering structures the damping matrix is also much more difficult to model from mechanical principles than the stiffness or mass matrix; and so it is rarely constructed.

The data D consists of FFT data in a number of disjoint frequency bands containing the modes that can be identified, i.e.,

$$D = \{D^{(r)} : r = 1, \dots, n_B\} \quad (20)$$

Correspondingly, as mentioned, the set of modal parameters that completely defines the distribution of D is given by

$$\mathbf{a} = [\varpi, \mathbf{v}] \quad (21)$$

where

$$\varpi = [\mathbf{f}, \Phi], \mathbf{v} = [\zeta, \mathbf{S}, \mathbf{S}_e] \quad (22)$$

with $\mathbf{f} = \{\mathbf{f}^{(r)}\}$; $\Phi = \{\Phi^{(r)}\}$; $\zeta = \{\zeta^{(r)}\}$; $\mathbf{S} = \{\mathbf{S}^{(r)}\}$; $\mathbf{S}_e = \{\mathbf{S}_e^{(r)}\}$; $\mathbf{f}^{(r)}$, $\Phi^{(r)}$, $\zeta^{(r)}$, $\mathbf{S}^{(r)}$, $\mathbf{S}_e^{(r)}$ denote the natural frequency, mode shape, damping ratio, power spectral density (PSD) of modal force and PSD of prediction error in a selected frequency band, respectively. These modal parameters can be identified by a Fast Bayesian FFT method [17][63][64][65].

Here, ϖ can be determined for given Θ_S because the natural frequencies $\mathbf{f}$ and mode shapes Φ are determined through the generalized eigenvalue equation:

$$\mathbf{K}(\Theta_S)\mathbf{u}_i = \omega_i^2 \mathbf{M}(\Theta_S)\mathbf{u}_i \quad (23)$$

where $\omega_i = 2\pi f_i(\Theta_S)$ with $f_i(\Theta_S)$ being the natural frequency of the i -th mode; the i -th mode shape $\Phi_i(\Theta_S)$ contains the observed components of the full mode shape $\mathbf{u}_i$.

Applying the standard form of (11), the posterior PDF of $\Theta = [\Theta_S, \Theta_P]$ given D is then given by

$$p(\Theta_S, \Theta_P | D) = A [\det \bar{\mathbf{C}}(\Theta_P)]^{-1/2} \exp\left\{-\frac{1}{2}[\tilde{\omega}(\Theta_S) - \hat{\omega}]^T \bar{\mathbf{C}}(\Theta_P)^{-1}[\tilde{\omega}(\Theta_S) - \hat{\omega}]\right\} \quad (24)$$

where $\boldsymbol{\theta}_s$ is related to the structural model $\tilde{w}(\cdot)$ and $\boldsymbol{\theta}_p$ is related to the modeling error standard derivations of ϖ ; A is a constant. The covariance matrix $\mathbf{C}_p(\boldsymbol{\theta}_p)$ in $\overline{\mathbf{C}}(\boldsymbol{\theta}_p)$ can be constructed by

$$\mathbf{C}_p(\boldsymbol{\theta}_p) = \begin{bmatrix} S_{f_1}^2 & & & \\ & \ddots & & \\ & & S_{f_m}^2 & \\ & & S_{\Phi_1}^2 \hat{\Phi}_1 \hat{\Phi}_1^T & \\ & & & \ddots & \\ & & & & S_{\Phi_m}^2 \hat{\Phi}_m \hat{\Phi}_m^T \end{bmatrix} \in R^{m(1+n) \times m(1+n)} \quad (25)$$

where $\{S_{f_i}\}$ and $\{S_{\Phi_i}\}$ corresponding to the natural frequency and mode shapes respectively are the structural model errors needed to be identified; m is the number of modes used in the model updating; $\hat{\Phi}_i \in R^n$ denotes the MPV of i th mode shape; n is the number of measured DOFs.

To relate the modeling error of natural frequency and mode shape in (25) to the posterior uncertainties of identified natural frequency and mode shape in Stage I respectively, it is defined that

$$S_{f_i} = \theta_{f_i} S_{f_i}^{(0)} (i = 1, \dots, m) \quad (26)$$

$$S_{\Phi_i} = \theta_{\Phi_i} S_{\Phi_i}^{(0)} (i = 1, \dots, m) \quad (27)$$

where $\boldsymbol{\theta}_p = [\{\theta_{f_i}\}, \{\theta_{\Phi_i}\}]$ is the dimensionless set of model parameters to be identified; $S_{f_i}^{(0)}$ is the nominal value equal to the standard derivation of i -th identified natural frequency;

$$S_{\Phi_i}^{(0)} = \frac{\sum_{j=2}^n \sqrt{q_{ij}}}{n-1} \quad (28)$$

and q_{ij} is the j -th eigenvalue of posterior covariance matrix of i -th identified mode shape with ascending order ($q_{ii}=0$ due to mode shape scaling constraint) [66].

Equivalently,

$$p(\boldsymbol{\theta} | D) \propto \exp[-L(\boldsymbol{\theta})] \quad (29)$$

where $\boldsymbol{\theta} = [\boldsymbol{\theta}_s, \boldsymbol{\theta}_p]$ and

$$L(\boldsymbol{\theta}) = \frac{1}{2} \ln[\det \overline{\mathbf{C}}(\boldsymbol{\theta}_p)] + \frac{1}{2} [\tilde{\boldsymbol{\omega}}(\boldsymbol{\theta}_s) - \hat{\boldsymbol{\omega}}]^T \overline{\mathbf{C}}(\boldsymbol{\theta}_p)^{-1} [\tilde{\boldsymbol{\omega}}(\boldsymbol{\theta}_s) - \hat{\boldsymbol{\omega}}] \quad (30)$$

is the NLLF for $\boldsymbol{\theta}$ given D .

4.1. Model reformulation

Determining the MPV of model parameters requires numerically minimizing the log-likelihood function. In this process, the determinant and inverse of the two covariance matrix $\mathbf{C}_D$ and $\mathbf{C}_p$ need to be evaluated repeatedly. However, due to the norm constraint of mode shapes, the two covariance matrix are ill-conditioned [63][66] since they are close to being singular. This issue can be resolved as follows.

First note that the covariance matrix is equal to the pseudo-inverse of the Hessian matrix [63][66], i.e., via the eigenvector representation ignoring the m zero eigenvalues of the Hessian matrix of identified modal parameters obtained in Stage I:

$$\mathbf{C}_D = \sum_{i=m+1}^{n_p} v_i \mathbf{b}_i \mathbf{b}_i^T = \mathbf{B} \begin{bmatrix} v_{m+1} & & \\ & \dots & \\ & & v_{n_p} \end{bmatrix} \mathbf{B}^T = \mathbf{B} \text{diag}(\mathbf{v}) \mathbf{B}^T \quad (31)$$

where $v_i = u_i^{-1}$ and u_i is the eigenvalue of the corresponding Hessian matrix in an ascending order;

$n_p = m + m \times n$; $\mathbf{b}_i \in R^{n_p}$ is the eigenvector corresponding to u_i ; $\mathbf{B} = [\mathbf{b}_{m+1}, \mathbf{b}_{m+2}, \dots, \mathbf{b}_{n_p}] \in R^{n_p \times (n_p - m)}$.

It can be proved that the posterior PDF in (24) can be written as a Gaussian distribution evaluated at $\mathbf{B}^T [\tilde{\boldsymbol{\omega}}(\boldsymbol{\theta}_s) - \hat{\boldsymbol{\omega}}]$ with covariance matrix $\text{diag}(\mathbf{v}) + \mathbf{B}^T \mathbf{C}_p(\boldsymbol{\theta}_p) \mathbf{B}$. The corresponding NLLF is then given by:

$$L(\boldsymbol{\theta}) = \frac{1}{2} \ln[\text{diag}(\mathbf{v}) + \mathbf{B}^T \mathbf{C}_p(\boldsymbol{\theta}_p) \mathbf{B}] + \frac{1}{2} [\tilde{\boldsymbol{\omega}}(\boldsymbol{\theta}_s) - \hat{\boldsymbol{\omega}}]^T \mathbf{B} [\text{diag}(\mathbf{v}) + \mathbf{B}^T \mathbf{C}_p(\boldsymbol{\theta}_p) \mathbf{B}]^{-1} \mathbf{B}^T [\tilde{\boldsymbol{\omega}}(\boldsymbol{\theta}_s) - \hat{\boldsymbol{\omega}}] \quad (32)$$

4.2. Computational issues

In the conventional Bayesian computational framework, the NLLF is approximated by a second order Taylor expansion around the MPV. When the problem is globally identifiable and all parameters have a MPV that is a regular minimum of the NLLF, i.e., with a zero partial derivative at the MPV, the resulting posterior PDF is approximated by a Gaussian PDF centered at the MPV and with a covariance matrix equal to the inverse of the Hessian of the NLLF at the MPV.

However, it is found that the value of $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p)$ that minimizes the NLLF need not correspond to a regular minimum. Some of the parameters, especially the modeling error variances, may have their MPV equal to the boundary value $\mathbf{0}$ of the admission region (the positive real line including zero). Special considerations are therefore needed for obtaining the posterior MPV of the parameters $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p)$.

Instead of minimizing numerically the NLLF with respect to the whole set of parameters $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_p)$, convergence and efficiency may be improved by optimizing $\boldsymbol{\theta}_s$ and $\boldsymbol{\theta}_p$ separately and then iterating until convergence. This is because for a given $\boldsymbol{\theta}_p$ optimizing $\boldsymbol{\theta}_s$ is the same as finding the MPV of structural parameters when there is no modeling error, although the posterior covariance is apparently increased by $\mathbf{C}_p(\boldsymbol{\theta}_p)$. This problem has a similar identifiability as the original problem without structural modeling error. On the other hand, for a given $\boldsymbol{\theta}_s$ optimizing $\boldsymbol{\theta}_p$ does not involve any structural analysis in the calculation of the NLLF and hence the process can be quite fast. Finally, one may start with $\boldsymbol{\theta}_p = \mathbf{0}$ as a simple initial guess.

As it is possible that the MPV of some parameters in $\boldsymbol{\theta}_p$ occurs at the boundary $\mathbf{0}$ at which the NLLF is not stationary, in the minimization of the NLLF the convergence criterion cannot be defined as the gradient being zero (within some tolerance). Rather, the MPV should be verified as the point where the value of the NLLF is less than its neighboring values. In this regard, algorithms such as simplex search are suitable while methods like Newton-Ralphs are not. In reality, due to the convergence criterion of the minimization algorithm, a parameter need not converge to a clear zero even if its MPV theoretically should occur at the boundary 0. A numerical criterion is required for ascertaining the nature of the MPV, i.e., whether the MPV of a particular parameter θ_{pi} occurs at the boundary. By comparing the magnitude of the first and second order term in the Taylor

series of the NLLF with respect to θ_{pi} , the decision can be determined based on the dimensionless ratio

$$r_i = \left| \frac{1}{2} \hat{L}^{(\theta_{pi}\theta_{pi})} \right|^{-1/2} \hat{L}^{(\theta_{pi})} \quad (33)$$

where $\hat{L}^{(\theta_{pi})}$ and $\hat{L}^{(\theta_{pi}\theta_{pi})}$ are the first and second partial derivative of the NLLF with respect to θ_{pi} and evaluated at the MPV. Note that $\hat{L}^{(\theta_{pi})} \geq 0$ and hence $r_i \geq 0$ since the MPV minimizes the NLLF; the same need not be true for $\hat{L}^{(\theta_{pi}\theta_{pi})}$ since the MPV may not be a regular minimum. A small value of r_i (compared to 1), say, $r_i < 1\%$, implies that the variation of the NLLF is dominated by the second order term in the neighborhood of the MPV and so θ_{pi} is a regular minimum. Otherwise the first order term dominates, suggesting that the MPV of θ_{pi} occurs at the boundary, which can also be checked together with whether it is close to zero. In this regard, it is preferable to work with dimensionless parameters, e.g., as a multiplier of some nominal value, so that their order of magnitude can be easily checked.

If the problem is not globally identifiable, the NLLF can either have multiple local (isolated) minima (i.e., locally identifiable) or even a manifold of stationary points (unidentifiable). Conventional optimization algorithms cannot be used for finding the critical points/regions that characterize the posterior PDF [67][68]. Despite the available advanced tools such as Markov Chain Monte Carlo (MCMC) [69][70][71], studies on unidentifiable cases have so far been mostly of academic nature.

5. Illustrative examples

In this section, two examples are used to illustrate the proposed method, including one with synthetic data example for method verification and another one with laboratory-scale shear frame data for verification/application with experimental complications.

5.1 Synthetic data

Consider a three-story shear building with uniform floor mass of 600 tons and lateral column stiffness (negligible mass) given in Table 1. The fundamental natural frequency is calculated to be

4.477Hz. Assume 1% classical damping ratio in all modes. The building is assumed to be subjected to ambient excitation modeled by independent and identically distributed (i.i.d.) Gaussian white noise in the x, y and rotational directions, with one-sided root PSDs of $3N/\sqrt{\text{Hz}}$, $3N/\sqrt{\text{Hz}}$ and $12N \cdot m/\sqrt{\text{Hz}}$, respectively. The acceleration data is contaminated by Gaussian noise with a root PSD of $2\mu\text{g}/\sqrt{\text{Hz}}$. On each floor, four corners are measured bi-axially. A total of 24 channels are measured in all the three floors. Twenty-minute data at a sampling rate of 200Hz is generated and used for modal identification. The PSD spectra of the data are shown in Figure 1, where clear peaks can be found for all the 9 possible structural modes below 35 Hz. Ambient modal identification is performed for each mode separately by the fast Bayesian FFT method [17][63][64][65]. A random sensor alignment error is introduced to the synthetic data (independent for different bi-axial sensors), which is uniformly distributed between +/- 10 degree.

Table 2 shows the modal parameters identified based on the simulated data. It is seen that the MPVs of natural frequency and damping ratio are close to the exact values. The modal assurance criteria (MAC) between the identified mode shape and actual mode shape without alignment error, identified mode shape and actual mode shape with alignment error are also shown in the table. The presence of alignment error reduces the MAC from almost unity (1.0000) by about 1% to 0.99. The identified mode shapes are shown in Figure 2, including three translational modes in X direction (modes 1,4,7), three translational modes in Y direction (modes 2,5,8) and three torsional modes (modes 3,6,9), which are of reasonable quality.

The columns labeled 'COV' show the posterior COVs (coefficient of variation = standard derivation / MPV) of the corresponding parameters. The COV of natural frequency is quite small (<0.1%), indicating a precise identification. The COV of damping ratio is larger, in the order of a few percent. To investigate the uncertainty of mode shape, the COV of mode shape is also calculated, which is equal to the square root sum of eigenvalues of the posterior covariance matrix of the corresponding mode shape. From the table, it is seen that the uncertainty of mode shape is very small and they are almost the same regardless of the presence of alignment errors. This suggests that the alignment error cannot be reflected by the uncertainty of the identified mode shapes, which is not surprising because the alignment error changes the mode shape value (hence response value) in terms of a projection rather than a noise. The alignment error however can cause bias in the identified mode shapes and modeling error in Stage II (see later).

Based on the modal parameters and the associated posterior covariance matrix obtained in the modal identification, model updating is performed by the proposed method. Assume that the mass in each floor is known and so only the stiffness will be updated. Table 1 shows the nominal values of stiffness in each column, based on which the following simplified model is used for identification:

$$\begin{aligned}
k_{xi1} &= k_{xi4} = 600\theta_{xi1} \\
k_{xi2} &= k_{xi3} = 600\theta_{xi2} \\
k_{yi1} &= k_{yi4} = 700\theta_{yi1} \\
k_{yi2} &= k_{yi3} = 700\theta_{yi2}
\end{aligned}
\quad [\text{kN/mm}] \quad (34)$$

where $i = 1, 2, 3$ denotes the floor number. The dimensionless model parameters to be updated are shown as follows

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_{x11} & \theta_{x12} & \theta_{y11} & \theta_{y12} \\ \theta_{x21} & \theta_{x22} & \theta_{y21} & \theta_{y22} \\ \theta_{x31} & \theta_{x32} & \theta_{y31} & \theta_{y32} \end{bmatrix} \quad (35)$$

which includes 12 model parameters. All the 9 modes identified are used for model updating. Two kinds of initial guesses for $\boldsymbol{\theta}$ are used. For the first kind, all elements are set to 1. For the second kind, the elements are set to be some random values around 1, giving a realization as shown below:

$$\boldsymbol{\theta} = \begin{bmatrix} 0.5503 & 0.7075 & 0.8368 & 0.8867 \\ 0.5987 & 0.7204 & 0.5543 & 1.2031 \\ 0.5454 & 0.5803 & 0.6351 & 0.7071 \end{bmatrix} \quad (36)$$

It is seen that even different initial guesses are given, the identified model parameters still converge to the same value (only the result from the random initial guess in Equation (36) is shown in Table 3, while the other results are omitted), where all the updated dimensionless model parameters are all close to 1. Given the same initial guess, when the modeling errors are not considered, i.e., assuming no modeling error, the MPV of model parameters for different initial guess can also converge to the same values, (only the result of random initial guess in Equation (36) is shown in Table 4, while others are omitted). Compared with the values in Table 3, these values are farther away from the exact value (i.e., 1). This is attributed to the modeling error caused by sensor misalignment. Modeling error is not accounted for in Table 4, leading to bias in the results.

The natural frequencies (MPV) obtained by modal identification, model updating by proposed method and model updating with structural modeling error not modelled are shown in Table 5. Except Mode 8, the ones obtained by proposed method tend to be closer to the ones identified from the acceleration data, while the ones obtained by the method with structural modeling error not modelled have larger discrepancy, although they still have similar order of magnitude. This discrepancy can also be reflected in the updated structural modeling errors as shown in Table 6, where the second and third rows show the nominal values of the structural modeling errors for natural frequency and mode shape, respectively. Recall that the natural frequency of Mode 8 obtained by proposed method has larger difference. Correspondingly, it is found that the structural modeling error of natural frequency corresponding to this mode is much larger than those in other modes. It is also interesting to see that the structural modeling error of mode shapes in all the modes are all larger than 10, indicating that some structural modeling errors are produced when the alignment errors are modelled in the data. This means that when there are structural modeling errors, using the proposed method can reflect the realistic situation and give more accurate updated model parameters.

To further verify the effect on the results whether the structural modeling error is modeled or not in the formulation, the data without alignment error are also used to perform the verification. Table 7 and Table 8 show the updated model parameters using the proposed method and the method with structural modeling error not considered, respectively. It is seen that the counterparts of MPVs in two tables are almost the same. The natural frequencies and mode shapes calculated based on the model parameters in these two tables are also quite close to each other (omitted here). This indicates that if there is no structural modeling error, whether the modeling errors are accounted for or not will not affect the results a lot. From the above observation, it can be concluded that the formulation that the structural modeling error not modelled is only one special case of the proposed method. In real application, structural modeling error usually exists and so the proposed method is expected to be more robust.

5.2 Experimental data

In this section, the proposed method is applied to experimental data acquired from a three-story shear frame situated in the laboratory. The data used is the same as that in [61]. The structure is supported by four columns at four corners. The weight of each floor is 25.2 kg and the mass of

each column is ignored. The heights of the three floors from bottom to top are 0.858m, 0.571m and 0.555m, respectively. The width and thickness of each floor are 25mm and 5 mm, respectively. In each floor, four corners were measured bi-axially and so eight un-axial sensors are required. Twenty four un-axial sensors are needed in the whole measurement. All the sensors are connected to a data acquisition system and controlled by a laptop. Twenty minute data was measured with a sampling rate of 2048 Hz and later decimated by 8 to 256Hz for modal identification.

Figure 3 shows the root PSD spectra of the measured data. The nine modes in the weak (x), strong (y) and torsional directions are marked by the numbers in the spectra. Table 9 shows the identified modal parameters and their posterior uncertainty. Similar observations can be found as those in the synthetic example. From the mode shapes (omitted here), it can be deduced that Modes 1, 2, 4 are translational modes in the weak (x) direction; Modes 3, 6, 8 are translational modes in the strong (y) direction; Modes 5, 7, 9 are the torsional modes. The second and third rows in Table 10 give the nominal values of the modeling errors, and they will be used as the standard values to evaluate the modeling error of model updating.

Structural stiffness is parametrized by $k_{xi} = \theta_{xi} k_{xi0} (i = 1, 2, 3)$, where k_{xi} denotes the stiffness in the weak (x) direction of the i-th floor; similarly, $k_{yi} = \theta_{yi} k_{yi0} (i = 1, 2, 3)$ denotes the stiffness in the strong (y) direction of the i-th floor. Here, $k_{xi0}, k_{yi0} (i = 1, 2, 3)$ are nominal values and given in Table 11. Structural mass is assumed to be known and so only the stiffness will be updated. The model parameters θ_s to be identified are shown as follows

$$\theta_s = \begin{bmatrix} \theta_{x1} & \theta_{y1} \\ \theta_{x2} & \theta_{y2} \\ \theta_{x3} & \theta_{y3} \end{bmatrix} \quad (37)$$

Table 12 shows the updated stiffness in the weak (x) and strong (y) directions. The non-dimensional stiffness parameters are generally close to 1, indicating that estimation of the nominal values of stiffness are reasonable. The only exception is θ_{x1} , which is 30% below the nominal value. This is likely to be attributed to the presence of a hole for each column in the first floor, which reduces significantly the stiffness in the weak direction but less so in the strong direction.

The fourth and fifth row in Table 10 show the identified modeling errors of natural frequency and mode shape for all nine modes. Values (MPV) vary among modes, from possibly zero (modes 1, 4, 5 & 7 frequencies) to significantly large values (e.g., modes 8 & 9 frequencies). The latter is consistent with the discrepancy in the identified frequencies in Stage I and the MPVs predicted in Stage II, as shown in Table 13. For modes 1, 4, 5 and 7, the natural frequencies in Stage I and Stage II are the same; the corresponding modeling errors of natural frequency are close to zero. For Modes 2, 3 and 6, the discrepancy in natural frequencies of the two stages is relatively small. The modeling errors of these three modes are larger than those in the former four mentioned modes. The discrepancy in MPV is quite large for the frequencies of modes 8 & 9 from the two stages; correspondingly the modeling error is also large. In this relatively clear-cut case the behavior of the proposed method in reflecting modeling error and reconciling between the identified (Stage I) and structural model predicted (Stage II) frequencies agrees with intuition. Nevertheless in general situations the behavior and trade-off in accordance with Bayes' rule is non-trivial and may not be directly or clearly reflected in simple statistics such as the MPVs considered here. Table 14 shows the counterpart results of Table 13 for the case when modeling error is not incorporated. Observations are mixed, e.g., the relative difference in frequency is no longer zero for modes 1, 4, 5 and 7, but it is now smaller for modes 8 and 9. It is non-trivial to establish, or perhaps unreasonable to expect, a direct intuitive relationship between the simple statistics of MPV and modeling error. In fact, a method (or model) that matches the frequencies in Stage I and II need not be a good one, because, e.g., the match can be obtained simply by using a more complicated model (over-fitting). Rather, it is the inner workings of the mathematics accordance with Bayes' rule that provides the correct trade-off in the presence of modeling error.

6. Conclusions

In this work, a two-stage Bayesian formulation for structural system identification considering modeling errors of natural frequencies and mode shapes has been developed. The modeling error focused in this work accounts for the discrepancy in the modal properties identified in Stage I (based on linear superposition in the frequency domain) and those predicted by structural model in Stage II. It is based on a recent framework that fundamentally follows Bayesian logic without adopting an additional discrepancy model to relate the identified modal properties (MPV) in Stage I to their model counterparts in Stage II as is conventionally done. Within the framework, a

Gaussian discrepancy model has been adopted, which significantly simplifies computations because the integral of two Gaussian PDFs in Eqs (5) and (7) can be analytically obtained. Other discrepancy models can be developed for specific applications based on physical modeling reasons, although it should be kept in mind that different models will lead to different identification uncertainties. The proposed theory has been illustrated with synthetic and laboratory data. The identification results between the proposed formulation with and without modeling error of natural frequencies and mode shapes taken into account have also been compared. Within the scope of the examples considered, it is found that the MPVs calculated from both formulations are similar when structural modeling error is not significant. Otherwise, the previous formulation that ignores modeling error can lead to significant bias. The discrepancy in natural frequencies between Stages I and II can be reflected by the modeling errors of natural frequencies obtained during the model updating. The model updating results are more robust when modeling the modeling error in the formulation.

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Appendix. Integral of the product of two Gaussian PDFs

This appendix derives the result for the integral of the product of two Gaussian PDFs:

$$\int \phi(\mathbf{x}; \mathbf{x}_1, \mathbf{C}_1) \phi(\mathbf{x}; \mathbf{x}_2, \mathbf{C}_2) d\mathbf{x} = \phi(\mathbf{x}_1; \mathbf{x}_2, \mathbf{C}_1 + \mathbf{C}_2) \quad (38)$$

where

$$\phi(\mathbf{x}; \boldsymbol{\mu}, \mathbf{C}) = (2\pi)^{-n/2} (\det \mathbf{C})^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right] \quad (39)$$

denotes the Gaussian PDF with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$ evaluated at $\mathbf{x}$.

The idea of the proof is to write the integrand as a Gaussian PDF of $\mathbf{x}$ multiplied by a factor so that integrating it gives simply the constant. Substituting (39), the integrand is given by

$$\begin{aligned} & \phi(\mathbf{x}; \mathbf{x}_1, \mathbf{C}_1) \phi(\mathbf{x}; \mathbf{x}_2, \mathbf{C}_2) \\ &= (2\pi)^{-n} (\det \mathbf{C}_1)^{-1/2} (\det \mathbf{C}_2)^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{x}_1)^T \mathbf{C}_1^{-1}(\mathbf{x} - \mathbf{x}_1) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_2)^T \mathbf{C}_2^{-1}(\mathbf{x} - \mathbf{x}_2)\right] \end{aligned} \quad (40)$$

The quadratic form in the exponent can be rewritten in the following form by completing square:

$$-\frac{1}{2}(\mathbf{x} - \mathbf{x}_1)^T \mathbf{C}_1^{-1}(\mathbf{x} - \mathbf{x}_1) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_2)^T \mathbf{C}_2^{-1}(\mathbf{x} - \mathbf{x}_2) = -\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \mathbf{C}_0^{-1}(\mathbf{x} - \mathbf{x}_0) - K \quad (41)$$

where $\mathbf{x}_0$, $\mathbf{C}_0$ and K are constant terms to be determined. Substituting (41) into (40), the integrand can be rewritten as

$$\begin{aligned} & \phi(\mathbf{x}; \mathbf{x}_1, \mathbf{C}_1) \phi(\mathbf{x}; \mathbf{x}_2, \mathbf{C}_2) \\ &= (2\pi)^{-n/2} (\det \mathbf{C}_1)^{-1/2} (\det \mathbf{C}_2)^{-1/2} \det(\mathbf{C}_0)^{1/2} \exp(-K) \phi(\mathbf{x}; \mathbf{x}_0, \mathbf{C}_0) \end{aligned} \quad (42)$$

Integrating with respect to $\mathbf{x}$ and noting that $\int \phi(\mathbf{x}; \mathbf{x}_0, \mathbf{C}_0) d\mathbf{x} = 1$ gives

$$\int \phi(\mathbf{x}; \mathbf{x}_1, \mathbf{C}_1) \phi(\mathbf{x}; \mathbf{x}_2, \mathbf{C}_2) d\mathbf{x} = (2\pi)^{-n/2} (\det \mathbf{C}_1)^{-1/2} (\det \mathbf{C}_2)^{-1/2} \det(\mathbf{C}_0)^{1/2} \exp(-K) \quad (43)$$

We now determine the terms $\mathbf{x}_0$, $\mathbf{C}_0$ and K . This can be done by direct algebra (which can be tedious) or more easily by comparing terms of derivatives. Considering the Hessian on both sides with respect to $\mathbf{X}$ yields

$$\mathbf{C}_0^{-1} = \mathbf{C}_1^{-1} + \mathbf{C}_2^{-1} \quad (44)$$

Considering the gradient on both sides gives

$$-(\mathbf{x} - \mathbf{x}_1)^T \mathbf{C}_1^{-1} - (\mathbf{x} - \mathbf{x}_2)^T \mathbf{C}_2^{-1} = -(\mathbf{x} - \mathbf{x}_0)^T \mathbf{C}_0^{-1} \quad (45)$$

Note that the terms of $\mathbf{X}$ on both sides automatically equate due to (44). Comparing the constant terms gives

$$\mathbf{x}_0 = \mathbf{C}_0(\mathbf{C}_1^{-1}\mathbf{x}_1 + \mathbf{C}_2^{-1}\mathbf{x}_2) \quad (46)$$

Finally, substituting $\mathbf{x} = \mathbf{x}_0$ into (41) gives

$$K = \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}_1)^T \mathbf{C}_1^{-1}(\mathbf{x}_0 - \mathbf{x}_1) + \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}_2)^T \mathbf{C}_2^{-1}(\mathbf{x}_0 - \mathbf{x}_2) \quad (47)$$

The determinant and exponential term in (43) can be further simplified. Using (44),

$$\begin{aligned} & (\det \mathbf{C}_1)^{-1/2} (\det \mathbf{C}_2)^{-1/2} (\det \mathbf{C}_0)^{1/2} \\ &= (\det \mathbf{C}_1 \mathbf{C}_0^{-1} \mathbf{C}_2)^{-1/2} = [\det \mathbf{C}_1 (\mathbf{C}_1^{-1} + \mathbf{C}_2^{-1}) \mathbf{C}_2]^{-1/2} = [\det(\mathbf{C}_1 + \mathbf{C}_2)]^{-1/2} \end{aligned} \quad (48)$$

On the other hand, for (46),

$$\begin{aligned} \mathbf{x}_0 - \mathbf{x}_1 &= \mathbf{C}_0(\mathbf{C}_1^{-1}\mathbf{x}_1 + \mathbf{C}_2^{-1}\mathbf{x}_2) - \mathbf{x}_1 = \mathbf{C}_0(\mathbf{C}_1^{-1}\mathbf{x}_1 + \mathbf{C}_2^{-1}\mathbf{x}_2 - \mathbf{C}_0^{-1}\mathbf{x}_1) \\ &= \mathbf{C}_0(\mathbf{C}_1^{-1}\mathbf{x}_1 + \mathbf{C}_2^{-1}\mathbf{x}_2 - \mathbf{C}_1^{-1}\mathbf{x}_1 - \mathbf{C}_2^{-1}\mathbf{x}_1) = \mathbf{C}_0 \mathbf{C}_2^{-1}(\mathbf{x}_2 - \mathbf{x}_1) \end{aligned} \quad (49)$$

Thus,

$$\begin{aligned} & (\mathbf{x}_0 - \mathbf{x}_1)^T \mathbf{C}_1^{-1}(\mathbf{x}_0 - \mathbf{x}_1) \\ &= (\mathbf{x}_2 - \mathbf{x}_1)^T \mathbf{C}_2^{-1} \mathbf{C}_0 \mathbf{C}_1^{-1} \mathbf{C}_0 \mathbf{C}_2^{-1}(\mathbf{x}_2 - \mathbf{x}_1) \\ &= (\mathbf{x}_2 - \mathbf{x}_1)^T \mathbf{C}_2^{-1} \mathbf{C}_0 \mathbf{C}_1^{-1} \mathbf{C}_1 \mathbf{C}_1^{-1} \mathbf{C}_0 \mathbf{C}_2^{-1}(\mathbf{x}_2 - \mathbf{x}_1) \\ &= (\mathbf{x}_2 - \mathbf{x}_1)^T (\mathbf{C}_2 \mathbf{C}_0^{-1} \mathbf{C}_1)^{-1} \mathbf{C}_1 (\mathbf{C}_2 \mathbf{C}_0^{-1} \mathbf{C}_1)^{-1}(\mathbf{x}_2 - \mathbf{x}_1) \\ &= (\mathbf{x}_2 - \mathbf{x}_1)^T (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{C}_1 (\mathbf{C}_1 + \mathbf{C}_2)^{-1}(\mathbf{x}_2 - \mathbf{x}_1) \end{aligned} \quad (50)$$

since

$$\mathbf{C}_2 \mathbf{C}_0^{-1} \mathbf{C}_1 = \mathbf{C}_2 (\mathbf{C}_1^{-1} + \mathbf{C}_2^{-1}) \mathbf{C}_1 = \mathbf{C}_1 + \mathbf{C}_2 \quad (51)$$

Similarly,

$$(\mathbf{x}_0 - \mathbf{x}_2)^T \mathbf{C}_2^{-1} (\mathbf{x}_0 - \mathbf{x}_2) = (\mathbf{x}_2 - \mathbf{x}_1)^T (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{C}_2 (\mathbf{C}_1 + \mathbf{C}_2)^{-1} (\mathbf{x}_2 - \mathbf{x}_1) \quad (52)$$

Substituting (50) and (52) into (47) and simplifying,

$$K = \frac{1}{2} (\mathbf{x}_1 - \mathbf{x}_2)^T (\mathbf{C}_1 + \mathbf{C}_2)^{-1} (\mathbf{x}_1 - \mathbf{x}_2) \quad (53)$$

Substituting into (43) and using (48) gives

$$\begin{aligned} & \int \phi(\mathbf{x}; \mathbf{x}_1, \mathbf{C}_1) \phi(\mathbf{x}; \mathbf{x}_2, \mathbf{C}_2) d\mathbf{x} \\ &= (2\pi)^{-n_1/2} [\det(\mathbf{C}_1 + \mathbf{C}_2)]^{-1/2} \exp\left[-\frac{1}{2} (\mathbf{x}_1 - \mathbf{x}_2)^T (\mathbf{C}_1 + \mathbf{C}_2)^{-1} (\mathbf{x}_1 - \mathbf{x}_2)\right] \\ &= \phi(\mathbf{x}_1; \mathbf{x}_2, \mathbf{C}_1 + \mathbf{C}_2) \end{aligned} \quad (54)$$

which completes the proof.

Tables and Figures

Table 1 Lateral stiffness of four supporting columns, synthetic data example

Column	<i>x</i> direction (kN/mm)	<i>y</i> direction (kN/mm)
1	600	700
2	600	700
3	600	700
4	600	700

Table 2 Identified modal parameters, synthetic data example

Mode	Natural frequency			Damping ratio			Mode shape			
	Exact (Hz)	MPV (Hz)	COV (%)	Exact (%)	MPV (%)	COV (%)	MAC ^a	COV (%)	MAC ^b	COV (%)
1	4.480	4.477	0.062	1.00	0.94	7.65	0.9904	0.0035	0.9999	0.0035
2	4.839	4.840	0.059	1.00	0.89	7.20	0.9903	0.0033	0.9999	0.0033
3	8.076	8.078	0.041	1.00	0.86	5.07	0.9905	0.0018	0.9999	0.0018
4	12.552	12.558	0.038	1.00	1.00	4.31	0.9905	0.0013	1.0000	0.0013
5	13.558	13.566	0.036	1.00	0.94	4.27	0.9889	0.0013	1.0000	0.0013
6	18.138	18.134	0.031	1.00	0.94	3.52	0.9894	0.0009	1.0000	0.0009
7	19.591	19.594	0.031	1.00	0.95	3.60	0.9898	0.001	1.0000	0.001
8	22.628	22.627	0.027	1.00	0.95	3.06	0.9897	0.0009	1.0000	0.0009
9	32.699	32.690	0.023	1.00	1.04	2.43	0.9900	0.0006	1.0000	0.0006

MAC^a: MAC between the identified mode shape and actual mode shape without alignment error

MAC^b: MAC between the identified mode shape and actual mode shape with alignment error

Table 3 Model parameters updated (MPVs), proposed method

Floor	θ_{xi1}	θ_{xi2}	θ_{yi1}	θ_{yi2}
i=1	0.9961	1.0035	1.0116	0.9924
i=2	0.9991	0.9962	1.0036	0.9946
i=3	1.0021	1.0018	1.0064	0.9952

Table 4 Model parameters updated (MPVs), without structural modeling error

Floor	θ_{xi1}	θ_{xi2}	θ_{yi1}	θ_{yi2}
i=1	0.9764	1.0277	1.0254	0.9763
i=2	0.9786	1.0211	1.0277	0.9719
i=3	0.9787	1.0228	1.0245	0.9728

Table 5 Natural frequency (MPV) obtained by modal identification, model updating by proposed method and model updating without structural modeling error. Excessive number of digits than usual are shown to investigate difference.

Natural frequency (Hz)	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9
Modal identification	4.4774	4.8398	8.0776	12.5583	13.5664	18.1344	19.5937	22.6274	32.6900
Model updating, proposed method	4.4781	4.8406	8.0765	12.5566	13.5645	18.1323	19.5912	22.6387	32.6943
Model updating, without modeling error	4.4814	4.8379	8.0799	12.5565	13.5509	18.1387	19.5807	22.6348	32.7027

Table 6 Structural modeling error: nominal values and identified values (MPVs) by proposed method

Modeling error	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9
$S_f^{(0)}$ (10^{-3} Hz)	2.787	2.867	3.290	4.799	4.944	5.660	5.976	6.031	7.444
$S_\Phi^{(0)}$ (10^{-3})	2.683	2.640	1.921	1.656	1.625	1.386	1.477	1.372	1.083
θ_f	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.585	0.000
θ_Φ	10.804	10.992	15.014	17.459	19.237	22.089	20.228	22.003	27.449

Table 7 Model parameters updated (MPVs), without alignment error, proposed method

Floor	θ_{xi1}	θ_{xi2}	θ_{yi1}	θ_{yi2}
i=1	0.9960	1.0018	1.0022	1.0027
i=2	1.0012	0.9988	1.0020	0.9983
i=3	0.9993	0.9983	1.0001	0.9991

Table 8 Model parameters updated (MPVs), without alignment error and modeling error

Floor	θ_{xi1}	θ_{xi2}	θ_{yi1}	θ_{yi2}
i=1	0.9980	1.0032	1.0012	1.0026
i=2	1.0003	0.9980	1.0024	0.9982
i=3	1.0002	0.9994	0.9998	0.9989

Table 9 Identified modal parameters (MPVs), laboratory frame

Mode	Natural frequency		Damping ratio		Mode shape
	MPV (Hz)	COV (%)	MPV (%)	COV (%)	1-EMAC (10^{-5})
1	0.808	0.080	0.341	24.41	44.14
2	3.522	0.026	0.169	15.31	0.72
3	4.822	0.014	0.072	19.87	0.35
4	5.905	0.016	0.118	14.13	0.33
5	7.107	0.018	0.169	10.79	0.10
6	17.659	0.010	0.135	7.68	0.90
7	26.380	0.011	0.226	4.89	0.14
8	31.488	0.006	0.070	7.90	0.29
9	45.863	0.006	0.117	5.06	0.05

Table 10 Modeling error: nominal values and identified values (MPVs) by proposed method

Modeling error	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9
$S_f^{(0)}$ (10^{-3} Hz)	0.643	0.900	0.681	0.969	1.276	1.813	2.860	1.732	2.691
$S_\Phi^{(0)}$ (10^{-3})	6.335	0.808	0.564	0.545	0.300	0.905	0.362	0.515	0.217
θ_f	0.000	34.310	62.917	0.000	0.000	201.331	0.000	1322.9	1159.6
θ_Φ	1.648	11.678	18.430	15.537	25.303	9.357	37.249	19.867	57.926

Table 11 Nominal values of interstory stiffness (N/mm), laboratory frame

Weak direction			Strong direction		
$k_{x_1 0}$	$k_{x_2 0}$	$k_{x_3 0}$	$k_{y_1 0}$	$k_{y_2 0}$	$k_{y_3 0}$
3.1	10.4	11.3	76.7	260.2	283.3

Table 12 Model parameters updated (MPVs), proposed method

Floor	θ_{xil}	θ_{yil}
i=1	0.7090	1.0913
i=2	1.0192	1.0397
i=3	1.0810	1.0060

Table 13 Natural frequency (MPV) obtained by modal identification, model updating by proposed method

Mode	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9
f (Hz): Stage I	0.8076	3.5219	4.8219	5.9046	7.1074	17.6593	26.3798	31.4877	45.8634
f (Hz): Stage II	0.8076	3.5528	4.8647	5.9046	7.1074	18.0244	26.3798	29.1963	42.7428
Relative difference (%)	0	0.87	0.88	0	0	2.03	0	7.85	7.30
MAC	0.9982	0.9990	0.9988	0.9992	0.9993	0.9992	0.9979	0.9988	0.9982

Table 14 Natural frequency (MPV) obtained by modal identification, model updating without modeling error

Mode	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9
f (Hz): Stage I	0.8076	3.5219	4.8219	5.9046	7.1074	17.6593	26.3798	31.4877	45.8634
f (Hz): Stage II	0.8018	3.5428	4.7545	5.8928	6.9468	18.8667	27.5967	30.9269	45.2428
Relative difference (%)	0.73	0.59	1.42	0.20	2.31	6.40	4.41	1.81	1.37
MAC	0.9982	0.9990	0.9982	0.9992	0.9990	0.9989	0.9987	0.9987	0.9980

Figure 1 PSD spectra of the measured data, synthetic data example

Figure 2 Identified mode shapes (MPV)

Figure 3 PSD spectra, laboratory frame