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ALGEBRA OF ENTIRE DIRICHLET SERIES WITH REAL FREQUENCIES

LIM JECK, LE HAI KHOI

ABSTRACT. We study the class $E(\Lambda, \mathbb{C})$ of all entire Dirichlet series with real frequencies Λ (which are generalizations of classical Dirichlet series as well as power series). Our method, based on Number Theory, allows us to get a link between the set of real frequencies and the set of counting numbers. So the following problems posed by C.O. Kiselman have been solved: (i) the criteria for $E(\Lambda, \mathbb{C})$ to be closed (invariant) under multiplication; (ii) the explicit description of the smallest subalgebra of entire functions that contains $E(\Lambda, \mathbb{C})$. Some open questions are provided.

1. INTRODUCTION

1.1. Basic notions and notation. Let $0 \leq (\lambda_n) \uparrow \infty$ be a given sequence of non-negative real numbers. Consider Dirichlet series with real frequencies $\Lambda = (\lambda_n)$

$$(1.1) \quad \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad a_n, z \in \mathbb{C}.$$

As is well known, if we set

$$D = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n},$$

then in case

$$(1.2) \quad L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} < \infty,$$

the series (1.1) represents an entire function in $\mathbb{C}$ if and only if $D = -\infty$ (see, e.g., [2, 4]).

On one hand, in case $\lambda_n = \log n$, $\forall n \geq 1$, series (1.1) becomes a *classical Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z},$$

which has many important applications in Number Theory.

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On the other hand, if $\lambda_n = n$, $\forall n \geq 1$, then series (1.1) becomes a *power series*

$$\sum_{n=1}^{\infty} a_n \zeta^n,$$

with respect to the variable $\zeta = e^{-z}$.

Thus Dirichlet series with real frequencies are direct generalizations of the classical Dirichlet series and power series.

It is worth noting that while classical Dirichlet series and power series have the property that a product of these series belongs to the same class, Dirichlet series with real frequencies, in general, lack this remarkable property. So researchers are interested in whether this important characteristic can be “preserved”.

In the present paper, the answer to this problem will be given.

Throughout this paper, the condition $L < \infty$ is always supposed to hold.

1.2. A statement of the problems. We form the following normed space of all entire Dirichlet series with frequencies Λ

$$(1.3) \quad E(\Lambda, \mathbb{C}) := \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty \right\},$$

whose norm is defined by the inner product

$$\langle f, g \rangle := \sum_{n=1}^{\infty} a_n \bar{b}_n, \quad f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}.$$

Clearly, $E(\Lambda, \mathbb{C})$ is a subclass of $\mathcal{O}(\mathbb{C})$, the space of entire functions endowed with the compact-open topology. It is worth noting that $E(\Lambda, \mathbb{C})$ is, in general, a proper subclass of $\mathcal{O}(\mathbb{C})$, because equality rarely occurs.

As is well known, $\mathcal{O}(\mathbb{C})$ is a Fréchet algebra with respect to the usual multiplication of functions, while $E(\Lambda, \mathbb{C})$ is, in general, not invariant under this operation induced from $\mathcal{O}(\mathbb{C})$. At the “Komplexa Seminariet” (“Complex Analysis Seminar”, Uppsala University, Sweden) in 1995, C.O. Kiselman posed the following questions about Dirichlet series with complex frequencies which, restricting to the case of entire Dirichlet series, can be stated as follows.

Question 1.1. *What are necessary and sufficient conditions on (λ_n) to make the class $E(\Lambda, \mathbb{C})$ invariant (closed) under multiplication?*

Question 1.2. *What is the smallest subalgebra of $\mathcal{O}(\mathbb{C})$, denoted by $A_{E(\Lambda, \mathbb{C})}$, which contains $E(\Lambda, \mathbb{C})$?*

The aim of this paper is to give answers to the questions above. Our method, based on Number Theory, surprisingly allows us to get a link between the set of real frequencies and the set of counting numbers.

The paper is organized as follows. Section 2 deals with some preliminary attempts to solve the questions. The class $E(\Lambda, \mathbb{C})$ of entire Dirichlet series and its additive class $E(M, \mathbb{C})$ are studied. We show, in particular, that the quantity L in (1.2) can be expressed in terms of counting numbers $N_{\lambda}(r)$.

In Section 3, which is the main part of the paper, several important lemmas on the set of counting numbers are proved. The results of this section allow us to obtain in Section 4 affirmative solutions to the problems raised in the paper. In Section 5 we define the class $H(\beta)$ of weighted Hilbert spaces of entire Dirichlet series, which has been studied in many papers. It is showed that although $H(\beta)$ is a proper subspace of $E(\Lambda, \mathbb{C})$, it is not necessarily closed under multiplication. The final Section 6 gives a summary of the results obtained in the paper: the criteria for $E(\Lambda, \mathbb{C})$ to be closed under multiplication and the explicit description of the smallest subalgebra of entire functions that contains $E(\Lambda, \mathbb{C})$. Some open questions about the class $H(\beta)$ are provided.

2. PRELIMINARY ATTEMPTS TO SOLVE THE QUESTIONS

It is obvious that depending on whether $\lambda_1 = 0$ or $\lambda_1 > 0$, the class $E(\Lambda, \mathbb{C})$ either does or does not contain constant functions. This important observation requires us to consider separately two cases for this class.

For any $f_1(z), \dots, f_p(z) \in E(\Lambda, \mathbb{C})$, a product $f_1(z) \dots f_p(z)$ is in $A_{E(\Lambda, \mathbb{C})}$, but not necessarily in $E(\Lambda, \mathbb{C})$. So it is natural to consider the smallest set M , containing Λ , such that $M + M = M$. In other words,

$$M = \{ \lambda_{k_1} + \dots + \lambda_{k_p} : \lambda_{k_j} \in \Lambda \ (1 \leq j \leq p), \ p \in \mathbb{N} \}.$$

If $M = \Lambda$ then $A_{E(\Lambda, \mathbb{C})} = E(\Lambda, \mathbb{C}) \subsetneq \mathcal{O}(\mathbb{C})$. This can happen, for instance, when $\Lambda = \{k\lambda_0\}_{k=0}^\infty$, where λ_0 is a fixed positive real number, or when $\Lambda = \mathbb{N} \cup \{0\}$.

However, in general, $M \neq \Lambda$. In this case we first note that since the set Λ is countable so is the set M . Then we can order the elements of M so that the obtained sequence $(\mu_j)_{j=1}^\infty$ satisfies the condition $(\mu_j) \uparrow +\infty$. We call the sequence (μ_j) the *additive closure* of (λ_n) . Suppose that the sequence $(\mu_j)_{j=1}^\infty$ behaves like the sequence $(\lambda_k)_{k=1}^\infty$ does, i.e., it satisfies the following condition

$$(2.1) \quad \limsup_{j \rightarrow \infty} \frac{\log j}{\mu_j} < \infty.$$

Then we denote

$$(2.2) \quad E(M, \mathbb{C}) := \left\{ f(z) = \sum_{j=1}^{\infty} a_j e^{-\mu_j z} : \limsup_{j \rightarrow \infty} \frac{\log |a_j|}{\mu_j} = -\infty \right\}.$$

Obviously $E(\Lambda, \mathbb{C}) \subseteq E(M, \mathbb{C}) \subseteq \mathcal{O}(\mathbb{C})$ and it is easy to verify that $E(M, \mathbb{C}) = A_{E(\Lambda, \mathbb{C})}$, the smallest subalgebra of $\mathcal{O}(\Omega)$ which contains $E(\Lambda, \Omega)$.

Thus, we obtain the following result.

Theorem 2.1. *The following assertions hold:*

- (i) *If $M = \Lambda$ then $A_{E(\Lambda, \mathbb{C})} = E(\Lambda, \mathbb{C})$.*
- (ii) *If $M \neq \Lambda$ and the sequence $(\mu_j)_{j=1}^\infty$ satisfies condition (2.1), then $A_{E(\Lambda, \mathbb{C})} = E(M, \mathbb{C})$.*

A question to ask is: Does the sequence $(\mu_j)_{j=1}^{\infty}$ always satisfy condition (2.1)?

We look at some examples.

Example 2.2. *Let*

$$\lambda_1 = 0, \lambda_2 = \frac{1}{2}\lambda_0, \lambda_k = (k-2)\lambda_0 \quad (k \geq 3),$$

where λ_0 is a fixed positive real number.

In this case $M = \left\{ \frac{k}{2}\lambda_0 : k = 0, 1, \dots \right\}$ clearly satisfies (2.1).

Example 2.3. *Let*

$$\lambda_k = \log p_k, \quad k = 1, 2, \dots$$

where p_k is the k^{th} prime number.

Then we can easily see that $\mu_k = \log(k+1)$, so

$$\lim_{j \rightarrow \infty} \frac{\log j}{\mu_j} = 1.$$

By the Prime Number Theorem, we have

$$\lim_{k \rightarrow \infty} \frac{p_k}{k \log k} = 1.$$

Taking log we get

$$\lim_{k \rightarrow \infty} (\log p_k - \log k - \log \log k) = 0,$$

which implies that

$$\lim_{k \rightarrow \infty} \frac{\log k}{\lambda_k} = \lim_{k \rightarrow \infty} \frac{\log k}{\log p_k} = 1.$$

Thus we have sufficient evidence to suspect that the additive closure $(\mu_j)_{j=1}^{\infty}$, always satisfies condition (2.1). In fact, we shall prove it later on.

Examples 2.2 and 2.3 are too simple in the sense that the corresponding set M can be enumerated explicitly and, therefore, condition (2.1) is easily checked.

A question can be asked: how to handle a set Λ that we do not know explicitly about the “numeration law” for the set M ? In this case a direct check of condition (2.1) of course is impossible and, therefore, we should find another method.

In general, checking of condition (2.1) can be done in the following way. Denote

$$N_{\lambda}(r) = \#\{k : |\lambda_k| \leq r\}, \quad r > 0.$$

Proposition 2.4. *Let $0 \leq (\lambda_n) \uparrow \infty$ be a given sequence of non-negative real numbers. Then we have*

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = \limsup_{r \rightarrow \infty} \frac{\log N_{\lambda}(r)}{r}.$$

Proof. Since (λ_n) is increasing, we have

$$\frac{\log N_\lambda(\lambda_k)}{\lambda_k} = \frac{\log k}{\lambda_k},$$

and so

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} \leq \limsup_{r \rightarrow \infty} \frac{\log N_\lambda(r)}{r}.$$

On the other hand, given $r > \lambda_1$, let k be the unique positive integer such that $\lambda_k \leq r < \lambda_{k+1}$. Then

$$\frac{\log N_\lambda(r)}{r} \leq \frac{\log k}{\lambda_k},$$

and hence

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} \geq \limsup_{r \rightarrow \infty} \frac{\log N_\lambda(r)}{r}.$$

From these estimates the desired result follows. $\square$

Now condition (1.2) can be rewritten as follows

$$(2.3) \quad \limsup_{r \rightarrow \infty} \frac{\log N_\lambda(r)}{r} < \infty.$$

Thus, in our case condition (2.1) is equivalent to the following

$$(2.4) \quad \limsup_{r \rightarrow \infty} \frac{\log N_\mu(r)}{r} < \infty.$$

Now we illustrate this idea in the following example.

Example 2.5. *Let*

$$\lambda_1 = 0, \quad \lambda_k = k^\alpha \lambda_0 \quad (k = 2, 3, \dots),$$

where $\lambda_0 > 0$ and $\alpha > 1$ are fixed.

It suffices to consider this example with $\lambda_0 = 1$, i.e., when we have a sequence of real numbers $\lambda_1 = 0, \lambda_k = k^\alpha \quad (k = 2, 3, \dots)$.

If $\alpha \in \mathbb{N}$ then the set M is not bigger than $\mathbb{N} \cup \{0\}$, and therefore, condition (2.1) is obviously satisfied.

Suppose that $\alpha \notin \mathbb{N}$. We shall show that in this case condition (2.4) holds. Take a sufficiently big positive number r . Then there exists $L = L(r) \in \mathbb{N}$ such that $\lambda_L = L^\alpha \leq r < \lambda_{L+1} = (L+1)^\alpha$, i.e. there are only L elements of Λ that belong to the segment $[0, r]$, which shows that $L(r) \sim r^{1/\alpha}$ as $r \rightarrow \infty$. Now we consider those terms of M represented by sums of more than one element from Λ belonging to this segment. It is clear that in these sums there does not exist any element λ_j with $j > L$. So, they all are of the form

$$\varepsilon_2 \lambda_2 + \dots + \varepsilon_L \lambda_L,$$

where $\varepsilon_j \ (j = 2, \dots, L)$ are non-negative integer numbers not equal to zero simultaneously.

We estimate possible values of these ε_j . Since $\varepsilon_j \lambda_j \leq r$ we have $\varepsilon_j \leq \left[\frac{r}{j^\alpha} \right]$, where $[x]$ denotes the integral part of x . Hence, we obtain

$$N_\mu(r) \leq \prod_{j=2}^L \left(\left[\frac{r}{j^\alpha} \right] + 1 \right) + L.$$

Noting that $r + j^\alpha \leq 2r$ we get

$$N_\mu(r) \leq \prod_{j=1}^L \frac{2r}{j^\alpha} = \frac{(2r)^L}{(L!)^\alpha} \leq \frac{(2r)^L}{L^{L\alpha/2}} \sim (2\sqrt{r})^{r^{1/\alpha}}.$$

From this estimate condition (2.4) follows.

3. IMPORTANT LEMMAS

In this section we provide several lemmas which play an essential role in the solutions to the problems. These results also have independent interest.

3.1. Case 1: $\lambda_1 > 0$. Besides the set M defined above, we also consider the multi-set $\overline{M}$, defined as follows.

$$\overline{M} = \overline{M}(\Lambda) := \{\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_p} : p \geq 0, 1 \leq k_1 \leq k_2 \leq \cdots \leq k_p\}.$$

For example, if $\Lambda = \mathbb{N}$, then $\overline{M} = \{0, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, \dots\}$. The multi-set $\overline{M}$ can also be identified with the space V of sequences $\mathbf{x} = (x_i)$ of non-negative integers that are eventually 0. So an equivalent definition of $\overline{M}$ is

$$\overline{M} = \left\{ \sum_{i=1}^{\infty} x_i \lambda_i : \mathbf{x} \in V \right\}.$$

It is easy to see that $\#\{\mu \in \overline{M} : \mu < r\}$ is finite for any $r > 0$, so we can number the elements of $\overline{M}$ in increasing order to obtain a sequence $(\bar{\mu}_j)_{j=1}^{\infty}$, with $\bar{\mu}_1 = 0$. We call the sequence $(\bar{\mu}_j)$ the *formal additive closure* of (λ_n) . To explicitly determine $\bar{\mu}_j$, consider the multiset

$$\left\{ \sum_{i=1}^j x_i \lambda_i : 0 \leq x_i \leq \frac{\lambda_j}{\lambda_i}, 1 \leq i \leq j \right\}.$$

This multiset is finite and easily computed, so we can sort the elements of the multiset and $\bar{\mu}_j$ would be the j th smallest element.

Note that

$$(3.1) \quad \bar{\mu}_j \leq \mu_j \leq \lambda_j$$

where we have included $0 \in M$ so that $\mu_1 = 0$. Denote

$$N_{\bar{\mu}}(r) = \#\{k : \bar{\mu}_k \leq r\}, \quad r > 0.$$

Now we shall show that if condition (1.2) holds, then so does condition (2.1). To show this, we devise 4 lemmas to compare $N_{\bar{\mu}}(r)$ for 2 different sequences.

Lemma 3.1. *Given 2 sequences of positive reals $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ tending to infinity such that $\lambda_k^2 = a\lambda_k^1$ for all k where a is a fixed positive real number. Let $(\bar{\mu}_j^1)_{j=1}^{\infty}$ and $(\bar{\mu}_j^2)_{j=1}^{\infty}$ be the formal additive closures of $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ respectively. Then*

$$N_{\bar{\mu}^2}(ar) = N_{\bar{\mu}^1}(r)$$

for any $r > 0$.

Proof. Define

$$T_1 = \left\{ \mathbf{x} \in V : \sum_{i=1}^{\infty} x_i \lambda_i^1 \leq r \right\}$$

$$T_2 = \left\{ \mathbf{x} \in V : \sum_{i=1}^{\infty} x_i \lambda_i^2 \leq ar \right\}$$

Then $N_{\bar{\mu}^1}(r) = |T_1|$ and $N_{\bar{\mu}^2}(ar) = |T_2|$. But note that $T_1 = T_2$, so $N_{\bar{\mu}^2}(ar) = N_{\bar{\mu}^1}(r)$. $\square$

Lemma 3.2. *Given 2 sequences of positive reals $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ tending to infinity such that $\lambda_k^1 \leq \lambda_k^2$ for all k . Let $(\bar{\mu}_j^1)_{j=1}^{\infty}$ and $(\bar{\mu}_j^2)_{j=1}^{\infty}$ be the formal additive closures of $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ respectively. Then*

$$N_{\bar{\mu}^2}(r) \leq N_{\bar{\mu}^1}(r)$$

for any $r > 0$.

Proof. Define

$$S_l = \left\{ \mathbf{x} \in V : \sum_{i=1}^{\infty} x_i \lambda_i^l \leq r \right\}$$

for $l = 1, 2$. Then $N_{\bar{\mu}^l}(r) = |S_l|$. Note that $S_2 \subseteq S_1$, thus $N_{\bar{\mu}^2}(r) \leq N_{\bar{\mu}^1}(r)$. $\square$

Lemma 3.3. *Given a sequence of positive reals $(\lambda_k^1)_{k=1}^{\infty}$ tending to infinity and fixed positive integers t, n , let $(\lambda_k^2)_{k=1}^{\infty}$ be such that $\lambda_k^2 = \lambda_k^1$ for $k \neq t$ and $\lambda_t^2 = n\lambda_t^1$. Let $(\bar{\mu}_j^1)_{j=1}^{\infty}$ and $(\bar{\mu}_j^2)_{j=1}^{\infty}$ be the formal additive closures of $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ respectively. Then*

$$nN_{\bar{\mu}^2}(r) \geq N_{\bar{\mu}^1}(r)$$

for any $r > 0$.

Proof. Define S_1, S_2 as above. Consider the map $f : S_1 \rightarrow S_2$ given by

$$f(\mathbf{x}) = \mathbf{y}$$

where $y_k = x_k$ for $k \neq t$ and $y_t = \lfloor x_t/n \rfloor$. Since

$$\sum_{i=1}^{\infty} y_i \lambda_i^2 \leq \sum_{i=1}^{\infty} x_i \lambda_i^1 \leq r,$$

the map is well-defined. But f is at most n -to-1, so $n|S_2| \geq |S_1|$ thus

$$nN_{\bar{\mu}^2}(r) \geq N_{\bar{\mu}^1}(r).$$

$\square$

Lemma 3.4. *Given 2 sequences of positive reals $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ tending to infinity such that $\lambda_k^2 \geq \lambda_k^1$ for all sufficiently large k . Let $(\bar{\mu}_j^1)_{j=1}^{\infty}$ and $(\bar{\mu}_j^2)_{j=1}^{\infty}$ be the formal additive closures of $(\lambda_k^1)_{k=1}^{\infty}$ and $(\lambda_k^2)_{k=1}^{\infty}$ respectively. Then there exists a positive constant C such that*

$$N_{\bar{\mu}^2}(r) \leq CN_{\bar{\mu}^1}(r)$$

for any $r > 0$.

Proof. Suppose $\lambda_k^2 \geq \lambda_k^1$ for all $k > K$. Define $(\lambda_k^3)_{k=1}^\infty$ such that $\lambda_k^3 = \lambda_k^2$ for all $k > K$ and $\lambda_k^3 = n_k \lambda_k^2 \geq \lambda_k^1$ for $k \leq K$ and appropriate integer constants n_k . Then by Lemma 3.3,

$$n_1 n_2 \dots n_K N_{\bar{\mu}^3}(r) \geq N_{\bar{\mu}^2}(r)$$

where $(\bar{\mu}_j^3)$ is the formal additive closure of (λ_k^3) . By Lemma 3.2, $N_{\bar{\mu}^3}(r) \leq N_{\bar{\mu}^1}(r)$. Thus we have

$$N_{\bar{\mu}^2}(r) \leq C N_{\bar{\mu}^1}(r)$$

where $C = n_1 n_2 \dots n_K$. $\square$

Now we can compare sequences with the sequence from Example 2.3 which allows us to prove the following important result.

Theorem 3.5. *Given an increasing sequence of positive reals $(\lambda_k)_{k=1}^\infty$ tending to infinity such that*

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\lambda_k} = L < \infty,$$

then if $(\bar{\mu}_j)_{j=1}^\infty$ is the formal additive closure of (λ_k) , we have

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\bar{\mu}_k} = L.$$

Proof. Firstly, since $\bar{\mu}_k \leq \lambda_k$, we have

$$\limsup_{r \rightarrow \infty} \frac{\log N_{\bar{\mu}}(r)}{r} \geq L.$$

So it suffices to show that

$$\limsup_{r \rightarrow \infty} \frac{\log N_{\bar{\mu}}(r)}{r} \leq L.$$

Define $\lambda_k^0 = \log p_k$ where (p_k) are all the primes in increasing order, then $\bar{\mu}_k^0 = \log k$ and

$$\lim_{r \rightarrow \infty} \frac{\log N_{\bar{\mu}^0}(r)}{r} = 1 \quad \text{and} \quad \limsup_{k \rightarrow \infty} \frac{\lambda_k^0}{\lambda_k} = L$$

where $(\bar{\mu}_j^0)$ is the formal additive closure of (λ_k^0) . Given any $\varepsilon > 0$, define $(\lambda_k^1)_{k=1}^\infty$ given by $\lambda_k^1 = \frac{\lambda_k^0}{L + \varepsilon}$. Note that $L \geq 0$. Then since

$$\limsup_{k \rightarrow \infty} \frac{\lambda_k^1}{\lambda_k} = \frac{L}{L + \varepsilon} < 1,$$

we have $\lambda_k^1 < \lambda_k$ for sufficiently large k . By Lemma 3.4, there exists a positive constant C such that

$$N_{\bar{\mu}}(r) \leq C N_{\bar{\mu}^1}(r)$$

where $(\bar{\mu}_j^1)$ is the formal additive closure of (λ_k^1) . By Lemma 3.1, we have $N_{\bar{\mu}^1}(r) = N_{\bar{\mu}^0}((L + \varepsilon)r)$. Thus

$$N_{\bar{\mu}}(r) \leq C N_{\bar{\mu}^0}((L + \varepsilon)r),$$

and so we have

$$\frac{\log N_{\bar{\mu}}(r)}{r} \leq \frac{\log N_{\bar{\mu}^0}((L + \varepsilon)r) + \log C}{r}.$$

But

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\log N_{\bar{\mu}^0}((L + \varepsilon)r) + \log C}{r} &= (L + \varepsilon) \lim_{r \rightarrow \infty} \frac{\log N_{\bar{\mu}^0}((L + \varepsilon)r)}{(L + \varepsilon)r} \\ &= L + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have

$$\limsup_{r \rightarrow \infty} \frac{\log N_{\bar{\mu}}(r)}{r} \leq L.$$

□

As an immediate consequence of Theorem 3.5, we have the following result about the additive closure of a given set of frequencies.

Corollary 3.6. *Given an increasing sequence of positive reals $(\lambda_k)_{k=1}^{\infty}$ tending to infinity such that*

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\lambda_k} = L < \infty.$$

Then if (μ_k) is the additive closure of (λ_k) , we have

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\mu_k} = L.$$

Proof. Since $\mu_k \leq \lambda_k$, we have

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\mu_k} \geq L.$$

By Theorem 3.5, we have

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\bar{\mu}_k} = L$$

where $(\bar{\mu}_k)$ is the formal additive closure of (λ_k) . Recall that $\bar{\mu}_k \leq \mu_k \leq \lambda_k$ from (3.1). Thus we have

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\mu_k} \leq L$$

and the result follows. □

3.2. Case 2: $\lambda_1 = 0$. If we take Λ excluding λ_1 , then the additive closure (μ_k) generated from it would be the same as before. Thus the above result still holds.

So we get the following improvement of Corollary 3.6 which is valid for both cases $\lambda_1 = 0$ and $\lambda_1 > 0$.

Corollary 3.7. *Given an increasing sequence of non-negative reals $(\lambda_k)_{k=1}^{\infty}$ tending to infinity such that*

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\lambda_k} = L < \infty.$$

Then if (μ_k) is the additive closure of (λ_k) , we have

$$\limsup_{k \rightarrow \infty} \frac{\log k}{\mu_k} = L.$$

4. SOLUTION TO THE PROBLEMS

From the previous section, we have

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = L \implies \limsup_{n \rightarrow \infty} \frac{\log n}{\mu_n} = L,$$

so we shall assume that Λ is closed under addition (otherwise we replace λ with its additive closure μ). Given that (λ_k) satisfies

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = L < \infty.$$

Recall that the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

represents an entire function if and only if

$$\lim_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty.$$

The product of two Dirichlet series will give us another Dirichlet series

$$\left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \right) \left(\sum_{n=1}^{\infty} b_n e^{-\lambda_n z} \right) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$$

where c_n is given by

$$c_n = \sum_{\lambda_j + \lambda_k = \lambda_n} a_j b_k.$$

Proposition 4.1. *Suppose that*

$$\lim_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = \lim_{n \rightarrow \infty} \frac{\log |b_n|}{\lambda_n} = -\infty.$$

Then we have

$$\lim_{n \rightarrow \infty} \frac{\log |c_n|}{\lambda_n} = -\infty.$$

Proof. Note that $\{|a_n|\}$ and $\{|b_n|\}$ are bounded. So suppose $|a_n|, |b_n| < A$ for all n . Without loss of generality, we can assume that λ_n is strictly increasing. Given any $M > 0$, there exists positive integer K such that

$$n > K \implies \frac{\log |a_n|}{\lambda_n}, \frac{\log |b_n|}{\lambda_n} < -M.$$

In other words,

$$n > K \implies |a_n|, |b_n| < e^{-M\lambda_n}.$$

There also exists a positive integer N such that $\lambda_N \geq 2\lambda_K$. Now for each $n > N$, let $J = J_n$ be the smallest positive integer such that $\lambda_n \leq 2\lambda_J$, so $J \geq K$. Then if $\lambda_j + \lambda_k = \lambda_n$, then one of j, k is at least J . Thus we have

for $n > N$,

$$\begin{aligned} |c_n| &\leq \sum_{\lambda_j + \lambda_k = \lambda_n} |a_j| |b_k| \leq \sum_{j=J}^n |a_j| A + \sum_{k=J}^n A |b_k| \\ &\leq nA \left(\sup_{j \geq J} |a_j| + \sup_{k \geq J} |b_k| \right) \leq 2nAe^{-M\lambda_J} \leq 2nAe^{-M\lambda_n/2}. \end{aligned}$$

So we have

$$\frac{\log |c_n|}{\lambda_n} \leq \frac{\log(2nAe^{-M\lambda_n/2})}{\lambda_n} = \frac{\log 2A}{\lambda_n} + \frac{\log n}{\lambda_n} - \frac{M}{2}.$$

Letting $n \rightarrow \infty$ in both sides of the last inequality, we have

$$\lim_{n \rightarrow \infty} \frac{\log |c_n|}{\lambda_n} \leq L - \frac{M}{2}.$$

Since M is arbitrary, we obtain

$$\lim_{n \rightarrow \infty} \frac{\log |c_n|}{\lambda_n} = -\infty.$$

□

As a consequence of Proposition 4.1, we have the following explicit description of the smallest subalgebra $A_{E(\Lambda, \mathbb{C})}$.

Corollary 4.2. *The smallest subalgebra $A_{E(\Lambda, \mathbb{C})}$ of $\mathcal{O}(\mathbb{C})$ which contains $E(\Lambda, \mathbb{C})$ is the class $E(M, \mathbb{C})$, where $M = \{\mu_j\}$ is the set consisting of the additive closure of (λ_k) .*

Proof. The smallest subalgebra of $\mathcal{O}(\mathbb{C})$ containing $E(\Lambda, \mathbb{C})$ must certainly contain $E(M, \mathbb{C})$. By Corollary 3.7, $E(M, \mathbb{C})$ is indeed a class of Dirichlet series. By the proposition above, $E(M, \mathbb{C})$ is closed under multiplication. $E(M, \mathbb{C})$ is also closed under addition and scalar multiplication, hence it is an algebra. □

As an immediate consequence of Corollary 4.2, we have necessary and sufficient conditions for $E(\Lambda, \mathbb{C})$ to be a subalgebra of $\mathcal{O}(\mathbb{C})$.

Corollary 4.3. *The Dirichlet class $E(\Lambda, \mathbb{C})$ is an algebra if and only if Λ is closed under addition.*

5. WEIGHTED HILBERT SPACES OF ENTIRE DIRICHLET SERIES

5.1. Weighted spaces. Let $\beta = (\beta_n)$ be a sequence of real positive numbers. To each β , we associate the following *weighted sequence space*

$$\ell_\beta^2 = \left\{ a = (a_n) \subset \mathbb{C} : \|a\|_{\ell_\beta^2} = \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\},$$

which is a Hilbert space with the inner product

$$(5.1) \quad \langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \beta_n^2, \quad (a_n), (b_n) \in \ell_\beta^2,$$

(see, e.g., [6]).

Such sequence spaces ℓ_β^2 have many important applications in studying operators on function spaces (see, e.g., [1, 5]).

Consider the following *function space* $\mathcal{H}(\beta)$ of entire Dirichlet series induced by weight β

$$(5.2) \quad H(\beta) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \text{ entire: } \|f\|_\beta := \|(a_n)\|_{\ell_\beta^2} < +\infty \right\}.$$

This space $H(\beta)$ is an inner product space, where

$$\langle f, g \rangle_\beta = \sum_{n=1}^{\infty} a_n \overline{b_n} \beta_n^2,$$

for any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ and $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$ in $H(\beta)$.

Depending on β , the induced space $H(\beta)$ may not be complete in its norm, and so it is not necessarily a Hilbert space. The following theorem from [3] provides a criterion on the weight β for $\mathcal{H}(\beta)$ to be complete.

Theorem 5.1. *The space $H(\beta)$ of entire Dirichlet series induced by a sequence of positive real numbers $\beta = (\beta_n)$, as defined in (5.2), is a Hilbert space if and only if the following condition holds,*

$$(5.3) \quad \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} = +\infty.$$

It is clear that if condition (5.3) holds, then the space $H(\beta)$ automatically becomes a Hilbert space of entire functions, so we can drop the word “entire” in (5.2).

5.2. Relationship between $H(\beta)$ and $E(\Lambda, \mathbb{C})$. We define a sequence space

$$E = \left\{ (a_n) : \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty, \text{ or } \lim_{n \rightarrow \infty} |a_n|^{1/\lambda_n} = 0 \right\}.$$

Then our space $E(\Lambda, \mathbb{C})$ can be described as follows:

$$E(\Lambda, \mathbb{C}) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : (a_n) \in E \right\}.$$

Now we put

$$\liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} = \beta_*, \quad \limsup_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} = \beta^*.$$

In [3] the following characterization of the relationship between ℓ_β^2 and E in terms of β_* and β^* was given.

Proposition 5.2. *There are three alternative possibilities*

- (1) $\ell_\beta^2 \subset E$, which is equivalent to $\beta_* = \infty$.
- (2) $E \subset \ell_\beta^2$, which is equivalent to $\beta^* < \infty$.
- (3) $\ell_\beta^2 \setminus E \neq \emptyset$ and $E \setminus \ell_\beta^2 \neq \emptyset$, which is equivalent to $\beta_* < \beta^* = \infty$.

As a consequence, these spaces never coincide.

It is easy to verify that the space $E(\Lambda, \mathbb{C})$ is never complete with respect to the norm above. Furthermore, by Proposition 5.2, with condition (5.3) $H(\beta)$ is a Hilbert subspace of $E(\Lambda, \mathbb{C})$. Therefore, in the sequel the condition $\beta_* = \infty$ is always supposed to hold.

Unfortunately, even for this Hilbert space $H(\beta)$, it is not closed under multiplication in general. Let us consider the following example.

Example 5.3. Let $\beta_n = n!$ and $\lambda_n = n$ starting from $n = 0$. Fix some positive real number $\alpha < 1$. Consider a Dirichlet series

$$(5.4) \quad f(z) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n z}, \quad a_n = \alpha^n / n!.$$

We show that $f^k \notin H(\beta)$ for some integer k .

Squaring the series, we have

$$\left(\sum_{n=0}^{\infty} a_n e^{-\lambda_n z} \right)^2 = \sum_{n=0}^{\infty} c_n e^{-\lambda_n z},$$

where now

$$c_n = \sum_{k=0}^n a_k a_{n-k}.$$

A direct calculation gives

$$c_n \beta_n = (2\alpha)^n.$$

Thus repeated squaring of the Dirichlet series (5.4) shows eventually that the obtained series is not in the space $H(\beta)$.

A necessary condition to have $H(\beta)$ closed under multiplication would be

$$\lim_{n \rightarrow \infty} \frac{\log(a_n \beta_n)}{\lambda_n} = -\infty$$

for the case $\beta_n = n!$ and $\lambda_n = n$, and it can be shown that if the above condition is imposed on $H(\beta)$, then it would be closed under multiplication.

6. CONCLUDING REMARKS AND OPEN PROBLEMS

6.1. The space $E(\Lambda, \mathbb{C})$. Summarizing the main results obtained in this paper, we formulate the criteria which are affirmative answers to Questions 1.1 and 1.2 of this paper.

Theorem 6.1. Let $0 \leq (\lambda_n) \uparrow \infty$ be a sequence of non-negative real numbers satisfying the condition $L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} < \infty$.

- (1) The class $E(\Lambda, \mathbb{C})$ is invariant (closed) under multiplication if and only if the sequence $\Lambda = (\lambda_n)$ is closed under addition.
- (2) The smallest subalgebra $A_{E(\Lambda, \mathbb{C})}$ of $\mathcal{O}(\mathbb{C})$ which contains $E(\Lambda, \mathbb{C})$ is the class $E(M, \mathbb{C})$, where $M = \{\mu_j\}$ is the set consisting of the additive closure of (λ_k) .

6.2. The space $H(\beta)$. Concerning the subspace $H(\beta)$, the following questions can be asked:

Question 6.2. *What conditions can be imposed on $\mathcal{H}(E, \beta)$ to turn it into an algebra?*

Question 6.3. *Can $\mathcal{H}(E, \beta)$ be extended into an algebra?*

To our knowledge, these Questions 6.2 and 6.3 are still open.

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