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Unconditionally Stable Fundamental Alternating-Direction-Implicit FDTD Method for Dispersive Media

Ding Yu Heh and Eng Leong Tan

1 Introduction

The finite-difference time-domain (FDTD) method [1,2] has been widely used for obtaining numerical solutions of various electromagnetic problems such as wave propagation, radiation and scattering. The method is related to Hofer's transmission line matrix (TLM) method [3–5] whereby both are time-domain based method. In TLM method, voltages and currents are solved within grids of transmission line models while the FDTD method solves directly the electric and magnetic fields within a Yee's cell. Both methods are also subjected to the stability criterion, which imposes a maximum constraint on the time step depending on the chosen spatial steps. If the spatial steps are small, this restriction may result in undesirably long simulation time. To remove the stability constraint, the unconditionally stable alternating-direction-implicit FDTD (ADI-FDTD) method [6,7] has been developed. For the TLM counterpart, the unconditionally stable TLM method based on ADI [8] and split-step (SS) [9] schemes has also been introduced. The introduction of these unconditionally stable time-domain based methods plays an important step toward improving the computation efficiency because the time step chosen is no longer restricted by the smallest spatial step. In this chapter, we shall focus on the development of novel unconditionally stable ADI-FDTD method, with extensions to model various dispersive media.

Due to the unconditional stability feature, there have been some extensions of the ADI-FDTD method for various dispersive media [10–15]. Unfortunately, although the ADI-FDTD method is unconditionally stable, its implementation in conventional form is rather complicated. Apart from solving tridiagonal systems, the right-hand sides (RHS) of the update equations involve many update coefficients and field variables (resultant from RHS matrix-operators) that require considerable arithmetic operations. Furthermore, one should not overlook the huge amount of memory indexing operations incurred throughout the whole procedures. The introduction of auxiliary terms for dispersive media into ADI-FDTD algorithm has made such overhead worse from the efficiency point of view where the arithmetic operations and memory indexing increase considerably.

Recently, a new algorithm of efficient ADI-FDTD method [16] which features higher efficiency has been proposed. Such efficient ADI algorithm is included within a family of fundamental implicit schemes, which feature similar fundamental updating structures that are in simplest forms with most efficient matrix-operator-free RHS [17]. We shall call this method the fundamental alternating-direction-implicit FDTD (FADI-FDTD)

method. According to Oxford Advanced dictionary, the word “fundamental” describes something that “cannot be divided” any further, e.g. a fundamental particle. The term “fundamental” used here is defined as the most basic form of ADI-FDTD that “cannot be reduced” further (while retaining its implicit left-hand sides). This is because its RHS is matrix-operator-free which has no more operator to be reduced.

In this chapter, we shall present the formulation of novel unconditionally stable FADI-FDTD method for various dispersive media. The generalized formulation for conventional ADI-FDTD is first shown for dispersive media, and will be extended into efficient FADI-FDTD method. Such generalized formulation is then applied to Debye, Lorentz, Drude and complex conjugate pole-residue pair models. Indeed, as to be compared later, the extension for full 3D dispersive media using novel FADI-FDTD method makes the resultant update equations much more concise and simpler than using conventional ADI-FDTD method. Subsequently, we shall demonstrate the application of novel FADI-FDTD method in the analysis of plasmonic waveguide. The characteristics of a surface plasmon waveguides with Au (gold) and Ag (silver) metal cladding, modeled as combination of Drude-Lorentz dispersive media will be analyzed. Further analysis of plasmonic waveguide grating filter will also be considered.

2 Generalized Formulation

For completeness, we include the formulation of both electric and magnetic dispersive media. The generalized formulation for dispersive media is first performed to the conventional ADI-FDTD method before being extended into FADI-FDTD method. To that end, the generalized frequency domain electric and magnetic current densities of the p -th pole dispersive media, $\tilde{\mathbf{J}}_p^g(\omega)$ and $\tilde{\mathbf{M}}_p^g(\omega)$ are related to frequency domain electric and magnetic fields $\tilde{\mathbf{E}}(\omega)$ and $\tilde{\mathbf{H}}(\omega)$ by

$$\begin{aligned}\tilde{\mathbf{J}}_p^g(\omega) &= j\omega\epsilon_0\chi_{e,p}(\omega)\tilde{\mathbf{E}}(\omega) \\ &= \zeta_{e,p}^g(\omega)\tilde{\mathbf{E}}(\omega)\end{aligned}\tag{1a}$$

$$\begin{aligned}\tilde{\mathbf{M}}_p^g(\omega) &= j\omega\mu_0\chi_{h,p}(\omega)\tilde{\mathbf{H}}(\omega) \\ &= \zeta_{h,p}^g(\omega)\tilde{\mathbf{H}}(\omega)\end{aligned}\tag{1b}$$

where $\chi_{e,p}(\omega)$ and $\chi_{h,p}(\omega)$ are frequency domain electric and magnetic susceptibility function of p -th pole dispersive media respectively. $\zeta_{e,p}^g(\omega)$ and $\zeta_{h,p}^g(\omega)$ are the generalized electric and magnetic frequency domain transfer functions that relates $\tilde{\mathbf{J}}_p^g(\omega)$ and $\tilde{\mathbf{M}}_p^g(\omega)$ to $\tilde{\mathbf{E}}(\omega)$ and $\tilde{\mathbf{H}}(\omega)$ respectively.

These frequency domain transfer functions can be decomposed into two parts to yield:

$$\begin{aligned}\zeta_{e,p}^g(\omega)\tilde{\mathbf{E}}(\omega) &= (\zeta_{e,p}(\omega) + \zeta'_{e,p}(\omega))\tilde{\mathbf{E}}(\omega) \\ &= \tilde{\mathbf{J}}_p(\omega) + \tilde{\mathbf{J}}'_p(\omega)\end{aligned}\tag{2a}$$

$$\begin{aligned}\zeta_{h,p}^g(\omega)\tilde{\mathbf{H}}(\omega) &= (\zeta_{h,p}(\omega) + \zeta'_{h,p}(\omega))\tilde{\mathbf{H}}(\omega) \\ &= \tilde{\mathbf{M}}_p(\omega) + \tilde{\mathbf{M}}'_p(\omega)\end{aligned}\tag{2b}$$

where

$$\tilde{\mathbf{J}}_p(\omega) = \zeta_{e,p}(\omega) \tilde{\mathbf{E}}(\omega) \quad (3a)$$

$$\tilde{\mathbf{J}}'_p(\omega) = \zeta'_{e,p}(\omega) \tilde{\mathbf{E}}(\omega) \quad (3b)$$

$$\tilde{\mathbf{M}}_p(\omega) = \zeta_{h,p}(\omega) \tilde{\mathbf{H}}(\omega) \quad (3c)$$

$$\tilde{\mathbf{M}}'_p(\omega) = \zeta'_{h,p}(\omega) \tilde{\mathbf{H}}(\omega) \quad (3d)$$

Note that in doing so, the generalized frequency domain electric current densities of the p -th pole dispersive media, $\tilde{\mathbf{J}}_p^g(\omega)$ is decomposed into $\tilde{\mathbf{J}}_p(\omega)$ and $\tilde{\mathbf{J}}'_p(\omega)$ while its magnetic counterpart is decomposed into $\tilde{\mathbf{M}}_p(\omega)$ and $\tilde{\mathbf{M}}'_p(\omega)$. The condition for the above decomposition will be shown later.

Taking the decomposition into consideration, we write down the Maxwell's curl equations as

$$\begin{aligned} \nabla \times \mathbf{H}(t) &= \epsilon_0 \epsilon_\infty \frac{\partial}{\partial t} \mathbf{E}(t) + \sigma \mathbf{E}(t) + \sum_p \mathcal{F}^{-1} \left\{ (\zeta_{e,p}(\omega) + \zeta'_{e,p}(\omega)) \mathbf{E}(\omega) \right\} \\ &= \epsilon_0 \epsilon_\infty \frac{\partial}{\partial t} \mathbf{E}(t) + \sigma \mathbf{E}(t) + \sum_p (\mathbf{J}_p(t) + \mathbf{J}'_p(t)) \end{aligned} \quad (4a)$$

$$\begin{aligned} -\nabla \times \mathbf{E}(t) &= \mu_0 \mu_\infty \frac{\partial}{\partial t} \mathbf{H}(t) + \sigma^* \mathbf{H}(t) + \sum_p \mathcal{F}^{-1} \left\{ (\zeta_{h,p}(\omega) + \zeta'_{h,p}(\omega)) \mathbf{H}(\omega) \right\} \\ &= \mu_0 \mu_\infty \frac{\partial}{\partial t} \mathbf{H}(t) + \sigma^* \mathbf{H}(t) + \sum_p (\mathbf{M}_p(t) + \mathbf{M}'_p(t)) \end{aligned} \quad (4b)$$

where ϵ_∞ and μ_∞ are permittivity and permeability at infinite frequency respectively.

We now state the condition for the decomposition in (2). To guarantee real-valued time signal in (4), the following condition must hold:

$$\mathbf{J}_p(t) + \mathbf{J}'_p(t) = 2 \operatorname{Re} \{ \mathbf{J}_p(t) \} \quad (5a)$$

$$\mathbf{M}_p(t) + \mathbf{M}'_p(t) = 2 \operatorname{Re} \{ \mathbf{M}_p(t) \} \quad (5b)$$

For complex-valued $\mathbf{J}_p(t)$, we note that $\mathbf{J}'_p(t) = \mathbf{J}_p^*(t)$ where “*” denotes its complex conjugate. On the other hand, for real-valued $\mathbf{J}_p(t)$, $\mathbf{J}'_p(t) = \mathbf{J}_p(t)$. As will be shown later, the condition in (5) is true and applicable for complex conjugate pole-residue pair, Debye, Lorentz and Drude dispersive models.

Using the condition in (5), the Maxwell's curl equations can now be rewritten as

$$\nabla \times \mathbf{H}(t) = \epsilon_0 \epsilon_\infty \frac{\partial}{\partial t} \mathbf{E}(t) + \sigma \mathbf{E}(t) + \sum_p 2 \operatorname{Re} \{ \mathbf{J}_p(t) \} \quad (6a)$$

$$-\nabla \times \mathbf{E}(t) = \mu_0 \mu_\infty \frac{\partial}{\partial t} \mathbf{H}(t) + \sigma^* \mathbf{H}(t) + \sum_p 2 \operatorname{Re} \{ \mathbf{M}_p(t) \}. \quad (6b)$$

Since (5) holds true, we need only consider (3a) and (3c). Converting (3a) and (3c) into time domain, we let the generalized discretization formulation in two substeps as

$$\mathbf{J}_p^{n+\frac{1}{2}} = k_{1p} \mathbf{J}_p^n + k_{2p} \mathbf{E}^{n+\frac{1}{2}} + k_{3p} \mathbf{E}^n \quad (7a)$$

$$\mathbf{M}_p^{n+\frac{1}{2}} = l_{1p} \mathbf{M}_p^n + l_{2p} \mathbf{H}^{n+\frac{1}{2}} + l_{3p} \mathbf{H}^n \quad (7b)$$

and

$$\mathbf{J}_p^{n+1} = k_{1p}\mathbf{J}_p^{n+\frac{1}{2}} + k_{2p}\mathbf{E}^{n+1} + k_{3p}\mathbf{E}^{n+\frac{1}{2}} \quad (8a)$$

$$\mathbf{M}_p^{n+1} = l_{1p}\mathbf{M}_p^{n+\frac{1}{2}} + l_{2p}\mathbf{H}^{n+1} + l_{3p}\mathbf{H}^{n+\frac{1}{2}}. \quad (8b)$$

Note that the discretization in two substeps is required in ADI-FDTD method.

We now discretize (6) into two substeps using ADI-FDTD method and combine it with semi-implicit approximation of

$$\frac{1}{2} \left(\mathbf{J}_p^n + \mathbf{J}_p^{n+\frac{1}{2}} \right) = \frac{1}{2} \left[(1 + k_{1p}) \mathbf{J}_p^n + k_{2p} \mathbf{E}^{n+\frac{1}{2}} + k_{3p} \mathbf{E}^n \right] \quad (9)$$

and

$$\frac{1}{2} \left(\mathbf{J}_p^{n+\frac{1}{2}} + \mathbf{J}_p^{n+1} \right) = \frac{1}{2} \left[(1 + k_{1p}) \mathbf{J}_p^{n+\frac{1}{2}} + k_{2p} \mathbf{E}^{n+1} + k_{3p} \mathbf{E}^{n+\frac{1}{2}} \right]. \quad (10)$$

Upon some manipulations, the generalized update equations of conventional ADI-FDTD method for dispersive media read:

For the first substep n to $n + \frac{1}{2}$:

$$E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} H_z^{n+\frac{1}{2}} = c_1 E_x^n - c_2 \frac{\partial}{\partial z} H_y^n - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{xp}^n \right\} \quad (11a)$$

$$E_y^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial z} H_x^{n+\frac{1}{2}} = c_1 E_y^n - c_2 \frac{\partial}{\partial x} H_z^n - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{yp}^n \right\} \quad (11b)$$

$$E_z^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial x} H_y^{n+\frac{1}{2}} = c_1 E_z^n - c_2 \frac{\partial}{\partial y} H_x^n - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{zp}^n \right\} \quad (11c)$$

$$H_x^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial z} E_y^{n+\frac{1}{2}} = d_1 H_x^n - d_2 \frac{\partial}{\partial y} E_z^n - d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{xp}^n \right\} \quad (11d)$$

$$H_y^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial x} E_z^{n+\frac{1}{2}} = d_1 H_y^n - d_2 \frac{\partial}{\partial z} E_x^n - d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{yp}^n \right\} \quad (11e)$$

$$H_z^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}} = d_1 H_z^n - d_2 \frac{\partial}{\partial x} E_y^n - d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{zp}^n \right\} \quad (11f)$$

$$J_{\xi p}^{n+\frac{1}{2}} = k_{1p} J_{\xi p}^n + k_{2p} E_{\xi}^{n+\frac{1}{2}} + k_{3p} E_{\xi}^n \quad \forall p, \quad \xi = x, y, z \quad (11g)$$

$$M_{\xi p}^{n+\frac{1}{2}} = l_{1p} M_{\xi p}^n + l_{2p} M_{\xi}^{n+\frac{1}{2}} + l_{3p} M_{\xi}^n \quad \forall p, \quad \xi = x, y, z \quad (11h)$$

For the second substep $n + \frac{1}{2}$ to $n + 1$:

$$E_x^{n+1} + c_2 \frac{\partial}{\partial z} H_y^{n+1} = c_1 E_x^{n+\frac{1}{2}} + c_2 \frac{\partial}{\partial y} H_z^{n+\frac{1}{2}} - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{xp}^{n+\frac{1}{2}} \right\} \quad (12a)$$

$$E_y^{n+1} + c_2 \frac{\partial}{\partial x} H_z^{n+1} = c_1 E_y^{n+\frac{1}{2}} + c_2 \frac{\partial}{\partial z} H_x^{n+\frac{1}{2}} - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{yp}^{n+\frac{1}{2}} \right\} \quad (12b)$$

$$E_z^{n+1} + c_2 \frac{\partial}{\partial y} H_x^{n+1} = c_1 E_z^{n+\frac{1}{2}} + c_2 \frac{\partial}{\partial x} H_y^{n+\frac{1}{2}} - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{zp}^{n+\frac{1}{2}} \right\} \quad (12c)$$

$$H_x^{n+1} + d_2 \frac{\partial}{\partial y} E_z^{n+1} = d_1 H_x^{n+\frac{1}{2}} + d_2 \frac{\partial}{\partial z} E_y^{n+\frac{1}{2}} - d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{xp}^{n+\frac{1}{2}} \right\} \quad (12d)$$

$$H_y^{n+1} + d_2 \frac{\partial}{\partial z} E_x^{n+1} = d_1 H_y^{n+\frac{1}{2}} + d_2 \frac{\partial}{\partial x} E_z^{n+\frac{1}{2}} - d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{yp}^{n+\frac{1}{2}} \right\} \quad (12e)$$

$$H_z^{n+1} + d_2 \frac{\partial}{\partial x} E_y^{n+1} = d_1 H_z^{n+\frac{1}{2}} + d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}} - d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{zp}^{n+\frac{1}{2}} \right\} \quad (12f)$$

$$J_{\xi p}^{n+1} = k_{1p} J_{\xi p}^{n+\frac{1}{2}} + k_{2p} E_{\xi}^{n+1} + k_{3p} E_{\xi}^{n+\frac{1}{2}} \quad \forall p, \quad \xi = x, y, z \quad (12g)$$

$$M_{\xi p}^{n+1} = l_{1p} M_{\xi p}^{n+\frac{1}{2}} + l_{2p} H_{\xi}^{n+1} + l_{3p} H_{\xi}^{n+\frac{1}{2}} \quad \forall p, \quad \xi = x, y, z \quad (12h)$$

where

$$c_1 = \frac{1 - \frac{\sigma \Delta t}{4\epsilon_0 \epsilon_{\infty}} - \frac{\Delta t}{4\epsilon_0 \epsilon_{\infty}} \sum_p 2 \text{Re}(k_{3p})}{1 + \frac{\sigma \Delta t}{4\epsilon_0 \epsilon_{\infty}} + \frac{\Delta t}{4\epsilon_0 \epsilon_{\infty}} \sum_p 2 \text{Re}(k_{2p})}, \quad c_2 = \frac{\frac{\Delta t}{2\epsilon_0 \epsilon_{\infty}}}{1 + \frac{\sigma \Delta t}{4\epsilon_0 \epsilon_{\infty}} + \frac{\Delta t}{4\epsilon_0 \epsilon_{\infty}} \sum_p 2 \text{Re}(k_{2p})}, \quad (13a)$$

$$d_1 = \frac{1 - \frac{\sigma^* \Delta t}{4\mu_0 \mu_{\infty}} - \frac{\Delta t}{4\mu_0 \mu_{\infty}} \sum_p 2 \text{Re}(l_{3p})}{1 + \frac{\sigma^* \Delta t}{4\mu_0 \mu_{\infty}} + \frac{\Delta t}{4\mu_0 \mu_{\infty}} \sum_p 2 \text{Re}(l_{2p})}, \quad d_2 = \frac{\frac{\Delta t}{2\mu_0 \mu_{\infty}}}{1 + \frac{\sigma^* \Delta t}{4\mu_0 \mu_{\infty}} + \frac{\Delta t}{4\mu_0 \mu_{\infty}} \sum_p 2 \text{Re}(l_{2p})}. \quad (13b)$$

To solve for E_x in the first substep, we need to substitute (11f) into (11a) to obtain the implicit electric field update equation as

$$\begin{aligned} E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} (d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}}) = & c_1 E_x^n - c_2 \frac{\partial}{\partial z} H_y^n - c_2 \sum_p \text{Re} \left\{ (1 + k_{1p}) J_{xp}^n \right\} + c_2 \frac{\partial}{\partial y} (d_1 H_z^n) \\ & - c_2 \frac{\partial}{\partial y} (d_2 \frac{\partial}{\partial x} E_y^n) - c_2 \frac{\partial}{\partial y} (d_2 \sum_p \text{Re} \left\{ (1 + l_{1p}) M_{xp}^n \right\}). \end{aligned} \quad (14)$$

For E_y , (11d) is substituted into (11b) and for E_z , (11e) is substituted into (11c). After the E fields are solved, H , J and M fields can be obtained explicitly via (11d)-(11h). The procedures are then repeated for the second substep. When the spatial differential operators are approximated using the second-order central differencing operated on a standard Yee's cell [1], the electric field components can be obtained by solving a set of tridiagonal matrices.

The generalized update equations of conventional ADI-FDTD for dispersive media can be formulated into compact matrix form (in two substeps) as

$$(\mathbf{I}_{18} - \mathbf{D}_2 \mathbf{A} + \mathbf{F}_l) \mathbf{u}^{n+\frac{1}{2}} = (\mathbf{D}_1 + \mathbf{D}_2 \mathbf{B} + \mathbf{F}_r) \mathbf{u}^n + \mathbf{D}_2 \mathbf{s}^{n+\frac{1}{2}} \quad (15a)$$

$$(\mathbf{I}_{18} - \mathbf{D}_2 \mathbf{B} + \mathbf{F}_l) \mathbf{u}^{n+1} = (\mathbf{D}_1 + \mathbf{D}_2 \mathbf{A} + \mathbf{F}_r) \mathbf{u}^{n+\frac{1}{2}} + \mathbf{D}_2 \mathbf{s}^{n+\frac{1}{2}} \quad (15b)$$

where

$$\mathbf{u} = [E_x, E_y, E_z, H_x, H_y, H_z, J_x^r, J_y^r, J_z^r, J_x^i, J_y^i, J_z^i, M_x^r, M_y^r, M_z^r, M_x^i, M_y^i, M_z^i]^T \quad (16)$$

$$\mathbf{s} = [-s_{ex}, -s_{ey}, -s_{ez}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \quad (17)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_3 & \mathbf{K}_1 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{K}_2 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{O}_3 & -\mathbf{K}_2 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ -\mathbf{K}_1 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \end{bmatrix} \quad (18)$$

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 & 0 \end{bmatrix} \quad (19)$$

$$\mathbf{D}_1 = \text{diag}(c_1, c_1, c_1, d_1, d_1, d_1, k_1^r, k_1^r, k_1^r, k_1^r, k_1^r, k_1^r, l_1^r, l_1^r, l_1^r, l_1^r) \quad (20)$$

$$\mathbf{D}_2 = \text{diag}(c_2, c_2, c_2, d_2, d_2, d_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad (21)$$

$$\mathbf{F}_l = \begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ -k_2^r \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ -k_2^i \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & -l_2^r \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & -l_2^i \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \end{bmatrix} \quad (22)$$

$$\mathbf{F}_r = \begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 & -c_2(1 + k_1^r) \mathbf{I}_3 & c_2 k_1^i \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & -d_2(1 + l_1^r) \mathbf{I}_3 & d_2 l_1^i \mathbf{I}_3 \\ k_3^r \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & -k_1^i \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ k_3^i \mathbf{I}_3 & \mathbf{O}_3 & k_1^r \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & l_3^r \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & -l_1^i \mathbf{I}_3 \\ \mathbf{O}_3 & l_3^i \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & l_1^r \mathbf{I}_3 & \mathbf{O}_3 \end{bmatrix}. \quad (23)$$

Note that any variables ρ^r represents $\text{Re}(\rho)$ and on the other hand ρ^i represents $\text{Im}(\rho)$. For simplicity, we assume only single pole of dispersive media for now. $\mathbf{I}_r$ and $\mathbf{O}_r$ represent identity and null matrices with dimension $r \times r$. We have also included electric current sources s_{ex} , s_{ey} and s_{ez} in the formulation.

A quick glance at (15) shows that its RHS still involves matrix operators $\mathbf{A}$ and $\mathbf{B}$ which incur considerable arithmetic operations and memory indexing, especially when one is solving the implicit update equations. Efforts have been made to further reduce such overheads by means of a more efficient ADI-FDTD method in lossless media proposed in [16,17]. Such efficient ADI algorithm is included within a family of fundamental implicit schemes, which feature similar fundamental updating structures that are in simplest forms with most efficient matrix-operator-free RHS. We now present the formulation of FADI-FDTD method for dispersive media.

Introducing an auxiliary variable $\hat{\mathbf{v}}$, (15) is rewritten as

$$\hat{\mathbf{v}}^n = (\mathbf{D}_1 + \mathbf{D}_2 \mathbf{B} + \mathbf{F}_r) \mathbf{u}^n \quad (24a)$$

$$(\mathbf{I}_{18} - \mathbf{D}_2 \mathbf{A} + \mathbf{F}_l) \mathbf{u}^{n+\frac{1}{2}} = \hat{\mathbf{v}}^n + \mathbf{D}_2 \mathbf{s}^{n+\frac{1}{2}} \quad (24b)$$

$$\hat{\mathbf{v}}^{n+\frac{1}{2}} = (\mathbf{D}_1 + \mathbf{D}_2 \mathbf{A} + \mathbf{F}_r) \mathbf{u}^{n+\frac{1}{2}} + \mathbf{D}_2 \mathbf{s}^{n+\frac{1}{2}} \quad (24c)$$

$$(\mathbf{I}_{18} - \mathbf{D}_2 \mathbf{B} + \mathbf{F}_l) \mathbf{u}^{n+1} = \hat{\mathbf{v}}^{n+\frac{1}{2}} \quad (24d)$$

Next, (24d) is expressed at one time step backward as

$$\hat{\mathbf{v}}^{n-\frac{1}{2}} = (\mathbf{I}_{18} - \mathbf{D}_2\mathbf{B} + \mathbf{F}_l) \mathbf{u}^n \quad (25)$$

and (24a) can be subsequently reduced into

$$\begin{aligned} \hat{\mathbf{v}}^n &= (\mathbf{D}_1 + \mathbf{D}_2\mathbf{B} + \mathbf{F}_r) \mathbf{u}^n \\ &= [(\mathbf{I}_{18} + \mathbf{D}_1) + \mathbf{F}_l + \mathbf{F}_r] \mathbf{u}^n - (\mathbf{I}_{18} - \mathbf{D}_2\mathbf{B} + \mathbf{F}_l) \mathbf{u}^n \\ &= [(\mathbf{I}_{18} + \mathbf{D}_1) + \mathbf{F}_l + \mathbf{F}_r] \mathbf{u}^n - \hat{\mathbf{v}}^{n-\frac{1}{2}} \end{aligned} \quad (26)$$

Similarly, by rewriting (24b) as

$$\hat{\mathbf{v}}^n = (\mathbf{I}_{18} - \mathbf{D}_2\mathbf{A} + \mathbf{F}_l) \mathbf{u}^{n+\frac{1}{2}} - \mathbf{D}_2\mathbf{s}^{n+\frac{1}{2}}, \quad (27)$$

(24c) is reducible to

$$\begin{aligned} \hat{\mathbf{v}}^{n+\frac{1}{2}} &= (\mathbf{D}_1 + \mathbf{D}_2\mathbf{A} + \mathbf{F}_r) \mathbf{u}^{n+\frac{1}{2}} + \mathbf{D}_2\mathbf{s}^{n+\frac{1}{2}} \\ &= [(\mathbf{I}_{18} + \mathbf{D}_1) + \mathbf{F}_l + \mathbf{F}_r] \mathbf{u}^{n+\frac{1}{2}} - \left[(\mathbf{I}_{18} - \mathbf{D}_2\mathbf{A} + \mathbf{F}_l) \mathbf{u}^{n+\frac{1}{2}} - \mathbf{D}_2\mathbf{s}^{n+\frac{1}{2}} \right] \\ &= [(\mathbf{I}_{18} + \mathbf{D}_1) + \mathbf{F}_l + \mathbf{F}_r] \mathbf{u}^{n+\frac{1}{2}} - \hat{\mathbf{v}}^n \end{aligned} \quad (28)$$

Finally, the FADI-FDTD update procedures read:

$$\hat{\mathbf{v}}^n = [(\mathbf{I}_{12} + \mathbf{D}_1) + \mathbf{F}_l + \mathbf{F}_r] \mathbf{u}^n - \hat{\mathbf{v}}^{n-\frac{1}{2}} \quad (29a)$$

$$(\mathbf{I}_{12} - \mathbf{D}_2\mathbf{A} + \mathbf{F}_l) \mathbf{u}^{n+\frac{1}{2}} = \hat{\mathbf{v}}^n + \mathbf{D}_2\mathbf{s}^{n+\frac{1}{2}} \quad (29b)$$

$$\hat{\mathbf{v}}^{n+\frac{1}{2}} = [(\mathbf{I}_{12} + \mathbf{D}_1) + \mathbf{F}_l + \mathbf{F}_r] \mathbf{u}^{n+\frac{1}{2}} - \hat{\mathbf{v}}^n \quad (29c)$$

$$(\mathbf{I}_{12} - \mathbf{D}_2\mathbf{B} + \mathbf{F}_l) \mathbf{u}^{n+1} = \hat{\mathbf{v}}^{n+\frac{1}{2}} \quad (29d)$$

where (29a) and (29b) form the first substep while (29c) and (29d) form the second. It is worth noting that the FADI-FDTD has matrix-operator-free RHS, i.e., there is no matrix operator $\mathbf{A}$ and $\mathbf{B}$ on the RHS of (29). The source excitation is also only needed at the first substep [18] which further simplifies the implementation compared to the conventional ADI-FDTD method.

We now provide the full update equations of FADI-FDTD by expanding (29). Letting

$$\hat{\mathbf{v}} = [\hat{e}_x, \hat{e}_y, \hat{e}_z, \hat{h}_x, \hat{h}_y, \hat{h}_z, \hat{j}_x^r, \hat{j}_y^r, \hat{j}_z^r, \hat{j}_x^i, \hat{j}_y^i, \hat{j}_z^i, \hat{m}_x^r, \hat{m}_y^r, \hat{m}_z^r, \hat{m}_x^i, \hat{m}_y^i, \hat{m}_z^i]^T, \quad (30)$$

and extending for arbitrary p dispersive pole, the full generalized update equations of FADI-FDTD method for dispersive media read:

For the first substep from n to $n + \frac{1}{2}$:

$$\hat{e}_\xi^n = (1 + c_1)E_\xi^n - \hat{e}_x^{n-\frac{1}{2}} - c_2 \sum_p \text{Re} \{ (1 + k_{1p})J_{\xi p}^n \}, \quad \xi = x, y, z \quad (31a)$$

$$\hat{h}_\xi^n = (1 + d_1)H_\xi^n - \hat{h}_x^{n-\frac{1}{2}} - d_2 \sum_p \text{Re} \{ (1 + l_{1p})M_{\xi p}^n \}, \quad \xi = x, y, z \quad (31b)$$

$$\hat{j}_{\xi p}^n = (1 + k_{1p})J_{\xi p}^n - \hat{j}_{\xi p}^{n-\frac{1}{2}} + (k_{3p} - k_{2p})E_\xi^n, \quad \forall p, \quad \xi = x, y, z \quad (31c)$$

$$\hat{m}_{\xi p}^n = (1 + l_{1p})M_{\xi p}^n - \hat{m}_{\xi p}^{n-\frac{1}{2}} + (l_{3p} - l_{2p})H_\xi^n, \quad \forall p, \quad \xi = x, y, z \quad (31d)$$

$$E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} H_z^{n+\frac{1}{2}} = \hat{e}_x^n - c_2 s_{ex} \quad (31e)$$

$$E_y^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial z} H_x^{n+\frac{1}{2}} = \hat{e}_y^n - c_2 s_{ey} \quad (31f)$$

$$E_z^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial x} H_y^{n+\frac{1}{2}} = \hat{e}_z^n - c_2 s_{ez} \quad (31g)$$

$$H_x^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial z} E_y^{n+\frac{1}{2}} = \hat{h}_x^n \quad (31h)$$

$$H_y^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial x} E_z^{n+\frac{1}{2}} = \hat{h}_y^n \quad (31i)$$

$$H_z^{n+\frac{1}{2}} - d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}} = \hat{h}_z^n \quad (31j)$$

$$J_{\xi p}^{n+\frac{1}{2}} = k_{2p}E_\xi^{n+\frac{1}{2}} + \hat{j}_{\xi p}^n, \quad \forall p, \quad \xi = x, y, z \quad (31k)$$

$$M_{\xi p}^{n+\frac{1}{2}} = l_{2p}H_\xi^{n+\frac{1}{2}} + \hat{m}_{\xi p}^n, \quad \forall p, \quad \xi = x, y, z. \quad (31l)$$

For second substep from $n + \frac{1}{2}$ to $n + 1$:

$$\hat{e}_\xi^{n+\frac{1}{2}} = (1 + c_1)E_\xi^{n+\frac{1}{2}} - \hat{e}_x^n - c_2 \sum_p \text{Re} \{ (1 + k_{1p})J_{\xi p}^{n+\frac{1}{2}} \}, \quad \xi = x, y, z \quad (32a)$$

$$\hat{h}_\xi^{n+\frac{1}{2}} = (1 + d_1)H_\xi^{n+\frac{1}{2}} - \hat{h}_x^n - d_2 \sum_p \text{Re} \{ (1 + l_{1p})M_{\xi p}^{n+\frac{1}{2}} \}, \quad \xi = x, y, z \quad (32b)$$

$$\hat{j}_{\xi p}^{n+\frac{1}{2}} = (1 + k_{1p})J_{\xi p}^{n+\frac{1}{2}} - \hat{j}_{\xi p}^n + (k_{3p} - k_{2p})E_\xi^{n+\frac{1}{2}}, \quad \forall p, \quad \xi = x, y, z \quad (32c)$$

$$\hat{m}_{\xi p}^{n+\frac{1}{2}} = (1 + l_{1p})M_{\xi p}^{n+\frac{1}{2}} - \hat{m}_{\xi p}^n + (l_{3p} - l_{2p})H_\xi^{n+\frac{1}{2}}, \quad \forall p, \quad \xi = x, y, z \quad (32d)$$

$$E_x^{n+1} + c_2 \frac{\partial}{\partial z} H_y^{n+1} = \hat{e}_x^{n+\frac{1}{2}} \quad (32e)$$

$$E_y^{n+1} + c_2 \frac{\partial}{\partial x} H_z^{n+1} = \hat{e}_y^{n+\frac{1}{2}} \quad (32f)$$

$$E_z^{n+1} + c_2 \frac{\partial}{\partial y} H_x^{n+1} = \hat{e}_z^{n+\frac{1}{2}} \quad (32g)$$

$$H_x^{n+1} + d_2 \frac{\partial}{\partial y} E_z^{n+1} = \hat{h}_x^{n+\frac{1}{2}} \quad (32h)$$

$$H_y^{n+1} + d_2 \frac{\partial}{\partial z} E_x^{n+1} = \hat{h}_y^{n+\frac{1}{2}} \quad (32i)$$

$$H_z^{n+1} + d_2 \frac{\partial}{\partial x} E_y^{n+1} = \hat{h}_z^{n+\frac{1}{2}} \quad (32j)$$

$$J_{\xi p}^{n+1} = k_{2p} E_{\xi}^{n+1} + \hat{j}_{\xi p}^{n+\frac{1}{2}}, \quad \forall p, \quad \xi = x, y, z \quad (32k)$$

$$M_{\xi p}^{n+1} = l_{2p} H_{\xi}^{n+1} + \hat{m}_{\xi p}^{n+\frac{1}{2}}, \quad \forall p, \quad \xi = x, y, z. \quad (32l)$$

Since J and M fields are not required in output, (31k) and (31l) can be further substituted into (32a), (32b), (32c) and (32d) for more efficiency and simplicity to yield

$$\begin{aligned} \hat{e}_{\xi}^{n+\frac{1}{2}} = & \left[1 + c_1 - c_2 \sum_p \operatorname{Re} \{ (1 + k_{1p}) k_{2p} \} \right] E_{\xi}^{n+\frac{1}{2}} - \hat{e}_{\xi}^n \\ & - c_2 \sum_p \operatorname{Re} \{ (1 + k_{1p}) \hat{j}_{\xi p}^n \} \end{aligned} \quad (33)$$

$$\begin{aligned} \hat{h}_{\xi}^{n+\frac{1}{2}} = & \left[1 + d_1 - d_2 \sum_p \operatorname{Re} \{ (1 + l_{1p}) l_{2p} \} \right] H_{\xi}^{n+\frac{1}{2}} - \hat{h}_{\xi}^n \\ & - c_2 \sum_p \operatorname{Re} \{ (1 + l_{1p}) \hat{m}_{\xi p}^n \} \end{aligned} \quad (34)$$

$$\hat{j}_{\xi p}^{n+\frac{1}{2}} = (k_{1p} k_{2p} + k_{3p}) E_{\xi}^{n+\frac{1}{2}} + k_{1p} \hat{j}_{\xi p}^n, \quad \forall p, \quad \xi = x, y, z \quad (35)$$

$$\hat{m}_{\xi p}^{n+\frac{1}{2}} = (l_{1p} l_{2p} + l_{3p}) H_{\xi}^{n+\frac{1}{2}} + l_{1p} \hat{m}_{\xi p}^n, \quad \forall p, \quad \xi = x, y, z \quad (36)$$

Similarly, (32k) and (32l) at one time step earlier can be further substituted into (31a), (31b), (31c) and (31d) to yield

$$\begin{aligned} \hat{e}_{\xi}^n = & \left[1 + c_1 - c_2 \sum_p \operatorname{Re} \{ (1 + k_{1p}) k_{2p} \} \right] E_{\xi}^n - \hat{e}_{\xi}^{n-\frac{1}{2}} \\ & - c_2 \sum_p \operatorname{Re} \{ (1 + k_{1p}) \hat{j}_{\xi p}^{n-\frac{1}{2}} \} \end{aligned} \quad (37)$$

$$\begin{aligned} \hat{h}_{\xi}^n = & \left[1 + d_1 - d_2 \sum_p \operatorname{Re} \{ (1 + l_{1p}) l_{2p} \} \right] H_{\xi}^n - \hat{h}_{\xi}^{n-\frac{1}{2}} \\ & - c_2 \sum_p \operatorname{Re} \{ (1 + l_{1p}) \hat{m}_{\xi p}^{n-\frac{1}{2}} \} \end{aligned} \quad (38)$$

$$\hat{j}_{\xi p}^n = (k_{1p} k_{2p} + k_{3p}) E_{\xi}^n + k_{1p} \hat{j}_{\xi p}^{n-\frac{1}{2}}, \quad \forall p, \quad \xi = x, y, z \quad (39)$$

$$\hat{m}_{\xi p}^n = (l_{1p} l_{2p} + l_{3p}) H_{\xi}^n + l_{1p} \hat{m}_{\xi p}^{n-\frac{1}{2}}, \quad \forall p, \quad \xi = x, y, z \quad (40)$$

To solve for E_x in the first substep implicitly, (31j) is substituted into (31e) to yield

$$E_x^{n+\frac{1}{2}} - c_2 \frac{\partial}{\partial y} (d_2 \frac{\partial}{\partial y} E_x^{n+\frac{1}{2}}) = \hat{e}_x^n - c_2 s_{ex} + c_2 \frac{\partial}{\partial y} \hat{h}_z^n. \quad (41)$$

For E_y , (31h) is substituted into (31f) and for E_z , (31i) is substituted into (31g). Other field components are solved explicitly. The procedures are then repeated for the second substep.

The advantages of the FADI-FDTD over the conventional ADI-FDTD can be summarized as follows:

1) Matrix-operator-free RHS in FADI-FDTD, c.f. (29). On the other hand, the RHS of conventional ADI-FDTD still involves matrix operators $\mathbf{A}$ and $\mathbf{B}$, c.f. (15).

2) Source excitation is only needed in first substep of FADI-FDTD, c.f. (29b), compared to both substeps of conventional ADI-FDTD, c.f. (15a) and (15b).

3) Implicit update equation of FADI-FDTD, c.f. (41) is much simpler compared to that of conventional ADI-FDTD, c.f. (14).

4) Implicit update equation of FADI-FDTD, c.f. (41) has much lesser spatial differential operators than that of conventional ADI-FDTD, c.f. (14). Thus, applying finite difference approximation to these spatial differential operators will incur less overheads in FADI-FDTD compared to conventional ADI-FDTD.

5) No dispersive terms J and M is required in implicit update equation of FADI-FDTD, c.f. (41). On the other hand, they are required in implicit update equation of conventional ADI-FDTD, c.f. (14).

6) The overall number of RHS terms in FADI-FDTD are lesser than that of conventional ADI-FDTD. This reduces the amount of arithmetic operations, memory indexing, and results in higher efficiency.

The generalized update equations of novel FADI-FDTD method in (31) and (32) are now applicable to various dispersive models as follows:

2.1 Complex Conjugate Pole-Residue Pair

For complex conjugate pole-residue pair model, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{r_p}{j\omega - a_p} + \frac{r_p^*}{j\omega - a_p^*} \quad (42a)$$

$$\chi_{h,p} = \frac{q_p}{j\omega - b_p} + \frac{q_p^*}{j\omega - b_p^*} \quad (42b)$$

where a_p and b_p are the p -th electric and magnetic poles respectively, r_p and q_p are the p -th pole electric and magnetic residues respectively.

From (1), the generalized electric and magnetic frequency domain transfer functions read:

$$\zeta_{e,p}^g(\omega) = j\omega\epsilon_0 \left(\frac{r_p}{j\omega - a_p} + \frac{r_p^*}{j\omega - a_p^*} \right) \quad (43a)$$

$$\zeta_{h,p}^g(\omega) = j\omega\mu_0 \left(\frac{q_p}{j\omega - b_p} + \frac{q_p^*}{j\omega - b_p^*} \right) \quad (43b)$$

In order to satisfy (5), $\zeta_{e,p}^g(\omega)$ and $\zeta_{h,p}^g(\omega)$ are decomposed into

$$\zeta_{e,p}(\omega) = \frac{j\omega\epsilon_0 r_p}{j\omega - a_p} \quad (44a)$$

$$\zeta'_{e,p}(\omega) = \frac{j\omega\epsilon_0 r_p^*}{j\omega - a_p^*} \quad (44b)$$

$$\zeta_{h,p}(\omega) = \frac{j\omega\mu_0 q_p}{j\omega - b_p} \quad (44c)$$

$$\zeta'_{h,p}(\omega) = \frac{j\omega\mu_0 q_p^*}{j\omega - b_p^*} \quad (44d)$$

and since (5) is true, we need only consider the relations in (3a) and (3c). $\tilde{\mathbf{J}}_p(\omega)$ and $\tilde{\mathbf{M}}_p(\omega)$ are now related to $\tilde{\mathbf{E}}(\omega)$ and $\tilde{\mathbf{H}}(\omega)$ by

$$\tilde{\mathbf{J}}_p(\omega) = \frac{j\omega\epsilon_0 r_p}{j\omega - a_p} \tilde{\mathbf{E}}(\omega) \quad (45a)$$

$$\tilde{\mathbf{M}}_p(\omega) = \frac{j\omega\mu_0 q_p}{j\omega - b_p} \tilde{\mathbf{H}}(\omega) \quad (45b)$$

(45) can then be converted into time domain as

$$\frac{\partial}{\partial t} \mathbf{J}_p(t) - a_p \mathbf{J}_p(t) = \epsilon_0 r_p \frac{\partial}{\partial t} \mathbf{E}(t) \quad (46a)$$

$$\frac{\partial}{\partial t} \mathbf{M}_p(t) - b_p \mathbf{M}_p(t) = \mu_0 q_p \frac{\partial}{\partial t} \mathbf{H}(t). \quad (46b)$$

Discretizing (46) using central difference approximation, the update coefficients in (7) and (8) is found to be

$$k_{1p} = \frac{1 + a_p \frac{\Delta t}{4}}{1 - a_p \frac{\Delta t}{4}}; \quad k_{2p} = \frac{r_p}{1 - a_p \frac{\Delta t}{4}}; \quad k_{3p} = -k_{2p}; \quad (47a)$$

$$l_{1p} = \frac{1 + b_p \frac{\Delta t}{4}}{1 - b_p \frac{\Delta t}{4}}; \quad l_{2p} = \frac{q_p}{1 - b_p \frac{\Delta t}{4}}; \quad l_{3p} = -l_{2p}; \quad (47b)$$

Once the update coefficients k_{1p} , k_{2p} , k_{3p} , l_{1p} , l_{2p} and l_{3p} are determined, the update equations of FADI-FDTD follows that of the generalized formulation detailed earlier.

2.2 Debye

For Debye model, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{\Delta\epsilon_p}{1 + j\omega\tau_{e,p}} \quad (48a)$$

$$\chi_{h,p} = \frac{\Delta\mu_p}{1 + j\omega\tau_{h,p}} \quad (48b)$$

where $\Delta\epsilon_p$ and $\Delta\mu_p$ are change in relative permittivity and permeability respectively due to the p -th Debye pole, $\tau_{e,p}$ and $\tau_{h,p}$ are the p -th pole electric and magnetic relaxation times respectively.

From (1), the generalized electric and magnetic frequency domain transfer functions read:

$$\zeta_{e,p}^g(\omega) = j\omega\epsilon_0 \left(\frac{\Delta\epsilon_p}{1 + j\omega\tau_{e,p}} \right) \quad (49a)$$

$$\zeta_{h,p}^g(\omega) = j\omega\mu_0 \left(\frac{\Delta\mu_p}{1 + j\omega\tau_{h,p}} \right) \quad (49b)$$

In order to satisfy (5), $\zeta_{e,p}^g(\omega)$ and $\zeta_{h,p}^g(\omega)$ are decomposed into

$$\zeta_{e,p}(\omega) = \frac{j\omega\epsilon_0\Delta\epsilon_p/2}{1 + j\omega\tau_{e,p}} \quad (50a)$$

$$\zeta'_{e,p}(\omega) = \frac{j\omega\epsilon_0\Delta\epsilon_p/2}{1 + j\omega\tau_{e,p}} \quad (50b)$$

$$\zeta_{h,p}(\omega) = \frac{j\omega\mu_0\Delta\mu_p/2}{1 + j\omega\tau_{h,p}} \quad (50c)$$

$$\zeta'_{h,p}(\omega) = \frac{j\omega\mu_0\Delta\mu_p/2}{1 + j\omega\tau_{h,p}} \quad (50d)$$

and since (5) is true, we need only consider the relations in (3a) and (3c). $\tilde{\mathbf{J}}_p(\omega)$ and $\tilde{\mathbf{M}}_p(\omega)$ are now related to $\tilde{\mathbf{E}}(\omega)$ and $\tilde{\mathbf{H}}(\omega)$ by

$$\tilde{\mathbf{J}}_p(\omega) = \frac{j\omega\epsilon_0\Delta\epsilon_p/2}{1 + j\omega\tau_{e,p}} \tilde{\mathbf{E}}(\omega) \quad (51a)$$

$$\tilde{\mathbf{M}}_p(\omega) = \frac{j\omega\mu_0\Delta\mu_p/2}{1 + j\omega\tau_{h,p}} \tilde{\mathbf{H}}(\omega) \quad (51b)$$

(45) can then converted into time domain as

$$\tau_{e,p} \frac{\partial}{\partial t} \mathbf{J}_p(t) + \mathbf{J}_p(t) = \frac{\epsilon_0\Delta\epsilon_p}{2} \frac{\partial}{\partial t} \mathbf{E}(t) \quad (52a)$$

$$\tau_{h,p} \frac{\partial}{\partial t} \mathbf{M}_p(t) + \mathbf{M}_p(t) = \frac{\mu_0\Delta\mu_p}{2} \frac{\partial}{\partial t} \mathbf{H}(t). \quad (52b)$$

Discretizing (46) using central difference approximation, the update coefficients in (7) and (8) is found to be

$$k_{1p} = \frac{1 - \frac{\Delta t}{4\tau_{e,p}}}{1 + \frac{\Delta t}{4\tau_{e,p}}}; \quad k_{2p} = \frac{\epsilon_0\Delta\epsilon_p/(2\tau_{e,p})}{1 + \frac{\Delta t}{4\tau_{e,p}}}; \quad k_{3p} = -k_{2p}; \quad (53a)$$

$$l_{1p} = \frac{1 - \frac{\Delta t}{4\tau_{h,p}}}{1 + \frac{\Delta t}{4\tau_{h,p}}}; \quad l_{2p} = \frac{\mu_0\Delta\mu_p/(2\tau_{h,p})}{1 + \frac{\Delta t}{4\tau_{h,p}}}; \quad l_{3p} = -l_{2p}; \quad (53b)$$

2.3 Lorentz

For Lorentz model, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{\Delta\epsilon_p \omega_{e,p}^2}{\omega_{e,p}^2 + 2j\omega\delta_{e,p} - \omega^2} \quad (54a)$$

$$\chi_{h,p} = \frac{\Delta\mu_p \omega_{h,p}^2}{\omega_{h,p}^2 + 2j\omega\delta_{h,p} - \omega^2} \quad (54b)$$

where $\Delta\epsilon_p$ and $\Delta\mu_p$ are change in relative permittivity and permeability respectively due to the p -th Lorentz pole, $\omega_{e,p}$ and $\omega_{h,p}$ are the electric and magnetic resonant frequencies of the p -th pole respectively, $\delta_{e,p}$ and $\delta_{h,p}$ are the electric and magnetic damping coefficients of the p -th pole respectively.

The Lorentz model is in fact a complex conjugate pole-residue pair model which can be rewritten as

$$\frac{\Delta\epsilon_p \omega_{e,p}^2}{\omega_{e,p}^2 + 2j\omega\delta_{e,p} - \omega^2} = \frac{j\gamma_{e,p}/2}{j\omega + \alpha_{e,p} + j\beta_{e,p}} + \frac{-j\gamma_{e,p}/2}{j\omega + \alpha_{e,p} - j\beta_{e,p}} \quad (55a)$$

$$\frac{\Delta\mu_p \omega_{h,p}^2}{\omega_{h,p}^2 + 2j\omega\delta_{h,p} - \omega^2} = \frac{j\gamma_{h,p}/2}{j\omega + \alpha_{h,p} + j\beta_{h,p}} + \frac{-j\gamma_{h,p}/2}{j\omega + \alpha_{h,p} - j\beta_{h,p}} \quad (55b)$$

where

$$\alpha_{e,p} = \delta_{e,p}; \quad \beta_{e,p} = \sqrt{\omega_{e,p}}; \quad \gamma_{e,p} = \frac{\Delta\epsilon_p \omega_{e,p}^2}{\beta_{e,p}}; \quad (56a)$$

$$\alpha_{h,p} = \delta_{h,p}; \quad \beta_{h,p} = \sqrt{\omega_{h,p}}; \quad \gamma_{h,p} = \frac{\Delta\mu_p \omega_{h,p}^2}{\beta_{h,p}}. \quad (56b)$$

Comparing (55) to (42), we find that

$$a_p = -\alpha_{e,p} - j\beta_{e,p}; \quad r_p = j\gamma_{e,p}/2; \quad (57a)$$

$$b_p = -\alpha_{h,p} - j\beta_{h,p}; \quad q_p = j\gamma_{h,p}/2; \quad (57b)$$

The update coefficients k_{1p} , k_{2p} , k_{3p} , l_{1p} , l_{2p} and l_{3p} can then be determined from (47) by replacing the poles and residues with the expressions in (57).

2.4 Drude

For Drude model, the frequency domain electric and magnetic susceptibility functions of the p -th pole are given as

$$\chi_{e,p} = \frac{-\omega_{e,p}^2}{\omega^2 - j\omega\nu_{e,p}} \quad (58a)$$

$$\chi_{h,p} = \frac{-\omega_{h,p}^2}{\omega^2 - j\omega\nu_{h,p}} \quad (58b)$$

where $\omega_{e,p}$ and $\omega_{h,p}$ are the p -th Drude pole electric and magnetic frequencies respectively. $\nu_{e,p}$ and $\nu_{h,p}$ are the inverse of the p -th pole electric and magnetic relaxation times respectively.

From (1), the generalized electric and magnetic frequency domain transfer functions read:

$$\begin{aligned}\zeta_{e,p}^g(\omega) &= j\omega\epsilon_0 \left(\frac{-\omega_{e,p}^2}{\omega^2 - j\omega\nu_{e,p}} \right) \\ &= \frac{\epsilon_0\omega_{e,p}^2}{j\omega + \nu_{e,p}}\end{aligned}\tag{59a}$$

$$\begin{aligned}\zeta_{h,p}^g(\omega) &= j\omega\mu_0 \left(\frac{-\omega_{h,p}^2}{\omega^2 - j\omega\nu_{h,p}} \right) \\ &= \frac{\epsilon_0\omega_{h,p}^2}{j\omega + \nu_{h,p}}\end{aligned}\tag{59b}$$

In order to satisfy (5), $\zeta_{e,p}^g(\omega)$ and $\zeta_{h,p}^g(\omega)$ are decomposed into

$$\zeta_{e,p}(\omega) = \frac{\epsilon_0\omega_{e,p}^2/2}{j\omega + \nu_{e,p}}\tag{60a}$$

$$\zeta'_{e,p}(\omega) = \frac{\epsilon_0\omega_{e,p}^2/2}{j\omega + \nu_{e,p}}\tag{60b}$$

$$\zeta_{h,p}(\omega) = \frac{\epsilon_0\omega_{h,p}^2/2}{j\omega + \nu_{h,p}}\tag{60c}$$

$$\zeta'_{h,p}(\omega) = \frac{\epsilon_0\omega_{h,p}^2/2}{j\omega + \nu_{h,p}}\tag{60d}$$

and since (5) is true, we need only consider the relations in (3a) and (3c). $\tilde{\mathbf{J}}_p(\omega)$ and $\tilde{\mathbf{M}}_p(\omega)$ are now related to $\tilde{\mathbf{E}}(\omega)$ and $\tilde{\mathbf{H}}(\omega)$ by

$$\tilde{\mathbf{J}}_p(\omega) = \frac{\epsilon_0\omega_{e,p}^2/2}{j\omega + \nu_{e,p}}\tilde{\mathbf{E}}(\omega)\tag{61a}$$

$$\tilde{\mathbf{M}}_p(\omega) = \frac{\epsilon_0\omega_{h,p}^2/2}{j\omega + \nu_{h,p}}\tilde{\mathbf{H}}(\omega)\tag{61b}$$

(45) can then converted into time domain as

$$\frac{\partial}{\partial t}\mathbf{J}_p(t) + \nu_{e,p}\mathbf{J}_p(t) = \frac{\epsilon_0\omega_{e,p}^2}{2}\mathbf{E}(t)\tag{62a}$$

$$\frac{\partial}{\partial t}\mathbf{M}_p(t) + \nu_{h,p}\mathbf{M}_p(t) = \frac{\mu_0\omega_{h,p}^2}{2}\mathbf{H}(t).\tag{62b}$$

Discretizing (46) using central difference approximation, the update coefficients in (7) and (8) is found to be

$$k_{1p} = \frac{1 - \frac{\nu_{e,p}\Delta t}{4}}{1 + \frac{\nu_{e,p}\Delta t}{4}}; \quad k_{2p} = \frac{\epsilon_0\omega_{e,p}^2\Delta t/8}{1 + \frac{\nu_{e,p}\Delta t}{4}}; \quad k_{3p} = k_{2p};\tag{63a}$$

$$l_{1p} = \frac{1 - \frac{\nu_{h,p}\Delta t}{4}}{1 + \frac{\nu_{h,p}\Delta t}{4}}; \quad l_{2p} = \frac{\mu_0\omega_{h,p}^2\Delta t/8}{1 + \frac{\nu_{h,p}\Delta t}{4}}; \quad l_{3p} = l_{2p}.\tag{63b}$$

Figure 1: Real, ϵ_{rr} and imaginary, ϵ_{ri} parts of the complex relative permittivity for metal Au (gold): Drude model and experimental data.

3 Application Examples of FADI-FDTD Method in Analysis of Optical Waveguide

Optical waveguide exhibiting plasmonic resonances [19–21] has attracted much interest recently as it allows guiding of light propagation in subwavelength region. The sub-wavelength light propagation in optical waveguide results in highly refined grids which undesirably prolong the simulation time of the Yee’s explicit FDTD method due to the CFL condition. Therefore, the FADI-FDTD method is a suitable candidate for such numerical experiment as it is very efficient and has unconditionally stable feature. In this section, we shall conduct the analysis of optical waveguide using FADI-FDTD method. The metal cladding in the optical waveguide is modeled using a combination of Drude-Lorentz dispersive model. The transmission coefficient of the optical waveguide is determined for photon energy range of 1.24 eV to 2.48 eV (wavelength 500 nm to 1000 nm). Two types of metal are considered for the cladding, which are Au (gold) and Ag (silver). On the other hand, air and SiO_2 are used separately as the material for the core.

The Drude model is commonly used to describe the complex relative permittivity for metals. The complex relative permittivity of multiple poles Drude model is expressed as

$$\epsilon_r(\omega) = \epsilon_\infty + \sum_p \frac{-\omega_{D,p}^2}{\omega^2 - j\omega\nu_p}. \quad (64)$$

In [22], the author first tried to fit the complex relative permittivity of metal Au (gold) with a single pole Drude model using the data tabulated from [23] for photon energy between 1.24 eV (wavelength 1000 nm) and 2.48 eV (500 nm). The relevant parameters of the Drude model is listed in the first row of Table 1. The real, ϵ_{rr} and imaginary, ϵ_{ri} parts of the complex relative permittivity are plotted in Fig. 1, along with the tabulated data. We first note that the imaginary part is negative for losses and absorption in

Figure 2: Real, ϵ_{rr} and imaginary, ϵ_{ri} parts of the complex relative permittivity for metal Au (gold): Drude-Lorentz model and experimental data.

Figure 3: Real, ϵ_{rr} and imaginary, ϵ_{ri} parts of the complex relative permittivity for metal Ag (silver): Drude-Lorentz model and experimental data.

metal. The real part is also negative for most metals at frequency below the plasma frequency. It can be seen that the Drude model does not fit the complex permittivity data accurately above energy level of 2 eV (600 nm).

To overcome such inaccuracy, additional Lorentz pole pairs can be added to take into account the interband transitions. In fact, the complex relative permittivity of several types of metal at optical frequency can be described more accurately by a combination of Drude-Lorentz model [24]. A Drude-Lorentz model with multiple poles are expressed as

$$\epsilon_r(\omega) = \epsilon_\infty + \sum_p \frac{-\omega_{D,p}^2}{\omega^2 - j\omega\nu_p} + \sum_p \frac{\Delta\epsilon_p\omega_{L,p}^2}{\omega_{L,p}^2 + 2j\omega\delta_p - \omega^2}. \quad (65)$$

Table 1: Parameters of Drude and Drude-Lorentz models for metal Au (gold)

Au	ϵ_∞	$\omega_{D,1}(\text{rad/s})$	$\nu_1(\text{rad/s})$	$\Delta\epsilon_1$	$\omega_{L,1}(\text{rad/s})$	$\delta_1(\text{rad/s})$
Drude	9.0685	1.3544×10^{16}	1.1536×10^{14}	-	-	-
Drude-Lorentz	5.9673	1.3280×10^{16}	1.0003×10^{14}	1.09	4.0845×10^{15}	3.2943×10^{14}

Table 2: Parameters of Drude-Lorentz model for metal Ag (silver)

Ag	ϵ_∞	$\omega_{D,1}(\text{rad/s})$	$\nu_1(\text{rad/s})$	$\Delta\epsilon_p(p=1,2)$	$\omega_{L,p}(p=1,2)(\text{rad/s})$	$\delta_p(p=1,2)(\text{rad/s})$
Drude-Lorentz	1	1.2583×10^{16}	7.2923×10^{13}	5.9245	1.2397×10^{15}	0.3519×10^{15}
	-	-	-	1.4013	6.8078×10^{15}	1.4334×10^{14}

The authors in [22] fitted the complex relative permittivity of metal Au (gold) with a combination of one Drude pole and one Lorentz pole pair for photon energy range between 1.24 eV and 2.48 eV. The relevant parameters of Drude-Lorentz model for metal Au are listed in the second row of Table 1. Fig. 2 now plots the real, ϵ_{rr} and imaginary, ϵ_{ri} parts of the complex relative permittivity for metal Au using Drude-Lorentz model, along with the same tabulated data in [23]. It can be seen now that the Drude-Lorentz model fits the data accurately beyond energy level of 2 eV, which validates the accuracy of the Drude-Lorentz model.

We next consider the complex relative permittivity of metal Ag (silver). [24] provided the fitting of complex relative permittivity for 11 types of metal, including Ag using a combination of single Drude pole and multiple Lorentz pole pairs for a wide photon energy range from 0.1 eV to 5 eV. The compilation of complex relative permittivity data are obtained from [25]. For a shorter energy range between 1.24 eV and 2.48 eV, we manage to reduce the order of the Drude-Lorentz model from one Drude pole and five Lorentz pole pairs in [24] to one Drude pole and two Lorentz pole pairs. Such reduced order of Drude-Lorentz model is more desirable to be used in FADI-FDTD method for simplicity and efficiency. The relevant parameters of Drude-Lorentz model for metal Ag are listed in Table 2. Fig. 3 shows the the real, ϵ_{rr} and imaginary parts ϵ_{ri} of the complex relative permittivity for metal Ag using Drude-Lorentz model, along with the tabulated data in [25]. It can be seen that our reduced order of Drude-Lorentz model fits the data well within photon energy of 1.24 eV and 2.48 eV.

For numerical demonstrations, we consider a 2-D optical waveguide with length of 10 μm and width of 0.4 μm where a core is confined between two metal claddings. The core is air (free space) having width of 0.2 μm and the two metal claddings are made of Au (gold) having Drude-Lorentz model parameters in Table 1. A transverse magnetic (TM) modulated Gaussian pulse with wavelengths 500 nm to 1000 nm (photon energy 1.24 eV to 2.48 eV) is excited in the core region and the pulse is allowed to propagate along the waveguide. Cell size Δ is chosen as 0.005 μm which corresponds to CPW of 100 at 5 μm and the time step Δt is specified as CFLN defined earlier. In this study, we compute the transmission coefficient of the optical waveguide by taking the ratio of Fourier transformed fields observed between two different points at the core centre separated by 5 μm along the waveguide length. The geometrical structure of the optical waveguide is adopted from [27]. The numerical experiment is run for various CFLNs to validate the accuracy FADI-FDTD method under different time steps.

Fig. 4 shows the magnitude of transmission coefficient of the optical waveguide from photon energy 1.24 eV to 2.48 eV for various CFLNs. Also shown in the figure is the

Figure 4: Magnitude of transmission coefficient of the optical waveguide from photon energy 1.24 eV to 2.48 eV for various CFLNs.

Table 3: Efficiency gain and relative error of FADI-FDTD method for various CFLNs

CFLN	2	4	6	8	10
Efficiency gain	1.67	3.33	5.00	6.66	8.33
Relative error	0.29%	0.59%	1.18%	1.91%	2.86%

Table 4: Relative error of efficient FADI-FDTD schemes for various CFLNs and CPWs

CFLN		2	4	6	8	10
CPW	50	0.37%	1.71%	4.04%	7.51%	12.54%
	100	0.29%	0.59%	1.18%	1.91%	2.86%

transmission coefficient computed using the Yee's explicit FDTD method to serve as reference. Since the Yee's explicit FDTD method is not unconditionally stable, it is run at its maximum time step limit of CFLN=1. The results are quite agreeable and close to each other. As the CFLN increases, the magnitude of transmission coefficient computed using FADI-FDTD method deviates further away from the reference. To illustrate the trade-off between accuracy and efficiency, Table 3 shows the efficiency gain and relative error of FADI-FDTD method. The efficiency gain is taken over the Yee's explicit FDTD and the relative error is computed as

$$err = \left| \frac{T_{FADI} - T_{explicit}}{T_{explicit}} \right| \quad (66)$$

where T_{FADI} and $T_{explicit}$ are the magnitude of transmission coefficients computed using FADI-FDTD and Yee's explicit FDTD methods respectively. The transmission coefficient is computed at centre photon energy of 1.9 eV. It can be seen that at CFLN=10, the FADI-FDTD method is 8.33 times faster than the Yee's explicit FDTD method at the expense of only 2.35 percent relative error. Table 4 further shows the relative error of FADI-FDTD method for various CFLNs under different CPWs. It can be seen that for the same CFLN but at a lower CPW of 50, the relative error increases.

Figure 5: Magnitude of transmission coefficient of the optical waveguide whereby its gold metal cladding is modeled separately as Drude-Lorentz and Drude model.

For comparison, Fig. 5 plots the the magnitude of transmission coefficient of the optical waveguide whereby its gold metal cladding is modeled separately as Drude-Lorentz and Drude model. The numerical experiment is conducted at CFLN=8. Significant difference of transmission coefficient magnitude can be seen above photon energy of 1.8 eV. This further indicates the inaccuracy of Drude model for gold. The Drude-Lorentz model should be employed for analysis of optical waveguide with gold as its metal cladding.

Previously we have considered Au-air-Au interface for the optical waveguide. We further investigate the transmission response of the optical waveguide using different metal and material types as its cladding and core. We first consider Ag-air-Ag interface, Fig. 6 plots the magnitude of transmission coefficients of optical waveguide with Ag-air-Ag interface and compares it with Au-air-Au interface. Ag (silver) is modeled as Drude-Lorentz model having parameters from Table 2. The numerical experiment is again conducted at CFLN=8. It can be seen that for Au-air-Au interface, no electromagnetic wave with photon energy above 2.4 eV can propagate beyond 5 μm of the optical waveguide, indicated by its zero transmission coefficient for photon energy above 2.4 eV. For Ag-air-Ag interface, the magnitude of transmission coefficient is relatively higher throughout the photon energy from 1.24 eV to 2.48 eV, which indicates that most electromagnetic wave can propagate through the optical waveguide. Finally, we replace the core in the optical waveguide with silicon dioxide (SiO_2) having constant refractive index of 1.444 [13]. Fig. 7 plots the magnitude of transmission coefficients of the optical waveguide with Au- SiO_2 -Au and Au-air-Au interfaces. We can see that for Au- SiO_2 -Au interface, the magnitude of transmission coefficient is reduced compared to that of Au-air-Au interface. No electromagnetic wave with photon energy above 2.2 eV can now propagate beyond 5 μm of the optical waveguide.

We next proceed to analyze plasmonic waveguide grating filter [26, 28], which is formed by periodic width variation of insulator as its core, and encapsulated by metal as its cladding. It is one of the basic building blocks for small size plasmonic circuit. In the following, various geometrical configurations of the plasmonic filters shall be adopted

Figure 6: Magnitude of transmission coefficients of the optical waveguide with Au-air-Au and Ag-air-Ag interfaces.

Figure 7: Magnitude of transmission coefficients of the optical waveguide with Au-air-Au and Au-SiO₂-Au interfaces.

from [28]. Fig. 8 shows the configurations of conventional plasmonic waveguide grating filter. The dimensions are: $W_1 = 0.1\mu m$, $W_2 = 0.2\mu m$, $L_p = 0.66\mu m$, $L_s = 0.292\mu m$. The number of periods is chosen as 19. We shall analyze such plasmonic waveguide grating filter in the wavelength region of $1.2\mu m$ to $2.4\mu m$, which corresponds to photon energy of 0.52 eV to 1.1 eV. The metal is chosen as Ag in which a Drude model is sufficient for the aforementioned photon energy range, having parameters: $\epsilon_\infty = 3.7$, $\omega_{D,1} = 1.3825 \times 10^{16}$ rad/s and $\nu_1 = 2.7347 \times 10^{13}$ rad/s. A TM modulated Gaussian pulse with the aforementioned photon energy range is excited in the core region and the pulse is allowed to propagate along the waveguide. Nonuniform cell size is chosen where $\Delta z = 0.004\mu m$ and $\Delta x = 0.005\mu m$. CFLN is set at 4. Fig. 9 shows the transmission coefficient of the conventional plasmonic waveguide grating filter from photon energy of 0.52 eV to 1.1 eV. For comparison, we also present the results obtained by Yee's explicit

Figure 8: Configurations of conventional plasmonic waveguide grating filter.

Figure 9: Magnitude of transmission coefficient of conventional plasmonic waveguide grating filter from photon energy 0.52 eV to 1.1 eV.

FDTD method. It can be seen that both results agree with each other satisfactorily and the FADI-FDTD method is thus validated. From Fig. 9, it is observed that with such configuration, the bandgap energy starts around 0.65 eV and ends around 0.9 eV. Considerable high sidelobes and ripples can be seen at the passband region.

To suppress the sidelobes level, it is proposed in [28] that apodization be introduced into the plasmonic waveguide grating filter. Fig. 10 shows the configurations of the apodized plasmonic waveguide grating filter, where linear apodization is applied at both input and output ports. CFLN is chosen at 10 henceforth for greater efficiency. Fig. 11 now plots the transmission coefficients of the apodized filter with linear apodization applied to 5 and 9 periods at both input and output ports. With apodization, we can see that the sidelobes level are suppressed and reduced where a smoother passband is depicted. Nevertheless, this comes at the expense of a narrower bandgap. Generally, higher number of periods will contribute to lower sidelobes level with narrower bandgap as can be seen from the comparison between apodized filter with 5 and 9 periods. Apart from apodization, the use of chirped grating can also achieve the same objective in

Figure 10: Configurations of apodized plasmonic waveguide grating filter.

Figure 11: Magnitude of transmission coefficient of apodized plasmonic waveguide grating filter from photon energy 0.52 eV to 1.1 eV.

Figure 12: Configurations of chirped plasmonic waveguide grating filter.

surpressing the sidelobes level. The configuration of a typical linear chirped plasmonic waveguide grating filter is shown in Fig. 12. Fig. 13 plots the transmission coefficients

Figure 13: Magnitude of transmission coefficient of chirped plasmonic waveguide grating filter from photon energy 0.52 eV to 1.1 eV.

Figure 14: Configurations of conventional plasmonic microcavity.

of the chirped filter with linear chirped applied to 5 and 9 periods at both input and output ports. We can also see that the sidelobes level can be suppressed and reduced. Similarly, higher number of periods will contribute to lower sidelobes level with narrower bandgap.

Finally, we consider a plasmonic microcavity in which a defect section is introduced in the middle section of the waveguide, both conventional and apodized, as depicted in Fig. 14 and Fig. 15, respectively. The defect length is chosen as $L = 0.688\mu m$ and the number of gratings is 18. Fig. 16 now shows the transmission coefficient of the conventional plasmonic microcavity, as well as the apodized microcavity in 6 and 8 periods. When a defect is introduced in the middle section of the waveguide, a transmission peak can be observed at its bandgap at around photon energy 0.78 eV. Sidelobes level suppression can also be achieved by apodization in the microcavity. As the number of period increases, the sidelobes level decreases at the expense of narrower bandgap. At the same time, the transmission peak at 0.78 eV increases as the apodization period increases.

Figure 15: Configurations of apodized plasmonic microcavity.

Figure 16: Magnitude of transmission coefficient of apodized plasmonic microcavity from photon energy 0.52 eV to 1.1 eV.

4 Conclusions

This chapter has presented the formulation of novel unconditionally stable FADI-FDTD method for dispersive media. A generalized formulation has been provided, which is applicable for various dispersive models such as Debye, Lorentz, Drude and complex conjugate pole-residue pair models. The extension for full 3D dispersive media using novel FADI-FDTD method has made the resultant update equations much more concise and simpler than using conventional ADI-FDTD method. To demonstrate the application of novel FADI-FDTD method, the analysis of plasmonic waveguide using FADI-FDTD method has been provided. The characteristics of a surface plasmon waveguides with Au (gold) and Ag (silver) metal cladding, modeled as combination of Drude-Lorentz dispersive media have been analyzed. Further analysis of plasmonic waveguide grating filter has also been considered.

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