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Pricing and Prioritization in a Duopoly with Self-Selecting, Heterogeneous, Time-Sensitive Customers under Low Utilization

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Time is often used as a differentiating factor in several service operations contexts by service providers (SPs) who prioritize their customers. We use a three-stage game to investigate the competition between two SPs providing service with relatively low utilization to *impatient* and *patient* customers. In the first stage, the SPs decide whether to offer *single service* in which customers are seen on a first-come-first-serve basis, or *differentiated service* with prioritization. In the second stage, they set their prices. In the third stage, customers self-select and make their purchase decisions. We use a novel approach to model customers' self-selection through an optimization problem with appropriate individual rationality (IR) and incentive compatibility (IC) conditions. In the pricing sub-game, we focus on scenarios in which all customers get a positive net utility from using the service. We analyze the following three kinds of service delivery by the SPs, and we characterize different types of equilibrium associated with them: (i) $\mathcal{SS}$ in which both SPs provide single service, (ii) $\mathcal{SD}$ in which one SP provides single service and the other SP provides differentiated service, and (iii) $\mathcal{DD}$ in which both of them provide differentiated service. We then use these results to determine the overall equilibrium. We derive conditions (on customer heterogeneity and the fraction of impatient customers) for $\mathcal{SS}$, $\mathcal{SD}$, or $\mathcal{DD}$ to result in the overall equilibrium.

Subject Classifications: Games/group decisions: Non-cooperative. Marketing: Pricing. Queues: Priority.

Area of Review: Operations and Supply Chains

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Keywords: service operations, duopoly, incentive compatibility, heterogeneity, prioritization

1. Introduction

Differentiated service offerings, with prioritization, are often used by service providers (SPs) to differentiate customers who are heterogeneous in their delay sensitivities. Some examples include (i) amusement/theme parks which allow customers to *jump queues* by purchasing high-priced priority tickets and have faced some backlash from the practice (Wallop 2010), (ii) *concierge medicine* in which physicians/general practitioners provide faster service and patients pay a premium in comparison to regular primary care (e.g., see Wieczner (2013) and Gavirneni and Kulkarni (2016)), and (iii) restaurants charging customers additional fees upfront when they make reservations which enable them to avoid waiting for tables later (e.g., see Knowledge@Wharton (2013) and Markovich (2014) who discuss about reservation fees). *However, will offering differentiated service instead of single service, in which all customers are seen on a first-come-first-serve (FCFS) basis, be the equilibrium even in the presence of competition with self-selecting, heterogeneous customers?* That is the main question we attempt to answer.

We consider a duopoly, involving a three-stage game, selling to *impatient* and *patient* customers. The first stage is the *service mode game* in which each SP decides whether to offer single or differentiated service. The second stage is the *pricing sub-game* in which the SPs set their prices.

The third stage is the *customer sub-game* in which customers self-select and make their choices based on the prices charged by the SPs and the delay cost they would expect to incur. Figure 1 illustrates the time-line of events involving the SPs and customers. We analyze the three-stage game because service mode is typically a longer term decision compared to price(s) that can be varied more easily. Nonetheless, we briefly discuss in §8 what happens in a two-stage game.

Figure 1: Time-line of events in the three-stage game

2. Literature Review

This research is related to the literature on queuing optimization and equilibrium consumer behavior in queuing systems. Hassin and Haviv (2003) examine this literature in detail. Among the research that they review, this work is closely related to Lederer and Li (1997). The model here is similar to that in Lederer and Li (1997) who also consider time-sensitive customers and price-plus-delay competition between firms. However, they assume that the number of firms is so large that the “full price” (price plus waiting cost) at equilibrium does not depend on the prices charged by the firms. We do not make this assumption, which fundamentally alters the nature of the problem and leads to different results. An important implication is as follows. Lederer and Li (1997) mainly consider (preemptive) priority queues because they are optimal for the firms in their setting. However, we find that there are scenarios in which prioritization is sub-optimal because (i) price charged by a firm affects the full price in our model so the firm decides it strategically, and (ii) customers self-select. Armony and Haviv (2003) are also closely related to this work because they consider a duopoly facing demand from two customer types with different delay sensitivities. However, they assume that all customers are served on a first-come-first-served basis, and do not consider priority queues. In this regard, whether a service provider should prioritize is a key question we analyze here, and which neither Lederer and Li (1997) nor Armony and Haviv (2003) consider.

Other related research works have considered consumer behavior and pricing in the context of service operations management. Luski (1976) was one of the pioneers to consider price-plus-delay competition in a duopoly and finds scenarios when identical firms charge different prices at

equilibrium. However, he does not consider prioritization of customers by service providers. Li and Lee (1994) also consider a similar setting but with non-identical firms and possible jockeying among customers, and show that the faster firm always charges a higher price at equilibrium. Li et al. (2012) extend the analysis to naive customers who do not know the firms' service rates but observe their queue lengths and use that information to make the decisions. Chen and Wan (2003) study the competition between two make-to-order firms in which the firms may provide service with different value and customers may have firm-dependent waiting costs. Some research papers have looked at pricing along with priority queues and/or customer heterogeneity. Rao and Peterson (1998) analyze the pricing of priority services for a service center with a fixed finite number of customers who maximize their own profits. Zhang et al. (2007) consider the optimal pricing of communication services for customers with different service valuations but same delay sensitivity. Afeche (2013) and Katta and Sethuraman (2005) model heterogeneity in both service valuation and delay sensitivity among customers. They optimize the service provider's revenue subject to individual rationality (IR) and incentive compatibility (IC) conditions among different customer classes. However, they do not analyze competition between service providers. Hsu et al. (2009) examine optimal pricing and scheduling subject to IC conditions in the presence of different quality-of-service guarantees for multiple customer classes, and show that randomized prioritization may be optimal. Afeche et al. (2013) model risk aversion in customers and analyze how time-sensitive services should then be priced. Pangburn and Stavroulaki (2008) investigate joint pricing and capacity decisions for a firm facing heterogeneous customers who are dispersed among different locations. It has to decide between offering segmented and pooled services. Segmented service examined by them is different from prioritized service we consider in one key aspect: there is no loss in economies of scale from prioritization because the entire capacity is used. Furthermore, they do not model competition. As in this work, Sainathan (2018) also compares differentiated service with *single service*. However, he considers an SP offering ancillary service in which all the customers have to be satisfied while minimizing the cost of service provision.

Some research works have examined prioritization under different kinds of customers, but without considering competition between service providers. Mendelson and Whang (1990) is a seminal paper that derives an optimal incentive compatible pricing mechanism for the $M/M/1$ queue in the presence of self-selecting, heterogeneous customers. Van Mieghem (2000) studies such pricing schemes when generalized $c\mu$ ($Gc\mu$) scheduling is used. Afeche and Mendelson (2004) derive and compare the revenue-maximizing and socially optimal equilibria under uniform pricing, preemptive, and non-preemptive priority auctions with an admission price. Dobson and Sainathan (2011)

compare prioritized and FCFS queues when prioritization, in the form of triaging by nurses who determine patients' acuity of illness, costs both time and money. Hassin and Haviv (1997) analyze equilibrium threshold strategic in which a customers decides whether to join high- or low-priority queue based on whether the total queue size is above/below a certain threshold value. Hassin and Haviv (2006) consider the question of who should be given priority. In their context, customers are of two types, they do not observe the queue length, and the service provider knows each customer's type. Furthermore, the service provider is restricted in terms of how she prices the service.

A central feature of this work is how we model customer self-selection by considering appropriate formulation of optimization problems involving corresponding individual rationality (IR) and incentive compatibility (IC) conditions (see the mathematical formulations of $\mathcal{CSS}$, $\mathcal{CSD}$, and $\mathcal{CDD}$ in §'s 4-6). Some research papers have analyzed optimal pricing in the context of service operations while ignoring IC constraints either because they consider aggregate demand (instead of individual customer choices) or they assume that the firm allocates customers who do not self-select. Some examples include So and Song (1998), Boyaci and Ray (2003), Allon and Federgruen (2009), and Anand et al. (2011).

Gibbens et al. (2000) comes close to our work because they also analyze whether two competing service providers should offer single/multiple service classes to customers having different delay sensitivities. However, there are some important differences. First, Gibbens et al. (2000) model heterogeneity by considering *infinite* types of customers, with delay sensitivities uniformly distributed between 0 and 1, while we consider *impatient* and *patient customers*. This difference makes customer self-selection in their paper more straightforward. Specifically, the purchase fractions in their paper are characterized more directly as solution(s) to one/more quadratic equations while they have to be found by solving optimization problems in this article (see $\mathcal{CSS}$, $\mathcal{CSD}$, and $\mathcal{CDD}$ in §'s 4-6). Second, Gibbens et al. (2000) do not consider prioritization, with high- and low-priority service classes, in which the fraction of customers choosing the high-priority service *not only affects its delay but also the delay of low-priority service*. Instead, they assume that the capacity gets equally divided between two service classes. Importantly, these modeling differences lead to significantly different results: while Gibbens et al. (2000) find that differentiated service is impossible under competition, we show that both $\mathcal{SD}$ and $\mathcal{DD}$ can be overall equilibrium. Our research work complements Gibbens et al. (2000).

To the best of our knowledge, this work is the first research to view *prioritization* as a service provider's *strategic choice* and consider it in the presence of competition. Most prior research either

take prioritization for granted (because it obviously performs better) or do not consider it at all¹. Indeed, Afeche (2013) mentions (in page 442) that incentive compatible and revenue-maximizing pricing under competition is only partially understood. Our work compares differentiated service (through prioritization) with single service, and analyzes how two key aspects, *heterogeneity in customers' delay sensitivities* and *competition between service providers*, affect an SP's decision whether to prioritize.

3. Model

Two identical service providers (SP 1 and SP 2) compete in selling a service to *impatient* (Type 1) and *patient* (Type 2) customers with *delay sensitivities* of c_1 and c_2 ($c_1 > c_2 > 0$) respectively. We use the ratio c_1/c_2 as a measure of (customer) *heterogeneity*. We assume $c_1 > 2c_2$ which simplifies the analysis and ensures there is sufficient heterogeneity between impatient and patient customers. The fraction of impatient (patient) customers is q ($1 - q$). All customers have the same valuation v for the service. They incur delay costs due to the time taken to get the service. Each SP either provides a single service class or differentiates her service by providing two service classes, high-priority (h) and low-priority (l). When SP i differentiates her service, we let δ_{ijk} denote the purchase fraction at SP i from Type j customers for service class k ($i, j = 1, 2; k = l, h$). For notational brevity, we slightly abuse the notation by letting δ_{ij} and δ_i denote the total purchase fraction at SP i from Type j customers and the total purchase fraction at SP i (from both customer types). Note that the purchase fractions are relative to the total market size which is normalized to one without loss of generality. We also let p_{ik} and p_i denote the prices charged by SP i under differentiated and single service respectively. We note that the purchase fractions are neither exogenous nor decided directly by an SP but they are instead determined endogenously by customer behavior and their *self-selection*. Next, we describe how delay costs are related to purchase fractions and how customers make their purchasing decisions.

The delay costs of a Type j customer purchasing single service, high-priority service, and low-priority service from SP i are given by $c_j(1 + \lambda\delta_i)$, $c_j(1 + \lambda\delta_{ih})$, and $c_j(1 + 2\lambda\delta_{ih} + \lambda\delta_{il})$ respectively, in which λ is a measure of the market size/congestion intensity. These expressions result from linear approximations of corresponding waiting times for FCFS queue, and high-priority and low-priority queues with preemptive priority, in an $M/M/1$ system, in which the arrival rate is λ and the service

¹Sainathan (2018) does compare prioritized and single services while Pangburn and Stavroulakis (2008) compare segmented and pooled services. However, as we explain above, their settings are different. Importantly, Sainathan (2018) analyzes ancillary services in which cost is minimized and single service becomes optimal due to low capacity, while segmented service analyzed in Pangburn and Stavroulakis (2008) is different from prioritized service here.

rate/capacity is normalized to one. We use delay costs which are linear in the purchase fractions, for better analytical tractability. We note that prior research works have also used such linear costs while examining congestion/negative externalities (e.g., see Gibbens et al. (2000) and Ahlin and Ahlin (2013)). Furthermore, our primary aim is to understand the nature of the competition (and resulting equilibria) between the SPs in this context instead of analyzing the queuing dynamics in detail. In this regard, the linear approximations and our analyses are better applicable when the utilization, given by λ , is relatively low, which is the main focus of this article. In §8, we discuss qualitatively how the results might change under high utilization.

We define a customer's anticipated *dis-utility* from a purchasing choice as the sum of its price and the delay cost he expects to incur, and his *net utility* from this choice as valuation v minus dis-utility. He selects the choice with the highest positive net utility; if all the choices result in non-positive net utilities, he does not purchase anything. Next, we describe how the SPs make their decisions and compete in this context.

The competition between the SPs involves three stages: *service mode game*, *pricing sub-game*, and *customer sub-game*. There are three possible outcomes of the service mode game: (i) both SPs provide single service ($\mathcal{SS}$), (ii) one of them provides single service while the other provides differentiated service ($\mathcal{SD}$), and (iii) both SPs provide differentiated service ($\mathcal{DD}$). Next, we analyze each of these outcomes. Although we provide the optimization formulations for customer sub-games, we mainly study the pricing sub-game equilibria. Furthermore, we only consider equilibria in which *the entire market is covered and customers get positive net utilities*, for the sake better analytical tractability and easier exposition. We later provide sufficient conditions for their existence.

4. $\mathcal{SS}$: Single Service by Both SPs

SP 1 and SP 2 charge prices p_1 and p_2 respectively. The equilibrium involving customer sub-game can then be formulated as the following optimization problem.

$$(\mathcal{CSS}) \quad \max_{\delta_{ij}; i, j=1,2} \sum_{i=1}^2 \sum_{j=1}^2 \delta_{ij}$$

$$s.t. \quad \delta_{ij} (p_i + c_j(1 + \lambda(\delta_{i1} + \delta_{i2}))) \leq \delta_{ij} (p_m + c_j(1 + \lambda(\delta_{m1} + \delta_{m2}))) ; i, j, m = 1, 2; i \neq m \quad (1)$$

$$\delta_{ij} (p_i + c_j(1 + \lambda(\delta_{i1} + \delta_{i2}))) \leq \delta_{ij} v \quad (2)$$

$$\delta_{11} + \delta_{21} \leq q; \delta_{12} + \delta_{22} \leq 1 - q; \delta_{ij} \geq 0; i, j = 1, 2. \quad (3)$$

Constraints in (1) and (2), respectively, imply that² if any customer purchases from a SP then

²Constraints in (3) are logical constraints

(i) he is better off than purchasing from the other SP and (ii) his dis-utility does not exceed the valuation for the service v . Thus, these are the incentive compatibility (IC) and individual rationality (IR) conditions respectively (for Type j customers). The objective is to maximize the total purchase fraction from both SPs. It ensures that customers will purchase until either all of them are satisfied, or both purchasing from SP 1 and SP 2 result in non-positive utilities. The optimal purchase fractions can be seen as an equilibrium outcome of a *self-selection game* in which customers of Type j ($j = 1, 2$) get a net utility of $v - p_i - c_j(1 + \delta_i)$ for purchasing from SP i ($i = 1, 2$) and a zero net utility for not buying anything. Hence, we denote these purchase fractions with superscript e . In the Appendix, we show that (CSS) results in unique δ_1^e and δ_2^e for any given prices. Hence, the profits³ of SP 1 and SP 2 given by $\lambda\delta_1^e p_1$ and $\lambda\delta_2^e p_2$ respectively, are *functions* of p_1 and p_2 . While similar results are obtained in prior research work, we use a different approach. In this regard, Luski (1976) and Gibbens et al. (2000) consider waiting cost distributions for customers, which enable the purchase fractions to be found in closed form so that showing their uniqueness is more straightforward. While Armony and Haviv (2003) do consider two customer types, they do not use an optimization model for the customer sub-game.

In the pricing sub-game, each SP sets her price (as a best response to her competitor's price) to optimize her profit. Theorem 1 provides the purchase fractions and prices under the equilibria, and their necessary and sufficient conditions.

Theorem 1 *Suppose $v > c_1 + 1.5\lambda c_1$. Then the pricing sub-game has at most one equilibrium, with $\delta_1^e = \delta_2^e = 0.5$, characterized as follows: (i) if $q \geq 0.5$ then $\mathcal{SS}_1^e$ occurs with $p_1^e = p_2^e = \lambda c_1$, (ii) if $0.5c_2/c_1 < q < 0.5$ then there is no equilibrium, and (iii) if $q \leq 0.5c_2/c_1$ then $\mathcal{SS}_2^e$ occurs with $p_1^e = p_2^e = \lambda c_2$.*

The result in Theorem 1 can be summarized as follows. If the valuation v is high enough so that the entire market is covered and customers get positive net utilities, then there can only be a *symmetric equilibrium*. Furthermore, a low (intermediate, high) fraction of impatient customers q results in a *low price* (*no, high price*) equilibrium. Although similar models have been used in prior research papers, we have not found a similar result regarding *two-price equilibria* to the best of our knowledge. In this regard, Luski (1976) derives conditions for unique asymmetric equilibrium, Gibbens et al. (2000) find the unique symmetric equilibrium when all customers are satisfied, and Armony and Haviv (2003) mainly provide numerical results pertaining to different price equilibria. Next, we consider $\mathcal{SD}$.

³The total demand for SP i is $\lambda\delta_i^e$ where λ measures the market size. The marginal cost of service provision is assumed to be zero for analytical simplicity.

5. $\mathcal{SD}$: Single and Differentiated Services

We assume without any loss of generality that SP 1 offers single service with price p_1 while SP 2 offers differentiated service with prices p_{2h} and p_{2l} . We only consider $p_{2h} > p_{2l}$ because otherwise, low-priority service is dominated by high-priority service and $\mathcal{SD}$ degenerates to $\mathcal{SS}$ ⁴. The equilibrium involving customer sub-game can be formulated as follows:

$$(\mathcal{CSD}) \quad \max_{\delta_{1j}, \delta_{2jk}; j=1,2; k=h,l} \delta_{11} + \delta_{12} + \delta_{21h} + \delta_{22h} + \delta_{21l} + \delta_{22l}$$

$$s.t. \quad \delta_{1j} (p_1 + c_j(1 + \lambda(\delta_{11} + \delta_{12}))) \leq \delta_{1j} (p_{2h} + c_j(1 + \lambda(\delta_{21h} + \delta_{22h}))) ; j = 1, 2 \quad (4)$$

$$\delta_{1j} (p_1 + c_j(1 + \lambda(\delta_{11} + \delta_{12}))) \leq \delta_{1j} (p_{2l} + c_j(1 + \lambda(2\delta_{21h} + 2\delta_{22h} + \delta_{21l} + \delta_{22l}))) ; j = 1, 2 \quad (5)$$

$$\delta_{1j} (p_1 + c_j(1 + \lambda(\delta_{11} + \delta_{12}))) \leq \delta_{1j} v ; j = 1, 2 \quad (6)$$

$$\delta_{2jh} (p_{2h} + c_j(1 + \lambda(\delta_{21h} + \delta_{22h}))) \leq \delta_{2jh} (p_1 + c_j(1 + \lambda(\delta_{11} + \delta_{12}))) ; j = 1, 2 \quad (7)$$

$$\delta_{2jh} (p_{2h} + c_j(1 + \lambda(\delta_{21h} + \delta_{22h}))) \leq \delta_{2jh} (p_{2l} + c_j(1 + \lambda(2\delta_{21h} + 2\delta_{22h} + \delta_{21l} + \delta_{22l}))) ; j = 1, 2 \quad (8)$$

$$\delta_{2jh} (p_{2h} + c_j(1 + \lambda(\delta_{21h} + \delta_{22h}))) \leq \delta_{2jh} v ; j = 1, 2 \quad (9)$$

$$\delta_{2jl} (p_{2l} + c_j(1 + \lambda(2\delta_{21h} + 2\delta_{22h} + \delta_{21l} + \delta_{22l}))) \leq \delta_{2jl} (p_{2h} + c_j(1 + \lambda(\delta_{21h} + \delta_{22h}))) ; j = 1, 2 \quad (10)$$

$$\delta_{2jl} (p_{2l} + c_j(1 + \lambda(2\delta_{21h} + 2\delta_{22h} + \delta_{21l} + \delta_{22l}))) \leq \delta_{2jl} (p_1 + c_j(1 + \lambda(\delta_{11} + \delta_{12}))) ; j = 1, 2 \quad (11)$$

$$\delta_{2jl} (p_{2l} + c_j(1 + \lambda(2\delta_{21h} + 2\delta_{22h} + \delta_{21l} + \delta_{22l}))) \leq \delta_{2jl} v ; j = 1, 2 \quad (12)$$

$$\delta_{11} + \delta_{21h} + \delta_{21l} \leq q ; \delta_{12} + \delta_{22h} + \delta_{22l} \leq 1 - q ; \delta_{1j}, \delta_{2jk} \geq 0 ; j = 1, 2 ; k = h, l. \quad (13)$$

Constraints in (4) and (5) are the IC conditions which ensure that Type j customers, by purchasing from SP 1, are better off than purchasing high- and low-priority services from SP 2 respectively. Constraints in (6) are the corresponding IR conditions. Similarly, constraints in (7)-(9) and (10)-(12) represent the IC and IR conditions for Type j customers purchasing high- and low-priority services from SP 2 respectively.

In the Appendix, we show that δ_1^e and δ_2^e are uniquely determined for any given prices. Furthermore, we show that although δ_{2h}^e and δ_{2l}^e can take multiple values, *they are also uniquely determined if customers' dis-utilities are minimized*. Since that is plausible, we assume it in the rest of our analysis. Hence, profits of SP 1 and SP 2, given by $\lambda\delta_1^e p_1$ and $\lambda\delta_{2h}^e p_{2h} + \lambda\delta_{2l}^e p_{2l}$ respectively, are *functions* of p_1 , p_{2h} , and p_{2l} . Next, we analyze the pricing sub-game.

We first introduce the terms *price orderings* and *price structure* to better characterize the outcomes of the pricing sub-game. A price ordering indicates the sequence of prices charged by SPs; e.g., $p_{2h} > p_1 > p_{2l}$ and $p_{2h} > p_{2l} = p_1$ are two different price orderings. A price structure indicates

⁴It is easy to show that such a pricing equilibrium under $\mathcal{SD}$ implies a corresponding one under $\mathcal{SS}$

how the prices affect impatient and patient customers' choices. For instance, when $p_{2h} > p_1 > p_{2l}$, *impatient customers being indifferent between high-priority service of SP 2 and single service of SP 1 while patient customers are indifferent between low-priority service of SP 2 and single service of SP 1* is a price structure. Equivalently, such a price structure can also be expressed mathematically as $p_{2h} + \lambda c_1 \delta_{2h} = p_1 + \lambda c_1 \delta_1$ and $p_{2l} + 2\lambda c_2 \delta_{2h} + \lambda c_2 \delta_{2l} = p_1 + \lambda c_2 \delta_1$. Note that since $p_{2h} > p_1 > p_{2l}$, these two equations automatically imply that impatient (patient) customers do not buy from low-priority (high-priority) service SP 2. In the Appendix, we consider different possible price orderings and price structures. We use simple necessary conditions to show that only these two price combinations can result in an equilibrium: (i) $p_{2h} > p_1 > p_{2l}$ with $p_{2h} + \lambda c_1 \delta_{2h} = p_1 + \lambda c_1 \delta_1$ and $p_{2l} + 2\lambda c_2 \delta_{2h} + \lambda c_2 \delta_{2l} = p_1 + \lambda c_2 \delta_1$, and (ii) $p_{2h} > p_1 = p_{2l}$ with impatient customers being indifferent among all three service choices. In the first case, *regular equilibrium* we simplify our analysis and make it more tractable by focusing only on *interior* equilibria, i.e., we do not analyze equilibria in boundary scenarios⁵ with $\delta_{2h} = q$ or $\delta_{2l} = 1 - q$. In the second case, we find that there are two further possibilities: (i) $\delta_{2h} = q$ (and $\delta_{2l} > 0$), *extreme equilibrium*, in which the high-priority service of SP 2 satisfies all impatient customers, and (ii) $\delta_{2l} = 0$, *reduced class equilibrium*⁶, in which the low-priority service of SP 2 does not satisfy any customer.

Equilibrium	Simple Conditions	Complex Conditions
SD_1^e (Regular)	$v > c_1 + \frac{6\lambda c_1^2 c_2}{12c_1 c_2 - c_1^2 - 9c_2^2},$ $c_1 < 3c_2,$ $\delta_{2h}^e < q, \delta_{2l}^e < 1 - q.$	If $\bar{\delta}_2 \leq 1 - 0.5q$ then $\lambda(1 - \bar{\delta}_2)(p_{2h}^e + \lambda c_1(2\bar{\delta}_2 - 1)) \leq \pi_1^e$ If $\bar{\delta}_2 > 1 - 0.5q$ and $p_{2h}^e + \lambda c_1 q \leq 4\lambda c_1 \delta_1^m$ then $\max\left(\frac{(p_{2h}^e + \lambda c_1 q)^2}{8c_1}, \lambda(1 - \bar{\delta}_2)(p_{2h}^e + \lambda c_1(2\bar{\delta}_2 - 1))\right) \leq \pi_1^e$ If $\bar{\delta}_2 > 1 - 0.5q$ and $p_{2h}^e + \lambda c_1 q > 4\lambda c_1 \delta_1^m$ then $\max\left(\frac{\lambda \delta_1^m (p_{2h}^e + \lambda c_1 q - 2\lambda c_1 \delta_1^m)}{\lambda(1 - \bar{\delta}_2)(p_{2h}^e + \lambda c_1(2\bar{\delta}_2 - 1))}\right) \leq \pi_1^e$
SD_2^e (Extreme)	$v > c_1 + 0.5\lambda c_1(1 + 3q)$ $+ \lambda c_2(1 - 3q), q < 1/3.$	If $c_1 < 3c_2$ then $q \leq \frac{c_2(c_1 - 2c_2)}{9c_1 c_2 - 2c_1^2 - 6c_2^2}$ If $9c_2 < c_1 < 10c_2$ then $q \leq \sqrt{\frac{c_2}{c_1}}$ If $10c_2 \leq c_1 \leq (6 + \sqrt{17})c_2$ then either $q \leq \frac{c_1 - \sqrt{(c_1 - 2c_2)(c_1 - 10c_2)}}{2(3c_1 - 5c_2)}$ or $\frac{c_1 + \sqrt{(c_1 - 2c_2)(c_1 - 10c_2)}}{2(3c_1 - 5c_2)} \leq q \leq \sqrt{\frac{c_2}{c_1}}$ If $c_1 > (6 + \sqrt{17})c_2$ then either $q \leq \frac{4c_2^2}{(c_1 - 3c_2)(c_1 - 7c_2)}$ or $\frac{c_1 + \sqrt{(c_1 - 2c_2)(c_1 - 10c_2)}}{2(3c_1 - 5c_2)} \leq q \leq \sqrt{\frac{c_2}{c_1}}$
SD_3^e (Reduced class)	$v > c_1 + \lambda c_1,$ $3c_2 \leq c_1 \leq 9c_2,$ $1/3 \leq q \leq 2/3$	

Table 1: Sufficient conditions for the existence of different types of equilibrium under $\mathcal{SD}$

⁵In the real world, such equilibria would be less likely since SP 1 would have an *expectation* of selling at least to some impatient and patient customers since she sets a price p_1 in between p_{2h} and p_{2l}

⁶The author thanks an anonymous reviewer for suggesting this concise term.

We develop sufficient conditions for the different types of equilibrium to exist. In this regard, we define $\bar{\delta}_2 \equiv (p_{2h}^e - p_{2l}^e)/(\lambda c_2)$ and $\delta_1^m \equiv \frac{c_1 - 3c_2}{2c_1 - 3c_2} \delta_{2h}^e + \frac{c_1 - c_2}{2c_1 - 3c_2} q$, and summarize the conditions in Table 1 for conciseness. The conditions are divided into (i) simple conditions that mostly result just from the basic nature of the corresponding equilibrium or some necessary conditions, and (ii) complex conditions that are derived based on large deviations from the equilibrium by the SPs.

Theorem 2 *The equilibria under $\mathcal{SD}$ are characterized as follows:*

- (i) $\mathcal{SD}_1^e$ with $\delta_1^e = \frac{6c_1c_2 - 6c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2}$, $\delta_{2h}^e = \frac{3c_1c_2 - 3c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2}$, $\delta_{2l}^e = \frac{3c_1c_2 - c_1^2}{12c_1c_2 - c_1^2 - 9c_2^2}$, $p_1^e = \lambda c_1 c_2 / (c_1 - c_2) \delta_1^e$, $p_{2h}^e = 2\lambda c_1 \delta_{2h}^e + 3\lambda c_2 \delta_{2l}^e$, and $p_{2l}^e = \lambda c_1 \delta_{2h}^e + 2\lambda c_2 \delta_{2l}^e$. Also, $\delta_1^e > 0.5 > \delta_2^e$ while $\pi_1^e < \pi_2^e$.
- (ii) $\mathcal{SD}_2^e$ with $\delta_1^e = 0.5(1 + q)$, $\delta_{2h}^e = q$, $\delta_{2l}^e = 0.5(1 - 3q)$, $p_1^e = p_{2l}^e = \lambda c_1 q + \lambda c_2(1 - 3q)$, $p_{2h}^e = 0.5\lambda c_1(1 + q) + \lambda c_2(1 - 3q)$, and $\pi_1^e < \pi_2^e$.
- (iii) $\mathcal{SD}_3^e$ with $\delta_1^e = 2/3$, $\delta_{2h}^e = 1/3$, $\delta_{2l}^e = 0$, $p_1^e = p_{2l}^e = \lambda c_1/3$, $p_{2h}^e = 2\lambda c_1/3$, and $\pi_1^e = \pi_2^e$.

An equilibrium exists if the corresponding conditions in Table 1 are satisfied. Furthermore, this equilibrium is unique.

Theorem 2 shows that there are three kinds of *asymmetric* equilibrium under $\mathcal{SD}$. In the regular equilibrium $\mathcal{SD}_1^e$, each SP satisfies both *impatient* and *patient* customers. While SP 1 uses a single service for this purpose, SP 2 uses high- and low-priority services to satisfy impatient and patient customers respectively. Interestingly, we find that SP 1 obtains a strictly higher purchase fraction than SP 2 from *pooling* the customers while SP 2 obtains a strictly higher profit than SP 1 by *differentiating* the customers. In the extreme equilibrium $\mathcal{SD}_2^e$, SP 1 satisfies only *patient* customers while SP 2 uses the high- and low-priority services to satisfy all *impatient* and some *patient* customers respectively. Again, we find that SP 2 obtains a strictly higher profit than SP 1 even though she sells to less customers. In the reduced class equilibrium $\mathcal{SD}_3^e$, SP 2 satisfies only *impatient* customers. Although SP 2 provides low-priority service, she prices it so high that it does not satisfy any customers, i.e., although there are two service classes, all the customers of SP 2 purchase her high-priority service. In this regard, the customer behavior and the resulting profit is akin to SP 2 just offering a single service (“high-priority service”). However, we note that the presence of low-priority service plays a crucial role in ensuring the existence of the equilibrium. It precludes SP 1 from increasing her price to make strictly better profits. Specifically, SP 1 cannot increase the price as much as she can when there is no low-priority service because some SP 2 customers now start purchasing the low-priority service thereby also reducing the fraction of customers who purchase high-priority service.

The relationship between the three kinds of equilibrium under $\mathcal{SD}$ can be intuitively summarized as follows. If customer heterogeneity (measured by c_1/c_2) is low then both SPs want to satisfy both types of customers and SD_1^e is likely. If the heterogeneity is high and q is low then SP 2 is likely to “own the market” for impatient customers and satisfy all of them resulting in SD_2^e . However, if the heterogeneity is high and q is also relatively high then SP 2 is willing to entirely forego selling to patient customers, which results in SD_3^e . Figure 2 uses the result in Theorem 2 to illustrate how q and c_1/c_2 affect the equilibrium under $\mathcal{SD}$.

Figure 2: Impact of c_1/c_2 and q on the equilibrium under $\mathcal{SD}$

6. $\mathcal{DD}$: Differentiated Service by Both SPs

Both SPs provide differentiated service. SP i ($i = 1, 2$) charges prices p_{ih} and p_{il} for high-priority and low-priority services respectively. As in §5, we only consider $p_{ih} > p_{il}$ because otherwise $\mathcal{DD}$ degenerates to $\mathcal{SD}$ (or $\mathcal{SS}$)⁷. The equilibrium involving customers' sub-game which can be

⁷It is easy to show that such a pricing equilibrium under $\mathcal{DD}$ implies a corresponding one under $\mathcal{SD}$ (or $\mathcal{SS}$)

formulated as follows⁸.

$$(CDD) \quad \max_{\delta_{ijk}; i,j=1,2; k=h,l} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=h,l} \delta_{ijk} \quad (14)$$

$$s.t. \quad \delta_{ijh} (p_{ih} + c_j(1 + \lambda\delta_{ih})) \leq \delta_{ijh} v; i, j = 1, 2 \quad (14)$$

$$\delta_{ijh} (p_{ih} + c_j(1 + \lambda\delta_{ih})) \leq \delta_{ijh} (p_{i'h} + c_j(1 + \lambda\delta_{i'h})); i, i', j = 1, 2; i \neq i' \quad (15)$$

$$\delta_{ijh} (p_{ih} + c_j(1 + \lambda\delta_{ih})) \leq \delta_{ijh} (p_{i'l} + c_j(1 + \lambda(2\delta_{i'h} + \delta_{i'l}))); i, i', j = 1, 2 \quad (16)$$

$$\delta_{ijl} (p_{il} + c_j(1 + \lambda(2\delta_{ih} + \delta_{il}))) \leq \delta_{ijl} v; i, j = 1, 2 \quad (17)$$

$$\delta_{ijl} (p_{il} + c_j(1 + \lambda(2\delta_{ih} + \delta_{il}))) \leq \delta_{ijl} (p_{i'l} + c_j(1 + \lambda(2\delta_{i'h} + \delta_{i'l}))); i, i', j = 1, 2; i \neq i' \quad (18)$$

$$\delta_{ijl} (p_{il} + c_j(1 + \lambda(2\delta_{ih} + \delta_{il}))) \leq \delta_{ijl} (p_{i'h} + c_j(1 + \lambda\delta_{i'h})); i, i', j = 1, 2 \quad (19)$$

$$\sum_{i=1}^2 \sum_{k=h,l} \delta_{i1k} \leq q; \sum_{i=1}^2 \sum_{k=h,l} \delta_{i2k} \leq 1 - q; \delta_{ijk} \geq 0; i, j = 1, 2; k = h, l. \quad (20)$$

Constraints in (14) and (17) respectively represent the IR conditions for Type j customers purchasing high-priority and low-priority services from SP i . Constraints in (15) and (16) ensure that Type j customers who purchase the high-priority service from a SP do better than purchasing high-priority service from the other SP and low-priority service from either SP respectively. Likewise, constraints in (18) and (19) guarantee that Type j customers who purchase the low-priority service from a SP do better than purchasing low-priority service from the other SP and high-priority service from either SP respectively.

In the Appendix we show that δ_1^e and δ_2^e are uniquely determined for any given prices. Furthermore, we show that $\delta_{ik}^e; i = 1, 2; k = h, l$ are also uniquely determined when *customers' dis-utilities are minimized*. Since that is plausible, we assume it in the rest of our analysis. Hence, any positive profit of SP i ($i = 1, 2$), given by $\lambda\delta_{ih}^e p_{ih} + \lambda\delta_{il}^e p_{il}$, is a *function* of the prices charged by her and the competing SP. Next, we analyze the pricing sub-game.

We again use the ideas of price ordering and price structure to discuss about the equilibrium outcomes of pricing sub-game. In the Appendix, we use simple necessary conditions to determine the existence of equilibrium under different possible price ordering and corresponding price structures. Interestingly, *we find that the equilibrium can exist only under the following price ordering and price structure: the SPs have to charge identical prices for high- and low-priority services, and the prices should be such that impatient customers prefer high-priority service and patient ones prefer low-priority service. Hence, we show that only a symmetric equilibrium can exist.* For brevity, we denote the prices for high- and low-priority services by p_h^e and p_l^e respectively, and

⁸For conciseness, we use aggregate purchase fractions, where necessary, in the formulation; e.g., δ_{ih} is used instead of $\delta_{i1h} + \delta_{i2h}$.

the corresponding purchase fraction (of each SP) as δ_h^e and δ_l^e . Note that the equilibrium price structure implies $0.5\lambda c_2 < p_h^e - p_l^e < 0.5\lambda c_1$. Theorem 3 characterizes the equilibrium.

Theorem 3 *Suppose $v > c_1 + 1.5\lambda c_1 q + 2\lambda c_2(1 - q)$. Then $\mathcal{DD}^e$ exists with $p_h^e = \lambda c_1 q + 2\lambda c_2(1 - q)$, $p_l^e = \lambda c_2(1 - q)$, $\delta_h^e = 0.5q$, and $\delta_l^e = 0.5(1 - q)$ if and only if $q < \frac{c_1 - 2c_2}{2(c_1 - c_2)}$.*

7. SS vs. SD vs. DD

We analyze the equilibrium outcomes when two SPs compete in the context of a three-stage game involving service delivery, pricing, and purchase fraction decisions. We will mainly focus on the outcome of the first-stage game, i.e., whether both SPs will just provide single service ($\mathcal{SS}$), one SP provides single service and the other SP provides differentiated service ($\mathcal{SD}$), or both SPs provide differentiated service ($\mathcal{DD}$). In this regard, if an outcome does not have an equilibrium (based on the analyses in §s 4-6), we ignore that outcome. Also, if there are multiple equilibria, we select the Pareto optimal equilibrium if it exists; otherwise, we specify all the equilibria. Theorem 4 provides the results for the overall equilibrium. In this characterization, we want to focus on the impact of q and customer heterogeneity (measured by c_1/c_2); we assume any conditions on v and equilibrium profit (e.g., see parts (i) and (ii) of Theorem 2) are already satisfied.

Theorem 4 *If $q \geq 0.5$ then $\mathcal{SS}_1^e$ is the equilibrium of the service mode game. Suppose $q < 0.5$; then assuming the conditions on v and π_i^e in Theorems 1-3 are satisfied, the equilibrium is characterized as follows:*

- (i) *If $\frac{c_1}{c_2} < 3$ and $q > \frac{3c_1c_2 - 3c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2}$ then $\mathcal{SD}_1^e$ is the equilibrium;*
- (ii) *If q , c_1 , and c_2 satisfy the corresponding conditions in Table 1 then $\mathcal{SD}_2^e$ is the equilibrium;*
- (iii) *If $3 \leq \frac{c_1}{c_2} \leq 9$ and $q \geq \frac{1}{3}$ then $\mathcal{SD}_3^e$ is the equilibrium;*
- (iv) *If $\frac{c_1 - 2c_2}{2(c_1 - c_2)} \leq q \leq \frac{c_2}{2c_1}$ and $q > \frac{c_2(c_1 - 2c_2)}{9c_1c_2 - 2c_1^2 - 6c_2^2}$ then $\mathcal{SS}_2^e$ is the equilibrium;*
- (v) *Otherwise, if $q < \frac{c_1 - 2c_2}{2(c_1 - c_2)}$ then $\mathcal{DD}^e$ is the equilibrium.*

Theorem 4 shows how the overall equilibrium changes with heterogeneity and q . If impatient customers form at least half of all the customers, both SPs provide single service with a *high price* equilibrium ($\mathcal{SS}_1^e$). Interestingly, we find that even if an equilibrium exists under $\mathcal{SD}$ ($\mathcal{SD}_3^e$ exists when $0.5 \leq q \leq 2/3$ and $3 \leq c_1/c_2 \leq 9$), *SP 2 chooses not to differentiate because that will result in a lower profit. The lower profit is a consequence of increased competition between SP 1 and SP 2 under $\mathcal{SD}$ in comparison to $\mathcal{SS}$, which can be seen from the lower prices.*

SD becomes the equilibrium for a relatively large region with low/intermediate heterogeneity. There are three possible cases here. If q takes intermediate values and heterogeneity is low ($c_1/c_2 < 3$) then there is a *regular* equilibrium SD_1^e in which the high-priority (low-priority) service of SP 2 satisfies some impatient (patient) customers. If q takes relatively low values and heterogeneity is low/intermediate then there is an *extreme* equilibrium SD_2^e in which the high-priority (low-priority) service of SP 2 satisfies *all impatient* (some patient) customers. If both q and heterogeneity take intermediate values ($1/3 \leq q < 1/2$, $3 \leq c_1/c_2 \leq 9$) then there is a *reduced class* equilibrium SD_3^e in which *the low-priority service of SP 2 does not satisfy any customer*. Although DD^e can also exist in the second and third cases, we again find that it is not the equilibrium since the profit of SP 1 is lower due to increased competition. *SP 1 would rather provide single service instead of differentiated service. Interestingly, even though SS_2^e can sometimes exist in the second case, we find that it is not an equilibrium since the profit of SP 2 increases from providing differentiated service instead of single service.*

Figure 3: Change of overall equilibrium with customer heterogeneity c_1/c_2 and fraction of impatient customers q

$\mathcal{SS}_2^e$ is the equilibrium only if both q and heterogeneity are low, *in scenarios in which it is the only possible equilibrium*. In this regard, when heterogeneity is high, $\mathcal{DD}^e$ is the equilibrium *even in scenarios even when $\mathcal{SS}_2^e$ exists because both SPs benefit from better customer differentiation*.

Figure 3 illustrates the results from Theorem 4. In the Appendix, we show that offering differentiated service is always better for a monopoly SP if she satisfies both patient and impatient customers. However, Figure 3 shows that $\mathcal{SS}_1^e$, $\mathcal{SS}_2^e$, $\mathcal{SD}_1^e$, $\mathcal{SD}_2^e$, and $\mathcal{SD}_3^e$ (in addition to $\mathcal{DD}^e$) can be equilibria in a duopoly even when both impatient and patient customers are satisfied. Next, we summarize the results from our analysis of the overall equilibrium, and qualitatively discuss their reasoning and implications. We also discuss about some modeling features and limitations.

8. Discussion

Table 2 summarizes the different kinds of equilibrium and the scenarios in which they generally become the overall equilibrium. If the fraction of impatient customers $q \geq 0.5$ then both SPs provide single service and charge high price ($\mathcal{SS}_1^e$). In this case, neither SP wants to differentiate because the price is already sufficiently high and the impact of competition will exceed that of differentiation from providing differentiated service. Suppose $q < 0.5$ and patient customers form a majority. If the heterogeneity c_1/c_2 is high and q is not very low, then both SPs provide differentiated service ($\mathcal{DD}^e$). Furthermore, for any q , $\mathcal{DD}^e$ becomes the overall equilibrium when c_1/c_2 is sufficiently high⁹. In these cases, since the heterogeneity is quite high, there is no equilibrium involving single service. If c_1/c_2 takes low/intermediate values then typically, SP 1 provides single service while SP 2 provides differentiated service. However, the nature of this equilibrium depends on the values of both c_1/c_2 and q .

If c_1/c_2 is low but q takes intermediate values, we have a regular equilibrium ($\mathcal{SD}_1^e$) in which both SPs satisfy both types of customers. Intuitively, since q is relatively large, both SPs want to satisfy impatient customers but since the heterogeneity is low, they also do not want to ignore patient customers.

If both c_1/c_2 and q take intermediate values, we have a reduced class equilibrium ($\mathcal{SD}_3^e$) in which SP 1 and SP 2 satisfy two-thirds and one-third of customers respectively. Although SP 2 offers differentiated service, she satisfies only impatient customers who select her high-priority service. Relatively high values of c_1/c_2 and q enable SP 2 to ignore patient customers. However, since these values are intermediate, SP 1 is also content to charge a lower price but satisfy both customer

⁹Mathematically, this result is because $\frac{c_1-2c_2}{2(c_1-c_2)} \rightarrow 0.5$ and $\frac{4c_2^2}{(c_1-3c_2)(c_1-7c_2)} \rightarrow 0$ as $c_1/c_2 \rightarrow \infty$.

types (when $q \neq 1/3$). This equilibrium is different from all the other kinds of equilibrium in that none of the customers purchase from a service class. That can pose some practical implementation challenges involving customers and SP 2. When customers can *see the queues*, they may feel bad and think *priority is not worth it* because everyone uses the high-priority service. In such cases, SP 2 can better promote and market the high-priority service so that customers have a better perception about it and feel better. When customers cannot directly see the queues, such measures are not necessary since customer perception is not a concern. SP 2 has to realize that the low-priority service is purely strategic in nature and its main purpose is to limit the competition from SP 1. She should not be unduly concerned that no one is purchasing the low-priority service.

We have an extreme equilibrium ($\mathcal{SD}_2^e$) in these cases: (i) c_1/c_2 is low and q is very low, (ii) c_1/c_2 is intermediate and q is low/very low, and (iii) c_1/c_2 is high and q is very low. In all these cases, SP 1 satisfies only patient customers while SP 2 satisfies both impatient and patient customers. In this regard, $\mathcal{SD}_2^e$ is the *counterpart* of $\mathcal{SD}_3^e$ in which SP 2 satisfies only impatient customers. They are two different kinds of *separating equilibrium* while $\mathcal{SD}_1^e$ is a *pooling equilibrium*. The intuition behind when $\mathcal{SD}_2^e$ becomes the overall equilibrium is as follows. Generally, the value of q should be low enough ($q < 1/3$) so that SP 2 can satisfy some patient customers in addition to satisfying all the impatient ones. Furthermore, when c_1/c_2 is low, q needs to be very low (and not just low) because both SPs would otherwise rather provide single service resulting in a low price equilibrium $\mathcal{SS}_2^e$; similarly, when c_1/c_2 is high, q needs to be very low (and not just low) because both SPs would otherwise rather provide differentiated service resulting in $\mathcal{DD}^e$. Next, we discuss about the modeling features and limitations.

Case	Heterogeneity	Fraction of Impatient Customers	Overall Equilibrium
1	—	≥ 0.5	$\mathcal{SS}_1^e$ (High price)
2	Low	Intermediate	$\mathcal{SD}_1^e$ (Regular)
3	Low Intermediate High	Very Low Low/Very Low Very Low	$\mathcal{SD}_2^e$ (Extreme)
4	Intermediate	Intermediate	$\mathcal{SD}_3^e$ (Reduced class)
5	Low	Low	$\mathcal{SS}_2^e$ (Low price)
6	High	Low/Intermediate	$\mathcal{DD}^e$ (Differentiated)

Table 2: Effect of heterogeneity and q on the overall equilibrium

We model competition using a three-stage game. We now consider what happens if we model it instead as a two-stage game in which each SP first decides on her price(s) followed by the customer sub-game. In the Appendix, we show that the differentiated equilibrium $\mathcal{DD}^e$ is the only possibility.

Hence, an equilibrium in which any SP provides single service does not exist in a two-stage game. Intuitively, the reasoning is as follows. Unlike in a three-stage game, an SP purely benefits from *delayed differentiation*; there is no negative impact from competition to consider. Although we analyze the two-stage game for completeness, we note that the three-stage game is more plausible since provision of differentiated service usually involves operational set-ups and changes that take longer and are harder to implement than just changing price(s).

In our model, we assume waiting costs are linear in the purchase fractions. This approximation works well for relatively low utilization. We briefly discuss qualitatively what is likely to happen if the utilization, i.e., λ is high and waiting costs become non-linear. When SPs are more congested, there is a higher difference in waiting times (and waiting costs) between high- and low-priority services. That enables the SPs to better differentiate impatient and patient customers. There is also less competition in a more congested system because both SPs have higher *net marginal waiting cost* (derivative of difference in waiting cost with respect to the purchase fraction of the service class at an SP) and they would therefore be willing to charge higher prices at equilibrium. However, offering differentiated service is likely to intensify the competition. In summary, we expect the overall general pattern here to be similar to Table 2 because similar tradeoffs (competition vs. differentiation) apply in both cases. Our model also makes other simplifying assumptions such as complete market coverage, two SPs, and two customer types.

9. Conclusion

Time is used as a differentiating factor by service providers (SPs) under different contexts. We consider differentiated service with two service classes in which some customers are prioritized, and analyze when offering it in the presence of self-selecting customers, who maximize their individual expected utilities, would be beneficial for SPs. The customers are heterogeneous in their delay sensitivities and belong to one of two types: impatient and patient. We model the competition between two SPs in the form of a three-stage game: in the first stage, the SPs decide whether to offer single/differentiated service; in the second stage, they decide their prices; and in the third stage, the customers self-select and purchase their best choices. A novel feature of this note involves how we model customer self-selection. We start with a simple customer utility model and we then model the customer purchase fractions as solutions of optimization problems involving appropriate individual rationality (IR) and incentive compatibility (IC) conditions. If there are multiple optimal solutions, we select the best solution with lowest customer dis-utility. We then show that profits of the SPs are uniquely determined for any given prices.

The key question we aim to answer is how the two SPs will deliver service at equilibrium: whether both of them will provide just single service ($\mathcal{SS}$), one of them provides single service and the other provides differentiated service ($\mathcal{SD}$), or both of them provide differentiated service ($\mathcal{DD}$). In this regard, we characterize six kinds of equilibrium: (i) *high price* equilibrium ($\mathcal{SS}_1^e$), (ii) *low price* equilibrium ($\mathcal{SS}_2^e$), (iii) *regular* equilibrium ($\mathcal{SD}_1^e$), (iv) *extreme* equilibrium ($\mathcal{SD}_2^e$), (v) *reduced class* equilibrium ($\mathcal{SD}_3^e$), and (vi) *differentiated* equilibrium in $\mathcal{DD}$ ($\mathcal{DD}^e$). We find that the overall equilibrium mainly depends on customer heterogeneity (measured by c_1/c_2) and the mix of patient/impatient customers (measured by q). Some potential future areas of research include modeling queuing aspects more in detail, analyzing partial market coverage, and examining oligopolistic competition with more than two customer types.

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Appendix

A. $\mathcal{SS}$

($\mathcal{CSS}$) clearly results in a unique value for $\delta_{11}^e + \delta_{12}^e + \delta_{21}^e + \delta_{22}^e$, the total equilibrium purchase fraction¹⁰, for any given prices p_1 and p_2 . However, are the equilibrium purchase fractions at each SP, i.e., δ_1^e and δ_2^e , also uniquely determined? Proposition 1 answers this question. In this regard, we note that Armony and Haviv (2003) prove a similar result; the contradiction argument we use in the proof is also somewhat similar to their argument. However, unlike Armony and Haviv (2003), we use an optimization problem formulation to model the customer sub-game ($\mathcal{CSS}$). For this reason and for the sake of completeness, we next state and prove Proposition 1.

Proposition 1 *The equilibrium purchase fractions δ_1^e and δ_2^e are unique for any (p_1, p_2) .*

Proof Suppose $p_1 = p_2$. Then the proposition clearly holds because the equilibrium purchase fractions are characterized as follows: (i) $\delta_1^e = \delta_2^e = \min((v - p_1 - c_1)/(\lambda c_1), 0.5)$ if $p_1 + c_1 + 0.5\lambda c_1(1 - q) \leq v$ and (ii) $\delta_1^e = \delta_2^e = \max(\min((v - p_1 - c_2)/(\lambda c_2), 0.5(1 - q)), 0)$ otherwise. Suppose $p_1 > p_2$ (without any loss of generality). We prove the result by contradiction. Let δ_1' and δ_1'' be two different equilibrium values of δ_1^e with $\delta_1' > \delta_1''$ (without any loss of generality). Using $'$ and $''$ to denote the other corresponding equilibrium values, we have $\delta_2' < \delta_2''$ (since $\delta_1' + \delta_2' = \delta_1'' + \delta_2''$). First, note that since $\delta_1' > 0$, due to (1), $p_1 \leq p_2 + \lambda c_j(\delta_2' - \delta_1')$ for either $j = 1$ or 2 . However, that means $\delta_1' < \delta_2'$ since $p_1 > p_2$. Hence, because both δ_1' and δ_2' are non-zero, $\lambda c_2(\delta_2' - \delta_1') \leq p_1 - p_2 \leq \lambda c_1(\delta_2' - \delta_1')$. If $p_1 - p_2 = \lambda c_2(\delta_2' - \delta_1') < \lambda c_2(\delta_2'' - \delta_1'') < \lambda c_1(\delta_2'' - \delta_1'')$ then $\delta_2^e < \delta_2''$ which leads to contradiction. So suppose $p_1 - p_2 > \lambda c_2(\delta_2' - \delta_1')$. However, in this case, some impatient customers have to be satisfied by SP 1 and so all patient customers are satisfied by SP 2, i.e., $\delta_{22}' = 1 - q$ which implies that $\delta_2' \geq 1 - q$. Hence, we need $\delta_2'' > 1 - q$ but for that to happen, some impatient customers need to be satisfied by SP 1. However, that is impossible because $p_1 - p_2 \leq \lambda c_1(\delta_2' - \delta_1') < \lambda c_1(\delta_2'' - \delta_1'')$ (as $\delta_2'' > \delta_2'$ and $\delta_1'' < \delta_1'$). ■

B. $\mathcal{SD}$

B.1 Customer Sub-game

Proposition 2 discusses about the uniqueness of the equilibrium outcome in the customer sub-game under $\mathcal{SD}$.

¹⁰It results from optimizing a bounded objective function within a compact space.

Proposition 2 *The equilibrium purchase fractions δ_1^e and δ_2^e are unique for any given (p_1, p_{2h}, p_{2l}) . Suppose $\delta_1^e, \delta_2^e > 0$. Then δ_{2h}^e and δ_{2l}^e are unique unless either (i) $p_1 > p_{2h}$, $\delta_2^e = 1 - q$ and $p_{2h} - p_{2l} = \lambda c_2 \delta_2^e$ or (ii) $p_1 < p_{2l}$, $\delta_1^e = 1 - q$, and $p_{2h} - p_{2l} = \lambda c_1 \delta_2^e$. In these two cases, there exist unique δ_{2h}^e and δ_{2l}^e , with $\delta_{2h}^e + \delta_{2l}^e = \delta_2^e$, which minimize customers' dis-utilities.*

Proof First, we show by contradiction that δ_1^e and δ_2^e are unique. We use ' and ' ' to denote two sets of optimal purchase fractions, and assume $\delta_1' < \delta_1''$ without any loss of generality. Since the objective $\delta_1' + \delta_2' = \delta_1'' + \delta_2''$, $\delta_2' > \delta_2''$. The proof follows by noting that the dis-utility of class j ($j = 1, 2$) customers *buying the best service class* at SP i strictly increases with the purchase fraction δ_i . Hence $p_1 + c_j + \lambda c_j \delta_1' > p_1 + c_j + \lambda c_j \delta_1''$ and $\min(p_{2h} + c_j + \lambda c_j \delta_{2h}', p_{2l} + c_j + 2\lambda c_j \delta_{2h}' + \lambda c_j \delta_{2l}') > \min(p_{2h} + c_j + \lambda c_j \delta_{2h}'', p_{2l} + c_j + 2\lambda c_j \delta_{2h}'' + \lambda c_j \delta_{2l}'')$. Note that the later inequality is just a consequence from $\delta_2' > \delta_2''$, and δ_{2jk}' 's and δ_{2jk}'' 's being feasible, but it does not depend on the values of the individual purchase fractions. Since $\delta_1'' > 0$, one of these two inequalities has to be satisfied:

$$p_1 + c_2 + \lambda c_2 \delta_1'' \leq \min(p_{2h} + c_2 + \lambda c_2 \delta_{2h}'', p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}'', v) \quad (21)$$

$$p_1 + c_1 + \lambda c_1 \delta_1'' \leq \min(p_{2h} + c_1 + \lambda c_1 \delta_{2h}'', p_{2l} + c_1 + 2\lambda c_1 \delta_{2h}'' + \lambda c_1 \delta_{2l}'', v) \quad (22)$$

Also, since $\delta_2' > 0$, one of these two inequalities has to be satisfied:

$$\min(p_{2h} + c_2 + \lambda c_2 \delta_{2h}', p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}' + \lambda c_2 \delta_{2l}') \leq p_1 + c_2 + \lambda c_2 \delta_1' \quad (23)$$

$$\min(p_{2h} + c_1 + \lambda c_1 \delta_{2h}', p_{2l} + c_1 + 2\lambda c_1 \delta_{2h}' + \lambda c_1 \delta_{2l}') \leq p_1 + c_1 + \lambda c_1 \delta_1' \quad (24)$$

Suppose (21) is satisfied. Then $p_1 + c_2 + \lambda c_2 \delta_1'' < \min(p_{2h} + c_2 + \lambda c_2 \delta_{2h}'', p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}'', v)$ which implies (23) is violated, (24) has to be satisfied, and (22) is violated. However, the inequality also implies $\delta_1' \geq 1 - q$ because otherwise setting $\delta_{12}' = \delta_1' + \epsilon$ retains feasibility while increasing the objective, thereby showing δ' 's are sub-optimal. Hence $\delta_1'' > 1 - q$ which implies $\delta_{11}'' > 0$. So (22) has to hold, which results in a contradiction.

A similar contradiction can be shown if we start with (22) being satisfied instead of (21). Next, we show that unique $\delta_1^e, \delta_2^e > 0$ also result in unique δ_{2h}^e and δ_{2l}^e unless either (i) $p_1 > p_{2h}$, $\delta_2^e = 1 - q$ and $p_{2h} - p_{2l} = \lambda c_2 \delta_2^e$ or (ii) $p_1 < p_{2l}$, $\delta_1^e = 1 - q$, and $p_{2h} - p_{2l} = \lambda c_1 \delta_2^e$. We consider different cases based on how the prices p_{2h} and p_{2l} are related because that affects how each customer type will choose between these two service classes and simplifies the analysis.

1. $p_{2h} - p_{2l} < \lambda c_2 \delta_2^e < \lambda c_1 \delta_2^e$: The high-priority service is priced too low so that everyone prefers it. Hence $\delta_{2l}^e = 0$ and $\delta_{2h}^e = \delta_2^e$.

2. $p_{2h} - p_{2l} > \lambda c_1 \delta_2^e > \lambda c_2 \delta_2^e$: The high-priority service is priced too high so that everyone prefers the low-priority service. Hence $\delta_{2h}^e = 0$ and $\delta_{2l}^e = \delta_2^e$.

3. $\lambda c_2 \delta_2^e \leq p_{2h} - p_{2l} \leq \lambda c_1 \delta_2^e$: We show the result by contradiction. Let $\delta_{2h}' < \delta_{2h}''$; since δ_2^e is unique, $\delta_{2l}' > \delta_{2l}''$. Since $\delta_{2h}'' > 0$, one of these two inequalities has to be satisfied:

$$p_{2h} + c_2 + \lambda c_2 \delta_{2h}'' \leq p_1 + c_2 + \lambda c_2 \delta_1^e \quad (25)$$

$$p_{2h} + c_1 + \lambda c_1 \delta_{2h}'' \leq \min(p_1 + c_1 + \lambda c_1 \delta_1^e, v) \quad (26)$$

Note that *only one equation* can be satisfied because if both are satisfied, $p_{2h} + c_j + \lambda c_j \delta_{2h}' < p_1 + c_j + \lambda c_j \delta_1^e$; $j = 1, 2$ which implies $\delta_1^e = 0$. Suppose (25) holds. Then $p_{2h} + c_2 + \lambda c_2 \delta_{2h}' < p_1 + c_2 + \lambda c_2 \delta_1^e$ which implies $\delta_{12}' = 0$ and $\delta_{11}' > 0$. However, that means $p_1 + c_1 + \lambda c_1 \delta_1^e \leq p_{2h} + c_1 + \lambda c_1 \delta_{2h}' < p_{2h} + c_1 + \lambda c_1 \delta_{2h}''$. This inequality and (25) together imply $p_1 > p_{2h}$. The inequality also yields $\delta_{21h}'' = 0$ and $\delta_{22h}'' = \delta_{2h}'' > 0$ which implies $p_{2h} - p_{2l} = \lambda c_2 \delta_2$. Furthermore, $\delta_{2h}'' + \delta_{2l}'' \leq 1 - q$ since only patient customers are satisfied (impatient customers strictly prefer SP 1 to high-priority service which itself is strictly preferred to low-priority service). However, $\delta_{2h}' + \delta_{2l}' \geq 1 - q$ since $\delta_{12}' = 0$ and $\delta_{11}' > 0$. Since $\delta_2^e = \delta_{2h}'' + \delta_{2l}'' = \delta_{2h}' + \delta_{2l}'$, we get $\delta_2^e = 1 - q$. Then δ_{2h}^e is the minimum value of all possible optimal solutions, and $\delta_{2l}^e = \delta_2^e - \delta_{2h}^e$. These values minimize customers' dis-utilities because for any given δ_2^e , the dis-utilities are strictly increasing in δ_{2h} .

Suppose (26) holds. Then $p_{2h} + c_1 + \lambda c_1 \delta_{2h}' < p_1 + c_1 + \lambda c_1 \delta_1^e$. Hence, $\delta_{11}' = 0$ and $\delta_{12}' = \delta_{12}' \leq 1 - q$. Also, since $\delta_1^e > 0$, $p_1 + c_2 + \lambda c_2 \delta_1^e \leq p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}' + \lambda c_2 \delta_{2l}' < p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}'' \leq p_{2h} + c_2 + \lambda c_2 \delta_{2h}''$. Hence $\delta_{22h}'' = \delta_{22l}'' = 0$. Since $\delta_{2h}'' > 0$, $\delta_{21h}'' > 0$ so that $p_1 + c_2 + \lambda c_2 \delta_1^e < p_{2h} + c_2 + \lambda c_2 \delta_{2h}'' < p_{2h} + c_1 + \lambda c_1 \delta_{2h}'' \leq v$ and $\delta_1^e \geq 1 - q$. Since we have already shown $\delta_1^e \leq 1 - q$, we get $\delta_1^e = 1 - q$. Hence, $\delta_{12}' = 1 - q$, and $\delta_{2l}' > 0$ implies $\delta_{21l}' > 0$ since $\delta_{22l}' = 0$. That is possible only when $p_{2h} - p_{2l} = \lambda c_1 \delta_2^e$. Finally, since $p_{2l} + c_1 + 2\lambda c_1 \delta_{2h}' + \lambda c_1 \delta_{2l}' = p_{2h} + c_1 + \lambda c_1 \delta_{2h}' < p_1 + c_1 + \lambda c_1 \delta_1^e$ but $p_1 + c_2 + \lambda c_2 \delta_1^e \leq p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}' + \lambda c_2 \delta_{2l}'$, we get $p_1 < p_{2l}$. As in the previous case, δ_{2h}^e is the minimum value of all possible optimal solutions while $\delta_{2l}^e = \delta_2^e - \delta_{2h}^e$. ■

B.2 Possible Pricing Structures at Equilibrium

When the prices (p_1 , p_{2h} , and p_{2l}) are distinct, we argue that we only need to consider the three pricing structures in Figure 4 at equilibrium. Note that Figure 4, in addition to the prices, also shows the dis-utilities of patient and impatient customers from purchasing different service classes. We use the idea that if a set of prices results in an equilibrium then lowering one of them will still result in an equilibrium *as long as the profits are unchanged*. The detailed reasoning is as follows.

First, some impatient customers are (at least marginally) willing to purchase the highest price option. Otherwise, there are no customers (impatient or patient) buying from this option, and the highest price can be reduced until impatient customers are marginally willing to purchase.

Figure 4: Possible equilibrium pricing structures under $\mathcal{SD}$ showing the prices and dis-utilities of impatient and patient customers

Similarly, some patient customers are (at least marginally) willing to purchase from the lowest price option. Also, for the same reason, some customers (either impatient or patient) are (at least marginally) willing to purchase the medium price option. There are three possibilities:

(i) Only impatient customers are willing: Impatient customers are indifferent between highest and medium price options but patient ones strictly prefer the lowest price option. Then the equilibrium pricing structure has to be as shown in part (a) of Figure 4; otherwise, the lowest price can be increased (by a small amount) to yield better profit for one of the SPs.

(ii) Both impatient and patient customers are willing: This condition automatically results in part (b) of Figure 4.

(iii) Only patient customers are willing: Patient customers are indifferent between lowest and medium price options but impatient ones strictly prefer the highest price option. Then the equilibrium pricing structure has to be as shown in part (c) of Figure 4; otherwise, the highest price can be increased (by a small amount) to yield better profit for one of the SPs.

Furthermore, using similar arguments, when there are only two distinct prices (that occurs either when $p_1 = p_{2h} > p_{2l}$ or $p_{2h} > p_{2l} = p_1$; note that there are always at least two distinct prices since $p_{2h} > p_{2l}$), either impatient customers or patient customers are indifferent between the pricing options. Next, we consider different price orderings and examine in greater detail which structure can possibly be an equilibrium in each of them.

B.2.1 $p_{2h} > p_1 > p_{2l}$

First, we show that we only need to consider the structure in part (b) of Figure 4. Consider part (a). If $\delta_{2l} > 1 - q$ then increase p_1 to $p_1 + \lambda c_1 \epsilon$ for a small $\epsilon > 0$. Then we find that $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ while $\tilde{\delta}_1 = \delta_1$ so that the profit of SP 1 increases. Hence $\delta_{2l} = 1 - q$; however, p_{2l} can now be increased by a small amount without any change in δ_{2h} or δ_{2l} so that the profit of SP 2 increases. Therefore, equilibrium is impossible in part (a).

Consider part (c). If $\delta_{2l} > 0$ then increasing p_1 to $p_1 + \lambda c_2 \epsilon$ increases the profit of SP 1 since δ_1 remains the same; note that $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$. Hence $\delta_{2l} = 0$ for equilibrium. Furthermore, $\delta_{2h} > q$ because otherwise p_{2h} can be increased (without any loss in sales of high-priority service) to increase the profit of SP 2. Also, since $p_{2h} + \lambda c_2 \delta_{2h} = p_1 + \lambda c_2 \delta_1$, $\delta_1 > \delta_{2h}$. Next, we show by contradiction an equilibrium is impossible. Suppose $\tilde{\delta}_1 = \delta_1 - \epsilon$; then $\tilde{p}_1 = p_1 + 2\lambda c_2 \epsilon$. For equilibrium, $\lambda \tilde{\delta}_1 \tilde{p}_1 \leq \lambda \delta_1 p_1$ for any small $\epsilon > 0$ which implies $p_1 \geq 2\lambda c_2 \delta_1$. Similarly, we find that if $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ then $\tilde{p}_{2h} = p_{2h} - 2\lambda c_2 \epsilon$ so that $\lambda \tilde{\delta}_{2h} \tilde{p}_{2h} \leq \lambda \delta_{2h} p_{2h}$ implies $p_{2h} \leq 2\lambda c_2 \delta_{2h}$. Hence $p_1 \geq 2\lambda c_2 \delta_1 > 2\lambda c_2 \delta_{2h} \geq p_{2h}$ so that $p_1 > p_{2h}$, a contradiction.

Consider part (b). We show that $\delta_{2h}, \delta_{2l} > 0$ at equilibrium. Suppose $\delta_{2h} = 0$. Then setting $\tilde{\delta}_{2h} = \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ implies $\tilde{\pi}_2 = \lambda(p_{2h} - \lambda c_1 \epsilon) + \lambda(p_{2l} - \lambda c_2 \epsilon)(\delta_{2l} - \epsilon) = \pi_2 + \lambda \epsilon(p_{2h} - p_{2l} - \lambda c_2 \delta_{2l}) - \lambda^2(c_1 - c_2)\epsilon^2 > \pi_2$ for small $\epsilon > 0$ since $p_{2h} - p_{2l} > \lambda c_2 \delta_2 = \lambda c_2 \delta_{2l}$. Hence $\delta_{2h} > 0$ for equilibrium. Suppose $\delta_{2l} = 0$. Then $p_{2h} \geq 2\lambda c_1 \delta_{2h}$; otherwise setting $\tilde{\delta}_{2h} = \delta_{2h} - \epsilon$ and $\tilde{p}_{2h} = \tilde{p}_{2l} = p_{2h} + 2\lambda c_1 \epsilon$ makes $\tilde{\pi}_2 > \pi_2$. That implies $p_{2l} > \lambda c_1 \delta_{2h}$ since $p_{2h} - p_{2l} < \lambda c_1 \delta_2 = \lambda c_1 \delta_{2h}$. Then setting $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = \epsilon$ still implies $\tilde{\pi}_2 = \lambda(p_{2h} - \lambda c_1 \epsilon)\delta_{2h} + \lambda(p_{2l} - 2\lambda c_2 \epsilon)\epsilon = \pi_2 + \lambda \epsilon(p_{2l} - \lambda c_1 \delta_{2h}) - 2\lambda^2 c_2 \epsilon^2 > \pi_2$ for small $\epsilon > 0$. Hence $\delta_{2l} > 0$ for equilibrium. Next, we focus on *interior equilibria* in which $\delta_{2h} < q$ and $\delta_{2l} < 1 - q$.

For such equilibria, set $\tilde{\delta}_1 = \delta_1 \pm \epsilon$. Then $\tilde{p}_1 + \lambda c_1 \tilde{\delta}_1 = p_{2h} + \lambda c_1 \tilde{\delta}_{2h}$, $\tilde{p}_1 + \lambda c_2 \tilde{\delta}_1 = p_{2l} + 2\lambda c_1 \tilde{\delta}_{2h} + \lambda c_1 \tilde{\delta}_{2l}$, and $\tilde{\delta}_1 + \tilde{\delta}_{2h} + \tilde{\delta}_{2l} = \delta_1 + \delta_{2h} + \delta_{2l}$. After some algebra, we can solve these equations to find that $\tilde{p}_1 = p_1 \mp \lambda c_1 c_2 / (c_1 - c_2) \epsilon$, $\tilde{\delta}_{2h} = \delta_{2h} \pm (c_1 - 2c_2) / (c_1 - c_2) \epsilon$, and $\tilde{\delta}_{2l} = \delta_{2h} \mp (2c_1 - 3c_2) / (c_1 - c_2) \epsilon$. Since $\tilde{\pi}_1 \leq \pi_1$ for any small $\epsilon > 0$ in both cases, we get $p_1 = \lambda c_1 c_2 / (c_1 - c_2) \delta_1$. Similarly, set $\tilde{\delta}_{2h} = \delta_{2h} + \alpha \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} + \beta \epsilon$; then $\tilde{\delta}_1 = \delta_1 - (\alpha + \beta) \epsilon$ and the prices are $\tilde{p}_{2h} = p_{2h} - \lambda c_1 (2\alpha + \beta) \epsilon$ and $\tilde{p}_{2l} = p_{2l} - \lambda c_2 (3\alpha + 2\beta) \epsilon$ since high-priority and low-priority services of SP 2 have to satisfy impatient and patient customers respectively. Furthermore, $\tilde{\pi}_2 \leq \pi_2$ implies

$$\begin{aligned} & \lambda(p_{2h} - \lambda c_1 (2\alpha + \beta) \epsilon) (\delta_{2h} + \alpha \epsilon) + \lambda(p_{2l} - \lambda c_2 (3\alpha + 2\beta) \epsilon) (\delta_{2l} + \beta \epsilon) \leq \lambda \delta_{2h} p_{2h} + \lambda \delta_{2l} p_{2l} \forall \alpha, \beta \\ \Leftrightarrow & \lambda \epsilon (\alpha(p_{2h} - 2\lambda c_1 \delta_{2h} - 3\lambda c_2 \delta_{2l}) + \beta(p_{2l} - \lambda c_1 \delta_{2h} - 2\lambda c_2 \delta_{2l})) \\ & - \lambda^2 \epsilon^2 (2c_1 \alpha^2 + (c_1 + 3c_2) \alpha \beta + 2c_2 \beta^2) \leq 0 \forall \alpha, \beta \end{aligned} \quad (27)$$

Hence $p_{2h} = 2\lambda c_1 \delta_{2h} + 3\lambda c_2 \delta_{2l}$ and $p_{2l} = \lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l}$ for equilibrium. Also, $p_1 + \lambda c_1 \delta_1 = p_{2h} + \lambda c_1 \delta_{2h}$, $p_1 + \lambda c_2 \delta_1 = p_{2l} + 2\lambda c_2 \delta_{2h} + \lambda c_2 \delta_{2l}$, and $\delta_1 + \delta_{2h} + \delta_{2l} = 1$. These three equations can be solved to obtain the purchase fractions. After some algebra, we obtain $\delta_1 = \frac{6c_1 c_2 - 6c_2^2}{12c_1 c_2 - c_1^2 - 9c_2^2}$, $\delta_{2h} = \frac{3c_1 c_2 - 3c_2^2}{12c_1 c_2 - c_1^2 - 9c_2^2}$, and $\delta_{2l}^e = \frac{3c_1 c_2 - c_1^2}{12c_1 c_2 - c_1^2 - 9c_2^2}$. Since $\delta_1 > 0$, we have $12c_1 c_2 - c_1^2 - 9c_2^2 > 0$ or $(6 - 3\sqrt{3})c_2 < c_1 < (6 + 3\sqrt{3})c_2$. The left inequality is automatically satisfied since $c_1 > c_2$. Also, since $\delta_{2l} > 0$, we need $3c_1 c_2 - c_1^2 > 0$ so that $c_1 < 3c_2 < (6 + 3\sqrt{3})c_2$. We then note that the expression

$2c_1\alpha^2 + (c_1 + 3c_2)\alpha\beta + 2c_2\beta^2 = 2c_1(\alpha + (c_1 + 3c_2)/(4c_1)\beta)^2 + (c_1 - c_2)(9c_2 - c_1)/(8c_1)\beta^2 \geq 0 \forall \alpha, \beta$, so that (27) is always satisfied.

Therefore, when $p_{2h} > p_1 > p_{2l}$, equilibrium is possible only with the pricing structure in (b). Furthermore, any interior equilibrium is possible only if $c_1 < 3c_2$, and it should have the prices and purchase fractions that were determined above.

B.2.2 $p_1 > p_{2h} > p_{2l}$

As in § B.2.1, we can show that an equilibrium is impossible for the pricing structure in part (a) of Figure 4. Similarly, for part (c), $\delta_{2l} = 0$ and $\delta_1 > q$. Hence profit of SP 2 = $\lambda\delta_2(p_1 + \lambda c_2\delta_1 - \lambda c_2\delta_2)$. Suppose we change the prices p_{2h} and p_{2l} so that the high-priority service only satisfies ϵ fraction of impatient customers and the low-priority service only satisfies $\delta_2 - \epsilon$ fraction of patient customers. Then we find $\tilde{p}_{2h} = p_1 + \lambda c_1\delta_1 - \lambda c_1\epsilon$ and $\tilde{p}_{2l} = p_1 + \lambda c_2(\delta_1 - \delta_2 - \epsilon)$. The new profit of SP 2 is given by

$$\begin{aligned} & \lambda\tilde{p}_{2h}\epsilon + \lambda\tilde{p}_{2l}(\delta_2 - \epsilon) \\ &= \lambda(p_1 + \lambda c_1\delta_1 - \lambda c_1\epsilon)\epsilon + \lambda(p_1 + \lambda c_2(\delta_1 - \delta_2 - \epsilon))(\delta_2 - \epsilon) \\ &> \lambda(p_1 + \lambda c_2\delta_1 - \lambda c_2\epsilon)\epsilon + \lambda(p_1 + \lambda c_2(\delta_1 - \delta_2 - \epsilon))(\delta_2 - \epsilon) \\ &= \lambda\delta_2(p_1 + \lambda c_2\delta_1 - \lambda c_2\delta_2). \end{aligned}$$

Since the profit of SP 2 is higher an equilibrium does not exist with the pricing structure in (c). Finally, we show equilibrium with the structure in (b) is also not possible. Again, $\delta_{2l} = 0$ because otherwise p_1 can be increased slightly, while maintaining the sales, to increase the profit of SP 1. Also, $\delta_{2h} = \delta_2 > 1 - q$ because otherwise p_{2h} and p_{2l} can both be increased slightly by the same amount to increase the profit of SP 2. Let $\tilde{\delta}_{2h} = \delta_{2h} + \alpha\epsilon$ and $\tilde{\delta}_{2l} = \beta\epsilon$ with $\beta > 0$. Then $\tilde{p}_{2h} = p_{2h} - \lambda c_1(2\alpha + \beta)\epsilon$ and $\tilde{p}_{2l} = p_{2l} - \lambda c_1(2\alpha + \beta)\epsilon - \lambda c_2(\alpha + \beta)\epsilon$. For equilibrium, $\lambda\tilde{p}_{2h}\tilde{\delta}_{2h} + \lambda\tilde{p}_{2l}\tilde{\delta}_{2l} \leq \lambda p_{2h}\delta_{2h}$ for any small $\epsilon > 0$ and $\forall \alpha < 0, \beta > 0$, which implies $p_{2h} \geq 2\lambda c_1\delta_{2h}$ and $p_{2l} \leq \lambda c_1\delta_{2h}$ so that $p_{2h} - p_{2l} \geq \lambda c_1\delta_{2h} > \lambda c_2\delta_{2h} = \lambda c_2\delta_2$, a contradiction since $p_{2h} - p_{2l} = \lambda c_2\delta_2$ in (b). Hence, it is impossible to have $p_1 > p_{2h} > p_{2l}$ (with any pricing structure) at equilibrium.

B.2.3 $p_{2h} > p_{2l} > p_1$

Consider the structure in part (a) of Figure 4. Then $\delta_{2l} = 0$ because otherwise p_1 can be increased slightly to increase the profit of SP 1 (then $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ so that $\tilde{\delta}_1 = \delta_1$). Also, $\delta_1 > 1 - q$ because otherwise p_1 can be increased while still satisfying all patient customers. Let $\tilde{\delta}_1 = \delta_1 - \epsilon$; then we find that $\tilde{p}_1 = p_1 + 2\lambda c_1\epsilon$. Requiring $\lambda\tilde{\delta}_1\tilde{p}_1 \leq \lambda\delta_1 p_1$ for any small $\epsilon > 0$ implies

$p_1 \geq 2\lambda c_1 \delta_1$. Now let $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = 0$; also, set $\tilde{p}_{2h} = p_{2h} - 2\lambda c_1 \epsilon = \tilde{p}_{2l}$. Requiring $\tilde{\pi}_2 \leq \pi_2$ implies $p_{2h} \leq 2\lambda c_1 \delta_{2h} < 2\lambda c_1 \delta_1 \leq p_1$ so that $p_{2h} < p_1$, a contradiction.

Consider the structure in part (b) of the figure. Then $\delta_1 + \delta_{2l} = 1 - q$ because if $\delta_1 + \delta_{2l} > 1 - q$ then p_1 can be increased slightly to increase the profit of SP 1. Let $\tilde{\delta}_1 = \delta_1 - \epsilon$; then we find that $\tilde{p}_1 = p_1 + 2\lambda c_2 \epsilon$. Requiring $\tilde{\pi}_1 \leq \pi_1$ for any small $\epsilon > 0$ implies $p_1 \geq 2\lambda c_2 \delta_1$. Now let $\tilde{\delta}_{2h} = \delta_{2h} = q$ and $\tilde{\delta}_{2l} = \delta_{2l} + \epsilon$; then $\tilde{p}_{2h} = p_{2h} + \lambda(c_1 - 2c_2)\epsilon$ and $\tilde{p}_{2l} = p_{2l} - 2\lambda c_2 \epsilon$. Requiring $\tilde{\pi}_2 \leq \pi_2$ for any small $\epsilon > 0$ implies, after some algebra, that $p_{2l} \leq 2\lambda c_2 \delta_{2l} + \lambda(2c_2 - c_1)\delta_{2h} < 2\lambda c_2 \delta_2 < 2\lambda c_2 \delta_1 \leq p_1$ so that $p_{2l} < p_1$, a contradiction (note that $\delta_2 < 2\delta_{2h} + \delta_{2l} < \delta_1$ since $p_{2l} > p_1$ and $p_{2l} + \lambda c_2(2\delta_{2h} + \delta_{2l}) = p_1 + \lambda c_2 \delta_1$).

Consider the structure in part (c) of the figure. Then $\delta_{2l} = 0$ because otherwise p_1 can be increased slightly to increase the profit of SP 1. Also, $\delta_{2h} > q$; otherwise p_{2h} can be increased slightly to increase the profit of SP 2. A contradiction can then be shown similar to the one with part (a) structure (we now get $p_{2h} \leq 2\lambda c_2 \delta_{2h} < 2\lambda c_2 \delta_1 \leq p_1$). *Hence, it is impossible to have $p_{2h} > p_{2l} > p_1$ (with any pricing structure) at equilibrium.*

B.2.4 $p_1 = p_{2h} > p_{2l}$

Suppose impatient customers are indifferent, i.e., $p_1 + \lambda c_1 \delta_1 = p_{2h} + \lambda c_1 \delta_{2h} = p_{2l} + 2\lambda c_1 \delta_{2h} + \lambda c_1 \delta_{2l}$. Then $\delta_{2l} \geq 1 - q$. If $\delta_{2l} > 1 - q$ then p_1 can be increased slightly (by $\lambda c_1 \epsilon$), while maintaining the sales ($\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ so that $\tilde{\delta}_1 = \delta_1$), to increase the profit of SP 1. If $\delta_{2l} = 1 - q$ then p_{2l} can be increased, while maintaining δ_{2h} and δ_{2l} at the same values, to increase the profit of SP 2.

Suppose patient customers are indifferent. Then $\delta_{2l} = 0$ because otherwise p_1 can be increased without affecting the sales to increase the profit of SP 1. Hence $\delta_1 = \delta_{2h} = 0.5$ and the profit of SP 2 is $0.5\lambda p_{2h}$. Suppose $q \geq 0.5$. Let the high-priority service of SP only satisfy $0.5 - \epsilon$ fraction of impatient customers and the low-priority service only satisfy ϵ fraction of patient customer, with a low $\epsilon > 0$. Then $\tilde{\delta}_{2h} = 0.5 - \epsilon$, $\tilde{\delta}_{2l} = \epsilon$, $\tilde{p}_{2h} = p_{2h} + \lambda c_1 \epsilon$, and $\tilde{p}_{2l} = p_{2h} - 0.5\lambda c_2 + \lambda c_2 \epsilon$. The new profit is $\lambda(p_{2h} + \lambda c_1 \epsilon)(0.5 - \epsilon) + \lambda(p_{2h} - 0.5\lambda c_2 + \lambda c_2 \epsilon)(\epsilon) = 0.5\lambda p_{2h} + \lambda^2(c_1 - c_2)\epsilon(0.5 - \epsilon) > 0.5\lambda p_{2h}$ so that there is no equilibrium. Suppose $q < 0.5$. Let $\tilde{\delta}_{2h} = \epsilon$ (only impatient customers) and $\tilde{\delta}_{2l} = 0.5 - \epsilon$ (only patient customers) so that $\tilde{p}_{2h} = p_{2h} + \lambda c_1(0.5 - \epsilon)$ and $\tilde{p}_{2l} = p_{2h} - \lambda c_2 \epsilon$. The new profit is $\lambda(p_{2h} + \lambda c_1(0.5 - \epsilon))(\epsilon) + \lambda(p_{2h} - \lambda c_2 \epsilon)(0.5 - \epsilon) = 0.5\lambda p_{2h} + \lambda^2(c_1 - c_2)\epsilon(0.5 - \epsilon) > 0.5\lambda p_{2h}$ so that there is no equilibrium. *Hence, it is impossible to have $p_1 = p_{2h} > p_{2l}$ (with any pricing structure) at equilibrium.*

B.2.5 $p_{2h} > p_{2l} = p_1$

I Suppose impatient customers are indifferent among the three options. If $\delta_{2l} > 1 - q$ then p_1 can be increased slightly (by $\lambda c_1 \epsilon$), while maintaining the sales, to increase the profit of SP 1. So $\delta_{2l} \leq 1 - q$. Suppose $\delta_{2l} > 0$. Then there are two possibilities:

(i) $\delta_{2h} < q$. Let $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ (only impatient customers) and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ (only patient ones); then $\tilde{p}_{2h} = p_{2h} - \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} - \lambda c_2 \epsilon$. The new profit is $\lambda(p_{2h} - \lambda c_1 \epsilon)(\delta_{2h} + \epsilon) + \lambda(p_{2l} - \lambda c_2 \epsilon)(\delta_{2l} - \epsilon) = \lambda \delta_{2h} p_{2h} + \lambda \delta_{2l} p_{2l} + \lambda \epsilon (p_{2h} - p_{2l} - \lambda c_1 \delta_{2h} - \lambda c_2 \delta_{2l} - \lambda(c_1 - c_2)\epsilon) > \lambda \delta_{2h} p_{2h} + \lambda \delta_{2l} p_{2l}$ for small $\epsilon > 0$ since $p_{2h} - p_{2l} = \lambda c_1 \delta_2 > \lambda c_1 \delta_{2h} + \lambda c_2 \delta_{2l}$. So there is no equilibrium.

(ii) $\delta_{2h} = q$. Then $\delta_1 = (1 + q)/2 > \delta_{2h}$ and $\delta_{2l} = (1 - 3q)/2$. Since $\delta_{2l} > 0, q < 1/3$ and $\delta_1 < 1 - q$. Let $\tilde{\delta}_{2h} = \delta_{2h} + \alpha \epsilon$ (satisfy only impatient customers) and $\tilde{\delta}_{2l} = \delta_{2l} + \beta \epsilon$ with $\alpha \leq 0$. Then $\tilde{p}_{2l} = p_{2l} - \lambda c_2 (3\alpha + 2\beta)\epsilon$; however, $\tilde{p}_{2h}$ depends on whether $\tilde{p}_{2l}$ is greater than p_1 and impatient customers' best preferences. If $3\alpha + 2\beta \geq 0$ then $\tilde{p}_{2h} = p_{2h} - \lambda c_1 (2\alpha + \beta)\epsilon$ and impatient customers are indifferent between SP 1 and high-priority service of SP 2; however, if $3\alpha + 2\beta \leq 0$ then $\tilde{p}_{2l} = p_{2h} - \lambda c_2 (3\alpha + 2\beta)\epsilon + \lambda c_1 (\alpha + \beta)\epsilon$ and impatient customers are indifferent between high- and low-priority services of SP 2. Since we require $\lambda \tilde{p}_{2h} \tilde{\delta}_{2h} + \lambda \tilde{p}_{2l} \tilde{\delta}_{2l} \leq \lambda p_{2h} \delta_{2h} + \lambda p_{2l} \delta_{2l}$, we obtain the following inequalities:

$$\begin{aligned} & \alpha(p_{2h} - 2\lambda c_1 \delta_{2h} - 3\lambda c_2 \delta_{2l}) + \beta(p_{2l} - \lambda c_1 \delta_{2h} - 2\lambda c_2 \delta_{2l}) \\ & - \lambda \epsilon (2c_1 \alpha^2 + (c_1 + 3c_2)\alpha\beta + 2c_2 \beta^2) \leq 0 \quad \forall \alpha \leq 0; \beta \geq -1.5\alpha \end{aligned} \quad (28)$$

$$\begin{aligned} & \alpha(p_{2h} - 3\lambda c_2 \delta_{2h} + \lambda c_1 \delta_{2h} - 3\lambda c_2 \delta_{2l}) + \beta(p_{2l} - 2\lambda c_2 \delta_{2h} + \lambda c_1 \delta_{2h} - 2\lambda c_2 \delta_{2l}) \\ & - \lambda \epsilon ((3c_2 - c_1)\alpha^2 + (5c_2 - c_1)\alpha\beta + 2c_2 \beta^2) \leq 0 \quad \forall \alpha \leq 0; \beta \leq -1.5\alpha \end{aligned} \quad (29)$$

Setting $\alpha = 0$ and $\beta > 0$ implies $p_{2l} \leq \lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l}$ while setting $\alpha = 0$ and $\beta < 0$ implies $p_{2l} \geq \lambda(2c_2 - c_1)\delta_{2h} + 2\lambda c_2 \delta_{2l}$. Furthermore, setting $\beta = -1.5\alpha$ in either (28) or (29), and requiring it to hold for any $\epsilon > 0$, results in the same inequality $p_{2h} - 1.5p_{2l} - 0.5\lambda c_1 \delta_{2h} \geq 0$. We note that $p_{2l} \leq \lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l} \Rightarrow p_{2h} - 1.5p_{2l} - 0.5\lambda c_1 \delta_{2h} = \lambda c_1 \delta_2 - 0.5p_{2l} - 0.5\lambda c_1 \delta_{2h} = 0.5\lambda c_1 \delta_{2h} + \lambda c_1 \delta_{2l} - 0.5p_{2l} \geq 0.5\lambda c_1 \delta_{2h} + \lambda c_1 \delta_{2l} - 0.5(\lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l}) = \lambda(c_1 - c_2)\delta_{2l} > 0$. Hence we require $\lambda(2c_2 - c_1)\delta_{2h} + 2\lambda c_2 \delta_{2l} \leq p_{2l} \leq \lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l}$. For any given δ_1, δ_{2h} , and δ_{2l} , since a higher p_{2l} implies a higher p_{2h} and p_1 thereby resulting *Pareto improving* higher profits for both SPs, we only consider $p_{2l} = \lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l} = \lambda c_1 q + \lambda c_2 (1 - 3q)$ in the rest of the analysis for better exposition and easier tractability. Then $p_{2h} = p_{2l} + \lambda c_1 \delta_2 = 0.5\lambda c_1 (1 + q) + \lambda c_2 (1 - 3q)$. Next, we show that (28) and (29) are together equivalent to

$$\alpha(c_1 - c_2)\delta_{2l} - \epsilon (2c_1 \alpha^2 + (c_1 + 3c_2)\alpha\beta + 2c_2 \beta^2) \leq 0 \quad \forall \alpha \leq 0, \beta. \quad (30)$$

Clearly, (30) implies (28) since $\lambda > 0$ times the left hand side of (30) is the left hand side of (28). It also implies (29) since λ times the left hand side of (30) minus the left hand side of (29) is $(3\alpha + 2\beta)\lambda(c_2 - c_1)\delta_{2h} - \alpha\lambda\epsilon(c_1 - c_2)(3\alpha + 2\beta) \geq 0 \forall \alpha \leq 0, 3\alpha + 2\beta \leq 0$. Suppose (30) does not hold. Then $\exists \alpha \leq 0, \beta$ such that $2c_1\alpha^2 + (c_1 + 3c_2)\alpha\beta + 2c_2\beta^2 < 0$. For any given α , this function is strictly convex in β so that its minimum value is obtained when $\beta = -\frac{c_1 + 3c_2}{4c_2} \cdot \alpha$. Substituting this value of β , the function becomes $\frac{(c_1 - c_2)(9c_2 - c_1)}{8c_2} \alpha^2$ which is negative if and only if $c_1 > 9c_2$. Furthermore, $\beta = -\frac{c_1 + 3c_2}{4c_2} \cdot \alpha \geq -3\alpha \geq -1.5\alpha \forall \alpha \leq 0, c_1 > 9c_2$. Hence (28) does not hold. Therefore, instead of checking if both (28) and (29), we can just check if (30) is satisfied for equilibrium to exist. We note that a similar result also applies for *larger deviations* by SP 2. If $\tilde{\delta}_{2h} = \delta_{2h} + x$ and $\tilde{\delta}_{2l} = \delta_{2h} + y$ then (30) becomes

$$x(c_1 - c_2)\delta_{2l} - (2c_1x^2 + (c_1 + 3c_2)xy + 2c_2y^2) \leq 0 \forall x \leq 0, y. \quad (31)$$

We will use inequality (31) later (in proving Theorem 2) to derive additional conditions for existence of equilibrium. Furthermore, setting $\tilde{\delta}_1 = \delta_1 - \epsilon$ implies $\tilde{p}_1 = p_1 + \lambda c_1 c_2 / (c_1 - c_2) \epsilon$; then $\tilde{\pi}_1 \leq \pi_1$ requires $p_1 \geq \lambda c_1 c_2 / (c_1 - c_2) \delta_1$ which results in $q(9c_1 c_2 - 2c_1^2 - 6c_2^2) \leq c_2(c_1 - 2c_2)$. We find that this inequality is automatically satisfied (for any $q \leq 1/3$) when $c_1 \geq 3c_2$. Hence, it can be simplified as $q \leq \frac{c_2(c_1 - 2c_2)}{9c_1 c_2 - 2c_1^2 - 6c_2^2} \forall c_1 < 3c_2$.

Suppose $\delta_{2l} = 0$. Then $\delta_{2h} = 1/3 \leq q$ and $\delta_1 = 2/3 \geq 1 - q$. The inequalities in (28) and (29) still apply for any $\alpha \leq 0$ and $\beta \geq 0$. Setting $\alpha = 0$ and $\beta > 0$ implies $p_{2l} \leq \lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l} = \lambda c_1 / 3$ (from (28)). Set $\alpha = -1$ and $\beta = 0$; that can be obtained by having $\tilde{p}_{2h} = \tilde{p}_{2l} = p_{2h} + 2\lambda c_1 \epsilon$. Then $\tilde{\pi}_2 \leq \pi_2 \forall \epsilon > 0$ requires $p_{2h} \geq 2\lambda c_1 \delta_{2h} = 2\lambda c_1 / 3$. Since $p_{2h} - p_{2l} = \lambda c_1 \delta_2 = \lambda c_1 / 3$, we have $p_{2h} = 2\lambda c_1 / 3$ and $p_{2l} = p_1 = \lambda c_1 / 3$. As before (when $\delta_{2h} = q$ and $\delta_{2l} > 0$), we can again show that inequalities (28) and (29) are together equivalent to (30). Hence, for equilibrium to exist, $\min_{\alpha \leq 0, \beta \geq 0} 2c_1\alpha^2 + (c_1 + 3c_2)\alpha\beta + 2c_2\beta^2 = \min_{\alpha \leq 0} \frac{(c_1 - c_2)(9c_2 - c_1)}{8c_2} \cdot \alpha^2 \geq 0$, i.e., $c_1 \leq 9c_2$. Next, we consider the deviation in SP 1. Set $\tilde{\delta}_1 = \delta_1 - \epsilon$. After some algebra, since $c_1 > 2c_2$, we find that $\tilde{\delta}_{2h} = \delta_{2h} - \frac{c_1 - 2c_2}{c_1 - c_2} \cdot \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} + \frac{2c_1 - 3c_2}{c_1 - c_2} \cdot \epsilon$, which implies $\tilde{p}_1 = p_1 + \lambda c_1 c_2 / (c_1 - c_2) \epsilon$. Then $\tilde{\pi}_1 \leq \pi_1$ for any $\epsilon > 0$ implies $p_1 = \lambda c_1 / 3 \geq \lambda c_1 c_2 / (c_1 - c_2) \delta_1$ or $c_1 \geq 3c_2$.

II Suppose patient customers are indifferent. Then $\delta_{2l} = 0$ because otherwise p_1 can be increased slightly to increase the profit of SP 1. Since $\delta_1 = 2\delta_{2h} + \delta_{2l} = 2\delta_{2h}$ and $\delta_1 + \delta_{2h} = 1$, we have $\delta_{2h} = 1/3$ and $\delta_1 = 2/3$. Also, $\delta_{2h} > q$ because p_{2h} can otherwise be increased to increase the profit of SP 2. Let $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} = 0$. Then set $\tilde{p}_{2h} = p_{2h} - 2\lambda c_2 \epsilon = \tilde{p}_{2l}$ so that $\lambda \tilde{p}_{2h} \tilde{\delta}_{2h} \leq \lambda p_{2h} \delta_{2h}$ implies $p_{2h} \leq 2\lambda c_2 \delta_{2h} = 2\lambda c_2 / 3$. Similarly, setting $\tilde{\delta}_1 = \delta_1 - \epsilon$ gives

$\tilde{p}_1 = p_1 + 2\lambda c_2 \epsilon$; then $\lambda \tilde{p}_1 \tilde{\delta}_1 \leq \lambda p_1 \delta_1$ implies $p_1 \geq 2\lambda c_2 \delta_1 = 4\lambda c_2/3 > 2\lambda c_2/3 \geq p_{2h}$ so that $p_1 > p_{2h}$, a contradiction.

Hence, when $p_{2h} > p_{2l} = p_1$, an equilibrium is possible only if impatient customers are different. There are two cases. The first case is an “extreme” equilibrium with $\delta_{2h} = q < 1/3$, $\delta_1 = 0.5(1+q)$, and $\delta_{2l} = 0.5(1-3q) > 0$. In this case, we consider only the “maximum” prices with $p_{2h} = 0.5\lambda c_1(1+q) + \lambda c_2(1-3q)$ and $p_{2l} = p_1 = \lambda c_1 q + \lambda c_2(1-3q)$. Furthermore, it requires $q \leq \frac{c_2(c_1-2c_2)}{9c_1c_2-2c_1^2-6c_2^2} \forall c_1 < 3c_2$. The second case is a reduced class equilibrium which requires $q \geq 1/3$, $p_{2h} = 2\lambda c_1/3$, $p_1 = p_{2l} = \lambda c_1/3$, $\delta_{2h} = 1/3$, $\delta_1 = 2/3$, $\delta_{2l} = 0$, and $3c_2 \leq c_1 \leq 9c_2$.

C. $\mathcal{DD}$

C.1 Customer Sub-game

Proposition 3 examines the uniqueness of equilibrium purchase fractions for any given prices.

Proposition 3 *The equilibrium purchase fractions δ_1^e and δ_2^e are unique for any given*

($p_{1h}, p_{1l}, p_{2h}, p_{2l}$). Suppose $\delta_1^e, \delta_2^e > 0$. There are three possibilities:

1. *If $p_{ih} - p_{il} < \lambda c_2 \delta_i^e$ or $p_{ih} - p_{il} > \lambda c_1 \delta_i^e$ for any $i = 1, 2$ then it's equivalent to SP i offering a single service class with $\delta_{il} = 0$ or $\delta_{ih} = 0$ respectively; the other $\delta_{ik}^e; i = 1, 2; k = h, l$ are obtained from the corresponding results under $\mathcal{SS}$ (Proposition 1) or $\mathcal{SD}$ (Proposition 2) as applicable.*
2. *If $\lambda c_2 \delta_i^e < p_{ih} - p_{il} < \lambda c_1 \delta_i^e$ then $\delta_{ik}^e; i = 1, 2; k = h, l$, are unique.*
3. *In all other cases, i.e., $p_{ih} - p_{il} = \lambda c_2 \delta_i^e$ or $\lambda c_1 \delta_i^e$ and $\lambda c_2 \delta_{i'}^e \leq p_{i'h} - p_{i'l} \leq \lambda c_1 \delta_{i'}^e; i, i' = 1, 2; i \neq i'$, there exist unique $\delta_{ik}^e; i = 1, 2; k = h, l$ which minimize customers' dis-utilities.*

Proof The proof for uniqueness of δ_1^e and δ_2^e is similar to the one in Proposition 2, and uses contradiction. We use the property that the dis-utility from the best service class at SP i is increasing in δ_i . Suppose $\delta_1' < \delta_1''$ so that $\delta_2' > \delta_2''$. Since $\delta_1' > 0$, one of these two inequalities has to be satisfied:

$$\begin{aligned} \min(p_{1h} + c_2 + \lambda c_2 \delta_{1h}'', p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}'' + \lambda c_2 \delta_{1l}'') \\ \leq \min(p_{2h} + c_2 + \lambda c_2 \delta_{2h}'', p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}'', v) \end{aligned} \quad (32)$$

$$\begin{aligned} \min(p_{1h} + c_1 + \lambda c_1 \delta_{1h}'', p_{1l} + c_1 + 2\lambda c_1 \delta_{1h}'' + \lambda c_1 \delta_{1l}'') \\ \leq \min(p_{2h} + c_1 + \lambda c_1 \delta_{2h}'', p_{2l} + c_1 + 2\lambda c_1 \delta_{2h}'' + \lambda c_1 \delta_{2l}'', v) \end{aligned} \quad (33)$$

Also, since $\delta'_2 > 0$, one of these two inequalities has to be satisfied:

$$\begin{aligned} \min(p_{2h} + c_2 + \lambda c_2 \delta'_{2h}, p_{2l} + c_2 + 2\lambda c_2 \delta'_{2h} + \lambda c_2 \delta'_{2l}) \\ \leq \min(p_{1h} + c_2 + \lambda c_2 \delta'_{1h}, p_{1l} + c_2 + 2\lambda c_2 \delta'_{1h} + \lambda c_2 \delta'_{1l}, v) \end{aligned} \quad (34)$$

$$\begin{aligned} \min(p_{2h} + c_1 + \lambda c_1 \delta'_{2h}, p_{2l} + c_1 + 2\lambda c_1 \delta'_{2h} + \lambda c_1 \delta'_{2l}) \\ \leq \min(p_{1h} + c_2 + \lambda c_2 \delta'_{1h}, p_{1l} + c_2 + 2\lambda c_2 \delta'_{1h} + \lambda c_2 \delta'_{1l}, v) \end{aligned} \quad (35)$$

Suppose (32) holds. Then (34) is violated, (35) has to hold, and (33) is violated. Also, (32) implies $\min(p_{1h} + c_2 + \lambda c_2 \delta'_{1h}, p_{1l} + c_2 + 2\lambda c_2 \delta'_{1h} + \lambda c_2 \delta'_{1l}) < \min(p_{2h} + c_2 + \lambda c_2 \delta'_{2h}, p_{2l} + c_2 + 2\lambda c_2 \delta'_{2h} + \lambda c_2 \delta'_{2l}, v)$. Hence $\delta'_1 \geq 1 - q$ because otherwise it can be increased by a low $\epsilon > 0$ (while keeping $\delta_2 = \delta'_2$) to get a better solution. This is done as follows: if $p_{1h} - p_{1l} < \lambda c_2 \delta'_1$ then set $\delta'_{12h} = \delta'_1 + \epsilon$ (and $\delta'_{12l} = \delta'_{11h} = \delta'_{11l} = 0$) while if $p_{1h} - p_{1l} > \lambda c_2 \delta'_1$, set $\delta'_{12l} = \delta'_1 + \epsilon$ (and $\delta'_{12h} = \delta'_{11h} = \delta'_{11l} = 0$). If $p_{1h} - p_{1l} = \lambda c_2 \delta'_1$ then set $\delta'_{12h} = \delta'_1 + \epsilon$ (and $\delta'_{12l} = \delta'_{11h} = \delta'_{11l} = 0$). However, unlike the other two cases, note that both δ'_{12h} and δ'_{12l} might have been positive before being set to their current values. Nevertheless the new solution is feasible because (32) holds and $p_{1h} + c_2 + \lambda c_2 (\delta'_1 + \epsilon) \leq p_{1h} + c_2 + \lambda c_2 \delta''_1 = \min(p_{1h} + c_2 + \lambda c_2 \delta''_{1h}, p_{1l} + c_2 + 2\lambda c_2 \delta''_{1h} + \lambda c_2 \delta''_{1l})$ in which the equality results from $p_{1h} - p_{1l} = \lambda c_2 \delta'_1 < \lambda c_2 \delta''_1$. Since $\delta'_1 \geq 1 - q$, we have $\delta''_1 > 1 - q$ which implies $\delta''_{11} > 0$ and (33) should hold, a contradiction. A similar contradiction can be shown if we start from (33) being satisfied instead of (32).

Suppose $\delta_1^e, \delta_2^e > 0$. If $p_{ih} - p_{il} < \lambda c_2 \delta_i^e < \lambda c_1 \delta_i^e$ for any SP i then *none of the customers purchase the low-priority service* from SP i because they strictly prefer her high-priority service. That is equivalent to SP i just offering a single service (with $p_i = p_{ih}$). A similar scenario occurs if $p_{ih} - p_{il} > \lambda c_1 \delta_i^e > \lambda c_2 \delta_i^e$. That covers the first possibility in the proposition.

Next, we consider examine the uniqueness of $\delta_{ik}^e; i = 1, 2; k = h, l$ under the second and third possibilities. Suppose $'$ and $''$ denote two sets of optimal purchase fractions and $\delta'_{1h} < \delta''_{1h}$ (without loss of generality) so that $\delta'_{1l} > \delta''_{1l}$. Since $\delta''_{1h} > 0$, one of these two inequalities has to be satisfied:

$$p_{1h} + c_2 + \lambda c_2 \delta''_{1h} \leq \min(p_{1l} + c_2 + 2\lambda c_2 \delta''_{1h} + \lambda c_2 \delta''_{1l}, p_{2l} + c_2 + 2\lambda c_2 \delta''_{2h} + \lambda c_2 \delta''_{2l}) \quad (36)$$

$$p_{1h} + c_1 + \lambda c_1 \delta''_{1h} \leq \min(p_{2h} + c_1 + \lambda c_1 \delta''_{2h}, v) \quad (37)$$

Similarly, since $\delta'_{1l} > 0$, one of these two inequalities has to be satisfied:

$$p_{1l} + c_2 + 2\lambda c_2 \delta'_{1h} + \lambda c_2 \delta'_{1l} \leq p_{2l} + c_2 + 2\lambda c_2 \delta'_{2h} + \lambda c_2 \delta'_{2l} \quad (38)$$

$$p_{1l} + c_1 + 2\lambda c_1 \delta'_{1h} + \lambda c_1 \delta'_{1l} \leq \min(p_{1h} + c_1 + \lambda c_1 \delta'_{1h}, p_{2h} + c_1 + \lambda c_1 \delta'_{2h}, v) \quad (39)$$

Suppose $\lambda c_2 \delta_i^e < p_{ih} - p_{il} < \lambda c_1 \delta_i^e; i = 1, 2$. Then (37) and (38) have to be satisfied. If $\delta_{2h}'' < \delta_{2h}'$ then (37) implies $p_{1h} + c_1 + \lambda c_1 \delta_{1h}' < \min(p_{2h} + c_1 + \lambda c_1 \delta_{2h}', v)$ so that $\delta_{21h}' = \delta_{2h}' = 0 > \delta_{2h}''$, a contradiction. Hence $\delta_{2h}'' \geq \delta_{2h}'$ which implies $\delta_{2l}' \geq \delta_{2l}''$. Therefore $\delta_{1l}' + \delta_{2l}' > \delta_{1l}'' + \delta_{2l}'' = 1 - q$. Note that when δ'' 's are the purchase fractions, all the patient customers are satisfied by the low-priority services from SPs 1 and 2 since $\lambda c_2 \delta_i^e < p_{ih} - p_{il}; i = 1, 2$, and $\delta_{1h}'' = \delta_{11h}'' > 0$. However, it is impossible to have $\delta_{1l}' + \delta_{2l}' > 1 - q$ since $\delta_{11l}' + \delta_{21l}' = 0$ as $p_{ih} - p_{il} < \lambda c_1 \delta_i^e; i = 1, 2$. Hence $\delta_{ik}^e; i = 1, 2; k = h, l$ are uniquely determined.

Next, we look at the other cases, i.e., $p_{ih} - p_{il} = \lambda c_2 \delta_i^e, \lambda c_1 \delta_i^e$ and $\lambda c_2 \delta_{i'}^e \leq p_{i'h} - p_{i'l} \leq \lambda c_1 \delta_{i'}^e; i, i' = 1, 2; i \neq i'$. As before, suppose $\delta_{1h}' < \delta_{1h}''$. If $\delta_{2h}' \leq \delta_{2h}''$ then having δ' 's instead of δ'' 's as purchase fractions results in lower dis-utilities for all customers. However, if $\delta_{2h}' > \delta_{2h}''$ then the customers would have the lowest dis-utilities with the set of purchase fractions given by $(\delta_{1h}', \delta_{1l}', \delta_{2h}'', \delta_{2l}'')$. However, the key question is whether these new purchase fractions are feasible. Next, we prove that they are feasible indeed.

We first note that since $\delta_{2h}' > 0$, one of these two inequalities has to be satisfied:

$$p_{2h} + c_2 + \lambda c_2 \delta_{2h}' \leq \min(p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}' + \lambda c_2 \delta_{1l}', p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}' + \lambda c_2 \delta_{2l}') \quad (40)$$

$$p_{2h} + c_1 + \lambda c_1 \delta_{2h}' \leq \min(p_{1h} + c_1 + \lambda c_1 \delta_{1h}', v) \quad (41)$$

Also, since $\delta_{1h}'' > 0$, either (36) or (37) is satisfied. The following two cases are possible:

1. (36) is satisfied. Then $p_{1h} - p_{1l} = \lambda c_2 \delta_1^e$. Also, (40) is violated since $p_{2h} + c_2 + \lambda c_2 \delta_{2h}' > p_{2h} + c_2 + \lambda c_2 \delta_{2h}'' \geq p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}'' \geq p_{1h} + c_2 + \lambda c_2 \delta_{1h}'' \geq p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}'' + \lambda c_2 \delta_{1l}'' > p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}' + \lambda c_2 \delta_{1l}'$. Hence $\delta_{22h}' = \delta_{22l}' = 0$. Since $\delta_2^e = \delta_{21}^e > 0$, all the patient customers are therefore satisfied by SP 1 so that $\delta_1^e \geq 1 - q$. Since (40) is violated, (41) has to be satisfied thereby violating (37), which implies $\delta_{11h}'' = 0$. Also, since $p_{1h} - p_{1l} < \lambda c_1 \delta_1^e$, $\delta_{11l}'' = 0$. So SP 1 satisfies only patient customers. If $\delta_{22l}'' > 0$ then we get $\delta_1^e < 1 - q$, a contradiction. Instead, $\delta_1^e = 1 - q$ and $\delta_{22l}'' = 0$ which implies $\delta_{2l}'' = \delta_{21l}'' > 0$ and $p_{2h} - p_{2l} = \lambda c_1 \delta_2^e$. From this analysis, we find that both δ' 's and δ'' 's satisfy the following property: SP 1 satisfies all patient customers while SP 2 satisfies only impatient customers. The new purchase functions are feasible since they also satisfy this property as $p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}' + \lambda c_2 \delta_{1l}' < p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}'' + \lambda c_2 \delta_{1l}'' = p_{1h} + c_2 + \lambda c_2 \delta_{1h}'' \leq p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}''$ and $p_{2h} + c_1 + \lambda c_1 \delta_{2h}' < p_{2h} + c_1 + \lambda c_1 \delta_{2h}' \leq \min(p_{1h} + c_1 + \lambda c_1 \delta_{1h}', v)$.

2. (37) is satisfied. The analysis is similar to the first case. We first note that (41) is violated and (40) has to be satisfied. Inequality (40) implies $p_{2h} - p_{2l} = \lambda c_2 \delta_2^e$. Also, it implies $p_{1l} + c_2 + 2\lambda c_2 \delta_{1h}'' + \lambda c_2 \delta_{1l}'' > p_{2h} + c_2 + \lambda c_2 \delta_{2h}' = p_{2l} + c_2 + 2\lambda c_2 \delta_{2h}'' + \lambda c_2 \delta_{2l}''$ and (36) is violated. Hence $\delta_{12h}'' = \delta_{12l}'' = 0$ and SP 1 satisfies only impatient customers. So SP 2 satisfies all patient customers

and $\delta_2^e \geq 1 - q$. Also, since (36) is violated, (37) has to be satisfied which implies (37) is violated. Hence $\delta'_{21h} = \delta'_{21l} = 0$ and SP 2 satisfies only patient customers so that $\delta_2^e \leq 1 - q$. If $\delta'_{12l} > 0$ then $\delta_2^e < 1 - q$ resulting in a contradiction. So $\delta_2^e = 1 - q$ and $\delta'_{12l} = 0$ which implies $\delta'_{1l} = \delta'_{11l} > 0$. Therefore, $p_{1h} - p_{1l} = \lambda c_1 \delta_1^e$ and SP 1 satisfies only impatient customers. From this analysis, we find that both δ' 's and δ'' 's satisfy the following property: SP 1 satisfies only impatient customers while SP 2 satisfies all the patient customers. The new purchase functions are feasible since they also satisfy this property as $p_{1h} + c_1 + \lambda c_1 \delta'_{1h} < p_{1h} + c_1 + \lambda c_1 \delta''_{1h} \leq \min(p_{2h} + c_1 + \lambda c_1 \delta''_{1h}, v)$ and $p_{2l} + c_2 + 2\lambda c_2 \delta''_{2h} + \lambda c_2 \delta''_{2l} < p_{2l} + c_2 + 2\lambda c_2 \delta'_{2h} + \lambda c_2 \delta'_{2l} \leq p_{1l} + c_2 + 2\lambda c_2 \delta'_{2h} + \lambda c_2 \delta'_{2l}$.

From our above analysis, the minimum $\delta_{ih,i=1,2}$ value among all the optimal solutions (of (CDD)) is selected as δ_{ih}^e while setting $\delta_{il}^e = \delta_i^e - \delta_{ih}^e$. This solution will be feasible, and it minimizes the customers' dis-utilities among all possible optimal solutions. ■

C.2 Possible Pricing Structures at Equilibrium

When the prices (p_{1h} , p_{2h} , p_{1l} , and p_{2l}) are distinct, we can use arguments similar to those in §B.2 to show that we only need to consider the four pricing structures in Figure 5 at equilibrium. These structures exhibit the following two key properties: (i) the price of each service class is set such that either impatient or patient customers become indifferent to purchasing from this service class or other service class(es) and (ii) if patient (impatient) customers can *marginally* purchase from a service class, they can also *marginally* purchase from lower (higher) priced service classes. What are the price structures when there are only three or two distinct prices? We answer this question later when we consider those price orderings in detail (see §'s C.2.4-C.2.7). Note that these price structures may not just be the same as those (with distinct prices) in $\mathcal{SD}$ (see Figure 4) or $\mathcal{SS}$ because some prices here are identical. Possible price orderings at equilibrium are as follows: (i) $p_{1h} > p_{2h} > p_{2l} > p_{1l}$, (ii) $p_{1h} > p_{2h} > p_{1l} > p_{2l}$, (iii) $p_{1h} > p_{1l} > p_{2h} > p_{2l}$, (iv) $p_{1h} > p_{2h} > p_{1l} = p_{2l}$, (v) $p_{1h} = p_{2h} > p_{1l} > p_{2l}$, (vi) $p_{1h} > p_{2h} = p_{1l} > p_{2l}$, and (vii) $p_{1h} = p_{2h} > p_{1l} = p_{2l}$. Any other price ordering is just a *mirror image* of one of these orderings with the labels for SPs 1 and 2 interchanged (e.g., $p_{2h} > p_{1h} > p_{2l} > p_{1l}$ is a mirror image of ordering (ii)).

C.2.1 $p_{1h} > p_{2h} > p_{2l} > p_{1l}$

Consider the structure in part (a) of Figure 5. First, we note that $\delta_{1h} = 0$; otherwise setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$, results in a feasible solution with the same objective value (for (CDD)) and lower dis-utility for all the customers. Also, $\delta_{1l} > 1 - q$ because otherwise

Figure 5: Possible equilibrium pricing structures under $\mathcal{DD}$ showing the prices and dis-utilities of impatient and patient customers

p_{1l} can be increased slightly (without affecting $\delta_{ik}; i = 1, 2; k = h, l$) to increase the profit of SP 1. Set $\tilde{\delta}_{2h} = \delta_{2h} - \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l}$. Then since $p_{1h} - p_{1l} = \lambda c_1 \delta_{1l} < \lambda c_1 (\delta_{1l} + \epsilon) = \lambda c_1 \tilde{\delta}_{1l}$, $\tilde{\delta}_{1h} = \delta_{1l} + \epsilon - (1 - q) > \epsilon$ and $\tilde{\delta}_{1l} = 1 - q$. As $\tilde{\delta}_{1h}$ is significantly higher than $\delta_{1h} = 0$ (by more than $\delta_{1l} - (1 - q)$), the prices $\tilde{p}_{2h} = p_{2h} + \lambda c_1 (\delta_{1l} - (1 - q)) + 2\lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + \lambda c_1 (\delta_{1l} - (1 - q)) + 3\lambda c_1 \epsilon$ are significantly higher than p_{2h} and p_{2l} so that $\tilde{\pi}_2 > \pi_2$. Note that the reason $\tilde{\delta}_{1l} = 1 - q$ is because $\tilde{p}_{2l} + \lambda c_2 (2\tilde{\delta}_{2h} + \tilde{\delta}_{2l}) = p_{2l} + \lambda c_2 (2\delta_{2h} + \delta_{2l}) + \lambda c_1 (\delta_{1l} - (1 - q)) + \lambda (3c_1 - 2c_2)\epsilon > p_{1l} + \lambda c_2 \delta_{1l} + \lambda c_1 (\delta_{1l} - (1 - q)) > p_{1l} + \lambda c_2 (2\delta_{1l} - (1 - q) + 2\epsilon) = p_{1l} + \lambda c_2 (2\tilde{\delta}_{1h} + \tilde{\delta}_{1l})$ for a small $\epsilon > 0$. Hence there is no equilibrium with part (a) structure.

Consider the structure in part (b). In this structure, $\delta_{1l} + \delta_{2l} \geq 1 - q$. If $\delta_{1l} + \delta_{2l} > 1 - q$ (note that it implies $\delta_{2l} > 0$ since $\delta_{1l} \leq 1 - q$) then setting $\tilde{p}_{1h} = p_{1h} + \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_2 \epsilon$ makes $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ while maintaining $\tilde{\delta}_{1h} = \delta_{1h}$ and $\tilde{\delta}_{1l} = \delta_{1l}$, thereby increasing the profit of SP 1. Hence $\delta_{1l} + \delta_{2l} = 1 - q$. Since $p_{1h} > p_{2h}$, $\delta_{1h} < \delta_{1l}$ and since $p_{1l} < p_{2l}$, $2\delta_{1h} + \delta_{1l} > 2\delta_{2h} + \delta_{2l}$ which implies $\delta_{1l} > \delta_{2l} \geq 0$. Set $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$ (and $\tilde{\delta}_{1h} = \delta_{1h}$); it is obtained by having $\tilde{p}_{1l} = p_{1l} + 2\lambda c_2 \epsilon$ (and $\tilde{p}_{1h} = p_{1h}$). Requiring $\tilde{\pi}_1 \leq \pi_1$ for any small $\epsilon > 0$ implies $p_{1l} \geq 2\lambda c_2 \delta_{1l}$. Similarly, setting $\tilde{\delta}_{2l} = \delta_{2l} + \epsilon$ (and $\tilde{\delta}_{2h} = \delta_{2h}$), we obtain $\tilde{p}_{2l} = p_{2l} - 2\lambda c_2 \epsilon$ (and $\tilde{p}_{2h} = p_{2h}$). Note that $c_1 > 2c_2$ implies $\tilde{p}_{2l} + \lambda c_1 (2\tilde{\delta}_{2h} + \tilde{\delta}_{2l}) > p_{2l} + \lambda c_1 (2\delta_{2h} + \delta_{2l})$ for any $\epsilon > 0$ so that the low-priority service does not satisfy any impatient customers after the price change. Requiring $\tilde{\pi}_2 \leq \pi_2$ for any small $\epsilon > 0$ implies $p_{2l} \leq 2\lambda c_2 \delta_{2l} < 2\lambda c_2 \delta_{1l} \leq p_{1l}$, a contradiction. Hence there is no equilibrium with part (b) structure.

Consider the structure in part (c). As with the structure in part (b), $\delta_{2h} > \delta_{1h} \geq 0$ and $\delta_{1l} > \delta_{2l} \geq 0$. Suppose $\delta_{2l} > 0$. Set $\tilde{\delta}_{1h} = \delta_{1h}$, $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$; they are obtained by having prices $\tilde{p}_{1h} = p_{1h} + \lambda c_1 \delta_{2l} + \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_2 \delta_{2l} + 2\lambda c_2 \epsilon$, which yield $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = 0$ since

$p_{2h} - p_{2l} < \lambda c_2(\delta_2 + \epsilon) < \lambda c_1(\delta_2 + \epsilon)$. Note that for small $\epsilon > 0$, $\tilde{p}_{1l} < p_{2h}$ so that the low-priority service of SP 1 satisfies only patient customers. Since $\tilde{p}_{1h}$ and $\tilde{p}_{1l}$ are *significantly higher* than p_{1h} and p_{1l} , we find $\tilde{\pi}_1 > \pi_1$. Hence $\delta_{2l} = 0$ for equilibrium. Then there are two cases:

1. $\delta_{1l} < 1 - q$: In this case, $\delta_{1h} = 0$ because otherwise a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Then $\delta_{2h} = 1 - \delta_{1l} > q$. Set $\tilde{\delta}_{2h} = q$ and $\tilde{\delta}_{2l} = \delta_{2h} - q$; these purchase fractions are obtained by prices $\tilde{p}_{2h} = p_{2h} + \lambda c_1(\delta_{2h} - q)$ and $\tilde{p}_{2l} = p_{2l} + \lambda c_2(\delta_{2h} - q) = p_{2h} - \lambda c_2 q$. Note that since $\lambda c_2 \tilde{\delta}_{2h} < \tilde{p}_{2h} - \tilde{p}_{2l} < \lambda c_1 \tilde{\delta}_{2h}$, low-priority service of SP 2 (high-priority service of SP 2) satisfies only patient (impatient) customers. Then $\tilde{\pi}_2 = \lambda(p_{2h} + \lambda c_1(\delta_{2h} - q))q + \lambda(p_{2h} - \lambda c_2 q)(\delta_{2h} - q) = \pi_2 + \lambda^2(c_1 - c_2)q(\delta_{2h} - q) > \pi_2$. *Hence, there is no equilibrium when $\delta_{1l} < 1 - q$.*

2. $\delta_{1l} = 1 - q$: In this case, it is possible that $\delta_{1h} > 0$; also, $\delta_{1h} + \delta_{2h} = q$. Set $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = \epsilon$; they are obtained by having prices $\tilde{p}_{2h} = p_{2h}$ and $\tilde{p}_{2l} = p_{2l} - 2\lambda c_2 \epsilon$. We find $\tilde{\pi}_2 > \pi_2$ for small $\epsilon > 0$ since $p_{2l} > p_{1l} \geq 0$. *Hence, there is no equilibrium when $\delta_{1l} = 1 - q$.*

Finally, consider the structure in part (d). We require $\delta_{1h} > q$ because if $\delta_{1h} = q$ then p_{1h} can be increased to increase the profit of SP 1. However, it is impossible to have $\delta_{1h} > q$ since an alternate solution with $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$ will be better since it will also be optimal (for (CDD)) and all customers will have lower dis-utilities. *Hence there is no equilibrium with part (d) structure.*

Therefore, there is no equilibrium in which $p_{1h} > p_{2h} > p_{2l} > p_{1l}$.

C.2.2 $p_{1h} > p_{2h} > p_{1l} > p_{2l}$

Consider the structure in part (a) of Figure 5. As in §C.2.1, $\delta_{1h} = 0$. Also, $\delta_{2l} > 1 - q$ because p_{2l} can otherwise be increased to increase the profit of SP 2. However, that implies setting $\tilde{p}_{1h} = p_{1h} + \lambda c_1 \epsilon > p_{1h}$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_1 \epsilon > p_{1l}$ increases the profit of SP 1 since $\tilde{\delta}_{1h} = \delta_{1h}$, $\tilde{\delta}_{1l} = \delta_{1l}$, $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$, and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$. *Hence there is no equilibrium with part (a) structure.*

Consider the structure in part (b). As in §C.2.1, $\delta_{1l} + \delta_{2l} \geq 1 - q$. There are two cases:

1. $\delta_{1l} + \delta_{2l} > 1 - q$: Suppose $\delta_{2l} < 1 - q$. Then let $x \equiv \delta_{1l} + \delta_{2l} - (1 - q)$; note that $0 < x < \delta_{1l}$. Set $\tilde{\delta}_{1h} = \delta_{1h} + x$ and $\tilde{\delta}_{1l} = \delta_{1l} - x$; they are obtained by prices $\tilde{p}_{1h} = p_{1h} - \lambda c_1 x$ and $\tilde{p}_{1l} = p_{1l} - \lambda c_2 x$, which yield $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = \delta_{2l}$. Note that $\tilde{p}_{1l} + \lambda c_1(2\tilde{\delta}_{1h} + \tilde{\delta}_{1l}) = p_{1l} + \lambda c_1(2\delta_{1h} + \delta_{1l}) + \lambda(c_1 - c_2)x > p_{1l} + \lambda c_1(2\delta_{1h} + \delta_{1l}) = p_{1h} + \lambda c_1 \delta_{1h} = \tilde{p}_{1h} + \lambda c_1 \tilde{\delta}_{1h}$ so that the low-priority service of SP 1 only satisfies patient customers. We find that $\tilde{\pi}_1 = \pi_1 + \lambda x(p_{1h} - p_{1l} - \lambda c_1 \delta_{1h} - \lambda c_2 \delta_{1l} - \lambda(c_1 - c_2)x) > \pi_1$ since $p_{1h} - p_{1l} = \lambda c_1 \delta_{1h} > \lambda c_1 \delta_{1h} + \lambda c_2 \delta_{1l} + \lambda(c_1 - c_2)x \forall x < \delta_{1l}$. A similar result is obtained when $\delta_{2l} = 1 - q$ by setting $\tilde{\delta}_{1h} = \delta_{1h} + x - \epsilon = \delta_{1h} - \epsilon$ and $\tilde{\delta}_{1l} = \delta_{1l} - x + \epsilon = \delta_{1l}$; in this case, the prices are

$\tilde{p}_{1h} = p_{1h} - \lambda c_1 \delta_{1l} + 2\lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} - \lambda c_2 \delta_{1l} + 2\lambda c_2 \epsilon$, which yield $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = 1 - q - \epsilon$. Hence, there is no equilibrium when $\delta_{1l} + \delta_{2l} > 1 - q$.

2. $\delta_{1l} + \delta_{2l} = 1 - q$: If $\delta_{2l} = 1 - q$ then set $\tilde{\delta}_{1h} = \delta_{1h}$ and $\tilde{\delta}_{1l} = \epsilon$; they are obtained by prices $\tilde{p}_{1h} = p_{1h}$ and $\tilde{p}_{1l} = p_{1l} - 2\lambda c_2 \epsilon$. Note $\tilde{p}_{1h} - \tilde{p}_{1l} < \lambda c_1 \tilde{\delta}_{1l}$ since $c_1 > 2c_2$ so that only patient customers are satisfied by the low-priority service of SP 1. Since $p_{1l} > p_{2l} \geq 0$, we find $\tilde{\pi}_1 > \pi_1$. Hence $\delta_{2l} < 1 - q$, i.e., $\delta_{1l} > 0$ for equilibrium. However, that implies $\delta_{1h} = 0$ because otherwise a better solution is obtained by $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Then set $\tilde{\delta}_{1h} = \epsilon$ and $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$; the corresponding prices are $\tilde{p}_{1h} = p_{1h} - 2\lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} - 2\lambda c_2 \epsilon$. We find that $\tilde{\pi}_1 = \lambda(p_{1h} - 2\lambda c_1 \epsilon) + \lambda(p_{1l} - 2\lambda c_2 \epsilon)(\delta_{1l} - \epsilon) = \pi_1 + \lambda\epsilon(p_{1h} - p_{1l} - 2\lambda c_2 \delta_{1l} - 2\lambda(c_1 - c_2)\epsilon) > \pi_1$ for small $\epsilon > 0$ since $p_{1h} - p_{1l} = \lambda c_1 \delta_{1l}$ and $c_1 > 2c_2$. Hence, there is no equilibrium when $\delta_{1l} + \delta_{2l} = 1 - q$.

Consider the structure in part (c). In this structure, $\delta_{1l} + \delta_{2l} \leq 1 - q$. There are two cases:

1. $\delta_{1l} + \delta_{2l} < 1 - q$: Then $\delta_{1h} = 0$ because otherwise a better solution is obtained by $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. So $\delta_{2h} > q$. If $\delta_{2l} > 0$ then setting $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$ (and $\tilde{\delta}_{1h} = \delta_{1h} = 0$) increases the price of low-priority service from SP 2 *significantly* so that $\tilde{\pi}_1 > \pi_1$. Note that $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = 0$, and the required purchase fractions are obtained by setting $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{1l} + \lambda c_2 \delta_{2l} + 2\lambda c_2 \epsilon$. Hence, $\delta_{2l} = 0$ for equilibrium. Then set $\tilde{\delta}_{1h} = \epsilon$ and $\tilde{\delta}_{1l} = \delta_{1l} - (1 + \alpha)\epsilon$ where $0 < \alpha \ll 1$; that results in $\tilde{\delta}_{2h} = \delta_{2h} + \alpha\epsilon$ and $\tilde{\delta}_{2l} = 0$. These purchase fractions are obtained by prices $\tilde{p}_{1h} = p_{1h} + \lambda c_1(\alpha - 1)\epsilon$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_2(2\alpha - 1)\epsilon$. Then we get

$$\begin{aligned} \tilde{\pi}_1 &= \lambda(p_{1h} + \lambda c_1(\alpha - 1)\epsilon) + \lambda(p_{1l} + \lambda c_2(2\alpha - 1)\epsilon)(\delta_{1l} - (1 + \alpha)\epsilon) \\ &= \pi_1 + \lambda\epsilon(p_{1h} - (1 + \alpha)p_{1l} + \lambda c_2(2\alpha - 1)\delta_{1l} - \lambda c_1(1 - \alpha)\epsilon - \lambda c_2(1 + \alpha)(2\alpha - 1)\epsilon) \\ &> \pi_1 \end{aligned}$$

for small $\alpha, \epsilon > 0$ since $[p_{1h} - (1 + \alpha)p_{1l} + \lambda c_2(2\alpha - 1)\delta_{1l}]_{\alpha=0} = p_{1h} - p_{1l} - \lambda c_2 \delta_{1l} > 0$. Hence, there is no equilibrium when $\delta_{1l} + \delta_{2l} < 1 - q$.

2. $\delta_{1l} + \delta_{2l} = 1 - q$: As in the part (b) structure (see the second case), $\delta_{1l} > 0$ because otherwise setting $\tilde{\delta}_{1l} = \epsilon$ (and charging price $p_{1l} - 2\lambda c_2 \epsilon$) results in $\tilde{\pi}_1 > \pi_1$. So $\delta_{2l} < 1 - q$. Set $\tilde{\delta}_{2h} = \delta_{2h} - \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} + \epsilon$; the corresponding prices are $\tilde{p}_{2h} = p_{2h} + 2\lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + 2\lambda c_2 \epsilon$, which yield $\tilde{\delta}_{1h} = \delta_{1h} + \epsilon$ and $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$. Note that since $\tilde{p}_{2h} - \tilde{p}_{2l} > p_{2h} - p_{2l} = \lambda c_2 \delta_{2l} = \lambda c_2 \tilde{\delta}_{2l}$, the high-priority service of SP 2 (and SP 1 as well) only satisfy impatient customers after the price changes. The new profit $\tilde{\pi}_2 = \lambda(p_{2h} + 2\lambda c_1 \epsilon)(\delta_{2h} - \epsilon) + \lambda(p_{2l} + 2\lambda c_2 \epsilon)(\delta_{2l} + \epsilon) = \pi_2 + \lambda\epsilon(2\lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l} - p_{2h} + p_{2l} - 2\lambda(c_1 - c_2)\epsilon) > \pi_2$ for small $\epsilon > 0$ since $2\lambda c_1 \delta_{2h} + 2\lambda c_2 \delta_{2l} > \lambda c_2(\delta_{2h} + \delta_{2l}) = p_{2h} - p_{2l}$. Hence, there is no equilibrium when $\delta_{1l} + \delta_{2l} = 1 - q$.

Finally, consider the structure in part (d). Similar to §C.2.1, we can show that *there is no equilibrium with part (d) structure*.

Therefore, there is no equilibrium in which $p_{1h} > p_{2h} > p_{1l} > p_{2l}$.

C.2.3 $p_{1h} > p_{1l} > p_{2h} > p_{2l}$

Consider the structure in part (a) of Figure 5. Using a similar reasoning as in §C.2.2, we can show that *there is no equilibrium with part (a) structure*.

Consider the structure in part (b). Then $\delta_{2l} = 0$; otherwise, setting $\tilde{p}_{1h} = p_{1h} + \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_1 \epsilon$ implies $\tilde{\delta}_{1k} = \delta_{1k}$; $k = h, l$ while $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ so that $\tilde{\pi}_1 > \pi_1$. Now set $\tilde{\delta}_{2h} = \delta_{2h} - \epsilon$ and $\tilde{\delta}_{2l} = \epsilon$; then $\tilde{p}_{2h} = p_{2h} + \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + \lambda c_1 \epsilon$. However, that implies $\tilde{\pi}_2 = \lambda(p_{2h} + \lambda c_1 \epsilon)(\delta_{2h} - \epsilon) + \lambda(p_{2l} + \lambda c_1 \epsilon)\epsilon = \pi_2 + \lambda\epsilon(\lambda c_1 \delta_{2h} - (p_{2h} - p_{2l})) > \pi_2$ since $p_{2h} - p_{2l} = \lambda c_2 \delta_{2h} < \lambda c_1 \delta_{2h}$. Hence, *there is no equilibrium with part (b) structure*.

Consider the structure in part (c). We find $\delta_{1h} = 0$ because otherwise a better solution is obtained by $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Hence $\delta_{1l} \geq q$. If $\delta_{1l} = q$ then both p_{1h} and p_{1l} can be increased slightly by the same amount without any loss in δ_1 so that $\tilde{\pi}_1 > \pi_1$. So $\delta_{1l} > q$. We note that SP 2 can obtain the same profit by setting a single price (for both high- and low-priority services) $p_{2e} = p_{2h} - \lambda c_2 \delta_{2l}$ which results in $\delta_{2e} = \delta_2$. Next, we show how SP 2 can obtain a better profit. Set $\tilde{\delta}_{2h} = \epsilon$ and $\tilde{\delta}_{2l} = \delta_2 - \epsilon$; they are obtained by prices $\tilde{p}_{2h} = p_{1h} - \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2e} - \lambda c_2 \epsilon$. Then we find $\tilde{\pi}_2 = \lambda(p_{1h} - \lambda c_1 \epsilon)\epsilon + \lambda(p_{2e} - \lambda c_2 \epsilon)(\delta_2 - \epsilon) = \pi_2 + \lambda\epsilon(p_{1h} - p_{2e} - \lambda c_2 \delta_2 - \lambda(c_1 - c_2)\epsilon) > \pi_2$ for small $\epsilon > 0$ since $p_{1h} + \lambda c_2 \delta_{1h} = p_{1h} > p_{1l} + \lambda c_2 \delta_{1l} = p_{2e} + \lambda c_2 \delta_2$. Hence, *there is no equilibrium with part (c) structure*.

Finally, consider the structure in part (d). Similar to §C.2.1, we can show that *there is no equilibrium with part (d) structure*.

Therefore, there is no equilibrium in which $p_{1h} > p_{1l} > p_{2h} > p_{2l}$.

C.2.4 $p_{1h} > p_{2h} > p_{1l} = p_{2l}$

All the pricing structures in Figure 4 are possible. A *new structure* with $p_{1h} + \lambda c_1 \delta_{1h} = p_{2h} + \lambda c_1 \delta_{2h} < p_{1l} + \lambda c_1 (2\delta_{1h} + \delta_{1l}) = p_{2l} + \lambda c_1 (2\delta_{2h} + \delta_{2l})$ and $p_{1h} + \lambda c_2 \delta_{1h} > p_{2h} + \lambda c_2 \delta_{2h} > p_{1l} + \lambda c_2 (2\delta_{1h} + \delta_{1l}) = p_{2l} + \lambda c_2 (2\delta_{2h} + \delta_{2l})$ is also possible. This structure was not possible earlier because the lowest price could just be increased slightly to obtain a higher profit. *That is not the case here since two competing SPs charge the same lowest price*. Any other pricing structure is not possible for reasons similar to those in §B.2. Next, we analyze whether each of the possible pricing structures can result in an equilibrium.

Consider the structure in part (a) of Figure 4. Then $\delta_{1h} = 0$; otherwise set $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$; then all customers have strictly lower dis-utility. Also, $0 < \delta_{2h} \leq q$, and $\delta_{1l} > 0$ for SP 1 to get a positive profit. Furthermore, since $p_{1l} = p_{2l}$, $\delta_{1l} = 2\delta_{2h} + \delta_{2l}$ so that $3\delta_{2h} + 2\delta_{2l} = 1$. There are two cases:

1. $\delta_{2h} + 2\delta_{2l} \geq q$: Set $\tilde{\delta}_{1h} = 0.5(q - \delta_{2h}) + \epsilon$ and $\tilde{\delta}_{1l} = \delta_{1l} + 0.5(\delta_{2h} - q) - 2\epsilon$. These purchase fractions are obtained by setting prices $\tilde{p}_{1h} = p_{2h} + \lambda c_1(\delta_{2h} - 2\epsilon) > p_{2h}$ (for small $\epsilon > 0$) and $\tilde{p}_{1l} = p_{1l} = p_{2l}$, which yields $\tilde{\delta}_{2h} = 0.5(q + \delta_{2h}) - \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} + 0.5(\delta_{2h} - q) + 2\epsilon$. Note that $\tilde{\delta}_{1h} + \tilde{\delta}_{2h} = q$, $\tilde{\delta}_{1h} + \tilde{\delta}_{1l} = \delta_{1l} - \epsilon$ and $2\tilde{\delta}_{1h} + \tilde{\delta}_{1l} = 2\tilde{\delta}_{2h} + \tilde{\delta}_{2l}$. Also, $p_{2h} - p_{2l} < \lambda c_1 \tilde{\delta}_2 = \lambda c_1(\delta_2 + \epsilon)$ so that all impatient customers are satisfied by high-priority services while all patient ones are satisfied by low-priority services. All the new purchase fractions are feasible; $\tilde{\delta}_{1l} > 0$ since $\delta_{1l} = 2\delta_{2h} + \delta_{2l} \geq 0.5(3\delta_{2h} + 2\delta_{2l}) = 0.5$ and $\tilde{\delta}_{2l} > 0$ since $\delta_{2h} + 2\delta_{2l} \geq q$. We find that $\tilde{\pi}_1 = \lambda \tilde{p}_{1h} \tilde{\delta}_{1h} + \lambda p_{1l} \tilde{\delta}_{1l} > \lambda p_{1l} \delta_{1l} = \pi_1$ since $\tilde{p}_{1h} > p_{2h} > p_{1l}$. Hence, there is no equilibrium when $\delta_{2h} + 2\delta_{2l} \geq q$.

2. $\delta_{2h} + 2\delta_{2l} < q$: The logic here is similar to the first case; the main difference is that we let $\tilde{\delta}_{2l}$ become zero since it cannot be negative. Set $\tilde{\delta}_{1h} = q - \delta_2 - \epsilon$ and $\tilde{\delta}_{1l} = 1 - q$; they are obtained by prices $\tilde{p}_{1h} = p_{2h} + \lambda c_1(2\delta_2 - q + 2\epsilon) = p_{1l} + \lambda c_1(3\delta_2 - q + 2\epsilon)$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_2(4\delta_2 + 4\epsilon - 1 - q)$, which yields $\tilde{\delta}_{2h} = \delta_2 + \epsilon$ and $\tilde{\delta}_{2l} = 0$. Note that $\tilde{\delta}_{1h} + \tilde{\delta}_{1l} = \delta_{1l} - \epsilon$. Also, note that $\tilde{p}_{1h} > p_{1l}$ since $3\delta_2 = 3\delta_{2h} + 3\delta_{2l} \geq 3\delta_{2h} + 2\delta_{2l} = 1 > q$ while $\tilde{p}_{1l} < p_{1l} = p_{2l}$ since $4\delta_2 + 4\epsilon - 1 - q = 4\delta_{2h} + 4\delta_{2l} - (3\delta_{2h} + 2\delta_{2l}) - q + 4\epsilon = \delta_{2h} + 2\delta_{2l} - q + 4\epsilon < 0$ for any small $\epsilon > 0$. The later result implies that the impatient customers are only satisfied by high-priority services (as in the first case). Furthermore, $p_{2h} - p_{2l} > \lambda c_2(\delta_2 + \epsilon)$ and $\tilde{p}_{1h} - \tilde{p}_{1l} = \lambda c_1(3\delta_2 - q + 2\epsilon) - \lambda c_2(4\delta_2 + 4\epsilon - 1 - q) > \lambda c_2((3\delta_2 - q + 3\epsilon) - (4\delta_2 + 4\epsilon - 1 - q)) = \lambda c_2 \tilde{\delta}_1$ for small $\epsilon > 0$ so that patient customers are only satisfied by low-priority services. We find that

$$\begin{aligned} \tilde{\pi}_1 &= \lambda \tilde{p}_{1h} \tilde{\delta}_{1h} + \lambda \tilde{p}_{1l} \tilde{\delta}_{1l} \\ &= \lambda (p_{1l} + \lambda c_1(3\delta_2 - q + 2\epsilon)) (q - \delta_2 - \epsilon) + \lambda (p_{1l} + \lambda c_2(4\delta_2 + 4\epsilon - 1 - q)) (1 - q) \\ &> \lambda \delta_{1l} p_{1l} = \pi_1 \end{aligned}$$

for any small $\epsilon > 0$ since $\lambda c_1(3\delta_2 - q)(q - \delta_2) + \lambda c_2(4\delta_2 - 1 - q)(1 - q) > \lambda c_1((3\delta_2 - q)(q - \delta_2) + (4\delta_2 - 1 - q)(1 - q)) = \lambda c_1(3\delta_2 - 1)(1 - \delta_2) \geq 0$. Hence, there is no equilibrium when $\delta_{2h} + 2\delta_{2l} < q$.

Consider the structure in part (b). In this structure, $\delta_{1h} + \delta_{2h} \leq q$, $\delta_{1h} < \delta_{2h}$ and $2\delta_{1h} + \delta_{1l} = 2\delta_{2h} + \delta_{2l}$ so that $\delta_{1l} > \delta_{2l}$. There are two cases:

1. Suppose $\delta_{1h} + \delta_{2h} > q$. Then $\delta_{1h} = 0$; otherwise, a better solution (with lower dis-utility to all customers) is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. So $\delta_{2h} > q$. Set

$\tilde{\delta}_{1h} = \delta_{1h} = 0$ and $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$; they are obtained by having prices $\tilde{p}_{1h} = p_{1h} + \lambda c_1 \delta_{2l} + \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} + \lambda c_2 \delta_{2l} + 2\lambda c_2 \epsilon$, which yields $\tilde{\delta}_{2h} = \delta_2 + \epsilon$ and $\tilde{\delta}_{2l} = 0$. Clearly, if $\delta_{2l} > 0$ then $\tilde{p}_{1l}$ is *significantly higher* than p_{1l} so that $\tilde{\pi}_1 > \pi_1$. So $\delta_{2l} = 0$; then $\tilde{\pi}_1 \leq \pi_1$ implies $p_{1l} \geq 2\lambda c_2 \delta_{1l}$. Set $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = \epsilon$; this is obtained by setting $\tilde{p}_{2h} = p_{2h} - \lambda c_2 \epsilon$ and $\tilde{p}_{2l} = p_{1l} - 2\lambda c_2 \epsilon$. Then $\tilde{\pi}_2 = \lambda(p_{2h} - \lambda c_2 \epsilon)\delta_{2h} + \lambda(p_{1l} - 2\lambda c_2 \epsilon)\epsilon > \lambda\delta_{2h}p_{2h}$ for small $\epsilon > 0$ since $p_{1l} \geq 2\lambda c_2 \delta_{1l} = 4\lambda c_2 \delta_{2h} > \lambda c_2 \delta_{2h}$. Hence, there is no equilibrium when $\delta_{1h} + \delta_{2h} > q$.

2. Suppose $\delta_{1h} + \delta_{2h} = q$. It is possible that $\delta_{1h} > 0$ here. If $\delta_{2l} > 0$, we can show similar to the first case that $\tilde{\pi}_1 > \pi_1$; note that $\tilde{p}_{1l} < p_{2h} < \tilde{p}_{1h}$ so that the low-priority (high-priority) service from SP 1 satisfies only patient (impatient) customers. Hence $\delta_{2l} = 0$ for equilibrium. Again, we adopt an approach similar to the first case. Setting $\tilde{\delta}_{1h} = \delta_{1h}$ and $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$ as before, $\tilde{\pi}_1 \leq \pi_1$ implies $p_{1l} \geq \lambda c_1 \delta_{1h} + 2\lambda c_2 \delta_{1l} > 0$. However, then setting $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = \epsilon$, obtained by prices $\tilde{p}_{2h} = p_{2h}$ (note that the price does not have to decrease since only impatient customers are satisfied by the high-priority service here) and $\tilde{p}_{2l} = p_{1l} - 2\lambda c_2 \epsilon$ implies $\tilde{\pi}_2 > \pi_2$ since $p_{1l} > 2\lambda c_2 \epsilon$ for small $\epsilon > 0$. Hence, there is no equilibrium when $\delta_{1h} + \delta_{2h} = q$.

Consider the structure in part (c). Note that $\delta_{1h} \geq q$ in this structure. If $\delta_{1h} > q$ a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Hence $\delta_{1h} = q$. However, that means p_{1h} can be increased slightly (without any reduction in δ_{1h}) to increase the profit of SP 1. Hence, there is no equilibrium with part (c) structure.

Consider the *new* structure with $p_{1h} + \lambda c_1 \delta_{1h} = p_{2h} + \lambda c_1 \delta_{2h}$, $\lambda c_2 \delta_i < p_{ih} - p_{il} < \lambda c_1 \delta_i$; $i = 1, 2$, and $2\delta_{1h} + \delta_{1l} = 2\delta_{2h} + \delta_{2l}$. Since $p_{1h} > p_{1l}$, $\delta_{1h} < \delta_{2h}$ which implies $\delta_{1l} > \delta_{2l} \geq 0$. Set $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$ (and $\tilde{\delta}_{1h} = \delta_{1h}$); they are obtained by prices $\tilde{p}_{1h} = p_{1h}$ and $\tilde{p}_{1l} = p_{1l} + 2\lambda c_2 \epsilon$. Then $\tilde{\pi}_1 \leq \pi_1$ implies $p_{1l} \geq 2\lambda c_2 \delta_{1l}$. Similarly, setting $\tilde{\delta}_{2l} = \delta_{2l} + \epsilon$ (and $\tilde{\delta}_{2h} = \delta_{2h}$) through prices $\tilde{p}_{2h} = p_{2h}$ and $\tilde{p}_{2l} = p_{2l} - 2\lambda c_2 \epsilon$, and requiring $\tilde{\pi}_2 \leq \pi_2$ implies $p_{2l} \leq 2\lambda c_2 \delta_{2l} < 2\lambda c_2 \delta_{1l} \leq p_{1l}$, a contradiction. Hence, there is no equilibrium with the new structure.

Therefore, there is no equilibrium in which $p_{1h} > p_{2l} > p_{1l} = p_{2l}$.

C.2.5 $p_{1h} = p_{2h} > p_{1l} > p_{2l}$

In addition to the three structures in Figure 4, a new structure *similar* to the one in §C.2.4 is possible. In this structure, $\delta_{1h} = \delta_{2h}$, $\lambda c_2 \delta_i < p_{ih} - p_{il} < \lambda c_1 \delta_i$; $i = 1, 2$, and $p_{1l} + \lambda c_2(2\delta_{1h} + \delta_{1l}) = p_{2l} + \lambda c_2(2\delta_{2h} + \delta_{2l})$. Next, we analyze whether each structure can result in an equilibrium.

Consider the structure in part (a). Then $\delta_{1h} = \delta_{2h} = 0$ because otherwise a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Also, if $\delta_{2l} = 1 - q$ then p_{2l} can be slightly increased to increase the profit of SP 2. So $\delta_{2l} > 1 - q$. Then set prices $\tilde{p}_{1k} = p_{1k} + \lambda c_1 \epsilon$; $k = h, l$.

We find that the purchase fractions of SP 1 remain the same; i.e., $\tilde{\delta}_{1h} = 0$ and $\tilde{\delta}_{1l} = \delta_{1l}$ while $\tilde{\delta}_{2h} = \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ so that $\tilde{\pi}_1 > \pi_1$. Hence, there is no equilibrium with part (a) structure.

Consider the structure in part (b). In this structure, $\delta_{2l} \leq 1 - q$. There are two cases:

1. $\delta_{2l} < 1 - q$: Then $\delta_{1h} = \delta_{2h} = 0$ because otherwise a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Also, $\delta_{1l} + \delta_{2l} = 1$ implies $\delta_{1l} > q$. Set $\tilde{\delta}_{2h} = \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$; they are obtained by prices $\tilde{p}_{2h} = p_{2h} - \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} - \lambda c_2 \epsilon$, which yields $\tilde{\delta}_{1h} = 0$ and $\tilde{\delta}_{1l} = \delta_{1l}$. We find $\tilde{\pi}_2 = \lambda(p_{2h} - \lambda c_1 \epsilon) + \lambda(p_{2l} - \lambda c_2 \epsilon)(\delta_{2l} - \epsilon) = \pi_2 + \lambda \epsilon(p_{2h} - p_{2l} - \lambda c_2 \delta_{2l} - \lambda(c_1 - c_2)\epsilon) > \pi_2$ for small $\epsilon > 0$ since $p_{2h} - p_{2l} > \lambda c_2 \delta_{2l}$. Hence, there is no equilibrium when $\delta_{2l} < 1 - q$.

2. $\delta_{2l} = 1 - q$: Then $\delta_{1h} = \delta_{2h} > 0$ for any equilibrium; otherwise, as in the first case, $\tilde{\pi}_2 > \pi_2$. Suppose $\delta_{1l} > 0$. Then set $\tilde{p}_{2h} = p_{2h} + \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + \lambda c_2 \epsilon$. We find that the purchase fractions of SP 2 remain the same; i.e., $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = 1 - q$ while $\tilde{\delta}_{1h} = \delta_{1h} + \epsilon$ and $\tilde{\delta}_{1l} = \delta_{1l} - \epsilon$, which implies $\tilde{\pi}_2 > \pi_2$. So $\delta_{1l} = 0$ for an equilibrium, which implies $\delta_{1h} = \delta_{2h} = 0.5q$. Next, we show that an equilibrium is still impossible. Set $\tilde{\delta}_{1h} = \delta_{1h} + \epsilon$ and $\tilde{\delta}_{1l} = 0$; they are obtained by prices $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{1h} - 2\lambda c_1 \epsilon$, which yields $\tilde{\delta}_{2h} = \delta_{2h} - \epsilon$, and $\tilde{\delta}_{2l} = 1 - q$. Requiring $\tilde{\pi}_1 \leq \pi_1$ implies $p_{1h} \leq 2\lambda c_1 \delta_{1h} = \lambda c_1 q$. Set $\tilde{\delta}_{2h} = \delta_{2h} - \epsilon$ and $\tilde{\delta}_{2l} = 1 - q$; the prices are $\tilde{p}_{2h} = p_{2h} + 2\lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + 2\lambda c_2 \epsilon$, which yield $\tilde{\delta}_{1h} = \delta_{1h} + \epsilon$ and $\tilde{\delta}_{1l} = 0$. We find that $\tilde{\pi}_2 > \pi_2$ for small $\epsilon > 0$ since $\lambda c_1 q + 2\lambda c_2(1 - q) > p_{1h} = p_{2h}$. Hence, there is no equilibrium when $\delta_{2l} = 1 - q$.

Consider the structure in part (c). In this structure, $\delta_{1h} = \delta_{2h} \geq 0.5q$ and $\delta_{2l} > \delta_{1l} \geq 0$. If $\delta_{1h} = \delta_{2h} > 0.5q$ then a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. So $\delta_{1h} = \delta_{2h} = 0.5q$. If $\delta_{1l} = 0$ setting $\tilde{\delta}_{1l} = \epsilon$ (and $\tilde{\delta}_{1h} = \delta_{1h}$) by having $\tilde{p}_{1l} = p_{1l} - 2\lambda c_2 \epsilon$ (and $\tilde{p}_{1h} = p_{1h}$) results in $\tilde{\pi}_1 > \pi_1$ for small $\epsilon > 0$ since $p_{1l} > p_{2l} \geq 0$. Hence $\delta_{1l} > 0$ for equilibrium. However, then set $\tilde{\delta}_{2h} = \delta_{2h}$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$; the prices are $\tilde{p}_{2h} = p_{2h} + \lambda c_1 \delta_{1l} + \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + \lambda c_2 \delta_{1l} + 2\lambda c_2 \epsilon$, which yields $\tilde{\delta}_{1h} = \delta_{1h} + \epsilon$ and $\tilde{\delta}_{1l} = 0$. Note that for small $\epsilon > 0$, $\tilde{p}_{2l} < p_{2l} + \lambda c_2 \delta_{2l} < p_{2l} + \lambda c_2 \delta_{2l} = p_{2h} = p_{1h}$ so that the low-priority service of SP 2 only satisfies patient customers after the price change. Since $\tilde{p}_{2h}$ and $\tilde{p}_{2l}$ are significantly higher than p_{2h} and p_{2l} respectively, we find $\tilde{\pi}_2 > \pi_2$. Hence, there is no equilibrium with part (c) structure.

Consider the new structure. For this structure, $\delta_{1h} = \delta_{2h} = 0.5q$ so that $\delta_{1l} + \delta_{2l} = 1 - q$. Also, $\delta_{1l} < \delta_{2l}$ since $p_{1l} > p_{2l}$. A contradiction is shown similar to §C.2.4. Considering $\tilde{\delta}_{1l} = \delta_{1l} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ (while keeping the purchase fractions of high-priority services the same) and requiring the corresponding new profits to be lower, we obtain $p_{1l} \leq 2\lambda c_2 \delta_{1l} < 2\lambda c_2 \delta_{2l} \leq p_{2l}$, a contradiction.

Therefore, there is no equilibrium in which $p_{1h} > p_{2l} > p_{1l} = p_{2l}$.

C.2.6 $p_{1h} > p_{2h} = p_{1l} > p_{2l}$

In this price ordering, only the three structures in Figure 4 are other possible. In any other structure, either p_{1h} or p_{2l} can be increased slightly (without any loss in the corresponding purchase fraction) to strictly increase the profit so there would be no equilibrium.¹¹

Consider the structure in part (a). Then $\delta_{1h} = 0$; otherwise, a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Also, $\delta_{2l} > 1 - q$ because p_{2l} can otherwise be increased slightly so that $\tilde{\pi}_2 > \pi_2$. Then set $\tilde{p}_{1k} = p_{1k} + \lambda c_1 \epsilon$; $k = l, h$, which yields $\tilde{\delta}_{1k} = \delta_{1k}$ while $\tilde{\delta}_{2h} = \delta_{2h} + \epsilon$ and $\tilde{\delta}_{2l} = \delta_{2l} - \epsilon$ so that $\tilde{\pi}_1 > \pi_1$. Hence, there is no equilibrium.

Consider the structure in part (b). In this structure, $\delta_{2l} \leq 1 - q$. There are two cases:

1. $\delta_{2l} < 1 - q$: Suppose $\delta_{2l} > 0$. Then $\delta_{1h} = 0$; otherwise, a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$; $i = 1, 2$. Set $\tilde{\delta}_{1h} = \delta_1 - \epsilon$ and $\tilde{\delta}_{1l} = 0$; then $\tilde{\delta}_{2h} = \delta_2 + \epsilon$ and $\tilde{\delta}_{2l} = 0$ since $p_{2h} - p_{2l} < \lambda c_2(\delta_2 + \epsilon) < \lambda c_1(\delta_2 + \epsilon)$. The new prices depend on the value of δ_1 . If $\delta_1 \leq q$ then $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{1l} + \lambda c_1 \delta_{2l} + 2\lambda c_1 \epsilon$ and SP 1 satisfies only impatient customers. However, if $\delta_1 > q$ then $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{1l} + \lambda c_2 \delta_{2l} + 2\lambda c_2 \epsilon$ (for small $\epsilon > 0$) and SP 1 satisfies both impatient and patient customers. In either case, since $\tilde{p}_{1h}$ is *significantly higher* than p_{1l} (note that p_{1h} does not matter since $\delta_{1h} = 0$), we find $\tilde{\pi}_1 > \pi_1$. Hence $\delta_{2l} = 0$ and $\delta_{1l} = \delta_{2h} = 0.5$. There are two possibilities:

a. $q \leq 0.5$: Set $\tilde{\delta}_{1h} = \epsilon$ and $\tilde{\delta}_{1l} = 0.5 - \epsilon$; these purchase fractions are obtained by prices $\tilde{p}_{1h} = p_{1h} - \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} - \lambda c_2 \epsilon$. Then $\tilde{\pi}_1 = \lambda(p_{1h} - \lambda c_1 \epsilon)\epsilon + \lambda(p_{1l} - \lambda c_2 \epsilon)(0.5 - \epsilon) = 0.5\lambda p_{1l} + \lambda\epsilon(p_{1h} - p_{1l} - 0.5\lambda c_2 - \lambda(c_1 - c_2)\epsilon) > \pi_1$ for small $\epsilon > 0$ since $\pi_1 = 0.5\lambda p_{1l}$ and $p_{1h} - p_{1l} = 0.5\lambda c_1 > 0.5\lambda c_2$. So there is no equilibrium. Note that a *better solution* is not possible (with the new prices since low-priority services (from both SP 1 and SP 2) can no longer satisfy any impatient customers. Also, note that $\tilde{p}_{1l} + \lambda c_1(2\tilde{\delta}_{1h} + \tilde{\delta}_{1l}) = p_{1l} + 0.5\lambda c_1 + \lambda(c_1 - c_2)\epsilon > \tilde{p}_{1h} + \lambda c_1 \tilde{\delta}_{1h}$ so the low-priority service of SP 1 satisfies only patient customers after the price change.

b. $q > 0.5$: The analysis is similar to part (a). But we set $\tilde{\delta}_{2h} = 0.5 - \epsilon$ and $\tilde{\delta}_{2l} = \epsilon$; these purchase fractions are obtained by prices $\tilde{p}_{2h} = p_{2h} + \lambda c_1 \epsilon$ and $\tilde{p}_{2l} = p_{2l} + \lambda c_2 \epsilon$. Then $\tilde{\pi}_2 = \lambda(p_{2h} + \lambda c_1 \epsilon)(0.5 - \epsilon) + \lambda(p_{2l} + \lambda c_2 \epsilon)\epsilon = 0.5\lambda p_{2h} + \lambda\epsilon(0.5\lambda c_1 - (p_{2h} - p_{2l}) - \lambda(c_1 - c_2)\epsilon) > \pi_2$ for small $\epsilon > 0$ since $\pi_2 = 0.5\lambda p_{2h}$ and $p_{2h} - p_{2l} = 0.5\lambda c_2 < 0.5\lambda c_1$. So there is no equilibrium. Again, similar to part (a), note that $\tilde{p}_{2l} + \lambda c_1(2\tilde{\delta}_{2h} + \tilde{\delta}_{2l}) = p_{2l} + \lambda c_1 - \lambda(c_1 - c_2)\epsilon > \tilde{p}_{2h} + \lambda c_1 \tilde{\delta}_{2h}$

¹¹Note that, without any loss of generality, we do not consider structures in which no one selects a service class and it still performs worse among both patient and impatient customers. In such cases, the price of this service class can be reduced so that a marginal (patient or impatient) customer is indifferent to it. If the former instance is an equilibrium, the later instance will also be an equilibrium.

for small $\epsilon > 0$ since $p_{2h} - p_{2l} < \lambda c_1 \delta_2 = 0.5\lambda c_1$, so that the low-priority service of SP 2 satisfies only patient customers after the price change.

2. $\delta_{2l} = 1 - q$: Then the low-priority service from SP 2 satisfies all patient customers. It is now possible that $\delta_{1h} > 0$. Initially, we note that SP 1 can obtain the same profit by setting a single price (for both high- and low-priority services) $p_{1e} \equiv p_{1l} + \lambda c_1 \delta_{1h}$. It can be easily verified that the corresponding purchase fraction $\delta_{1e} = \delta_1$ and the profit $\pi_{1e} = \lambda(p_{1l} + \lambda c_1 \delta_{1h})\delta_1 = \lambda p_{1l} \delta_{1l} + \lambda(p_{1l} + \lambda c_1 \delta_1)\delta_{1h} = \lambda p_{1l} \delta_{1l} + \lambda p_{1h} \delta_{1h} = \pi_1$. Set $\tilde{\delta}_{1h} = \delta_1 - \epsilon$ and $\tilde{\delta}_{1l} = 0$; they are obtained by prices $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{1e} + \lambda c_1(1 - q) + 2\lambda c_1 \epsilon$. Since $\tilde{p}_{1h}$ is significantly higher than p_{1e} , we find that $\tilde{\pi}_1 > \pi_1$ for small $\epsilon > 0$. *So there is no equilibrium.*

Consider the structure in part (c). The argument for no equilibrium is similar to the one in §C.2.4 (for part (c) structure).

Therefore, there is no equilibrium in which $p_{1h} > p_{2h} = p_{1l} > p_{2l}$.

C.2.7 $p_{1h} = p_{2h} > p_{1l} = p_{2l}$

There are only three possible structures we need to consider: (i) impatient customers are indifferent to all the four service classes, (ii) patient customers are indifferent to all of them, and (iii) no one is indifferent to all of them. Note that since the high- and low-priority services each have the same price, both impatient and patient customers will be indifferent towards each of them.

Consider the first structure in which $p_{ih} - p_{il} = \lambda c_1 \delta_i$, $\delta_{ih} \leq 0.5q$, and $\delta_{il} \geq 0.5(1 - q)$; $i = 1, 2$. Then $\delta_{ih} = 0$ because otherwise a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$. Hence $\delta_{il} = 0.5$. There are two cases:

(i) $q \leq 0.5$: The analysis here is similar to the corresponding scenario in §C.2.6 (when $\delta_{2l} < 1 - q$). Set $\tilde{\delta}_{1h} = \epsilon$ and $\tilde{\delta}_{1l} = 0.5 - \epsilon$; the prices $\tilde{p}_{1h} = p_{1h} - \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} - \lambda c_2 \epsilon$. Then $\tilde{\pi}_1 > \pi_1$. *Hence, there is no equilibrium.*

(ii) $q > 0.5$: Set $\tilde{\delta}_{1h} = 0.5 - \epsilon$ and $\tilde{\delta}_{1l} = 0$. Since $p_{2h} - p_{2l} < \lambda c_1 \tilde{\delta}_2$, we get $\tilde{\delta}_{2h} = q - 0.5 + \epsilon$ and $\tilde{\delta}_{2l} = 1 - q$. These purchase fractions are obtained through prices, $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{2h} + \lambda c_1(q - 1) + 2\lambda c_1 \epsilon = p_{1l} + \lambda c_1(q - 0.5) + 2\lambda c_1 \epsilon$. Since $\tilde{p}_{1h}$ is *significantly higher* than p_{1l} , we find that $\tilde{\pi}_1 > \pi_1$. *Hence, there is no equilibrium.* Note that $\tilde{p}_{1h} + \lambda c_2 \tilde{\delta}_{1h} = p_{1l} + \lambda c_1(q - 0.5) + \lambda c_2(0.5 - \epsilon) + 2\lambda c_1 \epsilon > p_{1l} + \lambda c_2(q - 0.5) + \lambda c_2 \epsilon + \lambda c_2(0.5 - \epsilon) + 2\lambda c_2 \epsilon = p_{1l} + \lambda c_2(q + 2\epsilon) = p_{1l} + \lambda c_2(2\tilde{\delta}_{2h} + \tilde{\delta}_{2l})$ for small $\epsilon > 0$ so SP 1 satisfies only impatient customers after the price change.

Consider the second structure in which $p_{ih} - p_{il} = \lambda c_2 \delta_i$, $\delta_{ih} \geq 0.5q$, and $\delta_{il} \leq 0.5(1 - q)$; $i = 1, 2$. Then $\delta_{il} = 0.5(1 - q)$ because otherwise a better solution is obtained by setting $\tilde{\delta}_{ih} = \delta_{ih} - \epsilon$ and $\tilde{\delta}_{il} = \delta_{il} + \epsilon$. Hence $\delta_{ih} = 0.5q$. Set $\tilde{\delta}_{1h} = 0.5q - \epsilon$ and $\tilde{\delta}_{1l} = 0.5(1 - q) + \epsilon$; they are obtained by

prices $\tilde{p}_{1h} = p_{1h} + 2\lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p_{1l} + 2\lambda c_2 \epsilon$, which yield $\tilde{\delta}_{2h} = 0.5q + \epsilon$ and $\tilde{\delta}_{2l} = 0.5(1 - q) - \epsilon$. Note that this solution cannot be further improved since only high-priority services (from SP 1 and SP 2) satisfy impatient and $\tilde{\delta}_{1h} + \tilde{\delta}_{2h} = q$ already. We find $\tilde{\pi}_1 = \lambda(p_{1h} + 2\lambda c_1 \epsilon)(0.5q - \epsilon) + \lambda(p_{1l} + 2\lambda c_2 \epsilon)(0.5(1 - q) + \epsilon) = \pi_1 + \lambda\epsilon(\lambda c_1 q + \lambda c_2(1 - q) - (p_{1h} - p_{1l}) - 2\lambda(c_1 - c_2)\epsilon) > \pi_1$ for small $\epsilon > 0$ since $p_{1h} - p_{1l} = 0.5\lambda c_2 < \lambda c_1 q + \lambda c_2(1 - q)$. Hence, there is no equilibrium.

Consider the third structure in which $\lambda c_2 \delta_i < p_{ih} - p_{il} < \lambda c_1 \delta_i; i = 1, 2$. Then $\delta_{ih} = 0.5q$ and $\delta_{il} = 0.5(1 - q)$ so that $\delta_i = 0.5$. Set $\tilde{\delta}_{1h} = 0.5q + \alpha\epsilon$ and $\tilde{\delta}_{1l} = 0.5(1 - q) + \beta\epsilon$; the prices are $\tilde{p}_{1h} = p_{1h} - 2\lambda c_1 \alpha\epsilon$ and $\tilde{p}_{1l} = p_{1l} - 4\lambda c_2 \alpha\epsilon - 2\lambda c_2 \beta\epsilon$, which yields $\tilde{\delta}_{1h} = 0.5q - \alpha\epsilon$ and $\tilde{\delta}_{1l} = 0.5(1 - q) - \beta\epsilon$. Note that the high-priority (low-priority) services still satisfy only impatient (patient) customers after the price change. Then $\tilde{\pi}_1 \leq \pi_1$ implies $\forall \alpha, \beta$

$$\lambda\epsilon(\alpha(p_{1h} - \lambda c_1 q - 2\lambda c_2(1 - q)) + \beta(p_{1l} - \lambda c_2(1 - q)) - (2\lambda c_1 \alpha^2 + 4\lambda c_2 \alpha\beta + 2\lambda c_2 \beta^2)\epsilon^2) \leq 0. \quad (42)$$

Hence $p_{1h} = \lambda c_1 q + 2\lambda c_2(1 - q)$ and $p_{1l} = \lambda c_2(1 - q)$ for an equilibrium to occur. Also, $0.5\lambda c_2 < p_{1h} - p_{1l} = \lambda c_1 q + \lambda c_2(1 - q) < 0.5\lambda c_1$. Since $c_2 < c_1$, it is always true that $0.5\lambda c_2 < \lambda c_2 < \lambda c_1 q + \lambda c_2(1 - q)$. Furthermore, $\lambda c_1 q + \lambda c_2(1 - q) < 0.5\lambda c_1$ implies $q < \frac{c_1 - 2c_2}{2(c_1 - c_2)}$.

D. Proofs of Theorems

We derive the sufficient conditions in Theorems 1-3 by using the following approach. Some simple conditions result either from the nature of the equilibrium or are based on *marginal deviations* by SPs. More complicated conditions are based on *large deviations* by SPs. These conditions together are sufficient because an equilibrium exists if *any deviation* (marginal or large) by the SPs is counterproductive. Note that some of them pertaining to $\mathcal{SD}$ may not be necessary because we do not consider the limited valuation v in large deviations. However, if v is large enough, they become necessary and sufficient conditions.

Proof of Theorem 1: First, we show by contradiction there can be no asymmetric equilibrium in which all customers get positive net utility. We let $p_1 > p_2$ without any loss of generality. Then either (i) impatient customers are indifferent or (ii) patient customers are indifferent; otherwise p_2 can be slightly increased to increase the profit of SP 2. Suppose impatient customers are indifferent; then $\delta_2 > 1 - q$ because otherwise p_2 can again be slightly increased to increase the profit of SP 2. Setting $\tilde{\delta}_1 = \delta_1 + \epsilon$ and requiring $\tilde{\pi}_1 < \pi_1$ implies $p_1 \leq 2\lambda c_1 \delta_1$; however, then setting $\tilde{\delta}_2 = \delta_2 - \epsilon$ and requiring $\tilde{\pi}_2 < \pi_2$ for any small $\epsilon > 0$ implies $p_2 \geq 2\lambda c_1 \delta_2 > 2\lambda c_1 \delta_1 \geq p_1$, a contradiction. Similarly, if patient customers are indifferent then $\delta_1 > q$, and we obtain $p_2 \geq 2\lambda c_2 \delta_2 > 2\lambda c_2 \delta_1 \geq p_1$, a contradiction. Hence $p_1 = p_2 \equiv p^e$ and $\delta_1 = \delta_2 \equiv \delta^e = 0.5$ at equilibrium.

Suppose $q \geq 0.5$. Then $\tilde{\delta}_1 = 0.5 \pm \epsilon$ implies $\tilde{p}_1 = p_1 \mp 2\lambda c_1 \epsilon$. Since $\tilde{\pi}_1 < \pi_1$ for small $\epsilon > 0$ in both cases, we obtain $p_1 = \lambda c_1$. Furthermore, the conditions are sufficient for an equilibrium since the (maximum)¹² profit of SP 1 for any purchase fraction x , given by $\lambda x (2\lambda c_1(1-x))$, is optimized when $x = 0.5$; $v > c_1 + 1.5\lambda c_1$ ensures all customers get positive net utility at equilibrium.

Suppose $q < 0.5$. Then $\tilde{\delta}_1 = 0.5 \pm \epsilon$ implies $\tilde{p}_1 = p_1 \mp 2\lambda c_2 \epsilon$. Since $\tilde{\pi}_1 < \pi_1$ for small $\epsilon > 0$ in both cases, we obtain $p_1 = \lambda c_2$. As long as $x > q$, the (maximum) profit of SP 1 for any purchase fraction x is given by $\lambda x (2\lambda c_2(1-x))$, which is optimized when $x = 0.5$. However, when $x \leq q$, SP 1 can choose to satisfy only impatient customers by charging a significantly higher price. Then the maximum profit of SP 1 is $\lambda x (\lambda c_2 + \lambda c_1(1-2x))$. An equilibrium exists if $0.5\lambda^2 c_2 \geq \max_{0 \leq x \leq q} \lambda^2 x (c_2 + c_1 - 2c_1 x)$. First, we note that $0.5\lambda^2 c_2 \geq \lambda^2 q (c_2 + c_1 - 2c_1 q)$ implies $q \leq 0.5c_2/c_1$. Also, since $0.5c_2/c_1 < (c_1 + c_2)/(4c_1)$, $\max_{0 \leq x \leq q} \lambda^2 x (c_2 + c_1 - 2c_1 x) = \lambda^2 q (c_2 + c_1 - 2c_1 q)$. So an equilibrium exists if $q \leq 0.5c_2/c_1$. However, if $q > 0.5c_2/c_1$, $\lambda c_2 + c_1 + \lambda c_1(1-q) < c_1 + \lambda c_2 + \lambda c_1(1 - 0.5c_2/c_1) = c_1 + 0.5\lambda c_2 + \lambda c_1 < c_1 + 1.5\lambda c_1 < v$ and $0.5\lambda^2 c_2 < \lambda^2 q (c_2 + c_1 - 2c_1 q)$ so that there is no equilibrium. ■

Proof of Theorem 2: I We first consider $\mathcal{SD}_1^e$ in which $p_{2h}^e > p_1^e > p_{2l}^e$. The prices, purchase fractions, and the condition $c_1 < 3c_2$ are based on the analysis in §B.2.1. The conditions $\delta_{2h} < q$ and $\delta_{2l} < 1 - q$ are required for an interior equilibrium. We find that $\delta_1^e > 0.5 \Leftrightarrow \frac{6c_1c_2 - 6c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2} > 0.5 \Leftrightarrow c_1^2 - 3c_2^2 > 0$ which is true since $c_1 > 2c_2$. We now compare the profits of SPs 1 and 2. We find that

$$\begin{aligned}
& \pi_1^e < \pi_2^e \\
& \Leftrightarrow \lambda^2 \frac{c_1 c_2}{c_1 - c_2} (\delta_1^e)^2 < \lambda^2 ((2c_1 \delta_{2h}^e + 3c_2 \delta_{2l}^e) \delta_{2h}^e + (c_1 \delta_{2h}^e + 2c_2 \delta_{2l}^e) \delta_{2l}^e) \\
& \Leftrightarrow \frac{c_1 c_2}{c_1 - c_2} (\delta_1^e)^2 < 2c_1 (\delta_{2h}^e)^2 + (c_1 + 3c_2) \delta_{2h}^e \delta_{2l}^e + 2c_2 (\delta_{2l}^e)^2 \\
& \Leftrightarrow 36c_2^2 (c_1 - c_2)^2 < 18c_2 (c_1 - c_2)^2 + 3(c_1 + 3c_2)(c_1 - c_2)(3c_2 - c_1) + 2c_1(3c_2 - c_1)^2 \\
& \Leftrightarrow (c_1 - 3c_2)^3 < 0,
\end{aligned}$$

which is true since $c_1 < 3c_2$. Next, we develop sufficient conditions for equilibrium to exist. For this purpose, we first consider *large* deviations by SP 1.

Suppose the purchase fraction of SP 1 decreases. Similar to the analysis in §B.2.1, we find that if the new purchase fraction is $\tilde{\delta}_1 < \delta_1^e$, then the price of SP 1 is $\tilde{p}_1 = \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - \tilde{\delta}_1)$, $\tilde{\delta}_{2h} = \delta_{2h}^e + \frac{c_1 - 2c_2}{c_1 - c_2} (\tilde{\delta}_1 - \delta_1^e)$ and $\tilde{\delta}_{2l} = \delta_{2l}^e + \frac{2c_1 - 3c_2}{c_1 - c_2} (\delta_1^e - \tilde{\delta}_1)$, *provided* $\tilde{p}_1 \leq p_{2h}$, $\tilde{\delta}_{2h} \geq 0$, $\tilde{\delta}_{2l} \leq 1 - q$, and $p_{2h} - p_{2l} \geq \lambda c_2 \tilde{\delta}_2$; we then find $\tilde{\pi}_1 = \lambda \tilde{\delta}_1 \tilde{p}_1 < \pi_1^e$. However, we need to consider which conditions

¹²The profit expression does not consider the valuation v which can reduce the price charged by SP 1 when $x < 0.5$.

will be binding and what happens when δ_1 is reduced further. First, we note that since $\delta_{2h} = 0.5\delta_1^e$ and $\frac{c_1-2c_2}{c_1-c_2} < \frac{1}{2} \forall c_1 < 3c_2$, $\tilde{\delta}_{2h} > 0$ always. The value of $\bar{\delta}_2 \equiv (p_{2h} - p_{2l})/(\lambda c_2)$ determines whether $\tilde{p}_1 = p_{2h}$ or $\tilde{\delta}_{2l} = 1 - q$ occurs first as δ_1 reduces. There are two possibilities:

(i) $\bar{\delta}_2 \leq 1 - 0.5q$: In this case, $\tilde{p}_1 = p_{2h}$ occurs first (or together with $\tilde{\delta}_{2l} = 1 - q$). At this point, $\tilde{\delta}_1 = \tilde{\delta}_{2h} = 1 - \bar{\delta}_2$ and $\tilde{\delta}_{2l} = 2\bar{\delta}_2 - 1$, and $\tilde{\pi}_1 < \pi_1^e$ still holds since $\tilde{p}_1 = \lambda c_1 c_2 / (c_1 - c_2)(2\delta_1^e - \tilde{\delta}_1)$. However, for any $\tilde{\delta}_1 < 1 - \bar{\delta}_2$, $p_{2h} - p_{2l} = \lambda c_2 \bar{\delta}_2 < \lambda c_2(1 - \tilde{\delta}_1) = \lambda c_2 \tilde{\delta}_{2l}$ so that $\tilde{p}_1 = p_{2h} + \lambda c_1(1 - 2\tilde{\delta}_1)$. Hence, if $\max_{0 \leq x < 1 - \bar{\delta}_2} \lambda x(p_{2h} + \lambda c_1 - 2\lambda c_1 x) \leq \pi_1^e$, we have an equilibrium. Since the function in the left side is continuous in x , the inequality is equivalent to $\max_{0 \leq x \leq 1 - \bar{\delta}_2} \lambda x(p_{2h}^e + \lambda c_1 - 2\lambda c_1 x) \leq \pi_1^e$. Furthermore, we find

$$\begin{aligned}
& p_{2h}^e + \lambda c_1 > 4\lambda c_1(1 - \bar{\delta}_2) \\
\Leftrightarrow & 2\lambda c_1 \delta_{2h}^e + 3\lambda c_2 \delta_{2l}^e > \lambda c_1(3 - 4c_1/c_2 \delta_{2h}^e - 4\delta_{2l}^e) \\
\Leftrightarrow & (2 + 4c_1/c_2)\delta_{2h}^e + (4 + 3c_2/c_1)\delta_{2l}^e > 3 \\
\Leftrightarrow & 8c_1^2 + 3c_1c_2 + 3c_2^2 > 36c_1c_2 - 3c_1^2 - 27c_2^2 \\
\Leftrightarrow & 11c_1^2 - 33c_1c_2 + 30c_2^2 > 0 \\
\Leftrightarrow & 11(c_1 - 1.5c_2)^2 + 5.25c_2^2 > 0,
\end{aligned}$$

which is true. Hence $\max_{0 \leq x \leq 1 - \bar{\delta}_2} \lambda x(p_{2h}^e + \lambda c_1 - 2\lambda c_1 x) = \lambda(1 - \bar{\delta}_2)(p_{2h} + \lambda c_1(2\bar{\delta}_2 - 1)) \leq \pi_1^e$ is required for the equilibrium.

(ii) $\bar{\delta}_2 > 1 - 0.5q$: In this case, $\tilde{\delta}_{2l} = 1 - q$ occurs first. At this point, $\tilde{\delta}_1 + \tilde{\delta}_{2h} = \delta_1^e + \delta_{2h}^e + \frac{2c_1-3c_2}{c_1-c_2}(\tilde{\delta}_1 - \delta_1^e) = q$ so that $\tilde{\delta}_1 = \delta_1^e - \frac{c_1-c_2}{2c_1-3c_2}(\delta_1^e + \delta_{2h}^e - q) = 2\delta_{2h}^e - \frac{c_1-c_2}{2c_1-3c_2}(3\delta_{2h}^e - q) = \frac{c_1-3c_2}{2c_1-3c_2}\delta_{2h}^e + \frac{c_1-c_2}{2c_1-3c_2}q \equiv \delta_1^m$. Since $\tilde{p}_1 = \lambda c_1 c_2 / (c_1 - c_2)(2\delta_1^e - \tilde{\delta}_1)$, $\tilde{\pi}_1 < \pi_1^e$ still holds. Furthermore, for any $1 - \bar{\delta}_2 \leq \tilde{\delta}_1 \leq \delta_1^m$, we obtain $\tilde{p}_1 = p_{2h}^e + \lambda c_1 q - 2\lambda c_1 \tilde{\delta}_1$ since SP 1 only satisfies impatient customers. Then using a logic similar to part (i), the condition for equilibrium becomes

$$\max \left(\max_{1 - \bar{\delta}_2 \leq x \leq \delta_1^m} \lambda x(p_{2h}^e + \lambda c_1 q - 2\lambda c_1 x), \lambda(1 - \bar{\delta}_2)(p_{2h} + \lambda c_1(2\bar{\delta}_2 - 1)) \right) \leq \pi_1^e$$

Furthermore, we find that $p_{2h}^e + \lambda c_1 q > 4\lambda c_1(1 - \bar{\delta}_2)$; the reasoning is as follows. After some algebra, the condition simplifies to

$$\begin{aligned}
& (2 + 4c_1/c_2)\delta_{2h}^e + (4 + 3c_2/c_1)\delta_{2l}^e > 4 - q \\
\Leftrightarrow & (2 + 4c_1/c_2)\delta_{2h}^e + (4 + 3c_2/c_1)\delta_{2l}^e \geq 4 - \delta_{2h}^e \\
\Leftrightarrow & 8c_1^2 + 3c_1c_2 + 3c_2^2 \geq 45c_1c_2 - 4c_1^2 - 33c_2^2 \\
\Leftrightarrow & (2c_1 - 3c_2)(c_1 - 2c_2) \geq 0,
\end{aligned}$$

which is true. Hence $\arg \max_{1-\delta_2 \leq x \leq \delta_1^m} \lambda x (p_{2h}^e + \lambda c_1 q - 2\lambda c_1 x)$ equals $\frac{p_{2h}^e + \lambda c_1 q}{4\lambda c_1}$ if $p_{2h}^e + \lambda c_1 q \leq 4\lambda c_1 \delta_1^m$ and equals δ_1^m otherwise. Requiring the corresponding profits to be less than π_1^e gives the conditions in Table 1.

Suppose the purchase fraction of SP 1 increases. Next, we show the profit of SP 1 always decreases by showing $\tilde{p}_1 \leq \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - \tilde{\delta}_1) \forall \tilde{\delta}_1 > \delta_1$. The inequality binds *provided* $\tilde{p}_1 \geq p_{2l}$, $\tilde{\delta}_{2h} \leq q$, $\tilde{\delta}_{2l} \geq 0$, and $p_{2h} - p_{2l} \leq \lambda c_1 \tilde{\delta}_2$. Let $\underline{\delta}_2 \equiv (p_{2h} - p_{2l}) / (\lambda c_1)$. There are three possibilities:

(i) $\tilde{p}_1 = p_{2l}$ occurs first (or together with other boundary conditions): At this point, $\tilde{\delta}_1 = 1 - \underline{\delta}_2$, $\tilde{\delta}_{2h} = 1 - 2\underline{\delta}_2 > \delta_{2h} > 0$, $\tilde{\delta}_{2l} = 3\underline{\delta}_2 - 1$, and $\tilde{p}_1 = \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - \tilde{\delta}_1)$. However, for any $\tilde{\delta}_1 > 1 - \underline{\delta}_2$, since $p_{2h} - p_{2l} > \lambda c_1 (1 - \tilde{\delta}_1) = \lambda c_1 \tilde{\delta}_2$, $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = \tilde{\delta}_2$. Hence, for such $\tilde{\delta}_1$, $\tilde{p}_1 \leq \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - (1 - \underline{\delta}_2)) - \lambda c_2 (1 - 2\underline{\delta}_2) + 2\lambda c_2 (1 - \underline{\delta}_2 - \tilde{\delta}_1)$ since SP 1 has to satisfy at least some patient customers. Note that if SP 1 satisfies any impatient customers, the price would be lower than the right side of the inequality since $\tilde{\delta}_1 > 1 - \underline{\delta}_2 > \underline{\delta}_2 > \tilde{\delta}_2$. Also, note that there is a *discontinuity* in $\tilde{p}_1$ at $1 - \underline{\delta}_2$ since the low-priority service of SP 2 satisfies all her customers (both impatient and patient) for any $\tilde{\delta}_1 > 1 - \underline{\delta}_2$. Hence $\tilde{p}_1 < \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - (1 - \underline{\delta}_2)) + \lambda c_1 c_2 / (c_1 - c_2) (1 - \underline{\delta}_2 - \tilde{\delta}_1) = \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - \tilde{\delta}_1)$ since $2c_2 > c_1 c_2 / (c_1 - c_2)$ as $c_1 > 2c_2$.

(ii) $\tilde{\delta}_{2h} = q$ occurs with $\tilde{p}_1 > p_{2l}$ (possibly together with $\tilde{\delta}_{2l} = 0$): At this point, $\tilde{\delta}_1 + \tilde{\delta}_{2l} = 1 - q$. For any higher values of $\tilde{\delta}_1$, a logic similar to part (i) applies. Since SP 1 has to satisfy at least some patient customers, we can show that $\tilde{p}_1 < \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - \tilde{\delta}_1)$.

(iii) $\tilde{\delta}_{2l} = 0$ occurs with $\tilde{p}_1 > p_{2l}$ and $\tilde{\delta}_{2h} < q$: Since $\tilde{\delta}_1 + \tilde{\delta}_{2h} = 1$, we have $\tilde{\delta}_1 > 1 - q$ so that SP 1 satisfies some patient customers too. At this point, suppose $\tilde{\delta}_1 \equiv z$. Then $\forall z < \tilde{\delta}_1 \leq 1 - \underline{\delta}_2$, $\tilde{p}_1 = \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - z) + 2\lambda c_1 (z - \tilde{\delta}_1) < \lambda c_1 c_2 / (c_1 - c_2) (2\delta_1^e - \tilde{\delta}_1)$. If $\tilde{\delta}_1 > 1 - \underline{\delta}_2$ then the price (as a function of $\tilde{\delta}_1$) will be even lower since all the customers of SP 2 (both impatient and patient) will be satisfied by just the low-priority service.

We now consider possible deviation by SP 2. Suppose $\tilde{\delta}_{2h} = \delta_{2h}^e + x$ and $\tilde{\delta}_{2l} = \delta_{2l}^e + y$. *Assuming the high- and low-priority services are priced to satisfy impatient and patient customers respectively*, we obtain $\tilde{p}_{2h} = p_{2h}^e - \lambda c_1 (2x + y)$ and $\tilde{p}_{2l} = p_{2l}^e - \lambda c_2 (3x + 2y)$. Similar to the analysis in §B.2.1, we then get

$$\begin{aligned}
\tilde{\pi}_2 &= \lambda (p_{2h}^e - \lambda c_1 (2x + y)) (\delta_{2h}^e + x) + \lambda (p_{2l}^e - \lambda c_2 (3x + 2y)) (\delta_{2l}^e + y) \\
&= \pi_2^e - \lambda^2 (2c_1 x^2 + (c_1 + 3c_2)xy + 2c_2 y^2) \\
&= \pi_2^e - \lambda^2 \left(\left(x + \frac{c_1 + 3c_2}{4c_1} y \right)^2 + \frac{(c_1 - c_2)(9c_2 - c_1)}{8c_1^2} y^2 \right) \\
&\leq \pi_2^e \forall x, y.
\end{aligned}$$

Hence, any such deviation is futile. We also need to consider what happens if the *assumption* above is violated. We only need to consider the following three possibilities¹³ which we show are equivalent to the *previous scenario with the assumption*:

(i) Customers purchase from a single service class: Then either impatient customers are indifferent between this service class and SP 1, which is equivalent to having $\tilde{\delta}_{2l} = 0$ in the previous scenario; or patient customers are indifferent between this service class and SP 1, which is equivalent to having $\tilde{\delta}_{2h} = 0$ in the previous scenario.

(ii) Impatient customers are indifferent between all three service classes: The profit here will be the same as the profit obtained by having a single service with price $\tilde{p}_1 = p_{2h} - \lambda c_1 \tilde{\delta}_{2l}$ and the purchase fraction of $\tilde{\delta}_2$. However, that is equivalent to having $\tilde{\delta}_{2h} = \tilde{\delta}_2$ and $\tilde{\delta}_{2l} = 0$ in the previous scenario.

(iii) Patient customers are indifferent between all three service classes: The profit here will be the same as the profit obtained by having a single service with price $\tilde{p}_1 = p_{2h} - \lambda c_2 \tilde{\delta}_{2l}$ and the purchase fraction of $\tilde{\delta}_2$. However, that is equivalent to having $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = \tilde{\delta}_2$ in the previous scenario.

II We consider $\mathcal{SD}_2^e$ in which $p_{2h}^e > p_{2l}^e = p_1^e$, $\delta_{2h}^e = q$, and $\delta_{2l}^e > 0$. The prices and purchase fractions are from the analysis in §B.2.5. The condition $q \leq \frac{c_2(c_1-2c_2)}{9c_1c_2-2c_1^2-6c_2^2} \forall c_1 < 3c_2$ is also from §B.2.5. The condition on v ensures all customers obtain positive net utility. Next, we first consider large deviations by SP 1 and then those by SP 2.

Suppose the purchase fraction of SP 1 decreases, i.e., $\tilde{\delta}_1 = \delta_1^e - x$ with $x > 0$. Then we find that $\tilde{p}_1 = p_1^e + \lambda x \cdot \frac{c_1c_2}{c_1-c_2}$ as long as $\tilde{\delta}_{2h} = \delta_{2h}^e - \frac{c_1-2c_2}{c_1-c_2} \cdot x \geq 0$, and $\tilde{\delta}_{2l} = \delta_{2l}^e + \frac{2c_1-3c_2}{c_1-c_2} \cdot x \leq 1 - q$, and $\tilde{p}_1 \leq p_{2h}^e$. The analysis in §B.2.5 (with the resulting condition on q stated above) implies that these deviations will result in $\tilde{\pi}_1 \leq \pi_1$. As x increases, it is possible that $\tilde{\delta}_1 \rightarrow 0$ without violating any condition on $\tilde{\delta}_{2h}$, $\tilde{\delta}_{2l}$, and $\tilde{p}_1$. Then SP 1 cannot do better for any $\tilde{\delta}_1 < \delta_1^e$. However, there can be three other possible *limiting* scenarios:

(a) $\tilde{\delta}_{2h} = 0$ occurs: At this point, $x = \frac{c_1-c_2}{c_1-2c_2}q$, $\tilde{p}_1 = p_1 + \lambda \cdot \frac{c_1c_2}{c_1-2c_2}q$, and the profit of SP 1 is strictly less than the equilibrium profit. Furthermore, for any $x \geq \frac{c_1-c_2}{c_1-2c_2}q$, $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = 1 - \tilde{\delta}_1 = 1 - \delta_1^e + x$ so that $\tilde{p}_1 = p_{2l}^e + \lambda c_2(1 - 2\delta_1^e + 2x) = p_1^e - \lambda c_2q + 2\lambda c_2x \geq p_1^e + \lambda \cdot \frac{c_1c_2}{c_1-c_2}x$. However, we find $\arg \max_{x \geq \frac{c_1-c_2}{c_1-2c_2}q} \lambda(\delta_1^e - x)(p_1^e - \lambda c_2q + 2\lambda c_2x) = \frac{c_1-c_2}{c_1-2c_2}q$ since the function is strictly concave in x and $\frac{\partial}{\partial x} \lambda(\delta_1^e - x)(p_1^e - \lambda c_2q + 2\lambda c_2x) \Big|_{x=\frac{c_1-c_2}{c_1-2c_2}q} = -\lambda(p_1^e - \lambda c_2q + 4\lambda c_2 \cdot \frac{c_1-c_2}{c_1-2c_2}q - 2\lambda c_2\delta_1^e) = -\lambda(\lambda c_1q - 5\lambda c_2q + 4\lambda c_2 \cdot \frac{c_1-c_2}{c_1-2c_2}q) < -\lambda^2(c_1 - c_2)q < 0$. Hence SP 1 can never do better in this scenario.

¹³Any other possibility would be dominated by one of them or the previous scenario with the assumption.

(b) $\tilde{\delta}_{2l} = 1 - q$ occurs before $\tilde{\delta}_{2h} = 0$: The analysis here is similar to the one in part (a). When $\tilde{\delta}_{2l} = 1 - q$, $\frac{1-3q}{2} + \frac{2c_1-3c_2}{c_1-c_2}x = 1 - q$ so that $x = \frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2}$; the profit of SP 1 here is strictly less than the equilibrium profit. At this point, $\tilde{\delta}_{2h} = \frac{q(3c_1-4c_2)-(c_1-2c_2)}{2(2c_1-3c_2)} > 0 \Rightarrow q > \frac{c_1-2c_2}{3c_1-4c_2}$ and $\tilde{\delta}_1 = \frac{(c_1-c_2)(1+q)}{2(2c_1-3c_2)}$. For any $\frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2} \leq x \leq \delta_1^e - 1 + \frac{c_1}{2c_2} \cdot (1-q)$, $\tilde{\delta}_{2l} = 1 - q$, and $\tilde{\delta}_{2h} = q - \tilde{\delta}_1 = q - \delta_1^e + x$ so that $\tilde{p}_1 = p_{2h}^e + \lambda c_1(q - 2\delta_1^e) + 2\lambda c_1 x = p_{2h}^e - \lambda c_1 + 2\lambda c_1 x = p_1^e - \lambda c_1 \frac{1+q}{2} + 2\lambda c_1 x \geq p_1^e + \lambda x \cdot \frac{c_1 c_2}{c_1 - c_2}$. When $x = \delta_1^e - 1 + \frac{c_1}{2c_2} \cdot (1-q)$, $p_{2h}^e - p_{2l}^e = \lambda c_1 \frac{1-q}{2} = 2\lambda c_2 \tilde{\delta}_2$. Hence, for any $x > \delta_1^e - 1 + \frac{c_1}{2c_2} \cdot (1-q)$, $\tilde{\delta}_{2h} = 1 - \tilde{\delta}_1$ and $\tilde{\delta}_{2l} = 0$ so that $\tilde{p}_1 = p_{2h}^e + \lambda c_1(1 - 2\delta_1^e) + 2\lambda c_1 x = p_{2h}^e - \lambda c_1 q + 2\lambda c_1 x$.

Suppose $c_1 \leq 3c_2$. Then $\arg \max_{\frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2} \leq x \leq \delta_1^e - 1 + \frac{c_1}{2c_2} \cdot (1-q)} \lambda(\delta_1^e - x)(p_{2h}^e - \lambda c_1 + 2\lambda c_1 x) = \frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2}$ since the function is strictly concave in x and $\frac{\partial}{\partial x} \lambda(\delta_1^e - x)(p_{2h}^e - \lambda c_1 + 2\lambda c_1 x) \Big|_{x=\frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2}} = -\lambda(p_{2h}^e - \lambda c_1 + 2\lambda c_1(1+q) \cdot \frac{c_1-c_2}{2c_1-3c_2} - 2\lambda c_1 \delta_1^e) = -\lambda(-\lambda c_1 \cdot \frac{3+q}{2} + 2\lambda c_1(1+q) \cdot \frac{c_1-c_2}{2c_1-3c_2} + \lambda c_2(1-3q)) \leq -\lambda(-\lambda c_1 \cdot \frac{3+q}{2} + \frac{4}{3}\lambda c_1(1+q) + \lambda c_1 \frac{1-3q}{3}) = -\lambda^2 c_1 \cdot \frac{1-q}{6} < 0$. Also, the necessary condition $q \leq \frac{c_2(c_1-2c_2)}{9c_1c_2-2c_1^2-6c_2^2}$ (see the start of the proof for part II), which automatically applies when $c_1 = 3c_2$ since $q < 1/3$, implies $q \leq \frac{c_1-2c_2}{c_1}$ since $8c_1c_2 - 2c_1^2 - 6c_2^2 = 2(c_1 - c_2)(3c_2 - c_1) \geq 0$. So $\frac{c_1}{2c_2} \cdot (1-q) \geq 1$ and it is impossible to have $x > \delta_1^e - 1 + \frac{c_1}{2c_2} \cdot (1-q) \geq \delta_1^e$. Hence SP 1 cannot do better.

Suppose $c_1 > 3c_2$. We first note that it is again impossible to have $x > \delta_1^e - 1 + \frac{c_1}{2c_2} \cdot (1-q)$ since $\frac{c_1}{2c_2} \cdot (1-q) > \frac{3}{2} \cdot \frac{2}{3} = 1$. Also, since $\tilde{\delta}_{2h} > 0$ at the boundary, $q > \frac{c_1-2c_2}{3c_1-4c_2} \geq \frac{3-2}{3 \cdot 3-4} = 0.2$. We require $\arg \max_{\frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2} \leq x} \lambda(\delta_1^e - x)(p_{2h}^e - \lambda c_1 + 2\lambda c_1 x)$ to be an *interior point* for SP 1 to do better. Equating the derivative with respect to x to zero, we obtain $x^* = \frac{-p_{2h}^e + \lambda c_1 + 2\lambda c_1 \delta_1^e}{4\lambda c_1} = \frac{3+q}{8} - \frac{c_2}{4c_1}(1-3q)$. Since x^* is in the interior, $x^* > \frac{1+q}{2} \cdot \frac{c_1-c_2}{2c_1-3c_2} > \frac{1+q}{4} = 0.5\delta_1^e$ so that $\delta_1^e - x^* < 0.5\delta_1^e$. Also, $x^* < \frac{3+q}{8}$ so that the corresponding price $p^* < p_{2h}^e - \lambda c_1 + 2\lambda c_1 \cdot \frac{3+q}{8} = \lambda c_1 \cdot \frac{1+3q}{4} + \lambda c_2(1-3q)$. Hence $\frac{p^*}{p_1^e} < \frac{\lambda c_1 \cdot \frac{1+3q}{4} + \lambda c_2(1-3q)}{\lambda c_1 q + \lambda c_2(1-3q)} < \frac{1+3q}{4q} < \frac{1+3 \cdot 0.2}{4 \cdot 0.2} = 2$ and the profit $\pi_1^* = \lambda(\delta_1^e - x^*)p^* < \lambda(0.5\delta_1^e)(2p_1^e) = \pi_1^e$. Therefore SP 1 cannot do better.

(c) $\tilde{p}_1 = p_{2h}^e$ occurs before $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = 1 - q$: At this point, $\tilde{\delta}_{2h} = \tilde{\delta}_1$ so that $q - \frac{c_1-2c_2}{c_1-c_2} \cdot x = \frac{1+q}{2} - x$. Then $x = \frac{1-q}{2} \cdot \frac{c_1-c_2}{c_2}$ and $\tilde{\delta}_{2h} = \tilde{\delta}_1 = \frac{qc_1-(c_1-2c_2)}{2c_2}$. Furthermore, $\tilde{\delta}_{2l} < 1 - q$ implies $\tilde{\delta}_{2h} > q$ or $q > \frac{c_1-2c_2}{c_1-c_2}$. However, that is impossible since $\frac{c_1-2c_2}{c_1-c_2} > \frac{c_2(c_1-2c_2)}{9c_1c_2-2c_1^2-6c_2^2} \forall 2c_2 < c_1 < 3c_2$ and $\frac{c_1-2c_2}{c_1-c_2} > 0.5 \forall c_1 > 3c_2$. Hence this scenario never occurs.

Suppose the purchase fraction of SP 1 increases, i.e., $\tilde{\delta}_1 = \delta_1^e + x$ with $x > 0$. Then $p_{2h}^e - p_{2l}^e > \lambda c_1 \tilde{\delta}_2 > \lambda c_2 \tilde{\delta}_2$ so that $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = \tilde{\delta}_2$. The price $\tilde{p}_1$ depends on whether SP 1 satisfies just patient customers or both impatient and patient customers. For any $0 < x \leq 1 - q - \frac{1+q}{2} = \frac{1-3q}{2}$, $\tilde{p}_1 = p_{2l}^e + \lambda c_2(1 - 2\delta_1^e) - 2\lambda c_2 x = p_1^e - \lambda c_2 q - 2\lambda c_2 x$ and SP 1 satisfies only patient customers. In this case, $\tilde{\pi}_1 = \lambda(\delta_1^e + x)(p_1^e - \lambda c_2 q - 2\lambda c_2 x)$. The derivative $\frac{\partial \tilde{\pi}_1}{\partial x} = \lambda(p_1^e - \lambda c_2 q - 4\lambda c_2 x - 2\lambda c_2 \delta_1^e) = \lambda^2(c_1 - 5c_2)q - 4\lambda^2 c_2 x$. If $c_1 \leq 5c_2$ then $\frac{\partial \tilde{\pi}_1}{\partial x} \Big|_{x=0} \leq 0$ so SP 1 cannot do better; note that if $x > \frac{1-3q}{2}$ then $\tilde{p}_1 = p_1^e - \lambda c_1 q - 2\lambda c_1 x < p_1^e - \lambda c_2 q - 2\lambda c_2 x$.

Suppose $c_1 > 5c_2$. We first note that if $x > \frac{1-3q}{2}$ then $\tilde{\pi}_1 = \lambda(\delta_1^e + x)(p_1^e - \lambda c_1 q - 2\lambda c_1 x)$; the derivative $\frac{\partial \tilde{\pi}_1}{\partial x} \Big|_{x=\frac{1-3q}{2}} = \lambda \left(p_1^e - \lambda c_1 q - 4\lambda c_1 \cdot \frac{1-3q}{2} - 2\lambda c_1 \delta_1^e \right) = \lambda(\lambda c_2(1-3q) - 2\lambda c_1(1-3q) - \lambda c_1(1+q)) < \lambda^2 c_1(-(1-3q) - (1+q)) = -2\lambda^2 c_1(1-q) < 0$. Hence, we only need to consider $x \leq \frac{1-3q}{2}$ to see if SP 1 can do better. There are two possibilities: (i) if $(c_1 - 5c_2)q - 4c_2 \cdot \frac{1-3q}{2} \leq 0$, i.e., $q \leq \frac{2c_2}{c_1+c_2}$ then x^* is obtained by equating the derivative to zero, and $x^* = \frac{(c_1-5c_2)q}{4c_2}$; (ii) if $q > \frac{2c_2}{c_1+c_2}$ then x^* is the *extreme point* $\frac{1-3q}{2}$.

In the first case, the profit $\pi_1^* = \lambda \left(\frac{1+q}{2} + \frac{(c_1-5c_2)q}{4c_2} \right) \left(p_1^e - \lambda c_2 q - \lambda \frac{(c_1-5c_2)q}{2} \right) = \lambda^2 \cdot \frac{(c_1 q + c_2(2-3q))^2}{8c_2}$.

An equilibrium exists only if

$$\begin{aligned} \pi_1^* &\leq \pi_1^e \\ \Leftrightarrow (c_1 q + c_2(2-3q))^2 &\leq 4c_2(1+q)(c_1 q + c_2(1-3q)) \\ \Leftrightarrow (c_1^2 - 10c_1 c_2 + 21c_2^2)q &\leq 4c_2^2 \\ \Leftrightarrow (c_1 - 3c_2)(c_1 - 7c_2)q &\leq 4c_2^2 \end{aligned} \quad (43)$$

Since $q \leq 1/3$ and $c_1 > 5c_2$, (43) holds $\forall c_1 \leq 9c_2$. Also, $\frac{2c_2}{c_1+c_2} \leq (>) \frac{4c_2^2}{(c_1-3c_2)(c_1-7c_2)} \forall 9c_2 < c_1 \leq (>)(6 + \sqrt{17})c_2$. Hence, the overall condition on q can be written as $q \leq \frac{2c_2}{c_1+c_2} \forall 5c_2 < c_1 \leq (6 + \sqrt{17})c_2$ and $q \leq \frac{4c_2^2}{(c_1-3c_2)(c_1-7c_2)} \forall c_1 > (6 + \sqrt{17})c_2$. In the second case, the profit is $\pi_1^* = \lambda(1-q)(p_1^e - \lambda c_2 q - \lambda c_2(1-3q)) = \lambda^2(c_1 - c_2)q(1-q)$. An equilibrium exists only if

$$\begin{aligned} \pi_1^* &\leq \pi_1^e \\ \Leftrightarrow 2(c_1 - c_2)q(1-q) &\leq (1+q)(c_1 q + c_2(1-3q)) \\ \Leftrightarrow (3c_1 - 5c_2)q^2 - c_1 q + c_2 &\geq 0 \end{aligned} \quad (44)$$

Inequality (44) automatically holds if $c_1 \leq 10c_2$; if $c_1 > 10c_2$ either $q \leq \frac{c_1 - \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$ or $q \geq \frac{c_1 + \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$. Hence, the overall condition on q can be written as $q > \frac{2c_2}{c_1+c_2} \forall 5c_2 < c_1 \leq 10c_2$; for any $c_1 > 10c_2$, $q > \frac{2c_2}{c_1+c_2}$ and either $q \leq \frac{c_1 - \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$ or $q \geq \frac{c_1 + \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$. Furthermore, we find that $\frac{2c_2}{c_1+c_2} = (<, >) \frac{c_1 - \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$ when $c_1 = (<, >)(6 + \sqrt{17})c_2$. Overall, the two cases yield the following additional conditions on q for equilibrium to exist: $q \leq \frac{c_1 - \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$ or $q \geq \frac{c_1 + \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$ for any $10c_2 \leq c_1 \leq (6 + \sqrt{17})c_2$, and $q \leq \frac{4c_2^2}{(c_1-3c_2)(c_1-7c_2)}$ or $q \geq \frac{c_1 + \sqrt{(c_1-2c_2)(c_1-10c_2)}}{2(3c_1-5c_2)}$ for any $c_1 > (6 + \sqrt{17})c_2$. Next, we consider deviations by SP 2.

We first note that if (31) in §B.2.5 is satisfied then SP 2 performs worse from *any deviation in which high- and low-priority services are priced so that impatient and patient customers, respectively, are indifferent between these services and the service of SP 1*. Any other deviation by SP 2 will also be counterproductive. There are just two other scenarios we need to consider.

(a) $\tilde{\delta}_{2h} > q$: For that to happen, high-priority service has to satisfy some patient customers. That implies the corresponding profit is the same as the profit obtained by having $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = \tilde{\delta}_2$. That profit is lower due to a *special case* of (31) with $x = -q$ and $y = \tilde{\delta}_2 - \frac{1-3q}{2}$.

(b) *Reduced class*: All customers of SP 2 purchase from a single service class. If SP 2 satisfies only impatient customers then $\tilde{\delta}_2 \leq q$ and the profit is lower due to a special case of (31) with $x = \tilde{\delta}_2 - q \leq 0$ and $y = -\frac{1-3q}{2}$. If SP 2 satisfies some patient customers then the profit is lower due to a special case of (31) with $x = -q$ and $y = \tilde{\delta}_2 - \frac{1-3q}{2}$.

Suppose $c_1 \leq 9c_2$. Then (31) is satisfied since the lowest value of left hand side of (31) for any given $x \leq 0$ is $x(c_1 - c_2)\delta_{2l}^e + \frac{(c_1 - c_2)(c_1 - 9c_2)}{8c_2}x^2 \leq 0$. So SP 1 cannot do better.

Suppose $c_1 > 9c_2$. As earlier, the lowest value of left hand side of (31) for any given $x \leq 0$ is still $x(c_1 - c_2)\delta_{2l}^e + \frac{(c_1 - c_2)(c_1 - 9c_2)}{8c_2}x^2$ which is strictly convex in x . The function is negative for any $-\frac{8c_2}{c_1 - 9c_2}\delta_{2l}^e < x < 0$, equals zero when $x = -\frac{8c_2}{c_1 - 9c_2}\delta_{2l}^e$, and is positive for any $x < -\frac{8c_2}{c_1 - 9c_2}\delta_{2l}^e$. When $x = -\frac{8c_2}{c_1 - 9c_2}\delta_{2l}^e$, $\tilde{\delta}_{2h} = \delta_{2h}^e + x = \frac{q(c_1 + 3c_2) - 4c_2}{c_1 - 9c_2}$, $\tilde{\delta}_{2l} = \delta_{2l}^e + y = \delta_{2l}^e - \frac{c_1 + 3c_2}{4c_2} \cdot x = \frac{3(1-3q)(c_1 - c_2)}{2(c_1 - 9c_2)}$, and $\tilde{\delta}_1 = \delta_1^e + \frac{c_1 - c_2}{4c_2} \cdot x = \frac{q(7c_1 - 15c_2) - (c_1 + 7c_2)}{2(c_1 - 9c_2)}$. If $0 < \tilde{\delta}_{2h} < \tilde{\delta}_1 \Rightarrow q > \max\left(\frac{4c_2}{c_1 + 3c_2}, \frac{c_1 - c_2}{5c_1 - 21c_2}\right) = \frac{c_1 - c_2}{5c_1 - 21c_2} \because c_1 > 9c_2$, and $\tilde{\delta}_{2l} < 1 - q \Rightarrow q > \frac{c_1 + 15c_2}{7c_1 + 9c_2}$ then there is no equilibrium since setting $\tilde{\delta}_{2h} = \tilde{\delta}_{2h} - \epsilon$ and $\tilde{\delta}_{2l} = \tilde{\delta}_{2l} + \frac{c_1 + 3c_2}{4c_2} \cdot \epsilon$ for small $\epsilon > 0$ will *ensure feasibility* while the corresponding profit will be strictly higher than π_2^e . Hence we require $q \leq \max\left(\frac{c_1 - c_2}{5c_1 - 21c_2}, \frac{c_1 + 15c_2}{7c_1 + 9c_2}\right)$, which is equivalent to $q \leq \frac{c_1 + 15c_2}{7c_1 + 9c_2} \forall 9c_2 < c_1 \leq 17c_2$ and $q \leq \frac{c_1 - c_2}{5c_1 - 21c_2} \forall c_1 > 17c_2$ for equilibrium to occur. Then as x increases (with the best $y = -\frac{c_1 + 3c_2}{4c_2} \cdot x$), the profit never exceeds π_2^e until we reach one of the following *limiting* scenarios:

(a) $\tilde{\delta}_{2l} = 1 - q$ occurs: Then $y = 1 - q - \delta_{2l}^e = \frac{1+q}{2}$ and $x = -\frac{4c_2}{c_1 + 3c_2} \cdot y = \frac{-2c_2(1+q)}{c_1 + 3c_2}$. After some algebra, we get $\tilde{\delta}_{2h} = \delta_{2h}^e + x = \frac{q(c_1 + c_2) - 2c_2}{c_1 + 3c_2}$ and $\tilde{\delta}_1 = \frac{2c_2(1+q)}{c_1 + 3c_2}$. For feasibility, we require $0 \leq \tilde{\delta}_{2h} \leq \tilde{\delta}_1 \Rightarrow \frac{2c_2}{c_1 + c_2} \leq q \leq \frac{4c_2}{c_1 - c_2}$. We only need to consider $x \leq \frac{-2c_2(1+q)}{c_1 + 3c_2}$ because otherwise the profit of SP 2 does not exceed π_2^e since (31) applies. For any such x , the best value of y is $\frac{1+q}{2}$ since $-\frac{c_1 + 3c_2}{4c_2} \cdot x \geq \frac{1+q}{2}$ and $3x + 2 \cdot \frac{1+q}{2} \geq -3q + 1 + q = 1 - 2q > 0$. Then the profit is $\tilde{\pi}_2 = \lambda(\delta_{2h}^e + x)(p_1^e + \lambda c_1 q - 2\lambda c_1 \delta_{2h}^e - 2\lambda c_1 x) + \lambda(1 - q)(p_1^e + \lambda c_2(q - \delta_{2h}^e - x - 2\delta_{2h}^e - 2x - (1 - q))) = \lambda^2(q + x)(c_2(1 - 3q) - 2c_1 x) + \lambda^2(1 - q)((c_1 - 4c_2)q - 3c_2 x)$. The derivative is $\frac{\partial \tilde{\pi}_2}{\partial x} = -2\lambda^2(c_1 q + c_2 + 2c_1 x)$. The best profit is obtained at the extreme point with $x^* = \frac{-2c_2(1+q)}{c_1 + 3c_2}$ if and only if $\frac{\partial \tilde{\pi}_2}{\partial x} \Big|_{x = \frac{-2c_2(1+q)}{c_1 + 3c_2}} = -2\lambda^2(c_1 q + c_2 - \frac{4c_1 c_2(1+q)}{c_1 + 3c_2}) = \frac{-2\lambda^2(c_1 q - 3c_2)(c_1 - c_2)}{c_1 + 3c_2} \geq 0$, i.e., $q \leq \frac{3c_2}{c_1}$. In this case, SP 2 cannot do better than the equilibrium profit. Suppose $q > \frac{3c_2}{c_1}$. Then equating the derivative to zero, $x^* = -\frac{q}{2} - \frac{c_2}{2c_1}$. The corresponding purchase fractions and prices are $\delta_{2h}^* = \delta_{2h}^e + x^* = \frac{q}{2} - \frac{c_2}{2c_1}$, $\delta_{2l}^* = 1 - q$, $p_{2h}^* = \lambda c_1 q + \lambda c_2(2 - 3q)$, and $p_{2l}^* = \lambda c_1 q + \lambda c_2\left(\frac{3c_2}{2c_1} - \frac{5q}{2}\right)$. An equilibrium exists if and only if $\pi_2^* = \lambda \delta_{2h}^* p_{2h}^* + \lambda \delta_{2l}^* p_{2l}^* \leq \pi_2^e$. After some algebra, we find this condition is equivalent

to $q \leq \sqrt{\frac{c_2}{c_1}}$. Then $q \leq \frac{c_1+15c_2}{7c_1+9c_2}$ since $\sqrt{\frac{c_2}{c_1}} \leq \frac{c_1+15c_2}{7c_1+9c_2}$. Hence this scenario occurs and equilibrium exists if $\frac{2c_2}{c_1+c_2} \leq q \leq \min\left(\sqrt{\frac{c_2}{c_1}}, \frac{4c_2}{c_1-c_2}\right)$.

(b) $\tilde{\delta}_{2h} = \tilde{\delta}_1$ occurs before $\tilde{\delta}_{2l} = 1-q$: At this point, $\tilde{\delta}_{2h} = \delta_{2h}^e + \frac{c_1-c_2}{4c_2} \cdot x = \tilde{\delta}_1 = \delta_1^e + x$ so that $x = -\frac{2c_2(1-q)}{c_1-5c_2}$. Then $\tilde{\delta}_{2h} = \tilde{\delta}_1 = \frac{c_1q-c_2(3q+2)}{c_1-5c_2}$. Since $\tilde{\delta}_{2l} < 1-q$, we obtain $q > \frac{4c_2}{c_1-c_2}$. As in part (a), we only need to consider $x \leq -\frac{2c_2(1-q)}{c_1-5c_2}$. There are two possibilities. If $\tilde{\delta}_1 = 1 - \delta_{2h}^e - x - \delta_{2l}^e - y \geq \tilde{\delta}_{2h} = \delta_{2h}^e + x$, i.e., $2x+y \leq 0.5(1-q)$ then $\tilde{p}_{2h} = p_1^e + \lambda c_1(\tilde{\delta}_1 - \tilde{\delta}_{2h}) = \lambda c_1 \cdot \frac{1+q}{2} + \lambda c_2(1-3q) - 2\lambda c_1 x - \lambda c_1 y \geq p_1^e$ and $\tilde{p}_{2l} = p_1^e + \lambda c_2(\tilde{\delta}_1 - 2\tilde{\delta}_{2h} - \tilde{\delta}_{2l}) = \lambda c_1 q + \lambda c_2(1-3q) - 3\lambda c_2 x - 2\lambda c_2 y$. If $2x+y \geq 0.5(1-q)$ then the expression for $\tilde{p}_{2h}$ remains the same but $\tilde{p}_{2h} \leq p_1^e$, and $\tilde{p}_{2l} = \tilde{p}_{2h} - \lambda c_2(\tilde{\delta}_{2h} + \tilde{\delta}_{2l}) = \lambda c_1 \cdot \frac{1+q}{2} + \lambda c_2 \cdot \frac{1-5q}{2} - \lambda(2c_1+c_2)x - \lambda(c_1+c_2)y$. In the first case, $\frac{\partial \tilde{\pi}_2}{\partial y} = -\lambda^2((c_1+3c_2)x + 4c_2y)$. For any $x \leq -\frac{2c_2(1-q)}{c_1-5c_2} < 0$, $2x - \frac{c_1+3c_2}{4c_2} \cdot x = -\frac{c_1-5c_2}{4c_2} \cdot x \geq 0.5(1-q)$ so that the best y for any x is $0.5(1-q) - 2x$. In the second case, note that $y > 0$ since $x < 0$ and $2x+y \geq 0.5(1-q) \geq 0$. The derivative $\frac{\partial \tilde{\pi}_2}{\partial y} = \lambda^2\left(-c_1(q+x) + c_1 \frac{1+q}{2} + c_2 \frac{1-5q}{2} - (2c_1+c_2)x - 2(c_1+c_2)y - \frac{(c_1+c_2)(1-3q)}{2}\right) = \lambda^2((c_1-c_2)q - (3c_1+c_2)x - 2(c_1+c_2)y) < 0$ since $(3c_1+c_2)x + 2(c_1+c_2)y > (1.5c_1+0.5c_2)(2x+y) > (c_1-c_2) \cdot 0.5(1-q) > (c_1-c_2)q$. Hence, the *overall best* y for any $x \leq -\frac{2c_2(1-q)}{c_1-5c_2}$ is $0.5(1-q) - 2x$. The corresponding profit is $\tilde{\pi}_2 = \lambda^2(q+x)(c_1q + c_2(1-3q)) + \lambda^2(1-2q-2x)((c_1-2c_2)q + c_2x)$. Then $\frac{\partial \tilde{\pi}_2}{\partial x} = \lambda^2(c_1q + c_2(1-3q) - 2(c_1-2c_2)q - 4c_2x + c_2(1-2q)) = \lambda^2(-c_1q + c_2(2-q) - 4c_2x)$. We note that $x \geq -0.5q$ because if $x < -0.5q$ then $y = 0.5(1-q) - 2x > 0.5(1+q)$ and $\tilde{\delta}_{2l} = \delta_{2l}^e + y > 1-q$ which is not feasible with the current price structure. However, $\frac{\partial \tilde{\pi}_2}{\partial x} \Big|_{x=-0.5q} = \lambda^2(-c_1q + c_2(4-q)) = \lambda^2(4c_2 - (c_1+c_2)q) < 0$ since $q > \frac{4c_2}{c_1-c_2} > \frac{4c_2}{c_1+c_2}$. So the best $x^* = -0.5q$ with the corresponding profit π_2^* . An equilibrium exists only if $\pi_2^* \leq \pi_2^e$. After some algebra, we find the condition is equivalent to $q \leq \sqrt{\frac{c_2}{c_1-c_2}}$. Finally, we consider $x \leq -0.5q$. The best y is still $\frac{1+q}{2}$ since $-\frac{c_1+3c_2}{4c_2} \cdot x \geq \frac{1+q}{2}$ and $3x + 2 \cdot \frac{1+q}{2} \geq -3q + 1 + q = 1-2q > 0$. The profit is $\tilde{\pi}_2 = \lambda^2(q+x)(c_2(1-3q) - 2c_1x) + \lambda^2((c_1-4c_2)q - 3c_2x)$. Then $x^* = -\frac{q}{2} - \frac{c_2}{2c_1}$. The profit is the same as the corresponding profit in part (a) so we require $q \leq \sqrt{\frac{c_2}{c_1}}$. Hence this scenario occurs and equilibrium exists if $\frac{4c_2}{c_1-c_2} < q \leq \sqrt{\frac{c_2}{c_1}}$.

(c) $\tilde{\delta}_{2h} = 0$ occurs before $\tilde{\delta}_{2l} = 1-q$ and $\tilde{\delta}_{2h} = \tilde{\delta}_1$: At this point $x = -q$ so $\tilde{\delta}_{2l} = \delta_{2l}^e - \frac{c_1+3c_2}{4c_2} \cdot x = \frac{c_1q+c_2(2-3q)}{4c_2}$. Since $\tilde{\delta}_{2l} < 1-q$, we require $q < \frac{2c_2}{c_1+c_2}$. In this scenario, there is clearly no deviation which can result in better profit for SP 2.

We so far only consider SP 2 offering two service classes. However, she can have all her customers select just one service class (reduced class). One way to do it will be to set $\tilde{p}_{2h} = \tilde{p}_{2l}$; another way is to set $\tilde{p}_{2h}$ *prohibitively high*. Next, we consider this option; we denote the price and purchase fraction of reduced class by $\tilde{p}_2$ and $\tilde{\delta}_2$. There are three possibilities.

(a) $\tilde{p}_2 = p_1^e$: Then $\tilde{\delta}_2 = 0.5$. Then the profit is $0.5\lambda^2(c_1q + c_2(1 - 3q)) < \pi_2^e = \lambda^2c_1q(1 - q) + 0.5\lambda^2c_2(1 - q)(1 - 3q)$ since $c_1q(2 - 3q) > c_2q(1 - 3q)$.

(b) $\tilde{p}_2 \neq p_1^e$ and the patient customers are indifferent after the deviation. Then $\tilde{\pi}_2 = \lambda^2\tilde{\delta}_2(c_1q + c_2(2 - 3q) - 2c_2\tilde{\delta}_2)$. If $q \leq \frac{2c_2}{c_1+c_2}$ then $\delta_2^* = \frac{c_1q+c_2(2-3q)}{4c_2}$. Requiring the corresponding profit $\pi_2^* \leq \pi_2^e$ implies $q \leq \frac{4c_2}{c_1+3c_2}$ which is true since $\frac{2c_2}{c_1+c_2} < \frac{4c_2}{c_1+3c_2}$. If $q > \frac{2c_2}{c_1+c_2}$ then $\delta_2^* = 1 - q$ and $\pi_2^* = \lambda^2(c_1 - c_2)q(1 - q) < \pi_2^e = \lambda^2c_1q(1 - q) + 0.5\lambda^2c_2(1 - q)(1 - 3q)$.

(c) $\tilde{p}_2 \neq p_1^e$ and impatient customers are indifferent after the deviation. Then $\tilde{\pi}_2 = \lambda^2\tilde{\delta}_2(c_1(1 + q) + c_2(2 - 3q) - 2c_1\tilde{\delta}_2)$. We first note that SP 2 will price so that it satisfies *only impatient customers* because otherwise $\tilde{p}_2 < p_1^e$ and SP 2 satisfies *all patient customers* which is sub-optimal since $\frac{\partial \tilde{\pi}_2}{\partial \tilde{\delta}_2} \leq 0 \forall \tilde{\delta}_2 \geq 1 - q$. Furthermore, $\delta_2^* = q$ since $\frac{\partial \tilde{\pi}_2}{\partial \tilde{\delta}_2} \Big|_{\tilde{\delta}_2=q} \geq 0$. Hence $\pi_2^* = \lambda^2q(c_1(1 - q) + c_2(1 - 3q)) < \pi_2^e = \lambda^2c_1q(1 - q) + 0.5\lambda^2c_2(1 - q)(1 - 3q)$ since $q < 0.5(1 - q) \forall q < \frac{1}{3}$.

We get the results in parts (a) to (d) by compiling the conditions on q from different cases. In this regard we note that $\frac{4c_2^2}{(c_1-3c_2)(c_1-7c_2)} < \sqrt{\frac{c_2}{c_1}} \forall c_1 > 9c_2$ since the condition implies $(c_1 - 9c_2)(c_1^3 - 11c_1^2c_2 + 43c_1c_2^2 + 49c_2^3) > 0$ which is true as $c_1^3 - 11c_1^2c_2 + 43c_1c_2^2 + 49c_2^3 = 274c_2^2 > 0$ when $c_1 = 9c_2$ and its partial derivative with respect to c_1 is $3c_1^2 - 22c_1c_2 + 43c_2^2 = 3(c_1 - \frac{11}{3}c_2)^2 + \frac{8}{3}c_2^2 > 0$.

III We consider $\mathcal{SD}_3^e$ in which $p_{2h}^e > p_{2l}^e = p_1^e$ and $\delta_{2l}^e = 0$. From the analysis in §B.2.5, we find that $q \geq 1/3$, $p_{2h}^e = 2\lambda c_1/3$, $p_{2l}^e = p_1^e = \lambda c_1/3$, $\delta_{2h}^e = 1/3$, $\delta_1^e = 2/3$, $\delta_{2l}^e = 0$, and $3c_2 \leq c_1 \leq 9c_2$ are necessary for equilibrium to exist. Next, we consider large deviations by SPs. First, we consider deviation by SP 1.

Suppose $\tilde{\delta}_1 > 2/3$. Then $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = \tilde{\delta}_2 = 1 - \tilde{\delta}_1$ since $p_{2h}^e - p_{2l}^e = \lambda c_1/3 > \lambda c_1\tilde{\delta}_2$. So $\tilde{p}_1 = \lambda c_1/3 + \lambda c_1(1 - 2\tilde{\delta}_1) < 0$. Hence $\tilde{\pi}_1 < 0 < \pi_1^e$.

Suppose $\tilde{\delta}_1 < 2/3$. Note that since $p_{2h}^e - p_{2l}^e = \lambda c_1/3 \geq \lambda c_2 \geq \lambda c_2\tilde{\delta}_2$, patient customers always get lower dis-utility from low-priority service of SP 2 than from her high-priority service. After some algebra (similar to §B.2.5), we find that $\tilde{\delta}_{2h} = \frac{1}{3} - \frac{c_1-2c_2}{c_1-c_2}(\frac{2}{3} - \tilde{\delta}_1)$, $\tilde{\delta}_{2l} = \frac{2c_1-3c_2}{c_1-c_2}(\frac{2}{3} - \tilde{\delta}_1)$, and $\tilde{p}_1 = \frac{\lambda c_1}{3} + \frac{\lambda c_1 c_2}{c_1 - c_2}(\frac{2}{3} - \tilde{\delta}_1)$ provided $\tilde{\delta}_{2h} \geq 0$ and $\tilde{\delta}_{2l} \leq 1 - q$. Then $\tilde{\pi}_1 = \frac{\lambda^2 c_1}{c_1 - c_2} \tilde{\delta}_1 (\frac{c_1 + c_2}{3} - c_2 \tilde{\delta}_1)$. Since $(c_1 + c_2)/(6c_2) \geq 2/3$ we obtain $\tilde{\pi}_1 < \frac{\lambda^2 c_1}{c_1 - c_2} \frac{2}{3} (\frac{c_1 + c_2}{3} - \frac{2c_2}{3}) = \pi_1^e$. Also, as $\tilde{\delta}_1$ reduces, we find $\tilde{\delta}_{2l}$ reaches $1 - q$ before $\tilde{\delta}_{2h}$ reaches zero. To see this, note that when $\tilde{\delta}_{2l} = 1 - q$, $\tilde{\delta}_{2h} = \frac{1}{3} - \frac{c_1-2c_2}{2c_1-3c_2}(1 - q) \geq \frac{1}{3} - \frac{c_1-2c_2}{2c_1-3c_2} \cdot \frac{2}{3} = \frac{c_2}{3(2c_1-3c_2)} > 0$. Furthermore, we find $\tilde{\delta}_1 = \frac{c_1-c_2}{2c_1-3c_2} \cdot q + \frac{c_1-3c_2}{3(2c_1-3c_2)}$ at this point. For any $\tilde{\delta}_1 \leq \frac{c_1-c_2}{2c_1-3c_2} \cdot q + \frac{c_1-3c_2}{3(2c_1-3c_2)}$, SP 1 satisfies only impatient customers so that $\tilde{p}_1 = p_{2h}^e + \lambda c_1(q - 2\tilde{\delta}_1) = 2\lambda c_1/3 + \lambda c_1q - 2\lambda c_1\tilde{\delta}_1$. Then the profit is $\tilde{p}_1 = \lambda^2 c_1 \tilde{\delta}_1 (2/3 + q - 2\tilde{\delta}_1)$. Next, we show that $q \leq 2/3$ is a *sufficient condition* so that $\tilde{\pi}_1 \leq \pi_1^e$. We first note that $\tilde{p}_1$ is maximized when $\tilde{\delta}_1 = \frac{1}{6} + \frac{q}{4}$. The resulting profit $2\lambda^2 c_1 (\frac{1}{6} + \frac{q}{4})^2 \leq \pi_1^e = \frac{2}{9} \lambda^2 c_1$ implies $q \leq 2/3$. Note that if $q > 2/3$ (and $c_1 \geq 2c_2$) then $\frac{1}{6} + \frac{q}{4} < \frac{c_1-c_2}{2c_1-3c_2} \cdot q + \frac{c_1-3c_2}{3(2c_1-3c_2)}$.

We now consider deviation by SP 2. The proof here is similar to the one in the first case of Theorem 2. As in that case, *we assume the high- and low-priority services are priced to satisfy impatient and patient customers respectively* (similarly, we can again show that violating this assumption does not make SP 2 better; we omit the details for conciseness). Suppose $\tilde{\delta}_{2h} = 1/3 + x$ and $\tilde{\delta}_{2l} = y \geq 0$. Then $\tilde{\pi}_2 = \lambda(\lambda c_1/3 + \lambda c_1(1/3 - 2x - y))(1/3 + x) + \lambda(\lambda c_1/3 - \lambda c_2(3x + 2y))y = \pi_2^e - \lambda^2(2c_1x^2 + (c_1 + 3c_2)xy + 2c_2y^2) \leq \pi_2^e$ since $2c_1x^2 + (c_1 + 3c_2)xy + 2c_2y^2 = 2c_1 \left(x + \frac{c_1 + 3c_2}{4c_1} \cdot y\right)^2 + \frac{(c_1 - c_2)(9c_2 - c_1)}{8c_1} \cdot y^2 \geq 0$.

The conditions $v > c_1 + \frac{6\lambda c_1^2 c_2}{12c_1 c_2 - c_1^2 - 9c_2^2}$ and $v > c_1 + \lambda c_1$ ensure that all customers are satisfied and obtain positive net utility at equilibrium in the first and second cases of Theorem 2 respectively. Note that we did not consider the *feasibility* of the purchase fractions or limited valuation v in the deviations. However, these factors will only further restrict the possible price deviations so that the new profit will still always be less than π_2^e .

Finally, we show uniqueness. Since $\mathcal{SD}_1^e$ requires $c_1 < 3c_2$ and $\mathcal{SD}_3^e$ requires $3c_2 \leq c_1 \leq 9c_2$ (see Table 1), both of them cannot exist together. Likewise, since $\mathcal{SD}_2^e$ requires $q < 1/3$ and $\mathcal{SD}_3^e$ requires $1/3 \leq q \leq 2/3$, both of them cannot exist together. Next, we show that $\mathcal{SD}_1^e$ and $\mathcal{SD}_2^e$ also cannot exist together. From Table 1 and Theorem 2, we find $\mathcal{SD}_1^e$ requires $c_1 < 3c_2$ and $\delta_{2h}^e = \frac{3c_1 c_2 - 3c_2^2}{12c_1 c_2 - c_1^2 - 9c_2^2} < q$ so that $\mathcal{SD}_2^e$ requires $q \leq \frac{c_2(c_1 - 2c_2)}{9c_1 c_2 - 2c_1^2 - 6c_2^2}$. However, for any $c_1 < 3c_2$, that implies $\frac{3c_1 c_2 - 3c_2^2}{12c_1 c_2 - c_1^2 - 9c_2^2} < \frac{c_2(c_1 - 2c_2)}{9c_1 c_2 - 2c_1^2 - 6c_2^2} \Leftrightarrow (3c_1 - 3c_2)(9c_1 c_2 - 2c_1^2 - 6c_2^2) < (c_1 - 2c_2)(12c_1 c_2 - c_1^2 - 9c_2^2) \Leftrightarrow c_1(c_1 - 3c_2)(5c_1 - 4c_2) > 0$ which is not true. ■

Proof of Theorem 3: First, we note that $\lambda c_2 \delta_i^e < p_{ih}^e - p_{il}^e = \lambda c_1 q + \lambda c_2(1 - q) < \lambda c_1 \delta_i^e; i = 1, 2$ if and only if $q < \frac{c_1 - 2c_2}{2(c_1 - c_2)}$. Without any loss of generality, we consider SP 1 to be the deviating firm. Also, we only consider deviations in which the *high-priority service has a strictly higher price than the low-priority service*; otherwise, it is equivalent to the *degenerate case* in which the purchase fraction of high-priority service is zero. Suppose $\tilde{\delta}_{1h} = \delta_{1h}^e + x = 0.5q + x$ and $\tilde{\delta}_{1l} = 0.5(1 - q) + y$. If the high- and low-priority services of SP 1 are priced so that impatient and patient customers, respectively, are indifferent between these services and those provided by SP 2, then $\tilde{p}_{1h} = p_{1h}^e - 2\lambda c_1 x$ and $\tilde{p}_{1l} = p_{1l}^e - 4\lambda c_2 x - 2\lambda c_2 y$. Profit is given by

$$\begin{aligned} \tilde{\pi}_1 &= \lambda(0.5q + x)(p_{1h}^e - 2\lambda c_1 x) + \lambda(0.5(1 - q) + y)(p_{1l}^e - 4\lambda c_2 x - 2\lambda c_2 y) \\ &= \pi_1^e + \lambda x(p_{1h}^e - \lambda c_1 q - 2\lambda c_2(1 - q)) + \lambda y(p_{1l}^e - \lambda c_2(1 - q)) - 2\lambda^2((c_1 - c_2)x^2 + c_2(x + y)^2) \\ &\leq \pi_1^e \quad \forall x, y. \end{aligned} \tag{45}$$

This result holds *provided* $\lambda c_2 \tilde{\delta}_2 \leq p_{2h}^e - p_{2l}^e \leq \lambda c_1 \tilde{\delta}_2$ *and SP 1 prices so that it uses high- and low-priority services to satisfy impatient and patient customers respectively*. There are more

possibilities; we will show that they are either infeasible or the profits of the deviating SP will be even less than the left hand side of (45) for certain x and y .

(i) $p_{2h}^e - p_{2l}^e \leq \lambda c_2 \tilde{\delta}_2$: This is not possible since $p_{2h}^e - p_{2l}^e = \lambda c_1 q + \lambda c_2(1 - q) > \lambda c_2 \tilde{\delta}_2 \ \forall \tilde{\delta}_2 \leq 1$.

(ii) $p_{2h}^e - p_{2l}^e > \lambda c_1 \tilde{\delta}_2$: Then $\tilde{\delta}_{2h} = 0$ and $\tilde{\delta}_{2l} = \tilde{\delta}_2$. Furthermore, since $p_{2h}^e - p_{2l}^e < \lambda c_1 \delta_2^e$, we have $\tilde{\delta}_2 < \delta_2^e = 0.5$ so that $\tilde{\delta}_1 > \delta_1^e = 0.5 > \frac{c_1 - 2c_2}{2(c_1 - c_2)} \geq q$. We need to consider three cases:

(a) Patient customers are indifferent between the high- and low-priority services of SP 1, and low-priority service of SP 2: Then $\tilde{p}_{1h} = p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - \lambda c_2 \tilde{\delta}_{1h}$ and $\tilde{p}_{1l} = p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - 2\lambda c_2 \tilde{\delta}_{1h} - \lambda c_2 \tilde{\delta}_{1l}$. Also, note that if $\tilde{p}_{1h} \neq p_{2l}^e$ then $\tilde{\delta}_2 \geq q$; the reasoning is as follows. If $\tilde{p}_{1h} < p_{2l}^e$ then all impatient customers are satisfied by the low-priority service of SP 2 so that $\tilde{\delta}_2 \geq q$. However, if $\tilde{p}_{1h} > p_{2l}^e$ then all impatient customers are satisfied by the high-priority service of SP 1 but that implies $\tilde{\delta}_2 > \tilde{\delta}_{1h} \geq q$. Hence, if $\tilde{p}_{1h} \neq p_{2l}^e$, the profit of SP 1 is

$$\begin{aligned} \tilde{\pi}_1 &= \lambda(p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - \lambda c_2 \tilde{\delta}_{1h})\tilde{\delta}_{1h} + \lambda(p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - 2\lambda c_2 \tilde{\delta}_{1h} - \lambda c_2 \tilde{\delta}_{1l})\tilde{\delta}_{1l} \\ &= \lambda(p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - \lambda c_2 \tilde{\delta}_{1h})\tilde{\delta}_1 \\ &\leq \lambda(p_{2l}^e + \lambda c_1 \tilde{\delta}_2 - \lambda c_1 q)q + \lambda(p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - \lambda c_2 q - \lambda c_2 \tilde{\delta}_1) \\ &< \lambda(p_{2h}^e - \lambda c_1 q)q + \lambda(p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - \lambda c_2 q - \lambda c_2 \tilde{\delta}_1) \end{aligned} \quad (46)$$

The right hand side of (46) is equal to the left hand side of (45) with $x = 0.5q$ and $y = \tilde{\delta}_1 - q - 0.5(1 - q)$; hence $\tilde{\pi}_1 \leq \pi_1^e$. If $\tilde{p}_{1h} = p_{2l}^e$ then $\tilde{\delta}_{1h} = \tilde{\delta}_2 = 0.5q$, $\tilde{\delta}_{1l} = 1 - q$, and $\tilde{p}_{1l} = \tilde{p}_{1h} - \lambda c_2 \tilde{\delta}_1 < 0$. Furthermore, since $\tilde{\delta}_{1h} = \delta_{1h}^e$ and $\tilde{p}_{1h} < p_{1h}^e$, we have $\tilde{\pi}_1 < \pi_1^e$.

(b) Impatient customers are indifferent between the high-priority service of SP 1 and low-priority service of SP 2 while patient customers are indifferent between the low-priority services of SPs 1 and 2: Then $\tilde{p}_{1h} = p_{2l}^e + \lambda c_1 \tilde{\delta}_2 - \lambda c_1 \tilde{\delta}_{1h}$ and $\tilde{p}_{1l} = p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - 2\lambda c_2 \tilde{\delta}_{1h} - \lambda c_2 \tilde{\delta}_{1l}$. If $\tilde{\delta}_{1h} \leq q$ then the profit of SP 1 is

$$\begin{aligned} \tilde{\pi}_1 &= \lambda(p_{2l}^e + \lambda c_1 \tilde{\delta}_2 - \lambda c_1 \tilde{\delta}_{1h})\tilde{\delta}_{1h} + \lambda(p_{2l}^e + \lambda c_2 \tilde{\delta}_2 - 2\lambda c_2 \tilde{\delta}_{1h} - \lambda c_2 \tilde{\delta}_{1l})\tilde{\delta}_{1l} \\ &< \lambda(p_{2h}^e - \lambda c_1 \tilde{\delta}_{1h})\tilde{\delta}_{1h} + \lambda(p_{2l}^e + \lambda c_2(1 - 3\tilde{\delta}_{1h} - 2\tilde{\delta}_{1l}))\tilde{\delta}_{1l} \\ &\leq \lambda(p_{2h}^e + \lambda c_1(q - 2\tilde{\delta}_{1h}))\tilde{\delta}_{1h} + \lambda(p_{2l}^e + \lambda c_2(2(q - \tilde{\delta}_{1h}) + 1 - q - \tilde{\delta}_{1l} - 2\tilde{\delta}_{1h} - \tilde{\delta}_{1l}))\tilde{\delta}_{1l} \end{aligned} \quad (47)$$

The right hand side of (47) is equal to the left hand side of (45) with $x = \tilde{\delta}_{1h} - 0.5q$ and $y = \tilde{\delta}_{1l} - 0.5(1 - q)$; hence $\tilde{\pi}_1 < \pi_1^e$. If $\tilde{\delta}_{1h} > q$ then we provide *transformations* for the purchase fractions of SP 1 to those with $\tilde{\delta}_{1h} \leq q$ which result in better profits; from (47) we know that even these profits are less than π_1^e . First, we note that $\tilde{p}_{1h} \leq p_{2l}^e$ for $\tilde{\delta}_{1h} > q$. If $\tilde{p}_{1h} = p_{2l}^e$ then set $\tilde{\delta}_{1h} = q$ and $\tilde{\delta}_{1l} = \tilde{\delta}_1 - q$ instead. If $\tilde{p}_{1h} < p_{2l}^e$ then $\tilde{\delta}_{1h} = \tilde{\delta}_1 \geq 1 - q$ and $\tilde{\delta}_{1l} = 0$ since the high-priority

service of SP 1 satisfies all patient customers. In this case, set $\tilde{\delta}_{1h} = \tilde{\delta}_1 - (1 - q)$ and $\tilde{\delta}_{1l} = 1 - q$ instead; *note that these purchase fractions may not be feasible*. In the interest of brevity, we omit the details on how these transformations result in better profits.

(c) Impatient customers are indifferent between the high- and low-priority services of SP 1, and low-priority service of SP 2: Then $\tilde{p}_{1h} = p_{2l}^e + \lambda c_1 \tilde{\delta}_2 - \lambda c_1 \tilde{\delta}_{1h}$ and $\tilde{p}_{1l} = p_{2l}^e + \lambda c_1 \tilde{\delta}_2 - 2\lambda c_1 \tilde{\delta}_{1h} - \lambda c_1 \tilde{\delta}_{1l}$. In this case, since $\tilde{p}_{1h} > \tilde{p}_{1l}$, high-priority service does not satisfy any patient customers so that $\tilde{\delta}_{1h} \leq q$. Then the result in (47) still holds so that $\tilde{\pi}_1 < \pi_1^e$.

(iii) $\lambda c_2 \tilde{\delta}_2 < p_{2h}^e - p_{2l}^e \leq \lambda c_1 \tilde{\delta}_2$: If the price structure is such that impatient customers are indifferent between high-priority services of SPs 1 and 2 while patient ones are indifferent between low-priority services of SPs 1 and 2 then (45) applies and $\tilde{\pi}_1 \leq \pi_1^e$. However, there are two other possibilities:

(a) Patient customers are indifferent between high- and low-priority services of SP 1, and low-priority service of SP 2: There can be three scenarios based on how impatient customers get satisfied. First, if all of them prefer high-priority service of SP 2, then $\tilde{\delta}_{2h} = q$. The profit of SP 1 is equal to the left hand side profit in (45) with $x = -0.5q$ and $y = \tilde{\delta}_1 - 0.5(1 - q)$ so that $\tilde{\pi}_1 \leq \pi_1^e$. Second, if all of them prefer high-priority service of SP 1 instead, then $\tilde{p}_{1h} + \lambda c_1 \tilde{\delta}_{1h} \leq p_{2h}^e$ and $\tilde{\delta}_{1h} \geq q$. Furthermore, the profit of SP 1 does not exceed the left hand side of (45) with $x = 0.5q$ and $y = \tilde{\delta}_1 - q - 0.5(1 - q)$ so that $\tilde{\pi}_1 \leq \pi_1^e$. Finally, if the high-priority services of both SPs 1 and 2 satisfy some impatient customers then the profit of SP 1 does not exceed the left hand side of (45) with $x = 0.5q - \tilde{\delta}_{2h}$ and $y = 0.5(1 - q) - \tilde{\delta}_{2l}$ so that $\tilde{\pi}_1 \leq \pi_1^e$ still applies.

(b) Impatient customers are indifferent between high- and low-priority services of SP 1, and high-priority service of SP 2: As in (a), there can again be three different scenarios but they are now based on how patient customers get satisfied. First, if the low-priority service of SP 2 satisfies all the patient customers then the profit of SP 1 is equal to the left hand side of (45) with $x = \tilde{\delta}_1 - 0.5q$ and $y = -0.5(1 - q)$ so that $\tilde{\pi}_1 \leq \pi_1^e$. Second, we note that it is impossible for the low-priority service of SP 1 to satisfy all the patient customers because that would imply $\tilde{\delta}_1 \geq 1 - q$ so that $\tilde{\delta}_2 \leq q$ and $p_{2h} - p_{2l} = \lambda c_1 q + \lambda c_2(1 - q) > \lambda c_1 q$.

Finally, we consider the scenario when the low-priority services of both SPs satisfy some patient customers. Note that $\tilde{p}_{1l} \geq p_{2l}^e$; otherwise $p_{2h}^e - p_{2l}^e > \lambda c_1 \tilde{\delta}_2$, a contradiction. If $\tilde{\delta}_{1l} + \tilde{\delta}_{2l} = 1 - q$ then the profit of SP 1 is equal to the left hand side of (45) with $x = \tilde{\delta}_{1h} - 0.5q$ and $y = \tilde{\delta}_{1l} - 0.5(1 - q)$ and $\tilde{\pi}_1 \leq \pi_1^e$. So suppose $\tilde{\delta}_{1l} + \tilde{\delta}_{2l} > 1 - q$. Also, suppose $\tilde{p}_{1l} > p_{2l}^e$; then the low-priority service of SP 2 does not satisfy any impatient customers so that $\tilde{\delta}_{2l} \leq 1 - q$. We show that the purchase fractions can then be *transformed* with $\tilde{\delta}_{1l} + \tilde{\delta}_{2l} = 1 - q$ for a better profit of SP 1 but even this profit

is still less than π_1^e . Set $\tilde{\delta}_{1h} = q - \tilde{\delta}_{2h}$ and $\tilde{\delta}_{1l} = 1 - q - \tilde{\delta}_{2l}$; these purchase fractions are obtained by having prices $\tilde{p}_{1h} = p_{2h}^e + \lambda c_1(q - 2\tilde{\delta}_{2h})$ and $\tilde{p}_{1l} = p_{2l}^e + \lambda c_2(2\delta_{2h} + \delta_{2l} - 2(q - \tilde{\delta}_{2h}) - (1 - q - \tilde{\delta}_{2l}))$. Then the new profit of SP 1 is higher (we omit the details for conciseness) but it is equal to the left hand side of (45) with $x = 0.5q - \tilde{\delta}_{2h}$ and $y = 0.5(1 - q) - \tilde{\delta}_{2l}$ which is less than π_1^e . Suppose $\tilde{p}_{1l} = p_{2l}^e$. If $\tilde{\delta}_{2l} \leq 1 - q$ then the analysis before still applies. If $\tilde{\delta}_{2l} > 1 - q$ then $\tilde{\delta}_1 + \tilde{\delta}_{2h} < q$; set $\tilde{\delta}_{1h} = \tilde{\delta}_1$ and $\tilde{\delta}_{1l} = 0$. They are obtained by having $\tilde{p}_{1h} = \tilde{p}_{1l} = p_{2h}^e + \lambda c_1(q - 2\tilde{\delta}_1)$. We can verify that the new profit of SP 1 is higher but it is equal to the left hand side of (45) with $x = \tilde{\delta}_1 - 0.5q$ and $y = -0.5(1 - q)$ which is less than π_1^e .

The condition $v > c_1 + 1.5\lambda c_1 q + 2\lambda c_2(1 - q)$ ensures all customers are satisfied and obtain positive net utility at equilibrium. As in the proof of Theorem 2, we note that the proof above does not consider the feasibility of the purchase fractions or limited valuation v in the deviations; however, these factors will only further reduce the profits from deviations. ■

Proof of Theorem 4: If $q \geq 0.5$, Theorem 1 implies $\mathcal{SS}_1^e$ exists in which $\pi_1^e = \pi_2^e = 0.5\lambda^2 c_1$. Since $q > 0.5 > \frac{c_1 - 2c_2}{2(c_1 - c_2)}$, DD^e do not exist from Theorem 3. We find from Theorem 2 that $\mathcal{SD}_1^e$, $\mathcal{SD}_2^e$, and $\mathcal{SD}_3^e$ are mutually exclusive. Also, $\mathcal{SD}_2^e$ does not exist and either (i) $\mathcal{SD}_1^e$ exists if $c_1/c_2 < 3$ or (ii) $\mathcal{SD}_3^e$ exists if $3 \leq c_1/c_2 \leq 9$. We show that the profit of SP 2 decreases in both cases. Clearly, in the second case, the profit is $2/9\lambda^2 c_1 < 0.5\lambda^2 c_1$. In the first case, we show the profits decrease as follows. First, we find that $\delta_2^e = \frac{6c_1c_2 - c_1^2 - 3c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2} < 1/2$ since $c_1^2 > 3c_2^2$ as $c_1 > 2c_2$. Furthermore, $p_{2h}^e = 2\lambda c_1\delta_{2h}^e + 3\lambda c_2\delta_{2l}^e = \frac{3\lambda c_1c_2(c_1 + c_2)}{12c_1c_2 - c_1^2 - 9c_2^2} < \lambda c_1$ since $c_1^2 - 9c_1c_2 + 12c_2^2 < 0 \forall 2 < c_1/c_2 < 3$. Hence the profit of SP 2 is strictly less than $0.5\lambda \cdot \lambda c_1 = 0.5\lambda^2 c_1$.

Suppose $q < 0.5$, and the conditions on v and equilibrium profit in Theorems 1-3 are all satisfied. Then $\mathcal{SS}_1^e$ does not exist. If $\frac{c_1}{c_2} < 3$ and $q > \frac{3c_1c_2 - 3c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2}$ then $\mathcal{SD}_1^e$ is an equilibrium from Theorem 2 (note that $\delta_{2l}^e < 1 - q$ automatically holds since $q < 0.5 \leq 1 - \delta_{2l}^e$ as $(c_1 - 3c_2)^2 \geq 0$). Also, it is the only possible equilibrium. $\mathcal{SS}_2^e$ does not exist since $q > \frac{3c_1c_2 - 3c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2} > \frac{c_2}{2c_1} \forall 2 < \frac{c_1}{c_2} < 3$. Similarly, DD^e does not exist since $q > \frac{3c_1c_2 - 3c_2^2}{12c_1c_2 - c_1^2 - 9c_2^2} > \frac{c_1 - 2c_2}{2(c_1 - c_2)} \forall \frac{c_1}{c_2} < 3$.

If $3 \leq \frac{c_1}{c_2} \leq 9$ and $q \geq \frac{1}{3}$ then $\mathcal{SD}_3^e$ is an equilibrium from Theorem 2 in which $\pi_2^e = 2/9\lambda^2 c_1$. Since $q \geq \frac{1}{3} > \frac{c_2}{2c_1} \forall \frac{c_1}{c_2} \geq 3$, $\mathcal{SS}_2^e$ does not exist. So the only other possible equilibrium is $\mathcal{DD}^e$ in which $p_{1h}^e = \lambda c_1 q + 2\lambda c_2(1 - q)$, $p_{1l}^e = \lambda c_2(1 - q)$, $\delta_{1h}^e = 0.5q$, and $\delta_{1l}^e = 0.5(1 - q)$. Also, for $\mathcal{DD}^e$ to exist, it is necessary that $\frac{c_1 - 2c_2}{2(c_1 - c_2)} > \frac{1}{3}$ or $\frac{c_1}{c_2} > 4$. We show that the profit of SP 1 is strictly less in $\mathcal{DD}^e$. After some algebra, this profit is $0.5\lambda^2(c_1q^2 + c_2(1 - q^2)) < \lambda^2 \left(\frac{c_1 + 3c_2}{8} \right) (\because q < 0.5) < \lambda^2 \left(\frac{c_1 + 3(c_1/4)}{8} \right) = \frac{7\lambda^2 c_1}{32} < \frac{2\lambda^2 c_1}{9}$.

Suppose q , c_1 , and c_2 satisfy the conditions in Table 1. Then $\mathcal{SD}_2^e$ exists with $\pi_1^e = 0.5\lambda^2(1 + q)(c_1q + c_2(1 - 3q))$ and $\pi_2^e = \lambda^2(c_1q(1 - q) + 0.5c_2(1 - q)(1 - 3q))$. There can only be two other

equilibria: (i) $\mathcal{SS}_2^e$ and (ii) $\mathcal{DD}^e$. $\mathcal{SS}_2^e$ is not an equilibrium if $\pi_2^e > 0.5\lambda^2 c_2$, i.e., $q < \frac{2c_1-4c_2}{2c_1-3c_2}$ which is true as follows. If $\frac{c_1}{c_2} \geq \frac{1}{3}$ then $q < \frac{1}{3} < \frac{2}{3} \leq \frac{2c_1-4c_2}{2c_1-3c_2}$. If $\frac{c_1}{c_2} < \frac{1}{3}$ then $q \leq \frac{c_2(c_1-2c_2)}{9c_1c_2-2c_1^2-6c_2^2} < \frac{2c_1-4c_2}{2c_1-3c_2} \forall c_1 > 2c_2$. $\mathcal{DD}^e$ exists only if $q < \frac{c_1-c_2}{2(c_1-c_2)}$. Then $\pi_1^e > 0.5\lambda^2 (c_1q^2 + c_2(1-q^2))$ since $q < \frac{c_1-c_2}{2(c_1-c_2)} < \frac{c_1-2c_2}{2c_2} \forall c_1 > 2c_2$. Hence SP 1 would rather provide single service instead of differentiated service, and $\mathcal{DD}^e$ is not an equilibrium.

If $\frac{c_1-2c_2}{2(c_1-c_2)} \leq q \leq \frac{c_2}{2c_1}$ then we find $\frac{c_1}{c_2} < 3$ and Theorem 3 implies $\mathcal{DD}^e$ does not exist. Furthermore, since $\frac{c_2}{2c_1} < \frac{3c_1c_2-3c_2^2}{12c_1c_2-c_1^2-9c_2^2}$, $\mathcal{SD}_1^e$ does not exist. If $q > \frac{c_2(c_1-2c_2)}{9c_1c_2-2c_1^2-6c_2^2}$, Theorem 2 part(ii) implies $\mathcal{SD}_2^e$ does not exist. Hence, only $\mathcal{SS}_2^e$ exists.

Finally, suppose none of the conditions we considered so far are satisfied and $q < \frac{c_1-2c_2}{2(c_1-c_2)}$. Theorem 3 implies $\mathcal{DD}^e$ exists. Also, we can verify $\mathcal{SD}_1^e$, $\mathcal{SD}_2^e$, and $\mathcal{SD}_3^e$ do not exist. So the only other possible equilibrium is $\mathcal{SS}_2^e$. However, the profit here is $0.5\lambda^2 c_2 < 0.5\lambda^2 (c_1q^2 + c_2(1-q^2))$ which is the profit in $\mathcal{DD}^e$. ■

E. Monopoly SP

Suppose an SP offers a single service and satisfies both customer types. We will show that her profit can be improved. Let the purchase fraction be δ ; then $\delta > 1 - q$. Also, let x be the disutility of impatient customers. Then the price $p = x - c_1 - \lambda c_1 \delta$ so that the profit is $\lambda \delta (x - \lambda c_1 \delta)$. Consider prices $p_h = x - c_1 - \lambda c_1 (\delta - (1 - q)) = p + \lambda c_1 (1 - q)$ and $p_l = x - c_1 - \lambda c_1 \delta + \lambda c_2 \delta - \lambda c_2 (2(\delta - (1 - q)) + 1 - q) = p - \lambda c_2 (\delta - (1 - q))$. Then $\delta_h = \delta - (1 - q)$ and $\delta_l = 1 - q$ so that $\lambda p_h \delta_h + \lambda p_l \delta_l = \lambda p \delta + \lambda^2 (c_1 - c_2) (1 - q) (\delta - (1 - q)) > \lambda p \delta$. If $p_l \geq 0$ then these prices strictly increases the profit. If $p_l < 0$ then set instead $p_l = p_h = p + \lambda c_1 (1 - q)$; then all customers will just choose the high-priority service so that $\lambda p_h \delta_h \geq \lambda p_h (\delta - (1 - q)) > \lambda p \delta$.

F. Two-Stage Game

The price(s) and service modes are jointly determined by the SPs in the first stage followed by the customer sub-game in the second stage. The equilibrium has to be one of the five types of equilibrium we characterize in the main article (see Table 2); otherwise, at least one of the SPs can deviate by charging different price(s) and perform better. Next, we show $\mathcal{DD}^e$ can be the only equilibrium because the SP providing single service can do better by differentiating her service in the other four types of equilibrium.

Suppose both SPs provide single service. If $q \leq 0.5$ set $\tilde{\delta}_{1h} = \epsilon$ and $\tilde{\delta}_{1l} = 0.5 - \epsilon$ by charging prices $\tilde{p}_{1h} = p^e + \lambda c_1 (0.5 - \epsilon)$ and $\tilde{p}_{1l} = p^e - \lambda c_2 \epsilon$. If $q > 0.5$ set $\tilde{\delta}_{1h} = 0.5 - \epsilon$ and $\tilde{\delta}_{1l} = \epsilon$ by

charging prices $\tilde{p}_{1h} = p^e + \lambda c_1 \epsilon$ and $\tilde{p}_{1l} = p^e - \lambda c_2(0.5 - \epsilon)$. In both cases, we can show that the new profit of SP 1 is strictly higher for low $\epsilon > 0$; we omit the details for conciseness.

Suppose SP 1 and SP 2 provide single and differentiated services respectively. First, we consider the regular equilibrium $\mathcal{SD}_1^e$. Set $\tilde{\delta}_{1h} = q - \delta_{2h}^e$, $\tilde{\delta}_{2h} = \delta_{2h}^e$, $\tilde{\delta}_{1l} = 1 - q - \delta_{2l}^e$, and $\tilde{\delta}_{2l} = \delta_{2l}^e$ by charging prices $\tilde{p}_{1h} = p_1^e + \lambda c_1 \tilde{\delta}_{1l}$ and $\tilde{p}_{1l} = p_1^e - \lambda c_2 \tilde{\delta}_{1h}$. Note that $\tilde{\delta}_{1h}, \tilde{\delta}_{1l} > 0$ and $\tilde{\delta}_{1h} + \tilde{\delta}_{1l} = \delta_1^e$. Again, we find SP 1 benefits from differentiating her service. Next, we consider extreme equilibrium $\mathcal{SD}_2^e$. Set $\tilde{\delta}_{1h} = \epsilon$, $\tilde{\delta}_{2h} = \delta_{2h}^e - \epsilon = q - \epsilon$, $\tilde{\delta}_{1l} = \delta_{1l}^e - 2\epsilon$, and $\tilde{\delta}_{2l} = \delta_{2l}^e + 2\epsilon$. Note that $\tilde{\delta}_{ik} > 0; i = 1, 2, \tilde{\delta}_{1h} + \tilde{\delta}_{2h} = q$ and $\tilde{\delta}_{1l} + \tilde{\delta}_{2l} = q$. The purchase fractions are obtained by charging prices $\tilde{p}_{1h} = p_{2h}^e + \lambda c_1(q - 2\epsilon)$ and $\tilde{p}_{1l} = p_{2l}^e + \lambda c_1(2\tilde{\delta}_{2h} + \tilde{\delta}_{2l} - 2\tilde{\delta}_{1h} - \tilde{\delta}_{1l}) = p_{2l}^e + \lambda c_1(2\delta_{2h}^e + \delta_{2l}^e - \delta_1^e) = p_{2l}^e = p_1^e$. Note that the low-priority service from either SP satisfies only patient customers since $p_{2h}^e - p_{2l}^e < \lambda c_1(\delta_2^e + \epsilon) = \lambda c_1 \tilde{\delta}_2$. The new profit of SP 1 is $\lambda(p_{2h}^e + \lambda c_1(q - 2\epsilon))\epsilon + \lambda p_1^e(\delta_1^e - 2\epsilon) = \pi_1^e + \lambda(p_{2h}^e + \lambda c_1 q - 2p_1^e)\epsilon - 2\lambda^2 c_1 \epsilon^2 > \pi_1^e$ for small $\epsilon > 0$ if $p_{2h}^e + \lambda c_1 q > 2p_1^e \Leftrightarrow 0.5\lambda c_1(1 + 3q) + \lambda c_2(1 - 3q) > 2\lambda c_1 q + 2\lambda c_2(1 - 3q) \Leftrightarrow c_1(1 - q) > 2c_2(1 - 3q)$ which is true since $c_1 > 2c_2$.

Consider reduced class equilibrium $\mathcal{SD}_3^e$. If $q = \frac{1}{3}$ then $\delta_{2h}^e = q$ and the deviation above for $\mathcal{SD}_2^e$ still applies and makes SP 1 better since $p_{2h}^e + \lambda c_1 q = \lambda c_1 > 2p_1^e = \frac{2\lambda c_1}{3}$. So suppose $q > \frac{1}{3}$. Then set $\tilde{\delta}_{1h} = q - \frac{1}{3}$, $\tilde{\delta}_{2h} = \delta_{2h}^e = \frac{1}{3}$, $\tilde{\delta}_{1l} = 1 - q - \epsilon$, and $\tilde{\delta}_{2l} = \epsilon$ obtained by charging prices $\tilde{p}_{1h} = p_{2h}^e + \lambda c_1(\frac{2}{3} - q) = p_1^e + \lambda c_1(1 - q)$ and $\tilde{p}_{1l} = p_{2l}^e + \lambda c_2(\frac{2}{3} + \epsilon - (2(q - \frac{1}{3}) + 1 - q - \epsilon)) = p_1^e + \lambda c_2(\frac{1}{3} - q + 2\epsilon)$. Note that as in $\mathcal{SD}_2^e$, the low-priority service from either SP still satisfies only patient customers since $p_{2h}^e - p_{2l}^e < \lambda c_1 \tilde{\delta}_2$ and $\tilde{p}_{1l} < p_{2l}^e$ for small $\epsilon > 0$. The new profit of SP 1 is

$$\begin{aligned}
& \lambda(p_1^e + \lambda c_1(1 - q)) \left(q - \frac{1}{3} \right) + \lambda \left(p_1^e + \lambda c_2 \left(\frac{1}{3} - q + 2\epsilon \right) \right) (1 - q - \epsilon) \\
&= \pi_1^e + \lambda^2(c_1 - c_2) \cdot \frac{(1 - q)(3q - 1)}{3} - \lambda \epsilon \left(p_1^e + \lambda c_2 \left(\frac{1}{3} - q \right) - 2\lambda c_2(1 - q) \right) - 2\lambda^2 c_2 \epsilon^2 \\
&= \pi_1^e + \lambda^2(c_1 - c_2) \cdot \frac{(1 - q)(3q - 1)}{3} - \lambda^2 \epsilon \cdot \frac{c_1 - c_2(5 - 3q)}{3} - 2\lambda^2 c_2 \epsilon^2 \\
&> \pi_1^e
\end{aligned}$$

for small $\epsilon > 0$. Finally, we note that $\mathcal{DD}^e$ is still an equilibrium (if it exists) since providing single service will be counterproductive for any SP (see proof of Theorem 3 for details).