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COMPOSITION OPERATORS BETWEEN HARDY SPACES ON LINEARLY CONVEX DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. We study composition operators acting between Hardy spaces $H^p(\Omega)$, where $\Omega \subset \mathbb{C}^2$ is a smoothly bounded, $\mathbb{C}$-linearly convex domain admitting the so-called F-type at all boundary points. This F-type domains contain certain convex domains of finite type and many cases of infinite type in the sense of Range. Criteria for boundedness and compactness of such composition operators are established. Our approach is based on the Cauchy-Leray kernel.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$. Let ρ be a defining function for Ω so that $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $b\Omega = \{z \in \mathbb{C}^2 : \rho(z) = 0\}$, $\nabla\rho \neq 0$ on $b\Omega$. The defining function ρ determines a family of approximating subdomains $\{\Omega_\varepsilon\}$, where $\Omega_\varepsilon = \{z \in \mathbb{C}^2 : \rho(z) < -\varepsilon\}$, $\varepsilon > 0$ sufficiently small, and so their boundaries $b\Omega_\varepsilon = \{z \in \mathbb{C}^2 : \rho(z) = -\varepsilon\}$. There exists a small $0 < \varepsilon_0 \ll 1$ such that for each $0 < \varepsilon < \varepsilon_0$, $b\Omega_\varepsilon$ is a smooth manifold which bounds Ω_ε.

Let $\mathcal{O}(\Omega)$ be the space of holomorphic functions in Ω, equipped with the usual topology of uniform convergence on compact subsets of Ω. The space of continuous functions on $b\Omega$ is denoted by $C(b\Omega)$.

For every $0 < p < \infty$, the Hardy space on Ω is defined as

$$H^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \|f\|_{H^p(\Omega)} = \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |f(w)|^p \, d\sigma_\varepsilon(w) \right)^{1/p} < \infty \right\},$$

where $d\sigma_\varepsilon$ is the induced area measure on $b\Omega_\varepsilon$.

A holomorphic self-map φ of Ω induces a linear *composition operator* C_φ on the space $\mathcal{O}(\Omega)$

$$C_\varphi u = u \circ \varphi, \quad u \in \mathcal{O}(\Omega).$$

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Researchers are interested in a relationship between function theoretic properties of the symbol φ and operator theoretic properties of C_φ. This study on various Banach spaces of holomorphic functions on the unit disk or the unit ball, such as (weighted) Hardy, Bergman and Fock spaces, et cetera, received a great attention during the past several decades. We refer the reader to the well-known monographs [3, 25, 28] and references therein for more detailed information.

In particular, for the unit disk of the complex plane, one of the important results states that every composition operator is bounded on the (weighted) Bergman spaces and Hardy spaces, by Littlewood's subordination principle (see, e.g., [20, 23]).

How about higher dimensions? Although a number of papers has been devoted to this case, it seems there is much to be done. For example, when $\Omega = \mathbb{B}_n$ is the unit ball in $\mathbb{C}^n$ (with $n \geq 2$), it is known that not every composition operator is bounded on the (weighted) Bergman spaces or Hardy spaces. Also there are counterexamples of φ for which C_φ is unbounded on Hardy space $H^2(\mathbb{B}_n)$ (see, e.g., [2, 1, 22]).

One of the more interesting topics is to find special conditions for smooth holomorphic self-map φ which guarantee boundedness and compactness of C_φ on Bergman spaces and Hardy spaces. For Hardy spaces, in particular, there are different approaches to characterize these properties of composition operators. One of the best approaches is using Carleson measure criteria for pullback measures. We list below some of the main discoveries since 1980's.

- In case $\Omega = \mathbb{B}_n$, in [11, 12], MacCluer formulated the characterization of boundedness and compactness in terms of a Carleson measure condition.

When φ is smooth up to the boundary, a necessary and sufficient condition for the boundedness of C_φ on $H^p(\mathbb{B}_n)$ is found by Wogen in [21, 22]. The condition involves a strict inequality between certain directional derivatives of φ.

The compactness result in [12] was also proved for the unit polydisk by Jafari [5].

- More generally, when Ω is a strongly pseudoconvex domain in $\mathbb{C}^n$ (for example, all balls, Heisenberg group, et cetera) or a pseudoconvex domain of finite commutator type in $\mathbb{C}^2$ (for example, $\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) = |z_1|^2 + |z_2|^{2m} - 1 < 0\}$, $m = 1, 2, \ldots$) with smooth boundary, the boundedness and compactness of $C_\varphi : H^p(\Omega) \to H^p(\Omega)$ is formulated again in terms of Carleson measure over $\overline{\Omega}$ by Song-Ying Li and Bernard Russo in [10]. The technical results applied in this paper are estimates for the Bergman kernel on strongly pseudoconvex domains in [4], on pseudoconvex domains of finite commutator type in $\mathbb{C}^2$ in [14], [13]. We notice that on such pseudoconvex domains, Carleson measures are constructed from the existence of a pseudometric on $b\Omega$.

The aim of this paper is to provide characterizations of boundedness and compactness for C_φ in terms of Cauchy-Leray kernel in $\mathbb{C}^2$, on a class of $\mathbb{C}$-linear convex domains of finite and infinite type in the sense of Range. These conditions of type are generalized by a geometric hypothesis named F-type (see Definition 2.5).

Our main result in the present paper is the following.

Main Theorem. *Let $0 < p \le q < \infty$. Suppose that Ω admits an F-type at all boundary points, that $\varphi\colon \Omega \to \Omega$ is a holomorphic self-map and denote by C_φ the composition operator from $H^p(\Omega)$ to $H^q(\Omega)$. Then the following assertions hold.*

(1) C_φ is bounded from $H^p(\Omega)$ to $H^q(\Omega)$ if and only if

$$\sup_{\zeta\in b\Omega}\left\{\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right)\right\}<\infty.$$

(2) If $C_\varphi\colon H^p(\Omega)\to H^q(\Omega)$ is compact, then

$$\sup_{\zeta\in b\Omega}\left\{\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\right\}$$
$$=\sup_{\zeta\in b\Omega}\left\{\inf_{T\in\mathcal{K}_{p,q}(\Omega)}\left[\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|T(K_\zeta^{\frac{1}{p}}(w))|^{q}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\right]\right\}.$$

(3) If $\sup_{\zeta\in b\Omega}\left\{\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\right\}=0$, then $C_\varphi\colon H^p(\Omega)\to H^q(\Omega)$ is compact.

For notations stated in the above theorem, we refer the reader to Section 2, where preliminaries of Theorem 2.9 and Theorem 2.10 are given.

The structure of the paper is as follows. Section 2 deals with preliminaries for $\mathbb{C}$-linearly convex domains admitting the F-type condition and the statements of main results, Theorem 2.9 and Theorem 2.10, whose proofs are given in Section 3 and Section 4, respectively.

Throughout this paper, we use the notations $\lesssim$ and $\gtrsim$ to denote inequalities up to a positive multiplicative constant, while $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$ and let ρ be a defining function for Ω. Let us define, for $\zeta\in b\Omega$ and $z\in\Omega$,

$$\Phi(\zeta,z)=\sum_{j=1}^{2}\frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j-z_j). \tag{2.1}$$

This function is called the *support function* of Ω since the complex hyperplane $\{z\in\mathbb{C}^2 : \Phi(\zeta,z)=0\}$ agrees with the complex tangent space $(T_\zeta(b\Omega))_{\mathbb{C}}$ to $b\Omega$ at ζ. For each fixed $\zeta\in b\Omega$, $\Phi(\zeta,\cdot)$ is holomorphic in Ω.

Definition 2.1 (Real convexity)**.** A domain Ω is said to be *strongly $\mathbb{R}$-linearly convex* if for any defining function for Ω, the following inequality holds

$$\operatorname{Re}\Phi(\zeta,z)\gtrsim|\zeta-z|^2,\quad\text{for any }\zeta\in b\Omega,\ z\in\overline{\Omega}.$$

In a weaker sense, Ω is $\mathbb{R}$-*linearly convex* if the above inequality is replaced by

$$\operatorname{Re}\Phi(\zeta, z) > 0, \quad \text{for any } \zeta \in b\Omega,\ z \in \overline{\Omega} \setminus \{\zeta\}.$$

Definition 2.2 (Complex convexity)**.** A domain Ω is said to be *strongly* $\mathbb{C}$-*linearly convex* if for any defining function for Ω, the following inequality holds

$$|\Phi(\zeta, z)| \gtrsim |\zeta - z|^2, \quad \text{for any } \zeta \in b\Omega,\ z \in \overline{\Omega}.$$

In a weaker sense, Ω is $\mathbb{C}$-*linearly convex* if the above inequality is replaced by

$$|\Phi(\zeta, z)| > 0, \quad \text{for any } \zeta \in b\Omega,\ z \in \overline{\Omega} \setminus \{\zeta\}.$$

It is clear that a $\mathbb{R}$-linearly (respectively, strongly $\mathbb{R}$-linearly) convex domain is $\mathbb{C}$-linearly (respectively, strongly $\mathbb{C}$-linearly) convex, while the converse in general is not true. Moreover, locally, a strongly $\mathbb{R}$-linearly (respectively, $\mathbb{C}$-linearly) convex is also $\mathbb{R}$-linearly (respectively, $\mathbb{C}$-linearly) convex.

It is also well-known that any strongly $\mathbb{C}$-linearly convex domain of class C^2 is strongly pseudoconvex. However, the converse is not true, say the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : \text{Im } z_2 > 2(\text{Re } z_1)^2 - (\text{Im } z_1)^2\}$$

is strongly pseudoconvex but not strongly $\mathbb{C}$-linearly convex.

We refer the reader to the survey [9, Section 6] or [19] for more details about linear convexity.

In the sequel, Ω is supposed to be a smoothly bounded, $\mathbb{C}$-linearly convex domain in $\mathbb{C}^2$ and $\rho(z)$ is a defining function of Ω.

Now we set

$$C(\zeta, z) = \frac{1}{2\pi i}\left[\sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta)\, d\zeta_j\right] \frac{1}{\Phi(\zeta, z)},\ \zeta \in b\Omega,\ z \in \Omega,$$

which is an $(1, 0)$-form of the ζ variable. The Cauchy-Leray kernel for the $\mathbb{C}$-linearly convex domain Ω is defined as

$$\Omega_0(C(\zeta, z)) = C(\zeta, z) \wedge (\overline{\partial}_\zeta C(\zeta, z)) = \sum_{j_0 \in \{1,2\}} \frac{A_{j_0}(\zeta)}{\Phi^2(\zeta, z)}\, d\zeta_1 \wedge\ d\zeta_2 \wedge\ d\overline{\zeta}_{j_0}.$$

This is a Cauchy-Fantappiè $(2, 1)$-form on $b\Omega \times \Omega$, where $A_{j_0}(\zeta)$ is a polynomial involving first and second derivatives in ζ of ρ.

To be shorter, for each $\zeta \in b\Omega$, we set

$$K_\zeta(z) = \sum_{j_0 \in \{1,2\}} \frac{A_{j_0}(\zeta)}{\Phi^2(\zeta, z)}.$$

Since $\Phi(\zeta, z)$ is holomorphic in z, $K_\zeta \in \mathcal{O}(\Omega)$ for every fixed $\zeta \in b\Omega$, so $(K_\zeta(z))^{\frac{1}{p}}$ is holomorphic on the domain Ω.

Example 2.3. Let us consider the following finite type complex ellipsoid

$$\Omega^m = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = |z_1|^{2m_1} + |z_2|^{2m_2} - 1 < 0\},$$

where $m = (m_1, m_2) \in (\mathbb{Z}^+)^2$. Then the support function of Ω^m is

$$\Phi(\zeta, z) = m_1|\zeta_1|^{2(m_1-1)}\overline{\zeta}_1(\zeta_1 - z_1) + m_2|\zeta_2|^{2(m_2-1)}\overline{\zeta}_2(\zeta_2 - z_2).$$

From this, it follows that the Cauchy-Leray kernel for Ω^m is

$$\begin{aligned}\Omega_0(C(\zeta, z)) &= \frac{1}{(m_1|\zeta_1|^{2(m_1-1)}\overline{\zeta}_1(\zeta_1 - z_1) + m_2|\zeta_2|^{2(m_2-1)}\overline{\zeta}_2(\zeta_2 - z_2))^2} \\ &\times \Big[m_1 m_2^2|\zeta_1|^{2(m_1-1)}\overline{\zeta}_1|\zeta_2|^{2(m_2-1)}\, d\zeta_1 \wedge\, d\zeta_2 \wedge\, d\overline{\zeta}_2 \\ &- \, m_2 m_1^2|\zeta_2|^{2(m_2-1)}\overline{\zeta}_2|\zeta_1|^{2(m_1-1)}\, d\zeta_1 \wedge\, d\zeta_2 \wedge\, d\overline{\zeta}_1\Big],\end{aligned}$$

and hence

$$K_\zeta(z) = \frac{m_1 m_2|\zeta_1|^{2(m_1-1)}|\zeta_2|^{2(m_2-1)}[m_2\overline{\zeta}_1 - m_1\overline{\zeta}_2]}{(m_1|\zeta_1|^{2(m_1-1)}\overline{\zeta}_1(\zeta_1 - z_1) + m_2|\zeta_2|^{2(m_2-1)}\overline{\zeta}_2(\zeta_2 - z_2))^2}. \tag{2.2}$$

We refer the reader to [7] for more details about Cauchy-Leray kernels in such complex ellipsoids in $\mathbb{C}^n$.

The following theorem is a basic result in Cauchy-Fantappiè theory.

Theorem 2.4. [18, Theorem 3.4, p. 171] *For any $u \in \mathcal{O}(\Omega) \cap C(b\Omega)$, we have*

$$u(z) = \int_{b\Omega} u(\zeta)\Omega_0(C(\zeta, z)),\ z \in \Omega.$$

Next we generalize the notion of geometric type on $b\Omega$ in the sense of Range. This was first introduced in [6] to study the tangential Cauchy-Riemann equations in $\mathbb{C}^2$.

Definition 2.5. ([6]) The function $F\colon [0, \infty) \to [0, \infty)$ is called a *type* if the following conditions are satisfied:

(1) F is smooth and increasing;
(2) $F(0) = 0$;
(3) $\displaystyle\int_0^\delta |\ln F(r^2)|\, dr < \infty$, for some small $\delta > 0$;
(4) $\dfrac{F(r)}{r}$ is increasing.

In what follows, the function F with the above properties is supposed to be given.

Definition 2.6. A (bounded, smooth $\mathbb{C}$-linearly convex) domain Ω in $\mathbb{C}^2$ is said to be *admitting an F-type* at the boundary point $P \in b\Omega$ if there are positive constants c, c' such that for all $\zeta \in b\Omega \cap B(P, c')$, we have

$$\rho(z) \gtrsim F(|z - \zeta|^2), \quad \text{for all } z \in B(\zeta, c) \text{ with } \Phi(\zeta, z) = 0.$$

Here the notation $B(w, r)$ means the Euclidean ball centered at w of radius $r > 0$.

We note that in case $F(t) = t^m$ ($m \in \mathbb{N}$), the F-type notion agrees with the finite type condition in the sense of Range in [16, 17].

Example 2.7. (a) ([18, p. 195]) Let $\Omega \subset \mathbb{C}^2$ be a bounded strictly convex domain with its smooth, strictly plurisubharmonic defining function ρ. For every $P \in b\Omega$, there exist positive constants c', c and C such that for all $\zeta \in \overline{\Omega} \cap B(P, c')$ we have

$$-\operatorname{Re}\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + C|\zeta - z|^2,$$

where $|\zeta - z| < c$.

Hence, when $\zeta \in b\Omega \cap B(P, c')$, $z \in \{|\zeta - z| < c\}$ and $\Phi(\zeta, z) = 0$, we have

$$\rho(z) \gtrsim F(|z - \zeta|^2),$$

with $F(t) = t$. So Ω is of F-type.

(b) ([24, Theorem 3.1]) Let us consider the complex ellipsoid

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) = |z_1|^{2m_1} + |z_2|^{2m_2} - 1 < 0\} \quad (m_1, m_2 \in \mathbb{N}).$$

For every $P \in b\Omega$, there exist positive constants c', c and C such that for all $\zeta \in \overline{\Omega} \cap B(P, c')$ we have

$$-\operatorname{Re}\Phi(\zeta, z) \geq -\rho(z) + \rho(\zeta) + C|\zeta - z|^{2m},$$

where $|\zeta - z| < c$, and $m = \max\{m_1, m_2\}$. Then Ω is a convex domain admitting an F-type, with $F(t) = t^m$.

(c) ([15, Proposition 1]) Let $\Omega \subset \mathbb{C}^2$ be a bounded convex domain with real analytic boundary, i.e., ρ is a real analytic function. For every $P \in b\Omega$, there exist positive constants c', c, C and a positive integer m_P such that for all $\zeta \in \overline{\Omega} \cap B(P, c')$ we have

$$-\operatorname{Re}\Phi(\zeta, z) \geq -\rho(z) + \rho(\zeta) + C|\zeta - z|^{2m_P},$$

for $|\zeta - z| < c$. Let $m = \sup_{P \in \Omega} m_P$, then Ω is a domain admitting an F-type, with $F(t) = t^m$.

(d) ([27, Lemma 3]) Let

$$\Omega^\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) = e^{1+\frac{2}{s}} \cdot e^{\frac{-1}{|z_1|^s}} + |z_2|^2 - 1 < 0\right\}.$$

Then there exists a constant $c > 0$ such that for all $\zeta, z \in \overline{\Omega}$ with $|\zeta - z| < c$

$$\operatorname{Re}\Phi(\zeta, z) \lesssim -\rho(\zeta) + \rho(z) - e^{1+\frac{2}{s}} \cdot e^{\frac{-1}{32|\zeta - z|^{2s}}},$$

for $0 < s < 1/2$. Hence Ω^∞ is a convex domain admitting an F-type, with $F(t) = e^{-1/32t^s}$.

The F-type condition implies the following important result.

Lemma 2.8. [6, Lemma 3.3, p. 112] *Let Ω be a smoothly bounded $\mathbb{C}$-linearly convex domain in $\mathbb{C}^2$ admitting an F-type at $P \in b\Omega$. Then there is a positive constant c such that the support function $\Phi(\zeta, z)$ satisfies the estimate*

$$|\Phi(\zeta, z)| \gtrsim |\rho(z)| + |\operatorname{Im}\Phi(\zeta, z)| + F(|z-\zeta|^2), \tag{2.3}$$

for every $\zeta \in b\Omega \cap B(P, c)$, and $z \in \overline{\Omega}$, $|z-\zeta| < c$.

By passing to the radial limits almost everywhere on $b\Omega$ (see, e.g., [8, Proposition 8.5.1]), we can extend φ to the closure $\overline{\Omega}$. We denote by dv_φ be the pullback measure from $b\Omega$ to $\overline{\Omega}$, that is

$$v_\varphi(D) = \sigma(\varphi^{-1}(D) \cap b\Omega) = \sigma\{w \in b\Omega : \varphi(w) \in D\},$$

for all open subsets $D \subset \overline{\Omega}$ such that $\varphi^{-1}(D) \cap b\Omega \neq \emptyset$.

For the reader's convenience, we separate the statement of the Main Theorem into two theorems and prove them respectively in Section 3 and Section 4.

The first result provides a characterization of boundedness for $C_\varphi \colon H^p(\Omega) \to H^q(\Omega)$.

Theorem 2.9. *Let $0 < p \leq q < \infty$. Suppose that Ω admits an F-type at all boundary points, $\varphi \colon \Omega \to \Omega$ is a holomorphic self-map. Then a composition operator C_φ is bounded from $H^p(\Omega)$ to $H^q(\Omega)$ if and only if*

$$\sup_{\zeta \in b\Omega} \left\{ \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |K_\zeta(\varphi(w))|^{\frac{q}{p}} \, d\sigma_\varepsilon(w) \right) \right\} < \infty. \tag{2.4}$$

Next we recall that a linear operator $T \colon H^p(\Omega) \to H^q(\Omega)$ is called compact if the image under T of any bounded subset of $H^p(\Omega)$ is a relatively compact subset (has compact closure) in $H^q(\Omega)$. Such an operator is necessarily bounded and so it is continuous. We also denote by $\mathcal{K}_{p,q}(\Omega)$ the set of all compact operators from $H^p(\Omega)$ to $H^q(\Omega)$.

The second result gives a characterization of compactness for $C_\varphi \colon H^p(\Omega) \to H^q(\Omega)$ in terms of the Cauchy-Leray kernel.

Theorem 2.10. *Let $0 < p \leq q < \infty$. Suppose that Ω admits an F-type at all boundary points, that $\varphi \colon \Omega \to \Omega$ is a holomorphic self-map and denote by C_φ the composition operator from $H^p(\Omega)$ to $H^q(\Omega)$. Then the following assertions hold.*

(1) If $C_\varphi \colon H^p(\Omega) \to H^q(\Omega)$ is compact, then

$$\begin{aligned} &\sup_{\zeta \in b\Omega} \left\{ \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |K_\zeta(\varphi(w))|^{\frac{q}{p}} \, d\sigma_\varepsilon(w) \right)^{\frac{1}{q}} \right\} \\ &= \sup_{\zeta \in b\Omega} \left\{ \inf_{T \in \mathcal{K}_{p,q}(\Omega)} \left[\sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |T(K_\zeta^{\frac{1}{p}}(w))|^q \, d\sigma_\varepsilon(w) \right)^{\frac{1}{q}} \right] \right\}. \end{aligned}$$

(2) If $\sup_{\zeta \in b\Omega} \left\{ \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |K_\zeta(\varphi(w))|^{\frac{q}{p}} \, d\sigma_\varepsilon(w) \right)^{\frac{1}{q}} \right\} = 0$, then $C_\varphi \colon H^p(\Omega) \to H^q(\Omega)$ is compact.

Finally, we note that in the proofs of both Theorem 2.9 and Theorem 2.10, ones of the important steps are using the Minkowski's Integral Inequality and the Henkin's coordinates. Therefore, we recall them here for the reader's convenience and for a complete exposition as well.

Lemma 2.11 (Minkowski's Integral Inequality)**.** [26, Appendix A1] *Suppose that* (S_1, μ_1) *and* (S_2, μ_2) *are two* σ*-finite measure spaces and* $F\colon S_1 \times S_2 \to \mathbb{R}$ *is measurable. Then for* $1 \le s \le \infty$*, we have*

$$\left[\int_{S_2} \left|\int_{S_1} F(\zeta, z)\, d\mu_1(\zeta)\right|^s d\mu_2(z)\right]^{\frac{1}{s}} \le \int_{S_1} \left(\int_{S_2} |F(\zeta, z)|^s\, d\mu_2(z)\right)^{\frac{1}{s}} d\mu_1(\zeta).$$

Lemma 2.12. [18, Lemma V.3.4] *There exist positive constants* M, a *and* $\eta \le \varepsilon$*, and for each* z *with* $dist(z, b\Omega) \le a$*, there is a smooth local coordinate system* $(x_1, x_2, x_3, x_4) = x = x(\zeta, z)$ *on the ball* $B(z, \eta)$ *such that*

$$\begin{cases} x_1(\zeta, z) = \rho(\zeta), \\ x(z, z) = (\rho(z), 0, 0, 0), \\ x_2(\zeta, z) = \operatorname{Im}(\Phi(\zeta, z)), \\ |x| < 1 \quad \text{for } \zeta \in B(z, \eta), \\ \|J_{\mathbb{R}}(x(., z))\| \le M \quad \text{and} \quad |\det J_{\mathbb{R}}(x(., z))| \ge \frac{1}{M}, \end{cases}$$

where $J_{\mathbb{R}}(x(., z))$ *is the real Jacobian of* $x(., z)$*.*

We also use the following general criterion of compactness for $C_\varphi\colon H^p(\Omega) \to H^q(\Omega)$ in the proof of Theorem 2.10.

Proposition 2.13. [10, Proposition 3.2] *Let* $0 < p < \infty$*,* Ω *be a bounded domain in* $\mathbb{C}^n$*, and* $\varphi : \Omega \to \Omega$ *a holomorphic self-map. Then* $C_\varphi\colon H^p(\Omega) \to H^q(\Omega)$ *is compact if and only if for each bounded sequence* $\{u_n\}$ *in* $H^p(\Omega)$ *that converges to* 0 *on compacta of* Ω*, the sequence* $C_\varphi(u_n)$ *converges to* 0 *in* $H^q(\Omega)$*-norm.*

3. Proof of Theorem 2.9

The proof of Theorem 2.9, which states that the operator C_φ is bounded from $H^p(\Omega)$ to $H^q(\Omega)$ if and only if

$$\sup_{\zeta \in b\Omega} \left\{ \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |K_\zeta(\varphi(w))|^{\frac{q}{p}}\, d\sigma_\varepsilon(w) \right) \right\} < \infty,$$

is presented in terms of $(H^p(\Omega), q)$-Carleson measures on $\overline{\Omega}$.

For any $0 < p \le q < \infty$, a Borel measure μ on $\overline{\Omega}$ is called an $(H^p(\Omega), q)$-Carleson measure if there is a positive constant C, such that for any $u \in H^p(\Omega)$,

$$\int_{\overline{\Omega}} |u(w)|^q\, d\mu(w) \le C \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |u(z)|^p\, d\sigma_\varepsilon(z) \right)^{\frac{q}{p}},$$

or in short

$$\|u\|_{L^q(\overline{\Omega},\mu)} \lesssim \|u\|_{H^p(\Omega)}.$$

Let us denote the $(H^p(\Omega), q)$-norm of μ by

$$\|\mu\|_{(H^p(\Omega),q)} = \sup_{\substack{u\in H^p(\Omega)\\ \|u\|_{H^p(\Omega)}\leq 1}} \left(\int_{\overline{\Omega}} |u(z)|^q \, d\mu(z) \right)^{\frac{1}{q}}.$$

Proposition 3.1. *Let $0 < p \leq q < \infty$. Suppose that the domain Ω satisfies the same assumptions as in Theorem 2.10, then a Borel measure μ on $\overline{\Omega}$ is $(H^p(\Omega), q)$-Carleson if and only if*

$$\sup_{\zeta\in b\Omega} \left\{ \int_{\overline{\Omega}} |K_\zeta(z)|^{\frac{q}{p}} \; d\mu(z) \right\} < \infty. \tag{3.1}$$

Proof. Let c be one of the positive constants provided by Lemma 2.8. For every fixed $\zeta \in b\Omega$, $(K_\zeta(z))^{\frac{1}{p}}$ belongs to $H^p(\Omega)$. Then we have

$$\|(K_\zeta)^{\frac{1}{p}}\|^p_{H^p(\Omega)} = \sup_{0<\varepsilon<\varepsilon_0} \int_{b\Omega_\varepsilon} \left| \sum_{j_0\in\{1,2\}} \frac{A_{j_0}(\zeta)}{\Phi^2(\zeta, z)} \right| d\sigma_\varepsilon(z) \lesssim \sup_{0<\varepsilon<\min\{c,\varepsilon_0\}} \int_{b\Omega_\varepsilon} \frac{d\sigma_\varepsilon(z)}{|\Phi(\zeta, z)|^2}. \tag{3.2}$$

To estimate the last integral in (3.2), we use the Henkin's coordinates in Lemma 2.12,

$$\begin{aligned} \int_{b\Omega_\varepsilon} \frac{d\sigma_\varepsilon(z)}{|\Phi(\zeta, z)|^2} &\lesssim \int_{|(x_2,x_3,x_4)|<c} \frac{dx_2\; dx_3\; dx_4}{[|\rho(z)| + x_2 + F(|(x_3, x_4)|^2)]|(x_2, x_3, x_4)|} \\ &\lesssim \int_{|(x_2,x_3,x_4)|<c} \frac{dx_2\; dx_3\; dx_4}{[x_2 + F(|(x_3, x_4)|^2)]|(x_2, x_3, x_4)|} \\ &\lesssim \int_{|(x_2,x_3,x_4)|<c} \frac{dx_2\; dx_3\; dx_4}{[x_2 + F(|(x_3, x_4)|^2)]|(x_3, x_4)|}. \end{aligned}$$

Then using $\mathbb{R}^2$-polar coordinates (r, θ) for (x_3, x_4) and integrating in x_2, the last integral is estimated as

$$\begin{aligned} \int_{|(x_2,x_3,x_4)|<c} \frac{dx_2\; dx_3\; dx_4}{[x_2 + F(|(x_3, x_4)|^2)]|(x_3, x_4)|} &\lesssim \int_{|(x_2,r,\theta)|<c} \frac{r\; dr\; dx_2}{[x_2 + F(r^2)]r} \\ &\lesssim \int_0^c |\ln F(r^2)| \; dr < \infty \end{aligned}$$

uniformly in $\zeta \in b\Omega$ and $0 < \varepsilon < \varepsilon_0$ by the third condition of F.

- If μ is an $(H^p(\Omega), q)$-Carleson measure, then we have

$$\begin{aligned} \int_{\overline{\Omega}} |K_\zeta(z)|^{\frac{q}{p}} \; d\mu(z) &= \int_{\overline{\Omega}} \left(|K_\zeta(z)|^{\frac{1}{p}} \right)^q \; d\mu(z) \\ &= \|(K_\zeta)^{\frac{1}{p}}\|^q_{L^q(\overline{\Omega},\mu)} \leq C\|(K_\zeta)^{\frac{1}{p}}\|^q_{H^p(\Omega)} < \infty \end{aligned}$$

uniformly in $\zeta \in b\Omega$. Taking supremum over $\zeta \in b\Omega$ yields the desired estimate.

- Conversely, let μ be a Borel measure on $\overline{\Omega}$ such that

$$\sup_{\zeta \in b\Omega} \left\{ \int_{\overline{\Omega}} |K_\zeta(z)|^{\frac{q}{p}} \ d\mu(z) \right\} < \infty.$$

We show that

$$\int_{\overline{\Omega}} |u(z)|^q \ d\mu(z) \le C \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |u(z)|^p \ d\sigma_\varepsilon(z) \right)^{\frac{q}{p}}$$

for all $u \in H^p(\Omega)$, $0 < p < \infty$.

Since u^p is holomorphic, the Cauchy-Leray Integral formula for the $\mathbb{C}$-linearly convex domain Ω gives

$$u^p(z) = \int_{b\Omega} u^p(\zeta)\Omega_0(C(\zeta, z)), \quad z \in \Omega.$$

Hence, for $z \in \Omega$, we have

$$|u(z)|^p \lesssim \int_{b\Omega} |u(\zeta)|^p \left| \sum_{j_0 \in \{1,2\}} \frac{A_{j_0}(\zeta)}{\Phi^2(\zeta, z)} \right| \ d\sigma(\zeta) \lesssim \int_{b\Omega} |u(\zeta)|^p |K_\zeta(z)| \ d\sigma(\zeta),$$

which implies

$$|u(z)| \le \left(C \int_{b\Omega} |u(\zeta)|^p |K_\zeta(z)| \ d\sigma(\zeta) \right)^{\frac{1}{p}}, \tag{3.3}$$

for some positive constant C.

To estimate the last integral in (3.3), we use the Minkowski's Integral Inequality, Lemma 2.11. More precisely, applying this inequality for two Borel measure spaces $(b\Omega_\varepsilon, d\sigma)$ and $(\Omega_\varepsilon, d\mu)$, we have

$$\begin{aligned} &\int_{\overline{\Omega}} |u(z)|^q \ d\mu(z) \\ &\le \int_{\overline{\Omega}} \left(\left[C \int_{b\Omega} |u(\zeta)|^p |K_\zeta(z)| \ d\sigma(\zeta) \right]^{\frac{q}{p}} \right) \ d\mu(z) \\ &\le C \left(\int_{b\Omega} \left[\int_{\overline{\Omega}} |K_\zeta(z)|^{\frac{q}{p}} \ d\mu(z) \right]^{\frac{p}{q}} |u(\zeta)|^p \ d\sigma(\zeta) \right)^{\frac{q}{p}} \\ &\le C \sup_{\zeta \in b\Omega} \left\{ \int_{\overline{\Omega}} |K_\zeta(z)|^{\frac{q}{p}} \ d\mu(z) \right\} \cdot \|u\|^q_{H^p(\Omega)} \le C \|u\|^q_{H^p(\Omega)}. \end{aligned} \tag{3.4}$$

The last estimate in (3.4) shows that μ is an $(H^p(\Omega), q)$-Carleson measure.

The proof of Proposition 3.1 is complete. □

Now let $0 < p \leq q < \infty$ and $\varphi\colon \Omega \to \Omega$ be a holomorphic self-map. Suppose that $C_\varphi\colon H^p(\Omega) \to H^q(\Omega)$ is bounded. This is equivalent to

$$\sup_{0<\varepsilon<\varepsilon_0} \int_{b\Omega_\varepsilon} |u(\varphi(w))|^q \, d\sigma_\varepsilon(w) \leq C\|u\|^q_{H^p(\Omega)},$$

for all $u \in H^p(\Omega)$.

Let us use the pullback measure dv_φ from $b\Omega$ to $\overline{\Omega}$,

$$v_\varphi(D) = \sigma(\varphi^{-1}(D) \cap b\Omega)$$

for all open subsets D. By a change of variables $z \mapsto \varphi(w)$, we have

$$\begin{aligned} \sup_{0<\varepsilon<\varepsilon_0} \int_{b\Omega_\varepsilon} |u(\varphi(w))|^q \, d\sigma_\varepsilon(w) &= \limsup_{\varepsilon\to 0^+} \int_{b\Omega_\varepsilon} |u(\varphi(w))|^q \, d\sigma_\varepsilon(w) \\ &= \int_{b\Omega} |u(\varphi(w))|^q \, d\sigma(w) = \int_{\Omega} |u(z)|^q \, dv_\varphi(z). \end{aligned}$$

Therefore, $C_\varphi\colon H^p(\Omega) \to H^q(\Omega)$ is bounded if and only if

$$\int_{\Omega} |u(z)|^q \, dv_\varphi(z) \leq C\|u\|^q_{H^p(\Omega)}. \tag{3.5}$$

Clearly, the inequality (3.5) holds for all $u \in H^p(\Omega)$ if and only if the measure v_φ is an $(H^p(\Omega), q)$-Carleson measure.

Applying Proposition 3.1, the boundedness holds if and only if

$$\sup_{\zeta\in b\Omega} \left[\int_{\overline{\Omega}} |K_\zeta(z)|^{\frac{q}{p}} \, dv_\varphi(z) \right\} < \infty,$$

or equivalently,

$$\sup_{\zeta\in b\Omega} \left\{ \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{b\Omega_\varepsilon} |K_\zeta(\varphi(w))|^{\frac{q}{p}} \, d\sigma_\varepsilon(w) \right) \right\} < \infty.$$

This completes the proof.

4. Proof of Theorem 2.10

Let $T\colon H^p(\Omega) \to H^q(\Omega)$ be a bounded operator. Recall $\mathcal{K}_{p,q}(\Omega)$ is the set of all compact operators from $H^p(\Omega)$ to $H^q(\Omega)$.

1) Suppose $C_\varphi\colon H^p(\Omega) \to H^q(\Omega)$ is compact. Then the essential norm

$$\|C_\varphi\|_e = \inf_{T\in\mathcal{K}_{p,q}(\Omega)} \|C_\varphi - T\|_{(H^p(\Omega)\to H^q(\Omega))} = 0.$$

We are going to verify that

$$\sup_{\zeta\in b\Omega}\left\{\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\right\}$$
$$=\sup_{\zeta\in b\Omega}\left\{\inf_{T\in\mathcal{K}_{p,q}(\Omega)}\left[\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|T(K_\zeta^{\frac{1}{p}}(w))|^{q}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\right]\right\}.$$

Since $K_\zeta^{\frac{1}{p}}(z)$ is bounded in $H^p(\Omega)$ (this fact has been proved in Section 3) and $C_\varphi\colon H^p(\Omega)\to H^q(\Omega)$ is bounded, for all $T\in\mathcal{K}_{p,q}(\Omega)$, we have

$$\|C_\varphi-T\|_{(H^p(\Omega)\to H^q(\Omega))}\gtrsim\|(C_\varphi-T)K_\zeta^{\frac{1}{p}}\|_{H^q(\Omega)}\gtrsim\|C_\varphi K_\zeta^{\frac{1}{p}}\|_{H^q(\Omega)}-\|TK_\zeta^{\frac{1}{p}}\|_{H^q(\Omega)}.$$

Taking the infimum over $T\in\mathcal{K}_{p,q}(\Omega)$, we have

$$\begin{aligned}\inf_{T\in\mathcal{K}_{p,q}(\Omega)}\left[\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|T(K_\zeta^{\frac{1}{p}}(w))|^{q}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\right] &= \inf_{T\in\mathcal{K}_{p,q}(\Omega)}\|TK_\zeta^{\frac{1}{p}}\|_{H^q(\Omega)}\\ &= \|C_\varphi K_\zeta^{\frac{1}{p}}\|_{H^q(\Omega)}\\ &= \sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}\end{aligned}$$

for all $\zeta\in b\Omega$.

2) Now suppose that

$$\sup_{0<\varepsilon<\varepsilon_0}\left(\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right)^{\frac{1}{q}}=0$$

for all $\zeta\in b\Omega$. We show that $C_\varphi\colon H^p(\Omega)\to H^q(\Omega)$ is compact.

For every $\delta>0$, we can find an $\varepsilon\in(0,\varepsilon_0)$ so that

$$\delta>\sup_{\zeta\in b\Omega}\left[\int_{b\Omega_\varepsilon}|K_\zeta(\varphi(w))|^{\frac{q}{p}}\,d\sigma_\varepsilon(w)\right]\gtrsim\sup_{\zeta\in b\Omega}\left[\int_{\Omega}|K_\zeta(z)|^{\frac{q}{p}}\,dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon}(z)\right].$$

So, by Proposition 2.13, take an arbitrary bounded sequence $\{u_n\}$ in $H^p(\Omega)$ that converges to 0 uniformly on compact subsets of Ω. Then by a boundedness of C_φ, with the above δ, there exists a positive integer N such that

$$\int_{\overline{\Omega}_\varepsilon}|u_n(z)|^q\,dv_\varphi(z)<\delta$$

for all $n>N$.

On the other hand, let $\chi_{\Omega\setminus\overline{\Omega}_\varepsilon}$ be the characteristic function of $\Omega\setminus\overline{\Omega}_\varepsilon$. By the Minkowski's Integral Inequality for $(\Omega, dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon})$ and $(b\Omega, d\sigma)$, we have

$$\begin{aligned}
\int_{\Omega\setminus\overline{\Omega}_\varepsilon} |u_n(z)|^q \, dv_\varphi(z) &= \int_\Omega |u_n(z)|^q \, dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon}(z) \\
&\lesssim \int_\Omega \left[\int_{b\Omega} |u_n(\zeta)|^p |K_\zeta(z)| \, d\sigma(\zeta)\right]^{\frac{q}{p}} \, dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon}(z) \\
&\lesssim \left(\int_{b\Omega} \left(\int_\Omega |K_\zeta(z)|^{\frac{q}{p}} \, dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon}(z)\right)^{\frac{p}{q}} |u_n(\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{q}{p}} \\
&\lesssim \sup_{\zeta\in b\Omega} \left(\int_\Omega |K_\zeta(z)|^{\frac{q}{p}} \, dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon}(z)\right) \cdot \left(\int_{b\Omega} |u_n(\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{q}{p}} \\
&\lesssim \sup_{\zeta\in b\Omega} \left(\int_\Omega |K_\zeta(z)|^{\frac{q}{p}} \, dv_\varphi|_{\Omega\setminus\overline{\Omega}_\varepsilon}(z)\right) \cdot \sup_{n\in\mathbb{N}}\{\|u_n\|^q_{H^p(\Omega)}\} \\
&\lesssim \delta \sup_{n\in\mathbb{N}}\{\|u_n\|^q_{H^p(\Omega)}\}.
\end{aligned}$$

Hence, for all $n > N$, we get

$$\begin{aligned}
\|C_\varphi(u_n)\|^q_{H^q(\Omega)} &= \sup_{0<\varepsilon'<\varepsilon_0} \int_{b\Omega_{\varepsilon'}} |u_n(\varphi(w))|^q \, d\sigma_{\varepsilon'}(w) \\
&= \limsup_{\varepsilon'\to 0} \int_{b\Omega_{\varepsilon'}} |u_n(\varphi(w))|^q \, d\sigma_{\varepsilon'}(w) \\
&= \int_\Omega |u_n(z)|^q \, dv_\varphi(z) \\
&= \int_{\overline{\Omega}_\varepsilon} |u_n(z)|^q \, dv_\varphi(z) + \int_{\Omega\setminus\overline{\Omega}_\varepsilon} |u_n(z)|^q \, dv_\varphi(z) \\
&\lesssim \delta + \delta \sup_{n\in\mathbb{N}}\{\|u_n\|^q_{H^p(\Omega)}\}.
\end{aligned}$$

The arbitrariness of δ, together with the last inequality and Proposition 2.13 yields the desired result.

Theorem 2.10 is proved completely.

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