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**GRADIENT-FREE DISTRIBUTED OPTIMIZATION AND
NASH EQUILIBRIUM SEEKING**

PANG YIPENG

SCHOOL OF ELECTRICAL & ELECTRONIC ENGINEERING

2020

**GRADIENT-FREE DISTRIBUTED
OPTIMIZATION AND NASH EQUILIBRIUM
SEEKING**

PANG YIPENG

School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy

2020

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised material and has not been submitted for a higher degree to any other University or Institution.

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HU GUOQIANG

Authorship Attribution Statement

This thesis contains material from 1 paper published in the following peer-reviewed journals and 3 papers published at conferences in which I am listed as an author.

Chapters 3, 4 and 5 are published as

Y. Pang, G. Hu, “Randomized Gradient-Free Distributed Optimization Methods for a Multi-Agent System with Unknown Cost Function,” *IEEE Transactions on Automatic Control*, vol. 65, no. 1, pp. 333-340, 2020.

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The contributions of the co-authors are as follows.

- Prof. Hu suggested the research topics, provided guidance along the research, reviewed the manuscripts and offered comments.
- I developed the algorithms, conducted the theoretical analysis and numerical simulations, prepared and revised the manuscripts.

25/08/2020

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Date

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PANG YIPENG

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Summary

With the prevalence of the concept of multi-agent system, there is a strong interest to investigate the optimization and game problems among multiple agents or decision-makers due to its wide range of applications in various fields, such as air traffic control, robotics, power systems, smart grids, smart buildings, *etc.* With the increase of the data size, computation and network complexity, solving these problems in a distributed manner has found its great advantages in terms of the efficiency and reliability compared to the traditional centralized methods. However, in most distributed algorithms, the knowledge on the gradient information of the cost functions is required, making the methods rather restrictive, especially for the problems where such information is not available. In this dissertation, several gradient-free techniques are developed to solve the distributed optimization and Nash equilibrium (NE) seeking problems in the multi-agent system. The following summarizes the specific considered problems and proposed algorithms.

A distributed optimization problem with a set constraint is considered first. The explicit form of the local cost function is assumed to be unknown, but its value can be measured by each local agent. In such problem settings, the gradient information of the cost functions is not available, and hence the gradient-based methods are no longer applicable. A randomized gradient-free (RGF) distributed projected gradient descent method is proposed based on the Gaussian smoothing technique. Specifically, an RGF oracle is developed locally with the use of a random variable to estimate the gradient information in a random direction. Under some conditions on

the step-size, an approximate convergence result to the optimal solution is derived.

To improve the convergence result, a distributed projected pseudo-gradient descent algorithm is proposed to solve the same set constrained distributed optimization problem in the second part of this dissertation. Specifically, the gradient information is estimated by a so-called pseudo-gradient operator, where the additional parameters and random variables are introduced. By doing so, an exact convergence to the optimal solution is achieved with proper settings of the parameters and the step-size. Moreover, we propose an extra step locally, known as ‘optimal averaging scheme’, after the decision variable updates at each time-step. This scheme relaxes the requirement on the step-size selection as compared to most distributed methods, but still obtains the same convergence result.

In many practical cases, the optimization problem is often in a highly dynamic environment, resulting in a time-varying objective, which leads to the study of online optimization framework. Thus, in the third part of this dissertation, we consider a distributed optimization problem where the local cost functions are allowed to vary with time. An online RGF distributed projected gradient descent algorithm is presented. The notions of both static and dynamic regrets are adopted to analyze the performance of the proposed algorithm. With some standard assumptions on the graph connectivity and cost functions, we are able to show that both regrets are sublinear, and their corresponding averages over time are approximately convergent to zero with a rate of $O(1/\sqrt{T})$, which is comparable to the state-of-the-art algorithms.

In the last part of this dissertation, we consider a multi-player non-cooperative static game in which the players have limited knowledge on their local cost functions. To find its NE, a gradient-free distributed algorithm is proposed, which does not rely on the knowledge of the explicit form of the players’ cost functions. With the assumption of the existence of the NE, the convergence of the proposed algorithm is rigorously studied for both diminishing and constant step-sizes, respectively. Specif-

ically, for the diminishing case, an exact convergence to the NE is attained, while for the constant case, an approximate convergence to the NE with a gap proportional to the step-size is achieved.

To summarize, this dissertation focuses on the research of gradient-free distributed algorithms in distributed optimization problems where the agents collaboratively achieve a system-level objective, and NE seeking problems where the players selfishly minimize their own cost functions. For the distributed optimization problems, both static and time-varying cost functions are investigated, respectively. The convergence results of the proposed algorithms are studied with rigorous proofs, the effectiveness is verified by comparing with the state-of-the-art methods in numerical simulations, and the performance is demonstrated through both numerical examples and practical applications in energy systems.

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Symbols and Acronyms

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_n$	The set of n -dimensional real column vectors.
$\mathbf{0}_n$	An n -dimensional vector with all elements equal to 0.
$\mathbf{1}_n$	An n -dimensional vector with all elements equal to 1.
I	The identity matrix with appropriate dimensions.
$A^\top$	The transpose of a matrix A .
$[A]_{ij}$	The element in the i -th row and j -th column of a matrix A .
$\ A\ $	The induced vector Euclidean norm of a matrix A .
$\ A\ _\infty$	The infinity norm of a matrix A .
$\ \mathbf{x}\ $	The standard Euclidean norm of a vector $\mathbf{x}$.
$\mathcal{P}_{\mathcal{X}}[\mathbf{x}]$	The projection of a vector $\mathbf{x}$ on the set $\mathcal{X}$.
$\nabla f(\mathbf{x})$	The gradient of a function f at the point $\mathbf{x}$.
$\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y})$	The partial derivative of a function f with respect to $\mathbf{x}$.
$\partial f(\mathbf{x})$	The subgradient of a function f at the point $\mathbf{x}$.

$\partial_{\mathbf{x}}f(\mathbf{x}, \mathbf{y})$	The subdifferential of a function f at $\mathbf{x}$ for any fixed $\mathbf{y}$: $f(\mathbf{z}, \mathbf{y}) \geq f(\mathbf{x}, \mathbf{y}) + \langle g, \mathbf{z} - \mathbf{x} \rangle$, $g \in \partial_{\mathbf{x}}f(\mathbf{x}, \mathbf{y})$
$\partial_{\mathbf{x}}^{\varepsilon}f(\mathbf{x}, \mathbf{y})$	The ε -subdifferential of a function f at $\mathbf{x}$ for any fixed $\mathbf{y}$: $f(\mathbf{z}, \mathbf{y}) \geq f(\mathbf{x}, \mathbf{y}) - \varepsilon + \langle g, \mathbf{z} - \mathbf{x} \rangle$, $g \in \partial_{\mathbf{x}}^{\varepsilon}f(\mathbf{x}, \mathbf{y})$.
$\mathbb{E}[\mathbf{x}]$	The expected value of $\mathbf{x}$.
$Cov(\mathbf{x}, \mathbf{y})$	The covariance of $\mathbf{x}$ and $\mathbf{y}$.
$\langle \mathbf{x}, \mathbf{y} \rangle$	The inner product of $\mathbf{x}$ and $\mathbf{y}$.
$\mathcal{N}_i^{\text{in}}$	The set of in-neighbors of agent/player i .
$\mathcal{N}_i^{\text{out}}$	The set of out-neighbors of agent/player i .
$ \mathcal{N} $	The number of elements in a set $\mathcal{N}$.
HVAC	Heating ventilation and air conditioning.
NE	Nash equilibrium.
RGF	Randomized gradient-free.

Chapter 1

Introduction

In this chapter, we provide an overview of this dissertation. Specifically, the background information is firstly introduced, followed by the motivation and objectives of our research. Then, the existing literature related to our work is thoroughly reviewed, followed by a highlight of our major contributions. Finally, the organization of the thesis is presented.

1.1 Background, Motivation, and Objectives

A multi-agent system is a distributed computing system which consists of a number of interacting computational entities. The key elements of a multi-agent system includes: the intelligent entity, known as *agent*; the aim of each agent represented by a cost function, known as *objective*; and the underlying communication topology, known as *network*. With the prevalence of the concept of multi-agent system, there is a strong interest to investigate the optimization and game problems among multiple agents or decision-makers, due to its wide range of applications in different fields, including air traffic control, robotics, power systems, smart grids, smart buildings, *etc* [1].

Mathematically, a general optimization problem can be defined as follows

$$\min f(\mathbf{x}), \quad s.t., \quad \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n,$$

where $\mathbf{x}$ is the decision variable with some constraints defined as a set $\mathcal{X}$, $f(\mathbf{x})$ is the cost function. Many scenarios regarding the coordination of multi-agent systems can be modeled as optimization problems where agents collaboratively optimize a global cost function. The global cost function consists of each agent's individual cost and is usually in the form of summation defined as follows

$$\min f(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}), \quad s.t., \quad \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n,$$

where $\mathbf{x}$ is the global decision variable of all agents constrained by a set $\mathcal{X}$, $f(\mathbf{x})$ is the global cost function and $f_i(\mathbf{x})$ is the local cost function of agent i . Based on the way of solving it, the optimization methods can be generally classified into two categories: centralized optimization and distributed optimization, as shown in Fig. 1.1. Specifically, the centralized optimization method solves the problem di-

Figure 1.1: Optimization methods in a multi-agent system.

rectly in a central unit by considering the costs of all agents together. This method is preferred for its straightforward formulation and easy implementation. However, it suffers from the global knowledge requirements and high computational costs. The distributed optimization method solves the problem where each agent considers its own local cost function, then collaboratively solves the problem by communicating

with its neighbors. This method is favored by its effectiveness in computational cost and reliability. With the increase of the data size, computation and network complexity, distributed optimization methods in multi-agent systems have found great advantages in terms of the efficiency and reliability over the traditional centralized methods. From the perspectives of the applications, many engineering problems can be formulated as distributed optimization problems in a multi-agent system, such as parameter estimation and detection [2,3], source localization in sensor networks [4,5], utility maximization [6], resource allocation [7,8], and multi-robot coordination [9–12]. Therefore, efficient distributed optimization algorithms are of great demand to these applications.

Despite of designing distributed optimization algorithms, which are specific to the problems where the cost functions are fixed with respect to time. In some situations, the uncertainties in the environment may influence the cost functions, which in turn, affect the performance of the designed algorithms. One approach to cater such issues is through stochastic methods, which has been studied in [13–15]. However, in many practical cases, especially when mobile agents are involved, the optimization problem is often in a highly dynamic environment, resulting in a time-varying objective. Therefore, the techniques used in fixed or static distributed optimization cannot be applied, leading to the study of online optimization framework, where the cost functions assigned to agents vary with time and these variations are revealed to agents only in hindsight.

Figure 1.2: Multi-player game.

Game theory is a power tool of analyzing the strategic interactions between rational decision-makers. It has found a great potential in various applications (*e.g.*, social

science, economics, electricity markets, power systems). For a general game, it consists of three basic elements: the players/decision-makers, their actions, and their cost/payoff functions, see Fig. 1.2 for a multi-player game. According to the intrinsic nature of the players' behaviors, the game is known as a non-cooperative game if the players are self-interested, and selfishly minimize/maximize their own cost/payoff functions by making decisions in response to other players' actions. In non-cooperative games, the Nash equilibrium (NE), named after John Forbes Nash Jr., is a solution to the game, which says that no player can improve its payoff by unilaterally changing its own action [16]. The Nash equilibrium (NE) is an important concept and often used to analyze the strategic interaction among the players. However, when the NE exists and how to find it are not trivial, especially in multi-player non-cooperative games. Moreover, many engineering problems can be modeled as non-cooperative games, such as power network control [17], energy management control [18–20], transportation network control [21–23], *etc.* Therefore, the NE seeking problems in different settings of non-cooperative games are of great interests to study.

Even though distributed optimization and non-cooperative game are similar in terms of the setups (*i.e.*, multiple agents/players, local cost functions, decisions), the solutions to the corresponding problems are rather different. The solution to the distributed optimization is a social optimum point of the aggregate of each agent's cost function, while the solution to the non-cooperative game is an equilibrium point where no one can improve its payoff by unilaterally changing its own action (*i.e.*, NE). The following simple numerical example illustrates the difference of these two concepts:

Example 1.1. *There are two agents (1 and 2), whose local cost functions are given by*

$$f_1(x_1, x_2) = (x_1 + 1)^2 + (x_2 + 1)^2, \quad f_2(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2,$$

where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are their local decision variables, respectively. By simple calculations, the optimal solution to the distributed optimization problem $\min_{x_1, x_2} (f_1 + f_2)$ is $(0, 0)$, while the Nash equilibrium of the non-cooperative game problem $\min_{x_1} f_1, \min_{x_2} f_2$ is $(-1, 1)$.

Clearly, from Example 1.1, it can be concluded that a social optimal point is not the same as a Nash equilibrium. For general distributed optimization and non-cooperative game problems with multiple agents and more general costs, it may not be possible to find their analytical solutions. Thus, it is of great interest to develop efficient algorithms to seek the solutions.

Currently, most of the distributed optimization and NE seeking algorithms are model-based methods, in which the knowledge on the agents' cost functions and constraints is required. For instance, one of the most common algorithms, the gradient-based approach, needs to compute the derivatives of the cost functions to guide the updates of the decision variables, which requires the agents to have direct access to the gradient or subgradient information of the local cost functions and the constraints. However, in some applications, the relation between the variables and the cost functions are unknown, the gradient information is not available for usage, or the derivative is not possible to determine [24]. For example, if considering the occupants' thermal comfort as the objective, the relation between the thermal comfort (output) and the control variables (input) such as the airflow, cooling temperature, recirculation rate, *etc.*, is difficult to model mathematically. On the other hand, the occupants' thermal comfort (output) can be easily obtained by collecting the occupants' feedback given any control input. In such cases where the cost function is unknown but one can obtain its output iteratively, the methods purely based on the measurements of the cost functions while requiring no gradient information are strongly motivated.

In view of the great demand for the efficient distributed optimization and NE seeking algorithms in the framework of multi-agent systems, and large interests on the

gradient-free settings, we aim to achieve the following objectives in this dissertation.

- *Formulate the gradient-free distributed optimization and NE seeking problems.*

In the settings of distributed optimization and NE seeking problems, we assume that the explicit form of the (possibly non-smooth) local cost function is unknown, but the value can be measured by the local agent/player. Specifically, for the distributed online optimization problem, the cost functions assigned to agents can vary with time and these variations are revealed to agents only in hindsight. It should be noted that, in these problem settings, the gradient information is no longer available for usage, or may not even exist.

- *Propose efficient gradient-free algorithms.*

Due to the lack of gradient information, it is desirable to propose some gradient-free techniques to estimate the gradient information based on the local measurements of the cost functions. Furthermore, since the agents/players are only accessible to the local information, they are supposed to exchange information with their neighbors to make decisions through an underlying communication graph. As we consider the optimization and NE seeking problems in the framework of multi-agent systems, the underlying communication network is not fully connected, which implies that agents/players need to make decisions based only on a limited set of information, such as the information from the neighbors. Thus, a distributed protocol is necessary to achieve the coordination among the agents/players.

- *Analyze the convergence properties of the gradient-free algorithms.*

It is well-known that the theoretical convergence analysis is an essential part for the evaluation of an algorithm. Different from the analysis for gradient-based algorithms where the gradient plays an important role in deriving the convergence results, the challenge of establishing the convergence results for the gradient-free algorithms lies in how to adapt the estimated gradient information into the theoretical analysis. In contrast to the deterministic gradient

term in gradient-based approaches, the characterization of the estimated gradient for gradient-free algorithms needs to invoke the probability theory due to the involvement of random variables. Thus, it is anticipated that the convergence results of the gradient-free algorithms should be established in the sense of expectation. Moreover, numerical simulations on testing the performance of the gradient-free algorithms from different influencing factors should be conducted to demonstrate the effectiveness.

1.2 Literature Review

This section provides an extensive review on the literature related to distributed static optimization, distributed online optimization, and NE seeking in non-cooperative games.

1.2.1 Distributed Optimization

Motivated by the optimization methods [25, 26] for a centralized optimization problem, recent works on distributed optimization of a sum of local cost functions have been reported in [27–47] and the references therein. Most commonly used distributed convex optimization methods include the gradient-based algorithm. For example, in unconstrained distributed optimization problems, a subgradient method and a subgradient-push algorithms were proposed in [27, 28], respectively. The work in [29] introduced a distributed subgradient algorithm in a random network. This method was improved in [30–32] to speed up the convergence rate to be linear. For constrained distributed optimization problems, projection-based methods are widely used, such as the standard projected subgradient method in [33], projected scaled-subgradient method in [34], directed-distributed projected subgradient method in [35], fixed step-size projected gradient descent method in [36], dual subgradient averaging method in [37], and push-sum dual averaging method in [38].

Besides, primal-dual subgradient-based methods are usually leveraged to cater the constrained optimization problems, such as the distributed primal-dual perturbed subgradient method in [39], the regularized primal-dual subgradient method in [40], and the distributed penalty primal-dual subgradient method in [41–43]. The work in [44] proposed an accelerated distributed Nesterov gradient descent for convex and smooth functions, while Yuan *et al.* in [45] developed an optimal distributed stochastic mirror descent for strongly convex functions. The work in [46] considered a distributed continuous-time optimization problem with non-convex velocity and non-uniform position constraints. A distributed algorithm with non-uniform gradient gains was presented to achieve finite-time convergence. More references on the topic of distributed optimization can be found in [47]. Overall, these works have extensively investigated the distributed optimization problems differing from each other in terms of constraints, communication topology, optimality conditions, *etc.*, and presented various optimization methods to achieve promising results.

For the gradient-free distributed optimization, the concept of gradient-free technique has been brought out since the development of the optimization theory [48], and successfully applied in the fields of aircraft design [49], hydrodynamics [50], medicine [51], and earth sciences [52]. Recent research on gradient-free schemes has been reported in [53–59] based on extremum seeking, and [60–70] based on smoothing techniques. The extremum seeking-based methods use the sinusoidal dither signals for perturbation such that the gradient of the cost function is extractable. This type of methods, in general, is a continuous-time control-based approach, and usually presumes some smoothness in the cost function. On the other hand, the smoothing-based methods estimate the gradient of the cost function based on two point values. This type of methods is usually designed in the discrete-time framework, and allows the problem to be non-smooth. Specifically, Shamir *et al.* in [60] investigated the performance of stochastic gradient descent method for non-smooth optimization problems. An averaging scheme was proposed to attain the minimax-optimal rates. On the other hand, Nesterov *et al.* in [61] provided an explicit way

of computing the estimated gradient information known as randomized gradient-free (RGF) oracle and investigated the convergence property for both convex and non-convex problems. This idea was extended to minimize a sum of non-smooth but Lipschitz continuous functions in [62–64], where the Gaussian smoothing technique was introduced to obtain the gradient-free oracle to replace the derivative in the standard subgradient methods. However, all these methods are endowed with a doubly-stochastic weighting matrix requirement, which is restrictive in a distributed setting. The doubly-stochastic assumption was removed in [65] by adopting a push-sum technique and [66, 67] by introducing a surplus variable, respectively. Duchi *et al.* in [68] introduced a smoothing technique which used two point gradient estimation to close the optimality gap between the final iterate and the optimal point by choosing appropriate parameters. This technique was extended to the distributed scenario in [69], but restricted by the doubly-stochastic assumption on the weighting matrix. The work in [70] proposed a gradient-free method to achieve the exact convergence with only row- or column-stochastic weighting matrix. The following table summarizes a review of the aforementioned distributed optimization algorithms.

Table 1.1: Review of distributed optimization algorithms

Reference	Algorithm	Adjacency Matrix	Step-size	Convergence
[62]	RGF-SG	D	decreasing	inexact
[63]	DGFA	D	constant	inexact
[64]	RGF-SG	D	diminishing	inexact
[65]	RGF-Push	C	decreasing	inexact
[69]	DZO	D	decreasing	exact
Chapter 3	RGF-DPGD	R, C	diminishing	inexact
Chapter 4	DPPGD	R, C	decreasing	exact

‘D’, ‘R’ and ‘C’ refer to ‘doubly-stochastic’, ‘row-stochastic’ and ‘column-stochastic’ conditions, respectively.

‘Decreasing’ and ‘diminishing’ refer to ‘ $\sum_{k=1}^{\infty} \alpha_k = \infty, \alpha_k \geq \alpha_{k+1}$ ’ and ‘ $\sum_{k=1}^{\infty} \alpha_k = \infty, \sum_{k=1}^{\infty} \alpha_k^2 < \infty$ ’.

1.2.2 Distributed Online Optimization

The topic of distributed online optimization has been investigated in [71–78] and the references therein. In [71], general gradient-based methods were proposed to solve both unconstrained and set constrained time-varying quadratic optimization problems from the perspective of continuous-time domain. For discrete-time methods, in [72], a distributed autonomous online learning based on dual averaging where agents communicate over a weighted strongly connected graph was investigated. A regret bound of $O(\ln T)$ was derived for strongly convex cost functions. The same problem was studied in [73], where a distributed online subgradient push-sum algorithm was presented without the doubly-stochastic requirement on the weighting matrix, and was found to achieve a regret bound of $O((\ln T)^2)$. As for the agents with general convex cost functions interacting over a switching network, distributed dual-subgradient averaging algorithms were studied in [74] and [75], showing a regret bound of $O(\sqrt{T})$. Besides, Mateos-Nunez *et al.* in [76] introduced distributed online subgradient descent algorithms with proportional-integral disagreement feedback, proving a regret bound of $O(\sqrt{T})$ for convex cost functions and $O(\ln T)$ for strongly convex cost functions. It is worth noting that the regret used in all the aforementioned methods is defined as the difference between the incurred network cost and the cost of the best fixed decision in hindsight, which is known as *static regret*. On the other hand, to study the scenario where the functions and the decision variables evolve simultaneously instead of single best fixed decision, the notion of *dynamic regret* was brought forward to characterize how much one regrets working in an online setting as opposed to the offline solution with full knowledge of past and future observations. With this concept, Shahrampour *et al.* in [77, 78] proposed a decentralized mirror descent method for an online optimization problem, where the minimizer follows an known linear dynamics corrupted by unknown unstructured noise, and established a regret bound as a function of the deviation of minimizer sequence.

It should be noted that all these methods make use of the full knowledge of the derivatives in the process of the optimization, which implicitly presumes that the derivatives can be obtained directly. However, there are many applications where the gradient information is not available to use, *e.g.*, bandit feedback settings, then these methods cannot be applied. That inspires the study of gradient-free techniques. The gradient-free techniques have been extensively applied in the field of static optimization as discussed in the previous part. However, for online optimization, gradient-free optimization schemes have received little attention. The relevant studies *e.g.*, [77–79], considered the scenario where gradient of the cost function is coupled with noise, and the stochastic gradient methods were applied. The following table summarizes a review of the aforementioned distributed online optimization algorithms.

Table 1.2: Review of distributed online optimization algorithms

Reference	Algorithm	Knowledge	Convexity	Regret Type	Rate
[72]	autonomous online learning	F	strongly	static	$O(\ln T)$
[73]	online subgradient push	F	strongly	static	$O((\ln T)^2)$
[76]	online subgradient descent	F	strongly	static	$O(\ln T)$
[76]	online subgradient descent	F	convex	static	$O(\sqrt{T})$
[74, 75]	dual subgradient averaging	F	convex	static	$O(\sqrt{T})$
[79]	Nesterov’s dual averaging	F	convex	static	$O(\sqrt{T})$
[77, 78]	decentralized mirror descent	F	convex	dynamic	$O(\sqrt{T})$
Chapter 5	oRGF-DPGD	B	convex	static dynamic	$O(\sqrt{T})$

‘F’ and ‘B’ refer to ‘full knowledge’ and ‘bandit feedback’, respectively.

1.2.3 Distributed NE Seeking

There are a large number of studies on Nash equilibrium computation in non-cooperative games, such as matrix games [80], potential games [81], zero-sum games [82, 83], aggregate games [84–88], generalized games [89–92], cluster games [93, 94], hybrid games [95], *etc.*, depending on how the players' cost functions are coupled and what types of constraints are involved. Regardless of the game types, the challenge of such problem settings is the requirement of global knowledge on all players' actions, which is not practical if the underlying communication network is not fully connected. In such cases, players have to make decisions based only on a limited set of information, such as the information from the neighbors. Therefore, a distributed information sharing protocol is usually adopted to disseminate the local information among players. For example, the work in [81] derived a systematic methodology of game design for distributed optimization in a state-based potential game, where a distributed learning algorithm was introduced. The work in [82] proposed a distributed saddle-point strategy in a zero-sum game and proved its convergence in undirected graphs. The same game setting was considered in [83], where a distributed subgradient-based algorithm was developed and the Nash equilibrium was established under a uniformly jointly strongly connected weight-balanced digraphs. In [84], generalized aggregative games with affine coupling constraints were considered, and a single-layer, fixed step, semi-decentralized algorithm was derived to compute the generalized aggregative equilibrium. Besides, in [85], a dynamic average consensus and primal-dual dynamics based seeking strategy was developed to find Nash equilibrium in set constrained aggregate games, and applied to address energy consumption control among a network of electricity users. This technique was also utilized in [86] to compute Nash equilibrium in set constrained aggregate games. The dynamic average consensus protocol was also adopted in [87] with the help of differential inclusions and differentiated projections for aggregative games, where the players' actions are coupled by linear constraints. This work was extended to the aggregative games of multiple heterogeneous Euler-Lagrange systems

in [88]. In [89], a continuous time generalized convex game with shared inequality constraints among players was considered, a leader-following consensus protocol with gradient descent method is applied to compute the generalized Nash equilibrium. This protocol was also employed in [90] to estimate the other players' actions for the generalized games, where the players' action sets are constrained by nonlinear inequality and linear equations. In [91], a distributed subgradient algorithm was presented for a generalized convex game with both convex inequality and set constraints. In [92], a forward-backward operator splitting method was introduced for a generalized convex game with the set of players' actions coupled by a globally shared affine constraints. In cluster games, to achieve simultaneous social cost minimization and Nash equilibrium, a dynamic average consensus protocol was implemented in [93] where the players' payoffs are assumed to be smooth. Zeng *et al.* in [94] considered cluster games with non-smooth payoffs, and applied a leader-following consensus protocol with projected differential inclusions to find generalized Nash equilibrium. In [95], the hybrid games with both continuous-time and discrete-time players were considered. A hybrid gradient play with a hybrid consensus protocol was utilized to compute the Nash equilibrium. In [96], the convergence rate was improved by introducing the Nesterov type acceleration for the gradient play in non-cooperative games with strongly monotone mappings. In [97], a projection-based gradient play term was applied with a dynamic average consensus protocol to solve the non-cooperative games with convex set constraints. In [98], a dynamic gradient play scheme was adopted with a dynamic average consensus protocol to reject the additive disturbances in the players' dynamics. Apart from the leader-following consensus and dynamic average consensus protocols, gossip-based averaging techniques were also commonly utilized in Nash equilibrium computations, such as [99–103]. Specifically, a gossip-based distributed algorithm with a diminishing step-size sequence was developed for aggregate games in [99], while the convergence to the Nash equilibrium was studied for both diminishing and constant step-sizes in [100, 101], respectively. In [102], the gossip-based algorithm is applied to a network graphical game where the undirected communication graph is a subset

of the undirected interference graph. This work was extended in [103], where both the communication and interference graphs are directed. In general, the aforementioned works are model-based approaches, *i.e.*, the implementation of the algorithms relies on the knowledge of the explicit form of the players' cost functions, such as the derivative computation.

There are non-model based approaches, which utilize the players' local measurements without the requirement on the information of the functional form. For example, the work in [104] considered a generalized convex game with both convex coupling inequality constraints and local set constraints. A finite-differencing method with two-way perturbations was proposed to approximate the partial gradient. The perturbation parameter needs to be chosen carefully to match the selected step-size. Different from that, the work in [105] proposed a distributed payoff-based algorithm for a class of convex games with and without coupling constraints. This technique was further extended in [106] where the algorithm convergence was proved under mere monotonicity assumption. Overall, the payoff-based learning strategy proposed in these two works enables players to sample their actions in a Gaussian distribution. Then, the mean of this distribution is iteratively updated using only local payoff values. Another typical non-model based approaches are extremum seeking-based methods, such as [107–112]. Specifically, the work in [107] proposed a continuous time multi-input stochastic extremum seeking algorithm for the Nash equilibrium seeking in non-cooperative games with general nonlinear cost functions. In [108], a discrete time stochastic extremum seeking method was presented in non-cooperative games where the players' cost functions are strictly convex, but the actions are subject to a linear dynamic constraint. The work in [109] developed an integrator-type extremum seeking algorithm in non-cooperative games with both quadratic payoffs and general non-quadratic payoffs as the output of a dynamic system. More extremum seeking algorithms have been proposed in potential games with unstable dynamics [110], dynamical constraints [111], and non-cooperative games with time-varying Nash equilibrium [57]. In general, the extremum seeking strategy makes use

of the cost value together with some sinusoidal dither signals for perturbation, such that the gradient of the cost function is extractable. The following table summarizes a review of the aforementioned non-model based NE seeking algorithms.

Table 1.3: Review of non-model based NE seeking algorithms

Reference	Game	Algorithm	System	Smoothness
[104]	GNEP	finite differencing	discrete time	non-smooth
[105]	GNEP	payoff-based	discrete time	smooth
[106]	NEP	payoff-based	discrete time	smooth
[107]	NEP	extremum seeking	continuous time	smooth
[108]	dynamics	extremum seeking	discrete time	smooth
[109]	dynamics	extremum seeking	continuous time	smooth
[110]	potential	extremum seeking	continuous time	smooth
[111]	dynamics	extremum seeking	continuous time	smooth
[57]	NEP	extremum seeking	continuous time	smooth
Chapter 6	NEP	RGF-DNES	discrete time	non-smooth

1.3 Major Contributions of the Thesis

In this dissertation, gradient-free algorithms have been proposed to solve distributed optimization and NE seeking problems. We would like to clarify that our gradient-free algorithms belong to the category of zero-order methods (*i.e.*, only the zero-order information is used, such as the function value), as opposed to the first-order methods in the literature where the gradient, subgradient, or instances of stochastic gradient (first-order) information is directly accessible. The major contributions of this dissertation are summarized as below.

- A set constrained distributed optimization problem with possibly non-smooth cost functions is considered. In this problem, the explicit expressions of the cost functions are assumed to be unknown, but the measurements can be made by the local agents. A gradient-free distributed optimization algorithm based on the smoothing technique is proposed, in which an RGF oracle is constructed locally to estimate the gradient. Different from the doubly-stochastic

requirement, the weighting matrix used in the algorithm only needs to be row or column stochastic, which enables the algorithm to be implemented in any strongly connected and fixed digraphs. To justify the effectiveness of the proposed algorithm, both the theoretical analysis and numerical simulations are conducted. The algorithm is proved to approximately converge to the optimal solution with an optimal gap related to a smoothing parameter, and its convergence rate is also studied.

- We consider the same set constrained distributed optimization problem. To improve the convergence results, we propose a gradient-free distributed optimization algorithm, in which a pseudo-gradient operator is developed to estimate the gradient and an optimal averaging scheme is introduced at each time-step to achieve the convergence to the exact optimal solution. This algorithm requires the weighing matrix to be only row or column stochastic instead of doubly-stochastic, making it convenient to implement in any strongly connected and fixed digraphs. The optimal averaging scheme allows the algorithm to converge to the exact optimal solution without the square-summable condition on the step-size, which increases the range of the step-size selection. The convergence of the algorithm is rigorously proved, and its convergence rate is studied with respect to different cases of the step-size.
- Noticing the time-varying nature of many practical problems, we consider a set constrained distributed online optimization problem, where the possibly non-differentiable cost functions are allowed to vary with time. Moreover, the analytical expressions of the cost functions are unknown, but the values can be measured and only revealed to the agents in hindsight. A gradient-free distributed online optimization algorithm is presented. Due to the time-varying nature of the optimal solution trajectory, the notions of both static and dynamic regrets are adopted to analyze the performance of the algorithm. With some standard assumptions on the graph connectivity and cost functions, we are able to show that both regrets are sublinear functions of time, and the

corresponding averages over time are approximately convergent to zero with a rate of $O(1/\sqrt{T})$, which is comparable to the state-of-the-art algorithms.

- A multi-player non-cooperative game is considered, in which the players have limited knowledge on their local cost functions. Due to the incomplete information on the players' actions, an average consensus protocol is adopted to disseminate the information among the players. To seek its NE, a gradient-free distributed algorithm is proposed, which does not rely on the knowledge of the explicit form of the players' cost functions. With some mild assumptions on the graph connectivity, cost functions and the existence of NE, the convergence results of the algorithm are derived for both diminishing and constant step-sizes, respectively. In particular, an exact convergence to the NE is established for the diminishing case, while an approximate convergence to the NE with a gap proportional to the step-size is attained for the constant case.

1.4 Organization of the Thesis

In the rest of the dissertation, we introduce some definitions and preliminary results of graph theory, game theory, Gaussian smoothing and RGF techniques in Chapter 2. In Chapter 3, we consider a set constrained distributed optimization problem, where the explicit expressions of the cost functions are unknown to agents. To solve the problem, an RGF distributed projected gradient descent method is proposed. To improve the convergence results, we develop a distributed projected pseudo-gradient descent algorithm in Chapter 4 for the same problem settings. Towards this end, the distributed online optimization problem has not been addressed yet. Thus, in Chapter 5, we consider a distributed online optimization problem, where the agents' cost functions are time-varying. An online RGF distributed projected gradient descent algorithm is presented. Up to now, the algorithms for both distributed static and online optimization problems have been studied. In Chapter 6, we turn our focus to a multi-player non-cooperative game, where the players have

limited knowledge on their cost functions. A gradient-free distributed NE seeking algorithm is proposed. Detailed theoretical analysis and numerical simulations for all proposed algorithms are provided to verify the performance of the algorithms in the corresponding chapters. Chapter 7 concludes this dissertation and gives some recommendations for further research.

Chapter 2

Preliminaries

In this chapter, some definitions and preliminary results of graph theory, game theory, Gaussian smoothing and randomized gradient-free (RGF) techniques are introduced.

2.1 Graph Theory

Graph theory is commonly used to model the topological relations between agents in a multi-agent system. A general graph can be classified into two categories: undirected graphs, where the edges connecting agents are symmetric, and directed graphs, where the edges connecting agents are asymmetric. Throughout the dissertation, we consider the directed graph, whose formal definition is given as follows.

For a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, the set of agents is $\mathcal{V} = \{1, 2, \dots, N\}$, and the set of ordered pairs, (i, j) , $i, j \in \mathcal{V}$ is $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ implying that the information can be transferred from agent i to agent j . The in-neighbors (respectively, out-neighbors) of agent i are denoted by $\mathcal{N}_i^{\text{in}} = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ (respectively, $\mathcal{N}_i^{\text{out}} = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$). Specifically, the agent i is included in both $\mathcal{N}_i^{\text{in}}$ and $\mathcal{N}_i^{\text{out}}$, respectively. It should be noted that $\mathcal{N}_i^{\text{in}} \neq \mathcal{N}_i^{\text{out}}$ in general.

2.2 Game Theory

Game theory is a power tool of analyzing the strategic interactions between rational decision-makers. With the emergence of multi-agent system, the multi-player non-cooperative game has received increasing attention. This type of games involves a number of players, who selfishly minimize their own cost functions by making decisions in response to other players' actions. Mathematically, the multi-player non-cooperative game is defined as follows.

A game denoted by $\Gamma(N, \{f_i\}, \{\Omega_i\}, \mathcal{G})$ consists of N players who are communicating with each other under a directed graph $\mathcal{G}$. Each player $i \in \mathcal{V}$ owns a cost function $f_i : \Omega \rightarrow \mathbb{R}$, where $\Omega = \Omega_i \times \Omega_{-i} \subset \mathbb{R}^N$ is the action set of all players, and $\Omega_i \subset \mathbb{R}$ (respectively, $\Omega_{-i} \subset \mathbb{R}^{N-1}$) denotes the action set of player i (respectively, all players except player i). Let $\mathbf{x} = (x_i, \mathbf{x}_{-i}) \in \Omega$ be the vector of all players' actions, where $x_i \in \Omega_i$ (respectively, $\mathbf{x}_{-i} \in \Omega_{-i}$) represents the action of player i (respectively, all players except player i).

An important concept in game theory, Nash equilibrium [16], named after John Forbes Nash Jr., is a proposed solution, which is formally defined below.

Definition 2.1. For a game $\Gamma(N, \{f_i\}, \{\Omega_i\}, \mathcal{G})$, Nash equilibrium is an action profile where no player can reduce its cost by unilaterally changing its own action, i.e., a vector $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*) \in \Omega$ is called a Nash equilibrium of game $\Gamma(N, \{f_i\}, \{\Omega_i\}, \mathcal{G})$ if and only if

$$f_i(x_i^*, \mathbf{x}_{-i}^*) \leq f_i(x_i, \mathbf{x}_{-i}^*), \quad \forall x_i \in \Omega_i, \quad \forall i \in \mathcal{V}.$$

2.3 Gaussian Smoothing and RGF

Gaussian smoothing and RGF techniques were initially proposed in [61] to solve derivative-free optimization problems. Some preliminary results about these techn-

ques are summarized as follows.

For a function $f(\mathbf{x}) : \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, its Gaussian smoothed function is defined as

$$f_\mu(\mathbf{x}) = \frac{1}{\kappa} \int_{\mathbb{R}^n} f(\mathbf{x} + \mu\xi) e^{-\frac{1}{2}\|\xi\|^2} d\xi,$$

with $\kappa = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|\xi\|^2} d\xi = (2\pi)^{n/2}$ and $\mu \geq 0$ is a smoothing parameter of function $f_\mu(\mathbf{x})$.

The RGF oracle of $f(\mathbf{x})$ can be designed by

$$g_\mu(\mathbf{x}) = \frac{f(\mathbf{x} + \mu\xi) - f(\mathbf{x})}{\mu} \xi,$$

where $\xi \in \mathbb{R}^n$ is uniformly generated over a unit ball.

Some basic properties of $f_\mu(\mathbf{x})$ and $g_\mu(\mathbf{x})$ are summarized in the following lemma.

Lemma 2.1. (see [61]) *Suppose that $f(\mathbf{x})$ is convex but not necessarily differentiable, its subgradient $\partial f(\mathbf{x})$ is bounded by a positive constant $\hat{D}$. Then, we have*

1) $f_\mu(\mathbf{x})$ is convex and differentiable, and it satisfies

$$f(\mathbf{x}) \leq f_\mu(\mathbf{x}) \leq f(\mathbf{x}) + \sqrt{n\mu}\hat{D};$$

2) the gradient $\nabla f_\mu(\mathbf{x})$ satisfies

$$\mathbb{E}[g_\mu(\mathbf{x})] = \nabla f_\mu(\mathbf{x});$$

3) the RGF oracle $g_\mu(\mathbf{x})$ satisfies

$$\mathbb{E}[\|g_\mu(\mathbf{x})\|] \leq \sqrt{\mathbb{E}[\|g_\mu(\mathbf{x})\|^2]} \leq (n+4)\hat{D},$$

where n is the dimension of $\mathbf{x}$.

The techniques mentioned above will be used in different problem settings in this dissertation. Therefore, the properties summarized in Lemma 2.1 will vary accordingly.

Chapter 3

Distributed Optimization via Randomized Gradient-Free Oracle

3.1 Introduction

As reviewed in Chapter 1, recent works on distributed optimization with a sum of local cost functions have been reported in [27–47] and the references therein. These works have extensively investigated the distributed optimization problems differing from each other in terms of constraints, communication topology, optimality conditions, *etc.*, and presented various optimization methods to achieve promising results.

The implementation of these methods require the agents to have direct access to the gradient or subgradient information of the local cost functions and the constraints. However, in some applications, the relation between the variables and the cost functions are unknown, the gradient information is not available for usage, or the derivative is not possible to determine. In such cases, the methods requiring no gradient information are of great demand to these applications. Recent studies on gradient-free optimization have been found in [60–65]. The work in [60] proposed

a stochastic gradient descent method, where strongly convex property for the cost function is assumed. This assumption is removed in [61], where the one-sided random gradient-free oracle was firstly developed. A centralized gradient-free method for an unconstrained optimization problem with different types of cost functions was discussed in details [61]. This work was extended to a distributed manner in [62, 63], where a standard subgradient method was used with the derivative part replaced by the oracles. The same idea was also applied to a push-sum algorithm in [65] for an unconstrained problem. In [64], the two-sided random gradient-free oracle was implemented and showed similar performance as its one-sided counterpart.

In this chapter, we introduce a randomized gradient-free method for a set constrained problem. As compared to related works in [62–65], instead of using a standard subgradient method, the methods in this chapter employ a state-of-the-art distributed gradient descent approach requiring no doubly-stochastic weighting matrix under a directed communication graph with one-sided randomized gradient-free strategy.

The main contributions of this chapter are summarized as follows.

- This chapter proposes a gradient-free distributed optimization method for a set constrained problem, where no explicit expressions but only local measurements of the cost functions are required, making it suitable for the applications where gradient computing is costly or not possible.
- In comparison with most distributed optimization methods where the doubly-stochastic matrix has been used (*e.g.*, [28, 29, 32, 33, 37, 39, 62–64]), the proposed method uses a row-stochastic matrix and a column stochastic matrix, making the algorithm convenient to implement for directed graphs. In fact, there is no guarantee for a directed graph to have its corresponding doubly-stochastic matrix. Moreover, finding a doubly-stochastic weight matrix for a directed graph is not a trivial task, especially for large communication network [113, 114]. As compared to the push-sum algorithms, *e.g.*, [65], the push-sum consensus involves variables in the denominator of the algorithm. This brings in the non-

linearity to the algorithm, and may cause the possible failure of the algorithm if the variables happen to be zero at some time-step.

- The detailed analysis on the convergence properties and the rate of convergence is also provided.

The rest of the chapter is organized as follows. Section 3.2 introduces the problem formulation. Section 3.3 presents the proposed randomized gradient-free optimization algorithm, followed by the detailed analysis on the convergence properties. Section 3.4 demonstrates the effectiveness of the algorithm through numerical simulations, followed by the conclusions in Section 3.5.

3.2 Problem Formulation

For a set constrained distributed optimization problem in a multi-agent system, the objective of the multi-agent system is to cooperatively solve the optimization problem defined in (3.1) with a set constraint $\mathcal{X} \subseteq \mathbb{R}^n$ as follows:

$$\min f(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}), \mathbf{x} \in \mathcal{X} \quad (3.1)$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex and closed set, f_i is a local cost function of agent i and $\mathbf{x} = [x_1, \dots, x_n]^T$ is a global decision vector. The explicit mathematical expression of f_i is unknown, but local agent i can measure the value of $f_i(\mathbf{x})$ by introducing the input $\mathbf{x}$ to the system f_i . The non-empty optimal solution set of (3.1) is denoted by $\mathcal{X}^*$ with optimal value $f^* = f(\mathbf{x}^*)$ for any $\mathbf{x}^* \in \mathcal{X}^*$.

We introduce a smoothed version of (3.1) to help us establish the gradient-free methods, given by

$$\min f_\mu(\mathbf{x}) = \sum_{i=1}^N f_{i,\mu^i}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X}, \quad (3.2)$$

where $f_{i,\mu^i}(\mathbf{x}) = \frac{1}{\kappa} \int_{\mathbb{R}^n} f_i(\mathbf{x} + \mu^i \xi) e^{-\frac{1}{2}\|\xi\|^2} d\xi$ is the Gaussian approximation of $f_i(\mathbf{x})$ [61, 62] with $\kappa = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|\xi\|^2} d\xi = (2\pi)^{n/2}$ and $\mu^i \geq 0$ is a smoothing parameter of function $f_{i,\mu^i}(\mathbf{x})$. The properties of the function $f_{i,\mu^i}(\mathbf{x})$ are presented in Lemma 3.1. Similarly, we denote the optimal solution set of problem (3.2) by $\mathcal{X}_\mu^*$ with optimal value $f_\mu^* = f_\mu(\mathbf{x}_\mu^*)$ for any $\mathbf{x}_\mu^* \in \mathcal{X}_\mu^*$.

Throughout this chapter, we suppose the following assumptions hold:

Assumption 3.1. *The directed graph is strongly connected.*

Assumption 3.2. *Each local cost function f_i is convex, but not necessarily differentiable. Its subgradient $\partial f_i(\mathbf{x})$ is bounded, i.e., $\forall \mathbf{x} \in \mathcal{X}$, there exists a positive constant $\hat{D}$ such that $\|\partial f_i(\mathbf{x})\| \leq \hat{D}$.*

3.3 Main Results

In this section, we will present the randomized gradient-free method for the optimization problem defined in (3.1), followed by the convergence analysis.

3.3.1 Randomized Gradient-Free Distributed Projected Gradient Descent (DPGD)

Motivated by the distributed gradient descent algorithm in [27, 35], we develop a randomized gradient-free distributed projected gradient descent (DPGD) method to solve the problem defined in (3.1), which is described as follows.

At the k -th step, each agent j delivers its state information $\mathbf{x}_k^j$ with a weighted auxiliary variable $[A_c]_{ij} \mathbf{y}_k^j$ to its out-neighbor $i \in \mathcal{N}_j^{\text{out}}$. Then, agent i updates its variables $\mathbf{x}_{k+1}^i$ and $\mathbf{y}_{k+1}^i$ with the information received from its in-neighbor $j \in \mathcal{N}_i^{\text{in}}$

as follows

$$\mathbf{x}_{k+1}^i = \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j + \epsilon \mathbf{y}_k^i - \alpha_k g_{\mu^i}(\mathbf{x}_k^i) \right], \quad (3.3a)$$

$$\mathbf{y}_{k+1}^i = \mathbf{x}_k^i - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j + \sum_{j=1}^N [A_c]_{ij} \mathbf{y}_k^j - \epsilon \mathbf{y}_k^i, \quad (3.3b)$$

where $g_{\mu^i}(\mathbf{x}_k^i)$ is the random gradient-free oracle given as

$$g_{\mu^i}(\mathbf{x}_k^i) = \frac{f_i(\mathbf{x}_k^i + \mu^i \xi_k^i) - f_i(\mathbf{x}_k^i)}{\mu^i} \xi_k^i, \quad (3.4)$$

$\xi^i \in \mathbb{R}^n$ is uniformly generated over a unit ball, A_r, A_c are the row-stochastic and column-stochastic adjacency matrices, respectively, *i.e.*, $\sum_{j=1}^N [A_r]_{ij} = 1$ for all $i \in \mathcal{V}$, and $\sum_{i=1}^N [A_c]_{ij} = 1$ for all $j \in \mathcal{V}$. For any directed graphs, they can be obtained by letting $[A_r]_{ij} = 1/|\mathcal{N}_i^{\text{in}}|$ and $[A_c]_{ij} = 1/|\mathcal{N}_j^{\text{out}}|$. ϵ is a small positive number. $\alpha_k > 0$ is a diminishing step-size satisfying

$$\sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty. \quad (3.5)$$

The procedures are summarized in Algorithm 3.1.

Algorithm 3.1 Randomized gradient-free DPGD

- 1: **Initialize:** $i \in \mathcal{V}$
 randomly select $\mathbf{x}_0^i, \mathbf{y}_0^i \in \mathcal{X}$
 randomly generate $\{\xi_k^i\}_{k \geq 0}$ over a unit ball independently
 - 2: **Iteration** ($k \geq 0$): $i \in \mathcal{V}$
 compute $g_{\mu^i}(\mathbf{x}_k^i)$ based on (3.4)
 update variables $\mathbf{x}_{k+1}^i$ based on (3.3a)
 update variables $\mathbf{y}_{k+1}^i$ based on (3.3b)
 - 3: **Output:** $i \in \mathcal{V}$
 $\mathbf{x}_k^i \rightarrow \mathbf{x}^*$
-

Compared with the standard subgradient method (*e.g.*, [62–64]) where a doubly-stochastic weighting matrix is used, our proposed algorithm introduced two weighting matrices, aiming to establish the average consensus property for any arbitrary

strongly connected graphs. However, a direct replacement of the doubly-stochastic matrix by a row-stochastic (or column-stochastic) matrix in those algorithms will cause the shift in the average. Thus, an auxiliary variable $\mathbf{y}_k^i$ is employed to offset the shift caused by the unbalanced (non-doubly-stochastic) weighting structure. The function of this auxiliary variable is to record the state change of the associated agent at each time, known as ‘surplus’. These variables collectively maintain the information of the average shift amount. For an agent in a directed graph, it has both in-neighbors and out-neighbors. A doubly-stochastic matrix can balance the information exchange for both in-neighbors and out-neighbors at the same time, while a row-stochastic (or column-stochastic) matrix can only balance the information exchange for one side neighbors (*i.e.*, either in-neighbors or out-neighbors). Hence, a row-stochastic and a column-stochastic weighting matrices are introduced to characterize the communication with the in-neighbors and out-neighbors, respectively. The parameter ϵ is used to specify the amount of surplus to update the state, which is important to the convergence (see [115] for the details).

3.3.2 Convergence Analysis

In this part, we provide detailed analysis on the convergence properties of the proposed algorithm. We denote the σ -field generated by the entire history of the random variables from step 0 to $k - 1$ by $\mathcal{F}_k$, *i.e.*,

$$\mathcal{F}_k = \begin{cases} \{\mathbf{x}_0^i, i \in \mathcal{V}\}, & k = 0, \\ \{(\mathbf{x}_0^i, i \in \mathcal{V}); (\xi_s^i, i \in \mathcal{V}); 0 \leq s \leq k - 1\}, & k \geq 1. \end{cases}$$

Before proceeding to the main results, we first provide some properties of the function $f_{i,\mu^i}(\mathbf{x})$, which is a variation of Lemma 2.1 stated in Chapter 2.

Lemma 3.1. *(see [61]) Suppose Assumption 3.2 holds. For each $i \in \mathcal{V}$, the following properties of the function $f_{i,\mu^i}(\mathbf{x}_k^i)$ are satisfied:*

1) $f_{i,\mu^i}(\mathbf{x}_k^i)$ is convex and differentiable, and it satisfies

$$f_i(\mathbf{x}_k^i) \leq f_{i,\mu^i}(\mathbf{x}_k^i) \leq f_i(\mathbf{x}_k^i) + \sqrt{n}\mu^i\hat{D};$$

2) the gradient $\nabla f_{i,\mu^i}(\mathbf{x}_k^i)$ satisfies

$$\mathbb{E}[g_{\mu^i}(\mathbf{x}_k^i)|\mathcal{F}_k] = \nabla f_{i,\mu^i}(\mathbf{x}_k^i);$$

3) the randomized gradient-free oracle $g_{\mu^i}(\mathbf{x}_k^i)$ satisfies

$$\mathbb{E}[\|g_{\mu^i}(\mathbf{x}_k^i)\||\mathcal{F}_k] \leq \sqrt{\mathbb{E}[\|g_{\mu^i}(\mathbf{x}_k^i)\|^2|\mathcal{F}_k]} \leq (n+4)\hat{D},$$

where n is the dimension of $\mathbf{x}$.

For the convenience of analysis, we write (3.3) in a compact form as

$$\mathbf{z}_{k+1}^i = \sum_{j=1}^{2N} [A]_{ij} \mathbf{z}_k^j + \vartheta_k^i, \quad (3.6)$$

where

$$\mathbf{z}_k^i = \begin{cases} \mathbf{x}_k^i, & i \in \{1, \dots, N\}, \\ \mathbf{y}_k^{i-N}, & i \in \{N+1, \dots, 2N\}, \end{cases}$$

$$\vartheta_k^i = \begin{cases} \mathbf{x}_{k+1}^i - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j - \epsilon \mathbf{y}_k^i, & i \in \{1, \dots, N\}, \\ \mathbf{0}_n, & i \in \{N+1, \dots, 2N\}, \end{cases}$$

$$A = \begin{bmatrix} A_r & \epsilon I \\ I - A_r & A_c - \epsilon I \end{bmatrix}.$$

Define $\bar{\mathbf{z}}_k = \frac{1}{N} \sum_{i=1}^{2N} \mathbf{z}_k^i = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_k^i + \frac{1}{N} \sum_{i=1}^N \mathbf{y}_k^i$, which is the average of $\mathbf{x}_k^i + \mathbf{y}_k^i$ over all agents at time-step k .

Next, we study the convergence of the weighting matrix A defined in (3.6).

Lemma 3.2. (Lemma 1 in [35]) Suppose Assumption 3.1 holds. A is the weighting matrix defined in (3.6), and the constant ϵ in A satisfies $\epsilon \in (0, \bar{\epsilon})$ with $\bar{\epsilon} = \frac{1}{(20+8N)^N} (1 - |\lambda_3|)^N$, where λ_3 is the third largest eigenvalue of A by setting $\epsilon = 0$. Then $\forall i, j \in \{1, \dots, 2N\}$, the entries $[A^k]_{ij}$ converge to their limits as $k \rightarrow \infty$ at a geometric rate, i.e.,

$$\left\| A^k - \begin{bmatrix} \frac{\mathbf{1}_N \mathbf{1}_N^T}{N} & \frac{\mathbf{1}_N \mathbf{1}_N^T}{N} \\ 0 & 0 \end{bmatrix} \right\|_{\infty} \leq \Gamma \gamma^k, \quad k \geq 1,$$

where $\Gamma > 0$ and $0 < \gamma < 1$ are some constants.

Remark 3.1. For the upper bound parameter $\bar{\epsilon}$, it is to ensure the average consensus for any arbitrary strongly connected directed topologies, which makes it rather conservative, but is hard to avoid in order to estimate the perturbation bound with respect to general matrices according to the matrix perturbation literature, see [116]. Thus, to choose a proper ϵ in real implementation, one needs to know the value of its upper bound $\bar{\epsilon}$. This implies that the structure of the communication network has also to be known as it involves λ_3 , which is not possible in distributed settings. One may try to estimate the range of the bound $\bar{\epsilon} = (\frac{1-|\lambda_3|}{20+8N})^N \in (0, \frac{1}{(8N)^N})$ and select ϵ in this range. Even though this estimation may not be good enough to confirm the exact value of $\bar{\epsilon}$, it is still useful in terms of providing some reference in the selection of ϵ .

Then, we characterize the bounds of the consensus terms $\mathbf{x}_k^i - \bar{\mathbf{z}}_k$ and $\mathbf{y}_k^i - \mathbf{0}_n$.

Lemma 3.3. Suppose Assumptions 3.1 and 3.2 hold. Let $\{\mathbf{z}_k^i\}_{k \geq 0}$ be the sequence generated by (3.6). Then, it holds that

1) for $i = \{1, \dots, N\}$ and $k \geq 1$

$$\begin{aligned} \mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] &\leq 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| \\ &+ \Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}] + \sum_{j=1}^N \mathbb{E}[\|\vartheta_{k-1}^j\| | \mathcal{F}_{k-1}]; \end{aligned}$$

2) for $i = \{N + 1, \dots, 2N\}$ and $k \geq 1$

$$\begin{aligned} \mathbb{E}[\|\mathbf{z}_k^i\| | \mathcal{F}_{k-1}] &\leq 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| \\ &\quad + \Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}], \end{aligned}$$

where $\Gamma > 0$ and $0 < \gamma < 1$ are some constants.

Proof: For $k \geq 1$, we have

$$\mathbf{z}_k^i = \sum_{j=1}^{2N} [A^k]_{ij} \mathbf{z}_0^j + \sum_{r=1}^{k-1} \sum_{j=1}^{2N} [A^{k-r}]_{ij} \vartheta_{r-1}^j + \vartheta_{k-1}^i.$$

by applying (3.6) recursively. Then we can obtain that

$$\bar{\mathbf{z}}_k = \frac{1}{N} \sum_{j=1}^{2N} \mathbf{z}_0^j + \frac{1}{N} \sum_{r=1}^{k-1} \sum_{j=1}^{2N} \vartheta_{r-1}^j + \frac{1}{N} \sum_{j=1}^{2N} \vartheta_{k-1}^j,$$

where we have used column-stochastic property of A , i.e., for $k \geq 1$, it holds that $\sum_{i=1}^{2N} [A^k]_{ij} = 1$.

For part 1), subtracting $\bar{\mathbf{z}}_k$ from $\mathbf{z}_k^i$ and taking the norm and conditional expectation on $\mathcal{F}_\ell$ from $\ell = 0$ to $k - 1$, we have that for $1 \leq i \leq N$ and $k \geq 1$,

$$\begin{aligned} \mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] &\leq \sum_{j=1}^{2N} \left\| [A^k]_{ij} - \frac{1}{N} \right\| \max_j \|\mathbf{z}_0^j\| \\ &\quad + \sum_{r=1}^{k-1} \sum_{j=1}^N \left\| [A^{k-r}]_{ij} - \frac{1}{N} \right\| \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}] \\ &\quad + \frac{N-1}{N} \mathbb{E}[\|\vartheta_{k-1}^i\| | \mathcal{F}_{k-1}] + \frac{1}{N} \sum_{j \neq i} \mathbb{E}[\|\vartheta_{k-1}^j\| | \mathcal{F}_{k-1}]. \end{aligned} \tag{3.7}$$

Noting that the last two terms

$$\frac{N-1}{N} \mathbb{E}[\|\vartheta_{k-1}^i\| | \mathcal{F}_{k-1}] + \frac{1}{N} \sum_{j \neq i} \mathbb{E}[\|\vartheta_{k-1}^j\| | \mathcal{F}_{k-1}] \leq \sum_{j=1}^N \mathbb{E}[\|\vartheta_{k-1}^j\| | \mathcal{F}_{k-1}],$$

and applying Lemma 3.2 to (3.7), we obtain the result of part 1).

For part 2), taking the norm and conditional expectation on $\mathcal{F}_\ell$ from $\ell = 0$ to $k - 1$ for $\mathbf{z}_k^i$, $N + 1 \leq i \leq 2N$ and $k > 1$, we have

$$\mathbb{E}[\|\mathbf{z}_k^i\| | \mathcal{F}_{k-1}] \leq \sum_{j=1}^{2N} \|[A^k]_{ij}\| \max_j \|\mathbf{z}_0^j\| + \sum_{r=1}^{k-1} \sum_{j=1}^N \|[A^{k-r}]_{ij}\| \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{k-1}]. \quad (3.8)$$

Applying Lemma 3.2 to (3.8), we complete the proof with similar arguments to part 1). ■

Next, we quantify the bound for the augmented randomized gradient-free oracle ϑ_k^j defined in (3.6).

Lemma 3.4. *Suppose Assumptions 3.1 and 3.2 hold. Let ϵ be the constant such that $\epsilon \leq \min(\bar{\epsilon}, \frac{1-\gamma}{2N\Gamma\gamma})$ where $\bar{\epsilon} = \frac{1}{(20+8N)^N} (1 - |\lambda_3|)^N$, $\Gamma > 0$ and $0 < \gamma < 1$ are some constants. Then, there exists some bounded constant $G > 0$, such that for all $k \geq 0$, the augmented randomized gradient-free oracle ϑ_k^j satisfies*

$$\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] \leq G\alpha_k,$$

where α_k is the non-increasing step-size used in (3.3).

Proof: We first show the following two inequalities hold for any $K \geq 2$

$$\sum_{k=1}^K \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] \leq \Phi_1 \sum_{k=1}^K \alpha_k + \Psi_1, \quad (3.9)$$

$$\sum_{k=1}^K \alpha_k \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] \leq \Phi_2 \sum_{k=1}^K \alpha_k^2 + \Psi_2, \quad (3.10)$$

where Φ_1, Φ_2, Ψ_1 and Ψ_2 are some bounded positive constants.

By the definition of ϑ_k^i , we have

$$\begin{aligned} \|\vartheta_k^i\| &\leq \left\| \mathbf{x}_{k+1}^i - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j \right\| + \|\epsilon \mathbf{y}_k^i\| \leq \epsilon \|\mathbf{y}_k^i\| \\ &\quad + \|\epsilon \mathbf{y}_k^i - \alpha_k g_{\mu^i}(\mathbf{x}_k^i)\| \leq \alpha_k \|g_{\mu^i}(\mathbf{x}_k^i)\| + 2\epsilon \|\mathbf{y}_k^i\| \end{aligned} \quad (3.11)$$

where the second inequality follows the projection's non-expansive property. Taking the conditional expectation on $\mathcal{F}_\ell$ from $\ell = k-1$ to k in (3.11) and applying Lemma 3.1-3) on $\mathbb{E}[\|g_{\mu^i}(\mathbf{x}_k^i)\| | \mathcal{F}_k]$, Lemma 3.3-2) on $\mathbb{E}[\|\mathbf{y}_k^i\| | \mathcal{F}_{k-1}]$, we have

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k] &\leq 4N^2 \epsilon \Gamma \gamma^k \max_j \|\mathbf{z}_0^j\| \\ &\quad + (n+4)N\hat{D}\alpha_k + 2N\epsilon\Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}]. \end{aligned} \quad (3.12)$$

Summing (3.12) from $k=1$ to K , and noting that the last term

$$\sum_{k=1}^K \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}] \leq \frac{\gamma}{1-\gamma} \sum_{k=1}^K \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k],$$

we can obtain that

$$\begin{aligned} \sum_{k=1}^K \sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k] &\leq (n+4)N\hat{D} \sum_{k=1}^K \alpha_k \\ &\quad + \frac{4N^2\epsilon\Gamma\gamma}{1-\gamma} \max_j \|\mathbf{z}_0^j\| + \frac{2N\epsilon\Gamma\gamma}{1-\gamma} \sum_{k=1}^K \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]. \end{aligned}$$

Rearranging the terms and doing some simple algebra, we can obtain the result of (3.9).

Similarly, it follows from (3.12) that

$$\begin{aligned} \sum_{k=1}^K \alpha_k \sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k] &\leq (n+4)N\hat{D} \sum_{k=1}^K \alpha_k^2 + 4N^2\epsilon\Gamma \max_j \|\mathbf{z}_0^j\| \sum_{k=1}^K \gamma^k \alpha_k \\ &\quad + 2N\epsilon\Gamma \sum_{k=1}^K \alpha_k \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}]. \end{aligned} \quad (3.13)$$

Noting that the step-size is diminishing, the last term holds that

$$\begin{aligned} \sum_{k=1}^K \alpha_k \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}] &\leq \sum_{k=1}^K \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_{r-1} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}] \\ &\leq \frac{\gamma}{1-\gamma} \sum_{k=1}^K \alpha_k \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]. \end{aligned}$$

By applying [66, Lemma 3-1)] on the second term of (3.13), we can obtain the result of (3.10).

Next, we show the convergence of $\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]$. Taking the limsup on both sides of (3.12), and noting that

$$\limsup_{k \rightarrow \infty} \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|\vartheta_{r-1}^j\| | \mathcal{F}_{r-1}] \leq \frac{\gamma}{1-\gamma} \limsup_{k \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k],$$

and $\lim_{k \rightarrow \infty} \gamma^k = 0$ and $\lim_{k \rightarrow \infty} \alpha_k = 0$, we obtain

$$\left(1 - \frac{2N\epsilon\Gamma\gamma}{1-\gamma}\right) \limsup_{k \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] \leq 0.$$

Therefore, $\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]$ converges to 0 due to $\epsilon \leq \frac{1-\gamma}{2N\Gamma\gamma}$.

Lastly, we prove the desired result by contradiction. Suppose the conclusion is not true. Then, there exists some k , such that $(\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$. Since $\alpha_k \neq 0$, there are two cases where $(\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$. *Case 1*: at some finite k_f , $\sum_{j=1}^N \mathbb{E}[\|\vartheta_{k_f}^j\| | \mathcal{F}_{k_f}] = \infty$; *Case 2*: $(\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$ when k goes to infinity. Next, we show that both cases lead to contradiction.

Case 1: Suppose at some finite k_f , $\sum_{j=1}^N \mathbb{E}[\|\vartheta_{k_f}^j\| | \mathcal{F}_{k_f}] = \infty$. From (3.10), we have $\sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] \leq \Phi_2 \sum_{k=1}^{\infty} \alpha_k^2 + \Psi_2 < \infty$. Since $\alpha_k \neq 0$, every term $\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]$ is bounded, which contradicts to $\sum_{j=1}^N \mathbb{E}[\|\vartheta_{k_f}^j\| | \mathcal{F}_{k_f}] = \infty$.

Case 2: Suppose $(\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]) / \alpha_k = \infty$ when k goes to infinity. Since the term $\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k]$ is convergent, it implies that there exists some finite $K > 0$ such that for all $k \geq K$, we have $\sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] > 2\Phi_1 \alpha_k$. Since $\sum_{k=1}^{\infty} \alpha_k = \infty$, we have $\Phi_1 \sum_{k=1}^{K-1} \alpha_k + \Psi_1 < \Phi_1 \sum_{k=K}^{\infty} \alpha_k = \infty$. Hence, we obtain

$$\sum_{k=1}^{\infty} \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] > \sum_{k=K}^{\infty} \sum_{j=1}^N \mathbb{E}[\|\vartheta_k^j\| | \mathcal{F}_k] > 2\Phi_1 \sum_{k=K}^{\infty} \alpha_k > \Phi_1 \sum_{k=1}^{\infty} \alpha_k + \Psi_1,$$

which contradicts to (3.9).

Therefore, combining both cases, we complete the proof. ■

With the above lemmas, we are ready to establish the main results consisting of two theorems – one for consensus and the other for optimality.

Theorem 3.1. *Suppose Assumptions 3.1 and 3.2 hold. Let $\{\mathbf{z}_k^i\}_{k \geq 0}$ be the sequence generated by (3.6) with a diminishing step-size sequence $\{\alpha_k\}_{k \geq 0}$ satisfying (3.5). Then, $\mathbf{z}_k^i$ satisfies*

1) For $i = \{1, \dots, N\}$,

$$\lim_{k \rightarrow \infty} \mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\|] = 0.$$

2) For $i = \{N+1, \dots, 2N\}$,

$$\lim_{k \rightarrow \infty} \mathbb{E}[\|\mathbf{z}_k^i\|] = 0.$$

Proof: For part (1), applying Lemma 3.4 to the result in Lemma 3.3-1) and taking

the total expectation, we have

$$\mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\|] \leq 2N\Gamma\gamma^k \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G\Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_{r-1} + G\alpha_{k-1}. \quad (3.14)$$

Noting that $\lim_{k \rightarrow \infty} \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_{r-1} = \frac{\gamma}{1-\gamma} \lim_{k \rightarrow \infty} \alpha_k = 0$ and $\lim_{k \rightarrow \infty} \gamma^k = 0$ from the results in [66, Lemma 4], we complete the proof of part (1).

For part (2), applying Lemma 3.4 to the result in Lemma 3.3-2) and taking the total expectation, we obtain the desired result by the same arguments as in part (1). ■

Remark 3.2. *Theorem 3.1 characterizes the consensus property of the algorithm; namely, all agents $\mathbf{x}_k^i$ (respectively $\mathbf{y}_k^i$), $i \in \mathcal{V}$ will converge to the same point $\bar{\mathbf{z}}_k$ (respectively $\mathbf{0}_n$) asymptotically to achieve the exact convergence.*

Theorem 3.2. *Suppose Assumptions 3.1 and 3.2 hold. Let $\{\mathbf{z}_k^i\}_{k \geq 0}$ be the sequence generated by (3.6) with a diminishing step-size sequence $\{\alpha_k\}_{k \geq 0}$ satisfying (3.5). Then, we have*

$$\lim_{k \rightarrow \infty} \mathbb{E}[f(\bar{\mathbf{z}}_k)] - f^* \leq \sqrt{n}\hat{D} \sum_{i=1}^N \mu^i.$$

Proof: Based on (3.6) and the fact that A is column-stochastic, we have

$$\bar{\mathbf{z}}_{k+1} = \frac{1}{N} \sum_{j=1}^{2N} \left[\sum_{i=1}^{2N} [A]_{ij} \right] \mathbf{z}_k^j + \frac{1}{N} \sum_{i=1}^{2N} \vartheta_k^i = \bar{\mathbf{z}}_k + \frac{1}{N} \sum_{i=1}^N \vartheta_k^i.$$

Thus, for any $\mathbf{x}_\mu^* \in \mathcal{X}_\mu^*$, we can derive that

$$\begin{aligned} \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_\mu^*\|^2 &= \|\bar{\mathbf{z}}_k - \mathbf{x}_\mu^*\|^2 + \left\| \frac{1}{N} \sum_{i=1}^N \vartheta_k^i \right\|^2 + \frac{2}{N} \sum_{i=1}^N \langle \vartheta_k^i, \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle \\ &= \|\bar{\mathbf{z}}_k - \mathbf{x}_\mu^*\|^2 + \frac{1}{N^2} \left\| \sum_{i=1}^N \vartheta_k^i \right\|^2 \end{aligned} \quad (3.15a)$$

$$- \frac{2\alpha_k}{N} \sum_{i=1}^N \langle \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle \quad (3.15b)$$

$$+ \frac{2}{N} \sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle. \quad (3.15c)$$

For the second term in (3.15a), taking the conditional expectation on $\mathcal{F}_k$ yields

$$\begin{aligned} \mathbb{E}\left[\left\|\sum_{i=1}^N \vartheta_k^i\right\|^2 \middle| \mathcal{F}_k\right] &\leq \sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\|^2 | \mathcal{F}_k] \\ &= \sum_{i=1}^N (\mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k])^2 + \sum_{i=1}^N \text{Cov}(\|\vartheta_k^i\|, \|\vartheta_k^i\|) \\ &\leq \left(\sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k]\right)^2 + V_{1,k} \leq G^2 \alpha_k^2 + V_{1,k}, \end{aligned} \quad (3.16)$$

where we have applied Lemma 3.4 on $\sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k]$ and used $\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y] + \text{Cov}(x, y)$. $V_{1,k} > 0$ is an upper bound of the covariance term $\sum_{i=1}^N \text{Cov}(\|\vartheta_k^i\|, \|\vartheta_k^i\|)$.

For (3.15b), taking the conditional expectation on $\mathcal{F}_k$ and applying Lemma 3.1-2) gives $\sum_{i=1}^N \langle \mathbb{E}[g_{\mu^i}(\mathbf{x}_k^i) | \mathcal{F}_k], \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle = \sum_{i=1}^N \langle \nabla f_{i,\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle$. Noting that

$$\begin{aligned} \nabla f_{i,\mu^i}(\mathbf{x}_k^i)^T (\bar{\mathbf{z}}_k - \mathbf{x}_\mu^*) &= \nabla f_{i,\mu^i}(\mathbf{x}_k^i)^T (\bar{\mathbf{z}}_k - \mathbf{x}_k^i) + \nabla f_{i,\mu^i}(\mathbf{x}_k^i)^T (\mathbf{x}_k^i - \mathbf{x}_\mu^*) \\ &\geq -\|\nabla f_{i,\mu^i}(\mathbf{x}_k^i)\| \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + f_{i,\mu^i}(\mathbf{x}_k^i) - f_{i,\mu^i}(\mathbf{x}_\mu^*) \\ &\geq -\|\nabla f_{i,\mu^i}(\mathbf{x}_k^i)\| \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + f_{i,\mu^i}(\mathbf{x}_k^i) - f_{i,\mu^i}(\bar{\mathbf{z}}_k) + f_{i,\mu^i}(\bar{\mathbf{z}}_k) - f_{i,\mu^i}(\mathbf{x}_\mu^*) \\ &\geq -(\|\nabla f_{i,\mu^i}(\mathbf{x}_k^i)\| + \|\nabla f_{i,\mu^i}(\bar{\mathbf{z}}_k)\|) \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + f_{i,\mu^i}(\bar{\mathbf{z}}_k) - f_{i,\mu^i}(\mathbf{x}_\mu^*) \\ &\geq -2(n+4)\hat{D} \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + f_{i,\mu^i}(\bar{\mathbf{z}}_k) - f_{i,\mu^i}(\mathbf{x}_\mu^*), \end{aligned}$$

where $f_{i,\mu^i}(\mathbf{x}_k^i) - f_{i,\mu^i}(\bar{\mathbf{z}}_k) \geq \langle \nabla f_{i,\mu^i}(\bar{\mathbf{z}}_k), \mathbf{x}_k^i - \bar{\mathbf{z}}_k \rangle \geq -\|\nabla f_{i,\mu^i}(\bar{\mathbf{z}}_k)\| \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|$ by convexity of $f_{i,\mu^i}(\mathbf{x})$; $\|\nabla f_{i,\mu^i}(\mathbf{x}_k^i)\|$ and $\|\nabla f_{i,\mu^i}(\bar{\mathbf{z}}_k)\|$ can be bounded using Lemma 3.1-3), particularly, $\|\nabla f_{i,\mu^i}(\cdot)\| = \|\mathbb{E}[g_{\mu^i}(\cdot) | \mathcal{F}_k]\| \leq \mathbb{E}[\|g_{\mu^i}(\cdot)\| | \mathcal{F}_k] \leq (n+4)\hat{D}$. Thus, we obtain

$$\sum_{i=1}^N \langle \mathbb{E}[g_{\mu^i}(\mathbf{x}_k^i) | \mathcal{F}_k], \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle \geq -2(n+4)\hat{D} \sum_{i=1}^N \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + f_\mu(\bar{\mathbf{z}}_k) - f_\mu^*. \quad (3.17)$$

Noting that for (3.15c), we have

$$\sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle = \sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1} \rangle \quad (3.18a)$$

$$+ \sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i \rangle \quad (3.18b)$$

$$+ \sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \mathbf{x}_{k+1}^i - \mathbf{x}_{\mu}^* \rangle. \quad (3.18c)$$

For (3.18a), we have

$$\begin{aligned} \sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1} \rangle &\leq \sum_{i=1}^N \|\vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i)\| \left\| \frac{1}{N} \sum_{i=1}^N \vartheta_k^i \right\| \\ &\leq \frac{1}{N} \left(\sum_{i=1}^N \|\vartheta_k^i\| \right)^2 + \frac{\alpha_k}{N} \sum_{i=1}^N \|\vartheta_k^i\| \sum_{i=1}^N \|g_{\mu^i}(\mathbf{x}_k^i)\|. \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_k$ gives

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}[\langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1} \rangle | \mathcal{F}_k] &\leq \frac{1}{N} \left(\sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k] \right)^2 \\ &\quad + \frac{\alpha_k}{N} \sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k] \sum_{i=1}^N \mathbb{E}[\|g_{\mu^i}(\mathbf{x}_k^i)\| | \mathcal{F}_k] + V_{2,k} \\ &\leq \frac{1}{N} G(G + (n+4)N\hat{D})\alpha_k^2 + V_{2,k}, \end{aligned} \quad (3.19)$$

where we have applied the results from Lemma 3.1-3 on $\mathbb{E}[\|g_{\mu^i}(\mathbf{x}_k^i)\| | \mathcal{F}_k]$ and Lemma 3.4 on $\sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k]$. $V_{2,k} > 0$ is an upper bound of the sum of the covariance terms $Cov(\sum_{i=1}^N \|\vartheta_k^i\|, \sum_{i=1}^N \|\vartheta_k^i\|)$ and $Cov(\sum_{i=1}^N \|\vartheta_k^i\|, \sum_{i=1}^N \|g_{\mu^i}(\mathbf{x}_k^i)\|)$.

For (3.18b), we have

$$\begin{aligned} \sum_{i=1}^N \langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i \rangle &\leq \sum_{i=1}^N \|\vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i)\| \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| \\ &\leq \sum_{i=1}^N (\|\vartheta_k^i\| + \alpha_k \|g_{\mu^i}(\mathbf{x}_k^i)\|) \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|. \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_k$ yields

$$\begin{aligned} & \sum_{i=1}^N \mathbb{E}[\langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i \rangle | \mathcal{F}_k] \\ & \leq (G + (n+4)\hat{D})\alpha_k \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| | \mathcal{F}_k] + V_{3,k}, \end{aligned} \quad (3.20)$$

where we have applied the results from Lemma 3.1-3) on $\mathbb{E}[\|g_{\mu^i}(\mathbf{x}_k^i)\| | \mathcal{F}_k]$ and Lemma 3.4 on $\sum_{i=1}^N \mathbb{E}[\|\vartheta_k^i\| | \mathcal{F}_k]$. $V_{3,k} > 0$ is an upper bound of the sum of the covariance terms $Cov(\sum_{i=1}^N \|\vartheta_k^i\|, \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|)$ and $Cov(\sum_{i=1}^N \|g_{\mu^i}(\mathbf{x}_k^i)\|, \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|)$.

For (3.18c), it follows from the projection's non-expansive property that

$$\langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \mathbf{x}_{k+1}^i - \mathbf{x}_\mu^* \rangle \leq 0. \quad (3.21)$$

Thus, taking the conditional expectation on $\mathcal{F}_k$ in (3.18) and substituting (3.19), (3.20) and (3.21), we obtain

$$\begin{aligned} & \sum_{i=1}^N \mathbb{E}[\langle \vartheta_k^i + \alpha_k g_{\mu^i}(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_\mu^* \rangle | \mathcal{F}_k] \leq \frac{1}{N} G(G + (n+4)N\hat{D})\alpha_k^2 + V_{2,k} + V_{3,k} \\ & \quad + (G + (n+4)\hat{D})\alpha_k \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| | \mathcal{F}_k]. \end{aligned} \quad (3.22)$$

Taking the conditional expectation on $\mathcal{F}_k$ in (3.15), and substituting (3.16), (3.17) and (3.22), we obtain that

$$\begin{aligned} 2\alpha_k(f_\mu(\bar{\mathbf{z}}_k) - f_\mu^*) & \leq 4(n+4)\hat{D} \sum_{i=1}^N \alpha_k \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\ & \quad + N(\|\bar{\mathbf{z}}_k - \mathbf{x}_\mu^*\|^2 - \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_\mu^*\|^2 | \mathcal{F}_k]) + 2V_{2,k} + 2V_{3,k} \\ & \quad + \frac{1}{N}(G^2\alpha_k^2 + V_{1,k}) + \frac{2}{N}G(G + (n+4)N\hat{D})\alpha_k^2 \\ & \quad + 2(G + (n+4)\hat{D}) \sum_{i=1}^N \alpha_k \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| | \mathcal{F}_k]. \end{aligned} \quad (3.23)$$

Taking the total expectation in (3.23) and summing up from $k = 0$ to $t - 1$, we have

$$\begin{aligned}
\sum_{k=0}^{t-1} \alpha_k (\mathbb{E}[f_\mu(\bar{\mathbf{z}}_k)] - f_\mu^*) &\leq \frac{NB_1}{2} + \frac{\hat{V}_1}{2N} + \hat{V}_2 + \hat{V}_3 \\
&+ 2(n+4)\hat{D} \sum_{i=1}^N \sum_{k=0}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|] \\
&+ (G + (n+4)\hat{D}) \sum_{i=1}^N \sum_{k=0}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{x}_{k+1}^i - \bar{\mathbf{z}}_{k+1}\|] \\
&+ \frac{G}{2N} (3G + 2(n+4)N\hat{D}) \sum_{k=0}^{t-1} \alpha_k^2, \tag{3.24}
\end{aligned}$$

where $B_1 = \mathbb{E}[\|\bar{\mathbf{z}}_0 - \mathbf{x}_\mu^*\|^2]$. $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$ are the upper bounds of $\sum_{k=0}^\infty V_{1,k}$, $\sum_{k=0}^\infty V_{2,k}$ and $\sum_{k=0}^\infty V_{3,k}$, respectively.

Considering $\sum_{k=1}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\|]$ and $\sum_{k=0}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{z}_{k+1}^i - \bar{\mathbf{z}}_{k+1}\|]$ for $1 \leq i \leq N$, from (3.14) we have

$$\begin{aligned}
\sum_{k=1}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\|] &\leq 2N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \sum_{k=1}^{t-1} \alpha_k \gamma^k \\
&+ G\Gamma \sum_{k=1}^{t-1} \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_k \alpha_{r-1} + G \sum_{k=1}^{t-1} \alpha_k \alpha_{k-1}, \\
\sum_{k=0}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{z}_{k+1}^i - \bar{\mathbf{z}}_{k+1}\|] &\leq 2N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \sum_{k=0}^{t-1} \alpha_k \gamma^{k+1} \\
&+ G\Gamma \sum_{k=0}^{t-1} \sum_{r=1}^k \gamma^{k-r+1} \alpha_k \alpha_{r-1} + G \sum_{k=0}^{t-1} \alpha_k^2.
\end{aligned}$$

Following the results from [66, Lemma 3], we have

$$\begin{aligned}
\sum_{k=0}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|] &\leq N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \left(\frac{\gamma^2}{1-\gamma^2} \right) + B_2 \\
&+ \left(\sum_{k=0}^{t-1} \alpha_k^2 \right) \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G \left(1 + \frac{\Gamma\gamma}{1-\gamma} \right) \right], \tag{3.25}
\end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{t-1} \alpha_k \mathbb{E}[\|\mathbf{x}_{k+1}^i - \bar{\mathbf{z}}_{k+1}\|] &\leq N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \left(\frac{\gamma^2}{1-\gamma^2} \right) \\ &\quad + \left(\sum_{k=0}^{t-1} \alpha_k^2 \right) \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right], \end{aligned} \quad (3.26)$$

where we have applied the results from [66, Lemma 3] and let $B_2 = \max_i \alpha_0 \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|]$. Substituting (3.25) and (3.26) to (3.24), we have

$$\begin{aligned} \sum_{k=0}^{t-1} \alpha_k (\mathbb{E}[f_\mu(\bar{\mathbf{z}}_k)] - f_\mu^*) &\leq \sum_{k=0}^{t-1} \alpha_k^2 \left[\frac{G(3G + 2(n+4)N\hat{D})}{2N} \right. \\ &\quad \left. + (G + 3(n+4)\hat{D})N \left(N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right) \right] \\ &\quad + \left[(G + 3(n+4)\hat{D})N^2\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \left(\frac{\gamma^2}{1-\gamma^2} \right) \right. \\ &\quad \left. + 2(n+4)N\hat{D}B_2 + \frac{NB_1}{2} + \frac{\hat{V}_1}{2N} + \hat{V}_2 + \hat{V}_3 \right]. \end{aligned} \quad (3.27)$$

Taking the limit as $t \rightarrow \infty$ and the fact that $\mathbb{E}[f_\mu(\bar{\mathbf{z}}_k)] \geq f_\mu^*$, $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, we have

$$\liminf_{k \rightarrow \infty} \mathbb{E}[f_\mu(\bar{\mathbf{z}}_k)] = f_\mu^*. \quad (3.28)$$

Dividing both sides of (3.23) by N and dropping the non-negative term $\frac{2\alpha_k}{N}(f_\mu(\bar{\mathbf{z}}_k) - f_\mu^*)$, we have

$$\mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_\mu^*\|^2 | \mathcal{F}_k] \leq \|\bar{\mathbf{z}}_k - \mathbf{x}_\mu^*\|^2 + Z(k) \quad (3.29)$$

where $Z(k)$ denotes the sum of all remaining terms. Following the analysis in (3.25) and (3.26) and the fact that $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, we can show that $\sum_{k=0}^{\infty} Z(k) < \infty$. Applying [25, Lemma 9 in Ch. 2] to (3.29), we obtain that the sequence $\|\bar{\mathbf{z}}_k - \mathbf{x}_\mu^*\|$ converges almost surely for any $\mathbf{x}_\mu^* \in \mathcal{X}_\mu^*$. In view of (3.28) and the continuity of f_μ , there exists a convergent subsequence of $\bar{\mathbf{z}}_k$, denoted by $\bar{\mathbf{z}}_{k_1}$, such that its limit (denoted by $\bar{\mathbf{z}}^*$) exists and belongs to $\mathcal{X}_\mu^*$, *i.e.*, $\lim_{k_1 \rightarrow \infty} \bar{\mathbf{z}}_{k_1} = \bar{\mathbf{z}}^* \in \mathcal{X}_\mu^*$ with

probability 1. Now, letting $\mathbf{x}_\mu^* = \bar{\mathbf{z}}^*$ and considering the convergent sequence $\|\bar{\mathbf{z}}_k - \bar{\mathbf{z}}^*\|$, it has a subsequence $\|\bar{\mathbf{z}}_{k_1} - \bar{\mathbf{z}}^*\|$ converging to 0 almost surely, which implies that it should also converge to 0 almost surely, *i.e.*, $\lim_{k \rightarrow \infty} \|\bar{\mathbf{z}}_k - \bar{\mathbf{z}}^*\| = 0$ with probability 1. Thus, we have $\bar{\mathbf{z}}_k$ converges to its limit $\lim_{k \rightarrow \infty} \bar{\mathbf{z}}_k = \bar{\mathbf{z}}^*$ almost surely, which implies that $\lim_{k \rightarrow \infty} \mathbb{E}[f_\mu(\bar{\mathbf{z}}_k)] = f_\mu^*$.

According to Lemma 3.1-1), for any $\mathbf{x}^* \in \mathcal{X}^*$ and $\mathbf{x}_\mu^* \in \mathcal{X}_\mu^*$, it can be obtained that $f_\mu^* = f_\mu(\mathbf{x}_\mu^*) \leq f_\mu(\mathbf{x}^*) \leq f(\mathbf{x}^*) + \sqrt{n}\hat{D} \sum_{i=1}^N \mu^i = f^* + \sqrt{n}\hat{D} \sum_{i=1}^N \mu^i$. Together with the fact that $f(\bar{\mathbf{z}}_k) \leq f_\mu(\bar{\mathbf{z}}_k)$, we obtain the desired result. ■

Remark 3.3. *Theorem 3.2 shows that the cost value of the multiagent system will converge to its optimal value with an error bound due to the use of gradient-free oracle instead of the true gradient information, which can be moderated by smaller μ^i .*

3.3.3 Convergence Rate

In this part, we discuss the convergence rate of the proposed algorithm under different cases of step-size. As discussed in Theorem 3.2, the algorithm is guaranteed to achieve an inexact convergence to a neighborhood of the optimal value with a bounded error. Hence, we are referring the convergence rate as the rate of convergence to the neighborhood of the optimal solution instead of the exact optimal solution. The following theorem summarizes the main result.

Theorem 3.3. *Suppose Assumptions 3.1 and 3.2 hold. Let $\{\mathbf{z}_k^i\}_{k \geq 0}$ be the sequence generated by (3.6) with step-size sequence $\alpha_k = \frac{\alpha}{(k+2)^a}$, where $\alpha > 0$ is a constant and $a \in (0, 1)$. Then we have*

$$1) \text{ for } a \in (0, \frac{1}{2}), \quad f_t - f^* \leq C_0 + O\left(\frac{1}{t^a}\right);$$

$$2) \text{ for } a = \frac{1}{2}, \quad f_t - f^* \leq C_0 + O\left(\frac{\ln t}{\sqrt{t}}\right);$$

3) for $a \in (\frac{1}{2}, 1)$, $f_t - f^* \leq C_0 + O\left(\frac{1}{t^{1-a}}\right)$;

where C_0 is a constant.

Proof: The proof follows the flow of the convergence rate analysis in [35]. Since $f_t = \min_{0 < k \leq t} \mathbb{E}[f(\bar{\mathbf{z}}_k)]$, we can write (3.27) as the following inequality with some positive constants C_0, C_1 and C_2 by dividing both sides by $\sum_{k=0}^{t-1} \alpha_k$:

$$f_t - f^* \leq C_0 + \frac{C_1}{\sum_{k=0}^{t-1} \alpha_k} + \frac{C_2 \sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k}, \quad (3.30)$$

where

$$\begin{aligned} C_0 &= \sqrt{n} \hat{D} \sum_{i=1}^N \mu^i, \\ C_1 &= (G + 3(n+4)\hat{D}) N^2 \Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \left(\frac{\gamma^2}{1-\gamma^2} \right) \\ &\quad + 2(n+4)N\hat{D}B_2 + \frac{NB_1}{2} + \frac{\hat{V}_1}{2N} + \hat{V}_2 + \hat{V}_3, \\ C_2 &= (G + 3(n+4)\hat{D}) N \left(N \Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right) \\ &\quad + \frac{G}{2N} (3G + 2(n+4)N\hat{D}). \end{aligned}$$

For $a \in (0, \frac{1}{2})$, it follows from (3.30) that

$$\begin{aligned} f_t - f^* &\leq C_0 + \frac{C_1(1-a)}{\alpha[(t+2)^{1-a} - 2^{1-a}]} + \frac{C_2\alpha(1-a)}{1-2a} \frac{(t+1)^{1-2a} - 1}{(t+2)^{1-a} - 2^{1-a}} \\ &= C_0 + O\left(\frac{1}{t^a}\right). \end{aligned}$$

For $a = \frac{1}{2}$, by the same arguments, (3.30) leads to

$$\begin{aligned} f_t - f^* &\leq C_0 + \frac{C_1}{2\alpha[\sqrt{t+2} - \sqrt{2}]} + \frac{\alpha C_2 \ln(t+1)}{2(\sqrt{t+2} - \sqrt{2})} \\ &= C_0 + O\left(\frac{\ln t}{\sqrt{t}}\right). \end{aligned}$$

For $a \in (\frac{1}{2}, 1)$, by the same arguments, (3.30) leads to

$$\begin{aligned} f_t - f^* &\leq C_0 + \frac{C_1(1-a)}{\alpha[(t+2)^{1-a} - 2^{1-a}]} \\ &\quad + \frac{C_2\alpha(1-a)}{2a-1} \frac{1 - \frac{1}{(t+1)^{2a-1}}}{(t+2)^{1-a} - 2^{1-a}} = C_0 + O\left(\frac{1}{t^{1-a}}\right). \end{aligned}$$

■

Remark 3.4. As can be seen in Theorem 3.3, the rate of convergence to the neighborhood of the optimal value is dependent on the selection of the step-size. Compared with all three cases, we have the fastest rate of convergence $O(\ln t / \sqrt{t})$ by choosing $\alpha_k = \alpha / \sqrt{k+2}$, which is the same as most distributed gradient method with a diminishing step-size (e.g., [27, 28, 35, 38]).

3.4 Numerical Simulation

3.4.1 Numerical Example

In this part, we consider a canonical distributed estimation problem in a multi-agent system with N agents under a strongly connected directed graph:

$$\min f(x) = \sum_{i=1}^N (\|\phi_i - M_i \mathbf{x}\|^2 + \lambda_i \|\mathbf{x}\|^2), \quad \mathbf{x} \in \mathcal{X},$$

where each agent is assumed to measure certain unknown parameter $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ with Gaussian noise ω_i , i.e., $\phi_i = M_i \mathbf{x} + \omega_i$. $M_i \in \mathbb{R}^{m \times n}$ is the measurement matrix of agent i and $\phi_i \in \mathbb{R}^m$ is the measurement data collected by agent i . The proposed randomized gradient-free distributed projected gradient descent method will be used to solve this problem. In the following, the influence of the number of agents N , smoothing parameter μ^i and step-size α_k on the convergence property were investigated, followed by the comparison with the state-of-the-art methods.

Throughout the simulation, we set $\epsilon = 0.1$, $\mu^i = 10^{-2}$, $\mathcal{X} = [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ unless specified.

3.4.1.1 Influence of the Number of Agents N

The simulation was conducted by setting $\alpha_k = 0.1/\sqrt{k+1}$. The number of agents was set at $N = 50, 100, 500$ and 1000 . The convergence result of the total optimality gap $\sum_{i=1}^N \|\mathbf{x}_k^i - \mathbf{x}^*\|$ was plotted in Fig. 3.1. As can be seen, these lines are almost parallel with the only shift in the vertical axis. This implies that the convergence speed is the same regardless of the number of agents, but the time to reach convergence is longer when the number of agents is larger. In fact, the larger the number of agents is, the more discrepancies between agents we have in the iteration due to the larger problem size. The total sum of these errors to the optimal solution is larger, hence it takes longer time to converge to the optimal solution.

Figure 3.1: Influence of the number of agents on the optimality ($\mathbf{x}_k^i \rightarrow \mathbf{x}^*$)

3.4.1.2 Influence of Smoothing Parameter μ^i

The simulation was conducted by setting $\mu^i = 1, 10^{-2}, 10^{-4}$ and $1/(k+1)$. The step-size α_k was set to $0.1/\sqrt{k+1}$. Then, the optimality gap $\sum_{i=1}^N \|\mathbf{x}_k^i - \mathbf{x}^*\|$ under these four cases was plotted in Fig. 3.2. As can be seen, the optimality gap is slightly smaller for smaller μ^i due to the additional penalty term $\sqrt{n}\hat{D} \sum_{i=1}^N \mu^i$ caused by Gaussian-smoothing as discussed in Theorem 3.2. For diminishing smoothing parameter sequence (dotted line), the optimality gap is smaller than its constant case (blue line), which implies that changing the constant smoothing parameter to a di-

minishing sequence can improve the convergence result. However, since the smoothing parameter is small in general, the impact of the smoothing parameter on the convergence is not significant.

Figure 3.2: Influence of μ^i on the optimality ($\mathbf{x}_k^i \rightarrow \mathbf{x}^*$)

3.4.1.3 Influence of Step-Size α_k

The simulation was conducted by setting $\alpha_k = \alpha/(k+1)^a$, where $\alpha = 0.1$, $a = 0, 0.2, 0.5, 0.7$ and 1 . The results for the influence of different α_k on both the consensus property and the optimality property were plotted in Fig. 3.3 and 3.4. As can be seen from Fig. 3.3, with the increase of a , the consensus gap indicated by $\sum_{i=1}^N \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|$ decreases faster, which implies a smaller error at each fixed time-step. This result can also be seen from Fig. 3.4 where the line showing the optimality gap $\sum_{i=1}^N \|\mathbf{x}_k^i - \mathbf{x}^*\|$ is relatively smoother with less oscillations for larger a . Furthermore, it can also be observed from Fig. 3.4 that the yellow line ($a = 0.5$) decreases faster than the rest of the cases, which refers to the best rate of convergence given by the step-size $\alpha_k = \alpha/\sqrt{k+1}$ as deduced in Theorem 3.3.

Figure 3.3: Influence of α_k on the consensus ($\mathbf{x}_k^i \rightarrow \bar{\mathbf{z}}_k$)

Figure 3.4: Influence of α_k on the optimality ($\mathbf{x}_k^i \rightarrow \mathbf{x}^*$)

3.4.1.4 Comparison with the State-of-the-Art Algorithms

We compared our randomized gradient-free distributed projected gradient descent method (RGF-DPGD) with the randomized gradient-free subgradient method (RGF-SG) in [62] and the directed-distributed projected subgradient method (D-DPS) in [35]. We set the step-size $\alpha_k = 0.1/\sqrt{k+1}$. The adjacency matrices A_c, A_r for our method and D-DPS were set to be the same while the adjacency matrix P in RGF-SG was set to be doubly-stochastic. The convergence of $\mathbf{x}_k^i$ to their optimal solutions $\mathbf{x}^*$ was plotted in Fig. 3.5. As can be seen, the randomized gradient-free methods achieve similar performance. The gradient-available method achieves the best performance as expected, since it has full access to the gradient information at each iteration. Overall, the three methods exhibit the same convergence speed, but our method is gradient-free and simpler in terms of the implementation in directed graphs, which does not require a doubly-stochastic adjacency matrix.

Figure 3.5: Comparison between RGF-DPGD, RGF-SG & D-DPS methods.

3.4.2 Temperature Control in HVAC Systems

In this part, we consider a 10-zone HVAC system where the temperature control is formulated as the following distributed Model Predictive Control problem:

$$\begin{aligned} \min J = & \sum_{j=1}^{10} \left(\sum_{k=1}^K \lambda \|T_k^j - T_{ref}^j\| + \sum_{k=0}^{K-1} \|m_{s,k}^j\| \right), \quad m_L \leq m_{s,k}^j \leq m_U \\ T_{k+1}^j = & \left(1 - \frac{\Delta t}{C^j} \left(\frac{1}{R_a^j} + \sum_{i \in \mathcal{N}^j} \frac{1}{R_{ij}^j} \right) \right) T_k^j + \frac{\Delta t}{C^j} P_d^j \\ & + \frac{\Delta t}{C^j} c_p m_{s,k}^j (T_{s,k}^j - T_k^j) + \frac{\Delta t}{C^j} \left(\frac{T_a}{R_a^j} + \sum_{i \in \mathcal{N}^j} \frac{T_k^i}{R_{ij}^j} \right), \end{aligned}$$

where T^j and m_s^j are the temperature (degrees) and air mass flow (kg/s) of zone j , respectively. C^j and P_d^j are the lumped mass of air and lumped thermal load including all the external factors, such as the occupancy, solar radiation of zone j , respectively. R_a^j and R_{ij} are the thermal resistances between the air from outside and zone j , and the neighboring zone i and zone j , respectively. c_p is the specific heat capacity of air. Δt and K are the sampling time and receding horizon, respectively. λ is a weighting parameter. m_L and m_U are the lower and upper bounds of the input (air mass flow), respectively. The proposed algorithm is implemented in to solve the problem.

In this simulation, we set $m_L = 0$, $m_U = 3$ kg/s, the initial zone temperature $T_0^j = 30$ degrees, the reference temperature for all zones $T_{ref}^1 = T_{ref}^2 = 24$, $T_{ref}^3 = T_{ref}^4 = 24.5$, $T_{ref}^5 = T_{ref}^6 = 25$, $T_{ref}^7 = T_{ref}^8 = 25.5$, $T_{ref}^9 = T_{ref}^{10} = 26$. The rest of the model parameters were set according to [117]. Figures 3.6 and 3.7 plotted the trajectories of the air mass flow and the zone temperature for all zones. It can be seen that all the inputs are kept within the constraints, while all the zone temperature can be maintained at their corresponding reference levels. Hence, the effectiveness of the proposed algorithm is verified.

Figure 3.6: Air mass flow (kg/s).

Figure 3.7: Zone temperature (degrees).

3.5 Conclusions

This chapter has presented a randomized gradient-free distributed optimization algorithm for a set constrained problem where the explicit expressions of the cost functions are unknown. A gradient-free oracle in replace of the true gradient information has been developed to guide the update of decision variables. The convergence property and rate analysis of the algorithm have been provided. Numerical simulations have been conducted to verify the performance as compared to the state-of-the-art algorithms.

Chapter 4

Distributed Optimization via Pseudo-Gradient Operator

4.1 Introduction

Distributed optimization of a sum of cost functions have been extensively studied over decades, such as the work in [27–47]. A common underlying assumption in all these methods is that the derivative term of the local cost functions and the constraints can be directly accessed. However, there are many applications in the fields of bio-chemistry, aircraft design, hydro-dynamics, earth sciences, *etc.*, where the relation between the variables and the cost functions are unknown, the gradient information is not available for usage, or the derivative is not possible to determine, these methods are no longer applicable. Hence, researchers start to draw attention to the gradient-free optimization.

As reviewed in Chapter 1, recent studies on this topic have been reported in [60–67]. Shamir *et al.* in [60] investigated the performance of stochastic gradient descent method for non-smooth optimization problems. An averaging scheme was proposed to attain the minimax-optimal rates. On the other hand, Nesterov *et al.* in [61]

provided an explicit way of computing the stochastic gradient information known as gradient-free oracle and investigated the convergence property for both convex and non-convex problems. This idea was extended to minimize a sum of non-smooth but Lipschitz continuous functions in [62–64], where the Gaussian smoothing technique was introduced to obtain the gradient-free oracle to replace the derivative in the standard subgradient methods. However, all these methods are endowed with a doubly-stochastic weighting matrix requirement, which is restrictive in a distributed setting. The doubly-stochastic assumption was removed in [65] by adopting a push-sum technique and [66, 67] by introducing a surplus variable, respectively. However, all these derivative-free methods are based on the Gaussian smoothing technique, where the introduced smoothing parameter imposes an additional penalty term along reaching the convergence. Thus, only an approximate convergence to a neighborhood of the optimal solution can be achieved with an error bound that cannot be eliminated. To achieve the convergence to the exact optimal solution, Duchi *et al.* in [68] introduced a smoothing technique which used two point gradient estimation to close the optimality gap between the final iterate and the optimal point by choosing appropriate parameters. This technique was extended to the distributed scenario in [69], but restricted by the doubly-stochastic assumption on the weighting matrix. The work in [70] proposed a gradient-free method to achieve the exact convergence with only row- or column-stochastic weighting matrix. In fact, all these optimization methods need to rely on the square-summable condition on the step-size to establish the exact convergence property. The square-summable condition is a typical setting to achieve the convergence to the exact optimal solution in distributed optimization algorithms, but is generally not practical in the actual applications due to the slow convergence rate.

In this chapter, we propose an exact gradient-free method to solve the set constrained distributed optimization problem without the square-summable condition on the step-size. Specifically, a pseudo-gradient operator is proposed to estimate the gradient and an optimal averaging scheme is introduced at each time-step to

achieve the convergence to the exact optimal solution.

The main contributions of this chapter are summarized as follows.

- A gradient-free distributed protocol is proposed to solve a distributed optimization problem with possibly non-smooth cost functions, which requires no knowledge on their explicit forms but only the local measurements. Moreover, different from the consensus-based approaches in most existing distributed optimization literature, *e.g.*, [30, 32, 33, 39, 41, 43–45, 62–64], the proposed algorithm removes the doubly-stochastic requirement on the weighting matrix, and hence enables the implementation in any strongly connected and fixed digraphs. Furthermore, in contrast to most gradient-free approaches, *e.g.*, [61–67], where only an approximate convergence was achieved, the proposed algorithm is able to obtain the asymptotic convergence to the exact optimal solution, and hence improves the convergence results.
- We introduce an optimal averaging scheme locally, after the decision variable updates at each time-step. It seems that this averaging scheme is straightforward in terms of the implementation, but we shall highlight that it is non-trivial, especially in terms of the theoretical results. Most existing literature, *e.g.*, [27–29, 33–35, 39, 41–43, 64, 67, 70] needs to rely on the square-summable condition of the step-size to achieve the asymptotic convergence to the exact optimal solution. However, by introducing the averaging scheme, the proposed algorithm enables the step-size to take any positive, non-summable and non-increasing sequence, but still obtains the convergence to the exact optimal solution, which increases the range of the step-size selection.
- The detailed theoretical analysis and numerical simulations are provided to justify the convergence of the proposed algorithm. For the theoretical results, in addition to the standard convergence analysis, we also investigate the convergence rate of the proposed algorithm with respect to different cases of the step-size. For the numerical simulations, we illustrate the performance of the

proposed algorithm from different perspectives, including the influence on the convergence results with respect to different factors, such as the topology of the communication graph, the number of agents, and the step-size and parameters selections. The effectiveness of the proposed method is verified by comparing with the state-of-the-art algorithms including a gradient-free push-sum protocol and a subgradient-based algorithm.

The rest of the chapter is organized as follows. Some notations and the problem formulation are given in Section 4.2. Section 4.3 covers the main results where the proposed algorithm is described first, followed by the detailed analysis of convergence properties. The numerical simulations are presented in Section 4.4 to illustrate the performance of the algorithm. Section 4.5 concludes the chapter.

4.2 Problem Formulation

For a set constrained distributed optimization problem in a multi-agent system, the objective is to cooperatively solve the following optimization problem:

$$\min f(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X}, \quad (4.1)$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex and closed set, and f_i is a local cost function of agent i and $\mathbf{x} = [x_1, \dots, x_n]^\top$ is a global decision vector. The explicit expression of the local cost function f_i is unknown, but the measurements can be made by agent i only. The optimal solution of (4.1) is denoted by $\mathbf{x}^*$ with optimal value $f^* = f(\mathbf{x}^*)$.

Remark 4.1. *From practical point of view, the settings of different set constraints $\mathcal{X}_i$ for different agents will be more realistic and interesting. In terms of the theoretical analysis, whether the set constraints are the same or not does not have a significant influence, as long as the optimal solution set still lies in the intersection of these set constraints (i.e., $\mathbf{x}^* \in \mathcal{X} = \cap_i^N \mathcal{X}_i$). Hence, similar to the work in [34, 35, 42, 62–64,*

[67](#), [69](#), [70](#)], we consider the identical set constraint $\mathcal{X}$ for simplicity.

Throughout this chapter, we suppose the following assumptions hold:

Assumption 4.1. *The directed graph is strongly connected.*

Assumption 4.2. *Each local cost function f_i is convex, but not necessarily differentiable. Its subgradient $\partial f_i(\mathbf{x})$ is bounded, i.e., $\forall \mathbf{x} \in \mathcal{X}$, there exists a positive constant $\hat{D}$ such that $\|\partial f_i(\mathbf{x})\| \leq \hat{D}$.*

4.3 Main Results

In this section, we will develop the projected pseudo-gradient descent method for the optimization problem defined in (4.1), followed by the detailed analysis on the convergence.

4.3.1 Distributed Projected Pseudo-Gradient Descent

The proposed distributed projected pseudo-gradient descent method for solving the optimization problem defined in (4.1) is described in details as follows.

At time-step k , each agent j delivers its state information $\mathbf{x}_k^j$ with a weighted auxiliary variable $[A_c]_{ij}\mathbf{y}_k^j$ to its out-neighbor $i \in \mathcal{N}_j^{\text{out}}$. Then, agent i updates its variables $\mathbf{x}_{k+1}^i$ and $\mathbf{y}_{k+1}^i$ with the information received from its in-neighbor $j \in \mathcal{N}_i^{\text{in}}$. Finally, each agent i adopts an optimal averaging scheme to trace the average of $\mathbf{x}_\ell^i$, $\ell = 0, 1, \dots, k+1$ weighted by the step-size sequence, defined by $\hat{\mathbf{x}}_{k+1}^i$. The

updating law is given as follows.

$$\mathbf{x}_{k+1}^i = \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j + \epsilon \mathbf{y}_k^i - \alpha_k \mathbf{g}^i(\mathbf{x}_k^i) \right], \quad (4.2a)$$

$$\mathbf{y}_{k+1}^i = \mathbf{x}_k^i - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j + \sum_{j=1}^N [A_c]_{ij} \mathbf{y}_k^j - \epsilon \mathbf{y}_k^i, \quad (4.2b)$$

$$\widehat{\mathbf{x}}_{k+1}^i = \widehat{\mathbf{x}}_k^i + \frac{\alpha_{k+1}}{\sum_{\ell=0}^{k+1} \alpha_{\ell}} (\mathbf{x}_{k+1}^i - \widehat{\mathbf{x}}_k^i), \quad (4.2c)$$

where A_r, A_c are the row-stochastic and column-stochastic weighting matrices, respectively, *i.e.*, $\sum_{j=1}^N [A_r]_{ij} = 1$ for all $i \in \mathcal{V}$, and $\sum_{i=1}^N [A_c]_{ij} = 1$ for all $j \in \mathcal{V}$. For any directed graphs, they can be obtained by letting $[A_r]_{ij} = 1/|\mathcal{N}_i^{\text{in}}|$ and $[A_c]_{ij} = 1/|\mathcal{N}_j^{\text{out}}|$. $\alpha_k > 0$ is a non-increasing step-size satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$. ϵ is a small positive number. $\mathbf{g}^i(\mathbf{x}_k^i)$ is a pseudo-gradient motivated from [68], given as

$$\mathbf{g}^i(\mathbf{x}_k^i) = \frac{1}{\beta_{2,k}} [f_i(\mathbf{x}_k^i + \beta_{1,k} \xi_{1,k}^i + \beta_{2,k} \xi_{2,k}^i) - f_i(\mathbf{x}_k^i + \beta_{1,k} \xi_{1,k}^i)] \xi_{2,k}^i, \quad (4.3)$$

$\beta_{1,k}, \beta_{2,k}$ are two positive non-increasing sequences with their ratio defined as

$$\tilde{\beta}_k = \beta_{2,k} / \beta_{1,k}. \quad (4.4)$$

$\xi_{1,k}^i$ and $\xi_{2,k}^i \in \mathbb{R}^n$ are two random variables satisfying the following assumption:

Assumption 4.3. (*Assumption F in [68]*) The random variables $\xi_{1,k}^i$ and $\xi_{2,k}^i \in \mathbb{R}^n$ are generated by any one of the following: (a) both $\xi_{1,k}^i$ and $\xi_{2,k}^i$ are standard normal in $\mathbb{R}^n$ with identity covariance; (b) both $\xi_{1,k}^i$ and $\xi_{2,k}^i$ are uniform on the ℓ_2 -ball of radius $\sqrt{n+2}$; (c) the distribution of $\xi_{1,k}^i$ is uniform on the ℓ_2 -ball of radius $\sqrt{n+2}$ and the distribution of $\xi_{2,k}^i$ is uniform on the ℓ_2 -ball of radius $\sqrt{n}$.

Similar to the gradient-free oracle in [61], at each time k , the pseudo-gradient operator (4.3) estimates the gradient in a random direction $\xi_{2,k}^i$ with a parameter $\beta_{2,k}$, but the function difference is taken at a perturbed point $\mathbf{x}_k^i + \beta_{1,k} \xi_{1,k}^i$ instead of $\mathbf{x}_k^i$, where the amount of perturbation is determined by the parameter $\beta_{1,k}$ and the random

variable $\xi_{1,k}^i$. As compared to the gradient-free oracle where the function difference is evaluated at $\mathbf{x}_k^i$ which may not be differentiable for non-smooth problems, the extra perturbation step in pseudo-gradient operator allows the function difference to be evaluated at a point which is less likely to be non-smooth. In fact, we can define a smoothed function of $f_i(\mathbf{x})$ based on the convolution of this perturbation, given by [68]

$$f_{i,\beta_{1,k}}(\mathbf{x}) = \mathbb{E}[f(\mathbf{x} + \beta_{1,k}\xi_{1,k}^i)] = \int_{\mathbb{R}^n} f_i(\mathbf{x} + \beta_{1,k}\xi_{1,k}^i) d\mu(\xi_{1,k}^i),$$

with the random variable $\xi_{1,k}^i \in \mathbb{R}^n$ having density μ with respect to Lebesgue measure¹. $\beta_{1,k}$ is a positive non-increasing sequence. Then, some properties of function $f_{i,\beta_{1,k}}(\mathbf{x})$ and the pseudo-gradient $\mathbf{g}^i(\mathbf{x}_k^i)$ are summarized in the following lemma:

Lemma 4.1. (see [68]) *Suppose Assumptions 4.2 and 4.3 hold. Then, for each $i \in \mathcal{V}$, the following properties of the function $f_{i,\beta_{1,k}}(\mathbf{x})$ are satisfied:*

1) $f_{i,\beta_{1,k}}(\mathbf{x})$ is convex and differentiable, and it satisfies

$$f_i(\mathbf{x}) \leq f_{i,\beta_{1,k}}(\mathbf{x}) \leq f_i(\mathbf{x}) + \beta_{1,k}\hat{D}\sqrt{n+2},$$

2) the pseudo-gradient $\mathbf{g}^i(\mathbf{x}_k^i)$ satisfies

$$\mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i)|\mathcal{F}_k] = \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k\hat{D}\mathbf{v},$$

3) there is a universal constant Q such that

$$\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\||\mathcal{F}_k] \leq \sqrt{\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\|^2|\mathcal{F}_k]} \leq Q\mathcal{T}_k,$$

where $\beta_{1,k}$ and $\tilde{\beta}_k$ are defined in (4.4), $\mathbf{v} \in \mathbb{R}^n$ is a vector satisfying $\|\mathbf{v}\| \leq n\sqrt{3n}/2$,

¹Here, we slightly abuse the notation of ξ for both a random variable and its instances.

and $\mathcal{T}_k = \hat{D} \sqrt{n \left[n \sqrt{\tilde{\beta}_k} + 1 + \ln n \right]}$. If $\tilde{\beta}_k$ is bounded, then $\mathcal{T}_k$ is bounded by a constant $\hat{\mathcal{T}}$.

The procedures are summarized in Algorithm 4.1.

Algorithm 4.1 DPPGD with optimal averaging scheme

- 1: **Initialize:** $i \in \mathcal{V}$
 randomly select $\mathbf{x}_0^i, \mathbf{y}_0^i \in \mathcal{X}$, $\hat{\mathbf{x}}_0^i \leftarrow \mathbf{x}_0^i$
 randomly generate $\{\xi_{1,k}^i\}_{k \geq 0}, \{\xi_{2,k}^i\}_{k \geq 0}$ independently
 - 2: **Iteration** ($k \geq 0$): $i \in \mathcal{V}$
 compute $\mathbf{g}^i(\mathbf{x}_k^i)$ based on (4.3)
 update variables $\mathbf{x}_{k+1}^i$ based on (4.2a)
 update variables $\mathbf{y}_{k+1}^i$ based on (4.2b)
 update variables $\hat{\mathbf{x}}_{k+1}^i$ based on (4.2c)
 - 3: **Output:** $i \in \mathcal{V}$
 $\hat{\mathbf{x}}_k^i \rightarrow \mathbf{x}^*$
-

Remark 4.2. The proposed method is a gradient-free algorithm where a pseudo-gradient operator $\mathbf{g}^i(\mathbf{x}_k^i)$ is used instead of the true gradient $\nabla f_i(\mathbf{x}_k^i)$. The row-stochastic A_r and column-stochastic A_c instead of doubly-stochastic weighting matrix make it possible to be implemented in any strongly connected and fixed directed graphs.

4.3.2 Convergence Analysis

In this part, we proceed to the analysis on the convergence properties of the proposed algorithm. We denote the σ -field generated by the entire history of the random variables from step 0 to $k-1$ by $\mathcal{F}_k$, i.e., $\mathcal{F}_k = \{(\mathbf{x}_0^i, i \in \mathcal{V}); (\xi_{1,s}^i, \xi_{2,s}^i, i \in \mathcal{V}); 0 \leq s \leq k-1\}$ with $\mathcal{F}_0 = \{\mathbf{x}_0^i, i \in \mathcal{V}\}$.

Equation (4.2) can be written as the following equivalent form

$$\mathbf{z}_{k+1}^i = \sum_{j=1}^{2N} [A]_{ij} \mathbf{z}_k^j + g_k^i, \quad (4.5)$$

where $\mathbf{z}_k^i = \mathbf{x}_k^i$ for $i \in \{1, \dots, N\}$, $\mathbf{z}_k^i = \mathbf{y}_k^{i-N}$ for $i \in \{N+1, \dots, 2N\}$, $g_k^i = \mathbf{x}_{k+1}^i - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j - \epsilon \mathbf{y}_k^i$ for $i \in \{1, \dots, N\}$, $g_k^i = \mathbf{0}_n$ for $i \in \{N+1, \dots, 2N\}$, and $A = \begin{bmatrix} A_r & \epsilon I \\ I - A_r & A_c - \epsilon I \end{bmatrix}$. Define $\bar{\mathbf{z}}_k = \frac{1}{N} \sum_{i=1}^{2N} \mathbf{z}_k^i = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_k^i + \frac{1}{N} \sum_{i=1}^N \mathbf{y}_k^i$, which is an average of $\mathbf{x}_k^i + \mathbf{y}_k^i$ over all agents at time-step k ; and

$$\hat{\mathbf{z}}_k = \frac{\sum_{\ell=0}^k \alpha_\ell \bar{\mathbf{z}}_\ell}{\sum_{\ell=0}^k \alpha_\ell}, \quad (4.6)$$

which is an average of $\bar{\mathbf{z}}$ weighted by the step-size sequence α_ℓ over time duration k .

The following lemma states an important result on the weighting matrix A defined in (4.5).

Lemma 4.2. (Lemma 1 in [35]) Suppose Assumption 4.1 holds. Let ϵ be the constant in the weighting matrix A such that $\epsilon \in (0, \bar{\epsilon})$ with $\bar{\epsilon} = (\frac{1-|\lambda_3|}{20+8N})^N$, where λ_3 is the third largest eigenvalue of the weighting matrix A by setting $\epsilon = 0$. Then $\forall i, j \in \{1, \dots, 2N\}$, the entries $[A^k]_{ij}$ converge to their limits as $k \rightarrow \infty$ at a geometric rate, i.e.,

$$\left\| A^k - \begin{bmatrix} \frac{\mathbf{1}_N \mathbf{1}_N^T}{N} & \frac{\mathbf{1}_N \mathbf{1}_N^T}{N} \\ 0 & 0 \end{bmatrix} \right\|_\infty \leq \Gamma \gamma^k, \quad k \geq 1,$$

where $\Gamma > 0$ and $0 < \gamma < 1$ are some constants.

Now, we can quantify the bounds of the consensus terms $\mathbf{x}_k^i - \bar{\mathbf{z}}_k$ and $\mathbf{y}_k^i - \mathbf{0}_n$ as shown in the following lemma:

Lemma 4.3. Suppose Assumptions 4.1, 4.2 and 4.3 hold. Let ϵ be the constant such that $\epsilon \leq \bar{\epsilon}$, where $\bar{\epsilon}$ is defined in Lemma 4.2. Let $\{\mathbf{z}_k^i\}_{k \geq 0}$ be the sequence generated by (4.5). Then, it holds that

1) for $i \in \{1, \dots, N\}$ and $k \geq 1$

$$\mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] \leq 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| + \Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}]$$

$$+ \sum_{j=1}^N \mathbb{E}[\|g_{k-1}^j\| | \mathcal{F}_{k-1}];$$

2) for $i = \{N+1, \dots, 2N\}$ and $k \geq 1$

$$\mathbb{E}[\|\mathbf{z}_k^i\| | \mathcal{F}_{k-1}] \leq 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| + \Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}],$$

where $\Gamma > 0$ and $0 < \gamma < 1$ are the constants defined in Lemma 4.2.

Proof: For $k \geq 1$, we have

$$\mathbf{z}_k^i = \sum_{j=1}^{2N} [A^k]_{ij} \mathbf{z}_0^j + \sum_{r=1}^{k-1} \sum_{j=1}^{2N} [A^{k-r}]_{ij} g_{r-1}^j + g_{k-1}^i. \quad (4.7)$$

by applying (4.5) recursively. Then we can obtain that

$$\bar{\mathbf{z}}_k = \frac{1}{N} \sum_{j=1}^{2N} \mathbf{z}_0^j + \frac{1}{N} \sum_{r=1}^{k-1} \sum_{j=1}^{2N} g_{r-1}^j + \frac{1}{N} \sum_{j=1}^{2N} g_{k-1}^j, \quad (4.8)$$

where we have used column-stochastic property of A , i.e., for $k \geq 1$, it holds that $\sum_{i=1}^{2N} [A^k]_{ij} = 1$.

For part (1), subtracting (4.8) from (4.7) and taking the norm and conditional expectation on $\mathcal{F}_\ell$ from $\ell = 0$ to $k-1$, we have that for $1 \leq i \leq N$ and $k \geq 1$,

$$\begin{aligned} \mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] &\leq \sum_{j=1}^{2N} \left\| [A^k]_{ij} - \frac{1}{N} \right\| \max_j \|\mathbf{z}_0^j\| \\ &\quad + \sum_{r=1}^{k-1} \sum_{j=1}^N \left\| [A^{k-r}]_{ij} - \frac{1}{N} \right\| \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}] \\ &\quad + \frac{N-1}{N} \mathbb{E}[\|g_{k-1}^i\| | \mathcal{F}_{k-1}] + \frac{1}{N} \sum_{j \neq i} \mathbb{E}[\|g_{k-1}^j\| | \mathcal{F}_{k-1}]. \end{aligned} \quad (4.9)$$

Noting that the last two terms

$$\begin{aligned} & \frac{N-1}{N} \mathbb{E}[\|g_{k-1}^i\| | \mathcal{F}_{k-1}] + \frac{1}{N} \sum_{j \neq i} \mathbb{E}[\|g_{k-1}^j\| | \mathcal{F}_{k-1}] \\ & \leq \frac{N-1}{N} \sum_{i=1}^N \mathbb{E}[\|g_{k-1}^i\| | \mathcal{F}_{k-1}] + \frac{1}{N} \sum_{j=1}^N \mathbb{E}[\|g_{k-1}^j\| | \mathcal{F}_{k-1}] = \sum_{j=1}^N \mathbb{E}[\|g_{k-1}^j\| | \mathcal{F}_{k-1}], \end{aligned}$$

and applying the property of $[A^k]_{ij}$ from Lemma 4.2 to (4.9), we complete the proof of part (1).

For part (2), taking the norm and conditional expectation on $\mathcal{F}_\ell$ from $\ell = 0$ to $k-1$ in (4.7) for $N+1 \leq i \leq 2N$ and $k > 1$, we have

$$\mathbb{E}[\|\mathbf{z}_k^i\| | \mathcal{F}_{k-1}] \leq \sum_{j=1}^{2N} \|[A^k]_{ij}\| \max_j \|\mathbf{z}_0^j\| + \sum_{r=1}^{k-1} \sum_{j=1}^N \|[A^{k-r}]_{ij}\| \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{k-1}]. \quad (4.10)$$

By applying the property of $[A^k]_{ij}$ from Lemma 4.2 to (4.10), the result holds with similar arguments to part (1). $\blacksquare$

It can be seen from Lemma 4.3 that the bound for the consensus terms is a function of the combined pseudo-gradient term g_k^i . Thus, in the following lemma, we give the property on the boundedness of g_k^i .

Lemma 4.4. *Suppose Assumptions 4.1, 4.2 and 4.3 hold. Let ϵ be the constant such that $\epsilon \leq \min(\bar{\epsilon}, \frac{1-\gamma}{2N\Gamma\gamma})$, where $\bar{\epsilon}$, Γ and γ are the constants defined in Lemma 4.2. Let $\tilde{\beta}_k$ defined in (4.4) be bounded. Then, there exists a bounded constant $G > 0$, such that for all $k \geq 0$, the augmented randomized gradient-free oracle g_k^j satisfies*

$$\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k] \leq G\alpha_k,$$

where α_k is the non-increasing step-size used in the algorithm.

Proof: By the projection's non-expansive property, we have

$$\left\| \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j + \epsilon \mathbf{y}_k^i - \alpha_k \mathbf{g}^i(\mathbf{x}_k^i) \right] - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j \right\| \leq \|\epsilon \mathbf{y}_k^i - \alpha_k \mathbf{g}^i(\mathbf{x}_k^i)\|.$$

Thus,

$$\begin{aligned} \|g_k^i\| &\leq \|\epsilon \mathbf{y}_k^i\| + \left\| \mathbf{x}_{k+1}^i - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_k^j \right\| \leq \epsilon \|\mathbf{y}_k^i\| + \|\epsilon \mathbf{y}_k^i - \alpha_k \mathbf{g}^i(\mathbf{x}_k^i)\| \\ &\leq 2\epsilon \|\mathbf{y}_k^i\| + \alpha_k \|\mathbf{g}^i(\mathbf{x}_k^i)\|. \end{aligned} \quad (4.11)$$

where the first inequality comes from the definition of g_k^i in (4.5). Taking the conditional expectation on $\mathcal{F}_\ell$ from $\ell = k-1$ to k in (4.11) and applying Lemma 4.1-3) on $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\| | \mathcal{F}_k]$, Lemma 4.3-2) on $\mathbb{E}[\|\mathbf{y}_k^i\| | \mathcal{F}_{k-1}]$, we have

$$\mathbb{E}[\|g_k^i\| | \mathcal{F}_k] \leq \alpha_k Q \hat{\mathcal{T}} + 4N\epsilon\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| + 2\epsilon\Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}].$$

Summing over $i = 1, \dots, N$, we have

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k] &\leq \alpha_k N Q \hat{\mathcal{T}} + 4N^2\epsilon\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| \\ &\quad + 2N\epsilon\Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}]. \end{aligned} \quad (4.12)$$

Thus, it follows from (4.12) that

$$\begin{aligned} \sum_{k=1}^K \alpha_k \sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k] &\leq N Q \hat{\mathcal{T}} \sum_{k=1}^K \alpha_k^2 + 4N^2\epsilon\Gamma \max_j \|\mathbf{z}_0^j\| \sum_{k=1}^K \gamma^k \alpha_k \\ &\quad + 2N\epsilon\Gamma \sum_{k=1}^K \alpha_k \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}]. \end{aligned} \quad (4.13)$$

Noting that the step-size is non-increasing, the last term holds that

$$\begin{aligned} \sum_{k=1}^K \alpha_k \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}] &\leq \sum_{k=1}^K \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_{r-1} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}] \\ &\leq \frac{\gamma}{1-\gamma} \sum_{k=1}^K \alpha_k \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k]. \end{aligned}$$

Combining this result with [66, Lemma 3], it follows from (4.13) that

$$\begin{aligned} \sum_{k=1}^K \alpha_k \sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k] &\leq NQ\hat{\mathcal{T}} \sum_{k=1}^K \alpha_k^2 + 4N^2\epsilon\Gamma \max_j \|\mathbf{z}_0^j\| \left(\frac{1}{2} \sum_{k=1}^K \alpha_k^2 + \frac{\gamma^2}{2(1-\gamma^2)} \right) \\ &\quad + \frac{2N\epsilon\Gamma\gamma}{1-\gamma} \sum_{k=1}^K \alpha_k \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k]. \end{aligned}$$

Rearranging the last term to the left hand side and noticing that $\epsilon \leq \frac{1-\gamma}{2N\Gamma\gamma}$, we can obtain that

$$\sum_{k=1}^K \alpha_k \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k] \leq \Phi \sum_{k=1}^K \alpha_k^2 + \Psi, \quad (4.14)$$

where $\Phi > 0$ and Ψ are some bounded constants given by

$$\begin{aligned} \Phi &= \frac{(1-\gamma)(NQ\hat{\mathcal{T}} + 2N^2\epsilon\Gamma \max_j \|\mathbf{z}_0^j\|)}{1 - (2N\epsilon\Gamma + 1)\gamma}, \\ \Psi &= \frac{2N^2\epsilon\Gamma \max_j \|\mathbf{z}_0^j\| \gamma^2}{(1 - (2N\epsilon\Gamma + 1)\gamma)(1 + \gamma)}. \end{aligned}$$

Next, we prove the desired result by contradiction. Suppose the conclusion is not true, then there exists some k , such that $(\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$. Since $\alpha_k \neq 0$, there are two cases where $(\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$. *Case 1*: at some finite k_f , $\sum_{j=1}^N \mathbb{E}[\|g_{k_f}^j\| | \mathcal{F}_{k_f}] = \infty$; *Case 2*: $(\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$ when k goes to infinity. Next, we show that both cases lead to contradiction.

Case 1: Suppose at some finite k_f , $\sum_{j=1}^N \mathbb{E}[\|g_{k_f}^j\| | \mathcal{F}_{k_f}] = \infty$. Since k_f is finite, we

can always find a finite constant $K > k_f$. From (4.14), we have

$$\sum_{k=1}^K \alpha_k \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k] \leq \Phi \sum_{k=1}^K \alpha_k^2 + \Psi < \infty,$$

which implies every term $\alpha_k \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k]$, $k = 1, \dots, K$ is bounded. Since $\alpha_k \neq 0$, then every term $\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k]$ is bounded for $k = 1, \dots, K$, which contradicts to $\sum_{j=1}^N \mathbb{E}[\|g_{k_f}^j\| | \mathcal{F}_{k_f}] = \infty$.

Case 2: Suppose $(\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k])/\alpha_k = \infty$ when k goes to infinity. Taking the limsup on both sides of (4.12), and noting that

$$\limsup_{k \rightarrow \infty} \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}] \leq \frac{\gamma}{1-\gamma} \limsup_{k \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k],$$

$\lim_{k \rightarrow \infty} \gamma^k = 0$ and $\lim_{k \rightarrow \infty} \alpha_k = \alpha_\infty$, we obtain

$$\left(1 - \frac{2N\epsilon\Gamma\gamma}{1-\gamma}\right) \limsup_{k \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k] \leq NQ\hat{\mathcal{T}}\alpha_\infty.$$

Since $\epsilon \leq \frac{1-\gamma}{2N\Gamma\gamma}$, we have $\limsup_{k \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k] \leq \frac{NQ\hat{\mathcal{T}}}{1 - \frac{2N\epsilon\Gamma\gamma}{1-\gamma}} \alpha_\infty$. Hence,

$$\limsup_{k \rightarrow \infty} \frac{\sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k]}{\alpha_k} = \frac{\limsup_{k \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|g_k^j\| | \mathcal{F}_k]}{\alpha_\infty} \leq \frac{NQ\hat{\mathcal{T}}}{1 - \frac{2N\epsilon\Gamma\gamma}{1-\gamma}} < \infty,$$

which leads to a contradiction.

Therefore, combining both cases, we can conclude that the desired result is true. ■

With the above lemmas, we are ready to establish the main results consisting of two theorems – one for consensus and the other for optimality. We first show the boundedness of $\limsup_{k \rightarrow \infty} \mathbb{E}[\|\hat{\mathbf{x}}_k^i - \hat{\mathbf{z}}_k\|]$ for $i \in \mathcal{V}$ depending on the selection of the step-size, followed by the boundedness of $\limsup_{k \rightarrow \infty} \mathbb{E}[f(\hat{\mathbf{z}}_k)] - f^*$ as $k \rightarrow \infty$ in the following two theorems.

Theorem 4.1. Suppose Assumptions 4.1, 4.2 and 4.3 hold. Let $\{\hat{\mathbf{x}}_k^i\}_{k \geq 0}$ be the sequence generated by (4.2c) with a positive and non-increasing step-size sequence $\{\alpha_k\}_{k \geq 0}$ satisfying $\lim_{k \rightarrow \infty} \alpha_k = \alpha_\infty$. Let ϵ be the constant such that $\epsilon \leq \min(\bar{\epsilon}, \frac{1-\gamma}{2N\Gamma\gamma})$, where $\bar{\epsilon}$, Γ and γ are the constants defined in Lemma 4.2. Let $\tilde{\beta}_k$ defined in (4.4) be bounded. Then, $\hat{\mathbf{x}}_k^i$ satisfies

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\|\hat{\mathbf{x}}_k^i - \hat{\mathbf{z}}_k\|] \leq \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right] \alpha_\infty.$$

where $G > 0$ is the constant defined in Lemma 4.4, and $\hat{\mathbf{z}}_k$ is defined in (4.6).

Proof: We first provide a bound for $\mathbb{E}[\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}]$, $k \geq 1$. Applying Lemma 4.4 to the result in Lemma 4.3-1), we have

$$\mathbb{E}[\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] \leq 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| + G\Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_{r-1} + G\alpha_{k-1}. \quad (4.15)$$

By the definitions of $\hat{\mathbf{x}}_k^i$ and $\hat{\mathbf{z}}_k$, we know that

$$\|\hat{\mathbf{x}}_k^i - \hat{\mathbf{z}}_k\| \leq \frac{\sum_{\ell=0}^k \alpha_\ell \|\mathbf{x}_\ell^i - \bar{\mathbf{z}}_\ell\|}{\sum_{\ell=0}^k \alpha_\ell}.$$

For $k \geq 1$, taking conditional expectation on $\mathcal{F}_{\ell-1}$ from $\ell = 1$ to k , and applying (4.15), we obtain that

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{x}}_k^i - \hat{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] &\leq \frac{B_0 + \sum_{\ell=1}^k \alpha_\ell \mathbb{E}[\|\mathbf{x}_\ell^i - \bar{\mathbf{z}}_\ell\| | \mathcal{F}_{\ell-1}]}{\sum_{\ell=0}^k \alpha_\ell} \\ &\leq \frac{1}{\sum_{\ell=0}^k \alpha_\ell} \left(B_0 + 2N\Gamma \max_j \|\mathbf{z}_0^j\| \sum_{\ell=1}^k \alpha_\ell \gamma^\ell + G \sum_{\ell=1}^k \alpha_\ell \alpha_{\ell-1} + G\Gamma \sum_{\ell=1}^k \sum_{r=1}^{\ell-1} \gamma^{\ell-r} \alpha_\ell \alpha_{r-1} \right), \end{aligned}$$

where $B_0 = \max_i \alpha_0 \|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|$ is bounded. Following the results in [66, Lemma 3], it can be obtained that

$$\mathbb{E}[\|\hat{\mathbf{x}}_k^i - \hat{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] \leq \frac{1}{\sum_{\ell=0}^k \alpha_\ell} \left[B_0 + 2N\Gamma \max_j \|\mathbf{z}_0^j\| \left(\frac{1}{2} \sum_{\ell=1}^k \alpha_\ell^2 + \frac{\gamma^2}{2(1-\gamma^2)} \right) \right]$$

$$\begin{aligned}
& + G \sum_{\ell=0}^k \alpha_{\ell}^2 + G\Gamma \left(\frac{\gamma}{1-\gamma} \sum_{\ell=1}^k \alpha_{\ell}^2 \right) \\
& \leq \frac{\sum_{\ell=0}^k \alpha_{\ell}^2}{\sum_{\ell=0}^k \alpha_{\ell}} \left[N\Gamma \max_j \|\mathbf{z}_0^j\| + G \left(1 + \frac{\Gamma\gamma}{1-\gamma} \right) \right] \\
& \quad + \frac{1}{\sum_{\ell=0}^k \alpha_{\ell}} \left[\frac{N\Gamma\gamma^2}{1-\gamma^2} \max_j \|\mathbf{z}_0^j\| + B_0 \right].
\end{aligned}$$

Taking the total expectation and letting $k \rightarrow \infty$, we complete the proof by noting the result of $\lim_{k \rightarrow \infty} \sum_{\ell=0}^k \alpha_{\ell}^2 / \sum_{\ell=0}^k \alpha_{\ell} = \alpha_{\infty}$ from [66, Lemma 4]. $\blacksquare$

Remark 4.3. Theorem 4.1 characterizes the consensus property of the algorithm; namely, all agents $\hat{\mathbf{x}}_k^i, i \in \mathcal{V}$ will converge to the same point $\hat{\mathbf{z}}_k$ with an error bounded by a constant depending on the limit of the step-size α_{∞} . If $\alpha_{\infty} = 0$, then the exact convergence can be achieved.

Theorem 4.2. Suppose Assumptions 4.1, 4.2 and 4.3 hold. Let $\{\hat{\mathbf{x}}_k^i\}_{k \geq 0}$ be the sequence generated by (4.2c) with a positive and non-increasing step-size sequence $\{\alpha_k\}_{k \geq 0}$ satisfying $\lim_{k \rightarrow \infty} \alpha_k = \alpha_{\infty}$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$. Let ϵ be the constant such that $\epsilon \leq \min(\bar{\epsilon}, \frac{1-\gamma}{2N\Gamma\gamma})$, where $\bar{\epsilon}$, Γ and γ are the constants defined in Lemma 4.2. Let $\beta_{1,k}$ and $\tilde{\beta}_k$ defined in (4.4) satisfy $\lim_{k \rightarrow \infty} \beta_{1,k} = 0$ and $\sum_{k=0}^{\infty} \tilde{\beta}_k < \infty$. Then, we have

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \mathbb{E}[f(\hat{\mathbf{z}}_k)] - f^* & \leq \alpha_{\infty}(B_1 + B_2) \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right] + \alpha_{\infty}B_4 \\
& + \alpha_{\infty} \sum_{k=0}^{\infty} \tilde{\beta}_k \left[\frac{N\hat{D}\|\mathbf{v}\|\epsilon\gamma}{1-\gamma} + B_3 + 2N^2\hat{D}\|\mathbf{v}\| \left(N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right) \right],
\end{aligned}$$

where B_1, B_2, B_3, B_4 are some positive constants, $G > 0$ is the constant defined in Lemma 4.4, $\mathbf{v} \in \mathbb{R}^n$ is a vector satisfying $\|\mathbf{v}\| \leq n\sqrt{3n}/2$, and $\hat{\mathbf{z}}_k$ is defined in (4.6).

Proof: Considering (4.5), and due to the fact that A is column-stochastic, we have

$$\bar{\mathbf{z}}_{k+1} = \frac{1}{N} \sum_{j=1}^{2N} \left[\sum_{i=1}^{2N} [A]_{ij} \right] \mathbf{z}_k^j + \frac{1}{N} \sum_{i=1}^{2N} g_k^i = \bar{\mathbf{z}}_k + \frac{1}{N} \sum_{i=1}^N g_k^i.$$

Thus, for any $\mathbf{x}_{\beta_{1,k}}^* \in \mathcal{X}_{\beta_{1,k}}^*$, we can derive that

$$\begin{aligned} \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{\beta_{1,k}}^*\|^2 &= \|\bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^*\|^2 + \left\| \frac{1}{N} \sum_{i=1}^N g_k^i \right\|^2 + \frac{2}{N} \sum_{i=1}^N \langle g_k^i, \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle \\ &= \|\bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^*\|^2 + \frac{1}{N^2} \left\| \sum_{i=1}^N g_k^i \right\|^2 \end{aligned} \quad (4.16a)$$

$$- \frac{2\alpha_k}{N} \sum_{i=1}^N \langle \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle \quad (4.16b)$$

$$+ \frac{2}{N} \sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle. \quad (4.16c)$$

Noting that for the second term in (4.16a), we take conditional expectation on $\mathcal{F}_k$, yielding

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{i=1}^N g_k^i \right\|^2 \middle| \mathcal{F}_k \right] &\leq \sum_{i=1}^N \mathbb{E}[\|g_k^i\|^2 | \mathcal{F}_k] = \sum_{i=1}^N (\mathbb{E}[\|g_k^i\| | \mathcal{F}_k])^2 + \sum_{i=1}^N \text{Cov}(\|g_k^i\|, \|g_k^i\|) \\ &\leq \left(\sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k] \right)^2 + V_{1,k}, \end{aligned}$$

where we have applied Lemma 4.4 on $\sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k]$ and used $\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y] + \text{Cov}(x, y)$. $V_{1,k} > 0$ is an upper bound of the covariance term $\sum_{i=1}^N \text{Cov}(\|g_k^i\|, \|g_k^i\|)$. Thus, we obtain

$$\mathbb{E} \left[\left\| \sum_{i=1}^N g_k^i \right\|^2 \middle| \mathcal{F}_k \right] \leq G^2 \alpha_k^2 + V_{1,k}, \quad (4.17)$$

Noting that for (4.16b), we take conditional expectation on $\mathcal{F}_k$ and apply Lemma 4.1-(2)

$$\sum_{i=1}^N \langle \mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i) | \mathcal{F}_k], \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle = \sum_{i=1}^N \langle \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k \hat{D}\mathbf{v}, \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle. \quad (4.18)$$

Noting that

$$\langle \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k \hat{D}\mathbf{v}, \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle$$

$$\begin{aligned}
&= \langle \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k \hat{D} \mathbf{v}, \bar{\mathbf{z}}_k - \mathbf{x}_k^i \rangle + \langle \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k \hat{D} \mathbf{v}, \mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^* \rangle \\
&\geq - (\|\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i)\| + \tilde{\beta}_k \|\mathbf{v}\| \hat{D}) \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\
&\quad + f_{i,\beta_{1,k}}(\mathbf{x}_k^i) - f_{i,\beta_{1,k}}(\mathbf{x}_{\beta_{1,k}}^*) - \tilde{\beta}_k \hat{D} \|\mathbf{v}\| \|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\| \\
&\geq - (\|\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i)\| + \tilde{\beta}_k \|\mathbf{v}\| \hat{D}) \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + (f_{i,\beta_{1,k}}(\mathbf{x}_k^i) - f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k)) \\
&\quad + f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k) - f_{i,\beta_{1,k}}(\mathbf{x}_{\beta_{1,k}}^*) - \tilde{\beta}_k \hat{D} \|\mathbf{v}\| \|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\| \\
&\geq - (\|\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i)\| + \|\nabla f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k)\| + \tilde{\beta}_k \|\mathbf{v}\| \hat{D}) \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\
&\quad + (f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k) - f_{i,\beta_{1,k}}(\mathbf{x}_{\beta_{1,k}}^*)) - \tilde{\beta}_k \hat{D} \|\mathbf{v}\| \|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\| \\
&\geq - (2Q\hat{\mathcal{T}} + \tilde{\beta}_k \|\mathbf{v}\| \hat{D}) \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + (f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k) - f_{i,\beta_{1,k}}(\mathbf{x}_{\beta_{1,k}}^*)) \\
&\quad - \tilde{\beta}_k \hat{D} \|\mathbf{v}\| \|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\|,
\end{aligned}$$

where we have used

$$f_{i,\beta_{1,k}}(\mathbf{x}_k^i) - f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k) \geq -\|\nabla f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k)\| \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|$$

due to the convexity of $f_{i,\beta_{1,k}}(\cdot)$ and

$$\|\nabla f_{i,\mu^i}(\cdot)\| = \|\mathbb{E}[\mathbf{g}^i(\cdot) | \mathcal{F}_k]\| \leq \mathbb{E}[\|\mathbf{g}^i(\cdot)\| | \mathcal{F}_k]$$

with $\mathbb{E}[\|\mathbf{g}^i(\cdot)\| | \mathcal{F}_k]$ bounded by applying Lemma 4.1-(3). Thus, we have

$$\begin{aligned}
\langle \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k \hat{D} \mathbf{v}, \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle &\geq f_{i,\beta_{1,k}}(\bar{\mathbf{z}}_k) - f_{i,\beta_{1,k}}(\mathbf{x}_{\beta_{1,k}}^*) \\
&\quad - (2Q\hat{\mathcal{T}} + \tilde{\beta}_k \|\mathbf{v}\| \hat{D}) \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| - \tilde{\beta}_k \hat{D} \|\mathbf{v}\| \|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\|.
\end{aligned} \tag{4.19}$$

For the term $\|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\|$, we can provide the following bound:

$$\begin{aligned}
\|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\| &= \left\| \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^N [A_r]_{ij} \mathbf{x}_{k-1}^j + \epsilon \mathbf{y}_{k-1}^i - \alpha_{k-1} \mathbf{g}^i(\mathbf{x}_{k-1}^i) \right] - \mathbf{x}_{\beta_{1,k}}^* \right\| \\
&\leq \left\| \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_{k-1}^j + \epsilon \mathbf{y}_{k-1}^i - \alpha_{k-1} \mathbf{g}^i(\mathbf{x}_{k-1}^i) - \mathbf{x}_{\beta_{1,k}}^* \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \left\| \sum_{j=1}^N [A_r]_{ij} \mathbf{x}_{k-1}^j - \mathbf{x}_{\beta_{1,k}}^* \right\| + \epsilon \|\mathbf{y}_{k-1}^i\| + \alpha_{k-1} \|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\
&\leq \sum_{j=1}^N [A_r]_{ij} \|\mathbf{x}_{k-1}^j - \mathbf{x}_{\beta_{1,k}}^*\| + \epsilon \|\mathbf{y}_{k-1}^i\| + \alpha_{k-1} \|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\
&\leq \sum_{j=1}^N [A_r]_{ij} \|\mathbf{x}_{k-1}^i - \mathbf{x}_{\beta_{1,k}}^*\| + \epsilon \|\mathbf{y}_{k-1}^i\| + \alpha_{k-1} \|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\
&\quad + \sum_{j=1}^N [A_r]_{ij} \|\mathbf{x}_{k-1}^i - \mathbf{x}_{k-1}^j\| \\
&= \|\mathbf{x}_{k-1}^i - \mathbf{x}_{\beta_{1,k}}^*\| + \epsilon \|\mathbf{y}_{k-1}^i\| + \alpha_{k-1} \|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\
&\quad + \sum_{j=1}^N [A_r]_{ij} (\|\mathbf{x}_{k-1}^i - \bar{\mathbf{z}}_{k-1}\| + \|\mathbf{x}_{k-1}^j - \bar{\mathbf{z}}_{k-1}\|).
\end{aligned}$$

Thus, applying the above relation recursively, and taking conditional expectation on $\mathcal{F}_k$, we have

$$\begin{aligned}
\|\mathbf{x}_k^i - \mathbf{x}_{\beta_{1,k}}^*\| &= \epsilon \sum_{\tau=0}^{k-1} \|\mathbf{y}_\tau^i\| + \sum_{\tau=0}^{k-1} \alpha_\tau \mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_\tau^i)\| | \mathcal{F}_\tau] \\
&\quad + \sum_{\tau=0}^{k-1} \sum_{j=1}^N [A_r]_{ij} (\|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\| + \|\mathbf{x}_\tau^j - \bar{\mathbf{z}}_\tau\|) + \|\mathbf{x}_0^i - \mathbf{x}_{\beta_{1,k}}^*\|.
\end{aligned} \tag{4.20}$$

Combining (4.19) and (4.20), and substituting to (4.18), we obtain

$$\begin{aligned}
\sum_{i=1}^N \langle \mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i) | \mathcal{F}_k], \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle &\geq -(Q\hat{\mathcal{T}} + (\tilde{\beta}_k \|\mathbf{v}\| + 1)\hat{D}) \sum_{i=1}^N \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\
&\quad + f_{\beta_{1,k}}(\bar{\mathbf{z}}_k) - f_{\beta_{1,k}}^* - \tilde{\beta}_k \hat{D} \|\mathbf{v}\| \left[\sum_{i=1}^N \|\mathbf{x}_0^i - \mathbf{x}_{\beta_{1,k}}^*\| + \epsilon \sum_{\tau=0}^{k-1} \sum_{i=1}^N \|\mathbf{y}_\tau^i\| \right. \\
&\quad \left. + NQ\hat{\mathcal{T}} \sum_{\tau=0}^{k-1} \alpha_\tau + 2N \sum_{\tau=0}^{k-1} \sum_{i=1}^N \|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\| \right],
\end{aligned} \tag{4.21}$$

where we have applied Lemma 4.1-(3) on $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_\tau^i)\| | \mathcal{F}_\tau]$. Noting that for term

(4.16c), we have

$$\sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle = \sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1} \rangle \quad (4.22a)$$

$$+ \sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i \rangle \quad (4.22b)$$

$$+ \sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \mathbf{x}_{k+1}^i - \mathbf{x}_{\beta_{1,k}}^* \rangle. \quad (4.22c)$$

For (4.22a), we have

$$\begin{aligned} \sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1} \rangle &\leq \sum_{i=1}^N \|g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i)\| \left\| \frac{1}{N} \sum_{i=1}^N g_k^i \right\| \\ &\leq \frac{1}{N} \left(\sum_{i=1}^N \|g_k^i\| \right)^2 + \frac{\alpha_k}{N} \sum_{i=1}^N \|g_k^i\| \sum_{i=1}^N \|\mathbf{g}^i(\mathbf{x}_k^i)\|. \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_k$, we obtain

$$\sum_{i=1}^N \mathbb{E}[\langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1} \rangle | \mathcal{F}_k] \leq \frac{G}{N} (G + NQ\hat{\mathcal{T}}) \alpha_k^2 + V_{2,k}, \quad (4.23)$$

where we have applied Lemma 4.1-(3) on the term $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\| | \mathcal{F}_k]$ and Lemma 4.4 on the term $\sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k]$. $V_{2,k} > 0$ is an upper bound of the sum of covariance terms $Cov(\sum_{i=1}^N \|g_k^i\|, \sum_{i=1}^N \|g_k^i\|)$ and $Cov(\sum_{i=1}^N \|g_k^i\|, \sum_{i=1}^N \|\mathbf{g}^i(\mathbf{x}_k^i)\|)$.

For (4.22b), we have

$$\begin{aligned} \sum_{i=1}^N \langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i \rangle &\leq \sum_{i=1}^N \|g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i)\| \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| \\ &\leq \sum_{i=1}^N (\|g_k^i\| + \alpha_k \|\mathbf{g}^i(\mathbf{x}_k^i)\|) \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|. \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_k$, we obtain

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}[\langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i \rangle | \mathcal{F}_k] \\ \leq (G + Q\hat{\mathcal{T}}) \alpha_k \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| | \mathcal{F}_k] + V_{3,k}, \end{aligned} \quad (4.24)$$

where we have applied Lemma 4.1-(3) on the term $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\| | \mathcal{F}_k]$ and Lemma 4.4 on the term $\sum_{i=1}^N \mathbb{E}[\|g_k^i\| | \mathcal{F}_k]$. $V_{3,k} > 0$ is an upper bound of the sum of covariance terms $Cov(\sum_{i=1}^N \|g_k^i\|, \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|)$ and $Cov(\sum_{i=1}^N \|\mathbf{g}^i(\mathbf{x}_k^i)\|, \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|)$.

For (4.22c), it follows from the projection's non-expansive property that

$$\langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \mathbf{x}_{k+1}^i - \mathbf{x}_{\beta_{1,k}}^* \rangle \leq 0. \quad (4.25)$$

Thus, taking the conditional expectation on $\mathcal{F}_k$ in (4.22) and substituting (4.23), (4.24) and (4.25), we obtain

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}[\langle g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i), \bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^* \rangle | \mathcal{F}_k] &\leq \frac{G(G + NQ\hat{\mathcal{T}})\alpha_k^2}{N} \\ &+ (G + Q\hat{\mathcal{T}}) \sum_{i=1}^N \alpha_k \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\| | \mathcal{F}_k] + V_{2,k} + V_{3,k}. \end{aligned} \quad (4.26)$$

Taking the conditional expectation on $\mathcal{F}_k$ in (4.16), and substituting (4.17), (4.21)

and (4.26), we obtain that

$$\begin{aligned}
2\alpha_k(f_{\beta_{1,k}}(\bar{\mathbf{z}}_k) - f_{\beta_{1,k}}^*) &\leq 2(2Q\hat{\mathcal{T}} + \tilde{\beta}_k\|\mathbf{v}\|\hat{D}) \sum_{i=1}^N \alpha_k \|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\
&+ \frac{1}{N}(G^2\alpha_k^2 + V_{1,k}) + N(\|\bar{\mathbf{z}}_k - \mathbf{x}_{\beta_{1,k}}^*\|^2 - \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{\beta_{1,k}}^*\|^2|\mathcal{F}_k]) \\
&+ 2\tilde{\beta}_k\hat{D}\|\mathbf{v}\| \left[\alpha_k \sum_{i=1}^N \|\mathbf{x}_0^i - \mathbf{x}_{\beta_{1,k}}^*\| + \epsilon\alpha_k \sum_{\tau=0}^{k-1} \sum_{i=1}^N \|\mathbf{y}_\tau^i\| \right. \\
&\quad \left. + NQ\hat{\mathcal{T}}\alpha_k \sum_{\tau=0}^{k-1} \alpha_\tau + 2N\alpha_k \sum_{\tau=0}^{k-1} \sum_{i=1}^N \|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\| \right] \\
&+ \frac{2G}{N}(G + NQ\hat{\mathcal{T}})\alpha_k^2 \\
&+ 2(G + Q\hat{\mathcal{T}}) \sum_{i=1}^N \alpha_k \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\||\mathcal{F}_k] + 2V_{2,k} + 2V_{3,k}.
\end{aligned} \tag{4.27}$$

Taking the total expectation in (4.27) and summing up from $k = 0$ to $t - 1$, we have

$$\begin{aligned}
&\sum_{k=0}^{t-1} \alpha_k (\mathbb{E}[f(\bar{\mathbf{z}}_k)] - f_{\beta_{1,k}}(\mathbf{x}^*)) \\
&\leq \sum_{k=0}^{t-1} (Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}) \sum_{i=1}^N \alpha_k \mathbb{E}[\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|]
\end{aligned} \tag{4.28a}$$

$$+ \sum_{k=0}^{t-1} (G + Q\hat{\mathcal{T}}) \sum_{i=1}^N \alpha_k \mathbb{E}[\|\mathbf{x}_{k+1}^i - \bar{\mathbf{z}}_{k+1}\|] \tag{4.28b}$$

$$+ \hat{D}\|\mathbf{v}\| \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \sum_{i=1}^N \mathbb{E}[\|\mathbf{x}_0^i - \mathbf{x}^*\|] \tag{4.28c}$$

$$+ \hat{D}\|\mathbf{v}\|\epsilon \sum_{i=1}^N \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \sum_{\tau=0}^{k-1} \mathbb{E}[\|\mathbf{y}_\tau^i\|] \tag{4.28d}$$

$$+ NQ\hat{D}\|\mathbf{v}\|\hat{\mathcal{T}} \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \sum_{\tau=0}^{k-1} \alpha_\tau \tag{4.28e}$$

$$+ 2N\hat{D}\|\mathbf{v}\| \sum_{i=1}^N \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \sum_{\tau=0}^{k-1} \mathbb{E}[\|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\|] \tag{4.28f}$$

$$+ \frac{G}{2N} \sum_{k=0}^{t-1} (3G + 2NQ\hat{\mathcal{T}})\alpha_k^2 \tag{4.28g}$$

$$+ \frac{N}{2} \mathbb{E}[\|\bar{\mathbf{z}}_0 - \mathbf{x}^*\|^2] + \frac{\hat{V}_1}{N} + 2\hat{V}_2 + 2\hat{V}_3, \quad (4.28h)$$

where $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$ are positive constants representing the upper bounds of some covariance terms.

Next, we divide both sides of (4.28) by $\sum_{k=0}^{t-1} \alpha_k$ and taking the limit superior as $t \rightarrow \infty$,

$$\limsup_{t \rightarrow \infty} \frac{(4.28a)}{\sum_{k=0}^{t-1} \alpha_k} \leq \alpha_\infty B_1 \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right], \quad (4.29)$$

where we have combined Lemma 4.3-1) with (4.28a) and denoted $(Q\hat{\mathcal{T}} + (\tilde{\beta}_k \|\mathbf{v}\| + 1)\hat{D})N$ by B_1 .

$$\limsup_{t \rightarrow \infty} \frac{(4.28b)}{\sum_{k=0}^{t-1} \alpha_k} \leq \alpha_\infty B_2 \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right], \quad (4.30)$$

where we have combined Lemma 4.3-1) with (4.28b) and denoted $(G + Q\hat{\mathcal{T}})N$ by B_2 .

$$\limsup_{t \rightarrow \infty} \frac{(4.28c)}{\sum_{k=0}^{t-1} \alpha_k} \leq N\hat{D}\|\mathbf{v}\| \max_i \mathbb{E}[\|\mathbf{x}_0^i - \mathbf{x}^*\|] \tilde{\beta}_\infty, \quad (4.31)$$

where $\tilde{\beta}_\infty$ denotes the limit of $\tilde{\beta}_k$.

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{(4.28d)}{\sum_{k=0}^{t-1} \alpha_k} &\leq \frac{N^2 \hat{D} \|\mathbf{v}\| \epsilon}{1-\gamma} \left[(1-\gamma) \max_j \mathbb{E}[\|\mathbf{y}_0^j\|] \right. \\ &\quad \left. + 2\Gamma\gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \right] \tilde{\beta}_\infty + \frac{N\hat{D}\|\mathbf{v}\|\epsilon\gamma}{1-\gamma} \alpha_\infty \sum_{k=0}^{\infty} \tilde{\beta}_k, \end{aligned} \quad (4.32)$$

where we have combined Lemma 4.3-2) with (4.28d).

$$\limsup_{t \rightarrow \infty} \frac{(4.28e)}{\sum_{k=0}^{t-1} \alpha_k} \leq B_3 \alpha_\infty \sum_{k=0}^{\infty} \tilde{\beta}_k, \quad (4.33)$$

where $NQ\hat{D}\|\mathbf{v}\|\hat{\mathcal{T}}$ is denoted by B_3 .

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{(4.28f)}{\sum_{k=0}^{t-1} \alpha_k} &\leq 2N^2 \hat{D}\|\mathbf{v}\| \left[\max_i \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|] \tilde{\beta}_\infty \right. \\ &\quad \left. + \left(N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right) \alpha_\infty \sum_{k=0}^{\infty} \tilde{\beta}_k \right], \end{aligned} \quad (4.34)$$

where we have combined Lemma 4.3-1) with (4.28f).

$$\limsup_{t \rightarrow \infty} \frac{(4.28g)}{\sum_{k=0}^{t-1} \alpha_k} \leq B_4 \lim_{t \rightarrow \infty} \frac{\sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k} = B_4 \alpha_\infty, \quad (4.35)$$

where $\frac{G}{2N}(3G + 2NQ\hat{\mathcal{T}})$ is denoted by B_4 .

Then, substituting the results of (4.29)-(4.35) to (4.28), we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\sum_{k=0}^{t-1} \alpha_k \mathbb{E}[f(\bar{\mathbf{z}}_k)]}{\sum_{k=0}^{t-1} \alpha_k} &\leq \limsup_{t \rightarrow \infty} \frac{\sum_{k=0}^{t-1} \alpha_k f_{\beta_{1,k}}(\mathbf{x}^*)}{\sum_{k=0}^{t-1} \alpha_k} \\ &\quad + \alpha_\infty (B_1 + B_2) \left[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right] + \alpha_\infty B_4 \\ &\quad + N\hat{D}\|\mathbf{v}\| \tilde{\beta}_\infty \left[\max_i \mathbb{E}[\|\mathbf{x}_0^i - \mathbf{x}^*\|] + N\epsilon \max_j \mathbb{E}[\|\mathbf{y}_0^j\|] \right. \\ &\quad \left. + \frac{2N\epsilon\Gamma\gamma}{1-\gamma} \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + 2N \max_i \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|] \right] \\ &\quad + \alpha_\infty \sum_{k=0}^{\infty} \tilde{\beta}_k \left[\frac{N\hat{D}\|\mathbf{v}\|\epsilon\gamma}{1-\gamma} + B_3 + 2N^2 \hat{D}\|\mathbf{v}\| \left(N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G + \frac{G\Gamma\gamma}{1-\gamma} \right) \right], \end{aligned} \quad (4.36)$$

The detailed derivations can be found in our previous work [70, Theorem 2].

According to Lemma 4.1-1), it obtains that

$$f^* \leq \frac{\sum_{k=0}^{t-1} \alpha_k f_{\beta_{1,k}}(\mathbf{x}^*)}{\sum_{k=0}^{t-1} \alpha_k} \leq f^* + \frac{\hat{D}\sqrt{n+2} \sum_{k=0}^{t-1} \alpha_k \beta_{1,k}}{\sum_{k=0}^{t-1} \alpha_k}.$$

Taking the limit superior as $t \rightarrow \infty$, we have

$$\limsup_{t \rightarrow \infty} \frac{\sum_{k=0}^{t-1} \alpha_k f_{\beta_{1,k}}(\mathbf{x}^*)}{\sum_{k=0}^{t-1} \alpha_k} = f^*.$$

by noting that $\lim_{t \rightarrow \infty} \sum_{k=0}^{t-1} \alpha_k \beta_{1,k} / \sum_{k=0}^{t-1} \alpha_k = \lim_{k \rightarrow \infty} \beta_{1,k}$ from [66, Lemma 4-2)] and $\lim_{k \rightarrow \infty} \beta_{1,k} = 0$. Moreover,

$$\mathbb{E}[f(\widehat{\mathbf{z}}_t)] \leq \frac{\sum_{k=0}^{t-1} \alpha_k \mathbb{E}[f(\bar{\mathbf{z}}_k)]}{\sum_{k=0}^{t-1} \alpha_k},$$

due to the convexity of f . Therefore, the desired result directly follows from (4.36) by noting that $\lim_{k \rightarrow \infty} \tilde{\beta}_k = \tilde{\beta}_\infty = 0$. $\blacksquare$

Remark 4.4. Theorem 4.2 shows that the cost value of the multi-agent system will finally converge to a neighborhood of its optimal value with an error bounded by some terms, which are dependent on the step-size α_k and parameters $\beta_{1,k}, \beta_{2,k}$. Appropriate choice of the step-size and parameters will lead to exact convergence to the optimum. In particular, if the step-size α_k is set to $1/(k+1)^a$, where $a \in (0, 1)$; the parameters $\beta_{1,k}, \beta_{2,k}$ are set to $1/(k+1)^{p_1}$ and $1/(k+1)^{p_2}$, respectively, where $p_1 > 0$ and $p_2 - p_1 > 1$; then $\alpha_\infty = 0$ and $\sum_{k=0}^\infty \tilde{\beta}_k < \infty$, which means all the error terms will converge to 0.

Different from most subgradient methods where the square-summable step-size condition (i.e., $\sum_{k=0}^\infty \alpha_k^2 < \infty$) is utilized, the proposed algorithm is able to obtain the exact convergence result as in Theorem 4.2 without the square-summable step-size condition. It should be noted that the convergence of those algorithms are established with respect to the original decision variable $\mathbf{x}_k^i$ of each agent $i \in \mathcal{V}$, where the optimality is based on the results in convex optimization [25, Lemma 11 in Ch. 2]. Instead, this chapter proposed a scheme to average the decision variable along the time $0 \leq \ell \leq k$, and the convergence analysis is conducted with respect to the time-averaged decision variable $\widehat{\mathbf{x}}_k^i$ of each agent $i \in \mathcal{V}$, which helps remove the square-summable step-size condition. It is found that even though the exact convergence to the optimal solution cannot be guaranteed for the original decision variable $\mathbf{x}_k^i$, it can be guaranteed for the averaged decision variable $\widehat{\mathbf{x}}_k^i$. An intuitive explanation is as follows.

The feature of a non-summable but square-summable step-size is that it decreases

not too fast or too slow. It should decrease fast enough to avoid a very large update by the gradient term, which may cause over-correction; It cannot decrease too fast, which makes the update by the gradient term too small. For most subgradient algorithms without averaging scheme, if the step-size is not square-summable, the trajectory of the agent's decision variable will oscillate around the optimal solution since the step-size does not decrease fast enough to eliminate the overshoot. However, the proposed averaging scheme, which averages the decision variable along its trajectory, has an effect of neutralizing the overshoot along the trajectory. This effect is visualized in a simple convex optimization setup as follows.

Considering a simple convex optimization problem

$$\min f(x), \quad x \in \mathbb{R}^n.$$

The typical gradient-based optimization algorithms are usually in the following form

$$x_{k+1} = x_k - \alpha_k g_k,$$

where g_k is the gradient or subgradient of f at x_k . That gives

$$x_k - x^* = x_0 - x^* - (\alpha_0 g_0 + \alpha_1 g_1 + \dots + \alpha_{k-1} g_{k-1}). \quad (4.37)$$

By introducing our proposed optimal averaging scheme

$$\hat{x}_k = \frac{\sum_{\ell=0}^k \alpha_\ell x_\ell}{\sum_{\ell=0}^k \alpha_\ell},$$

we can obtain that

$$\begin{aligned} \hat{x}_k - x^* &= \frac{\alpha_0(x_0 - x^*) + \sum_{\ell=1}^k \alpha_\ell(x_\ell - x^*)}{\sum_{\ell=0}^k \alpha_\ell} = x_0 - x^* - \frac{\sum_{\ell=1}^k \alpha_\ell \sum_{r=0}^{\ell-1} \alpha_r g_r}{\sum_{\ell=0}^k \alpha_\ell} \\ &= x_0 - x^* - \frac{\alpha_1(\alpha_0 g_0) + \dots + \alpha_k(\alpha_0 g_0 + \alpha_1 g_1 + \dots + \alpha_{k-1} g_{k-1})}{\sum_{\ell=0}^k \alpha_\ell} \end{aligned}$$

$$\begin{aligned}
&= x_0 - x^* - \left[\left(\frac{\sum_{\ell=1}^k \alpha_\ell}{\sum_{\ell=0}^k \alpha_\ell} \right) \alpha_0 g_0 + \dots + \left(\frac{\alpha_k}{\sum_{\ell=0}^k \alpha_\ell} \right) \alpha_{k-1} g_{k-1} \right] \\
&= x_0 - x^* - (\eta_0^k \alpha_0 g_0 + \eta_1^k \alpha_1 g_1 + \dots + \eta_{k-1}^k \alpha_{k-1} g_{k-1}), \tag{4.38}
\end{aligned}$$

where $\eta_l^k = \sum_{\ell=l}^k \alpha_\ell / \sum_{\ell=0}^k \alpha_\ell$. Comparing the terms in (4.37) for typical gradient-based optimization algorithms and (4.38) for our proposed method, it can be seen that the ratio sequence $\{\eta_0^k, \eta_1^k, \eta_1^k, \dots, \eta_{k-1}^k\}$ is getting smaller and smaller, which implies that the proposed averaging scheme is able to produce a similar effect to the square-summable condition even if the step-size α_k in (4.38) is not square-summable.

4.3.3 Convergence Rate

In this part, we analyze the speed of convergence of the proposed algorithm for various step-size cases. The results are summarized in the following theorem.

Theorem 4.3. *Suppose Assumptions 4.1, 4.2 and 4.3 hold. Let $\{\hat{\mathbf{x}}_k^i\}_{k \geq 0}$ be the sequence generated by (4.2c) with a step-size sequence $\alpha_k = \frac{\alpha}{(k+2)^a}$, where $\alpha > 0$ is a constant and $a \in [0, 1]$. Let the parameters $\beta_{1,k}, \beta_{2,k}$ be set to $1/(k+2)^{p_1}$ and $1/(k+2)^{p_2}$, respectively, where $p_1 > 1$ and $p = p_2 - p_1 > 1$. Let ϵ be the constant such that $\epsilon \leq \min(\bar{\epsilon}, \frac{1-\gamma}{2N\Gamma\gamma})$, where $\bar{\epsilon}$, Γ and γ are the constants defined in Lemma 4.2. Then, we have*

- 1) for $a = 0$, $\mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* \leq C\alpha + O\left(\frac{1}{t}\right)$;
- 2) for $a \in (0, \frac{1}{2})$, $\mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* \leq O\left(\frac{1}{t^a}\right)$;
- 3) for $a = \frac{1}{2}$, $\mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* \leq O\left(\frac{\ln t}{\sqrt{t}}\right)$;
- 4) for $a \in (\frac{1}{2}, 1)$, $\mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* \leq O\left(\frac{1}{t^{1-a}}\right)$;
- 5) for $a = 1$, $\mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* \leq O\left(\frac{1}{\ln t}\right)$

where C is a constant.

Proof: By noting that

$$\mathbb{E}[f(\hat{\mathbf{z}}_t)] \leq \frac{\sum_{k=0}^{t-1} \alpha_k \mathbb{E}[f(\bar{\mathbf{z}}_k)]}{\sum_{k=0}^{t-1} \alpha_k}$$

from the convexity of f , and

$$\frac{\sum_{k=0}^{t-1} \alpha_k f_{\beta_{1,k}}(\mathbf{x}^*)}{\sum_{k=0}^{t-1} \alpha_k} \leq f^* + \frac{\hat{D}\sqrt{n+2} \sum_{k=0}^{t-1} \alpha_k \beta_{1,k}}{\sum_{k=0}^{t-1} \alpha_k}$$

from Lemma 4.1-1), we divide both sides of (4.28) by $\sum_{k=0}^{t-1} \alpha_k$, which gives

$$\begin{aligned} \mathbb{E}[f(\hat{\mathbf{z}}_t)] &\leq f^* + \frac{C_0}{\sum_{k=0}^{t-1} \alpha_k} + \frac{C_1 \sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k} + \frac{C_2 \sum_{k=0}^{t-1} \alpha_k \beta_{1,k}}{\sum_{k=0}^{t-1} \alpha_k} \\ &\quad + \frac{C_3 \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k}{\sum_{k=0}^{t-1} \alpha_k} + \frac{C_4 (\sum_{k=0}^{t-1} \alpha_k^2) (\sum_{k=0}^{t-1} \tilde{\beta}_k)}{\sum_{k=0}^{t-1} \alpha_k}, \end{aligned}$$

where C_0, C_1, C_2, C_3 and C_4 some constants.

1) For $a = 0$, the step-size is a constant, i.e., $\alpha_k = \alpha$. Then, we have

$$\begin{aligned} \mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* &\leq \frac{C_0}{t\alpha} + C_1\alpha + \frac{C_2\alpha(1 - \frac{1}{(t+1)^{p_1-1}})}{t(p_1 - 1)} + \frac{C_3\alpha(1 - \frac{1}{(t+1)^{p-1}})}{t(p - 1)} \\ &\quad + \frac{C_4\alpha(1 - \frac{1}{(t+1)^{p-1}})}{p - 1} = C\alpha + O\left(\frac{1}{t}\right). \end{aligned}$$

2) For $a \in (0, \frac{1}{2})$, we have

$$\begin{aligned} \mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* &\leq \frac{C_0(1-a)}{\alpha[(t+2)^{1-a} - 2^{1-a}]} + \frac{C_1\alpha(1-a)}{1-2a} \frac{(t+1)^{1-2a} - 1}{(t+2)^{1-a} - 2^{1-a}} \\ &\quad + \frac{C_2(1-a)(1 - \frac{1}{(t+1)^{a+p_1-1}})}{\alpha[(t+2)^{1-a} - 2^{1-a}](a+p_1-1)} + \frac{C_3(1-a)(1 - \frac{1}{(t+1)^{a+p-1}})}{\alpha[(t+2)^{1-a} - 2^{1-a}](a+p-1)} \\ &\quad + \frac{C_4\alpha(1-a)(1 - \frac{1}{(t+1)^{p-1}})}{(1-2a)(p-1)} \frac{(t+1)^{1-2a} - 1}{(t+2)^{1-a} - 2^{1-a}} = O\left(\frac{1}{t^{1-a}}\right) + O\left(\frac{1}{t^a}\right) = O\left(\frac{1}{t^a}\right). \end{aligned}$$

3) For $a = \frac{1}{2}$, we have

$$\begin{aligned} \mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* &\leq \frac{C_0}{2\alpha[\sqrt{t+2} - \sqrt{2}]} + \frac{\alpha C_1 \ln(t+1)}{2(\sqrt{t+2} - \sqrt{2})} \\ &+ \frac{C_2(1 - \frac{1}{(t+1)^{p_1-0.5}})}{\alpha[\sqrt{t+2} - \sqrt{2}](2p_1 - 1)} + \frac{C_3(1 - \frac{1}{(t+1)^{p-0.5}})}{\alpha[\sqrt{t+2} - \sqrt{2}](2p - 1)} \\ &+ \frac{\alpha C_4 \ln(t+1)(1 - \frac{1}{(t+1)^{p-1}})}{2(\sqrt{t+2} - \sqrt{2})(p-1)} = O\left(\frac{1}{\sqrt{t}}\right) + O\left(\frac{\ln t}{\sqrt{t}}\right) = O\left(\frac{\ln t}{\sqrt{t}}\right). \end{aligned}$$

4) For $a \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned} \mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* &\leq \frac{C_0(1-a)}{\alpha[(t+2)^{1-a} - 2^{1-a}]} + \frac{C_1\alpha(1-a)}{2a-1} \frac{1 - \frac{1}{(t+1)^{2a-1}}}{(t+2)^{1-a} - 2^{1-a}} \\ &+ \frac{C_2(1-a)(1 - \frac{1}{(t+1)^{a+p_1-1}})}{\alpha[(t+2)^{1-a} - 2^{1-a}](a+p_1-1)} + \frac{C_3(1-a)(1 - \frac{1}{(t+1)^{a+p-1}})}{\alpha[(t+2)^{1-a} - 2^{1-a}](a+p-1)} \\ &+ \frac{C_4\alpha(1-a)(1 - \frac{1}{(t+1)^{p-1}})}{(2a-1)(p-1)} \frac{1 - \frac{1}{(t+1)^{2a-1}}}{(t+2)^{1-a} - 2^{1-a}} = O\left(\frac{1}{t^{1-a}}\right). \end{aligned}$$

5) For $a = 1$, we have

$$\begin{aligned} \mathbb{E}[f(\hat{\mathbf{z}}_t)] - f^* &\leq \frac{C_0 + \alpha^2 C_1(1 - \frac{1}{t+1})}{\alpha[\ln(t+2) - \ln 2]} + \frac{C_2(1 - \frac{1}{(t+1)^{p_1}})}{\alpha p_1[\ln(t+2) - \ln 2]} \\ &+ \frac{C_3(1 - \frac{1}{(t+1)^p})}{\alpha p[\ln(t+2) - \ln 2]} + \frac{C_4(1 - \frac{1}{t+1})(1 - \frac{1}{(t+1)^{p-1}})}{\alpha(p-1)[\ln(t+2) - \ln 2]} = O\left(\frac{1}{\ln t}\right). \end{aligned}$$

■

Remark 4.5. Theorem 4.3 shows that the proposed optimal averaging scheme allows us to achieve the optimal rate of convergence $\mathcal{O}(\ln k / \sqrt{k})$, in which the optimal step-size $\alpha_k = 1/\sqrt{1+k}$ does not satisfy the square-summable condition.

4.4 Numerical Simulation

4.4.1 Numerical Example

In this part, we investigate the performance of the proposed algorithm through a numerical example. In particular, we consider a non-smooth test problem in a multi-agent system with N agents originated from [61]:

$$\min f(\mathbf{x}) = \sum_{i=1}^N \left(l_i |x_1 - 1| + \sum_{d=1}^{n-1} |1 + x_{d+1} - 2x_d|^2 \right), \quad \mathbf{x} \in \mathcal{X},$$

where $\mathbf{x} = [x_1, \dots, x_n]^\top \in \mathcal{X} \subseteq \mathbb{R}^n$, $l_i, i = 1, 2, \dots, N$ is a positive constant.

In the simulation, the performance of the proposed algorithm is investigated from the following perspectives: the topology of the communication graph (in Section IV-A), the step-size and parameters selections (in Section IV-B), the number of agents (in Section IV-C), and comparison with both state-of-the-art gradient-free algorithm and gradient-based algorithm (in Section IV-D). We characterize the performance by the following merit: the optimality gap defined by $\hat{\mathbf{x}}_k^i - \mathbf{x}^*$ for agent $i \in \mathcal{V}$. Throughout the simulation, we let $[A_r]_{ij} = 1/|\mathcal{N}_i^{\text{in}}|$ and $[A_c]_{ij} = 1/|\mathcal{N}_j^{\text{out}}|$, where $|\mathcal{N}|$ denotes the number of elements in $\mathcal{N}$. l_i is randomly set in $[0.5, 1.5]$.

4.4.1.1 Influence of Communication Topology

In this part, we tested the performance of the algorithm under three different graphs as shown in Fig. 4.1. As can be seen, the number of edges in the graph is increasing from $\mathcal{G}_1, \mathcal{G}_2$ to $\mathcal{G}_3$. In this experiment, we set the dimension of the problem $n = 2$, the number of agents $N = 10$, the step-size $\alpha_k = 0.1/\sqrt{1+k}$, parameters $\beta_{1,k} = 0.1/(1+k)^{1.5}$, $\beta_{2,k} = 0.1/(1+k)^{2.5}$. The plot to show consensus and optimality results was shown in Fig. 4.2. As can be seen, the convergence result is better for the graph with more edges, which is reasonable because more information exchanges

are allowed.

Figure 4.1: Three different communication topologies.

Figure 4.2: Influence of communication topology on the convergence property.

4.4.1.2 Influence of the Number of Agents N

Next, we tested the performance of the algorithm for the cases of different number of agents N under the communication graph as shown in Fig. 4.3. Specifically, we set the number of agents $N = 10, 20, 30$ and 40 . The rest of the parameters were set the same as in Section IV-A. The convergence result was plotted in Fig. 4.4. It can be observed that the convergence time for the case with larger number of agents is longer, which is as expected due to the larger number of error terms in the decision variable updates.

Figure 4.3: Communication topology.

Figure 4.4: Influence of the Number of agents on the convergence property.

4.4.1.3 Influence of Step-Size α_k and Parameters $\beta_{1,k}, \beta_{2,k}$

In this part, we set the dimension of the problem $n = 1$, the number of agents $N = 10$ under the directed graph $\mathcal{G}_1$ shown in Fig. 4.1-(a). Then, we investigated the performance of the algorithm for the cases of different step-size α_k and two positive parameter sequences $\beta_{1,k}, \beta_{2,k}$, respectively.

To test the influence of the step-size on the convergence, we set the step-size $\alpha_k = 0.1/(1+k)^a$, where $a = 0, 0.2, 0.5, 0.7$ and 1 . It should be noted that the step-size α_k is not square-summable for $a = 0, 0.2, 0.5$. Two positive sequences were set to $\beta_{1,k} = 1/(1+k)^{1.5}$ and $\beta_{2,k} = 1/(1+k)^{2.5}$. The plots of both consensus and optimality results were shown in Figs. 4.5-(a) and 4.5-(b). As can be seen from both figures, both the consensus and optimality errors decrease for diminishing step-size, which are consistent with our findings in Theorems 4.1 and 4.2. Moreover, it can be observed that faster convergence result is attained with slower diminishing step-size (*i.e.*, smaller a), but larger errors (oscillations in the plot) are incurred. The optimal rate of convergence is achieved at $a = 0.5$, which coincides with our analysis in Theorem 4.3.

To test the influence of the two positive parameter sequences on the convergence, we set $\beta_{1,k} = 1/(1+k)^{1.5}$, $\tilde{\beta}_k = \beta_{2,k}/\beta_{1,k} = 1/(1+k)^b$, where $b = 1, 3, 5, 7$ and 9 . The step-size α_k was set to $0.1/\sqrt{k+1}$. The convergence result under these five

(a) Consensus ($\hat{\mathbf{x}}_k^i \rightarrow \hat{\mathbf{z}}_k$) (b) Optimality ($\hat{\mathbf{z}}_k \rightarrow \mathbf{x}^*$)Figure 4.5: Influence of step-size α_k on the convergence property.

cases was plotted in Fig. 4.6. As can be seen, typical b values (ranging from 1 to 3) do not have much influence on the speed of convergence. However, it can also be observed that when b is increasing, the convergence performance is downgraded.

Figure 4.6: Influence of $\tilde{\beta}_k$ on the convergence property.

4.4.1.4 Comparison with the State-Of-The-Art Algorithms

In the final experiment, we compared our proposed method with the state-of-the-art algorithms, including the randomized gradient-free push-sum protocol (RGF-Push) proposed in [65] using diminishing smoothing parameter and a subgradient-based method (D-DPS) proposed in [35]. All these three methods can work for directed graphs. We set the dimension of the problem $n = 2$, the number of agents $N = 10$ under the directed graph $\mathcal{G}_1$ shown in Fig. 4.1-(a). The step-size was set to $\alpha_k = 0.1/(k + 1)^{0.5}$. The convergence results of all three methods were shown in Fig. 4.7. As can be seen, our proposed method shows a similar performance to the RGF-Push protocol, where both methods exhibit a theoretical convergence rate of $\ln k/\sqrt{k}$. The gradient-based algorithm (D-DPS) outperforms the two gradient-free methods as expected due to the use of the true gradient information. For the advantages of our proposed algorithm over push-sum protocol, our proposed method only performs linear operations in the decision variable updates, while the nonlinear operation (*i.e.*, dividing the decision variable by an auxiliary variable) is required in the push-sum protocol. Hence, our proposed method is simpler in terms of the computation and more reliable without the risk of being divided by zero.

Figure 4.7: Comparison between D-DPS, RGF-Push and the proposed method.

4.4.2 Temperature Control in HVAC Systems

In this part, we consider a 10-zone HVAC system where the temperature control is formulated as the following distributed Model Predictive Control problem:

$$\begin{aligned} \min J = & \sum_{j=1}^{10} \left(\sum_{k=1}^K \lambda \|T_k^j - T_{ref}^j\| + \sum_{k=0}^{K-1} \|m_{s,k}^j\| \right), \quad m_L \leq m_{s,k}^j \leq m_U \\ T_{k+1}^j = & \left(1 - \frac{\Delta t}{C^j} \left(\frac{1}{R_a^j} + \sum_{i \in \mathcal{N}^j} \frac{1}{R_{ij}} \right) \right) T_k^j + \frac{\Delta t}{C^j} P_d^j \\ & + \frac{\Delta t}{C^j} c_p m_{s,k}^j (T_{s,k}^j - T_k^j) + \frac{\Delta t}{C^j} \left(\frac{T_a}{R_a^j} + \sum_{i \in \mathcal{N}^j} \frac{T_k^i}{R_{ij}} \right), \end{aligned}$$

where T^j and m_s^j are the temperature (degrees) and air mass flow (kg/s) of zone j , respectively. C^j and P_d^j are the lumped mass of air and lumped thermal load including all the external factors, such as the occupancy, solar radiation of zone j , respectively. R_a^j and R_{ij} are the thermal resistances between the air from outside and zone j , and the neighboring zone i and zone j , respectively. c_p is the specific heat capacity of air. Δt and K are the sampling time and receding horizon, respectively. λ is a weighting parameter. m_L and m_U are the lower and upper bounds of the input (air mass flow), respectively. The proposed algorithm is implemented in to solve the problem.

In this simulation, we set $m_L = 0$, $m_U = 3$ kg/s, the initial zone temperature $T_0^j = 30$ degrees, the reference temperature for all zones $T_{ref}^1 = T_{ref}^2 = 24$, $T_{ref}^3 = T_{ref}^4 = 24.5$, $T_{ref}^5 = T_{ref}^6 = 25$, $T_{ref}^7 = T_{ref}^8 = 25.5$, $T_{ref}^9 = T_{ref}^{10} = 26$. The rest of the model parameters were set according to [117]. Figures 4.8 and 4.9 plotted the trajectories of the air mass flow and the zone temperature for all zones. It can be seen that all the inputs are kept within the constraints, while all the zone temperature can be maintained at their corresponding reference levels. Hence, the effectiveness of the proposed algorithm is verified.

Figure 4.8: Air mass flow (kg/s).

Figure 4.9: Zone temperature (degrees).

4.5 Conclusions

This chapter has considered a set constrained distributed optimization problem with possibly non-smooth cost functions. A distributed projected pseudo-gradient descent algorithm with an optimal averaging scheme has been proposed to solve the problem. In particular, a pseudo-gradient operator has been developed locally to estimate the gradient information to guide the decision variable updates. The proposed optimal averaging scheme has enabled the step-size to take any positive, non-summable and non-increasing sequence. Theoretical analysis on the convergence of the proposed algorithm including the convergence rate has been provided. To illustrate its performance, the proposed algorithm has been tested in a non-smooth problem. The convergence properties have been investigated from different perspectives, such as communication topology, number of agents, step-size and parameters selections, while its effectiveness has been verified by comparing with the state-of-the-art algorithms.

Chapter 5

Distributed Online Optimization with Time-Varying Costs

5.1 Introduction

Many scenarios regarding the coordination of multi-agent systems can be modeled as optimization problems where agents collaboratively optimize a global cost function. The main characteristic of this problem is that agents are allowed to exchange information with their neighbors over a communication graph. One type of such optimization problems is the well-known consensus-based distributed optimization problem, where the global objective is often cast as a sum of local cost functions with each one assigned to an agent in the network, to achieve the convergence to the set of minimizers of the global cost function. The problem has various applications, such as parameter estimation and detection [2,3], source localization in sensor networks [4,5], utility maximization [6], resource allocation [7,8], path-planning of mobile robots [9]. The distributed algorithms for such problem have been widely studied, where recent works have been reported in [28,32,33,35,36,41] and the references therein.

The cost functions considered in the above mentioned works are fixed with respect

to time. However, in some situations, the uncertainties in the environment may influence the cost functions, which in turn, affect the performance of the designed algorithms. One approach to cater such issues is through stochastic methods, which has been studied in [13–15]. However, in many practical cases, especially when mobile agents are involved, the optimization problem is often in a highly dynamic environment, resulting in a time-varying objective. Thus, the techniques used in fixed or static distributed optimization cannot be formally applied, leading to the study of online optimization framework, where the cost functions assigned to agents vary with time and these variations are revealed to agents only in hindsight.

Recent works on distributed online optimization have been studied in [71–79]. In [71], general gradient-based methods were proposed to solve both unconstrained and set constrained time-varying quadratic optimization problems from the perspective of continuous-time domain. For discrete-time methods, in [72], a distributed autonomous online learning based on dual averaging where agents communicate over a weighted strongly connected graph was investigated. A regret bound of $O(\ln T)$ was derived for strongly convex cost functions. The same problem was studied in [73], where a distributed online subgradient push-sum algorithm was presented without the doubly-stochastic requirement on the weighting matrix, and was found to achieve a regret bound of $O((\ln T)^2)$. As for the agents with general convex cost functions interacting over a switching network, distributed dual-subgradient averaging algorithms were studied in [74] and [75], showing a regret bound of $O(\sqrt{T})$. Besides, Mateos-Nunez *et al.* in [76] introduced distributed online subgradient descent algorithms with proportional-integral disagreement feedback, proving a regret bound of $O(\sqrt{T})$ for convex cost functions and $O(\ln T)$ for strongly convex cost functions. The work in [79] presented two decentralized stochastic methods based on Nesterov’s dual averaging techniques, and achieved the regret bound of $\mathcal{O}(\sqrt{T})$ for Lipschitz continuous convex functions. It is worth noting that the regret used in all the aforementioned methods is defined as the difference between the incurred network cost and the cost of the best fixed decision in hindsight, which is known as

static regret. On the other hand, to study the scenario where the functions and the decision variables evolve simultaneously instead of a single best fixed decision, the notion of *dynamic regret* was brought forward to characterize how much one regrets working in an online setting as opposed to the offline solution with full knowledge of past and future observations. With this concept, Shahrampour *et al.* in [77, 78] proposed a decentralized mirror descent method for an online optimization problem, where the minimizer follows an known linear dynamics corrupted by unknown unstructured noise, and established a regret bound as a function of the deviation of minimizer sequence.

It should be noted that all these methods make use of the derivatives in the process of the optimization, which implicitly presumes that the derivatives can be obtained directly. However, there are many applications where the gradient information is not available to use, then these methods cannot be applied. This inspires the study of gradient-free optimization, which has been studied in the field of static optimization, *e.g.*, [61–67, 70], to list a few. A Gaussian-smoothing technique was firstly proposed in [61] to solve the general unconstrained optimization problem. This technique was applied to a distributed optimization problem in [62, 63] and further improved in [64, 65]. The variants of this technique were also adopted in [66, 67, 70] to solve the static distributed optimization problems. For online optimization, the works in [77–79] considered the scenarios where the gradient of the cost function is coupled with noise, hence proposed stochastic gradient methods. However, gradient-free optimization schemes have received little attention in online optimization problems in general.

In this chapter, we investigate an online optimization problem where the explicit form of the cost function is not available, but its value is measurable, and only revealed after the decision is made at each time-step. Thus, one cannot calculate the derivative based on the expression of the cost function, leading to the gradient-free settings. Motivated by our previous work in [66, 67], we propose an online randomized gradient-free distributed projected gradient descent (oRGF-DPGD) algorithm,

in which a randomized gradient-free oracle is built locally to approximate the local function derivative, followed by the update of the state variables at each time-step. With some standard assumptions on the graph connectivity and the local cost functions, we are able to provide the bounds for both static and dynamic regrets as a small error term plus a sublinear function of the time duration. The following summarizes the major contributions of this chapter.

- To the best of our knowledge, this work is the first attempt to address the distributed online optimization problem with time-varying cost functions in gradient-free settings. Specifically, the proposed algorithm solves the distributed optimization problem where the cost function can be time-varying; and its implementation does not require the explicit expression of the cost function, but only the measurements at each time-step, making it suitable for those applications where finding the gradient is costly or not practical. Thus, this gradient-free feature naturally subsumes the scenario where the gradient can be computed but coupled with unstructured noise, *e.g.*, [77–79].
- Different from most consensus-based approaches where the weighting matrix is doubly-stochastic [72, 74–78], we associate the communication graph with a row-stochastic and a column-stochastic matrices. This setting is less restrictive, and is feasible to any directed graphs [113, 114]. In addition, this algorithm is allowed to take any positive and non-increasing step-size as compared to those using diminishing and square summable step-size [72, 73]. The wider range of step-size selection implies a wider range of stability.
- Both the static and the dynamic regrets are studied in this chapter. Moreover, for the static regret, we establish a bound consisting of a small error term plus a sublinear function of the time duration; for the dynamic regret, we provide a bound consisting of a small error term plus a product of a term depending on the variation of the optimal solution sequence and a sublinear function of the time duration. With appropriate assumptions, we are able to show that

both types of regrets are sublinear, and their averages over the time duration are convergent to a small neighborhood of zero with the convergence rate of $O(1/\sqrt{T})$, respectively, which is comparable to the state-of-the-art algorithms with available gradient information *e.g.*, [74–76, 79].

In the subsequent sections, the problem studied in this chapter is introduced in Section 5.2. The main results are presented in Section 5.3, which includes the detailed description of the proposed algorithm and the convergence analysis. We verify the effectiveness of the proposed algorithm by some numerical simulations in Section 5.4. Section 5.5 concludes the chapter.

5.2 Problem Formulation

For each agent $i \in \mathcal{V}$, it is associated with a set of local cost functions $\{f_i^t\}_{t \geq 0}$, whose explicit expressions are assumed to be unknown, but the measurement f_i^t of agent i 's decision at time-step t (*i.e.*, $f_i^t(\mathbf{x}^i(t))$) can be made by agent i at time-step t . The objective of the multi-agent system is to cooperatively solve the following constrained optimization problem at each time-step t :

$$\min f^t(\mathbf{x}(t)) = \sum_{i=1}^N f_i^t(\mathbf{x}(t)), \quad \mathbf{x}(t) \in \mathcal{X}, \quad (5.1)$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex and compact set.

As the cost function is time-varying, the optimal solution to (5.1) may vary at different time-step t . Different from the typical algorithms for optimization problem with time-invariant cost functions, the algorithm for online optimization problem is not necessary to reach the optimal solution at every time-step t ; however, its performance should be quantified by comparing with its offline counterpart over a period of time, which leads to the concept of regret.

Denoting the best fixed decision of problem (5.1) over a time duration $T \in \mathbb{N}$ by $\mathbf{x}_s^*$, *i.e.*, $\mathbf{x}_s^* = \arg \min_{\mathbf{x} \in \mathcal{X}} \sum_{t=0}^T f^t(\mathbf{x})$, then the *static regret* associated with agent $i \in \mathcal{V}$ can be formally defined as follows:

$$\mathcal{R}_i^s(T) = \sum_{t=0}^T \mathbb{E}[f^t(\mathbf{x}^i(t))] - \sum_{t=0}^T f^t(\mathbf{x}_s^*),$$

where $\mathbf{x}^i(t)$ is the decision variable of agent i at time t . The static regret is the most commonly used benchmark in online optimization literatures (see [72–76]), which characterizes the difference between the incurred network cost and the cost of the best fixed decision in hindsight. On the other hand, to study the scenario where the functions and the decision variables evolve simultaneously instead of a single best fixed decision, the notion of *dynamic regret* was brought forward. At time-step t , the optimal solution to (5.1) is denoted by $\mathbf{x}_d^*(t)$, *i.e.*, $\mathbf{x}_d^*(t) = \arg \min_{\mathbf{x} \in \mathcal{X}} f^t(\mathbf{x})$. Thus, the dynamic regret associated with agent $i \in \mathcal{V}$ can be formally defined as follows:

$$\mathcal{R}_i^d(T) = \sum_{t=0}^T \mathbb{E}[f^t(\mathbf{x}^i(t))] - \sum_{t=0}^T f^t(\mathbf{x}_d^*(t)).$$

The dynamic regret measures how much network cost being incurred in an online setting as opposed to the offline solution with full knowledge of past and future observations. In general, an online optimization algorithm performs well if the static or dynamic regret is sublinear with respect to T . However, the sublinear dynamic regret is usually not possible unless the optimal solution sequence $\{\mathbf{x}_d^*(t)\}_{t=0}^T$ does not change significantly with time. Thus, we define the deviation of the consecutive optimal solution sequence as follows:

$$\Theta_T = \sum_{t=0}^T \|\mathbf{x}_d^*(t+1) - \mathbf{x}_d^*(t)\|. \quad (5.2)$$

To establish the gradient-free method, we introduce a smoothed version of (5.1)

given by

$$\min f_{\mu}^t(\mathbf{x}(t)) = \sum_{i=1}^N f_{i,\mu^i}^t(\mathbf{x}(t)), \quad \mathbf{x}(t) \in \mathcal{X},$$

where $f_{i,\mu^i}^t(\mathbf{x}(t))$ is the Gaussian approximation of $f_i^t(\mathbf{x}(t))$ [61]

$$f_{i,\mu^i}^t(\mathbf{x}(t)) = \frac{1}{\kappa} \int_{\mathbb{R}^n} f_i^t(\mathbf{x}(t) + \mu^i \xi) e^{-\frac{1}{2}\|\xi\|^2} d\xi,$$

with $\kappa = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|\xi\|^2} d\xi = (2\pi)^{n/2}$ and $\mu^i \geq 0$ is a smoothing parameter of function $f_{i,\mu^i}^t(\mathbf{x}(t))$. Based on [61], the randomized gradient-free oracle of $f_i^t(\mathbf{x}(t))$ at time-step t can be designed as follows:

$$g_{\mu^i}^t(\mathbf{x}(t)) = \frac{f_i^t(\mathbf{x}(t) + \mu^i \xi^i) - f_i^t(\mathbf{x}(t))}{\mu^i} \xi^i,$$

where $\xi^i \in \mathbb{R}^n$ is uniformly generated over a unit ball. The properties of the functions $g_{\mu^i}^t(\mathbf{x}(t))$ and $f_{i,\mu^i}^t(\mathbf{x}(t))$ are presented in Lemma 5.1.

The following assumptions are made throughout the chapter:

Assumption 5.1. *The directed graph $\mathcal{G}$ is strongly connected.*

Assumption 5.2. *Each local cost function f_i^t at time-step t is convex, but not necessarily differentiable. Its subgradient $\partial f_i^t(\mathbf{x}(t))$ is bounded, i.e., $\forall \mathbf{x}(t) \in \mathcal{X}$, there exists a positive constant $\hat{D}$ such that $\|\partial f_i^t(\mathbf{x}(t))\| \leq \hat{D}$.*

5.3 Main Results

In this section, we present an online randomized gradient-free distributed projected gradient descent method for the problem defined in (5.1), followed by the convergence analysis.

5.3.1 Online Randomized Gradient-Free DPGD Method

The main steps of randomized gradient-free distributed projected gradient descent method are described in this part.

At time-step t , each agent j delivers its state information $\mathbf{x}^j(t)$ with a weighted auxiliary variable $[A_c]_{ij}\mathbf{y}^j(t)$ to its out-neighbor $i \in \mathcal{N}_j^{\text{out}}$. Then, at time-step $t + 1$, agent i updates its variables $\mathbf{x}^i(t + 1)$ and $\mathbf{y}^i(t + 1)$ with the information received from its in-neighbor $j \in \mathcal{N}_i^{\text{in}}$ as follows

$$\mathbf{x}^i(t + 1) = \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^N [A_r]_{ij} \mathbf{x}^j(t) + \epsilon \mathbf{y}^i(t) - \alpha(t) g_{\mu^i}^t(\mathbf{x}^i(t)) \right], \quad (5.3a)$$

$$\mathbf{y}^i(t + 1) = \mathbf{x}^i(t) - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}^j(t) + \sum_{j=1}^N [A_c]_{ij} \mathbf{y}^j(t) - \epsilon \mathbf{y}^i(t), \quad (5.3b)$$

where $g_{\mu^i}^t(\mathbf{x}^i(t))$ is the random gradient-free oracle, given by

$$g_{\mu^i}^t(\mathbf{x}^i(t)) = \frac{f_i^t(\mathbf{x}^i(t) + \mu^i \xi^i(t)) - f_i^t(\mathbf{x}^i(t))}{\mu^i} \xi^i(t), \quad (5.4)$$

A_r, A_c are the row-stochastic and column-stochastic adjacency matrices, respectively, *i.e.*, $\sum_{j=1}^N [A_r]_{ij} = 1$ for all $i \in \mathcal{V}$, and $\sum_{i=1}^N [A_c]_{ij} = 1$ for all $j \in \mathcal{V}$, $\alpha(t) > 0$ is a non-increasing step-size, and ϵ is a small positive number. The procedures are summarized in Algorithm 5.1.

Algorithm 5.1 Online randomized gradient-free DPGD

- 1: **Initialize:** $i \in \mathcal{V}$
 arbitrarily select $\mathbf{x}^i(0), \mathbf{y}^i(0) \in \mathcal{X}$
 randomly generate $\{\xi^i(t)\}_{t \geq 0}$ over a unit ball independently
 - 2: **Iteration** ($t \geq 0$): $i \in \mathcal{V}$
 compute $g_{\mu^i}^t(\mathbf{x}^i(t))$ based on (5.4)
 update variables $\mathbf{x}^i(t + 1)$ based on (5.3a)
 update variables $\mathbf{y}^i(t + 1)$ based on (5.3b)
 - 3: **Output:** $i \in \mathcal{V}$
 $\mathbf{x}^i(t)$
-

For the convenience of analysis, we may write (5.3) in a compact form as

$$\mathbf{z}^i(t+1) = \sum_{j=1}^{2N} [A]_{ij} \mathbf{z}^j(t) + g^i(t), \quad (5.5)$$

where $\mathbf{z}^i(t) = \mathbf{x}^i(t)$ for $i \in \{1, \dots, N\}$ and $\mathbf{z}^i(t) = \mathbf{y}^{i-N}(t)$ for $i \in \{N+1, \dots, 2N\}$; $g^i(t) = \mathbf{x}^i(t+1) - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}^j(t) - \epsilon \mathbf{y}^i(t)$ for $i \in \{1, \dots, N\}$ and $g^i(t) = \mathbf{0}_n$ for $i \in \{N+1, \dots, 2N\}$; and $A = \begin{bmatrix} A_r & \epsilon I \\ I - A_r & A_c - \epsilon I \end{bmatrix}$. As $\mathcal{X}$ is compact and convex, we denote $\rho = \sup_{\mathbf{x} \in \mathcal{X}} \|\mathbf{x}\|$. Thus, $\|\mathbf{x}^i(t)\|$ or $\|\mathbf{z}^i(t)\| \leq \rho$, for $i \in \{1, \dots, N\}$, $t \geq 0$.

5.3.2 Convergence Analysis

In this part, we provide the detailed analysis on the convergence properties of our proposed algorithm. We denote the σ -field generated by the entire history of the random variables from step 0 to $t-1$ by $\mathcal{F}_t$, *i.e.*,

$$\mathcal{F}_t = \begin{cases} \{\mathbf{x}_0^i, i \in \mathcal{V}\}, & t = 0, \\ \{(\mathbf{x}_0^i, i \in \mathcal{V}); (\xi^i(r), i \in \mathcal{V}); 0 \leq r \leq t-1\}, & t \geq 1. \end{cases}$$

Before proceeding to the main results, we first introduce some important lemmas.

The following lemma provides some properties of the functions $g_{\mu^i}^t(\mathbf{x})$ and $f_{i,\mu^i}^t(\mathbf{x})$, which is a variation of Lemma 2.1 stated in Chapter 2.

Lemma 5.1. (see [61]) *Suppose Assumption 5.2 holds. For each $i \in \mathcal{V}$, the following properties of the functions $g_{\mu^i}^t(\mathbf{x}(t))$ and $f_{i,\mu^i}^t(\mathbf{x}(t))$ are satisfied.*

1) *Function $f_{i,\mu^i}^t(\mathbf{x}(t))$ is convex, and it satisfies*

$$f_i^t(\mathbf{x}(t)) \leq f_{i,\mu^i}^t(\mathbf{x}(t)) \leq f_i^t(\mathbf{x}(t)) + \sqrt{n} \hat{\mu} \hat{D},$$

where $\hat{\mu} = \max_{i \in \mathcal{V}} \mu_i$.

2) Function $f_{i,\mu^i}^t(\mathbf{x}(t))$ is differentiable and its gradient satisfies

$$\nabla f_{i,\mu^i}^t(\mathbf{x}(t)) = \mathbb{E}[g_{\mu^i}^t(\mathbf{x}(t))],$$

and is Lipschitz continuous with a Lipschitz constant $\hat{L}$, i.e.,

$$\|\nabla f_{i,\mu^i}^t(\mathbf{x}(t)) - \nabla f_{i,\mu^i}^t(\mathbf{y}(t))\| \leq \hat{L}\|\mathbf{x}(t) - \mathbf{y}(t)\|,$$

where $\hat{L} = \max_{i \in \mathcal{V}} \frac{\sqrt{n}\hat{D}}{\mu^i}$.

3) The random gradient-free oracle $g_{\mu^i}^t(\mathbf{x}(t))$ satisfies

$$\mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}(t))\|] \leq \sqrt{\mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}(t))\|^2]} \leq (n+4)\hat{D}.$$

An important result on the weighting matrix A is presented in the following lemma:

Lemma 5.2. (Lemma 1 in [35]) Suppose Assumption 5.1 holds. A is the weighting matrix defined in (5.5). Let the constant ϵ in A be chosen such that $\epsilon \in (0, \bar{\epsilon})$. Then $\forall i, j \in \{1, \dots, 2N\}$, the entries $[A^t]_{ij}$ converge to their limits as $t \rightarrow \infty$ at a geometric rate, i.e.,

$$\left\| A^t - \begin{bmatrix} \frac{\mathbf{1}_N \mathbf{1}_N^T}{N} & \frac{\mathbf{1}_N \mathbf{1}_N^T}{N} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|_{\infty} \leq \Gamma \gamma^t, \quad t \geq 1,$$

where $\bar{\epsilon} = (\frac{1-|\lambda_3|}{20+8N})^N$, λ_3 is the third largest eigenvalue of A by setting $\epsilon = 0$. $\Gamma > 0$ and $0 < \gamma < 1$ are some constants.

Now, we denote that $\bar{\mathbf{z}}(t) = \frac{1}{N} \sum_{i=1}^{2N} \mathbf{z}^i(t)$. Then, we can obtain from (5.5) that

$$\begin{aligned} \bar{\mathbf{z}}(t+1) &= \frac{1}{N} \sum_{i=1}^{2N} \sum_{j=1}^{2N} [A]_{ij} \mathbf{z}^j(t) + \frac{1}{N} \sum_{i=1}^{2N} g^i(t) \\ &= \frac{1}{N} \sum_{j=1}^{2N} \mathbf{z}^j(t) + \frac{1}{N} \sum_{i=1}^{2N} g^i(t) \end{aligned}$$

$$= \bar{\mathbf{z}}(t) + \frac{1}{N} \sum_{i=1}^{2N} g^i(t), \quad (5.6)$$

where we have used column-stochastic property of A , *i.e.*, for $t \geq 1$, it holds that $\sum_{i=1}^{2N} [A^t]_{ij} = 1$.

Similar to the common assumption on the boundness of the gradient/subgradient in distributed optimization problems, the following lemma provides a bound on the augmented gradient-free oracle $g^i(t)$:

Lemma 5.3. *Suppose Assumptions 5.1 and 5.2 hold. Let the constant ϵ be chosen such that $\epsilon \leq \min(\bar{\epsilon}, \frac{1-\gamma}{2N\Gamma\gamma})$. Then, there exists some bounded constant $G > 0$, such that for all $t \geq 0$, the augmented gradient-free oracle $g^j(t)$ satisfies*

$$\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t] \leq G\alpha(t),$$

where $\bar{\epsilon} > 0$, $\Gamma > 0$ and $0 < \gamma < 1$ are the constants defined in Lemma 5.2. $\alpha(t)$ is the non-increasing step-size.

Proof: Step 1: For any $K \geq 2$, the following relation holds

$$\sum_{t=1}^K \alpha(t) \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t] \leq \Phi \sum_{t=1}^K \alpha^2(t) + \Psi, \quad (5.7)$$

where Φ and Ψ are positive constants.

By the projection's non-expansive property, we have

$$\begin{aligned} \left\| \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^N [A_r]_{ij} \mathbf{x}^j(t) + \epsilon \mathbf{y}^i(t) - \alpha(t) g_{\mu^i}^t(\mathbf{x}^i(t)) \right] - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}^j(t) \right\| \\ \leq \|\epsilon \mathbf{y}^i(t) - \alpha(t) g_{\mu^i}^t(\mathbf{x}^i(t))\|, \end{aligned}$$

which implies

$$\begin{aligned}
\|g^i(t)\| &\leq \left\| \mathbf{x}^i(t+1) - \sum_{j=1}^N [A_r]_{ij} \mathbf{x}^j(t) \right\| + \|\epsilon \mathbf{y}^i(t)\| \\
&\leq \|\epsilon \mathbf{y}^i(t) - \alpha(t) g_{\mu^i}^t(\mathbf{x}^i(t))\| + \epsilon \|\mathbf{y}^i(t)\| \\
&\leq \alpha(t) \|g_{\mu^i}^t(\mathbf{x}^i(t))\| + 2\epsilon \|\mathbf{y}^i(t)\|.
\end{aligned} \tag{5.8}$$

For the boundedness of $\|\mathbf{y}^i(t)\|$, it follows from (5.5) that for $t \geq 1$

$$\mathbf{z}^i(t) = \sum_{j=1}^{2N} [A^t]_{ij} \mathbf{z}_0^j + \sum_{r=1}^{t-1} \sum_{j=1}^{2N} [A^{t-r}]_{ij} g^j(r-1) + g^i(t-1). \tag{5.9}$$

Thus, we can obtain from (5.9) that for $i = \{N+1, \dots, 2N\}$ and $t > 1$

$$\mathbb{E}[\|\mathbf{z}^i(t)\| | \mathcal{F}_{t-1}] \leq \sum_{j=1}^{2N} \|[A^t]_{ij}\| \rho + \sum_{r=1}^{t-1} \sum_{j=1}^N \|[A^{t-r}]_{ij}\| \mathbb{E}[\|g^j(r-1)\| | \mathcal{F}_{r-1}]. \tag{5.10}$$

Applying Lemma 5.2 to (5.10) yields

$$\mathbb{E}[\|\mathbf{y}^i(t)\| | \mathcal{F}_{t-1}] \leq 2N\rho\Gamma\gamma^t + \Gamma \sum_{r=1}^{t-1} \gamma^{t-r} \sum_{j=1}^N \mathbb{E}[\|g^j(r-1)\| | \mathcal{F}_{r-1}].$$

Thus, with the above result, it can be obtained from (5.8) that

$$\mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] \leq 4N\rho\epsilon\Gamma\gamma^t + (n+4)\hat{D}\alpha(t) + 2\epsilon\Gamma \sum_{r=1}^{t-1} \gamma^{t-r} \sum_{j=1}^N \mathbb{E}[\|g^j(r-1)\| | \mathcal{F}_{r-1}],$$

where Lemma 5.1-(3) is used. Thus, we have

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] &\leq 4N^2\rho\epsilon\Gamma\gamma^t + (n+4)N\hat{D}\alpha(t) \\
&\quad + 2N\epsilon\Gamma \sum_{r=1}^{t-1} \gamma^{t-r} \sum_{j=1}^N \mathbb{E}[\|g^j(r-1)\| | \mathcal{F}_{r-1}].
\end{aligned} \tag{5.11}$$

Thus, it follows from (5.11) that

$$\begin{aligned} \sum_{t=1}^K \alpha(t) \sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] &\leq 4N^2 \rho \epsilon \Gamma \sum_{t=1}^K \gamma^t \alpha(t) + (n+4)N\hat{D} \sum_{t=1}^K \alpha^2(t) \\ &\quad + 2N\epsilon \Gamma \sum_{t=1}^K \alpha(t) \sum_{r=1}^{t-1} \gamma^{t-r} \sum_{j=1}^N \mathbb{E}[\|g^j(r-1)\| | \mathcal{F}_{r-1}]. \end{aligned} \quad (5.12)$$

Noting that

$$\sum_{t=1}^K \gamma^t \alpha(t) \leq \frac{1}{2} \sum_{t=1}^K \alpha^2(t) + \frac{\gamma^2}{2(1-\gamma^2)},$$

and the step-size is non-increasing, it holds that

$$\sum_{t=1}^K \alpha(t) \sum_{r=1}^{t-1} \gamma^{t-r} \sum_{j=1}^N \mathbb{E}[\|g^j(r-1)\| | \mathcal{F}_{r-1}] \leq \frac{\gamma}{1-\gamma} \sum_{t=1}^K \alpha(t) \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t].$$

Applying the above results, it follows from (5.12) that

$$\begin{aligned} \sum_{t=1}^K \alpha(t) \sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] &\leq (n+4)N\hat{D} \sum_{t=1}^K \alpha^2(t) \\ &\quad + 4N^2 \rho \epsilon \Gamma \left(\frac{1}{2} \sum_{t=1}^K \alpha^2(t) + \frac{\gamma^2}{2(1-\gamma^2)} \right) \\ &\quad + \frac{2N\epsilon \Gamma \gamma}{1-\gamma} \sum_{t=1}^K \alpha(t) \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t]. \end{aligned}$$

Thus, the result (5.7) follows by moving the last term to the left hand side and $\epsilon \leq \frac{1-\gamma}{2N\Gamma\gamma}$.

Step 2: Prove the final result by contradiction.

Suppose the statement is not true. Then, $(\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t])/\alpha(t) = \infty$ holds for some t . Since $\alpha(t) \neq 0$, there are two cases where $(\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t])/\alpha(t) = \infty$. *Case 1:* at some finite t_f , $(\sum_{j=1}^N \mathbb{E}[\|g^j(t_f)\| | \mathcal{F}_{t_f}])/\alpha(t_f) = \infty$; *Case 2:* when t goes to infinity, $(\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t])/\alpha(t) = \infty$. Next, we show that both cases lead to

contradiction.

Case 1: If there exists some finite t_f such that $\sum_{j=1}^N \mathbb{E}[\|g^j(t_f)\| | \mathcal{F}_{t_f}] = \infty$, then we can always find a finite constant $K > t_f$ in (5.7) such that

$$\sum_{t=1}^K \alpha(t) \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t] \leq \Phi \sum_{t=1}^K \alpha^2(t) + \Psi < \infty,$$

which implies every term $\alpha(t) \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t]$, $t = 1, \dots, K$ is bounded. Since $\alpha(t) \neq 0$, every term $\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t]$ is bounded for $t = 1, \dots, K$, which contradicts to $\sum_{j=1}^N \mathbb{E}[\|g^j(t_f)\| | \mathcal{F}_{t_f}] = \infty$.

Case 2: Suppose $(\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t])/\alpha(t) = \infty$ when t goes to infinity. Taking the limit superior on both sides of (5.11) and noting that $\epsilon \leq \frac{1-\gamma}{2N\Gamma\gamma}$, we can prove that

$$\limsup_{t \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t] \leq \frac{(n+4)N\hat{D}}{1 - \frac{2N\epsilon\Gamma\gamma}{1-\gamma}} \lim_{t \rightarrow \infty} \alpha(t) < \infty.$$

Then we have

$$\limsup_{t \rightarrow \infty} \frac{\sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t]}{\alpha(t)} = \frac{\limsup_{t \rightarrow \infty} \sum_{j=1}^N \mathbb{E}[\|g^j(t)\| | \mathcal{F}_t]}{\lim_{t \rightarrow \infty} \alpha(t)} \leq \frac{(n+4)N\hat{D}}{1 - \frac{2N\epsilon\Gamma\gamma}{1-\gamma}} < \infty,$$

leading to a contradiction. This completes the entire proof. $\blacksquare$

With the above lemmas, we are ready to establish the main results. We first provide a bound for $\mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|]$, $1 \leq i \leq N$.

Theorem 5.1. *Suppose Assumptions 5.1 and 5.2 hold. Let $\{\mathbf{z}^i(t)\}_{t \geq 0}$ be the sequence generated by (5.5) with non-increasing step-size sequence $\{\alpha(t)\}_{t \geq 0}$. Then, for $i = \{1, \dots, N\}$, $\mathbf{z}^i(t)$ satisfies*

$$\mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|] \leq 2N\rho\hat{\Gamma}\gamma^t + G\hat{\Gamma} \sum_{r=1}^t \gamma^{t-r} \alpha(r-1),$$

where $\hat{\Gamma} = \max\{\Gamma, 1\} > 0, 0 < \gamma < 1$ and $G > 0$ are some constants.

Proof: We can obtain from (5.6) that

$$\bar{\mathbf{z}}(t) = \frac{1}{N} \sum_{j=1}^{2N} \mathbf{z}_0^j + \frac{1}{N} \sum_{r=1}^t \sum_{j=1}^{2N} g^j(r-1). \quad (5.13)$$

Subtracting (5.13) from (5.9) and taking the norm, we have that for $1 \leq i \leq N$ and $t \geq 1$,

$$\begin{aligned} \|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\| &\leq \sum_{j=1}^{2N} \left\| [A^t]_{ij} - \frac{1}{N} \right\| \rho + \sum_{r=1}^{t-1} \sum_{j=1}^N \left\| [A^{t-r}]_{ij} - \frac{1}{N} \right\| \|g^j(r-1)\| \\ &\quad + \frac{N-1}{N} \|g^i(t-1)\| + \frac{1}{N} \sum_{j \neq i} \|g^j(t-1)\|. \end{aligned}$$

Noting that

$$\begin{aligned} &\frac{N-1}{N} \|g^i(t-1)\| + \frac{1}{N} \sum_{j \neq i} \|g^j(t-1)\| \\ &\leq \frac{N-1}{N} \sum_{i=1}^N \|g^i(t-1)\| + \frac{1}{N} \sum_{j=1}^N \|g^j(t-1)\| = \sum_{j=1}^N \|g^j(t-1)\|. \end{aligned}$$

By applying Lemma 5.2 and denoting $\hat{\Gamma} = \max\{\Gamma, 1\}$, we can obtain that

$$\begin{aligned} \|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\| &\leq 2N\rho\Gamma\gamma^t + \sum_{j=1}^N \|g^j(t-1)\| + \Gamma \sum_{r=1}^{t-1} \gamma^{t-r} \sum_{j=1}^N \|g^j(r-1)\| \\ &\leq 2N\rho\hat{\Gamma}\gamma^t + \hat{\Gamma} \sum_{r=1}^t \gamma^{t-r} \sum_{j=1}^N \|g^j(r-1)\|. \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_\ell$ from $\ell = 0$ to $t-1$ and using the result of Lemma 5.3 yields

$$\mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\| | \mathcal{F}_{t-1}] \leq 2N\rho\hat{\Gamma}\gamma^t + G\hat{\Gamma} \sum_{r=1}^t \gamma^{t-r} \alpha(r-1). \quad (5.14)$$

Taking the total expectation for (5.14), we complete the proof. ■

Remark 5.1. Theorem 5.1 characterizes the consensus property of the algorithm. Denoting $\lim_{t \rightarrow \infty} \alpha(t)$ by $\tilde{\alpha}$, then it is not difficult to show that

$$\lim_{t \rightarrow \infty} \sum_{r=1}^t \gamma^{t-r} \alpha(r-1) = \frac{\tilde{\alpha}}{1-\gamma},$$

which means $\limsup_{t \rightarrow \infty} \mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|] \leq (G\hat{\Gamma}\tilde{\alpha})/(1-\gamma)$; namely, all agents $\mathbf{z}^i(t), i \in \mathcal{V}$ approximately converge to $\bar{\mathbf{z}}(t)$ with the error gap proportional to the limit of the step-size $\tilde{\alpha}$. If $\tilde{\alpha} = 0$, then the exact convergence can be attained.

Next, we provide a bound for the static regret $\mathcal{R}_i^s(T)$, which is summarized in the following theorem.

Theorem 5.2. Suppose Assumptions 5.1 and 5.2 hold. Let $\{\mathbf{z}^i(t)\}_{t \geq 0}$ be the sequence generated by (5.5) with the step-size sequence $\alpha(t) = 1/\sqrt{t+1}$. Then, for any $i \in \mathcal{V}$ and time duration $T > 0$, the static regret $\mathcal{R}_i^s(T)$ satisfies

$$\mathcal{R}_i^s(T) \leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \mathcal{C}_1^s + \mathcal{C}_2^s\sqrt{T+1},$$

where

$$\mathcal{C}_1^s = 2(G + 2(n+5)\hat{D})N^2\rho\hat{\Gamma}/(1-\gamma)$$

$$\mathcal{C}_2^s = \mathcal{B}_T^s + N\hat{V}^s + G(3G + 2(n+4)N\hat{D})/N + 2(G + 2(n+5)\hat{D})NG\hat{\Gamma}/\gamma(1-\gamma),$$

$\hat{V}^s$ is a positive constant, $\mathcal{B}_T^s = \sum_{t=0}^T (\mathcal{B}_{cov,1}^s/2N + \mathcal{B}_{cov,2}^s + \mathcal{B}_{cov,3}^s)$, $\mathcal{B}_{cov,1}^s$, $\mathcal{B}_{cov,2}^s$ and $\mathcal{B}_{cov,3}^s$ are the upper bounds of some covariance terms.

Proof: Based on (5.5) and the fact that A is column-stochastic, we have

$$\bar{\mathbf{z}}(t+1) = \bar{\mathbf{z}}(t) + \frac{1}{N} \sum_{i=1}^N g^i(t).$$

Thus, we can derive that

$$\begin{aligned}\|\bar{\mathbf{z}}(t+1) - \mathbf{x}_s^*\|^2 &= \|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2 + \left\| \frac{1}{N} \sum_{i=1}^N g^i(t) \right\|^2 + \frac{2}{N} \sum_{i=1}^N \langle g^i(t), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle \\ &= \|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2 + \frac{1}{N^2} \left\| \sum_{i=1}^N g^i(t) \right\|^2\end{aligned}\quad (5.15a)$$

$$- \frac{2\alpha(t)}{N} \sum_{i=1}^N \langle g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle \quad (5.15b)$$

$$+ \frac{2}{N} \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle. \quad (5.15c)$$

For the second term in (5.15a), taking the conditional expectation on $\mathcal{F}_t$ yields

$$\begin{aligned}\mathbb{E} \left[\left\| \sum_{i=1}^N g^i(t) \right\|^2 \middle| \mathcal{F}_t \right] &\leq \sum_{i=1}^N \mathbb{E}[\|g^i(t)\|^2 | \mathcal{F}_t] \\ &= \sum_{i=1}^N (\mathbb{E}[\|g^i(t)\|^2 | \mathcal{F}_t]) + \sum_{i=1}^N \text{Cov}(\|g^i(t)\|, \|g^i(t)\|) \\ &\leq \left(\sum_{i=1}^N \mathbb{E}[\|g^i(t)\|^2 | \mathcal{F}_t] \right) + \mathcal{B}_{cov,1}^s \leq G^2 \alpha^2(t) + \mathcal{B}_{cov,1}^s,\end{aligned}\quad (5.16)$$

where we have applied Lemma 5.3 on $\sum_{i=1}^N \mathbb{E}[\|g^i(t)\|^2 | \mathcal{F}_t]$ and used $\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y] + \text{Cov}(x, y)$. The upper bound of the covariance term $\sum_{i=1}^N \text{Cov}(\|g^i(t)\|, \|g^i(t)\|)$ is denoted by $\mathcal{B}_{cov,1}^s > 0$.

For (5.15b), taking the conditional expectation on $\mathcal{F}_t$ and applying Lemma 5.1-(2) gives $\sum_{i=1}^N \langle \mathbb{E}[g_{\mu^i}^t(\mathbf{x}^i(t)) | \mathcal{F}_t], \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle = \sum_{i=1}^N \langle \nabla f_{i,\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle$. Noting that

$$\begin{aligned}\langle \nabla f_{i,\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle &= \langle \nabla f_{i,\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}^i(t) \rangle + \langle \nabla f_{i,\mu^i}^t(\mathbf{x}^i(t)), \mathbf{x}^i(t) - \mathbf{x}_s^* \rangle \\ &\geq -\|\nabla f_{i,\mu^i}^t(\mathbf{x}^i(t))\| \|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| + f_{i,\mu^i}^t(\mathbf{x}^i(t)) - f_{i,\mu^i}^t(\mathbf{x}_s^*) \\ &\geq -(n+4)\hat{D}\|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| + f_i^t(\mathbf{x}^i(t)) - f_{i,\mu^i}^t(\mathbf{x}_s^*) \\ &\geq -(n+4)\hat{D}\|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| + f_i^t(\mathbf{x}^i(t)) - f_i^t(\bar{\mathbf{z}}(t)) + f_i^t(\bar{\mathbf{z}}(t)) - f_{i,\mu^i}^t(\mathbf{x}_s^*) \\ &\geq -(n+5)\hat{D}\|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| + f_i^t(\bar{\mathbf{z}}(t)) - f_{i,\mu^i}^t(\mathbf{x}_s^*),\end{aligned}$$

where $\|\nabla f_{i,\mu^i}^t(\mathbf{x}^i(t))\|$ can be bounded using Lemma 5.1-(3), particularly,

$$\|\nabla f_{i,\mu^i}^t(\mathbf{x}^i(t))\| = \|\mathbb{E}[g_{\mu^i}^t(\mathbf{x}^i(t))|\mathcal{F}_t]\| \leq \mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}^i(t))\||\mathcal{F}_t] \leq (n+4)\hat{D};$$

and

$$f_i^t(\mathbf{x}^i(t)) - f_i^t(\bar{\mathbf{z}}(t)) \geq \langle \partial f_i^t(\bar{\mathbf{z}}(t)), \mathbf{x}^i(t) - \bar{\mathbf{z}}(t) \rangle \geq -\hat{D}\|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\|$$

based on Assumption 5.2. Thus, we obtain

$$\begin{aligned} & \sum_{i=1}^N \langle \mathbb{E}[g_{\mu^i}^t(\mathbf{x}^i(t))|\mathcal{F}_t], \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle \\ & \geq -(n+5)\hat{D} \sum_{i=1}^N \|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| + f^t(\bar{\mathbf{z}}(t)) - f_{\mu}^t(\mathbf{x}_s^*). \end{aligned} \quad (5.17)$$

Noting that for (5.15c), we have

$$\begin{aligned} & \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle \\ & = \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1) \rangle \end{aligned} \quad (5.18a)$$

$$+ \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1) \rangle \quad (5.18b)$$

$$+ \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \mathbf{x}^i(t+1) - \mathbf{x}_s^* \rangle. \quad (5.18c)$$

For (5.18a), we have

$$\begin{aligned} & \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1) \rangle \\ & \leq \sum_{i=1}^N \|g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t))\| \left\| \frac{1}{N} \sum_{i=1}^N g^i(t) \right\| \end{aligned}$$

$$\leq \frac{1}{N} \left(\sum_{i=1}^N \|g^i(t)\| \right)^2 + \frac{\alpha(t)}{N} \sum_{i=1}^N \|g^i(t)\| \sum_{i=1}^N \|g_{\mu^i}^t(\mathbf{x}^i(t))\|.$$

Taking the conditional expectation on $\mathcal{F}_t$ gives

$$\begin{aligned} & \sum_{i=1}^N \mathbb{E}[\langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1) \rangle | \mathcal{F}_t] \\ & \leq \frac{1}{N} \left(\sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] \right)^2 + \frac{\alpha(t)}{N} \sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] \sum_{i=1}^N \mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}^i(t))\| | \mathcal{F}_t] + \mathcal{B}_{cov,2}^s \\ & \leq \frac{1}{N} G(G + (n+4)N\hat{D})\alpha^2(t) + \mathcal{B}_{cov,2}^s, \end{aligned} \quad (5.19)$$

where we have applied Lemma 5.1-(3) on $\mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}^i(t))\| | \mathcal{F}_t]$ and Lemma 5.3 on $\sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t]$. $\mathcal{B}_{cov,2}^s > 0$ is an upper bound of the sum of the covariance terms $Cov(\sum_{i=1}^N \|g^i(t)\|, \sum_{i=1}^N \|g^i(t)\|)$ and $Cov(\sum_{i=1}^N \|g^i(t)\|, \sum_{i=1}^N \|g_{\mu^i}^t(\mathbf{x}^i(t))\|)$.

For (5.18b), we have

$$\begin{aligned} & \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1) \rangle \\ & \leq \sum_{i=1}^N \|g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t))\| \|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\| \\ & \leq \sum_{i=1}^N (\|g^i(t)\| + \alpha(t)\|g_{\mu^i}^t(\mathbf{x}^i(t))\|) \|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\|. \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_t$ yields

$$\begin{aligned} & \sum_{i=1}^N \mathbb{E}[\langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1) \rangle | \mathcal{F}_t] \\ & \leq \sum_{i=1}^N (\mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] + \alpha(t)\mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}^i(t))\| | \mathcal{F}_t]) \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\| | \mathcal{F}_t] + \mathcal{B}_{cov,3}^s \\ & \leq (G + (n+4)\hat{D})\alpha(t) \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\| | \mathcal{F}_t] + \mathcal{B}_{cov,3}^s, \end{aligned} \quad (5.20)$$

where we have applied Lemma 5.1-(3) on $\mathbb{E}[\|g_{\mu^i}^t(\mathbf{x}^i(t))\| | \mathcal{F}_t]$ and Lemma 5.3 on

$\sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t]$. $\mathcal{B}_{cov,3}^s > 0$ is an upper bound of the sum of the covariance terms $Cov(\sum_{i=1}^N \|g^i(t)\|, \|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\|)$ and $Cov(\sum_{i=1}^N \|g_{\mu^i}^t(\mathbf{x}^i(t))\|, \|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\|)$.

For (5.18c), it follows from the projection's non-expansive property that

$$\langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \mathbf{x}^i(t+1) - \mathbf{x}_s^* \rangle \leq 0. \quad (5.21)$$

Thus, taking the conditional expectation on $\mathcal{F}_t$ in (5.18) and substituting (5.19), (5.20) and (5.21), we obtain

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}[\langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{x}^i(t)), \bar{\mathbf{z}}(t) - \mathbf{x}_s^* \rangle | \mathcal{F}_t] &\leq \frac{1}{N}G(G + (n+4)N\hat{D})\alpha^2(t) \\ &+ (G + (n+4)\hat{D})\alpha(t) \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\| | \mathcal{F}_t] + \mathcal{B}_{cov,2}^s + \mathcal{B}_{cov,3}^s. \end{aligned} \quad (5.22)$$

Taking the conditional expectation on $\mathcal{F}_t$ in (5.15), and substituting (5.16), (5.17) and (5.22), we obtain that

$$\begin{aligned} \alpha(t)(f^t(\bar{\mathbf{z}}(t)) - f_{\mu}^t(\mathbf{x}_s^*)) &\leq \frac{\mathcal{B}_{cov,1}^s}{2N} + \mathcal{B}_{cov,2}^s + \mathcal{B}_{cov,3}^s + (n+5)\hat{D} \sum_{i=1}^N \alpha(t)\|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| \\ &+ \frac{N}{2}(\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2 - \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}_s^*\|^2 | \mathcal{F}_t]) + \frac{G}{2N}(3G + 2(n+4)N\hat{D})\alpha^2(t) \\ &+ (G + (n+4)\hat{D}) \sum_{i=1}^N \alpha(t)\mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\| | \mathcal{F}_t]. \end{aligned} \quad (5.23)$$

Since $\alpha(t)$ is positive, dividing both sides of (5.23) by $\alpha(t)$, we have

$$\begin{aligned} f^t(\bar{\mathbf{z}}(t)) - f_{\mu}^t(\mathbf{x}_s^*) &\leq \frac{1}{\alpha(t)} \left(\frac{\mathcal{B}_{cov,1}^s}{2N} + \mathcal{B}_{cov,2}^s + \mathcal{B}_{cov,3}^s \right) + (n+5)\hat{D} \sum_{i=1}^N \|\mathbf{x}^i(t) - \bar{\mathbf{z}}(t)\| \\ &+ \frac{N}{2\alpha(t)}(\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2 - \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}_s^*\|^2 | \mathcal{F}_t]) + \frac{G}{2N}(3G + 2(n+4)N\hat{D})\alpha(t) \\ &+ (G + (n+4)\hat{D}) \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}^i(t+1)\| | \mathcal{F}_t]. \end{aligned}$$

Taking the total expectation, and combining the result in Theorem 5.1, we have

$$\begin{aligned} \mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f_\mu^t(\mathbf{x}_s^*) &\leq \frac{1}{\alpha(t)} \left(\frac{\mathcal{B}_{cov,1}^s}{2N} + \mathcal{B}_{cov,2}^s + \mathcal{B}_{cov,3}^s \right) + 2(G + (2n + 9)\hat{D})N^2\rho\hat{\Gamma}\gamma^t \\ &\quad + \frac{N}{2\alpha(t)} (\mathbb{E}[\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2] - \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}_s^*\|^2]) + \frac{G}{2N} (3G + 2(n+4)N\hat{D})\alpha(t) \\ &\quad + (G + (2n + 9)\hat{D})NG\hat{\Gamma} \sum_{r=1}^{t+1} \gamma^{t-r}\alpha(r-1). \end{aligned}$$

Summing up from $t = 0$ to T , and denoting by $\mathcal{B}_T^s = \sum_{t=0}^T (\mathcal{B}_{cov,1}^s/2N + \mathcal{B}_{cov,2}^s + \mathcal{B}_{cov,3}^s)$, we obtain

$$\begin{aligned} \sum_{t=0}^T (\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f_\mu^t(\mathbf{x}_s^*)) &\leq \frac{\mathcal{B}_T^s}{\alpha(T)} + 2(G + (2n + 9)\hat{D})N^2\rho\hat{\Gamma} \sum_{t=0}^T \gamma^t \\ &\quad + (G + (2n + 9)\hat{D})NG\hat{\Gamma} \sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r}\alpha(r-1) \\ &\quad + \sum_{t=0}^T \frac{N}{2\alpha(t)} (\mathbb{E}[\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2] - \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}_s^*\|^2]) \\ &\quad + \frac{G}{2N} (3G + 2(n+4)N\hat{D}) \sum_{t=0}^T \alpha(t). \end{aligned} \quad (5.24)$$

where we have used the fact that $\alpha(t)$ is non-increasing. It is obvious that $\mathbb{E}[\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2]$ is bounded due to the bounded domain. Denoting its bound by $\hat{V}^s$, we have

$$\begin{aligned} &\sum_{t=0}^T \frac{1}{\alpha(t)} (\mathbb{E}[\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2] - \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{x}_s^*\|^2]) \\ &\leq \frac{\mathbb{E}[\|\bar{\mathbf{z}}(0) - \mathbf{x}_s^*\|^2]}{\alpha(0)} + \sum_{t=1}^T \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(t-1)} \right) \mathbb{E}[\|\bar{\mathbf{z}}(t) - \mathbf{x}_s^*\|^2] \\ &\leq \frac{\hat{V}^s}{\alpha(0)} + \hat{V}^s \sum_{t=1}^T \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(t-1)} \right) \leq \frac{2\hat{V}^s}{\alpha(T)}. \end{aligned}$$

Substituting the above relation to (5.24) and using the result of $f_\mu^t(\mathbf{x}) \leq f^t(\mathbf{x}) +$

$\sqrt{n}N\hat{\mu}\hat{D}$ from Lemma 5.1-(1), yields

$$\begin{aligned}
\sum_{t=0}^T (\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_s^*)) &\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} \\
&+ \frac{\mathcal{B}_T^s + N\hat{V}^s}{\alpha(T)} + 2(G + (2n+9)\hat{D})N^2\rho\hat{\Gamma} \sum_{t=0}^T \gamma^t \\
&+ (G + (2n+9)\hat{D})NG\hat{\Gamma} \sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r}\alpha(r-1) \\
&+ \frac{G}{2N}(3G + 2(n+4)N\hat{D}) \sum_{t=0}^T \alpha(t). \tag{5.25}
\end{aligned}$$

Considering the static regret $\mathcal{R}_i^s(T)$

$$\begin{aligned}
\mathcal{R}_i^s(T) &= \sum_{t=0}^T \left(\mathbb{E}[f^t(\mathbf{z}^i(t))] - f^t(\mathbf{x}_s^*) \right) \\
&= \sum_{t=0}^T \left(\mathbb{E}[f^t(\mathbf{z}^i(t)) - f^t(\bar{\mathbf{z}}(t))] \right) + \sum_{t=0}^T \left(\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_s^*) \right) \\
&\leq N\hat{D} \sum_{t=0}^T \mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|] + \sum_{t=0}^T \left(\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_s^*) \right) \\
&\leq 2N^2\rho\hat{D}\hat{\Gamma} \sum_{t=0}^T \gamma^t + N\hat{D}G\hat{\Gamma} \sum_{t=0}^T \sum_{r=1}^t \gamma^{t-r}\alpha(r-1) + \sum_{t=0}^T \left(\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_s^*) \right),
\end{aligned}$$

where the first inequality follows from $f_i^t(\mathbf{z}^i(t)) - f_i^t(\bar{\mathbf{z}}(t)) \leq \partial f_i^t(\mathbf{z}^i(t))^T(\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)) \leq \hat{D}\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|$ based on Assumption 5.2, and the second inequality is a consequence of Theorem 5.1. Combining the result of (5.25), we have

$$\begin{aligned}
\mathcal{R}_i^s(T) &\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \frac{\mathcal{B}_T^s + N\hat{V}^s}{\alpha(T)} + 2(G + 2(n+5)\hat{D})N^2\rho\hat{\Gamma} \sum_{t=0}^T \gamma^t \\
&+ (G + 2(n+5)\hat{D})NG\hat{\Gamma} \sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r}\alpha(r-1) \\
&+ \frac{G}{2N}(3G + 2(n+4)N\hat{D}) \sum_{t=0}^T \alpha(t) \\
&\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \frac{\mathcal{B}_T^s + N\hat{V}^s}{\alpha(T)} + \frac{2(G + 2(n+5)\hat{D})N^2\rho\hat{\Gamma}}{1-\gamma}
\end{aligned}$$

$$+ \left(\frac{(G + 2(n + 5)\hat{D})NG\hat{\Gamma}}{\gamma(1 - \gamma)} + \frac{G(3G + 2(n + 4)N\hat{D})}{2N} \right) \sum_{t=0}^T \alpha(t)$$

where we have used the result that $\sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r} \alpha(r-1) \leq \frac{1}{\gamma(1-\gamma)} \sum_{t=0}^T \alpha(t)$ in the second inequality. Let $\alpha(t) = 1/\sqrt{t+1}$. Noting that $\sum_{t=0}^T \frac{1}{\sqrt{t+1}} \leq 1 + \int_0^{T+1} \frac{dt}{\sqrt{t+1}} \leq 2\sqrt{T+1}$, we have

$$\begin{aligned} \mathcal{R}_i^s(T) &\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \sqrt{T+1}(\mathcal{B}_T^s + N\hat{V}^s) \\ &\quad + \sqrt{T+1} \left(\frac{2(G + 2(n + 5)\hat{D})NG\hat{\Gamma}}{\gamma(1 - \gamma)} + \frac{G(3G + 2(n + 4)N\hat{D})}{N} \right) \\ &\quad + \frac{2(G + 2(n + 5)\hat{D})N^2\rho\hat{\Gamma}}{1 - \gamma}, \end{aligned}$$

which immediately leads to the desired result. $\blacksquare$

Remark 5.2. Theorem 5.2 quantifies the bound of the static regret $\mathcal{R}_i^s(T)$. It is obvious that $\mathcal{C}_1^s$ and $\mathcal{C}_2^s$ are bounded when T approaches ∞ , as long as $\mathcal{B}_T^s$ is not increasing too fast with respect to T , i.e., $\lim_{T \rightarrow \infty} \mathcal{B}_T^s/\sqrt{T} = 0$. Then it can be further obtained that $\mathcal{R}_i^s(T)/T = \sqrt{n}N\hat{\mu}\hat{D} + O(1/\sqrt{T})$. It implies that if the cumulative covariance sum is not increasing too fast, the average static regret over a period of time T will be bounded by two parts. The first part is the penalty due to the use of gradient-free oracle instead of the true gradient information, which can be moderated by choosing small smoothing parameter $\hat{\mu}$. The second part measures how fast the proposed algorithm converges, which will be in the order of $O(1/\sqrt{T})$. As the time duration T goes to infinity, the average static regret approximately converges to zero with an error bounded by a constant depending on the smoothing parameters $\hat{\mu}$.

Last, we establish a bound for the dynamic regret $\mathcal{R}_i^d(T)$, which is summarized in the following theorem.

Theorem 5.3. Suppose Assumptions 5.1 and 5.2 hold. Let $\{\mathbf{z}^i(t)\}_{t \geq 0}$ be the sequence generated by (5.5) with the step-size sequence $\alpha(t) = 1/\sqrt{t+1}$. Then, for

any $i \in \mathcal{V}$ and time duration $T > 0$, the dynamic regret $\mathcal{R}_i^d(T)$ satisfies

$$\mathcal{R}_i^d(T) \leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \mathcal{C}_1^d + \mathcal{C}_2^d\sqrt{T+1},$$

where

$$\begin{aligned} \mathcal{C}_1^d &= (2N\rho G\hat{\Gamma} + 4N^2\rho^2\hat{L}\hat{\Gamma} + 2(n+5)N^2\rho\hat{D}\hat{\Gamma})/(1-\gamma) \\ \mathcal{C}_2^d &= N\hat{V}^d + 2N\rho\Theta_T + N\mathcal{B}_T^d + N(n+4)^2\hat{D}^2/2 + (G^2\hat{\Gamma} \\ &\quad + 2N\rho\hat{L}G\hat{\Gamma} + (n+5)N\hat{D}G\hat{\Gamma})/\gamma(1-\gamma), \end{aligned}$$

$\hat{V}^d$ is a positive constant, Θ_T is defined in (5.2), $\mathcal{B}_T^d = \sum_{t=0}^T(\mathcal{B}_{cov,1}^d + \mathcal{B}_{cov,2}^d)$, $\mathcal{B}_{cov,1}^d$ and $\mathcal{B}_{cov,2}^d$ are the upper bounds of some covariance terms.

Proof: Let us define a positive scalar function $V(t)$ as

$$V(t) = \frac{1}{2}\langle \bar{\mathbf{z}}(t) - \mathbf{x}_d^*(t), \bar{\mathbf{z}}(t) - \mathbf{x}_d^*(t) \rangle,$$

where $\bar{\mathbf{z}}(t)$ is defined in (5.6) and $\mathbf{x}_d^*(t)$ is the optimal solution of (5.1) at time-step t . Then, it can be obtained that

$$\begin{aligned} \Delta V(t) &= V(t+1) - V(t) \\ &= -\frac{1}{2}\|\bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t)\|^2 + \frac{1}{2}\langle \mathbf{x}_d^*(t+1) + \mathbf{x}_d^*(t) \\ &\quad - 2\bar{\mathbf{z}}(t+1), \mathbf{x}_d^*(t+1) - \mathbf{x}_d^*(t) \rangle + \langle \bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t), \bar{\mathbf{z}}(t+1) - \mathbf{x}_d^*(t) \rangle \\ &\leq -\frac{1}{2}\|\bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t)\|^2 + 2\rho\|\mathbf{x}_d^*(t+1) - \mathbf{x}_d^*(t)\| \\ &\quad + \langle \bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t), \bar{\mathbf{z}}(t+1) - \mathbf{x}_d^*(t) \rangle, \end{aligned} \tag{5.26}$$

where we have used $\rho = \sup_{\mathbf{x} \in \mathcal{X}} \|\mathbf{x}\|$. Then, we will bound the dynamic regret $\mathcal{R}_i^d(T)$ in the following two steps.

Step 1. Bound of function $\Delta V(t)$:

According to (5.6), we can further expand the last term in (5.26) as follows

$$\begin{aligned} \langle \bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t), \bar{\mathbf{z}}(t+1) - \mathbf{x}_d^*(t) \rangle &= \left\langle \frac{1}{N} \sum_{i=1}^N g^i(t), \bar{\mathbf{z}}(t+1) - \mathbf{x}_d^*(t) \right\rangle \\ &= \frac{1}{N} \sum_{i=1}^N \langle g^i(t), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle \end{aligned} \quad (5.27a)$$

$$+ \frac{1}{N} \sum_{i=1}^N \langle g^i(t), \mathbf{z}^i(t+1) - \mathbf{x}_d^*(t) \rangle. \quad (5.27b)$$

For (5.27a), we have

$$\frac{1}{N} \sum_{i=1}^N \langle g^i(t), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle \leq \frac{1}{N} \sum_{i=1}^N \|g^i(t)\| \|\bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1)\|.$$

Taking the conditional expectation on $\mathcal{F}_t$ on both sides, we have

$$\begin{aligned} &\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle g^i(t), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle \middle| \mathcal{F}_t \right] \\ &\leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t] \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1)\| | \mathcal{F}_t] + \mathcal{B}_{cov,1}^d, \end{aligned}$$

where $\mathcal{B}_{cov,1}^d > 0$ is an upper bound of the covariance term $Cov(g^i(t), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1))$. Applying Lemma 5.3 on $\sum_{i=1}^N \mathbb{E}[\|g^i(t)\| | \mathcal{F}_t]$ and taking the total expectation, we have

$$\begin{aligned} &\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle g^i(t), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle \right] \\ &\leq \frac{1}{N} G\alpha(t) \mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1)\|] + \mathcal{B}_{cov,1}^d \\ &\leq \frac{1}{N} G\alpha(t) \left(2N\rho\hat{\Gamma}\gamma^{t+1} + G\hat{\Gamma} \sum_{r=1}^{t+1} \gamma^{t-r+1} \alpha(r-1) \right) + \mathcal{B}_{cov,1}^d \\ &\leq 2\rho G\hat{\Gamma}\alpha(t)\gamma^{t+1} + \frac{G^2\hat{\Gamma}}{N} \alpha(t) \sum_{r=1}^{t+1} \gamma^{t-r+1} \alpha(r-1) + \mathcal{B}_{cov,1}^d, \end{aligned} \quad (5.28)$$

where we have applied Theorem 5.1.

For (5.27b), we have

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \langle g^i(t), \mathbf{z}^i(t+1) - \mathbf{x}_d^*(t) \rangle \\ &= \frac{1}{N} \sum_{i=1}^N \langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{z}^i(t+1) - \mathbf{x}_d^*(t) \rangle \end{aligned} \quad (5.29a)$$

$$+ \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \mathbf{z}^i(t+1) \rangle. \quad (5.29b)$$

For (5.29a), it follows from the projection's non-expansive property that

$$\langle g^i(t) + \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{z}^i(t+1) - \mathbf{x}_d^*(t) \rangle \leq 0. \quad (5.30)$$

For (5.29b), it can be further expanded as

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \mathbf{z}^i(t+1) \rangle \\ &= \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \end{aligned} \quad (5.31a)$$

$$+ \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1) \rangle \quad (5.31b)$$

$$+ \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle. \quad (5.31c)$$

For (5.31a), it can be further expanded as

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \\ &= \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)(g_{\mu^i}^t(\mathbf{z}^i(t)) - \nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t))), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \\ &+ \frac{1}{N} \sum_{i=1}^N \langle \alpha(t)\nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_t$ and applying Lemma 5.1-(2) on $\mathbb{E}[g_{\mu^i}^t(\mathbf{z}^i(t))|\mathcal{F}_t]$,

we have

$$\begin{aligned}
& \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \middle| \mathcal{F}_t \right] \\
&= \frac{1}{N} \sum_{i=1}^N \langle \alpha(t) (\nabla f_{i,\mu^i}^t(\mathbf{z}^i(t)) - \nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t))), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \\
&\quad + \frac{1}{N} \sum_{i=1}^N \langle \alpha(t) \nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \\
&\leq \frac{1}{N} \sum_{i=1}^N \alpha(t) \|\nabla f_{i,\mu^i}^t(\mathbf{z}^i(t)) - \nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t))\| \|\mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t)\| \\
&\quad + \frac{1}{N} \sum_{i=1}^N \langle \alpha(t) \nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle.
\end{aligned}$$

Noting that $\|\mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t)\| \leq 2\rho$ and applying Lemma 5.1-(2) on $\|\nabla f_{i,\mu^i}^t(\mathbf{z}^i(t)) - \nabla f_{i,\mu^i}^t(\bar{\mathbf{z}}(t))\|$, we have

$$\begin{aligned}
& \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \middle| \mathcal{F}_t \right] \\
&\leq \frac{2\rho \hat{L} \alpha(t)}{N} \sum_{i=1}^N \|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\| + \frac{1}{N} \sum_{i=1}^N \alpha(t) (f_{i,\mu^i}^t(\mathbf{x}_d^*(t)) - f_{i,\mu^i}^t(\bar{\mathbf{z}}(t))) \\
&\leq \frac{2\rho \hat{L} \alpha(t)}{N} \sum_{i=1}^N \|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\| + \frac{1}{N} \alpha(t) (f_{\mu}^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))),
\end{aligned}$$

where we have used the convex property of $f_{i,\mu^i}^t(\mathbf{x})$ in the first inequality and Lemma 5.1-(1) in the second inequality. Taking the total expectation and applying

Theorem 5.1 on $\mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|]$, we have

$$\begin{aligned}
& \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \mathbf{x}_d^*(t) - \bar{\mathbf{z}}(t) \rangle \right] \\
& \leq \frac{2\rho \hat{L} \alpha(t)}{N} \sum_{i=1}^N \left(2N\rho \hat{\Gamma} \gamma^t + G\hat{\Gamma} \sum_{r=1}^t \gamma^{t-r} \alpha(r-1) \right) \\
& \quad + \frac{1}{N} \alpha(t) (f_{\mu}^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))) \\
& \leq 4N\rho^2 \hat{L} \hat{\Gamma} \alpha(t) \gamma^t + 2\rho \hat{L} G \hat{\Gamma} \alpha(t) \sum_{r=1}^t \gamma^{t-r} \alpha(r-1) \\
& \quad + \frac{1}{N} \alpha(t) (f_{\mu}^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))).
\end{aligned} \tag{5.32}$$

For (5.31b), it can be obtained that

$$\begin{aligned}
& \frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1) \rangle \\
& \leq \frac{1}{2} \left\| \frac{1}{N} \sum_{i=1}^N \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)) \right\|^2 + \frac{1}{2} \|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2 \\
& \leq \frac{1}{2N} \alpha^2(t) \sum_{i=1}^N \|g_{\mu^i}^t(\mathbf{z}^i(t))\|^2 + \frac{1}{2} \|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2.
\end{aligned}$$

Taking the total expectation and applying Lemma 5.1-(3) on $\mathbb{E}[g_{\mu^i}^t(\mathbf{z}^i(t))\|^2]$ yields

$$\begin{aligned}
& \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1) \rangle \right] \\
& \leq \frac{(n+4)^2 \hat{D}^2}{2} \alpha^2(t) + \frac{1}{2} \mathbb{E}[\|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2].
\end{aligned} \tag{5.33}$$

For (5.31c), it can be obtained that

$$\begin{aligned}
& \frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle \\
& \leq \frac{1}{N} \sum_{i=1}^N \alpha(t) \|g_{\mu^i}^t(\mathbf{z}^i(t))\| \|\bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1)\|
\end{aligned}$$

Taking the total expectation and applying Lemma 5.1-(3) on $\mathbb{E}[g_{\mu^i}^t(\mathbf{z}^i(t))\|^2]$ and Theorem 5.1 on $\mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1)\|]$ yields

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle \alpha(t) g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1) \rangle \right] \\ & \leq (n+4) \hat{D} \alpha(t) \left(2N \rho \hat{\Gamma} \gamma^{t+1} + G \hat{\Gamma} \sum_{r=1}^{t+1} \gamma^{t-r+1} \alpha(r-1) \right) + \mathcal{B}_{cov,2}^d \quad (5.34) \\ & \leq 2(n+4) N \rho \hat{D} \hat{\Gamma} \alpha(t) \gamma^{t+1} + (n+4) \hat{D} G \hat{\Gamma} \alpha(t) \sum_{r=1}^{t+1} \gamma^{t-r+1} \alpha(r-1) + \mathcal{B}_{cov,2}^d, \end{aligned}$$

where $\mathcal{B}_{cov,2}^d > 0$ is an upper bound of the covariance term $Cov(g_{\mu^i}^t(\mathbf{z}^i(t)), \bar{\mathbf{z}}(t+1) - \mathbf{z}^i(t+1))$. Taking the total expectation for (5.29), and combining the results of (5.30) and (5.31) with substitutions of (5.32), (5.33) and (5.34), we have

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \langle g^i(t), \mathbf{z}^i(t+1) - \mathbf{x}_d^*(t) \rangle \right] \\ & \leq 4N \rho^2 \hat{L} \hat{\Gamma} \alpha(t) \gamma^t + 2\rho \hat{L} G \hat{\Gamma} \alpha(t) \sum_{r=1}^t \gamma^{t-r} \alpha(r-1) \\ & \quad + \frac{1}{N} \alpha(t) (f_{\mu}^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))) + \frac{(n+4)^2 \hat{D}^2}{2} \alpha^2(t) \\ & \quad + \frac{1}{2} \mathbb{E}[\|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2] + 2(n+4) N \rho \hat{D} \hat{\Gamma} \alpha(t) \gamma^{t+1} \\ & \quad + (n+4) \hat{D} G \hat{\Gamma} \alpha(t) \sum_{r=1}^{t+1} \gamma^{t-r+1} \alpha(r-1) + \mathcal{B}_{cov,2}^d \\ & \leq (4N \rho^2 \hat{L} \hat{\Gamma} + 2(n+4) N \rho \hat{D} \hat{\Gamma}) \alpha(t) \gamma^t \\ & \quad + (2\rho \hat{L} G \hat{\Gamma} + (n+4) \hat{D} G \hat{\Gamma}) \alpha(t) \sum_{r=1}^{t+1} \gamma^{t-r} \alpha(r-1) \\ & \quad + \frac{1}{N} \alpha(t) (f_{\mu}^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))) + \frac{(n+4)^2 \hat{D}^2}{2} \alpha^2(t) \\ & \quad + \frac{1}{2} \mathbb{E}[\|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2] + \mathcal{B}_{cov,2}^d. \quad (5.35) \end{aligned}$$

Taking the total expectation for (5.27), and substituting (5.28) and (5.35) into

(5.27a) and (5.27b), we have

$$\begin{aligned}
& \mathbb{E}[\langle \bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t), \bar{\mathbf{z}}(t+1) - \mathbf{x}_d^*(t) \rangle] \\
& \leq 2\rho G\hat{\Gamma}\alpha(t)\gamma^{t+1} + \frac{G^2\hat{\Gamma}}{N}\alpha(t)\sum_{r=1}^{t+1}\gamma^{t-r+1}\alpha(r-1) \\
& \quad + (4N\rho^2\hat{L}\hat{\Gamma} + 2(n+4)N\rho\hat{D}\hat{\Gamma})\alpha(t)\gamma^t \\
& \quad + (2\rho\hat{L}G\hat{\Gamma} + (n+4)\hat{D}G\hat{\Gamma})\alpha(t)\sum_{r=1}^{t+1}\gamma^{t-r}\alpha(r-1) \\
& \quad + \frac{1}{N}\alpha(t)(f_\mu^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))) + \frac{(n+4)^2\hat{D}^2}{2}\alpha^2(t) \\
& \quad + \frac{1}{2}\mathbb{E}[\|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2] + \mathcal{B}_{cov,1}^d + \mathcal{B}_{cov,2}^d \\
& \leq (2\rho G\hat{\Gamma} + 4N\rho^2\hat{L}\hat{\Gamma} + 2(n+4)N\rho\hat{D}\hat{\Gamma})\alpha(t)\gamma^t \\
& \quad + \left(\frac{G^2\hat{\Gamma}}{N} + 2\rho\hat{L}G\hat{\Gamma} + (n+4)\hat{D}G\hat{\Gamma}\right)\alpha(t)\sum_{r=1}^{t+1}\gamma^{t-r}\alpha(r-1) \\
& \quad + \frac{1}{N}\alpha(t)(f_\mu^t(\mathbf{x}_d^*(t)) - f^t(\bar{\mathbf{z}}(t))) + \frac{(n+4)^2\hat{D}^2}{2}\alpha^2(t) \\
& \quad + \frac{1}{2}\mathbb{E}[\|\bar{\mathbf{z}}(t) - \bar{\mathbf{z}}(t+1)\|^2] + \mathcal{B}_{cov,1}^d + \mathcal{B}_{cov,2}^d. \tag{5.36}
\end{aligned}$$

Taking the total expectation for (5.26), and substituting (5.36) into it, we can obtain

$$\begin{aligned}
\mathbb{E}[\Delta V(t)] & \leq -\frac{1}{2}\mathbb{E}[\|\bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t)\|^2] + 2\rho\|\mathbf{x}_d^*(t+1) - \mathbf{x}_d^*(t)\| \\
& \quad + \mathbb{E}[\langle \bar{\mathbf{z}}(t+1) - \bar{\mathbf{z}}(t), \bar{\mathbf{z}}(t+1) - \mathbf{x}_d^*(t) \rangle] \\
& \leq 2\rho\|\mathbf{x}_d^*(t+1) - \mathbf{x}_d^*(t)\| + \frac{1}{N}\alpha(t)(f_\mu^t(\mathbf{x}_d^*(t)) - \mathbb{E}[f^t(\bar{\mathbf{z}}(t))]) \\
& \quad + (2\rho G\hat{\Gamma} + 4N\rho^2\hat{L}\hat{\Gamma} + 2(n+4)N\rho\hat{D}\hat{\Gamma})\alpha(t)\gamma^t \\
& \quad + \left(\frac{G^2\hat{\Gamma}}{N} + 2\rho\hat{L}G\hat{\Gamma} + (n+4)\hat{D}G\hat{\Gamma}\right)\alpha(t)\sum_{r=1}^{t+1}\gamma^{t-r}\alpha(r-1) \\
& \quad + \frac{(n+4)^2\hat{D}^2}{2}\alpha^2(t) + \mathcal{B}_{cov,1}^d + \mathcal{B}_{cov,2}^d. \tag{5.37}
\end{aligned}$$

It is obvious that $V(t) = \frac{1}{2}\|\bar{\mathbf{z}}(t) - \mathbf{x}_d^*(t)\|^2$ is bounded due to the bounded domain.

Denoting the bound of $V(t)$ by $\hat{V}^d$, we have

$$\begin{aligned} -\sum_{t=0}^T \frac{\mathbb{E}[\Delta V(t)]}{\alpha(t)} &= \sum_{t=0}^T \frac{\mathbb{E}[V(t)] - \mathbb{E}[V(t+1)]}{\alpha(t)} \\ &\leq \frac{\mathbb{E}[V(0)]}{\alpha(0)} + \sum_{t=1}^T \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(t-1)} \right) \mathbb{E}[V(t)] \\ &\leq \frac{\hat{V}^d}{\alpha(0)} + \hat{V}^d \sum_{t=1}^T \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(t-1)} \right) = \frac{\hat{V}^d}{\alpha(T)}, \end{aligned}$$

where we have used the fact that $\alpha(t)$ is positive and non-increasing. Dividing both sides of (5.37) by $\alpha(t)$, summing up from $t = 0$ to T and combining the above relation, we have

$$\begin{aligned} \sum_{t=0}^T (\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f_\mu^t(\mathbf{x}_d^*(t))) &\leq \frac{N\hat{V}^d}{\alpha(T)} + \frac{2N\rho\Theta_T}{\alpha(T)} + \frac{N(n+4)^2\hat{D}^2}{2} \sum_{t=0}^T \alpha(t) \\ &\quad + (2N\rho G\hat{\Gamma} + 4N^2\rho^2\hat{L}\hat{\Gamma} + 2(n+4)N^2\rho\hat{D}\hat{\Gamma}) \sum_{t=0}^T \gamma^t \\ &\quad + (G^2\hat{\Gamma} + 2N\rho\hat{L}G\hat{\Gamma} + (n+4)N\hat{D}G\hat{\Gamma}) \sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r} \alpha(r-1) \\ &\quad + \frac{N}{\alpha(T)} \sum_{t=0}^T (\mathcal{B}_{cov,1}^d + \mathcal{B}_{cov,2}^d), \end{aligned}$$

where we have used that $\alpha(t)$ is positive and non-increasing.

Step 2. Bound of dynamic regret $\mathcal{R}_i^d(T)$:

Denoting $\sum_{t=0}^T (\mathcal{B}_{cov,1}^d + \mathcal{B}_{cov,2}^d)$ by $\mathcal{B}_T^d$ and using the result of $f_\mu^t(\mathbf{x}) \leq f^t(\mathbf{x}) + \sqrt{n}N\hat{\mu}\hat{D}$ from Lemma 5.1-(1), we can obtain

$$\begin{aligned} \sum_{t=0}^T (\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_d^*(t))) &\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} \\ &\quad + \frac{N\hat{V}^d + 2N\rho\Theta_T + N\mathcal{B}_T^d}{\alpha(T)} + \frac{N(n+4)^2\hat{D}^2}{2} \sum_{t=0}^T \alpha(t) \end{aligned}$$

$$\begin{aligned}
& + (2N\rho G\hat{\Gamma} + 4N^2\rho^2\hat{L}\hat{\Gamma} + 2(n+4)N^2\rho\hat{D}\hat{\Gamma}) \sum_{t=0}^T \gamma^t \\
& + G(G\hat{\Gamma} + 2N\rho\hat{L}\hat{\Gamma} + (n+4)N\hat{D}\hat{\Gamma}) \sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r} \alpha(r-1). \tag{5.38}
\end{aligned}$$

Considering the dynamic regret $\mathcal{R}_i^d(T)$

$$\begin{aligned}
\mathcal{R}_i^d(T) &= \sum_{t=0}^T \left(\mathbb{E}[f^t(\mathbf{z}^i(t))] - f^t(\mathbf{x}_d^*(t)) \right) \\
&= \sum_{t=0}^T \left(\mathbb{E}[f^t(\mathbf{z}^i(t)) - f^t(\bar{\mathbf{z}}(t))] \right) + \sum_{t=0}^T \left(\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_d^*(t)) \right) \\
&\leq N\hat{D} \sum_{t=0}^T \mathbb{E}[\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|] + \sum_{t=0}^T \left(\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_d^*(t)) \right) \\
&\leq 2N^2\rho\hat{D}\hat{\Gamma} \sum_{t=0}^T \gamma^t + N\hat{D}G\hat{\Gamma} \sum_{t=0}^T \sum_{r=1}^t \gamma^{t-r} \alpha(r-1) \\
&\quad + \sum_{t=0}^T \left(\mathbb{E}[f^t(\bar{\mathbf{z}}(t))] - f^t(\mathbf{x}_d^*(t)) \right),
\end{aligned}$$

where the first inequality follows from $f_i^t(\mathbf{z}^i(t)) - f_i^t(\bar{\mathbf{z}}(t)) \leq \langle \partial f_i^t(\mathbf{z}^i(t)), \mathbf{z}^i(t) - \bar{\mathbf{z}}(t) \rangle \leq \hat{D}\|\mathbf{z}^i(t) - \bar{\mathbf{z}}(t)\|$ based on Assumption 5.2, and the second inequality is a consequence of Theorem 5.1. Combining the result of (5.38), we have

$$\begin{aligned}
\mathcal{R}_i^d(T) &\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \frac{N\hat{V}^d + 2N\rho\Theta_T + N\mathcal{B}_T^d}{\alpha(T)} \\
&\quad + (G^2\hat{\Gamma} + 2N\rho\hat{L}G\hat{\Gamma} + (n+5)N\hat{D}G\hat{\Gamma}) \sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r} \alpha(r-1) \\
&\quad + (2N\rho G\hat{\Gamma} + 4N^2\rho^2\hat{L}\hat{\Gamma} + 2(n+5)N^2\rho\hat{D}\hat{\Gamma}) \sum_{t=0}^T \gamma^t \\
&\quad + \frac{N(n+4)^2\hat{D}^2}{2} \sum_{t=0}^T \alpha(t) \\
&\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \frac{N\hat{V}^d + 2N\rho\Theta_T + N\mathcal{B}_T^d}{\alpha(T)} \\
&\quad + \left(\frac{N(n+4)^2\hat{D}^2}{2} + \frac{G^2\hat{\Gamma} + 2N\rho\hat{L}G\hat{\Gamma} + (n+5)N\hat{D}G\hat{\Gamma}}{\gamma(1-\gamma)} \right) \sum_{t=0}^T \alpha(t)
\end{aligned}$$

$$+ \frac{2N\rho G\hat{\Gamma} + 4N^2\rho^2\hat{L}\hat{\Gamma} + 2(n+5)N^2\rho\hat{D}\hat{\Gamma}}{1-\gamma},$$

where we have used the result that $\sum_{t=0}^T \sum_{r=1}^{t+1} \gamma^{t-r} \alpha(r-1) \leq \frac{1}{\gamma(1-\gamma)} \sum_{t=0}^T \alpha(t)$ in the second inequality. Let $\alpha(t) = 1/\sqrt{t+1}$. Noting that $\sum_{t=0}^T \frac{1}{\sqrt{t+1}} \leq 1 + \int_0^{T+1} \frac{dt}{\sqrt{t+1}} \leq 2\sqrt{T+1}$, we have

$$\begin{aligned} \mathcal{R}_i^d(T) &\leq (T+1)\sqrt{n}N\hat{\mu}\hat{D} + \sqrt{T+1}(N\hat{V}^d + 2N\rho\Theta_T + N\mathcal{B}_T^d) \\ &\quad + \sqrt{T+1} \left(N(n+4)^2\hat{D}^2 + \frac{2G^2\hat{\Gamma} + 4N\rho\hat{L}G\hat{\Gamma} + 2(n+5)N\hat{D}G\hat{\Gamma}}{\gamma(1-\gamma)} \right) \\ &\quad + \frac{2N\rho G\hat{\Gamma} + 4N^2\rho^2\hat{L}\hat{\Gamma} + 2(n+5)N^2\rho\hat{D}\hat{\Gamma}}{1-\gamma}, \end{aligned}$$

which immediately leads to the desired result. $\blacksquare$

Remark 5.3. Theorem 5.3 quantifies the bound of the dynamic regret $\mathcal{R}_i^d(T)$. It is obvious that $\mathcal{C}_1^d$ is bounded when T approaches ∞ . For $\mathcal{C}_2^d$, it is generally unbounded due to the terms Θ_T and $\mathcal{B}_T^d$ when T approaches ∞ . However, if both Θ_T and $\mathcal{B}_T^d$ are not increasing too fast with respect to T , i.e., $\lim_{T \rightarrow \infty} \Theta_T/\sqrt{T} = 0$ and $\lim_{T \rightarrow \infty} \mathcal{B}_T^d/\sqrt{T} = 0$, then it can be further obtained that $\mathcal{R}_i^d(T)/T = \sqrt{n}N\hat{\mu}\hat{D} + O(1/\sqrt{T})$. It implies that if the variation of the optimal solution sequence and the cumulative covariance sum are not increasing too fast, the average dynamic regret over a period of time T will be bounded by two parts. The first part is the penalty due to the use of gradient-free oracle instead of the true gradient information, which can be moderated by choosing small smoothing parameter $\hat{\mu}$. The second part measures how fast the proposed algorithm converges, which will be in the order of $O(1/\sqrt{T})$. As the time duration T goes to infinity, the average dynamic regret approximately converges to zero with an error bounded by a constant depending on the smoothing parameters $\hat{\mu}$.

5.4 Numerical Simulation

5.4.1 Numerical Example

In this part, we use a numerical example to verify the derived properties. In particular, we consider the following online distributed optimization problem in a multi-agent system with 10 agents under a directed communication graph:

$$\min f^t(x) = \sum_{i=1}^{10} f_i^t(x), x \in \mathcal{X},$$

where $f_i^t(x) = a_i x^2 - 2b_i x + c_i \left(\frac{2 \sin(0.008t)}{t}\right)^2$ at time-step t , $\mathcal{X} = [-1, 1]$, $a_i, b_i, c_i > 0$ and $\sum_{i=1}^{10} a_i = \sum_{i=1}^{10} b_i = \sum_{i=1}^{10} c_i = 10$. Our proposed online randomized gradient-free DPGD method will be used to solve this problem. For the weighting matrix A_r and A_c , we let $[A_r]_{ij} = 1/|\mathcal{N}_i^{\text{in}}|$ and $[A_c]_{ij} = 1/|\mathcal{N}_j^{\text{out}}|$. Throughout the simulation, we set $\epsilon = 0.1$, $\hat{\mu} = 10^{-4}$ and the step-size $\alpha(t) = 1/\sqrt{1+t}$.

For any $t > 0$, it is obvious that the best fixed decision for a given time duration $T > 0$ is $\mathbf{x}_s^* = \frac{1}{T} \sum_{t=1}^T \frac{2 \sin(0.008t)}{t}$; and the optimal solution at time-step t is $\mathbf{x}_d^*(t) = \frac{2 \sin(0.008t)}{t}$, which is time-varying. As t increases, the deviation of the optimal solution $\mathbf{x}_d^*(t)$ decreases gradually. We applied the proposed algorithm (Algorithm 5.1) to the problem. The trajectories of each agent's decision variable and the optimal solution were plotted in Fig. 5.1. As can be seen from Fig. 5.1, all agents converge to a single trajectory, which verifies the consensus property established in Theorem 5.1. Besides, the consensus trajectory formed by all agents gradually coincides with the trajectory of the optimal solution, which verifies the optimality property established in Theorems 5.2 and 5.3.

On the other hand, the average static regret $\mathcal{R}_i^s(T)/T$ and dynamic regret $\mathcal{R}_i^d(T)/T$ were shown in Figs. 5.2 and 5.3, respectively. It can be observed that both average static and dynamic regrets descend to zero asymptotically, which is consistent with

Figure 5.1: Trajectories of $x^i(t)$.

our analysis in Remarks 5.2 and 5.3.

Figure 5.2: Trajectories of $\mathcal{R}_i^s(t)/t$. Figure 5.3: Trajectories of $\mathcal{R}_i^d(t)/t$.

5.4.2 Temperature Control in HVAC Systems

In this part, we consider a 10-zone HVAC system where the temperature control is formulated as the following distributed Model Predictive Control problem:

$$\min J^t = \sum_{j=1}^{10} \left(\sum_{t=1}^T \lambda \|T^j(t) - T_{ref}^j\| + \sum_{k=0}^{K-1} \|m_s^j(t)\| \right), \quad m_L \leq m_s^j(t) \leq m_U$$

$$T^j(t+1) = \left(1 - \frac{\Delta t}{C^j} \left(\frac{1}{R_a^j} + \sum_{i \in \mathcal{N}^j} \frac{1}{R_{ij}} \right) \right) T^j(t) + \frac{\Delta t}{C^j} P_d^j + \frac{\Delta t}{C^j} c_p m_s^j(t) (T_{s,t}^j - T^j(t)) + \frac{\Delta t}{C^j} \left(\frac{T_{a,t}}{R_a^j} + \sum_{i \in \mathcal{N}^j} \frac{T^i(t)}{R_{ij}} \right),$$

where T^j and m_s^j are the temperature (degrees) and air mass flow (kg/s) of zone j , respectively. C^j and P_d^j are the lumped mass of air and lumped thermal load including all the external factors, such as the occupancy, solar radiation of zone j , respectively. R_a^j and R_{ij} are the thermal resistances between the air from outside and zone j , and the neighboring zone i and zone j , respectively. c_p is the specific heat capacity of air. Δt and K are the sampling time and receding horizon, respectively. λ is a weighting parameter. m_L and m_U are the lower and upper bounds of the input (air mass flow), respectively. The proposed algorithm is implemented in to solve the problem.

In this simulation, we set $m_L = 0$, $m_U = 3$ kg/s, the initial zone temperature $T_0^j = 30$ degrees, the reference temperature for all zones $T_{ref}^1 = T_{ref}^2 = 24$, $T_{ref}^3 = T_{ref}^4 = 24.5$, $T_{ref}^5 = T_{ref}^6 = 25$, $T_{ref}^7 = T_{ref}^8 = 25.5$, $T_{ref}^9 = T_{ref}^{10} = 26$ from $t = 0$ to 2.5 hours and 1 degree off from $t = 2.5$ to 5 hours. Hence the cost function is time-varying. The rest of the model parameters were set according to [117]. Figures 5.4 and 5.5 plotted the trajectories of the air mass flow and the zone temperature for all zones. It can be seen that all the inputs are kept within the constraints, while all the zone temperature can be maintained at their corresponding reference levels. Hence, the effectiveness of the proposed algorithm is verified.

Figure 5.4: Air mass flow (kg/s).

Figure 5.5: Zone temperature (degrees).

5.5 Conclusions

In this chapter, we have investigated the problem of online optimization where the cost functions can be time-varying. To address this problem, we have developed an online randomized gradient-free distributed projected gradient descent algorithm in a multi-agent system where the underlying communication network is directed. The implementation of this algorithm does not require the explicit expressions of the cost functions, but only the measurements. A local randomized gradient-free oracle is built as a replacement of the gradient information in guiding the update of the decision variables. With some standard assumptions on graph connectivity and the cost function, we have characterized the bounds of both static and dynamic regrets for any agent as a small error term plus a sublinear function of the time duration. Finally, numerical simulations have been conducted to verify the effectiveness of the algorithm.

Chapter 6

Distributed Nash Equilibrium Seeking in Non-Cooperative Games

6.1 Introduction

Over the decades, game theory, as a power tool of analyzing the strategic interactions between rational decision-makers, has found its great potential in various application fields such as social science, economics, electricity markets, power systems, to list a few. An important concept in game theory, Nash equilibrium, named after John Forbes Nash Jr., is a proposed solution in non-cooperative games involving two or more players. Recently, with the emergence of multi-agent system, Nash equilibrium seeking in multi-player non-cooperative games has received increasing attention. More precisely, this type of games involves a number of players, who selfishly minimize their own cost functions by making decisions in response to other players' actions.

Recently, a large number of studies on Nash equilibrium computation in non-cooperative

games have been reported, such as [80–84, 88, 91, 92, 96, 98] to list a few. The challenge of such problem settings is the requirement of global knowledge on all players' actions, which is not practical if the underlying communication network is not fully connected. In such cases, players have to make decisions based only on a limited set of information, such as the information from the neighbors. Therefore, a distributed information sharing protocol is usually adopted to disseminate the local information among players. For example, a dynamic average consensus protocol was adopted in [85], where a primal-dual dynamic based seeking strategy was developed to find Nash equilibrium in set constrained aggregate games. It was also utilized in [87] with the help of differential inclusions and differentiated projections for aggregative games, where the players' actions are coupled by linear constraints. The dynamic average consensus protocol was also proposed to achieve simultaneous social cost minimization and Nash equilibrium in a class of N -coalition games in [93]. Different from these works, the work in [89] considered a continuous time generalized convex game with shared inequality constraints among players, and proposed a leader-following consensus protocol with gradient descent method to compute the generalized Nash equilibrium. This protocol was also employed in [90] to estimate the other players' actions for the generalized games, where the players' action sets are constrained by nonlinear inequality and linear equations. Apart from the leader-following consensus and dynamic average consensus protocols, gossip-based averaging techniques were also commonly utilized in Nash equilibrium computations, such as [99, 101]. Most of the existing literature including the aforementioned works are model-based approaches, *i.e.*, the implementation of the algorithms relies on the knowledge of the explicit form of the players' cost functions, such as the derivative computation. However, the requirement of the knowledge on the explicit expression of players' cost functions is restrictive in the cases where the input/output relationship is difficult to model.

There are non-model based approaches, which utilize the players' local measurements without the requirement on the information of the functional form. For

example, the work in [104] considered a generalized convex game with both convex coupling inequality constraints and local set constraints. A finite-differencing method with two-way perturbations was proposed to approximate the partial gradient. The perturbation parameter needs to be chosen carefully to match the selected step-size. Different from that, the work in [105] proposed a distributed payoff-based algorithm for a class of convex games with and without coupling constraints. This technique was further extended in [106] where the algorithm convergence was proved under mere monotonicity assumption. Overall, the payoff-based learning strategy proposed in these two works enables players to sample their actions in a Gaussian distribution. Then, the mean of this distribution is iteratively updated using only local payoff values. Another typical non-model based approaches are extremum seeking-based methods, such as [57, 107–111]. Specifically, the work in [107] proposed a continuous time multi-input stochastic extremum seeking algorithm for the Nash equilibrium seeking in non-cooperative games with general nonlinear cost functions. In [108], a discrete time stochastic extremum seeking method was presented in non-cooperative games where the players' cost functions are strictly convex, but the actions are subject to a linear dynamic constraint. The work in [109] developed an integrator-type extremum seeking algorithm in non-cooperative games with both quadratic payoffs and general non-quadratic payoffs as the output of a dynamic system. More extremum seeking algorithms have been proposed in potential games with unstable dynamics [110], dynamical constraints [111], and non-cooperative games with time-varying Nash equilibrium [57]. In general, the extremum seeking strategy makes use of the cost value together with some sinusoidal dither signals for perturbation, such that the gradient of the cost function is extractable. Even though all the aforementioned works need no explicit model information during the implementation, they assume the players' cost functions to be smooth to some extent, which can be restrictive if the players' cost functions are generally non-differentiable. This motivates the study of gradient-free technique, which is free of the knowledge on the explicit expressions of the players' cost functions and applicable to non-differentiable problems. In fact, gradient-free algorithms have been studied in distributed opti-

mization problems [62–64, 66, 67, 70]. However, they have received little attention in non-cooperative games.

In this chapter, we focus on the research of non-model based Nash equilibrium seeking methods. Specifically, a gradient-free distributed algorithm is proposed to solve the Nash equilibrium seeking problem in a multi-player non-cooperative static game under a directed communication graph. As compared to the existing literature, the major contributions of this chapter are twofold.

- The proposed algorithm does not rely on the knowledge of the explicit form of the players' cost functions. Different from non-model based approaches such as payoff-based learning [105, 106] and extremum seeking [57, 107–111], the proposed algorithm allows the cost functions to be non-smooth. Unlike the finite-differencing method in [104] where the perturbation parameter needs to match the step-size, the proposed algorithm establishes the convergence to the Nash equilibrium with only the requirement of a small smoothing parameter.
- The convergence of the proposed algorithm to the Nash equilibrium is rigorously studied for both diminishing and constant step-sizes, respectively. Specifically, for the diminishing case, an exact convergence to the Nash equilibrium is attained, while for the constant case, an approximate convergence to the Nash equilibrium with the gap proportional to the step-size is achieved.

The chapter is organized as follows. The problem is defined in Section 6.2. Main procedures of the proposed algorithm are described in Section 6.3. The convergence analysis of the proposed algorithm for both diminishing step-size and constant step-size is presented in Section 6.4. In Section 6.5, the performance of the proposed algorithm is illustrated through a numerical example. Section 6.6 concludes the chapter.

6.2 Problem Formulation

Consider a multi-player non-cooperative game, denoted by $\Gamma(N, \{f_i\}, \{\Omega_i\}, \mathcal{G})$, with N players under a directed communication graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. A matrix A associated with the directed graph $\mathcal{G}$ is known as the adjacency matrix, which is designed such that $[A]_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $[A]_{ij} = 0$ otherwise. In particular, the set $\mathcal{E}$ includes (i, i) for all $i \in \mathcal{V}$. If player j is not an out-neighbor of player i (i.e., $j \notin \mathcal{N}_i^{\text{out}}$), then player j does not have direct access to player i 's action. Game Γ is played such that for given $\mathbf{x}_{-i} \in \Omega_{-i}$, the objective of each player $i \in \mathcal{V}$ is to minimize its own cost function, i.e.,

$$\min_{x_i \in \Omega_i} f_i(x_i, \mathbf{x}_{-i}), \quad i \in \mathcal{V}. \quad (6.1)$$

It should be highlighted that the solution set of player i to the problem (6.1) is dependent on the other players' action $\mathbf{x}_{-i}$, which may not be directly accessible. Thus, the objective is to develop a distributed strategy such that all players' actions converge to a Nash equilibrium under the communication graph $\mathcal{G}$.

The following standard assumptions are made throughout the chapter.

Assumption 6.1. *The directed graph $\mathcal{G}$ is strongly connected and its associated adjacency matrix A is doubly-stochastic, i.e., $\sum_{j=1}^N [A]_{ij} = 1$ for all $i \in \mathcal{V}$, and $\sum_{i=1}^N [A]_{ij} = 1$ for all $j \in \mathcal{V}$.*

Assumption 6.2. *For each player $i \in \mathcal{V}$, its action set Ω_i is non-empty, convex and compact. The cost function $f_i(x_i, \mathbf{x}_{-i})$ is convex in x_i for every $\mathbf{x}_{-i}$, and jointly continuous in $\mathbf{x}$ but not necessarily differentiable. Also, $f_i(x_i, \mathbf{x}_{-i})$ is Lipschitz continuous in x_i (respectively, $\mathbf{x}_{-i}$) for every fixed $\mathbf{x}_{-i}$ (respectively, x_i), i.e., $\forall x_i, y_i \in \Omega_i$ (respectively, $\forall \mathbf{x}_{-i}, \mathbf{y}_{-i} \in \Omega_{-i}$), there exists a positive constant D_1 (respectively, D_2) such that $\|f_i(x_i, \mathbf{x}_{-i}) - f_i(y_i, \mathbf{x}_{-i})\| \leq D_1 \|x_i - y_i\|$ (respectively, $\|f_i(x_i, \mathbf{x}_{-i}) - f_i(x_i, \mathbf{y}_{-i})\| \leq D_2 \|\mathbf{x}_{-i} - \mathbf{y}_{-i}\|$).*

Remark 6.1. *Both Assumptions 6.1 and 6.2 are standard and commonly assumed in*

distributed Nash equilibrium seeking problems. In particular, Assumption 6.2 implies that game $\Gamma(N, \{f_i\}, \{\Omega_i\}, \mathcal{G})$ admits a Nash equilibrium [118, 119], [120, Prop.2.2].

6.3 Gradient-Free Distributed NE Seeking

In this section, we describe our proposed distributed Nash equilibrium (NE) seeking algorithm in details.

At time k , each player $i \in \mathcal{V}$ maintains an estimate of all players' actions, denoted by $\mathbf{y}_k^i = [y_{1,k}^i, \dots, y_{N,k}^i]^\top \in \mathbb{R}^N$, where $y_{j,k}^i, j \in \mathcal{V}$ represents player i 's estimate of player j 's action. Hence, at time k , every player $l \in \mathcal{V}$ passes its estimate of all players' actions $y_{j,k}^l, j \in \mathcal{V}$ and its own action $x_{l,k}$ to its out-neighbors. Then, each player $i \in \mathcal{V}$, on receiving the information from its in-neighbors, updates its own action and the estimate of all players' actions (including the estimate of its own local action) based on the following updating laws:

$$x_{i,k+1} = \mathcal{P}_{\Omega_i}[x_{i,k} - \alpha_k g_{\mu^i}^i(\mathbf{y}_k^i)], \quad (6.2a)$$

$$y_{j,k+1}^i = \sum_{l \in \mathcal{N}_i^{\text{in}}} [A]_{il} y_{j,k}^l + \delta_i [A]_{ij} (x_{j,k} - y_{j,k}^i), \quad j \in \mathcal{V} \quad (6.2b)$$

where $g_{\mu^i}^i(\mathbf{y}_k^i)$ is the randomized gradient-free oracle

$$g_{\mu^i}^i(\mathbf{y}_k^i) = \frac{f_i(y_{i,k}^i + \mu^i \xi_k^i, \mathbf{y}_{-i,k}^i) - f_i(y_{i,k}^i, \mathbf{y}_{-i,k}^i)}{\mu^i} \xi_k^i, \quad (6.3)$$

The parameter $\delta_i > 0$ is a constant parameter, and $\alpha_k \geq 0$ is a step-size sequence. The initial values $x_{i,0}$ and $y_{j,0}^i$ for $i, j \in \mathcal{V}$ can be any real numbers. The adjacency matrix A is doubly-stochastic as supposed in Assumption 6.1. It should be noted that the design of a doubly-stochastic adjacency matrix A for a given directed graph is non-trivial. The detailed procedures can be referred to the work in [114], where two distributed strategies (imbalance-correcting algorithm and load-pushing algorithm)

have been developed to construct such matrix under different conditions. With the well-constructed matrix A , each player i selects the parameter δ_i such that $0 \leq \delta_i[A]_{ij} < 2[A]_{ii}$ for all $j \in \mathcal{V}$. The above mentioned procedures are summarized in Algorithm 6.1.

Algorithm 6.1 Gradient-free distributed NE seeking

- 1: **Initialize:** $i \in \mathcal{V}$
 arbitrarily generate $x_{i,0} \in \mathbb{R}, \mathbf{y}_0^i \in \mathbb{R}^N$
 randomly generate $\{\xi_k^i\}_{k \geq 0} \sim \mathcal{N}(0, 1)$ independently
 set δ_i such that $0 \leq \delta_i[A]_{ij} < 2[A]_{ii}, \forall j \in \mathcal{V}$
 - 2: **Iteration** ($k \geq 0$): $i \in \mathcal{V}$
 compute $g_{\mu^i}^i(\mathbf{y}_k^i)$ based on (6.3)
 update variables $x_{i,k+1}$ based on (6.2a)
 update variables $y_{j,k+1}^i$ based on (6.2b)
 - 3: **Output:** $i \in \mathcal{V}$
 $x_{i,k} \rightarrow x^*$
-

Remark 6.2. In (6.2b), it should be noted that $[A]_{ij} = 0$ if player j is NOT an in-neighbor of player i , which implies that player i updates the estimate of player j 's action only based on the estimates $y_{j,k}^l$ from its in-neighbors $l \in \mathcal{N}_i^{\text{in}}$. On the other hand, if player j is an in-neighbor of player i , then $[A]_{ij} \neq 0$ giving rise to an additional error term $\delta_i[A]_{ij}(x_{j,k} - y_{j,k}^i)$ in the update of the estimate on player j 's action.

6.4 Convergence Analysis

In this section, we study the convergence of the algorithm to the Nash equilibrium for the scenarios of diminishing step-size and constant step-size, respectively. We let $\mathcal{F}_k$ denote the σ -field generated by the entire history of the random variables from step 0 to $k - 1$, i.e.,

$$\mathcal{F}_k = \begin{cases} \{x_{i,0}, y_{j,0}^i, i, j \in \mathcal{V}\}, & k = 0, \\ \{x_{i,0}, y_{j,0}^i, \xi_s^i, i, j \in \mathcal{V}; 0 \leq s \leq k - 1\}, & k \geq 1. \end{cases}$$

Since the cost function $f_i(x_i, \mathbf{x}_{-i})$ may not be partially differentiable in x_i as in Assumption 6.2, we introduce a Gaussian-smoothed version of the cost function $f_i(x_i, \mathbf{x}_{-i})$ given by [61]

$$f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) = \frac{1}{\kappa} \int_{\mathbb{R}} f_i(x_i + \mu^i \xi^i, \mathbf{x}_{-i}) e^{-\frac{1}{2} \|\xi^i\|^2} d\xi^i,$$

with $\xi^i \in \mathbb{R}$ is a normally distributed random variable, $\kappa = \int_{\mathbb{R}} e^{-\frac{1}{2} \|\xi^i\|^2} d\xi^i = (2\pi)^{1/2}$, and $\mu^i \geq 0$ is a smoothing parameter of function $f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$. Then, the randomized gradient-free oracle of $f_i(x_i, \mathbf{x}_{-i})$ can be designed as [61]

$$g_{\mu^i}^i(x_i, \mathbf{x}_{-i}) = \frac{f_i(x_i + \mu^i \xi^i, \mathbf{x}_{-i}) - f_i(x_i, \mathbf{x}_{-i})}{\mu^i} \xi^i.$$

From the results in [61], some properties of functions $g_{\mu^i}^i(x_i, \mathbf{x}_{-i})$ and $f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$ are summarized in the following lemma, which is a variation of Lemma 2.1 stated in Chapter 2.

Lemma 6.1. (see [61]) *Suppose Assumption 6.2 holds. The functions $g_{\mu^i}^i(x_i, \mathbf{x}_{-i})$ and $f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$, $\forall i \in \mathcal{V}$ satisfy the following properties:*

1. *The function $f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$ is convex in x_i due to the convexity of f_i in x_i . Moreover, $f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$ satisfies*

$$f_i(x_i, \mathbf{x}_{-i}) \leq f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) \leq f_i(x_i, \mathbf{x}_{-i}) + \mu^i D_1.$$

2. *The function $f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$ is partially differentiable in x_i and its partial derivative with respect to x_i satisfies*

$$\nabla_{x_i} f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) = \mathbb{E}[g_{\mu^i}^i(x_i, \mathbf{x}_{-i})],$$

and is Lipschitz continuous in x_i with a constant $L_1 = \max_{i \in \mathcal{V}} \frac{D_1}{\mu^i}$, and Lips-

chitz continuous in $\mathbf{x}_{-i}$ with a constant $L_2 = \max_{i \in \mathcal{V}} \frac{D_2}{\mu^i}$, i.e.,

$$\begin{aligned}\|\nabla_{x_i} f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) - \nabla_{x_i} f_{i,\mu^i}(y_i, \mathbf{x}_{-i})\| &\leq L_1 \|x_i - y_i\|, \\ \|\nabla_{x_i} f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) - \nabla_{x_i} f_{i,\mu^i}(x_i, \mathbf{y}_{-i})\| &\leq L_2 \|\mathbf{x}_{-i} - \mathbf{y}_{-i}\|.\end{aligned}$$

Further, $\nabla_{x_i} f_{i,\mu^i}(x_i, \mathbf{x}_{-i})$ always belongs to some ε -subdifferential of function $f_i(x_i, \mathbf{x}_{-i})$, i.e.,

$$\nabla_{x_i} f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) \in \partial_{x_i}^\varepsilon f_i(x_i, \mathbf{x}_{-i}), \quad \varepsilon = \mu^i D_1.$$

Specifically, we have $\nabla_{x_i} f_{i,0}(x_i, \mathbf{x}_{-i}) \in \partial_{x_i} f_i(x_i, \mathbf{x}_{-i})$, when μ^i tends to 0¹.

3. The random gradient-free oracle $g_{\mu^i}^i(x_i, \mathbf{x}_{-i})$ satisfies

$$\mathbb{E}[\|g_{\mu^i}^i(x_i, \mathbf{x}_{-i})\|] \leq \sqrt{\mathbb{E}[\|g_{\mu^i}^i(x_i, \mathbf{x}_{-i})\|^2]} \leq \mathcal{B},$$

where $\mathcal{B} = \sqrt{n + 4D_1}$, n is the dimension of x_i .

To facilitate the gradient-free method, we formulate a smoothed version of game Γ with the Gaussian-smoothed cost function f_{i,μ^i} , denoted by $\Gamma_\mu(N, \{f_{i,\mu^i}\}, \{\Omega_i\}, \mathcal{G})$. Under Assumption 6.2, Lemma 6.1 implies that the smoothed cost functions f_{i,μ^i} have similar properties to f_i stated in Assumption 6.2. Hence, game $\Gamma_\mu(N, \{f_{i,\mu^i}\}, \{\Omega_i\}, \mathcal{G})$ admits a Nash equilibrium for the same reasoning as in Remark 6.1. Next, the equivalence of the original game Γ and its smoothed version Γ_μ under certain conditions is introduced in the following result:

Lemma 6.2. *Suppose Assumption 6.2 holds. Games Γ and Γ_μ are equivalent and share the same Nash equilibria when the smoothing parameter $\mu^i, \forall i \in \mathcal{V}$ tends to 0.*

Proof: From Remark 6.1 and previous discussion, Assumption 6.2 implies the existence of Nash equilibrium in both games Γ and Γ_μ . Moreover, applying Squeeze

¹In this paper, we slightly abuse the notation μ^i to represent the sequence μ_k^i just for easy presentation without the loss of generality. We mean μ^i tending to 0 by $\lim_{k \rightarrow \infty} \mu_k^i = 0$

Theorem to Lemma 6.1-1), we have

$$\lim_{\mu^i \rightarrow 0} f_{i,\mu^i}(x_i, \mathbf{x}_{-i}) = f_i(x_i, \mathbf{x}_{-i}), \quad \forall i \in \mathcal{V}.$$

Then, games Γ and Γ_μ share the same number of players, cost functions, action sets and communication graph. Hence the result holds. $\blacksquare$

Next, we make some definitions on the game mappings of games $\Gamma(N, \{f_i\}, \{\Omega_i\}, \mathcal{G})$ and $\Gamma_\mu(N, \{f_{i,\mu^i}\}, \{\Omega_i\}, \mathcal{G})$. For game Γ , since the cost function f_i in game Γ is not necessarily differentiable, so the game mapping of game Γ refers to a set-valued map $\mathbf{F}(\mathbf{x})$, which is defined as the map of the subdifferentials of all players' cost functions:

$$\mathbf{F}(\mathbf{x}) = \prod_{i \in \mathcal{V}} \partial_{x_i} f_i(x_i, \mathbf{x}_{-i}).$$

If the cost function f_i is differentiable, then the game mapping $\mathbf{F}(\mathbf{x})$ reduces to a single-valued map. For game Γ_μ , since the cost function f_{i,μ^i} is differentiable, so the game mapping of game Γ_μ refers to a single-valued map F_μ , which is defined by stacking the partial derivatives of all smoothed cost functions:

$$F_\mu(\mathbf{x}) = [\nabla_{x_1} f_{1,\mu^1}(x_1, \mathbf{x}_{-1}), \dots, \nabla_{x_N} f_{N,\mu^N}(x_N, \mathbf{x}_{-N})]^\top.$$

When μ^i tends to 0 for all $i \in \mathcal{V}$, we denote the game mapping F_μ by $F_0(\mathbf{x})$, *i.e.*,

$$F_0(\mathbf{x}) = \lim_{\mu^i \rightarrow 0, \forall i \in \mathcal{V}} F_\mu(\mathbf{x}) = [\nabla_{x_1} f_{1,0}(x_1, \mathbf{x}_{-1}), \dots, \nabla_{x_N} f_{N,0}(x_N, \mathbf{x}_{-N})]^\top.$$

Thus, based on Lemma 6.1-2), we have $F_0(\mathbf{x}) \in \mathbf{F}(\mathbf{x})$.

Next, we introduce an important property related to the adjacency matrix A summarized in the following lemma:

Lemma 6.3. *Suppose Assumption 6.1 holds. Let $\delta_l > 0, l \in \mathcal{V}$ be selected such that $0 \leq \delta_l[A]_{li} < 2[A]_{ll}$ for all $i \in \mathcal{V}$, where A is the adjacency matrix. Then, there*

exists a constant $C > 0$ and $\gamma \in (0, 1)$ such that, for $k \geq 1$, the matrix $\tilde{A}_i$ given by

$$[\tilde{A}_i]_{lm} = \begin{cases} [A]_{lm} & \text{if } l \neq m \\ |[A]_{ll} - \delta_l[A]_{li}| & \text{if } l = m \end{cases}$$

holds that $\|\tilde{A}_i^k\|_\infty < C\gamma^k$.

Proof: We first show that all the row sums of $\tilde{A}_i$ are always less than or equal to 1. For any $l \in \mathcal{N}$, if $0 \leq \delta_l[A]_{li} \leq [A]_{ll}$, since A is doubly-stochastic, we have $\sum_{n=1}^N [\tilde{A}_i]_{ln} = 1 - \delta_l[A]_{li} \leq 1$; if $[A]_{ll} < \delta_l[A]_{li} < 2[A]_{ll}$, similarly, we have $\sum_{n=1}^N [\tilde{A}_i]_{ln} = 1 + \delta_l[A]_{li} - 2[A]_{ll} < 1$. Hence, we always have $\sum_{n=1}^N [\tilde{A}_i]_{ln} \leq 1$ for any $l \in \mathcal{N}$ where the equal sign holds only if $[A]_{li} = 0$.

Next, we show that all the eigenvalues of $\tilde{A}_i$ have magnitude less than or equal to 1. Let λ be an eigenvalue of the matrix $\tilde{A}_i$, and let $\mathbf{v} = [v_1, \dots, v_N]^\top$ be a corresponding eigenvector. Then we have $\lambda \mathbf{v} = \tilde{A}_i \mathbf{v}$, i.e., for each row $j \in \mathcal{V}$

$$\lambda v_j = [\tilde{A}_i]_{j1}v_1 + [\tilde{A}_i]_{j2}v_2 + \dots + [\tilde{A}_i]_{jN}v_N.$$

Suppose the k -th entry of $\mathbf{v}$ has the maximal absolute value (denoted by $|\hat{v}|$) among all $|v_j|$, $j \in \mathcal{V}$. Then, letting $j = k$ in the above equation, and noting that all the entries of $\tilde{A}_i$ are non-negative and the row sums are less than or equal to 1, we have

$$|\lambda||\hat{v}| = |\lambda||v_k| = \left| \sum_{n=1}^N [\tilde{A}_i]_{kn}v_n \right| \leq \left(\sum_{n=1}^N [\tilde{A}_i]_{kn} \right) |\hat{v}| \leq |\hat{v}|, \quad (6.4)$$

which leads to $|\lambda| \leq 1$ as $|\hat{v}| > 0$.

Next, we show $|\lambda| \neq 1$ by contradiction. Suppose $|\lambda| = 1$, then the relation (6.4) is true if the equal signs in both inequalities are satisfied, which implies the following properties:

- 1) (first equal sign) if $[\tilde{A}_i]_{kn} \neq 0$, then $|v_n| = |\hat{v}|$;

2) (second equal sign) $[A]_{ki} = 0$.

If player k has an in-neighbour, say player $n \neq k$, then $[\tilde{A}_i]_{kn} = [A]_{kn} \neq 0$. From property 1), we have $|v_n| = |\hat{v}|$. Thus, the n -th entry of $\mathbf{v}$ also has the maximal absolute value. That means if player $n \in \mathcal{V}$ is a direct in-neighbor of the player $k \in \mathcal{V}$ with $|v_k| = |\hat{v}|$, then $|v_n| = |\hat{v}|$. Since the graph is strongly connected, thus we can always find a path for each player $n \in \mathcal{V}$ linking to player k , *i.e.*, $n \rightarrow \cdots \rightarrow k$. Thus, from the above analysis, we have $|v_n| = \cdots = |v_k| = |\hat{v}|$ along this path. Therefore, we have $|v_n| = |\hat{v}|$ for all $n \in \mathcal{V}$. From property 2), $|v_n| = |\hat{v}|$ for all $n \in \mathcal{V}$ implies that $[A]_{ni} = 0$ for all $n \in \mathcal{V}$, which is impossible due to the strong connectivity of the graph. Therefore, the eigenvalues of $\tilde{A}_i$ can only have magnitude strictly less than 1, *i.e.*, $|\lambda| < 1$.

Finally, we represent $\tilde{A}_i^k$ in the Jordan canonical form for some P_m , J_m and Q_m . Since all the eigenvalues of $\tilde{A}_i$ have magnitude smaller than 1, then the diagonal entries in J_m are smaller than 1, for all m . Thus, there exists a constant $C > 0$ and $\gamma \in (0, 1)$ such that

$$\|\tilde{A}_i^k\| = \left\| \sum_{m=1}^N P_m J_m^k Q_m \right\| \leq \sum_{m=1}^N \|P_m\| \|Q_m\| \|J_m^k\| \leq C\gamma^k,$$

which completes the proof. ■

Remark 6.3. *Similar to the result in [121, Corollary 1], constants C and γ in Lemma 6.3 depend on the minimum weight (denoted by ϕ) that each player gives to its own value and the values of its neighbors (*i.e.*, if $[A]_{ij} > 0$, then $[A]_{ij} \geq \phi$), the number of players N , and the parameters $\{\delta_l\}_{l \in \mathcal{V}}$ selected by all players. Moreover, for larger N and smaller ϕ , constant γ gets closer to 1, implying a slower convergence rate; for $\delta_l \rightarrow 0$ or $\delta_l[A]_{li} \rightarrow 2[A]_{li}$, $\forall l \in \mathcal{V}$, matrix $\tilde{A}_i$ reduces to A and constant γ gets closer to 1, implying a slower convergence rate.*

6.4.1 Diminishing Step-Size

In this part, we adopt the diminishing step-size sequence in the proposed algorithm, i.e., the step-size sequence α_k satisfies that $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$.

Now, we present the result on the consensus property: for any $i \in \mathcal{V}$, each player l 's estimate of player i 's action $y_{i,k}^l, l \in \mathcal{V}$ converges to player i 's real action $x_{i,k}$ as k goes to infinity, which is formally stated in the following theorem.

Theorem 6.1. *Suppose Assumptions 6.1 and 6.2 hold. Let $\{x_{i,k}\}_{k \geq 0}, \{y_{i,k}^l\}_{k \geq 0}, i, l \in \mathcal{V}$ be the sequences generated by (6.2a) and (6.2b), respectively, with a step-size sequence $\{\alpha_k\}_{k \geq 0}$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, and a positive constant $\delta_l, l \in \mathcal{V}$ satisfying $0 \leq \delta_l[A]_{li} < 2[A]_{ll}$ for all $i \in \mathcal{V}$, where A is the adjacency matrix. Then, we have*

$$\lim_{k \rightarrow \infty} \mathbb{E}[\|x_{i,k} - y_{i,k}^l\|] = 0, \quad i, l \in \mathcal{V}.$$

Proof: It can be obtained from (6.2a) and (6.2b) that

$$\begin{aligned} x_{i,k+1} &= \mathcal{P}_{\Omega_i}[x_{i,k} - \alpha_k g_{\mu^i}^i(\mathbf{y}_k^i)], \\ y_{i,k+1}^l &= \sum_{m=1}^N [A]_{lm} y_{i,k}^m + \delta_l [A]_{li} (x_{i,k} - y_{i,k}^l). \end{aligned}$$

Then, taking the subtraction and applying the norm

$$\begin{aligned} \|x_{i,k+1} - y_{i,k+1}^l\| &\leq \left\| \sum_{m=1}^N [A]_{lm} (x_{i,k} - y_{i,k}^m) - \delta_l [A]_{li} (x_{i,k} - y_{i,k}^l) \right\| + \alpha_k \|g_{\mu^i}^i(\mathbf{y}_k^i)\| \\ &= \left\| \sum_{m=1}^N [\tilde{A}_i]_{lm} (x_{i,k} - y_{i,k}^m) \right\| + \alpha_k \|g_{\mu^i}^i(\mathbf{y}_k^i)\| \\ &\leq \sum_{m=1}^N [\tilde{A}_i]_{lm} \|x_{i,k} - y_{i,k}^m\| + \alpha_k \|g_{\mu^i}^i(\mathbf{y}_k^i)\|, \end{aligned}$$

where the first inequality follows from the projection's non-expansive property, and

the equality holds by the definition of $\tilde{A}_i$ as in Lemma 6.3. Hence, we obtain that

$$\begin{aligned} \|x_{i,k} - y_{i,k}^l\| &\leq \sum_{m=1}^N [\tilde{A}_i^k]_{lm} \|x_{i,0} - y_{i,0}^m\| + \sum_{r=1}^{k-1} \sum_{m=1}^N [\tilde{A}_i^{k-r}]_{lm} \alpha_{r-1} \|g_{\mu^i}^i(\mathbf{y}_{r-1}^i)\| \\ &\quad + \alpha_{k-1} \|g_{\mu^i}^i(\mathbf{y}_{k-1}^i)\|. \end{aligned}$$

Then, taking the total expectation, it follows from Lemmas 6.1-3) and 6.3 that

$$\mathbb{E}[\|x_{i,k} - y_{i,k}^l\|] \leq NC\hat{\sigma}\gamma^k + NC\mathcal{B} \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_{r-1} + \mathcal{B}\alpha_{k-1}, \quad (6.5)$$

where $\hat{\sigma} = \max_{i \in \mathcal{V}} \sigma_i$ and $\sigma_i = \max_{m \in \mathcal{V}} |x_{i,0} - y_{i,0}^m|$. Taking the limit $k \rightarrow \infty$ and noting that $\lim_{k \rightarrow \infty} \alpha_k = 0$, the desired result follows from [66, Lemma 4-1)]. ■

Remark 6.4. Theorem 6.1 is a characterization of the consensus property of the algorithm. For any $i \in \mathcal{V}$, each player l 's estimate of player i 's action $y_{i,k}^l, l \in \mathcal{V}$ converges to player i 's real action $x_{i,k}$ as k goes to infinity.

Next, we introduce a result from [25, Lemma 11 in Ch. 2] to facilitate the convergence analysis.

Lemma 6.4. (see [25, Lemma 11 in Ch. 2]) Let u_k, v_k, w_k, η_k be non-negative random variables satisfying that

$$\begin{aligned} \mathbb{E}[u_{k+1} | \mathcal{F}_k] &\leq (1 + \eta_k)u_k - v_k + w_k \quad a.s., \\ \sum_{k=0}^{\infty} \eta_k &< \infty \quad a.s., \quad \sum_{k=0}^{\infty} w_k < \infty \quad a.s., \end{aligned}$$

where $\mathbb{E}[u_{k+1} | \mathcal{F}_k]$ denotes the conditional expectation for the given $u_0, \dots, u_k, v_0, \dots, v_k, w_0, \dots, w_k, \eta_0, \dots, \eta_k$, and 'a.s.' means 'almost surely'. Then,

1) $\{u_k\}_{k \geq 0}$ converges a.s.;

2) $\sum_{k=0}^{\infty} v_k < \infty$ a.s.

Now, we make a mild assumption on the uniqueness of the Nash equilibrium in game Γ as follows.

Assumption 6.3. *The game mapping $\mathbf{F}$ of game Γ is strictly monotone on Ω , i.e., for any $\mathbf{x}, \mathbf{y} \in \Omega$, $\mathbf{x} \neq \mathbf{y}$, $\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y}) \in \mathbf{F}$, we have $\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle > 0$.*

Remark 6.5. *Assumption 6.3 ensures the uniqueness of the Nash equilibrium in game Γ and is commonly adopted in many existing works, e.g., [94, 100–103].*

Now, we are ready to establish the convergence of all players' actions to the unique Nash equilibrium of game Γ , which is formally stated in the following theorem.

Theorem 6.2. *Suppose Assumptions 6.1, 6.2 and 6.3 hold. Let $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ be the action profile at the unique Nash equilibrium of game Γ . Let $\{x_{i,k}\}_{k \geq 0}, \{y_{i,k}^l\}_{k \geq 0}$, $i, l \in \mathcal{V}$ be the sequences generated by (6.2a) and (6.2b), respectively, with a step-size sequence $\{\alpha_k\}_{k \geq 0}$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, and a positive constant $\delta_l, l \in \mathcal{V}$ satisfying $0 \leq \delta_l[A]_{li} < 2[A]_{ll}$ for all $i \in \mathcal{V}$, where A is the adjacency matrix. Then, the sequence $\{\mathbf{x}_k\}_{k \geq 0}$ converges to $\mathbf{x}^*$ almost surely when the smoothing parameter $\mu^i, \forall i \in \mathcal{V}$ tends to 0.*

Proof: Noting that there might be multiple Nash equilibria in game Γ_μ , we let $\mathbf{x}_\mu^* = (x_{i,\mu^i}^*, \mathbf{x}_{-i,\mu^{-i}}^*)$ be an action profile at one of them. Applying [101, Lemma 1] yields

$$x_{i,\mu^i}^* = \mathcal{P}_{\Omega_i}[x_{i,\mu^i}^* - \alpha_k \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*)], \quad i \in \mathcal{V}.$$

Thus, subtracting (6.2a) by the above equation, and taking the norm

$$\begin{aligned} \|x_{i,k+1} - x_{i,\mu^i}^*\|^2 &\leq \|(x_{i,k} - x_{i,\mu^i}^*) - \alpha_k (g_{\mu^i}^i(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*))\|^2 \\ &= \|x_{i,k} - x_{i,\mu^i}^*\|^2 + \alpha_k^2 \|g_{\mu^i}^i(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*)\|^2 \\ &\quad - 2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, g_{\mu^i}^i(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*) \rangle, \end{aligned}$$

where we have applied the projection's non-expansive property. Taking the conditional expectation on $\mathcal{F}_k$, we obtain

$$\begin{aligned}
\mathbb{E}[\|x_{i,k+1} - x_{i,\mu^i}^*\|^2 | \mathcal{F}_k] &\leq \|x_{i,k} - x_{i,\mu^i}^*\|^2 + \alpha_k^2 \mathbb{E}[\|g_{\mu^i}^i(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*)\|^2 | \mathcal{F}_k] \\
&\quad - 2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, \nabla_{x_i} f_{i,\mu^i}(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*) \rangle \\
&= \|x_{i,k} - x_{i,\mu^i}^*\|^2 + \alpha_k^2 \mathbb{E}[\|g_{\mu^i}^i(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*)\|^2 | \mathcal{F}_k] \\
&\quad - 2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, \nabla_{x_i} f_{i,\mu^i}(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_k) \rangle \\
&\quad - 2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_k) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*) \rangle. \tag{6.6}
\end{aligned}$$

It is noted that

$$\mathbb{E}[\|g_{\mu^i}^i(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*)\|^2 | \mathcal{F}_k] \leq 2\mathbb{E}[\|g_{\mu^i}^i(\mathbf{y}_k^i)\|^2 | \mathcal{F}_k] + 2\|\nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_\mu^*)\|^2 \leq 4\mathcal{B}^2$$

and

$$\begin{aligned}
&- 2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, \nabla_{x_i} f_{i,\mu^i}(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_k) \rangle \\
&= -2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, \nabla_{x_i} f_{i,\mu^i}(\mathbf{y}_k^i) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{y}_k^i, \mathbf{x}_{-i,k}) \rangle \\
&\quad - 2\alpha_k \langle x_{i,k} - x_{i,\mu^i}^*, \nabla_{x_i} f_{i,\mu^i}(\mathbf{y}_k^i, \mathbf{x}_{-i,k}) - \nabla_{x_i} f_{i,\mu^i}(\mathbf{x}_k) \rangle \\
&\leq 2L_1\alpha_k \|x_{i,k} - x_{i,\mu^i}^*\| \|y_k^i - x_{i,k}\| + 2L_2\alpha_k \|x_{i,k} - x_{i,\mu^i}^*\| \|\mathbf{y}_{-i,k}^i - \mathbf{x}_{-i,k}\| \\
&\leq L_1(1 + \|x_{i,k} - x_{i,\mu^i}^*\|^2)\alpha_k \|y_k^i - x_{i,k}\| + L_2(1 + \|x_{i,k} - x_{i,\mu^i}^*\|^2)\alpha_k \|\mathbf{y}_{-i,k}^i - \mathbf{x}_{-i,k}\| \\
&\leq \hat{L}(1 + \|x_{i,k} - x_{i,\mu^i}^*\|^2) \left(NC\hat{\sigma}\alpha_k\gamma^k + NC\mathcal{B} \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_k \alpha_{r-1} + \mathcal{B}\alpha_k \alpha_{k-1} \right),
\end{aligned}$$

where $\hat{L} = L_1 + \sqrt{N-1}L_2$, we have applied $2\|\mathbf{a}\| \leq 1 + \|\mathbf{a}^2\|$ in the second inequality and (6.5) in the third inequality. Thus, combining the above results to (6.6) and summing over $i \in \mathcal{V}$

$$\begin{aligned}
\mathbb{E}[\|\mathbf{x}_{k+1} - \mathbf{x}_\mu^*\|^2 | \mathcal{F}_k] &\leq (1 + \eta_k) \|\mathbf{x}_k - \mathbf{x}_\mu^*\|^2 + w_k \\
&\quad - 2\alpha_k \langle \mathbf{x}_k - \mathbf{x}_\mu^*, F_\mu(\mathbf{x}_k) - F_\mu(\mathbf{x}_\mu^*) \rangle, \tag{6.7}
\end{aligned}$$

where

$$\begin{aligned}\eta_k &= \hat{L} \left(NC\hat{\sigma}\alpha_k\gamma^k + NC\mathcal{B} \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_k \alpha_{r-1} + \mathcal{B}\alpha_k \alpha_{k-1} \right) \\ w_k &= 4N\mathcal{B}^2\alpha_k^2 + \hat{L} \left(N^2C\hat{\sigma}\alpha_k\gamma^k + N^2C\mathcal{B} \sum_{r=1}^{k-1} \gamma^{k-r} \alpha_k \alpha_{r-1} + N\mathcal{B}\alpha_k \alpha_{k-1} \right).\end{aligned}$$

Taking the limit $\mu^i \rightarrow 0$ for $\forall i \in \mathcal{V}$, it follows from Lemma 6.2 that $\lim_{\mu^i \rightarrow 0, \forall i \in \mathcal{V}} \mathbf{x}_\mu^* = \mathbf{x}^*$. Thus, it can be obtained from (6.7) that

$$\begin{aligned}\mathbb{E}[\|\mathbf{x}_{k+1} - \mathbf{x}^*\|^2 | \mathcal{F}_k] &\leq (1 + \eta_k) \|\mathbf{x}_k - \mathbf{x}^*\|^2 + w_k \\ &\quad - 2\alpha_k \langle \mathbf{x}_k - \mathbf{x}^*, F_0(\mathbf{x}_k) - F_0(\mathbf{x}^*) \rangle.\end{aligned}\tag{6.8}$$

Following the results in [66, Lemma 3] and the step-size $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, we have

$$\sum_{k=0}^{\infty} \eta_k < \infty, \quad \sum_{k=0}^{\infty} w_k < \infty.$$

Applying Lemma 6.4 to (6.8), we can obtain that $\|\mathbf{x}_k - \mathbf{x}^*\|$ converges almost surely, and $\sum_{k=0}^{\infty} \alpha_k \langle \mathbf{x}_k - \mathbf{x}^*, F_0(\mathbf{x}_k) - F_0(\mathbf{x}^*) \rangle < \infty$. From Assumption 6.3 and the fact that $F_0(\mathbf{x}) \in \mathbf{F}(\mathbf{x})$, we have $\langle \mathbf{x}_k - \mathbf{x}^*, F_0(\mathbf{x}_k) - F_0(\mathbf{x}^*) \rangle \geq 0$. Together with the step-size $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, we obtain

$$\liminf_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{x}^*.$$

Since $\|\mathbf{x}_k - \mathbf{x}^*\|$ converges almost surely, we obtain the desired result. $\blacksquare$

Remark 6.6. Theorem 6.2 shows that the players' action profile $\mathbf{x}_k$ will converge to the unique Nash equilibrium $\mathbf{x}^*$ of game Γ by selecting the diminishing smoothing parameter sequence.

6.4.2 Constant Step-Size

In this part, we suppose the step-size $\alpha_k = \alpha$, which is a positive constant.

A similar result to Theorem 6.1 on the consensus property can be established. Instead of achieving the exact convergence, for any $i \in \mathcal{V}$, each player l 's estimate of player i 's action $y_{i,k}^l, l \in \mathcal{V}$ approximately converges to player i 's real action $x_{i,k}$ with an error proportional to the step-size. The following theorem formally states the result.

Theorem 6.3. *Suppose Assumptions 6.1 and 6.2 hold. Let $\{x_{i,k}\}_{k \geq 0}, \{y_{i,k}^l\}_{k \geq 0}, i, l \in \mathcal{V}$ be the sequences generated by (6.2a) and (6.2b), respectively, with a constant step-size sequence $\alpha_k = \alpha$, and a positive constant $\delta_l, l \in \mathcal{V}$ satisfying $0 \leq \delta_l[A]_{li} < 2[A]_{li}$ for all $i \in \mathcal{V}$, where A is the adjacency matrix. Then, we have*

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\|x_{i,k} - y_{i,k}^l\|] \leq \left(\frac{\gamma N C \mathcal{B}}{1 - \gamma} + \mathcal{B} \right) \alpha.$$

Proof: Following same arguments as in Theorem 6.1, the result holds by taking the limsup on both sides of (6.5). ■

Next, we introduce a slightly stronger assumption compared to Assumption 6.3 on the uniqueness of the Nash equilibrium in game Γ .

Assumption 6.4. *The game mapping $\mathbf{F}$ of game Γ is strongly monotone on Ω with a constant $\chi > 0$, i.e., for any $\mathbf{x}, \mathbf{y} \in \Omega$, $\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y}) \in \mathbf{F}$, we have $\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \chi \|\mathbf{x} - \mathbf{y}\|^2$.*

Remark 6.7. *Assumption 6.4 also ensures the uniqueness of the Nash equilibrium in game Γ and is commonly supposed in the literature on Nash equilibrium seeking, [96, 100, 101, 104].*

Now, we are ready to characterize the approximate convergence of all players' actions to the Nash equilibrium of game Γ , which is formally stated in the following theorem.

Theorem 6.4. *Suppose Assumptions 6.1, 6.2 and 6.4 hold. Let $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ be the action profile at the unique Nash equilibrium of game Γ . Let $\{x_{i,k}\}_{k \geq 0}, \{y_{i,k}^l\}_{k \geq 0}, i, l \in \mathcal{V}$ be the sequences generated by (6.2a) and (6.2b), respectively, with a constant step-size sequence $\alpha_k = \alpha$ satisfying the following condition*

$$0 < 2\chi\alpha - \hat{L}\left(\frac{\gamma N C \mathcal{B}}{1-\gamma} + \mathcal{B}\right)\alpha^2 < 1, \quad (6.9)$$

and a positive constant $\delta_l, l \in \mathcal{V}$ satisfying $0 \leq \delta_l[A]_{li} < 2[A]_{ii}$ for all $i \in \mathcal{V}$, where A is the adjacency matrix. Then, with the smoothing parameter $\mu^i, \forall i \in \mathcal{V}$ tending to 0, the sequence $\{\mathbf{x}_k\}_{k \geq 0}$ satisfies

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\|\mathbf{x}_k - \mathbf{x}^*\|^2] \leq \frac{[4N\mathcal{B}^2 + \hat{L}(\frac{\gamma N^2 C \mathcal{B}}{1-\gamma} + N\mathcal{B})]\alpha}{2\chi - \hat{L}(\frac{\gamma N C \mathcal{B}}{1-\gamma} + \mathcal{B})\alpha}.$$

Proof: From Assumption 6.4 and the fact that $F_0(\mathbf{x}) \in \mathbf{F}(\mathbf{x})$, we have $\langle \mathbf{x}_k - \mathbf{x}^*, F_0(\mathbf{x}_k) - F_0(\mathbf{x}^*) \rangle \geq \chi \|\mathbf{x}_k - \mathbf{x}^*\|$. Following the same arguments as in Theorem 6.2, and applying the above results to (6.8), we obtain

$$\mathbb{E}[\|\mathbf{x}_{k+1} - \mathbf{x}^*\|^2 | \mathcal{F}_k] \leq (1 + \eta_k - 2\chi\alpha) \|\mathbf{x}_k - \mathbf{x}^*\|^2 + w_k,$$

where

$$\begin{aligned} \eta_k &= \hat{L} \left(N C \hat{\sigma} \alpha \gamma^k + N C \mathcal{B} \alpha^2 \sum_{r=1}^{k-1} \gamma^{k-r} + \mathcal{B} \alpha^2 \right) \\ w_k &= 4N\mathcal{B}^2 \alpha^2 + \hat{L} \left(N^2 C \hat{\sigma} \alpha \gamma^k + N^2 C \mathcal{B} \alpha^2 \sum_{r=1}^{k-1} \gamma^{k-r} + N\mathcal{B} \alpha^2 \right). \end{aligned}$$

Taking the total expectation, followed by the limsup on both sides, we complete the proof based on the step-size condition (6.9). $\blacksquare$

Remark 6.8. *In general, if the constant step-size α is set small, then the step-size condition (6.9) can be satisfied. Theorem 6.4 shows that all players' actions approximately converge to the Nash equilibrium of game Γ with an error depending*

on the step-size α , the number of players N , the cost function parameters D_1, D_2 ($\hat{L}$ and $\mathcal{B}$ are functions of D_1, D_2) and the communication topology γ . It should also be noted that if the step-size α is small, then the error bound is close to 0.

6.5 Numerical Simulations

In this section, we demonstrate the performance of the proposed algorithm by a numerical example. Consider an energy consumption game of N players for Heating Ventilation and Air Conditioning (HVAC) system (see [122]), where the cost function of each player i can be modeled by the following quadratic function:

$$f_i(\mathbf{x}) = a_i(x_i - x_i^r)^2 + \left(b \sum_{j=1}^N x_j + c\right)x_i, \quad x_i \in \Omega_i,$$

where $a_i > 0, b > 0, c$ and x_i^r are constants for $i \in \mathcal{V}$. It is easy to verify that Assumptions 2, 3 and 4 are satisfied. Throughout the simulation, we let $a_i = 1$ for $i \in \mathcal{V}$, $b = 0.1$, $c = 10$ and $\Omega_i = [-30, 30]$ for $i \in \mathcal{V}$. In the following simulation, we investigate the effectiveness of the proposed algorithm from the perspectives of network topology and number of players, followed by a comparison with the gradient-based counterpart.

6.5.1 Network Topology

In this part, we first consider $N = 5$ players under three different communication graphs as shown in Fig. 6.1. Obviously, all these digraphs are strongly connected, hence Assumption 1 is satisfied. Constant x_i^r for $i \in \{1, \dots, 5\}$ is set to 10, 15, 20, 25, 30, respectively. For the implementation of the algorithm, the step-size α_k is set to $0.1/\sqrt{k+1}$. Besides, we let the smoothing parameter sequence to be diminishing, e.g., $\mu_k^i = 10^{-2}/k+1$ and $\delta_i = 0.5$ for $i \in \{1, \dots, 5\}$. The initial values of all players' actions $\mathbf{x}_0$ and the estimates of all players' actions $\mathbf{y}_0^i$ from

Figure 6.1: Three different network topologies.

player i for $i \in \{1, \dots, 5\}$ are all set to 0. The relative errors of all players' actions ($\|\mathbf{x}_k - \mathbf{x}^*\|/\|\mathbf{x}^*\|$) produced by the proposed gradient-free method with diminishing step-size for three different network topologies are plotted in Fig. 6.2. As can be

Figure 6.2: The relative errors of all players' actions produced by the proposed gradient-free method ($\alpha_k = 0.1/\sqrt{k+1}$) under three different communication networks.

observed, convergence can be achieved under all three network topologies. Specifically, the performance is better for the graph with more edges due to the increased number of communication channels.

6.5.2 Number of Players

In this part, we increase the number of players to $N = 10, 20, 30$ and 40 under a strongly connected communication graph as shown in Fig. 6.3. We set $x_i^r = 2i$ for $i \in \{1, \dots, N\}$. The rest of parameters are set the same as in section 6.5-A. It is

Figure 6.3: Network topology for N players.

shown in Fig. 6.4 that the relative errors of all players' actions ($\|\mathbf{x}_k - \mathbf{x}^*\|/\|\mathbf{x}^*\|$) produced by the proposed gradient-free method with diminishing step-size for $N = 10, 20, 30$ and 40 . As can be seen, the algorithm is scalable to different number of

Figure 6.4: The relative errors of all players' actions produced by the proposed gradient-free method ($\alpha_k = 0.1/\sqrt{k+1}$) with different number of players.

players, and the convergence result is better for smaller number of players, which is as expected.

6.5.3 Gradient-Free vs. Gradient-Based Algorithm

In this part, we compare the performance of the proposed gradient-free algorithm with its gradient-based counterpart. Specifically, the gradient-based algorithm adopts the same updating laws as in (6.2a) and (6.2b), but the gradient-free oracle is replaced with the true gradient information. We consider the same problem settings as in section 6.5-A under the communication graph as shown in Fig. 6.1-(a). For the implementation of the algorithm, the step-size α_k is set to $0.1/\sqrt{k+1}$ and 0.1 , respectively. Figs. 6.5 and 6.6 present the players' actions generated by the proposed gradient-free algorithm with both diminishing step-size and constant step-

size, respectively. For the gradient-based counterpart, the convergence results of the players' actions with both diminishing step-size $\alpha_k = 0.1/\sqrt{k+1}$ and constant step-size $\alpha_k = 0.1$ are plotted in Figs. 6.7 and 6.8, respectively.

Figure 6.5: The plot of all players' actions produced by the proposed gradient-free method with diminishing step-size $\alpha_k = 0.1/\sqrt{k+1}$.

Figure 6.6: The plot of all players' actions produced by the proposed gradient-free method with constant step-size $\alpha_k = 0.1$.

Figure 6.7: The plot of all players' actions produced by the gradient-based method with diminishing step-size $\alpha_k = 0.1/\sqrt{k+1}$.

Comparing Figs. 6.5 and 6.7 for diminishing step-size, and Figs. 6.6 and 6.8 for constant step-size, it can be observed that the convergence speed of the gradient-based algorithm is generally faster than its gradient-free counterpart for both di-

Figure 6.8: The plot of all players' actions produced by the gradient-based method with constant step-size $\alpha_k = 0.1$.

minishing and constant step-size scenarios. This result is reasonable because the gradient-based algorithm has direct access to the true gradient, where the structure information is included. On the other hand, the faster speed implies more aggressive updates in the process, leading to a relatively larger overshoot, which can be moderated by a smaller step-size.

6.6 Conclusions

We have developed a gradient-free distributed Nash equilibrium seeking algorithm for non-cooperative games among a group of players under a directed and strongly connected communication graph. The proposed algorithm does not require the knowledge on the explicit analytical expression of the cost function and allows the problem to be non-smooth. The convergence of the proposed algorithm to the Nash equilibrium has been rigorously studied for both diminishing and constant step-sizes, respectively. Specifically, by choosing a diminishing smoothing parameter, we have shown the convergence to the exact Nash equilibrium for diminishing step-size, and the neighborhood of the Nash equilibrium for constant step-size, in which the gap is proportional to the step-size. Finally, we have illustrated the performance of the algorithm through a numerical example in the application of HVAC system.

Chapter 7

Conclusions and Recommendations

7.1 Conclusions

In this dissertation, the distributed optimization and Nash equilibrium (NE) seeking problems in the framework of multi-agent system have been studied. In particular, we have considered a set constrained distributed static optimization problem first. Two gradient-free distributed optimization algorithms have been proposed to solve the problem. Then, a distributed online optimization problem has been investigated, where the cost functions are time-varying. To solve the problem, an online gradient-free distributed algorithm has been developed. Lastly, a multi-player non-cooperative game has been studied, where the players are assumed to have limited knowledge on their cost functions. A gradient-free distributed Nash equilibrium seeking algorithm has been presented. The convergence properties of all proposed algorithms have been justified from both theoretical proofs and numerical simulations. Details of the achieved results are listed as below.

- In Chapter [3](#), we have considered a set constrained distributed optimization

problem with possibly non-smooth cost functions. In this problem, we have assumed the explicit expressions of the cost functions are unknown, but the values can be measured locally by the agents. A randomized gradient-free (RGF) based distributed optimization algorithm has been proposed to solve the problem. This algorithm is equipped with row/column stochastic weighting matrices, which removes the requirement of doubly-stochastic condition. The convergence properties of the algorithm has been rigorously proved. It has been shown that the algorithm is able to converge to the neighborhood of the optimal solution with a gap depending on the smoothing parameter. The effectiveness of the algorithm has been verified through numerical simulations.

- The same set constrained distributed optimization problem has been considered in Chapter 4. To improve the convergence results, we have developed a gradient-free distributed optimization algorithm, where the pseudo-gradient operator is constructed locally to estimate the gradient. An optimal averaging scheme has been introduced in the updating procedures. The convergence results including the convergence rate analysis of the algorithm has been derived. It has been shown that the algorithm achieves the convergence to the exact optimal solution without the square-summable condition on the step-size, which increases the range of the step-size selection. The performance of the algorithm has also been illustrated in the numerical simulations.
- Chapter 5 has formulated a distributed online optimization, where the cost functions are allowed to vary with time, but the values are only revealed to the agents after the decisions are made at each time-step. For this problem, a gradient-free distributed online optimization algorithm based on RGF has been presented. To analyze the performance of the algorithm, the notions of both static and dynamic regrets have been introduced. With some mild assumptions, we have characterized both regrets as sublinear functions of time, and their corresponding averages over time are approximately convergent to zero with a rate that is comparable to the state-of-the-art algorithms.

- In Chapter 6, a multi-player non-cooperative game has been investigated, where the players have been assumed to have limited knowledge on their local cost functions. An average consensus protocol has been leveraged to estimate the information among the players, and a gradient-free distributed algorithm has been proposed to find the NE. The convergence properties of the algorithm for both diminishing and constant step-sizes have been established, respectively. Specifically, we have attained an exact convergence to the NE for the diminishing case, and an approximate convergence to the NE with a gap proportional to the step-size for the constant case.

7.2 Recommendations for Further Research

Besides the achievements presented in the dissertation, there are still many concerns in the distributed optimization and NE seeking remaining to be solved. Further research on this topic can be attempted in the following aspects.

7.2.1 Convexity

In distributed optimization, the gradient descent technique takes the advantages of its well-established properties and straightforward implementation, and hence is most commonly adopted in optimization problems. However, the performance of this technique is restricted to the convexity of the problem due to the existence of multiple minima. Most literature presumes the cost functions to be convex, implying a unique solution in the problem, which makes the analysis much easier. Finding the global minimum in a general optimization problem is known as *global optimization*, which is a hard problem in general.

In fact, almost all the applications in our living world are non-convex. Thus, those gradient-based methods generally cannot be directly applied in such cases, or have

to be conditionally applied with some assumptions on the problem or the optimal solution set, see [123]. There are non-gradient based methods which can be used for global optimization problems, such as search-based methods [124–128], particle swarm optimization methods (PSO) [129], ant colony optimization methods [130] and genetic algorithms (GA) [131]. These methods are more problem specific and require much work of parameters tunings, which makes the analysis of these methods difficult. Moreover, these methods need to access the value of the global cost function, which is supposed to be unknown in the settings of distributed optimization. Hence, it is difficult to implement these methods in a distributed manner.

Therefore, it is of great importance to investigate some novel techniques with guaranteed performance which can be applied for general distributed non-convex optimization problems.

7.2.2 Convergence

Currently, the convergence result established by almost all existing works on distributed optimization and NE seeking is in the sense of asymptotic: either asymptotically convergent to a neighborhood of the optimal solution/NE, or asymptotically convergent to the exact optimal solution/NE. Theoretically, asymptotic convergence is good, but is not much useful in real applications since the algorithm has to stop at a particular time-step. What the asymptotic convergence can achieve is that for a specific accuracy criteria, it can only be guaranteed that the algorithm is able to meet this criteria at a particular time-step which can be estimated by the rate analysis. However, it will be more interesting and expected with more application backgrounds if one can achieve the exact or true optimal solution at some finite time depending on the initial conditions, known as *finite-time convergence* or at a fixed time independent of the initial conditions, known as *fixed-time convergence*.

Finite-time and fixed-time convergence have been extensively studied in control

fields of continuous-time multi-agent systems with dynamic systems using sliding mode control *e.g.*, [132–136]. Lyapunov tools are commonly used to design the control protocols and analyze the performances. However, for discrete-time distributed optimization and NE seeking problems, very little attention has been received for the finite-time or fixed-time convergence.

Therefore, it is of great interests to investigate some novel distributed optimization and NE seeking algorithms which can reach the optimal solution in finite time or fixed time steps.

7.2.3 Robustness

A well-designed control law for a given system can work perfectly well in the nominal condition, but may fail if some parameters of the system change, which may be due to the aging effect or environmental changes. The ability of an algorithm to resist the change of a system without adapting its initial stable configurations is known as *robustness*, which is well-known in many application fields including statistics, economics, control theory, computer science, *etc.* The algorithm with good robustness is crucial, especially for the applications which are directly related to the humans' health and safety.

Most literature considered robustness of algorithms with respect to disturbance rejection, such as [11, 89] and the references therein. Besides, there are works discussing the robustness of algorithms from the perspective of network unreliability, which includes [137] with time-varying network, [138] with quantized information exchange, [139] with data transmission delays, and [104] with package dropouts and link breaks. The robustness of these algorithms usually comes with the cost of the performance downgrade, because very little information of the undesirable source is utilized during the design of these algorithms. In fact, with the emergence of Big Data, much information from the undesirable source can be collected, analyzed, and

then leveraged in the design of robust algorithms to improve the performance.

Therefore, it is a trend to develop distributed optimization and NE seeking algorithms which are both robust and effective.

Author's Publications

Journal Papers:

1. **Y. Pang**, G. Hu, "Randomized Gradient-Free Distributed Optimization Methods for a Multi-Agent System with Unknown Cost Function," *IEEE Transactions on Automatic Control*, vol. 65, no. 1, pp. 333-340, 2020.
2. **Y. Pang**, G. Hu, "Distributed Nash Equilibrium Seeking with Limited Cost Function Knowledge via A Consensus-Based Gradient-Free Method," accepted in *IEEE Transactions on Automatic Control*, to appear.
3. **Y. Pang**, G. Hu, "Randomized Gradient-Free Distributed Online Optimization via a Dynamic Regret Analysis," submitted to *IEEE Transactions on Automatic Control*.
4. **Y. Pang**, G. Hu, "Gradient-Free Distributed Optimization with Exact Convergence," submitted to *Automatica*.
5. **Y. Pang**, G. Hu, "Nash Equilibrium Seeking in N -Coalition Games via a Gradient-Free Method," submitted to *Automatica*.
6. **Y. Pang**, G. Hu, "Gradient-Free Nash Equilibrium Seeking in N -Cluster Games with Uncoordinated Constant Step-Sizes," submitted to *IEEE Transactions on Neural Networks and Learning Systems*.

Conference Papers:

1. **Y. Pang**, G. Hu, "A Distributed Optimization Method with Unknown Cost Function in a Multi-Agent System via Randomized Gradient-Free Method," in *2017 11th Asian Control Conference (ASCC)*, IEEE, 2017, pp. 144-149.
2. **Y. Pang**, G. Hu, C. J. Spanos, "Distributed Model Predictive Control of a HVAC System via Projected Subgradient Method," in *2018 IEEE 14th International Conference on Control & Automation (ICCA)*, IEEE, 2018, pp. 878-883.
3. **Y. Pang**, G. Hu, "Exact Convergence of Gradient-Free Distributed Optimization Method in a Multi-Agent System," in *2018 IEEE 57th Conference on Decision Control (CDC)*, IEEE, 2018, pp. 5728-5733.
4. **Y. Pang**, G. Hu, "A Distributed Optimization Method with Unknown Cost Function in Multi-Agent Systems via a Learning-Based Method," in *2019 IEEE 15th International Conference on Control & Automation (ICCA)*, IEEE, 2019, pp. 1050-1055.
5. **Y. Pang**, G. Hu, "Randomized Gradient-Free Distributed Online Optimization with Time-Varying Cost Functions," in *2019 IEEE 58th Conference on Decision Control (CDC)*, IEEE, 2019, pp. 4510-4515.
6. J. Wang, **Y. Pang**, G. Hu, "Social Profit Optimization of Competitive Electricity Market: A Leader-following Approach," in *Chinese Control Conference (CCC)*, 2019, pp. 7494-7499.
7. S. Zhang, **Y. Pang**, G. Hu, "Trajectory-Tracking Control of Robotic System via Proximal Policy Optimization," in *IEEE International Conference on Cybernetics and Intelligent Systems (CIS) and Robotics, Automation and Mechatronics (RAM)*, IEEE, 2019, pp. 380-385.

8. **Y. Pang**, G. Hu, “A Gradient-Free Distributed Nash Equilibrium Seeking Method with Uncoordinated Step-Sizes,” accepted in *2020 IEEE 59th Conference on Decision Control (CDC)*, to appear.

Bibliography

- [1] M. P. Singh, *Multiagent systems : a theoretical framework for intentions, know-how, and communications*. Springer-Verlag, 1994.
- [2] R. Nowak, “Distributed EM algorithms for density estimation and clustering in sensor networks,” *IEEE Transactions on Signal Processing*, vol. 51, no. 8, pp. 2245–2253, 2003.
- [3] S. Ram, V. Veeravalli, and A. Nedic, “Distributed and Recursive Parameter Estimation in Parametrized Linear State-Space Models,” *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 488–492, 2010.
- [4] V. Lesser, C. L. Ortiz, and M. Tambe, *Distributed Sensor Networks : a Multiagent Perspective*. Springer US, 2003.
- [5] M. Rabbat and R. Nowak, “Decentralized source localization and tracking [wireless sensor networks],” in *2004 IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 3, 2004, pp. iii–921–4.
- [6] D. P. Palomar and M. Chiang, “Alternative Distributed Algorithms for Network Utility Maximization: Framework and Applications,” *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2254–2269, 2007.
- [7] M. Chiang, P. Hande, T. Lan, and C. W. Tan, “Power Control in Wireless Cellular Networks,” *Foundations and Trends® in Networking*, vol. 2, no. 4, pp. 381–533, 2007.

- [8] C. Shen, T.-H. Chang, K.-Y. Wang, Z. Qiu, and C.-Y. Chi, “Distributed Robust Multicell Coordinated Beamforming With Imperfect CSI: An ADMM Approach,” *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 2988–3003, 2012.
- [9] M. C. De Gennaro and A. Jadbabaie, “Decentralized Control of Connectivity for Multi-Agent Systems,” in *2006 IEEE 45th Conference on Decision and Control (CDC)*, 2006, pp. 3628–3633.
- [10] X. Dong and G. Hu, “Time-Varying Formation Tracking for Linear Multiagent Systems With Multiple Leaders,” *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3658–3664, 2017.
- [11] Z. Feng, C. Sun, and G. Hu, “Robust Connectivity Preserving Rendezvous of Multirobot Systems Under Unknown Dynamics and Disturbances,” *IEEE Transactions on Control of Network Systems*, vol. 4, no. 4, pp. 725–735, 2017.
- [12] Y. Wu, W. Xia, M. Cao, and X.-M. Sun, “Reach control problem for affine multi-agent systems on simplices,” *Automatica*, vol. 107, pp. 264–271, 2019.
- [13] P. Mertikopoulos, E. V. Belmega, R. Negrel, and L. Sanguinetti, “Distributed Stochastic Optimization via Matrix Exponential Learning,” *IEEE Transactions on Signal Processing*, vol. 65, no. 9, pp. 2277–2290, 2017.
- [14] A. Agarwal and J. C. Duchi, “Distributed delayed stochastic optimization,” in *2012 IEEE 51st Conference on Decision and Control (CDC)*, 2012, pp. 5451–5452.
- [15] S. S. Ram, A. Nedić, and V. V. Veeravalli, “Distributed Stochastic Subgradient Projection Algorithms for Convex Optimization,” *Journal of Optimization Theory and Applications*, vol. 147, no. 3, pp. 516–545, 2010.
- [16] J. Nash, “Non-Cooperative Games,” *The Annals of Mathematics*, vol. 54, no. 2, p. 286, 1951.

- [17] C. Saraydar, N. Mandayam, and D. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Transactions on Communications*, vol. 50, no. 2, pp. 291–303, 2002.
- [18] A.-H. Mohsenian-Rad, V. W. S. Wong, J. Jatskevich, R. Schober, and A. Leon-Garcia, "Autonomous Demand-Side Management Based on Game-Theoretic Energy Consumption Scheduling for the Future Smart Grid," *IEEE Transactions on Smart Grid*, vol. 1, no. 3, pp. 320–331, dec 2010.
- [19] Q. Zhu, J. Zhang, P. W. Sauer, A. Dominguez-Garcia, and T. Basar, "A game-theoretic framework for control of distributed renewable-based energy resources in smart grids," in *2012 American Control Conference (ACC)*. IEEE, jun 2012, pp. 3623–3628.
- [20] H. Yin, C. Zhao, M. Li, C. Ma, and M.-Y. Chow, "A Game Theory Approach to Energy Management of An EngineGenerator/Battery/Ultracapacitor Hybrid Energy System," *IEEE Transactions on Industrial Electronics*, vol. 63, no. 7, pp. 4266–4277, jul 2016.
- [21] M. G. Bell, "A game theory approach to measuring the performance reliability of transport networks," *Transportation Research Part B: Methodological*, vol. 34, no. 6, pp. 533–545, aug 2000.
- [22] E. Altman and L. Wynter, "Equilibrium, Games, and Pricing in Transportation and Telecommunication Networks," *Networks and Spatial Economics*, vol. 4, no. 1, pp. 7–21, mar 2004.
- [23] H. M. Amar and O. A. Basir, "A Game Theoretic Solution for the Territory Sharing Problem in Social Taxi Networks," *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 7, pp. 2114–2124, jul 2018.
- [24] O. Kramer, D. E. Ciaurri, and S. Koziel, "Derivative-Free Optimization." Springer, Berlin, Heidelberg, 2011, pp. 61–83.

- [25] B. T. Polyak, “Introduction to Optimization,” *Optimization Software, Inc, New York*, 1987.
- [26] S. P. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge University Press, mar 2004.
- [27] C. Xi, Q. Wu, and U. A. Khan, “On the distributed optimization over directed networks,” *Neurocomputing*, vol. 267, pp. 508–515, 2017.
- [28] A. Nedic and A. Olshevsky, “Distributed Optimization Over Time-Varying Directed Graphs,” *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 601–615, 2015.
- [29] I. Lobel and A. Ozdaglar, “Distributed Subgradient Methods for Convex Optimization Over Random Networks,” *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1291–1306, 2011.
- [30] W. Shi, Q. Ling, G. Wu, and W. Yin, “EXTRA: An Exact First-Order Algorithm for Decentralized Consensus Optimization,” *SIAM Journal on Optimization*, vol. 25, no. 2, pp. 944–966, 2015.
- [31] C. Xi, Q. Wu, and U. A. Khany, “Fast distributed optimization over directed graphs,” in *2016 American Control Conference (ACC)*. IEEE, 2016, pp. 6507–6512.
- [32] J. Xu, S. Zhu, Y. C. Soh, and L. Xie, “Convergence of Asynchronous Distributed Gradient Methods over Stochastic Networks,” *IEEE Transactions on Automatic Control*, 2017.
- [33] A. Nedic, A. Ozdaglar, and P. Parrilo, “Constrained Consensus and Optimization in Multi-Agent Networks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 922–938, 2010.
- [34] V. S. Mai and E. H. Abed, “Distributed optimization over weighted directed graphs using row stochastic matrix,” in *2016 American Control Conference (ACC)*, 2016, pp. 7165–7170.

- [35] C. Xi and U. A. Khan, “Distributed Subgradient Projection Algorithm over Directed Graphs,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3986–3992, 2016.
- [36] Q. Liu, S. Yang, and Y. Hong, “Constrained Consensus Algorithms with Fixed Step Size for Distributed Convex Optimization Over Multi-agent Networks,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4259–4265, 2017.
- [37] J. C. Duchi, A. Agarwal, and M. J. Wainwright, “Dual Averaging for Distributed Optimization: Convergence Analysis and Network Scaling,” *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 592–606, 2012.
- [38] K. I. Tsianos, S. Lawlor, and M. G. Rabbat, “Push-Sum Distributed Dual Averaging for convex optimization,” in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 5453–5458.
- [39] T.-H. Chang, A. Nedic, and A. Scaglione, “Distributed Constrained Optimization by Consensus-Based Primal-Dual Perturbation Method,” *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1524–1538, 2014.
- [40] D. Yuan, D. W. C. Ho, and S. Xu, “Regularized Primal-Dual Subgradient Method for Distributed Constrained Optimization,” *IEEE Transactions on Cybernetics*, vol. 46, no. 9, pp. 2109–2118, 2016.
- [41] M. Zhu and S. Martinez, “On Distributed Convex Optimization Under Inequality and Equality Constraints,” *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 151–164, 2012.
- [42] L. Xiao, “Distributed Subgradient Algorithm for Multi-Agent Convex Optimization with Global Inequality and Equality Constraints,” *Applied and Computational Mathematics*, vol. 5, no. 5, p. 213, 2016.
- [43] I. Masubuchi, T. Wada, T. Asai, T. H. L. Nguyen, Y. Ohta, and Y. Fujisaki, “Distributed Multi-Agent Optimization Based on an Exact Penalty Method

- with Equality and Inequality Constraints,” *SICE Journal of Control, Measurement, and System Integration*, vol. 9, no. 4, pp. 179–186, 2016.
- [44] G. Qu and N. Li, “Accelerated distributed Nesterov Gradient Descent for convex and smooth functions,” in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, 2017, pp. 2260–2267.
- [45] D. Yuan, Y. Hong, D. W. Ho, and G. Jiang, “Optimal distributed stochastic mirror descent for strongly convex optimization,” *Automatica*, vol. 90, pp. 196–203, 2018.
- [46] P. Lin, W. Ren, C. Yang, and W. Gui, “Distributed optimization with non-convex velocity constraints, nonuniform position constraints and nonuniform stepsizes,” *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2575–2582, 2018.
- [47] T. Yang, X. Yi, J. Wu, Y. Yuan, D. Wu, Z. Meng, Y. Hong, H. Wang, Z. Lin, and K. H. Johansson, “A survey of distributed optimization,” *Annual Reviews in Control*, vol. 47, pp. 278–305, 2019.
- [48] J. Matyas, “Random Optimization,” *Automation and Remote control*, vol. 26, no. 2, pp. 246–253, 1965.
- [49] A. L. Marsden, M. Wang, J. E. Dennis, and P. Moin, “Trailing-edge noise reduction using derivative-free optimization and large-eddy simulation,” *Journal of Fluid Mechanics*, vol. 572, p. 13, feb 2007.
- [50] R. Duvigneau and M. Visonneau, “Hydrodynamic design using a derivative-free method,” *Structural and Multidisciplinary Optimization*, vol. 28, no. 2-3, pp. 195–205, sep 2004.
- [51] R. Oeuvray and M. Bierlaire, “A New Derivative-Free Algorithm for the Medical Image Registration Problem,” *International Journal of Modelling and Simulation*, vol. 27, no. 2, pp. 115–124, jan 2007.

- [52] D. Echeverría Ciaurri, O. Isebor, and L. Durlofsky, “Application of derivative-free methodologies to generally constrained oil production optimization problems,” *Procedia Computer Science*, vol. 1, no. 1, pp. 1301–1310, may 2010.
- [53] K. Kvaternik and L. Pavel, “An analytic framework for decentralized extremum seeking control,” in *2012 American Control Conference (ACC)*, 2012, pp. 3371–3376.
- [54] J.-M. Xu and Y. C. Soh, “Distributed Extremum Seeking Control of networked large-scale systems under constraints,” in *2013 IEEE 52nd Conference on Decision and Control (CDC)*, 2013, pp. 2187–2192.
- [55] J. Poveda and N. Quijano, “Distributed extremum seeking for real-time resource allocation,” in *2013 American Control Conference*, 2013, pp. 2772–2777.
- [56] S. Dougherty and M. Guay, “Extremum-seeking control of distributed systems using consensus estimation,” in *2014 IEEE 53rd Conference on Decision and Control (CDC)*, 2014, pp. 3450–3455.
- [57] M. Ye and G. Hu, “A distributed extremum seeking scheme for networked optimization,” in *2015 IEEE 54th Conference on Decision and Control (CDC)*, 2015, pp. 4928–4933.
- [58] A. Menon and J. S. Baras, “Collaborative extremum seeking for welfare optimization,” in *2014 IEEE 53rd Conference on Decision and Control (CDC)*, 2014, pp. 346–351.
- [59] M. Ye and G. Hu, “Distributed Extremum Seeking for Constrained Networked Optimization and Its Application to Energy Consumption Control in Smart Grid,” *IEEE Transactions on Control Systems Technology*, vol. 24, no. 6, pp. 2048–2058, 2016.
- [60] O. Shamir and T. Zhang, “Stochastic Gradient Descent for Non-smooth Optimization: Convergence Results and Optimal Averaging Schemes,” in *Pro-*

- ceedings of the 30th International Conference on Machine Learning*, 2013, pp. 71–79.
- [61] Y. Nesterov and V. Spokoiny, “Random Gradient-Free Minimization of Convex Functions,” *Foundations of Computational Mathematics*, vol. 17, no. 2, pp. 527–566, 2017.
- [62] D. Yuan and D. W. C. Ho, “Randomized Gradient-Free Method for Multiagent Optimization Over Time-Varying Networks,” *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 6, pp. 1342–1347, 2015.
- [63] J. Li, C. Wu, Z. Wu, and Q. Long, “Gradient-free method for nonsmooth distributed optimization,” *Journal of Global Optimization*, vol. 61, no. 2, pp. 325–340, 2015.
- [64] X.-M. Chen and C. Gao, “Strong consistency of random gradient-free algorithms for distributed optimization,” *Optimal Control Applications and Methods*, vol. 38, no. 2, pp. 247–265, 2017.
- [65] D. Yuan, S. Xu, and J. Lu, “Gradient-free method for distributed multi-agent optimization via push-sum algorithms,” *International Journal of Robust and Nonlinear Control*, vol. 25, no. 10, pp. 1569–1580, 2015.
- [66] Y. Pang and G. Hu, “A distributed optimization method with unknown cost function in a multi-agent system via randomized gradient-free method,” in *2017 11th Asian Control Conference (ASCC)*, 2017, pp. 144–149.
- [67] Y. Pang and G. Hu, “Randomized Gradient-Free Distributed Optimization Methods for a Multi-Agent System with Unknown Cost Function,” *IEEE Transactions on Automatic Control*, vol. 65, no. 1, pp. 333–340, 2020.
- [68] J. C. Duchi, M. I. Jordan, M. J. Wainwright, and A. Wibisono, “Optimal Rates for Zero-Order Convex Optimization: The Power of Two Function Evaluations,” *IEEE Transactions on Information Theory*, vol. 61, no. 5, pp. 2788–2806, 2015.

- [69] D. Yuan, D. W. C. Ho, and S. Xu, “Zeroth-Order Method for Distributed Optimization With Approximate Projections,” *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 2, pp. 284–294, 2016.
- [70] Y. Pang and G. Hu, “Exact Convergence of Gradient-Free Distributed Optimization Method in a Multi-Agent System,” in *2018 IEEE 58th Conference on Decision and Control (CDC)*, 2018, pp. 5728–5733.
- [71] C. Sun, M. Ye, and G. Hu, “Distributed Time-varying Quadratic Optimization for Multiple Agents under Undirected Graphs,” *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3687–3694, 2017.
- [72] F. Yan, S. Sundaram, S. Vishwanathan, and Y. Qi, “Distributed Autonomous Online Learning: Regrets and Intrinsic Privacy-Preserving Properties,” *IEEE Transactions on Knowledge and Data Engineering*, vol. 25, no. 11, pp. 2483–2493, 2013.
- [73] M. Akbari, B. Gharesifard, and T. Linder, “Distributed Online Convex Optimization on Time-Varying Directed Graphs,” *IEEE Transactions on Control of Network Systems*, vol. 4, no. 3, pp. 417–428, 2017.
- [74] S. Hosseini, A. Chapman, and M. Mesbahi, “Online distributed optimization via dual averaging,” in *2013 IEEE 52nd Conference on Decision and Control (CDC)*, 2013, pp. 1484–1489.
- [75] S. Hosseini, A. Chapman, and M. Mesbahi, “Online Distributed Convex Optimization on Dynamic Networks,” *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3545–3550, 2016.
- [76] D. Mateos-Nunez and J. Cortes, “Distributed Online Convex Optimization Over Jointly Connected Digraphs,” *IEEE Transactions on Network Science and Engineering*, vol. 1, no. 1, pp. 23–37, 2014.

- [77] S. Shahrampour and A. Jadbabaie, “An online optimization approach for multi-agent tracking of dynamic parameters in the presence of adversarial noise,” in *2017 American Control Conference (ACC)*, 2017, pp. 3306–3311.
- [78] S. Shahrampour and A. Jadbabaie, “Distributed Online Optimization in Dynamic Environments Using Mirror Descent,” *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 714–725, mar 2018.
- [79] S. Lee, A. Nedic, and M. Raginsky, “Stochastic Dual Averaging for Decentralized Online Optimization on Time-Varying Communication Graphs,” *IEEE Transactions on Automatic Control*, vol. 62, no. 12, pp. 6407–6414, 2017.
- [80] J. Shamma and G. Arslan, “Dynamic fictitious play, dynamic gradient play, and distributed convergence to Nash equilibria,” *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 312–327, 2005.
- [81] N. Li and J. R. Marden, “Designing Games for Distributed Optimization,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 7, no. 2, pp. 230–242, 2013.
- [82] B. Gharesifard and J. Cortes, “Distributed convergence to Nash equilibria in two-network zero-sum games,” *Automatica*, vol. 49, no. 6, pp. 1683–1692, 2013.
- [83] Y. Lou, Y. Hong, L. Xie, G. Shi, and K. H. Johansson, “Nash Equilibrium Computation in Subnetwork Zero-Sum Games With Switching Communications,” *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 2920–2935, 2016.
- [84] G. Belgioioso and S. Grammatico, “A Douglas-Rachford Splitting for Semi-decentralized Equilibrium Seeking in Generalized Aggregative Games,” in *2018 IEEE 58th Conference on Decision and Control (CDC)*, 2018, pp. 3541–3546.

- [85] M. Ye and G. Hu, “Game Design and Analysis for Price-Based Demand Response: An Aggregate Game Approach,” *IEEE Transactions on Cybernetics*, vol. 47, no. 3, pp. 720–730, 2017.
- [86] S. Liang, P. Yi, and Y. Hong, “Distributed Nash equilibrium seeking of a class of aggregative games,” in *2017 13th IEEE International Conference on Control & Automation (ICCA)*, 2017, pp. 58–63.
- [87] Z. Deng and X. Nian, “Distributed Generalized Nash Equilibrium Seeking Algorithm Design for Aggregative Games Over Weight-Balanced Digraphs,” *IEEE Transactions on Neural Networks and Learning Systems*, vol. 30, no. 3, pp. 695–706, 2019.
- [88] Z. Deng and S. Liang, “Distributed algorithms for aggregative games of multiple heterogeneous EulerLagrange systems,” *Automatica*, vol. 99, pp. 246–252, 2019.
- [89] C. Sun and G. Hu, “Distributed Nash Equilibrium Seeking for Generalized Convex Games with Shared Constraints,” *Journal of Physics: Conference Series*, vol. 1016, no. 1, p. 012012, 2018.
- [90] K. Lu, G. Jing, and L. Wang, “Distributed Algorithms for Searching Generalized Nash Equilibrium of Noncooperative Games,” *IEEE Transactions on Cybernetics*, vol. 49, no. 6, pp. 2362–2371, 2019.
- [91] M. Zhu and E. Frazzoli, “On distributed equilibrium seeking for generalized convex games,” in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 4858–4863.
- [92] P. Yi and L. Pavel, “An operator splitting approach for distributed generalized Nash equilibria computation,” *Automatica*, vol. 102, pp. 111–121, 2019.
- [93] M. Ye, G. Hu, and F. Lewis, “Nash equilibrium seeking for N-coalition non-cooperative games,” *Automatica*, vol. 95, pp. 266–272, 2018.

- [94] X. Zeng, J. Chen, S. Liang, and Y. Hong, “Generalized Nash equilibrium seeking strategy for distributed nonsmooth multi-cluster game,” *Automatica*, vol. 103, pp. 20–26, 2019.
- [95] M. Ye, “Nash Equilibrium Seeking for Games in Hybrid Systems,” in *2018 15th International Conference on Control, Automation, Robotics and Vision (ICARCV)*. IEEE, 2018, pp. 140–145.
- [96] T. Tatarenko, W. Shi, and A. Nedic, “Accelerated Gradient Play Algorithm for Distributed Nash Equilibrium Seeking,” in *2018 IEEE 58th Conference on Decision and Control (CDC)*, 2018, pp. 3561–3566.
- [97] D. Yue and Z. Meng, “Distributed Nash equilibrium seeking for constrained games,” in *2018 37th Chinese Control Conference (CCC)*. IEEE, 2018, pp. 9660–9665.
- [98] A. R. Romano and L. Pavel, “Dynamic Gradient Play for NE Seeking with Disturbance Rejection,” in *2018 IEEE 58th Conference on Decision and Control (CDC)*, 2018, pp. 346–351.
- [99] J. Koshal, A. Nedic, and U. V. Shanbhag, “A gossip algorithm for aggregative games on graphs,” in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 4840–4845.
- [100] F. Salehisadaghiani and L. Pavel, “Distributed Nash equilibrium seeking by gossip in games on graphs,” in *2016 IEEE 55th Conference on Decision and Control (CDC)*, 2016, pp. 6111–6116.
- [101] F. Salehisadaghiani and L. Pavel, “Distributed Nash equilibrium seeking: A gossip-based algorithm,” *Automatica*, vol. 72, pp. 209–216, 2016.
- [102] F. Salehisadaghiani and L. Pavel, “Distributed Nash equilibrium seeking in networked graphical games,” *Automatica*, vol. 87, pp. 17–24, 2018.

- [103] F. Salehisadaghiani and L. Pavel, “Nash Equilibrium Seeking with Non-doubly Stochastic Communication Weight Matrix,” in *Game Theory for Networks. GameNets 2017*. Springer, Cham, 2017, pp. 3–15.
- [104] M. Zhu and E. Frazzoli, “Distributed robust adaptive equilibrium computation for generalized convex games,” *Automatica*, vol. 63, pp. 82–91, 2016.
- [105] T. Tatarenko and M. Kamgarpour, “Learning Generalized Nash Equilibria in a Class of Convex Games,” *IEEE Transactions on Automatic Control*, vol. 64, no. 4, pp. 1426–1439, 2019.
- [106] T. Tatarenko and M. Kamgarpour, “Learning Nash Equilibria in Monotone Games,” in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 3104–3109.
- [107] S. J. Liu and M. Krstic, “Stochastic Nash equilibrium seeking for games with general nonlinear payoffs,” *SIAM Journal on Control and Optimization*, vol. 49, no. 4, pp. 1659–1679, 2011.
- [108] M. S. Stankovic, K. H. Johansson, and D. M. Stipanovic, “Distributed Seeking of Nash Equilibria With Applications to Mobile Sensor Networks,” *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 904–919, 2012.
- [109] P. Frihauf, M. Krstic, and T. Basar, “Nash Equilibrium Seeking in Noncooperative Games,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1192–1207, 2012.
- [110] M. Ye and G. Hu, “Solving potential games with unstable dynamics,” in *Proceedings of the 33rd Chinese Control Conference*, 2014, pp. 8182–8187.
- [111] M. Ye and G. Hu, “Solving Potential Games With Dynamical Constraint,” *IEEE Transactions on Cybernetics*, vol. 46, no. 5, pp. 1156–1164, 2016.
- [112] M. Ye and G. Hu, “Distributed optimization for systems with time-varying quadratic objective functions,” in *2015 IEEE 54th Conference on Decision and Control (CDC)*, 2015, pp. 3285–3290.

- [113] B. Gharesifard and J. Cortes, “When does a digraph admit a doubly stochastic adjacency matrix?” in *Proceedings of the 2010 American Control Conference*, 2010, pp. 2440–2445.
- [114] B. Gharesifard and J. Cortés, “Distributed Strategies for Generating Weight-Balanced and Doubly Stochastic Digraphs,” *European Journal of Control*, vol. 18, no. 6, pp. 539–557, 2012.
- [115] K. Cai and H. Ishii, “Average consensus on general strongly connected digraphs,” *Automatica*, vol. 48, no. 11, pp. 2750–2761, 2012.
- [116] R. Bhatia, *Matrix Analysis*, ser. Graduate Texts in Mathematics. New York, NY: Springer New York, 1997, vol. 169.
- [117] Y. Pang, G. Hu, and C. J. Spanos, “Distributed Model Predictive Control of a HVAC System via a Projected Subgradient Method,” in *2018 IEEE 14th International Conference on Control and Automation (ICCA)*. IEEE, jun 2018, pp. 878–883.
- [118] G. Debreu, “A Social Equilibrium Existence Theorem,” *Proceedings of the National Academy of Sciences*, vol. 38, no. 10, pp. 886–893, 1952.
- [119] I. L. Glicksberg, “A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points,” *Proceedings of the American Mathematical Society*, vol. 3, no. 1, p. 170, 1952.
- [120] P. Jacquot and C. Wan, “Nonsmooth Aggregative Games with Coupling Constraints and Infinitely Many Classes of Players,” *arXiv preprint arXiv:1806.06230*, 2018.
- [121] A. Nedić, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis, “Distributed Subgradient Methods and Quantization Effects,” in *2008 IEEE 47th Conference on Decision and Control (CDC)*, 2008, pp. 4177–4184.

- [122] M. Ye and G. Hu, “Distributed Nash Equilibrium Seeking by a Consensus Based Approach,” *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4811–4818, 2017.
- [123] M. Zhu and S. Martinez, “An Approximate Dual Subgradient Algorithm for Multi-Agent Non-Convex Optimization,” *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1534–1539, 2013.
- [124] R. Hooke and T. A. Jeeves, ““ Direct Search” Solution of Numerical and Statistical Problems,” *Journal of the ACM*, vol. 8, no. 2, pp. 212–229, 1961.
- [125] F. J. Solis and R. J.-B. Wets, “Minimization by Random Search Techniques,” *Mathematics of Operations Research*, vol. 6, no. 1, pp. 19–30, 1981.
- [126] R. M. Lewis, V. Torczon, and M. W. Trosset, “Direct search methods: then and now,” *Journal of Computational and Applied Mathematics*, vol. 124, no. 1-2, pp. 191–207, 2000.
- [127] D. Popovic and A. Teel, “Direct search methods for nonsmooth optimization,” in *2004 IEEE 43rd Conference on Decision and Control (CDC)*, 2004, pp. 3173–3178.
- [128] S. Kim and D. Zhang, “Convergence properties of direct search methods for stochastic optimization,” in *Proceedings of the 2010 Winter Simulation Conference*. IEEE, 2010, pp. 1003–1011.
- [129] J. Kennedy and R. Eberhart, “Particle swarm optimization,” in *Proceedings of ICNN’95 - International Conference on Neural Networks*, vol. 4, 1995, pp. 1942–1948.
- [130] M. Dorigo, M. Birattari, and T. Stutzle, “Ant colony optimization,” *IEEE Computational Intelligence Magazine*, vol. 1, no. 4, pp. 28–39, 2006.
- [131] D. E. D. E. Goldberg and D. E., *Genetic algorithms in search, optimization, and machine learning*. Addison-Wesley Longman Publishing Co., Inc., 1989.

- [132] J. Cortés, “Finite-time convergent gradient flows with applications to network consensus,” *Automatica*, vol. 42, no. 11, pp. 1993–2000, nov 2006.
- [133] Z.-H. Guan, F.-L. Sun, Y.-W. Wang, and T. Li, “Finite-Time Consensus for Leader-Following Second-Order Multi-Agent Networks,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 59, no. 11, pp. 2646–2654, nov 2012.
- [134] X. Liu, M. Z. Q. Chen, H. Du, and S. Yang, “Further results on finite-time consensus of second-order multi-agent systems without velocity measurements,” *International Journal of Robust and Nonlinear Control*, vol. 26, no. 14, pp. 3170–3185, sep 2016.
- [135] Y. Cao, W. Ren, D. W. Casbeer, and C. Schumacher, “Finite-Time Connectivity-Preserving Consensus of Networked Nonlinear Agents With Unknown Lipschitz Terms,” *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1700–1705, jun 2016.
- [136] J.-G. Dong, “Finite-time connectivity preservation rendezvous with disturbance rejection,” *Automatica*, vol. 71, pp. 57–61, sep 2016.
- [137] A. Jadbabaie, Jie Lin, and A. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, jun 2003.
- [138] M. Rabbat and R. Nowak, “Quantized incremental algorithms for distributed optimization,” *IEEE Journal on Selected Areas in Communications*, vol. 23, no. 4, pp. 798–808, apr 2005.
- [139] U. Münz, A. Papachristodoulou, and F. Allgöwer, “Delay robustness in consensus problems,” *Automatica*, vol. 46, no. 8, pp. 1252–1265, aug 2010.