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Exact Convergence of Gradient-Free Distributed Optimization Method in a Multi-Agent System*

Yipeng Pang and Guoqiang Hu

***Abstract*— In this paper, a gradient-free algorithm is proposed for a set constrained distributed optimization problem in a multi-agent system under a directed communication network. For each agent, a pseudo-gradient is designed locally and utilized instead of the true gradient information to guide the decision variables update. Compared with most gradient-free optimization methods where a doubly-stochastic weighting matrix is usually employed, this algorithm uses a row-stochastic matrix plus a column-stochastic matrix, and is able to achieve exact asymptotic convergence to the optimal solution.**

***Index Terms*— Distributed optimization, multi-agent system, gradient-free optimization**

I. Introduction

Gradient-free optimization schemes can be traced back to the age of developing optimization theory, such as the work in [1]. Recent researches on this topic have been reported in [2]–[8], where centralized gradient-free methods have been investigated in [3]. It was extended to a distributed version in [4], [5] and further improved in [6], [7]. The idea of the method is computing some stochastic gradient information based on the measurements of the function plus some random variables, to replace the true gradient in the standard distributed optimization algorithm. In [8], this idea was extended to a state-of-the-art gradient-based algorithm reported in [9], which relaxed the requirement of doubly stochastic adjacency matrix. However, all these derivative-free methods can only establish the weak convergence results to a neighborhood of the optimal solution with an error bounded by some parameters which cannot be eliminated. In [10], [11], a smoothing technique was developed to solve the non-smooth optimization problem, where two point gradient estimation was used to close the optimality gap between the final iterate and the optimal point by choosing appropriate step-size. This technique was extended to distributed case in [12] in a directed communication graph, but restricted by the assumption of doubly-stochastic weighting matrix.

In this paper, motivated by the gradient-descent method in [13], we adopt the idea from [11] to construct a pseudo-gradient, and propose a directed-distributed projected pseudo-gradient descent (D-DPPGD) method to solve the set constrained distributed optimization problem without computing the true gradient. It is worth noting that our method achieves the same convergence result as [13] but with no gradient information requirements.

The major contributions of this paper are summarized as follows. 1). This paper proposes a distributed optimization algorithm which does not require any explicit expressions of the cost functions but only local measurements, making it suitable for those applications where finding the gradient is costly or not practical. 2). In contrast to the randomized gradient-free methods in [3]–[8] where inexact convergence to the neighborhood of the optimal solution was achieved, this algorithm establishes the exact convergence to the optimal solution. 3). Unlike the consensus-based approaches in [4]–[6], [14]–[20], this method does not require the adjacency matrix to be doubly-stochastic, which makes it possible to be implemented in any directed graphs, since finding a doubly-stochastic adjacency matrix for a directed graph is not guaranteed [21], [22].

The rest of the paper is organized as follows. In Section II, the notations and problem formulation are firstly introduced, followed by the main results in Section III, where the proposed algorithm is elaborated first, then the convergence properties are carefully analyzed in details. The conclusion is in Section IV.

II. Notations and Problem Formulation

Throughout the paper, we use $\mathbb{R}$ and $\mathbb{R}^n$ to denote the set of real numbers and n-dimensional column vectors, respectively, and $\mathbf{1}_n$ $(\mathbf{0}_n)$ to represent an n-dimensional vector with all elements equal to one (zero). For a matrix A, we denote the element in the i-th row and j-th column of A by $[A]_{ij}$, its transpose by A^T, and the induced vector Euclidean norm by $\|A\|$. For a vector $\mathbf{x}$, $\|\mathbf{x}\|$ denotes the standard Euclidean norm. For a function f, we use $\nabla f(\mathbf{x})$ $(\partial f(\mathbf{x}))$ to represent its gradient (subgradient) at the point $\mathbf{x}$. We write $\mathbb{E}[x]$ and $Cov(x, y)$ to denote the expected value of x and covariance value of x and y, respectively. $\mathcal{P}_{\mathcal{X}}[\mathbf{x}]$ represents the projection of a vector $\mathbf{x}$ on the set $\mathcal{X}$, *i.e.,* $\mathcal{P}_{\mathcal{X}}[\mathbf{x}] = \arg\min_{\hat{\mathbf{x}}\in\mathcal{X}} \|\hat{\mathbf{x}} - \mathbf{x}\|^2$.

For a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of agents, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of ordered pairs, (i, j), $i, j \in \mathcal{V}$, where agent i is able to send information to agent j. We denote the set of agent i's in-neighbors by $\mathcal{N}_i^{\text{in}} = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ and out-neighbors by $\mathcal{N}_i^{\text{out}} = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. Specifically, we allow both $\mathcal{N}_i^{\text{in}}$ and $\mathcal{N}_i^{\text{out}}$ to contain agent i itself, and $\mathcal{N}_i^{\text{in}} \neq \mathcal{N}_i^{\text{out}}$ in general. The objective of the multi-agent system is to cooperatively solve the following set constrained optimization problem:

$$\min f(\mathbf{x}) = \textstyle\sum_{i=1}^{N} f_i(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X}, \tag{1}$$

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where $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex and closed set, and f_i is a local cost function of agent i and $\mathbf{x} \in \mathbb{R}^n$ is a global decision vector. The explicit expression of the local cost function f_i is unknown, but the measurements can be made by agent i only. The optimal solution of (1) is denoted by $\mathbf{x}^\star$ with optimal value $f^\star = f(\mathbf{x}^\star)$.

We introduce a smoothed version of (1), given by

$$\min f_{\beta_{1,k}}(\mathbf{x}) = \sum_{i=1}^{N} f_{i,\beta_{1,k}}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X},$$

where $f_{i,\beta_{1,k}}(\mathbf{x})$ is a smoothed function of $f_i(\mathbf{x})$ [10], [11]

$$f_{i,\beta_{1,k}}(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}+\beta_{1,k}\xi)] = \int_{\mathbb{R}^n} f_i(\mathbf{x}+\beta_{1,k}\xi) d\mu^i(\xi),$$

with the random variable $\xi \in \mathbb{R}^n$ having density μ^i with respect to Lebesgue measure[1]. $\beta_{1,k}$ is a positive non-increasing sequence. The properties of the function $f_{i,\beta_{1,k}}(\mathbf{x})$ are presented in Lemma 3.

In this paper, we make the following assumptions:

Assumption 1: The directed graph is strongly connected.

Assumption 2: Each local cost function f_i is convex, but not necessarily differentiable. Its subgradient $\partial f_i(\mathbf{x})$ is bounded, *i.e.,* $\forall \mathbf{x} \in \mathcal{X}$, there exists a positive constant $\hat{D}$ such that $\|\partial f_i(\mathbf{x})\| \leq \hat{D}$.

III. Main Results

In this section, we will develop the projected pseudo-gradient descent method for the optimization problem defined in (1), followed by the convergence analysis.

A. D-DPPGD Method

The D-DPPGD method for solving the optimization problem defined in (1) is described as follows.

At the k-th step, each agent j delivers its state information $\mathbf{x}_k^j$ with a weighted auxiliary variable $[A_c]_{ij}\mathbf{y}_k^j$ to its out-neighbor $i \in \mathcal{N}_j^{\text{out}}$. Then, agent i updates its variables $\mathbf{x}_{k+1}^i$ and $\mathbf{y}_{k+1}^i$ with the information received from its in-neighbor $j \in \mathcal{N}_i^{\text{in}}$ as follows

$$\mathbf{x}_{k+1}^i = \mathcal{P}_{\mathcal{X}}\Big[\sum_{j=1}^N [A_r]_{ij}\mathbf{x}_k^j + \epsilon \mathbf{y}_k^i - \alpha_k \mathbf{g}^i(\mathbf{x}_k^i)\Big], \tag{2a}$$

$$\mathbf{y}_{k+1}^i = \mathbf{x}_k^i - \sum_{j=1}^N [A_r]_{ij}\mathbf{x}_k^j + \sum_{j=1}^N [A_c]_{ij}\mathbf{y}_k^j - \epsilon \mathbf{y}_k^i, \tag{2b}$$

where $\mathbf{g}^i(\mathbf{x}_k^i)$ is a pseudo-gradient [11], given as

$$\mathbf{g}^i(\mathbf{x}_k^i) = \tfrac{1}{\beta_{2,k}}[f_i(\mathbf{x}_k^i + \beta_{1,k}\xi_{1,k}^i + \beta_{2,k}\xi_{2,k}^i) - f_i(\mathbf{x}_k^i + \beta_{1,k}\xi_{1,k}^i)]\xi_{2,k}^i, \tag{3}$$

A_r, A_c are the row-stochastic and column-stochastic adjacency matrices, respectively, *i.e.,* $\sum_{j=1}^N [A_r]_{ij} = 1$ for all $j \in \mathcal{V}$, and $\sum_{i=1}^N [A_c]_{ij} = 1$ for all $i \in \mathcal{V}$. For any directed graphs, they can be obtained by letting $[A_r]_{ij} = 1/|\mathcal{N}_i^{\text{in}}|$ and $[A_c]_{ij} = 1/|\mathcal{N}_j^{\text{out}}|$. $\alpha_k > 0$ is a diminishing step-size satisfying

$$\sum_{k=0}^\infty \alpha_k = \infty, \quad \sum_{k=0}^\infty \alpha_k^2 < \infty. \tag{4}$$

ϵ is a small positive number. $\beta_{1,k}$, $\beta_{2,k}$ are two positive non-increasing sequences with their ratio defined as

$$\tilde{\beta}_k = \beta_{2,k}/\beta_{1,k}. \tag{5}$$

[1]Here, we slightly abuse the notation of ξ for both a random variable and its instances.

$\xi_{1,k}^i$ and $\xi_{2,k}^i \in \mathbb{R}^n$ are random variables satisfying the following assumption:

Assumption 3: (Assumption F in [11]) The random variables $\xi_{1,k}^i$ and $\xi_{2,k}^i \in \mathbb{R}^n$ are generated by any one of the following: (a) both $\xi_{1,k}^i$ and $\xi_{2,k}^i$ are standard normal in $\mathbb{R}^n$ with identity covariance; (b) both $\xi_{1,k}^i$ and $\xi_{2,k}^i$ are uniform on the ℓ_2-ball of radius $\sqrt{n+2}$; (c) the distribution of $\xi_{1,k}^i$ is uniform on the ℓ_2-ball of radius $\sqrt{n+2}$ and the distribution of $\xi_{2,k}^i$ is uniform on the ℓ_2-ball of radius $\sqrt{n}$.

Remark 1: The proposed algorithm (2) is a gradient-free algorithm where a psuedo-gradient operator $\mathbf{g}^i(\mathbf{x}_k^i)$ is used instead of the true gradient $\nabla f_i(\mathbf{x}_k^i)$. The row-stochastic A_r and column-stochastic A_c instead of doubly-stochastic adjacency matrix make it possible to be implemented in any directed graphs.

For the convenience of analysis, we may write (2) in a compact form as

$$\mathbf{z}_{k+1}^i = \sum_{j=1}^{2N} [A]_{ij}\mathbf{z}_k^j + g_k^i, \tag{6}$$

where $\mathbf{z}_k^i = \mathbf{x}_k^i$ for $i \in \{1,\ldots,N\}$, $\mathbf{z}_k^i = \mathbf{y}_k^{i-N}$ for $i \in \{N+1,\ldots,2N\}$, $g_k^i = \mathbf{x}_{k+1}^i - \sum_{j=1}^N [A_r]_{ij}\mathbf{x}_k^j - \epsilon \mathbf{y}_k^i$ for $i \in \{1,\ldots,N\}$, $g_k^i = \mathbf{0}_n$ for $i \in \{N+1,\ldots,2N\}$, and $A = \begin{bmatrix} A_r & \epsilon I \\ I - A_r & A_c - \epsilon I \end{bmatrix}$. Define

$$\bar{\mathbf{z}}_k = \tfrac{1}{N}\sum_{i=1}^{2N} \mathbf{z}_k^i = \tfrac{1}{N}\sum_{i=1}^N \mathbf{x}_k^i + \tfrac{1}{N}\sum_{i=1}^N \mathbf{y}_k^i$$

as an average of $\mathbf{x}_k^i + \mathbf{y}_k^i$ over all agents at time-step k.

B. Convergence Analysis

In this part, we proceed to the analysis on the convergence properties of the proposed algorithm. We denote the σ-field generated by the entire history of the random variables from step 0 to $k-1$ by $\mathcal{F}_k$, *i.e.,* $\mathcal{F}_k = \{(\mathbf{x}_0^i, i \in \mathcal{V}); (\xi_{1,s}^i, \xi_{2,s}^i, i \in \mathcal{V}); 0 \leq s \leq k-1\}$ with $\mathcal{F}_0 = \{\mathbf{x}_0^i, i \in \mathcal{V}\}$. We first quantify the bound of the consensus terms $\mathbf{x}_k^i - \bar{\mathbf{z}}_k$ and $\mathbf{y}_k^i - \mathbf{0}_n$ by some terms in the following lemma:

Lemma 1: Suppose Assumptions 1, 2 and 3 hold. Let $\{\mathbf{z}_k^i\}_{k\geq 0}$ be the sequence generated by (6). Then, it holds that

1) for $i = \{1,\ldots,N\}$ and $k \geq 1$

$$\begin{aligned}\mathbb{E}[\|\mathbf{z}_k^i - \bar{\mathbf{z}}_k\| | \mathcal{F}_{k-1}] \leq\ & 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| \\ &+ \Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}] \\ &+ \sum_{j=1}^N \mathbb{E}[\|g_{k-1}^j\| | \mathcal{F}_{k-1}];\end{aligned}$$

2) for $i = \{N+1,\ldots,2N\}$ and $k \geq 1$

$$\begin{aligned}\mathbb{E}[\|\mathbf{z}_k^i\| | \mathcal{F}_{k-1}] \leq\ & 2N\Gamma\gamma^k \max_j \|\mathbf{z}_0^j\| \\ &+ \Gamma \sum_{r=1}^{k-1} \gamma^{k-r} \sum_{j=1}^N \mathbb{E}[\|g_{r-1}^j\| | \mathcal{F}_{r-1}],\end{aligned}$$

where $\Gamma > 0$ and $0 < \gamma < 1$ are some constants.

Proof: For $k \geq 1$, we have

$$\mathbf{z}_k^i = \sum_{j=1}^{2N} [A^k]_{ij}\mathbf{z}_0^j + \sum_{r=1}^{k-1}\sum_{j=1}^{2N} [A^{k-r}]_{ij} g_{r-1}^j + g_{k-1}^i. \tag{7}$$

by applying (6) recursively. Then we can obtain that

$$\bar{\mathbf{z}}_k = \tfrac{1}{N}\sum_{j=1}^{2N} \mathbf{z}_0^j + \tfrac{1}{N}\sum_{r=1}^{k-1}\sum_{j=1}^{2N} g_{r-1}^j + \tfrac{1}{N}\sum_{j=1}^{2N} g_{k-1}^j, \tag{8}$$

where we have used column-stochastic property of A, *i.e.,* for $k \geq 1$, it holds that $\sum_{i=1}^{2N} [A^k]_{ij} = 1$.

For part (1), subtracting (8) from (7) and taking the norm and conditional expectation on $\mathcal{F}_\ell$ from $\ell=0$ to $k-1$, we have that for $1\le i\le N$ and $k\ge 1$,

$$\begin{aligned}\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\||\mathcal{F}_{k-1}] \le& \textstyle\sum_{j=1}^{2N}\left\|[A^k]_{ij}-\frac{1}{N}\right\|\max_j\|\mathbf{z}_0^j\| \\ &+\textstyle\sum_{r=1}^{k-1}\sum_{j=1}^{N}\left\|[A^{k-r}]_{ij}-\frac{1}{N}\right\|\mathbb{E}[\|g_{r-1}^j\||\mathcal{F}_{r-1}] \\ &+\tfrac{N-1}{N}\mathbb{E}[\|g_{k-1}^i\||\mathcal{F}_{k-1}]+\tfrac{1}{N}\textstyle\sum_{j\neq i}\mathbb{E}[\|g_{k-1}^j\||\mathcal{F}_{k-1}].\end{aligned}\tag{9}$$

Noting that the last two terms

$$\begin{aligned}&\tfrac{N-1}{N}\mathbb{E}[\|g_{k-1}^i\||\mathcal{F}_{k-1}]+\tfrac{1}{N}\textstyle\sum_{j\neq i}\mathbb{E}[\|g_{k-1}^j\||\mathcal{F}_{k-1}]\\ \le&\tfrac{N-1}{N}\textstyle\sum_{i=1}^{N}\mathbb{E}[\|g_{k-1}^i\||\mathcal{F}_{k-1}]+\tfrac{1}{N}\textstyle\sum_{j=1}^{N}\mathbb{E}[\|g_{k-1}^j\||\mathcal{F}_{k-1}]\\ =&\textstyle\sum_{j=1}^{N}\mathbb{E}[\|g_{k-1}^j\||\mathcal{F}_{k-1}],\end{aligned}$$

and applying the property of $[A^k]_{ij}$ from Lemma 1-(b) in [13] to (9), we complete the proof of part (1).

For part (2), taking the norm and conditional expectation on $\mathcal{F}_\ell$ from $\ell=0$ to $k-1$ in (7) for $N+1\le i\le 2N$ and $k>1$, we have

$$\begin{aligned}\mathbb{E}[\|\mathbf{z}_k^i\||\mathcal{F}_{k-1}]\le&\textstyle\sum_{j=1}^{2N}\left\|[A^k]_{ij}\right\|\max_j\|\mathbf{z}_0^j\|\\ &+\textstyle\sum_{r=1}^{k-1}\sum_{j=1}^{N}\left\|[A^{k-r}]_{ij}\right\|\mathbb{E}[\|g_{r-1}^j\||\mathcal{F}_{k-1}].\end{aligned}\tag{10}$$

Applying Lemma 1-(b) in [13] to (10), the result holds with similar arguments to part (1). □

The following lemma gives a bound for the augmented pseudo-gradient operator g_k^i defined in (6), which will be used in the proof of convergence.

Lemma 2: Suppose Assumptions 1, 2 and 3 hold. Let ϵ be the constant such that $\epsilon\le\frac{1-\gamma}{2N\Gamma\gamma}$, where $\Gamma>0$ and $0<\gamma<1$ are some constants. Let $\tilde{\beta}_k$ defined in (5) be bounded. Then, there exists a bounded constant $G>0$, such that for all $k\ge 0$,

$$\textstyle\sum_{j=1}^{N}\mathbb{E}[\|g_k^j\||\mathcal{F}_k]\le G\alpha_k,$$

where α_k is the step-size used in the algorithm.

Proof: The proof follows similar flow to Lemma 5 in [13] and is omitted here due to the space limit. □

With the above lemmas, we are ready to establish the main results consisting of two theorems – one for consensus and the other for optimality. We first show the boundedness of $\limsup_{k\to\infty}\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\|]$ for $1\le i\le N$, followed by the boundedness of $\lim_{k\to\infty}\mathbb{E}[f(\bar{\mathbf{z}}_k)]-f^\star$ as $k\to\infty$.

Theorem 1: Suppose Assumptions 1, 2 and 3 hold. Let $\{\mathbf{z}_k^i\}_{k\ge 0}$ be the sequence generated by (6) with a diminishing step-size sequence $\{\alpha_k\}_{k\ge0}$ satisfying (4). Let $\tilde{\beta}_k$ defined in (5) be bounded. Then, $\mathbf{z}_k^i$ satisfies

1) For $i=\{1,\dots,N\}$

$$\lim_{k\to\infty}\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\|]=0.$$

2) For $i=\{N+1,\dots,2N\}$

$$\lim_{k\to\infty}\mathbb{E}[\|\mathbf{z}_k^i\|]=0.$$

Proof: For part (1), applying Lemma 2 to the result in Lemma 1-(1) and taking the total expectation, we have

$$\begin{aligned}\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\|]\le&\,2N\Gamma\gamma^k\max_j\mathbb{E}[\|\mathbf{z}_0^j\|]\\ &+G\Gamma\textstyle\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_{r-1}+G\alpha_{k-1}.\end{aligned}\tag{11}$$

Thus, it follows from (11) that for any $K>0$

$$\begin{aligned}\textstyle\sum_{k=1}^{K}\alpha_k\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\|]\le&\,2N\Gamma\max_j\mathbb{E}[\|\mathbf{z}_0^j\|]\textstyle\sum_{k=1}^{K}\alpha_k\gamma^k\\ &+G\Gamma\textstyle\sum_{k=1}^{K}\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_k\alpha_{r-1}+G\sum_{k=1}^{K}\alpha_k\alpha_{k-1}.\end{aligned}$$

Following the results from Lemma 3 in [8] on $\sum_{k=1}^{K}\alpha_k\gamma^k$, $\sum_{k=1}^{K}\alpha_k\alpha_{k-1}$ and $\sum_{k=1}^{K}\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_k\alpha_{r-1}$, the above inequality can be further simplified as

$$\begin{aligned}\textstyle\sum_{k=1}^{K}\alpha_k\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\|]\le&\,N\Gamma\max_j\mathbb{E}[\|\mathbf{z}_0^j\|](\textstyle\sum_{k=1}^{K}\alpha_k^2\\ &+\tfrac{\gamma^2}{1-\gamma^2})+G(\tfrac{\Gamma\gamma}{1-\gamma}+1)\textstyle\sum_{k=0}^{K}\alpha_k^2.\end{aligned}$$

Taking $K\to\infty$ and noting that $\sum_{k=0}^{\infty}\alpha_k^2<\infty$, we have $\sum_{k=1}^{\infty}\alpha_k\mathbb{E}[\|\mathbf{z}_k^i-\bar{\mathbf{z}}_k\|]<\infty$. Together with the fact that $\sum_{k=0}^{\infty}\alpha_k=\infty$, we complete the proof of part (1).

For part (2), applying Lemma 2 to the result in Lemma 1-(2) and taking the total expectation, we have

$$\mathbb{E}[\|\mathbf{z}_k^i\|]\le 2N\Gamma\gamma^k\max_j\mathbb{E}[\|\mathbf{z}_0^j\|]+G\Gamma\textstyle\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_{r-1}.\tag{12}$$

Following the same reasoning as in part (1), we can obtain the desired result. □

Remark 2: Theorem 1 characterizes the consensus property of the algorithm; namely, all agents $\mathbf{x}_k^i$ (respectively $\mathbf{y}_k^i$), $i\in\mathcal{V}$ will converge to the same point $\bar{\mathbf{z}}_k$ (respectively $\mathbf{0}_n$) asymptotically to achieve the exact convergence.

Theorem 2: Suppose Assumptions 1, 2 and 3 hold. Let $\{\mathbf{z}_k^i\}_{k\ge0}$ be the sequence generated by (6) with a diminishing step-size sequence $\{\alpha_k\}_{k\ge0}$ satisfying (4). Let $\beta_{1,k}$ and $\tilde{\beta}_k$ defined in (5) satisfy $\lim_{k\to\infty}\beta_{1,k}=0$ and $\sum_{k=0}^{\infty}\tilde{\beta}_k<\infty$. Then, we have

$$\lim_{k\to\infty}\mathbb{E}[f(\bar{\mathbf{z}}_k)]=f^\star.$$

Proof: See Appendix A. □

Remark 3: Theorem 2 shows that the cost value of the multi-agent system will finally converge to its exact optimal value with appropriate choice of the step-size α_k and parameters $\beta_{1,k},\beta_{2,k}$. For instance, if the step-size α_k is set to $1/(k+1)^a$, where $a\in(0,1)$; the parameters $\beta_{1,k},\beta_{2,k}$ are set to $1/(k+1)^{p_1}$ and $1/(k+1)^{p_2}$, respectively, where $p_1>0$ and $p_2-p_1>1$; then $\alpha_\infty=0$, $\tilde{\beta}_\infty=0$ and $\sum_{k=0}^{\infty}\tilde{\beta}_k<\infty$, the exact convergence to the optimal value can be achieved.

IV. Conclusions

In this paper, we have developed a gradient-free distributed optimization algorithm in a multi-agent system where the underlying communication network is directed. We construct a pseudo-gradient for each agent in replace of the gradient information in the state update. We have analyzed the convergence properties in details and rigorously proved the exact convergence of this algorithm to the optimal solution.

APPENDIX

A. Proof of Theorem 2

Before the proof of Theorem 2, we first provide some properties of function $f_{i,\beta_{1,k}}(\mathbf{x})$ summarized in the following Lemma:

Lemma 3: ([11]) Suppose Assumptions 2 and 3 hold. Then, for each $i\in\mathcal{V}$, the following properties of the function $f_{i,\beta_{1,k}}(\mathbf{x})$ are satisfied:

1) $f_{i,\beta_{1,k}}(\mathbf{x})$ is convex and differentiable, and it satisfies

$$f_i(\mathbf{x})\le f_{i,\beta_{1,k}}(\mathbf{x})\le f_i(\mathbf{x})+\beta_{1,k}\hat{D}\sqrt{n+2},$$

2) the pseudo-gradient $\mathbf{g}^i(\mathbf{x}_k^i)$ satisfies

$$\mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i)|\mathcal{F}_k] = \nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k \hat{D}\mathbf{v},$$

3) there is a universal constant Q such that

$$\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\||\mathcal{F}_k] \leq \sqrt{\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\|^2|\mathcal{F}_k]} \leq Q\mathcal{T}_k,$$

where $\beta_{1,k}$ and $\tilde{\beta}_k$ are defined in (5), $\mathbf{v} \in \mathbb{R}^n$ is a vector satisfying $\|\mathbf{v}\| \leq n\sqrt{3n}/2$, and $\mathcal{T}_k = \hat{D}\sqrt{n\left[n\sqrt{\tilde{\beta}_k} + 1 + \ln n\right]}$. If $\tilde{\beta}_k$ is bounded, then $\mathcal{T}_k$ is bounded by a constant $\hat{\mathcal{T}}$.

Next, we proceed to the proof of Theorem 2. Considering (6), and the fact that A is column-stochastic, we have

$$\begin{aligned}\bar{\mathbf{z}}_{k+1} &= \tfrac{1}{N}\textstyle\sum_{j=1}^{2N}\left[\sum_{i=1}^{2N}[A]_{ij}\right]\mathbf{z}_k^j + \tfrac{1}{N}\sum_{i=1}^{2N} g_k^i \\ &= \bar{\mathbf{z}}_k + \tfrac{1}{N}\textstyle\sum_{i=1}^{N} g_k^i.\end{aligned}$$

Thus, we can derive that

$$\begin{aligned}\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}^\star\|^2 &= \|\bar{\mathbf{z}}_k - \mathbf{x}^\star\|^2 + \left\|\tfrac{1}{N}\textstyle\sum_{i=1}^{N} g_k^i\right\|^2 \\ &\quad + \tfrac{2}{N}\textstyle\sum_{i=1}^{N} {g_k^i}^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \\ &= \|\bar{\mathbf{z}}_k - \mathbf{x}^\star\|^2 + \tfrac{1}{N^2}\left\|\textstyle\sum_{i=1}^{N} g_k^i\right\|^2 \quad (13a)\\ &\quad - \tfrac{2\alpha_k}{N}\textstyle\sum_{i=1}^{N} \mathbf{g}^i(\mathbf{x}_k^i)^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \quad (13b)\\ &\quad + \tfrac{2}{N}\textstyle\sum_{i=1}^{N}(g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star). \quad (13c)\end{aligned}$$

Noting that for the second term in (13a), we take conditional expectation on $\mathcal{F}_k$, yielding

$$\begin{aligned}&\mathbb{E}[\|\textstyle\sum_{i=1}^{N} g_k^i\|^2|\mathcal{F}_k] \leq \sum_{i=1}^{N}\mathbb{E}[\|g_k^i\|^2|\mathcal{F}_k] \\ &= \textstyle\sum_{i=1}^{N}(\mathbb{E}[\|g_k^i\||\mathcal{F}_k])^2 + \sum_{i=1}^{N} Cov(\|g_k^i\|, \|g_k^i\|) \\ &\leq \left(\textstyle\sum_{i=1}^{N}\mathbb{E}[\|g_k^i\||\mathcal{F}_k]\right)^2 + V_1,\end{aligned}$$

where we have applied Lemma 2 on $\sum_{i=1}^{N}\mathbb{E}[\|g_k^i\||\mathcal{F}_k]$ and used $\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y] + Cov(x,y)$. $V_1 > 0$ is an upper bound of the covariance term $\sum_{i=1}^{N} Cov(\|g_k^i\|, \|g_k^i\|)$. Thus, we obtain

$$\mathbb{E}[\|\textstyle\sum_{i=1}^{N} g_k^i\|^2|\mathcal{F}_k] \leq G^2\alpha_k^2 + V_1, \quad (14)$$

Noting that for (13b), we take conditional expectation on $\mathcal{F}_k$ and apply Lemma 3-(2)

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}\mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i)|\mathcal{F}_k]^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \\ &\quad = \textstyle\sum_{i=1}^{N}(\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k\hat{D}\mathbf{v})^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star).\end{aligned} \quad (15)$$

Noting that

$$\begin{aligned}&(\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k\hat{D}\mathbf{v})^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \\ =&(\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k\hat{D}\mathbf{v})^T(\bar{\mathbf{z}}_k - \mathbf{x}_k^i) \\ &+ (\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k\hat{D}\mathbf{v})^T(\mathbf{x}_k^i - \mathbf{x}^\star) \\ \geq& -\|\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i)\|\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| - \tilde{\beta}_k\hat{D}\|\mathbf{v}\|\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\ &+ f_{i,\beta_{1,k}}(\mathbf{x}_k^i) - f_{i,\beta_{1,k}}(\mathbf{x}^\star) - \tilde{\beta}_k\hat{D}\|\mathbf{v}\|\|\mathbf{x}_k^i - \mathbf{x}^\star\| \\ \geq& -\left(Q\hat{\mathcal{T}} + \tilde{\beta}_k\|\mathbf{v}\|\hat{D}\right)\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| + (f_i(\mathbf{x}_k^i) - f_i(\bar{\mathbf{z}}_k)) \\ &+ (f_i(\bar{\mathbf{z}}_k) - f_{i,\beta_{1,k}}(\mathbf{x}^\star)) - \tilde{\beta}_k\hat{D}\|\mathbf{v}\|\|\mathbf{x}_k^i - \mathbf{x}^\star\|,\end{aligned}$$

where we have used $f_{i,\beta_{1,k}}(\mathbf{x}_k^i) \geq f_i(\mathbf{x}_k^i)$ based on Lemma 3-(1);

$$\|\nabla f_{i,\mu^i}(\mathbf{x}_k^i)\| = \|\mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i)|\mathcal{F}_k]\| \leq \mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\||\mathcal{F}_k]$$

with $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\||\mathcal{F}_k]$ bounded by applying Lemma 3-(3); and

$$f_i(\mathbf{x}_k^i) - f_i(\bar{\mathbf{z}}_k) \geq \partial f_i(\bar{\mathbf{z}}_k)^T(\mathbf{x}_k^i - \bar{\mathbf{z}}_k) \geq -\hat{D}\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|$$

based on Assumption 2. Thus, we have

$$\begin{aligned}&(\nabla f_{i,\beta_{1,k}}(\mathbf{x}_k^i) + \tilde{\beta}_k\hat{D}\mathbf{v})^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \geq f_i(\bar{\mathbf{z}}_k) - f_{i,\beta_{1,k}}(\mathbf{x}^\star) \\ &- \left(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\right)\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| - \tilde{\beta}_k\hat{D}\|\mathbf{v}\|\|\mathbf{x}_k^i - \mathbf{x}^\star\|,\end{aligned} \quad (16)$$

For the term $\|\mathbf{x}_k^i - \mathbf{x}^\star\|$, we can provide the following bound:

$$\begin{aligned}&\|\mathbf{x}_k^i - \mathbf{x}^\star\| \\ =&\left\|\mathcal{P}_{\mathcal{X}}\left[\textstyle\sum_{j=1}^{N}[A_r]_{ij}\mathbf{x}_{k-1}^j + \epsilon\mathbf{y}_{k-1}^i - \alpha_{k-1}\mathbf{g}^i(\mathbf{x}_{k-1}^i)\right] - \mathbf{x}^\star\right\| \\ \leq&\left\|\textstyle\sum_{j=1}^{N}[A_r]_{ij}\mathbf{x}_{k-1}^j + \epsilon\mathbf{y}_{k-1}^i - \alpha_{k-1}\mathbf{g}^i(\mathbf{x}_{k-1}^i) - \mathbf{x}^\star\right\| \\ \leq&\left\|\textstyle\sum_{j=1}^{N}[A_r]_{ij}\mathbf{x}_{k-1}^j - \mathbf{x}^\star\right\| + \epsilon\|\mathbf{y}_{k-1}^i\| + \alpha_{k-1}\|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\ \leq&\textstyle\sum_{j=1}^{N}[A_r]_{ij}\|\mathbf{x}_{k-1}^j - \mathbf{x}^\star\| + \epsilon\|\mathbf{y}_{k-1}^i\| + \alpha_{k-1}\|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\ \leq&\textstyle\sum_{j=1}^{N}[A_r]_{ij}\|\mathbf{x}_{k-1}^i - \mathbf{x}^\star\| + \epsilon\|\mathbf{y}_{k-1}^i\| + \alpha_{k-1}\|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\ &\quad + \textstyle\sum_{j=1}^{N}[A_r]_{ij}\|\mathbf{x}_{k-1}^i - \mathbf{x}_{k-1}^j\| \\ =&\|\mathbf{x}_{k-1}^i - \mathbf{x}^\star\| + \epsilon\|\mathbf{y}_{k-1}^i\| + \alpha_{k-1}\|\mathbf{g}^i(\mathbf{x}_{k-1}^i)\| \\ &\quad + \textstyle\sum_{j=1}^{N}[A_r]_{ij}(\|\mathbf{x}_{k-1}^i - \bar{\mathbf{z}}_{k-1}\| + \|\mathbf{x}_{k-1}^j - \bar{\mathbf{z}}_{k-1}\|).\end{aligned}$$

Thus, applying the above relation recursively, and taking conditional expectation on $\mathcal{F}_k$, we have

$$\begin{aligned}&\|\mathbf{x}_k^i - \mathbf{x}^\star\| = \epsilon\textstyle\sum_{\tau=0}^{k-1}\|\mathbf{y}_\tau^i\| + \sum_{\tau=0}^{k-1}\alpha_\tau\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_\tau^i)\||\mathcal{F}_\tau] \\ &+ \textstyle\sum_{\tau=0}^{k-1}\sum_{j=1}^{N}[A_r]_{ij}(\|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\| + \|\mathbf{x}_\tau^j - \bar{\mathbf{z}}_\tau\|) + \|\mathbf{x}_0^i - \mathbf{x}^\star\|.\end{aligned} \quad (17)$$

Combining (16) and (17), and substituting to (15), we obtain

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}\mathbb{E}[\mathbf{g}^i(\mathbf{x}_k^i)|\mathcal{F}_k]^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \\ \geq& -\left(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\right)\textstyle\sum_{i=1}^{N}\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\| \\ &+ f(\bar{\mathbf{z}}_k) - f_{\beta_{1,k}}(\mathbf{x}^\star) - \tilde{\beta}_k\hat{D}\|\mathbf{v}\|\Big[\textstyle\sum_{i=1}^{N}\|\mathbf{x}_0^i - \mathbf{x}^\star\| \\ &+ \epsilon\textstyle\sum_{\tau=0}^{k-1}\sum_{i=1}^{N}\|\mathbf{y}_\tau^i\| + NQ\hat{\mathcal{T}}\sum_{\tau=0}^{k-1}\alpha_\tau \\ &+ 2N\textstyle\sum_{\tau=0}^{k-1}\sum_{i=1}^{N}\|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\|\Big],\end{aligned} \quad (18)$$

where we have applied Lemma 3-(3) on $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_\tau^i)\||\mathcal{F}_\tau]$. Noting that for term (13c), we have

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}(g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star) \\ =&\textstyle\sum_{i=1}^{N}(g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1}) \quad (19a)\\ &+ \textstyle\sum_{i=1}^{N}(g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i) \quad (19b)\\ &+ \textstyle\sum_{i=1}^{N}(g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i))^T(\mathbf{x}_{k+1}^i - \mathbf{x}^\star). \quad (19c)\end{aligned}$$

For (19a), we have

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}(g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1}) \\ &\leq \textstyle\sum_{i=1}^{N}\|g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i)\|\|\frac{1}{N}\sum_{i=1}^{N} g_k^i\| \\ &\leq \tfrac{1}{N}\left(\textstyle\sum_{i=1}^{N}\|g_k^i\|\right)^2 + \tfrac{\alpha_k}{N}\textstyle\sum_{i=1}^{N}\|g_k^i\|\sum_{i=1}^{N}\|\mathbf{g}^i(\mathbf{x}_k^i)\|.\end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_k$, we obtain

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}\mathbb{E}[(g_k^i + \alpha_k\mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_k - \bar{\mathbf{z}}_{k+1})|\mathcal{F}_k] \\ &\qquad\leq \tfrac{G}{N}(G + NQ\hat{\mathcal{T}})\alpha_k^2 + V_2,\end{aligned} \quad (20)$$

where we have applied Lemma 3-(3) on $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\||\mathcal{F}_k]$ and Lemma 2 on $\sum_{i=1}^{N}\mathbb{E}[\|g_k^i\||\mathcal{F}_k]$. $V_2 > 0$ is an upper bound of the sum of covariance terms $Cov(\sum_{i=1}^{N}\|g_k^i\|, \sum_{i=1}^{N}\|g_k^i\|)$ and $Cov(\sum_{i=1}^{N}\|g_k^i\|, \sum_{i=1}^{N}\|\mathbf{g}^i(\mathbf{x}_k^i)\|)$.

For (19b), we have

$$\begin{aligned}&\sum_{i=1}^N (g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i)\\ &\quad\le \sum_{i=1}^N \|g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i)\|\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|\\ &\quad\le \sum_{i=1}^N (\|g_k^i\| + \alpha_k\|\mathbf{g}^i(\mathbf{x}_k^i)\|)\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|.\end{aligned}$$

Taking the conditional expectation on $\mathcal{F}_k$, we obtain

$$\begin{aligned}&\sum_{i=1}^N \mathbb{E}[(g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i)|\mathcal{F}_k]\\ &\le (G + Q\hat{\mathcal{T}})\alpha_k \sum_{i=1}^N \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\||\mathcal{F}_k] + V_3,\end{aligned} \tag{21}$$

where we have applied Lemma 3-(3) on $\mathbb{E}[\|\mathbf{g}^i(\mathbf{x}_k^i)\||\mathcal{F}_k]$ and Lemma 2 on $\sum_{i=1}^N \mathbb{E}[\|g_k^i\||\mathcal{F}_k]$. $V_3 > 0$ is an upper bound of the sum of covariance terms $Cov(\sum_{i=1}^N \|g_k^i\|, \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|)$ and $Cov(\sum_{i=1}^N \|\mathbf{g}^i(\mathbf{x}_k^i)\|, \|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\|)$.

For (19c), it follows from Lemma 1-(a) in [15] that

$$(g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i))^T(\mathbf{x}_{k+1}^i - \mathbf{x}^\star) \le 0. \tag{22}$$

Thus, taking the conditional expectation on $\mathcal{F}_k$ in (19) and substituting (20), (21) and (22), we obtain

$$\begin{aligned}&\sum_{i=1}^N \mathbb{E}[(g_k^i + \alpha_k \mathbf{g}^i(\mathbf{x}_k^i))^T(\bar{\mathbf{z}}_k - \mathbf{x}^\star)|\mathcal{F}_k] \le \tfrac{G(G+NQ\hat{\mathcal{T}})\alpha_k^2}{N}\\ &+ (G + Q\hat{\mathcal{T}})\sum_{i=1}^N \alpha_k \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\||\mathcal{F}_k] + V_2 + V_3.\end{aligned} \tag{23}$$

Taking the conditional expectation on $\mathcal{F}_k$ in (13), and substituting (14), (18) and (23), we obtain that

$$\begin{aligned}&2\alpha_k(f(\bar{\mathbf{z}}_k) - f_{\beta_{1,k}}(\mathbf{x}^\star))\\ &\le 2\big(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\big)\sum_{i=1}^N \alpha_k\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|\\ &+ \tfrac{1}{N}(G^2\alpha_k^2 + V_1) + N(\|\bar{\mathbf{z}}_k - \mathbf{x}^\star\|^2 - \mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}^\star\|^2|\mathcal{F}_k])\\ &+ 2\tilde{\beta}_k\hat{D}\|\mathbf{v}\|\Big[\alpha_k\sum_{i=1}^N \|\mathbf{x}_0^i - \mathbf{x}^\star\| + \epsilon\alpha_k\sum_{\tau=0}^{k-1}\sum_{i=1}^N \|\mathbf{y}_\tau^i\|\\ &+ NQ\hat{\mathcal{T}}\alpha_k\sum_{\tau=0}^{k-1}\alpha_\tau + 2N\alpha_k\sum_{\tau=0}^{k-1}\sum_{i=1}^N \|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\|\Big]\\ &+ \tfrac{2G}{N}(G + NQ\hat{\mathcal{T}})\alpha_k^2\\ &+ 2(G + Q\hat{\mathcal{T}})\sum_{i=1}^N \alpha_k\mathbb{E}[\|\bar{\mathbf{z}}_{k+1} - \mathbf{x}_{k+1}^i\||\mathcal{F}_k] + 2V_2 + 2V_3.\end{aligned} \tag{24}$$

Taking the total expectation in (24) and summing up from $k = 0$ to $t - 1$, we have

$$\begin{aligned}&\sum_{k=0}^{t-1} \alpha_k(\mathbb{E}[f(\bar{\mathbf{z}}_k)] - f_{\beta_{1,k}}(\mathbf{x}^\star))\\ &\le \sum_{k=0}^{t-1}\big(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\big)\sum_{i=1}^N \alpha_k\mathbb{E}[\|\mathbf{x}_k^i - \bar{\mathbf{z}}_k\|] &\text{(25a)}\\ &+ \sum_{k=0}^{t-1}(G + Q\hat{\mathcal{T}})\sum_{i=1}^N \alpha_k\mathbb{E}[\|\mathbf{x}_{k+1}^i - \bar{\mathbf{z}}_{k+1}\|] &\text{(25b)}\\ &+ \hat{D}\|\mathbf{v}\|\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k\sum_{i=1}^N \mathbb{E}[\|\mathbf{x}_0^i - \mathbf{x}^\star\|] &\text{(25c)}\\ &+ \hat{D}\|\mathbf{v}\|\epsilon\sum_{i=1}^N\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k\sum_{\tau=0}^{k-1}\mathbb{E}[\|\mathbf{y}_\tau^i\|] &\text{(25d)}\\ &+ NQ\hat{D}\|\mathbf{v}\|\hat{\mathcal{T}}\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k\sum_{\tau=0}^{k-1}\alpha_\tau &\text{(25e)}\\ &+ 2N\hat{D}\|\mathbf{v}\|\sum_{i=1}^N\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k\sum_{\tau=0}^{k-1}\mathbb{E}[\|\mathbf{x}_\tau^i - \bar{\mathbf{z}}_\tau\|] &\text{(25f)}\\ &+ \tfrac{G}{2N}\sum_{k=0}^{t-1}(3G + 2NQ\hat{\mathcal{T}})\alpha_k^2 &\text{(25g)}\\ &+ \tfrac{N}{2}\mathbb{E}[\|\bar{\mathbf{z}}_0 - \mathbf{x}^\star\|^2] + \tfrac{V_1}{N} + 2V_2 + 2V_3. &\text{(25h)}\end{aligned}$$

For (25a), substituting (11), we have

$$\begin{aligned}&\text{(25a)} \le \big(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\big)N\alpha_0\max_i \mathbb{E}[\|\mathbf{z}_0^i - \bar{\mathbf{z}}_0\|]\\ &+ \sum_{k=1}^{t-1}\big(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\big)N\alpha_k\Big[2N\Gamma\gamma^k\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\\ &\quad + G\Gamma\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_{r-1} + G\alpha_{k-1}\Big].\end{aligned}$$

Noting that $\tilde{\beta}_k$ is bounded, then $\big(Q\hat{\mathcal{T}} + (\tilde{\beta}_k\|\mathbf{v}\| + 1)\hat{D}\big)N$ is bounded, and can be denoted by $B_1 > 0$. Thus, the above inequality can be simplified as

$$\begin{aligned}&\text{(25a)} \le B_1\Big[G\sum_{k=1}^{t-1}(\Gamma\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_k\alpha_{r-1} + \alpha_k\alpha_{k-1})\\ &+ 2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\sum_{k=1}^{t-1}\alpha_k\gamma^k + \alpha_0\max_i \mathbb{E}[\|\mathbf{z}_0^i - \bar{\mathbf{z}}_0\|]\Big].\end{aligned}$$

Following the results from Lemma 3 in [8] on $\sum_{k=1}^{t-1}\alpha_k\gamma^k$, $\sum_{k=1}^{t-1}\alpha_k\alpha_{k-1}$ and $\sum_{k=1}^{t-1}\sum_{r=1}^{k-1}\gamma^{k-r}\alpha_k\alpha_{r-1}$, the above inequality can be further simplified as

$$\begin{aligned}&\text{(25a)} \le B_1\Big[2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\Big(\tfrac{1}{2}\sum_{k=1}^{t-1}\alpha_k^2 + \tfrac{\gamma^2}{2(1-\gamma^2)}\Big)\\ &+ G\Big(\tfrac{\Gamma\gamma}{1-\gamma}\sum_{k=1}^{t-1}\alpha_k^2 + \sum_{k=0}^{t-1}\alpha_k^2\Big) + \alpha_0\max_i \mathbb{E}[\|\mathbf{z}_0^i - \bar{\mathbf{z}}_0\|]\Big]\\ &\le \big(\sum_{k=0}^{t-1}\alpha_k^2\big)B_1\Big[N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G\Big(1 + \tfrac{\Gamma\gamma}{1-\gamma}\Big)\Big]\\ &+ B_1N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\Big(\tfrac{\gamma^2}{1-\gamma^2}\Big) + B_1\alpha_0\max_i \mathbb{E}[\|\mathbf{z}_0^i - \bar{\mathbf{z}}_0\|].\end{aligned}$$

Taking the limit $t \to \infty$ and noting that $\sum_{k=1}^\infty \alpha_k^2 < \infty$, we have

$$\lim_{t\to\infty}\text{(25a)} < \infty \tag{26}$$

For (25b), substituting (11) and denoting the upper bound of $(G + Q\hat{\mathcal{T}})N$ by $B_2 > 0$, we have

$$\begin{aligned}&\text{(25b)} \le B_2\Big[2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\sum_{k=0}^{t-1}\alpha_k\gamma^{k+1}\\ &\quad + G\Gamma\sum_{k=0}^{t-1}\sum_{r=1}^{k}\gamma^{k-r+1}\alpha_k\alpha_{r-1} + G\sum_{k=0}^{t-1}\alpha_k^2\Big].\end{aligned}$$

Following the results from Lemma 3 in [8] on $\sum_{k=0}^{t-1}\alpha_k\gamma^{k+1}$ and $\sum_{k=0}^{t-1}\sum_{r=1}^{k}\gamma^{k-r+1}\alpha_k\alpha_{r-1}$, the above inequality can be simplified as

$$\begin{aligned}&\text{(25b)} \le B_2\Big[2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\Big(\tfrac{1}{2}\sum_{k=0}^{t-1}\alpha_k^2 + \tfrac{\gamma^2}{2(1-\gamma^2)}\Big)\\ &\quad + G\Gamma\Big(\tfrac{\gamma}{1-\gamma}\sum_{k=0}^{t-1}\alpha_k^2\Big) + G\Big(\sum_{k=0}^{t-1}\alpha_k^2\Big)\Big]\\ &\quad = \Big(\sum_{k=0}^{t-1}\alpha_k^2\Big)B_2\Big[N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G\Big(1 + \tfrac{\Gamma\gamma}{1-\gamma}\Big)\Big]\\ &\quad + B_2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\Big(\tfrac{\gamma^2}{1-\gamma^2}\Big).\end{aligned}$$

Taking the limit $t \to \infty$ and noting that $\sum_{k=1}^\infty \alpha_k^2 < \infty$, we obtain

$$\lim_{t\to\infty}\text{(25b)} < \infty. \tag{27}$$

For (25c), we have

$$\text{(25c)} \le N\hat{D}\|\mathbf{v}\|\max_i \mathbb{E}[\|\mathbf{x}_0^i - \mathbf{x}^\star\|]\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k.$$

Taking the limit $t \to \infty$ and noting that $\sum_{k=0}^\infty \alpha_k\tilde{\beta}_k \le \sum_{k=0}^\infty \alpha_k^2 + \sum_{k=0}^\infty \tilde{\beta}_k^2 \le \sum_{k=0}^\infty \alpha_k^2 + (\sum_{k=0}^\infty \tilde{\beta}_k)^2 < \infty$, we obtain

$$\lim_{t\to\infty}\text{(25c)} < \infty. \tag{28}$$

For (25d), substituting (12), we have

$$\begin{aligned}&\text{(25d)} \le \hat{D}\|\mathbf{v}\|\epsilon\sum_{i=1}^N\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k\\ &\quad\times\sum_{\tau=0}^{k-1}\Big[2N\Gamma\gamma^\tau\max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + G\Gamma\sum_{r=1}^{\tau-1}\gamma^{\tau-r}\alpha_{r-1}\Big]\\ &\le \tfrac{N\hat{D}\|\mathbf{v}\|\epsilon}{1-\gamma}\sum_{k=0}^{t-1}\alpha_k\tilde{\beta}_k\Big[2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|] + \gamma\sum_{\tau=1}^{k-1}\alpha_\tau\Big]\\ &\le \tfrac{N\hat{D}\|\mathbf{v}\|\epsilon}{1-\gamma}\Big[2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\sum_{k=0}^{t-1}\tilde{\beta}_k(\alpha_k + \gamma\sum_{\tau=1}^{k-1}\alpha_\tau^2)\Big]\\ &\le \tfrac{N\hat{D}\|\mathbf{v}\|\epsilon}{1-\gamma}\Big[2N\Gamma\max_j \mathbb{E}[\|\mathbf{z}_0^j\|]\sum_{k=0}^{t-1}\tilde{\beta}_k(\alpha_k + \gamma\sum_{\tau=1}^{t-1}\alpha_\tau^2)\Big].\end{aligned}$$

Taking the limit $t \to \infty$ and noting that $\sum_{k=0}^{\infty} \alpha_k \tilde{\beta}_k < \infty$, $\sum_{k=0}^{\infty} \tilde{\beta}_k < \infty$ and $\sum_{\tau=0}^{\infty} \alpha_\tau^2 < \infty$, we obtain

$$\lim_{t\to\infty} \text{(25d)} < \infty. \tag{29}$$

For (25e), denoting the upper bound of $NQ\hat{D}\|\mathbf{v}\|\hat{\mathcal{T}}$ by $B_3 > 0$, we have

$$\begin{aligned}\text{(25e)} &\le B_3 \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \sum_{\tau=0}^{k-1} \alpha_\tau \le B_3 \sum_{k=0}^{t-1} \tilde{\beta}_k \sum_{\tau=0}^{k-1} \alpha_\tau^2 \\ &\le B_3 \sum_{k=0}^{t-1} \tilde{\beta}_k \sum_{\tau=0}^{t-1} \alpha_\tau^2.\end{aligned}$$

Taking the limit $t \to \infty$ and noting that $\sum_{k=0}^{\infty} \tilde{\beta}_k < \infty$ and $\sum_{\tau=0}^{\infty} \alpha_\tau^2 < \infty$, we obtain

$$\lim_{t\to\infty} \text{(25e)} < \infty. \tag{30}$$

For (25f), substituting (11), we have

$$\begin{aligned}\text{(25f)} &\le 2N^2\hat{D}\|\mathbf{v}\| \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \max_i \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|] \\ &\quad + 2N^2\hat{D}\|\mathbf{v}\| \sum_{k=0}^{t-1} \tilde{\beta}_k \sum_{\tau=1}^{k-1} \alpha_\tau \Big[2N\Gamma\gamma^\tau \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \\ &\quad + G\Gamma \sum_{r=1}^{\tau-1} \gamma^{\tau-r} \alpha_{r-1} + G\alpha_{\tau-1} \Big] \\ &\le 2N^2\hat{D}\|\mathbf{v}\| \max_i \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|] \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \\ &\quad + 2N^2\hat{D}\|\mathbf{v}\| \sum_{k=0}^{t-1} \tilde{\beta}_k \Big[2N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \sum_{\tau=1}^{k-1} \alpha_\tau \gamma^\tau \\ &\quad + G\Gamma \sum_{\tau=1}^{k-1} \sum_{r=1}^{\tau-1} \gamma^{\tau-r} \alpha_\tau \alpha_{r-1} + G \sum_{\tau=1}^{k-1} \alpha_\tau \alpha_{\tau-1} \Big].\end{aligned}$$

Following the results from Lemma 3 in [8] on $\sum_{\tau=1}^{k-1} \alpha_\tau \alpha_{\tau-1}$ and $\sum_{\tau=1}^{k-1} \sum_{r=1}^{\tau-1} \gamma^{\tau-r} \alpha_\tau \alpha_{r-1}$, the above inequality can be simplified as

$$\begin{aligned}\text{(25f)} &\le 2N^2\hat{D}\|\mathbf{v}\| \max_i \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|] \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \\ &\quad + 2N^2\hat{D}\|\mathbf{v}\| \sum_{k=0}^{t-1} \tilde{\beta}_k \Big[2N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \Big(\tfrac{1}{2} \sum_{\tau=1}^{k-1} \alpha_\tau^2 \\ &\quad + \tfrac{\gamma^2}{2(1-\gamma^2)} \Big) + G\Gamma \Big(\tfrac{\gamma}{1-\gamma} \sum_{\tau=1}^{k-1} \alpha_\tau^2 \Big) + G \Big(\sum_{\tau=0}^{k-1} \alpha_\tau^2 \Big) \Big] \\ &\le 2N^2\hat{D}\|\mathbf{v}\| \max_i \mathbb{E}[\|\mathbf{x}_0^i - \bar{\mathbf{z}}_0\|] \sum_{k=0}^{t-1} \alpha_k \tilde{\beta}_k \\ &\quad + 2N^2\hat{D}\|\mathbf{v}\| \sum_{k=0}^{t-1} \tilde{\beta}_k \Big[\Big(\sum_{\tau=0}^{k-1} \alpha_\tau^2 \Big) \Big[N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \\ &\quad + G\Big(1 + \tfrac{\Gamma\gamma}{1-\gamma}\Big) \Big] + N\Gamma \max_j \mathbb{E}[\|\mathbf{z}_0^j\|] \Big(\tfrac{\gamma^2}{1-\gamma^2} \Big) \Big].\end{aligned}$$

Taking the limit $t \to \infty$ and noting that $\sum_{k=0}^{\infty} \alpha_k \tilde{\beta}_k < \infty$, $\sum_{k=0}^{\infty} \tilde{\beta}_k < \infty$ and $\sum_{\tau=0}^{\infty} \alpha_\tau^2 < \infty$, we obtain

$$\lim_{t\to\infty} \text{(25f)} < \infty. \tag{31}$$

For (25g), denoting the upper bound of $\frac{G}{2N}(3G + 2NQ\hat{\mathcal{T}})$ by $B_4 > 0$, we have (25g) $\le B_4 \sum_{k=0}^{t-1} \alpha_k^2$. Taking the limit $t \to \infty$ and noting that $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, we obtain

$$\lim_{t\to\infty} \text{(25g)} < \infty. \tag{32}$$

Now, considering (25), we take the limit $t \to \infty$. Then, substituting (26)-(32) gives

$$\sum_{k=0}^{\infty} \alpha_k (\mathbb{E}[f(\bar{\mathbf{z}}_k)] - f_{\beta_{1,k}}(\mathbf{x}^\star)) < \infty. \tag{33}$$

Together with the fact that $\sum_{k=0}^{\infty} \alpha_k = \infty$, we have

$$\lim_{k\to\infty} \mathbb{E}[f(\bar{\mathbf{z}}_k)] = \lim_{k\to\infty} f_{\beta_{1,k}}(\mathbf{x}^\star).$$

According to Lemma 3-(1) that

$$f^\star \le \lim_{k\to\infty} f_{\beta_{1,k}}(\mathbf{x}^\star) \le f^\star + \lim_{k\to\infty} \beta_{1,k} \hat{D} N \sqrt{n+2},$$

with $\lim_{k\to\infty} \beta_{1,k} = 0$. we obtain the desired result.

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