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DYNAMICS OF OPINION FORMATION ON COMPLEX SOCIAL NETWORKS

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School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University

in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

2020

Statement Of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

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Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

22/08/2020

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Xiao Gaoxi

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Authorship Attribution Statement

This thesis contains material from 3 paper published/accepted in the following peer-reviewed journal and conference in which I am listed as an author. Specifically, Chapter 3 is published as papers as follows:

1. Vu Xuan Nguyen and Gaoxi Xiao. Global consensus making on multiplex scale-free networks. In *2017 6th International Conference on Computer Science and Network Technology (ICCSNT)*, pages 347–351. IEEE, 2017.
2. Vu Xuan Nguyen, Gaoxi Xiao, Xin-Jian Xu, Guoqi Li, and Zhen Wang. Opinion formation on multiplex scale-free networks. *EPL (Europhysics Letters)*, 121(2):26002, 2018.

The contributions of the co-authors are as follows:

1. A/Prof. Xiao and I conceived initial concepts and experiments.
2. I conducted the experiments and analyzed the results.
3. A/Prof. Xiao, Prof. Xu, Prof. Li, and Prof. Wang supervised the work and commented on the results.
4. I and A/Prof. Xiao wrote the first draft of the manuscripts.
5. A/Prof. Xiao, Prof. Xu, Prof. Li, and Prof. Wang contributed to the revision of the manuscript.

Chapter 4 is published as the paper:

1. Vu Xuan Nguyen, Gaoxi Xiao, Xin-Jian Xu, Qingchu Wu, and Cheng-Yi Xia. Dynamics of opinion formation under majority rules on complex social networks. *Scientific Reports*, 10(1):1-9, 2020.

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2. I conducted the experiments and analyzed the results.
3. A/Prof. Xiao, Pro. Xu, Prof. Wu, and Pro. Xia supervised the work and commented on the results.
4. I and A/Prof. Xiao wrote the first draft of the manuscripts.
5. A/Pro. Xiao, Prof. Xu, Pro. Wu, and Prof. Xia contributed to the revision of the manuscript.

22 August 2020
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Summary

Complexity science is an emerging interdisciplinary research field that has been grabbing a great deal of attention over the past few decades. The field mainly studies collective dynamics of complex systems and networks emerging from interactions among their interconnected components. Such complex systems and networks are known to be ubiquitous in a huge variety of areas ranging from biology, neurology, health, and medicine to sociology, economics, and communication networks, etc.

The rise of modern network sciences and the advances in studies on dynamical processes on complex networks and systems have provided the insightful understanding and novel approaches to computational sociology. Such studies pose interesting questions whose solutions may make impacts on economics, sociology, and politics, etc. By basing opinion dynamics on mathematical models, the existing studies, to a significant extent, have exhibited observations resembling real-life phenomena and revealed key factors playing important roles in the construction of social structures and the formation of public opinions.

In this thesis, four problems on dynamics of opinion formation have been investigated which, to the best of our knowledge, have been largely missed in existing studies. The four problems include:

1. In the modern world, people may be active on multiple social networks such as Reddit, Twitter or Facebook. As a result, single social networks have evolved to multiplex networks. Understanding the dynamics of opinion formation on multiplex networks is of both research interest and application values. We considered the rules proposed in the bounded confidence opinion dynamics models and examine the effects of the interplay between layers of a multiplex network on the evolution of public opinions governed by such rules. The

results show that the interactions of individuals on multiple different layers of a multiplex network may diminish or enhance the opinion diversity depending on the tolerance threshold of each layer.

2. Majority rule is one of the most popular tendencies in human behaviors in choosing either of two alternatives. Following the classical majority rule, one tends to adopt the opinion shared by the majority of the connections s/he has. Under such a regime, it is shown in most of the existing studies that a complete consensus is achieved across the population in relatively dense networks. However, we figure out that in sparse networks, the dynamics may be very different where multiple steady states of co-existence could emerge. Moreover, we examine a modified majority rule where different social influences of different individuals are taken into account. It is revealed that under such a rule, once again multi-steady state of coexistence could emerge in sparse networks, yet due to very different reasons from those for the case under the classic majority rule.
3. Classical bounded-confidence models mostly deal with pairwise interactions where the two involving agents have equal influences on each other. In real life, human contacts may be more complex and sophisticated. We examine opinion dynamics under the effects of bias in social interactions. Theoretical and simulation results show that the unbalance in interpersonal contacts may lead to macroscopic polarization and/or the emergence of extremism in the entire opinion system. Influences of a few other factors on the emergence and prevalence of extremism are also discussed.
4. Most of existing opinion dynamics models aim to reveal influences of a certain key factor in opinion formation. Dynamics of opinion formation under the

interplay of multiple social rules and principles are largely unknown. We examine a simple model where individuals interact with each other under the influences of the two rules of interpersonal consensus making and majority orientation. We show that some interesting and complex system dynamics shall then emerge.

Our contributions as listed above shall help provide deeper insights into opinion formation and evolution in human societies. A few directions for future research are also discussed.

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Chapter 1

Introduction

1.1 Motivations

Many real-life complex systems and networks such as social networks, the Internet, computer networks, biological food chains, intra-cellular proteins, etc., can be modeled as networks with non-linearly interacting units defined as nodes that are connected via links representing interactions between them [1–3]. The studies on these networks form an inter-disciplinary scientific field called *complex networks*, aiming at revealing emergent properties of the entire systems which are not held by any of the individual nodes.

Due to the pervasion of complex networks into various areas ranging from statistical physics, computer science to neurobiology, sociology, biology, and so on [1, 4], complex networks are emerging as a hot area and rapidly growing with the help of the discovered knowledge of network theory and the fast growth in computational technologies. Being firstly presented by Euler in the 1780s in terms of the solution to the Seven Bridges of Königsberg problem [5], network theory has shown its potential for solving different problems in different fields. An idea of random networks allowing the treatment of a family of networks rather than a specific one was proposed by Rapoport in 1957 [6]. In the 1960s, Paul Erdős and Alfréd Rényi (ER) conducted

an extensive and systematic research on the same concept of random networks by combining graph theory and probability theory [7].

The remarkable revolution of network science started at the end of the 20th century with the discovery that for many real-life networks, such as the World Wide Web [8] and the Internet [9], the nodal degree distributions follow a power law. Such networks are called scale-free networks and have been widely studied over the past few decades. The scientists' attention has also been extended to social and biological networks. Unlike the classic network models as that in the Seven Bridges of Königsberg problem, social networks and biological networks are always changing over time by evolutionary and environmental processes in connection with changes in mindset, habitat, and technology, etc. The emergence of the studies on dynamic/adaptive complex networks has been enriching the research in the area [10–12].

Since 2010, the multiplex properties observed in many real-life networks have opened a new frontier of complex networks, known as multiplex networks, with a lot of questions remaining open [13]. The concept of multiplex networks comes from the fact that many real-life networks can be modeled as a set of correlated networks called layers, where each of the layers is composed of a set of elementary units and a set of links representing a different type of interactions. For an interesting and important special class of multiplex networks, nodes on different layers may refer to the same entities in different real-life systems. One example of multiplex networks is a population of people who are connected to others via different types of social ties corresponding to different social networks such as education, business, and religion. The study on such networks may reveal interesting macroscopic properties that are not captured in single, non-interacting networks.

Extensive researches have been carried out on both epidemic-like [14] and decision-based spreading processes [15–19] in both single-layer (or monoplex [20]) and multiplex networks over the last decade [21]. While most existing studies heavily rely on mathematical models to explore and conjecture system dynamics on the macroscopic level, limited attempts have also been made to validate opinion models using actual data gathered from online social networks platforms. For example, [22] analyzed sentiments in posts downloaded from Twitter to observe the changing of public opinions towards some electronic products over a period of time and then developed a data-specific model. While the exploratory models-based approach has attracted a great deal of attention, empirical studies, though interesting, have witnessed less devotion mainly because of the limited data availability. Despite of the efforts and progress that have been made, many problems remain open even for relatively simple single-layer networks. As to their extensions to multiplex networks, in many cases, people may not even have realized how complex the system dynamics may become, let alone analyzing such system dynamics.

The studies in this thesis shall focus on a specific area of complex networks which may be conveniently named as “decision-based spreading process”. It studies how individuals in a social system may form up their opinions, and more importantly, change their opinions due to various influences they may be exposed to through their social connections. Our studies shall be mainly composed of two parts, on extending certain models from single-layer networks to multilayer networks and on examining the effects of making some seemingly minor or even trivial (yet reasonable) extensions to a few well-known models on single-layer networks, respectively. We shall show that (i) some well-studied models, once being applied to multiplex networks with some relatively simple inter-layer dynamics being taken into account, may reveal some surprisingly complex system dynamics which does not exist in single-layer

networks; and (ii) even slight extensions and/or changes to some well-established opinion formation rules may lead to significant changes to system dynamic behaviors which may help explain certain observations we may easily make on opinion formation in social systems.

1.2 Main Contributions of the Thesis

As aforementioned, we study on opinion formation and opinion evolution dynamics on both single-layer and multilayer networks. The major contributions of the thesis are summarized as follows.

In Chapter 3, we shall carry out studies on dynamics of opinion formation on two-layer multiplex networks. On each layer, the agents hold a type of opinion to a subject associated with the respective layer. The interlayer interactions mimic the compromise phenomenon in real-life social networks where one's viewpoint on a subject is influenced by his/her opinion on other subjects, resulting in their decisions on compromising or neglecting others' opinions. On each layer, we adopt the very well-known Deffuant model (to be surveyed in Chapter 2) where two directly connected individuals may compromise to make a consensus if the difference between their opinions is not larger than a certain preset tolerance threshold; or make no changes otherwise. We extend the model to duplex networks where two individuals make a consensus only if the total differences in their opinions on both the layers is not larger than the sum of the tolerance thresholds on the two layers. It is found that in a duplex network composed of two identical layers, an increase in tolerance threshold in a layer declines the opinion diversity on the other layer and the two critical tolerance thresholds follow a *one-sum* rule; that is, each of the layers reaches a global consensus if the sum of the two tolerance thresholds on the two layers is

approximately equal to 1, a double of critical bound on isolated networks. It is also found that if one of the layers has a relatively small tolerance threshold, the presence of the multiplexity in the network always increases the diversity of opinion clusters on the other layer compared to that in isolated single-layer networks. Otherwise, the effects of decreasing or increasing the opinion diversity caused by the multiplexity may become nonmonotonic in respect of the change in both the tolerance thresholds. Further investigations reveal that in duplex networks with two non-identical layers, the one-sum rule no longer holds. The presence of the coupling is either a diminishing or facilitating factor for the opinion diversity on the other layer depending on the strength of the layer-layer coupling and the magnitude of the tolerance ranges.

In Chapter 4, we study opinion dynamics on complex social networks where each individual holding a binary opinion on a certain subject may change her/his view to match the opinion of the majority. Two rules of interactions between individuals, termed as *classic majority rule* and *influence majority rule*, respectively, are imposed on synthetic social networks. The former one, as its name suggests, is a well-known existing model that has been extensively studied. It allows each individual to change her/his opinion following the majority of her/his immediate neighbors. The latter rule is a slight extension of the former rule where we allow each individual to count the influences of all her/his neighbors and follow the opinion with stronger overall influences. The influence of each individual, in this study, is conveniently counted as the number of social connections this individual has. Our study reveals that in a population with a dense connectivity, all individuals tend to converge to having a single global consensus after a sufficiently long evolution. In a relatively sparsely connected network, however, the system may exhibit rich properties where the coexistence of different opinions, and more interestingly, multiple steady states of coexistence can be observed. Closer observations and detailed analyses reveal that

low-degree and high-degree nodes may play different roles in formulating the final steady state, including multi-steady states, of the systems under different opinion evolution rules. Such observations may help understand the complex dynamics of opinion evolution and coexistence in social systems.

In Chapter 5, we study how simple social interaction dynamics may allow emergence of extremism in social networks, one of the most appealing topics of opinion dynamics in computational sociophysics in recent decades. Specifically, most of the existing studies presume that the initial existence of certain groups of opinion extremities and the intrinsic stubbornness in individuals' characteristics are the key factors allowing the tenacity or even prevalence of such extreme opinions. We present a modified bounded-confidence model showing that without an explicit classification of extremists and their distinct attributes, extremism may still arise from the seemingly "normal" social interactions between individuals when such interactions tend to achieve biased consensus between individuals. That is, right-wing opinion holders may agree on an opinion that is to a certain extent biased towards right, while left-wing opinion holders' consensus is more or less biased towards left. With such types of consensus making, several clusters of diversified opinions are gradually formed up in a general trend of shifting toward the utmost opinions (most opposing and most agreeable), resulting in the emergence of extremism and the concurrent dwindling of moderate views. Further, we consider the case where bias tends to be stronger in consensus making happening closer to the extremities of the opinion space. It is found that such case allows moderate opinions a greater chance to survive compared to that of the case where the bias extent is universal across the opinion space. As to the extreme opinion holders' lower tolerances towards different opinions, which arguably may exist in many real-life social systems, they significantly dwarf extreme opinion communities rather than helping them to prevail.

Finally, we examine the evolution of opinions under the influences of both the majority and the consensus making rules. The two rules may indeed co-exist and together, affect the opinion dynamics and opinion distribution of a social system. Specifically, we propose a simple model where each social entity may be under the influences of both of the two rules in changing his/her current opinion. It is shown that in a network sustaining a low tolerance level in consensus making, the larger proportion the majority-oriented individuals is, the more fragmented the final opinion distribution becomes. Conversely, when there exists a relatively high tolerance threshold in consensus making, the existence of the majority-oriented individuals may help promote achieving a global consensus.

1.3 Organization of the Thesis

The remainder of the thesis is organized as follows:

- Chapter 2 starts itself by providing definitions and concepts of complex networks, followed by a review of the main opinion formation models.
- Chapter 3 examines opinion formation following the consensus making rule on multiplex networks. It is revealed that the system dynamics becomes much more complex than that in single-layer networks.
- Chapter 4 investigates the multiple co-existences of opinions in the steady state in sparse complex social networks driven by majority rules. It is shown that the multiple co-existences emerge due to different reasons while under different majority game rules.
- Chapter 5 examines the effects of bias in interpersonal interactions and its role in emergence of extremism in social networks. Effects of a few other factors

are also evaluated.

- Chapter 6 studies opinion dynamics under the interplay of the majority and the consensus making rules. It is shown that the co-existence of multiple rules may significantly change the system dynamics.
- Chapter 7 concludes the thesis and proposes a few directions for future research.

Chapter 2

Literature Review

2.1 Complex Networks

There are different definitions of complex networks. Typically, a complex network refers to a large-scale network or system which is made of two main components of nodes (vertices) and links (edges) with non-trivial topological features [23–25]. The structure of complex networks may have significant impacts on dynamical processes. Hence, multiple attempts have been made to model real-life complex networks properly without a loss of critical features. In the following part, we present a few artificial network models that are widely adopted in the studies on complex networks.

2.1.1 Network Models

2.1.1.1 Random Networks

A random network is a complex network in which the presence of any link connecting two nodes is of randomness. Intensive and systematic researches on random networks were first carried out by Paul Erdős and Alfréd Rényi, well-known for their classic random network model named as ER model [7, 26, 27].

In the ER network model, a link is set up between any pairs of nodes with a probability p . Let k denote the number of links a node has, termed as nodal degree.

With a fixed average degree, the nodal degrees of an ER network follow a Poisson distribution when network size N goes to infinity [26]:

$$P(k) = \binom{N}{k} p^k (1-p)^{N-k} \approx \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}, \quad (2.1)$$

where $\langle k \rangle = p(N-1)$ is the average degree.

The probability p has a major impact on the structure of ER networks. Specifically, there exists a critical probability $p_c = 1/N$; if $p > p_c$, a connected network with a single giant component is formed; if $p < p_c$, the network is composed of a finite number of finite-sized connected clusters [7, 26, 28, 29].

ER networks have a diameter (the shortest distance between the two most distant nodes in a network) $d = \frac{\ln(N)}{\ln(\langle k \rangle)}$ [1, 27, 30, 31], and the average shortest path length of the network is $l \sim \frac{\ln(N)}{\ln(\langle k \rangle)}$ [31, 32] for $\langle k \rangle > 1$. As logarithmic functions of network size, both diameter and average shortest path length are rather short compared to the network size and increase slowly as the network gets bigger. Networks with such properties of network diameter and average shortest path length are said to have a *small-world* property. The small-world concept describes the fact that in most networks, even with a very large size, there is a relatively short average path length between any two nodes.

The ER model has been proven to be a simple yet effective network model in the study of random graphs and stochastic processes. However, in general it does not successfully represent large-scale networks and systems. For instance, it cannot reveal the clustering effects where friends of a friend are usually also friends [33]. Hence, it is not a surprise that other network models have been proposed, most noticeably including Watts-Strogatz network model and scale-free network model, etc.

2.1.1.2 Watts-Strogatz Network Model

The most well-known small-world network model was proposed by Duncan J. Watts and Steven H. Strogatz, and hence known as Watts-Strogatz or WS model. The model becomes so popular that sometimes when people use the name “small-world networks”, they are actually referring to WS model. The model nicely resembles two noticeable features than can be observed in many real-life networks such as typical power grid systems [34]: a small average shortest path length and a high clustering coefficient [35–37].

Figure 2.1: A WS network with $\beta = 0$.

The model has two critical parameters, the number of links K and the rewiring probability β . The network generation algorithm starts with a ring with N nodes, where each node has K edges linked to its nearest neighbors on either side. Once the ring is utterly formed, these links are randomly rewired with a probability β . Note that when $\beta = 1$, the network turns to be an ER random network.

2.1.1.3 Scale-free Networks

Scale-free networks refer to networks or systems in which the nodal degrees follow a power-law distribution independently of the network size. They have been grabbing

Figure 2.2: A WS network with $\beta = 1$.

huge attention in modern complex network sciences because of their ubiquity in many natural and manmade systems and networks, including protein networks [38], metabolic networks [39], brain functional networks [40], webs of human sexual contacts [41], scale-free networks in cell biology [42], the Internet [9, 43, 44], and the World-Wide-Web [8, 45, 46]. Though it is still being debated why similar nodal degree distributions exist in so many real-life systems, scale-free networks are safely among the most interesting and most important network models, if not the single most important one, in modern network research.

Figure 2.3: Visualization of a scale-free network of 500 nodes. Size of each node reflects its nodal degree.

Figure 2.3 shows a visualization of a scale-free network with a size of 500 nodes.

The presence of a few high-degree nodes (called hubs) is a vital attribute of scale-free networks compared to other network models. These hubs handle a large number of connections in the network and therefore have a significant influence on the evolution of memes occurring on it. Figure 2.4 shows an example where 5 hubs in a network containing 500 nodes are linked to nearly 20% of the whole population. As mentioned, the degree distribution of such networks follows a power-law function

$$P(k) \sim k^{-\gamma}, \quad (2.2)$$

where γ is the degree exponent (see Fig. 2.5).

Figure 2.4: Hub nodes and their first neighbors. The hubs are colored with blue and their first neighbors are colored with red. The network visualization is done using Gephi [47].

We here briefly present two of the most well-known scale-free network models widely adopted in existing studies.

A. BA Model

The Barabási-Albert (BA) model, also known as BA model, was proposed by A. Barabási and R. Albert [4, 43]. The BA model generates scale-free

Figure 2.5: Nodal degree distribution (with $\gamma \approx 2$) of the scale-free network in Fig. 2.4. A power-law distribution can be easily observed in the log-log scale presentation of degree distribution. At high nodal degrees, a heavy tail can be observed due to fluctuations of a small number of hub nodes at different nodal degrees.

networks by *growth* and *preferential attachment*: the network starts with m_0 nodes called *seeds*; at each time step, a node is attached to the network by setting m links to m of the existing nodes. The probability that the newly-added node is connected to an existing node is proportional to the degree of the existing node. After t time steps, a network of $m_0 + t$ and mt links is obtained with a degree distribution

$$P(k) \approx \frac{2m^2}{k^3}. \quad (2.3)$$

BA networks have an approximate exponent of 3 and an approximate average degree of $2m$.

B. UCM

For cases where strict non-correlation is needed in simulation, an uncorrelated configuration model (UCM) proposed by Catanzaro et al. [48] has been widely used. The algorithm can generate random networks with arbitrary degree distributions. Specifically, while using the UCM method to generate a scale-free

network, each node i of the network of N nodes is assigned a degree k_i extracted from the probability distribution $P(k) \sim k^\gamma$, subject to the constraints $m \leq k_i \leq N^{1/2}$ and $\sum k_i$ is even. The network is constructed by randomly connecting the links attached to each vertex until the network is formed.

The UCM becomes widely adopted since it overcomes a few shortcomings of BA models. For instance, the exponent of BA networks is largely fixed at 3 while UCM can generate scale-free networks with various exponents. Also, it helps achieve strict randomness while it was proved that BA model, popular as it is, cannot strictly eliminate correlations in network links.

2.1.1.4 Multiplex Networks

The multiplexity can be observed in various complex networks where the functioning of a node affects the functioning of the corresponding nodes on other layers. For example, people are connected on different online social networks such as Twitter, LinkedIn, and Facebook that forms multiple communication channels between two individuals. Another example is the networks of transport where people in different cities are linked by various types of transport [49] (see Fig. 2.7). In such networks, it is common that there are various processes occurring concurrently or alternately on different layers and the dynamics of one process may be affected by that of the others. Understanding such dynamical processes under the effects of multiplexity essentially remains as a challenge.

The necessary first step in the investigation of multiplex networks is to propose a mathematical representation that can describe utterly the connections between nodes in the same layer (intra-layer) and different layers (inter-layer), respectively. It can naturally describe such multiplexes based on the basic concept of graph. Since a graph (monoplex network) with a size of N is typically presented as a pair

Figure 2.6: An example of a 3-layer interconnected network. The network visualization is done using MuxViz [50].

Figure 2.7: The multilayer transport network of London [51].

$G = (V, E)$, where V is the set of nodes labeled by $i, i = 1, 2, \dots, N$ and $E \subseteq V \times V$ is the set of edges, the multiplex network, therefore, can be naturally presented as the set $G = (V, E)$ made of L layers $G_1, G_2, \dots, G_L$, where each layer is a graph $G_l = (V, E_l), l = 1, 2, \dots, L; E = \cup_{l=1}^L E_l$.

Such an approach is simple and effective in presenting multi-layer networks where each node has inter-network connections only with its corresponding nodes on the other layers. Besides, a few attempts to obtain a consistent mathematical representation of multiplex networks by adopting tensorial approach were presented in [52, 53]. Yet another approach to investigate multiplex networks is to transform them into weighted monoplex networks [52, 54]. While aggregating different layers in a multilayer network to a single-layer network allows inheriting techniques and methods obtained in the studies on single networks, such an approach, however, may lead to the loss of information of the inherent multiplex nature of such networks and consequentially allow generation of some wrong conclusions [20, 55].

The studies on multiplex networks have been heavily focusing on two aspects: topological structures and measures of multiplex networks, and dynamics of spreading processes. The article [20] provides a systematic and magnificent review, especially on evaluating interaction strength between layers (aiming at measuring the correlations between constituent layers of the multiplex network, which has also been addressed in [49, 56, 57]) and the dynamics of diffusive processes (mainly focusing on epidemic and opinion spreading). Other hot topics on dynamics of multiplex networks include system resilience, i.e., robustness of networks and systems under random failure or intentional attack [58, 59], and synchronization [60], etc.

2.1.2 A Few Measures of Complex Networks

With the fast developments of modern network sciences over the past few decades, many measures, many of which are sourced from statistical physics, have been proposed for analyzing and exploring the topological features of complex networks. A survey of such measures can be found in [61]. We hereby briefly present a few measures that are extensively utilized in existing studies and are most closely related to our studies reported in this thesis.

A. Nodal Degree Distribution

As mentioned above, the nodal degree (or degree in short) distribution is the probability distribution of the degrees of all nodes in a network. The degree distribution $P(k)$ denotes the probability that a randomly chosen node has a degree k . It plays an important role in characterizing network topology that is known to have major influences on the dynamics processes executed on the networks. The most well-known degree distributions of complex networks include Poisson distribution [26] and power-law distribution [43] which are mainly employed in this thesis.

B. Community Structure

Community or modularity structure of complex networks reflects the property that a certain group of nodes may have a denser connectivity among themselves than with other nodes. Such groups are termed as modules or communities [62–65]. Community structure can be easily observed in many real-life networks and systems such as social networks [66], metabolic networks [67], pollination networks [68], and protein-protein interaction networks [69]. Detecting and evaluating communities in a large-scale complex network is both

interesting and challenging. A few methods have been developed in recent years [63, 70, 71].

C. Assortativity

In real-life networks, high-degree nodes may tend to be connected to high-degree nodes or low-degree nodes with unequal probabilities. The former case is known to be an assortative network, while the latter one a disassortative network [72, 73]. Some examples of assortative and disassortative networks include physics coauthorship, film actor collaborations, neural networks, and the World Wide Web, etc. [72]. If all nodes of a network are randomly connected without any bias in degree correlation, the network is considered to be an uncorrelated network (also known as random network or uncorrelated random network) [48]. The network assortativity can be measured by assortativity coefficient [72].

2.2 Opinion Formation Models

Being intensively studied in recent years, opinion formation (or opinion evolution or opinion dynamics) has become an area of interest and importance in studies on complex networks and systems [20, 74–80]. Such studies mainly aim to obtain a deeper understanding of opinion evolution in societies where social entities may interact through the social connections in between and consequently affect each other’s public opinions.

Many opinion dynamics models have been proposed. An comprehensive review of these models can be found in [81–87]. Recent studies have been making efforts to propose models that have a closer link with the real world typically by taking new aspects and concepts into account. Examples of such studies include the investiga-

tion of multidimensional opinion space inspired by the fact that people may hold inter-correlated opinions on various subjects [88], the interdependence between consensus and coherence [89], effects of social power on the evolution of opinions [90,91], and compromise making [80], etc.

The opinion dynamics models could be largely classified into continuous and discrete-valued opinion models. A brief survey for each of them is presented in the following sections.

2.3 Continuous-Valued Opinion Models

2.3.1 Deffuant Model

Deffuant model [92] is one of the most popular bounded-confidence models. The model employs the parameter of tolerance range. Such a parameter, to a certain extent, reflects the concept of *selective exposure*, which describes one’s tendency to involve himself/herself in social interactions that support his/her opinions [93].

Besides the intensive studies on the original Deffuant models, the past few years have witnessed increasing efforts to further the understanding of the Deffuant mechanisms by introducing modifications to the model. For instance, [94] investigated a generalization of an asymmetric version of the Deffuant model with an interaction kernel, while the authors of [95] generalized the Deffuant model by assuming that each agent can choose multiple neighbors in each opinion update step. Some also proposed discrete versions of the Deffuant model, e.g., [96] to study the dynamics of binary opinions on multiple issues.

The models in such studies exhibit rich system dynamics emerging from the opinion update rules that sometimes surprisingly resemble observations in human societies. Studies have also been dedicated to achieving a deeper understanding of

the Deffuant model based on numerical experiments. To name a few, [97] performed extensive numerical experiments on a leader–follower opinion dynamics model. [98] simulated the Deffuant model on a ring with two modifications: the repelling mechanism and the circular opinions. The authors in [99] investigated the dependence of the convergence time on various parameters, e.g., confidence bound, the network size, and their willingness to compromise. Another simulation study in [100] examined the effects of the opinion-dependent weighting factor on the formation of steady-state opinion clusters. While many studies conjecture the equilibrium convergence of the Deffuant model when all agents’ opinions are pairwise either the same or confidence bound apart, by numerical evidences, a new study by Deffuant et al. showed a new stationary state where the opinions held by moderates keep fluctuating without clustering [101]. While more sophisticated models are being introduced to resemble real-life observations and reveal the possible mechanism behind, mathematical analyses on the Deffuant model and its variants are still limited. By restricting the Deffuant model to some simplified assumptions, e.g., the well-mixing assumption which largely ignores the underlying network topology, some theoretical results have been obtained. Specifically, the convergence of the homogeneous Deffuant model, where all agents have the same confidence bound, has been proved in [102]. Convergence analysis of the asymmetric Deffuant model, in which only one individual changes his/her opinion in unidirectional pairwise interactions, was conducted in [103]. Convergence rate of the asymmetric Deffuant model was established in [104]. Noticeably, the authors in [105] took a further step by establishing convergence properties of the Deffuant model in the heterogeneous confidence bound case by viewing the opinion update network as a control system.

In this thesis, we mainly adopt the idea of pair-wise consensus making in the Deffuant model to investigate dynamics of continuous opinion systems. Hence, we

here present a more detailed insight into the Deffuant model while briefly introducing other well-known continuous opinion models.

2.3.1.1 Model Definition

At the beginning of the opinion evolution, each agent $i \in G$ is assigned a continuous-valued opinion o that follows a certain distribution, typically a uniform distribution limited in a closed interval $[0, 1]$. If two individuals with sufficiently close opinions, namely, their opinions differ of an amount less than or equal to a tolerance range d (also known as tolerance threshold. Hereafter, these two names may be used interchangeably as long as no confusion would be caused): $|o_i - o_j| \leq d$, they tend to adjust their opinions towards each other after interactions. The equations describing such opinion dynamics are as follows:

$$\begin{aligned} o_i(t+1) &= o_i(t) + \mu(o_j(t) - o_i(t)), \\ o_j(t+1) &= o_j(t) + \mu(o_i(t) - o_j(t)), \end{aligned} \tag{2.4}$$

where $o_i(t)$ denotes the opinion held by agent i at time t . The existing studies reveal that setting μ at a smaller value only affects the speed of convergence while the final steady state would be the same as that when $\mu = 0.5$ [92, 106]. Therefore, like that in most existing studies, we use $\mu = 0.5$ in our work for convenience unless otherwise specified.

The classic model describes opinion formation in a continuous time domain, where opinion updating events may randomly happen in a network following a Poisson process with a rate λ . To be specific, at time $t + \Delta t$, a node is active with a probability $\lambda \Delta t$. For the sake of simplicity, in most studies, Δt and λ are set to be 1. The methods for selecting the nodes to be involved in interactions may be different in different existing results. In [107], it was proposed that for each interaction, a

node is randomly selected and then one of its neighbors is also selected at random.

Typically, people adopt the number of opinion clusters and size of the biggest clusters [107] to measure the degree of consensus or opinion diversity in the system. The evolution process stops once the network reaches the quasi-steady state [84] at which all agents' opinions remain unchanged or change within a small fluctuation range. To identify and measure the opinion clusters, links connecting all pairs of nodes with opposing opinions are removed. Afterwards, the Dulmage-Mendelsohn decomposition algorithm [108] is applied to obtain the number of disconnected components that represent surviving opinion clusters in the final state.

2.3.1.2 Theoretical Approach for Analyzing Opinion Dynamics

In the Deffuant model, the next state of an opinion system depends on its current state. Therefore, one may consider opinions as states in a Markov process. By keeping record of opinion states at time step t , one can compute the states at time step $t + \Delta t$ and finally reproduce the temporal evolution of opinion systems. Let $p_k(o, t)$ denote the probability of a randomly selected node having a nodal degree $k, k = k_1, \dots, k_n$, and opinion o at time t . Let $p(k)$ denote the probability density of degree k . $p_k(o, t)$ can be expressed in terms of $p(k)$ and $p(o|k, t)$: $p_k(o, t) = p(k)p(o|k, t)$, where $p(o|k, t)$ denotes the relative density of nodes having opinion o in the set of k - degree nodes. In static networks, $p(k, t)$ becomes $p(k)$.

The dynamical equation describing the evolution of opinion can be written as a rate equation:

$$\frac{\partial p_k(o, t)}{\partial t} = -p_k(o, t) \int_{0 < |o' - o| \leq d} \theta(o', t) do' + \int_{0 < |o' - o| \leq \frac{d}{2}} p_k(o', t) \theta(2o - o', t) do', \quad (2.5)$$

where $\theta(o, t)$ denotes the probabilities of hitting a node with opinion o by following the selected link at time t . Likewise, let $\theta_k(o, t)$ denote the probability of hitting

a node with opinion o and degree k by following the selected link at time t . In uncorrelated or weakly-correlated networks, $\theta_k(o, t)$ and $\theta(o, t)$ are calculated as follows:

$$\begin{aligned}\theta_k(o, t) &= \frac{kp(k)p(o|k, t)}{\sum_k kp(k)} \\ &= \frac{kp(k)p(o|k, t)}{\langle k \rangle},\end{aligned}\tag{2.6}$$

$$\begin{aligned}\theta(o, t) &= \frac{\sum_k kp(k)p(o|k, t)}{\langle k \rangle} \\ &= \sum_k \theta_k(o, t),\end{aligned}\tag{2.7}$$

where $\langle k \rangle$ is the average degree of the network [72, 109].

In mean-field study, fully-mixed populations are employed. As such, each node may have connections with all the others, Eq. (2.5) becomes [110]:

$$\frac{\partial p(o, t)}{\partial t} = -p(o, t) \int_{0 < |o' - o| \leq d} \theta(o', t) do' + \int_{0 < |o' - o| \leq \frac{d}{2}} p(o', t) \theta(2o - o', t) do'. \tag{2.8}$$

The integrals in Eq. (2.8) can be numerically integrated using trapezoidal method by discretizing continuous o into equally-spaced discrete values with a reasonably small increment over the opinion axis. By iterative calculations of Eq. (2.8), the temporal process of opinion formation can be reproduced. Figure 2.8 shows an example of the temporal evolution of opinion systems in an infinite-sized network characterized by nodal degree obeying a Poisson distribution with a mean of 10.

2.3.1.3 A Few Existing Observations on the Classical Deffuant Model

Existing studies found that the tolerance threshold (also interpreted as confidence bound, tolerance range, or uncertainty threshold [92, 107]) plays an important role in opinion formation processes. The numerical experiments reveal an interesting result, that is, the number of coexisting opinion clusters in the steady state is approximately

Figure 2.8: An example of the temporal opinion evolution in an infinite network with a tolerance bound $d = 0.5$.

$1/(2 * tolerance\ range)$ [92]. A notion of critical confidence bound is also captured to refer to a threshold beyond which an opinion system eventually converges to a single cluster sharing the same opinion that equals the average of *first impression* [92,111,112]. A few analytical frameworks have been developed using the mean-field approach [102,110,113]. The model has also been deliberately modified by replacing continuous-valued opinions with discretized values [83].

The dynamics of opinion formation largely depends on parameter d . As a result, a change in d changes agent-to-agent interactions and eventually drives the entire population to enter a global steady state in which all individuals' opinions remain unchanged.

(a) Initial state

(b) $d = 0.1$

(c) $d = 0.2$

(d) $d = 0.5$

Figure 2.9: The formation of opinion clusters in different cases of tolerance range d .

Figure 2.9 shows a visual example of opinion clusters formed in the final steady

state. The diversity of opinion clusters largely depends on d that represents the degree of tolerance held by the population. It is found that at $d = 0.5$, the only macroscopic opinion cluster containing all the agents survives eventually. A polarized-to-consensus transition point has also been figured out as d is around 0.25 [107]. Figures 2.10 and 2.11 show the number of coexisting clusters and the size of the largest (denoted by S_1) and the second-largest clusters (denoted by S_2) normalized by the network size as a function of tolerance range in a scale-free network with a size $N = 2000$ and an average degree of 10. A detailed investigation conducted on ER random networks can be found in [107].

Figure 2.10: The number of opinion clusters as a function of tolerance range.

2.3.2 A Few Other Continuous Opinion Models

The classical DeGroot model [114] is among the first proposed continuous models. In the model, at each iteration, each agent (voter) updates his/her opinion by averaging the votes (in real number values) of his/her neighbors. It is shown that over time, all opinions converge to a consensus. An extended study shows that the complete consensus in a large society can be achieved if and only if the influence of the most influential individual disappears as the society grows [115]. Due to the linear nature

Figure 2.11: The size of the two biggest clusters as a function of tolerance range.

of interactions, the model can be conveniently analyzed.

The Hegselmann-Krause (HK) model [116], like the Deffuant model, is also a member of the family of bounded-confidence models. It is similar to the DeGroot model in adopting the averaging scheme to update agents' current opinions. However, at each iteration, each agent moves to the arithmetic mean of all agents with opinion within his/her tolerance threshold. For simplicity, most of the existing studies perform analyses on the homogeneous HK model [117–120]. For the case with heterogeneous confidence bound, the convergence properties of the HK model were conjectured in [121]. The convergence of a special case of the heterogeneous HK model where the confidence bound is either 0 or 1 has been established in [122].

While the above continuous models mainly focus on the configuration of opinion systems, the seceder model [123] and its variants are widely used to study the formation and evolution of biological and social groups driven by local interactions. For instance, [123] demonstrated a simple local three-individual collision mechanism giving rise to the formation of hierarchical organizations. [124] turned the original seceder model into a model of network evolution to further examine the community

structures in social networks. The authors in [125] took a further step to generalize the original seceder model by adopting local polling groups with variable sizes and introducing higher-dimensional opinion vectors to explore the self-organized segregation of societies.

2.4 Discrete-valued Opinion Models

2.4.1 Majority Game on Binary-State Networks

Majority game (or majority rule) is among the most popular models of dynamics of binary-state systems. Following the rule, people may be affected by those who are socially connected to them (e.g., friends, family members, and colleagues) and tend to adopt the opinion that the majority of their connected individuals hold. To reflect such “majority effects”, a well-known *majority dynamics model* (also known as local *majority rule* or *majority voter model*) has been proposed [126–130]. Under the rule, each individual holds an initial opinion. During the temporal evolution, the individuals iteratively update their states to match the opinion of the majority of her/his immediate neighbors. In this thesis, we term this rule as the *classic majority rule*, to differentiate it from the new majority rule we shall introduce later in Chapter 4.

2.4.1.1 Description of State Update Rule

Still let $G = (V, E)$ be an undirected graph representing a social network, where V is the set of vertices (nodes) representing individuals and E is the set of edges (links) representing social connections. Denote the set of neighbors of a vertex $v \in V$ as $\eta(v)$ and the nodal degree of v as k_v , $k_v = |\eta(v)|$. At time step t , each node $v \in V$ holds a state $S_v(t) \in \{0, 1\}$ representing one’s opinion on a particular subject.

At the start of the opinion dynamics, each node v is endowed with a state $S_v(0)$ drawn from $\{0, 1\}$ at a certain distribution as assigned. At each time step, a node is randomly chosen and its state is updated following the majority rule. To be specific, a vertex shall adopt the opinion that occupies the majority of its adjacent nodes. When there is a tie, i.e., when the numbers of adjacent nodes holding each of the two opinions equal each other, the node shall retain its current opinion. Iteratively, the nodes update their states over time until an equilibrium is achieved where no vertex needs to further change its state.

Mathematically, the rule can be described in Eqs. (2.9) as follows [127]:

$$S_v(t) = \operatorname{argmax}_{s \in \{0,1\}} |\{u | S_u(t-1) = s, u \in \eta(v)\}|. \quad (2.9)$$

2.4.1.2 A Few Existing Results in Majority Rule Studies

Most of the studies on majority rules reveal an ultimate fate of all agents converging to a global consent. By studying majority dynamics on infinite regular tree graphs, the authors in [126] derive conditions which lead to convergence to unanimity. The study in [127] extends this work in several directions. It partitions the graph nodes into *social types* and nodes of the same social type play the same role in the geometry of the graph. It shows that having a large number of voters of each type implies aggregation and that without this condition aggregation may not occur. The work is then extended beyond the binary alternatives in general outcome spaces. Berger [131] gives an example of a series of graphs in which majority dynamics results in the adoption, by all individuals, of the opinion of the individuals in a constant size group, provided they all agree. A widely studied family of Gossip-based protocols on networks is carried out [132–134], aiming to analyze distributed, repeated algorithms for the aggregation of information on networks.

2.4.2 A Few Other Discrete Opinion Models

The well-known voter model originally studied independently by Holley and Liggett (1975) [135] and Clifford and Sudbury (1973) [136] was among the earliest proposed discrete opinion dynamics models. In the model, each individual holds an opinion of two discrete values. At each iteration, some randomly chosen individual interacts with one of his/her neighbors, who is also chosen at random, and adopts his/her opinion. Many variants of voter model have been extensively examined [137–139]. The voter model could be adopted as a fundamental framework for studies on social contagion.

The Sznajd model [79] is another well-studied discrete model. Based on a simple Ising spin model, the authors studied opinion evolution in which each member of a community can take only two attitudes. It is found that in a closed (isolated) community, the community has to evolve to either of only two possibilities of the steady state: dictatorship or stalemate. The model has been extended in other studies [140, 141].

2.5 Summary

In this chapter, we present a general background of complex networks and models of opinion dynamics, mainly on what we employ in the following chapters. Some relevant analyses of algorithmic opinion models were also provided. We will hereafter present the chapters covering our main contributions.

Chapter 3

Opinion Formation on Multiplex Networks

3.1 Introduction

As pointed out in Chapter 2, existing studies on opinion dynamics have largely been based on isolated single networks while many modern real-life social networks have witnessed the concurrent activeness of their members on various social networks. Such observations have motivated recent studies on various social dynamics in multiplex networks [142–144].

In computational sociology, some first-step work has also been proposed to discover new properties of opinion formation on multiplex networks under the influence of the interplay between constituent layers. The authors in [84] developed an analytical framework to investigate the Deffuant model on a one-dimensional multiplex network in which each layer is a network on $\mathbb{Z}$ domain. The work was rigorously conducted to draw an interesting conclusion that under a specific configuration of multiplexity, the existence of interconnected layers impedes the convergence of opinions to unanimity. However, the conclusion may not hold in cases where networks have more dimensions and more complex structures. For instance, another consensus model validated in allocation problems; in contrast, concludes that more dimensions

of opinions lead to better chances for consensus [145].

In this chapter, we examine opinion dynamics under the inter-layer interplay on duplex scale-free networks. Each layer represents a single social network on which each agent holds an opinion on a subject associated with that layer. It is found that in duplex networks of identical layers, a weakened-enhanced opinion diversity transition exists and varies as the function of the magnitudes of the tolerance ranges of both layers in a non-monotonic way. An extensive investigation is also carried out to examine the impact of the inter-layer coupling on the opinion formation.

3.2 Duplex Opinion Dynamics Model

The duplex network is a pair $G = (V, E)$ consisting of 2 layers G_1 and G_2 , each of which is a scale-free network $G_i = (V, E_i)$, where V is the set of vertices representing individuals in a social network and $E_i \subseteq V \times V$ is the set of edges on layer i , $i = 1, 2$, representing social connections. The edge set of G , denoted by E , consists of 2 sets of edges on the two layers, respectively, implying two types of social relationships: $E = E_1 \cup E_2$. The synthetic scale-free networks are constructed using the generative model [48]. Given an average degree $\langle k \rangle$, the scaling exponent of the power-law function can be obtained by numerically solving the following equation:

$$\sum_k k p(k) = \langle k \rangle, \quad (3.1)$$

where

$$p(k) = \frac{k^{-\gamma}}{\zeta(\gamma, k_{\min}) - \zeta(\gamma, k_{\max})}, \quad (3.2)$$

where $k_{\min}$ and $k_{\max}$ are the minimum and maximum degrees, respectively, and $\zeta(\gamma, k)$ is the Hurwitz ζ function [146, 147].

The connectivity in a layer is independent of that in the other layer. However,

we anticipate that the overlap of the links may have significant effects on the opinion formation. This inter-layer interplay is quantified as a link overlap parameter denoted by η which can be obtained as follows:

$$\eta = \frac{|E_1 \cap E_2|}{|E_1|}. \quad (3.3)$$

In order to evaluate the effects of the overlap parameter, we deliberately keep the degree of all nodes on both the layers to be the same. To be specific, given a network G_1 , a scheme for constructing a layer G_2 that meets a desired η is described as follows: in case of $\eta = 1$, the two layers are of exactly the same topology; otherwise, perform the following rewiring process: starting with an initial $G_2 \equiv G_1$; randomly choosing nodes $A, B, C, D \in V$ where links $\{A, B\}, \{C, D\} \in E_2$ and $\{A, C\}, \{B, D\} \notin E_2$, then replacing the links $\{A, B\}, \{C, D\}$ with $\{A, C\}, \{B, D\}$, respectively in E_2 . Repeat such rewiring operations until the pre-specified η is met. All nodes in the resulting layer G_2 have the same nodal degree as their corresponding nodes have in G_1 .

We here introduce a rule of pair-wise interaction between individuals on each layer (termed as *intralayer* interaction [52]) under the influence of the multiplexity. Firstly, we recall the rule of compromise making in a single network following the Deffuant model: initially, each node in the network is assigned a continuous-valued opinion drawn from a uniform distribution on the region $[0,1]$. At time step t , if a randomly selected link $e = \{u, v\}$ is active, the opinions held by u and v at time step $t + 1$ are determined by the following rule: if $|o(u, t) - o(v, t)| \leq d$, where $o(u, t)$ and $o(v, t)$ denote the opinions held by the nodes u and v , respectively, at time step t , and d is the tolerance bound homogeneous across the population, then the two

nodes make consensus:

$$\begin{aligned} o(u, t+1) &= o(u, t) + \mu(o(v, t) - o(u, t)), \\ o(v, t+1) &= o(v, t) + \mu(o(u, t) - o(v, t)), \end{aligned} \tag{3.4}$$

where $\mu \in (0, 1/2]$ is convergence parameter; otherwise, they maintain their current opinions. According to most of the existing studies, μ only affects the convergence time [92, 145]. We thus fix $\mu = 1/2$ in the remainder of this chapter.

At the beginning, on each of the layers, say l , each node v holds an opinion value, denoted by $o_l(v)$, drawn from a uniform distribution over the region $[0,1]$. At time step t , the two layers are active with equal probabilities. On the active layer, randomly choose a node, say u , and then pick one of its neighbors, say v , at random. The rule of inter-layer compromise making is described as follows: if u and v have a connection only on the active layer, e.g., layer 1, they behave the same as that in a single-layer network. In other words, if they are not connected on the other layer, i.e., layer 2, their opinions on layer 2 do not interfere in the opinion exchange process occurring on the current layer. Otherwise, they make consensus on layer 1 if and only if the following condition holds:

$$\begin{aligned} \frac{|o_1(u, t) - o_1(v, t)| + |o_2(u, t) - o_2(v, t)|}{2} &\leq \frac{d_1 + d_2}{2} \\ \Leftrightarrow |o_1(u, t) - o_1(v, t)| + |o_2(u, t) - o_2(v, t)| &\leq d_1 + d_2, \end{aligned} \tag{3.5}$$

where $d_i, i \in \{1, 2\}$, are the confidence bounds on layer i which are independent of each other. Equation 3.5 reflects such cases where connected individuals weigh the total differences between their opinions on a few different issues to decide whether they would like to compromise to make a consensus. The opinions evolve until both layers reach a quasi-steady state at which all agents either maintain their opinions or change their opinions by an amount less than a small fluctuation range (set at 10^{-3} in our study) in any future updates. The condition holds when on each layer,

all pairs of neighboring nodes either share approximately identical opinions or would never make local consensus according to the defined rule.

Equation (3.5) reflects a simple case where connected individuals weigh the total differences between their opinions on a few different issues to decide whether they would like to compromise to make a consensus. In the present model, component layers are supposed to play the same role, i.e., they are un-weighted layers. Therefore, the opinion spreading processes on both layers occur concurrently and similarly. For this reason, we mainly discuss the opinion dynamics on layer 1 under the effects of layer 2's opinion system in comparison to the cases of those in isolated single-layer networks.

3.3 Results and Discussions

To quantify the degree of consensus as well as the opinion diversity, we employ the quantity of the number of opinion clusters coexisting in the steady state. Such a cluster number on each layer is calculated by firstly removing links connecting two agents holding different opinions and then computing the number of disconnected components in each layer by using the Dulmage-Mendelsohn decomposition algorithm [108]. Note that the term *global consensus* is defined in the strictest sense, implying that in the final state, the entire population concurs and the number of surviving cluster, therefore, equals 1. Generally, for some cases, even when a giant fraction of the population already converges to a major opinion cluster, several minor groups, though of small sizes sometimes containing only a single node (called isolated agents or outliers in some existing studies [148]), still exist in the final state, and maintain their opinions. The corresponding critical confidence bound, hence, may vary according to criteria for global consensus.

3.3.1 Global Consensus Formation on Duplex Network of Duplicated Layers

We start with the simplest case where the two layers are identical. Figure 3.1 depicts the diversity of opinion systems as d_2 changes. Similarly to the case of monoplex networks, the number of opinion groups declines as d_1 increases. The formation of opinion clusters on layer 1 is affected significantly by the presence of layer 2 characterized by various d_2 . When d_2 is very small, e.g. $d_2 = 0.01$, the number of surviving clusters approaches the size of the network when d_1 approaches 0. However, the fragmentation of the opinion system is severer since when $d_2 \rightarrow 0$, two nodes on layer 1 make a local consensus only if $|o_1(u, t) - o_1(v, t)| + |o_2(u, t) - o_2(v, t)| \leq d_1$, meaning the closeness in the nodes' opinions needs to be below a value less than its layer's own tolerance bound d_1 , making the individuals harder to make compromise in pair-wise communication sessions and finally diverse to different opinion groups. Note that, though the conclusion above, and all the conclusions reported in the next a few paragraphs, hold in the ER random networks, in an ER network with the same average nodal degree as that of a SF network, the number of clusters tends to be relatively smaller as there are fewer low-degree nodes in the ER network than in the SF network.

Noticeably, the consensus-to-polarized transition point (a point on the tolerance range below which the final state of the population's opinion jumps from having a single largest cluster to having two co-existent dominant largest clusters, as termed in [107]) shifts to the left along d_1 axis as d_2 increases and vanishes when d_2 is sufficiently large, e.g. $d_2 \geq 0.4$ (see Fig. 3.2). It is because under the influence of layer 2 with great openness characterized by a large d_2 , a giant cluster on layer 1 containing a large proportion of the population tends to be formed even at small

Figure 3.1: Number of clusters at steady state as a function of d_1 on duplex network with a size of 5000 and various average nodal degrees in three cases of d_2 : 0.01 (diamonds), 0.2 (triangles) and 0.4 (circles). Plot for single network case (squares) is added for reference. The results are averaged over 10 independent realizations.

values of d_1 . This also causes the extinction of the second-largest clusters and the proliferation of microscopic clusters with opinion values distributing over the entire opinion spectrum as seen in Fig. 3.3(a).

Figure 3.2: Sizes of biggest (solid lines) and second biggest (dash lines) clusters in the final state on single network (black) and layer 1 of duplex network with $d_2 = 0.01$ (red), 0.2 (green) and 0.4 (blue), respectively. All networks have a size of 5000 and an average degree of 10.

Figure 3.3: Opinion distribution on a single network with $d_1 = 0.1$ (d) and layer 1 of a duplex network at steady state with $d_1 = 0.1$ and various d_2 : 0.4 (a), 0.2 (b) and 0.01 (c). The network size is 5000 nodes and $\langle k \rangle = 10$. The inset in (a) is the same plot zoomed into small values of opinion density.

A question comes naturally that given a layer 2 with a certain degree of openness quantified by d_2 , at which confidence bound layer 1 reaches a complete consensus. In single networks, the critical threshold d^c is 0.5 [111]. In duplex networks, the phenomenon becomes more complex since the tolerance range of one layer may interfere in the opinion formation on the other layer coupled with it. d_1^c , therefore, may change accordingly. We proceed to determine the critical bound d_1^c for various d_2 . A trial-and-error approach is adopted. Specifically, for each d_2 , d_1^{lower} and d_1^{upper} are set at 0.01 and 1 respectively; the process of opinion exchanging with a tolerance range $d_1 = \frac{d_1^{\text{lower}} + d_1^{\text{upper}}}{2}$ starts until the opinion systems on both layers enter the steady state. The number of opinion clusters on layer 1, denoted by N_{clusters} , is then calculated: if $N_{\text{clusters}} > 1$, d_1^{lower} is reset at d_1 ; otherwise, d_1^{upper} adopts d_1 as a new boundary. Then the entire process starts over until d_1^{lower} deviates from d_1^{upper} by an

amount less than 0.01. The corresponding value of d_1 , being equal to $\frac{d_1^{\text{lower}} + d_1^{\text{upper}}}{2}$, is considered as the critical tolerance range d_1^c .

As shown in Fig. 3.4, critical bound d_1^c descends monotonically with respect to the increase of d_2 . The result shows the sum $d_1^c + d_2$ is consistent and around 0.9. This sum reaches 1, a double of critical confidence bound in an isolated network, as the network size approaches infinity. It can be understood that the full coupling of two layers strengthens the reciprocal tie of the layers, leading to a manner that a great receptiveness of layer 2 characterized by a large value of d_2 makes layer 1 easier to reach a complete consensus and vice versa.

Figure 3.4: Critical confidence bound on layer 1 as a function of d_2 . For each d_2 , the corresponding d_1^c is the average of 10 realizations. Errors bars represent 95% confidence intervals.

3.3.2 Effects of Partial Coupling on Opinion Diversity in Duplex Networks

To get a first insight on the effects of η , we plot the number of surviving opinion clusters on layer 1 in the steady state as a function of d_1 and d_2 in different cases of η . The cluster formation on the layer 1, therefore, is not influenced by d_2 as observed in Fig. 3.5(a). For $\eta = 1$, the layers are fully coupled. As a result, d_2 has a significant

effect on the formation of opinion clusters on layer 1. To be specific, the number of clusters declines as either d_1 or d_2 increases but roughly remains unchanged along the lines $d_1 + d_2 = \text{constant}$. The phenomenon becomes much more complicated when $\eta = 0.5$ that will be discussed in detail later.

Figure 3.5: Number of coexisting opinion clusters on layer 1 at steady state as a function of tolerance ranges d_1 and d_2 when $\eta = 0$ (a), 0.5 (b), and 1 (c), respectively. The number of clusters is shown in color scale. Each layer is a scale-free network with a size $N = 5000$, an average degree $\langle k \rangle = 10$, a minimum degree of 4, and a cutoff degree of 70. Dash lines are boundaries of critical regions in which a single opinion cluster is formed on layer 1.

Figure 3.6: The effect of multiplexity on the diversity of opinion clusters in comparison to the opinion dynamics on single networks.

Also seen in Fig. 3.1, for a certain value of d_2 , there exists a corresponding

value of d_1 , below (above) which the number of opinion clusters on layer 1 is smaller (larger) than that in the single network. We term such a point as weaken-enhance transition on the $d_1 - d_2$ plane. As shown in Fig. 3.6, the *weakened diversity* region implies that the multiplexity facilitates the consensus formation; in other words, the inter-layer interplay declines the diversity of the opinion system compared to the case in single-layer networks. *Enhanced diversity* indicates the region where the multiplexity makes the number of clusters larger than that in a single network. Lastly, *global consensus* shows the region where the duplex network reaches a global consensus, i.e., only a single giant cluster of opinion survives. The figure shows that for small values of d_2 , the multiplexity always increases the opinion diversity. As d_2 increases, the weaken-enhance transition shifts to the right along the d_1 -axis. Noticeably, as d_1 increases beyond 0.25, the transition point starts to plunge until d_1 reaches around 0.4. It is because in duplex networks, the consensus-to-polarized transition tends to vanish at large values of d_2 as aforementioned, while in single networks, this point clearly exists at $d_1 \approx 0.25$, causing a surge in the number of opinion clusters since the second-largest cluster is broken into minor parts [107]. Note that in multiple rounds of numerical simulations, the boundaries between different regions appear to be largely steady, with differences of only small fluctuations in between.

We here turn to a deeper look at the effects of overlap factor η by examining the survival opinion clusters on layer 1 under fixed values of d_2 for various η . When d_2 is small, e.g. 0.2 in Fig. 3.7(a), the system behaves in a non-monotonic way: for small values of d_1 , a tight inter-layer coupling, i.e. a larger η , reduces the number of opinion clusters on layer 1, but for large values of d_1 , the larger η is, the greater the number of clusters becomes; that is, under the influence of layer 2 with a small d_2 , the inter-layer interplay becomes an enhancing factor of the opinion diversity on layer 1.

Extensive simulations carried out on networks with various sizes confirm that when $d_2 \geq 0.5$, this non-monotonic behavior vanishes and the number of opinion clusters decreases with the increase of η for all $d_1 \in (0, 0.5)$ (see Figs. 3.7(b) and 3.7(c)). For these cases, it can be explained that a layer with a large tolerance range tends to weaken the opinion diversity on its coupled layer. Furthermore, the stronger the inter-layer coupling is, the stronger the effect becomes, regardless of the confidence bound on the layer in question.

Figure 3.8 shows the effects of d_2 for different cases of η . In loosely-coupled duplex networks, the probability that two nodes have connections on both layers is low, causing a layer to rarely interfere in decision making on the other layer. As a result, the effect of d_2 on the number of opinion clusters on layer 1 is not obvious as seen in Fig. 3.8(a). However, it is found that when η increases beyond 0.5, the system starts exhibiting a monotonic behavior: an increase of d_2 causes a decline in the opinion diversity irrespective of d_1 .

By adopting the same trial-and-error approach, we determine d_1^c in different cases of link overlap. Scale-free networks with a size of 10000 and an average degree

Figure 3.7: Number of steady-state opinion clusters on layer 1 of duplex network with a size $N = 5000$ for $d_2 = 0.2$ (a) and $d_2 = 0.5$ (b), and $d_2 = 0.6$ (c). Each subplot shows results in three cases of η : 0.2 (squares), 0.6 (triangles) and 1 (circles).

Figure 3.8: Number of steady-state opinion clusters on layer 1 of duplex network with a size $N = 5000$ for $\eta = 0.2$ (a), and 0.5 (b), and 0.6 (c). Each subplot depicts results in different cases of d_2 : 0.01 (squares), 0.2 (triangles), 0.4 (circles), and 0.6 (diamonds).

of 10 are employed. The results are averaged over 10 different realizations, each corresponding to a randomly generated network. An interesting observation is shown in Fig. 3.9: for a range of d_2 less than 0.5 , the increase in the overlap of the two layers enlarges critical tolerance range in layer 1, meaning that layer 1 requires a higher threshold to reach a global consensus. This can be understood that an increase of η intensifies the impact of layer 2's opinion system on opinion formation on layer 1. Therefore, with a d_2 less than the critical threshold in isolated networks, a larger critical bound on layer 1 is needed as a mechanism to compensate potential fragmentation on layer 2 due to a small d_2 . However, for $d_2 > 0.5$, d_1^c no longer declines with an increase in d_2 . It is found that for $d_2 > 0.5$, d_1^c remains at around 0.5 since there are always some minor opinion groups insisting on their different opinions from that in the giant cluster.

3.4 Conclusion

We studied the dynamics of opinion formation on duplex scale-free networks. It is found that the presence of inter-layer interplay either enhances or weakens the

Figure 3.9: Critical bound d_1^c as a function of d_2 for various values of link overlap parameter: $\eta = 0.2$ (red), $\eta = 0.4$ (green) and $\eta = 0.8$ (blue). Errors bars represent 95% confidence intervals.

diversity of opinion systems compared to the case on non-interacting single networks. These effects depend on the relative magnitudes of tolerance ranges associated with the layers. The global consensus state is gained if the sum of two tolerance ranges is not smaller than 1, a double of the critical range in opinion systems on single networks.

When the overlap is sufficiently larger, an increase in the tolerance range on a layer decreases the diversity of opinions on the other layer monotonically. It is also revealed that the one-sum rule no longer holds when duplex networks are not fully coupled. Following that, an open-minded social network with a large tolerance cannot complement the global consensus formation on the other network, i.e., a layer can reach a complete consensus only if its associated confidence bound is sufficiently large, which is at least around 0.5 in duplex scale-free networks.

Our study in this chapter reveals the interesting dynamics of opinion formation and global consensus making in multiplex social networks. It also demonstrates the potential complexity the system dynamics may have when a well-known, extensively studied model is extended from a single-layer network to a multiplex network. Con-

sidering that most of the social systems are actually inter-correlated to each other, we may expect that many important observations are yet to be made in this area.

In this study, the two layers are assumed to be active with an equal probability. However, in real-life, relative frequencies of being active may vary from layer to layer. Future studies may consider such differences as a parameter in the quantification of inter-layer interplay that may reveal new properties in opinion formation.

Though most conclusions reported in this chapter hold in both synthetic ER and SF networks, some other network topologies may lead to significant differences in opinion dynamics and cluster formation. For examples, some conclusions may not hold in networks with strong community structures. Investigations on dynamics of opinion formation on other network topologies shall also be of our future research interest.

Chapter 4

Multiple Coexistences of Steady States in Opinion Systems under Majority Rules

4.1 Introduction

As briefly introduced in Chapter 2, the majority rule, simple as it is, exhibits rich dynamical properties which have inspired scientists to propose and investigate quite a few variant models. The works by Mossel and Tamuz et al. [127, 128] focus on the effective aggregation of information on two-state systems under majority dynamics and conditions that drive the social systems to converge to unanimity. It also addresses other questions such as whether an opinion supported by the initial majority remains as the majority in the final state. Studies have also been carried out to develop different theoretical frameworks with different configurations of underlying networks. Some examples include the studies by Kanoria and Montanary on majority dynamics on regular tree graphs [126], by Howard on 3-regular trees [149], and by Fontes, Schonmann, and Sidoravicius [150] on $\mathbb{Z}^d$, etc.

In most studies, complete consensus state at which all individuals adopt the same opinion is observed at the end of the opinion evolution process. The study in [130] shows that under the majority dynamics, when the initial opinions of agents on Erdős-Rényi random graphs with degrees $\Omega(\sqrt{n})$ are independently and identically

distributed, the agents eventually converge to the initial majority opinion, with a constant probability. It is also argued in [151] that independently of underlying graphs, there is always a group formed by half of the agents that can obliterate the opposite opinion. Likewise, the authors in [127] show that if the initial population is sufficiently biased towards a certain opinion, this opinion may finally become the unanimous preference of the entire population. Some other variants of majority dynamics can be found in [129, 152–158].

In our study, for the first time to the best of our knowledge, we find that under the classic majority rule, the systems’ behaviors may be different in densely and sparsely connected networks, respectively. Specifically, the existence of a large proportion of low-degree nodes may sustain the coexistence of two competing opinions in equilibrium, which is not observed in dense networks. These low-degree nodes are connected to each other forming up separating boundaries between clusters of opposite opinion holders. As the setting up of such separating boundaries is sensitive to the initial allocations of different opinion holders, multi-steady states of opinion coexistence can be observed.

Further considering that when people choose to follow the “majority” opinion, what matters most may not be the number of holders of each of the opinions, but how influential these opinion holders are, we propose a new majority game model, termed as *influence majority rule*. To be specific, we let each individual follow the opinion with stronger overall influences among her/his immediate neighbors. To simplify the discussions, we let the overall influence on an individual be equal to the sum of the influences that her/his different opinion-holding neighbors have, and the influence of each neighbor be equal to the number of connections this neighbor has. In a social network model where each individual is represented as a vertex and the connections s/he has are represented as network edges, this means that for each

node, we count and sum up the nodal degrees of those nodes being adjacent to this node holding opinions 0 and 1, respectively, and let this node adopt the opinion with a larger sum value. Other influence majority models certainly can also be proposed; studies on these models, however, are out of the scope of this thesis and shall be carried out in our future work.

Under the influence majority rule, as will be presented later, at first sight, it may seem that the observations are similar to those under the classic majority rule. That is, in dense networks, complete consensus would be achieved at the end of opinion evolution, while in sparse networks, multi-steady states of coexistence would be achieved. Closer observations, however, reveal that the seemingly similar final states have different structures and they emerge due to different reasons: unlike that under the classic majority rule where low-degree nodes form up separating boundaries between clusters holding opposite opinions, under the influence majority rule, it is high-degree hub nodes that connect their surrounding low-degree nodes to form into communities holding different opinions.

The rest of this chapter is organized as follows. Section 4.2 presents a formal description of the system model and the opinion formation rules, followed by some brief discussions on theoretical analyses. Simulation results and discussions are presented in Sections 4.3 and 4.4. Finally, Section 4.5 concludes the chapter and briefly discusses some possible future research as well.

4.2 Models and Analyses

4.2.1 Opinion Update Rules

Let $G = (V, E)$ be an undirected graph representing a social network, where V is the set of vertices (nodes) representing individuals and E is the set of edges (links) representing social connections. Denote the set of neighbors of a vertex $v \in V$ as

$\eta(v)$ and the nodal degree of v as k_v , $k_v = |\eta(v)|$. At time step t , each node $v \in V$ holds a state $S_v(t) \in \{0, 1\}$ representing one's opinion on a particular subject.

At the start of the opinion dynamics, each node v is endowed with a state $S_v(0)$ drawn from $\{0, 1\}$ at a certain distribution as assigned. At each time step, a node is randomly chosen and its state is updated following either of the two majority rules. Iteratively, the nodes update their states over time until an equilibrium is achieved where no vertex needs to further change its state.

Under the classic majority rule, a vertex shall adopt the opinion that occupies the majority of its adjacent nodes. When there is a tie, i.e., when the numbers of adjacent nodes holding each of the two opinions equal each other, the node shall retain its current opinion. When under the influence majority rule, a vertex shall sum up the degrees of those adjacent nodes holding opinions 0 and 1, respectively. It shall adopt the opinion that has a larger value of sum. When there is a tie, the node shall retain its current opinion.

Mathematically, the two different rules can be respectively described in Eqs. (4.1) and (4.2) as follows:

$$S_v(t) = \operatorname{argmax}_{s \in \{0,1\}} |\{u | S_u(t-1) = s, u \in \eta(v)\}|, \quad (4.1)$$

$$S_v(t) = \operatorname{argmax}_{s \in \{0,1\}} \sum_{w \in \{u | S_u(t-1)=s, u \in \eta(v)\}} k_w. \quad (4.2)$$

For the sake of convenience, we hereafter term the classic majority rule and the influence majority rule as CMR and IMR, respectively.

4.2.2 System Dynamics Equations

Denote the relative densities of k -degree nodes holding states 0 and 1 at time t as $\rho_{0,k}(t)$ and $\rho_{1,k}(t) = 1 - \rho_{0,k}(t)$, respectively. In an uncorrelated random network, the temporal evolution of the system governed by CMR can be approximately described by the following dynamics equations:

$$\frac{d\rho_{0,k}(t)}{dt} = -\rho_{0,k}(t) \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{m} \theta_0(t)^m \theta_1(t)^{k-m} + \rho_{1,k}(t) \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{m} \theta_1(t)^m \theta_0(t)^{k-m}, \quad (4.3)$$

where $\theta_0(t)$ ($\theta_1(t)$) is the probability that a randomly chosen link points to a 0-state node (1-state node):

$$\begin{aligned} \theta_0(t) &= \frac{\sum_k k P(k) \rho_{0,k}(t)}{\sum_k k P(k)} \\ &= \frac{\sum_k k P(k) \rho_{0,k}(t)}{\langle k \rangle}, \quad \langle k \rangle \text{ is the average degree,} \end{aligned} \quad (4.4)$$

$$\theta_1(t) = 1 - \theta_0(t). \quad (4.5)$$

To derive the dynamics equation for IMR, let D_l be a multiset having cardinality l drawn from an ordinary set $U = \mathbb{N} \cap [k_{\min}, k_{\max}]$. D_l can be represented by a set of ordered pairs: $D_l = \{(k, n_{D_l}(k)) \mid k \in U, n_{D_l}(k) \in \mathbb{N}\}$, where $n_{D_l}(k)$ denotes the number of occurrences of the element k in D_l and hence $\sum_{k \in \text{Supp}(D_l)} n_{D_l}(k) = l$. Here $\text{Supp}(D_l) = \{k \in U \mid n_{D_l}(k) > 0\}$ is the support set of D_l formed from distinct elements of D_l . Denote the total sum of elements of D_l as $\sum_{D_l} = \sum_{k \in \text{Supp}(D_l)} k n_{D_l}(k)$ and define function $\Gamma(\cdot)$ of D_l as $\Gamma(D_l) = \prod_{k \in \text{Supp}(D_l)} n_{D_l}(k)!$. The dynamics equation of a system governed by IMR can be expressed as follows:

$$\begin{aligned} \frac{d\rho_{0,k}(t)}{dt} = & -\rho_{0,k}(t) k! \sum_{m=0}^{k-1} \sum_{D_m < \sum} \frac{1}{\Gamma(D_m) \Gamma(D_{k-m})} \prod_{k \in \text{Supp}(D_m)} \theta_{0,k}^{n_{D_m}(k)}(t) \prod_{k \in \text{Supp}(D_{k-m})} \theta_{1,k}^{n_{D_{k-m}}(k)}(t) \\ & + \rho_{1,k}(t) k! \sum_{m=0}^{k-1} \sum_{D_m < \sum} \frac{1}{\Gamma(D_m) \Gamma(D_{k-m})} \prod_{k \in \text{Supp}(D_m)} \theta_{1,k}^{n_{D_m}(k)}(t) \prod_{k \in \text{Supp}(D_{k-m})} \theta_{0,k}^{n_{D_{k-m}}(k)}(t), \end{aligned} \quad (4.6)$$

where $\theta_{0,k_n}(t)$ ($\theta_{1,k_n}(t)$) is the probability that a randomly chosen link points to a

node having degree k and state 0 (1) at time t :

$$\theta_{0,k_n}(t) = \frac{kP(k)\rho_{0,k}(t)}{\langle k \rangle}, \quad (4.7)$$

$$\theta_{1,k_n}(t) = \frac{kP(k)}{\langle k \rangle} - \theta_{0,k_n}(t). \quad (4.8)$$

Note that when the network has a Delta-Dirac degree distribution, Eq. (4.6) is reduced to Eq. (4.3).

4.3 Temporal Convergence to Unanimity in Dense Networks

We investigate the evolution of the system state under the two majority rules, mainly on underlying scale-free networks [159] generated from the configuration method [48]. Specifically, let $F_0(t)$ and $F_1(t)$ denote the fractions of nodes holding states 0 and 1 at time step t , respectively. At $t = 0$, $F_0(0) \leq 0.5$ of the population are randomly chosen to be endowed with state 0 and the remaining nodes are endowed with state 1. The system starts to evolve under either of the two rules until the final steady state is reached. We refer to the state with a larger (smaller) proportion in a specified graph as the majority (minority) state of that graph.

We first verify Eq. (4.3) by carrying simulations on scale-free and Erdős-Rényi networks. Figure 4.1 shows the decline of the initial minority state over time, finally leading to the unanimity among all individuals in the steady state. Reasonably good matches between analytical and simulation results can be observed. Extensive simulations also show that the denser the networks are (quantified by average degree $\langle k \rangle$), the faster the state converges. This observation is in line with the conclusions in [129]. It implies that having more social connections may facilitate exposing a node to the community holding the majority opinion and promote the updating of its state to match the dominant state of its neighbors, resulting in a faster concurrence

Figure 4.1: Temporal fraction of 0-state nodes under classic majority rule on scale-free (SF) and Erdős-Rényi (ER) networks with a size of $N = 20000$, $k_{\min} = 3$, and $k_{\text{cutoff}} = 70$. The simulation results are averaged over 50 independent realizations with error bars representing the standard deviation.

on a denser network.

Under IMR, for a general network with a wide range of nodal degrees, the space of multisets becomes massive and consequently, the calculations of Eq. (4.6) may require heavy computations. For simplicity, we present both simulation and analytical results on ER networks with a relatively narrow nodal degree range where k varies from 5 to 10. As shown in Fig. 4.2, there exists a reasonably good match between theoretical analysis and numerical simulation results, and the vanishing of the initial minority is observed.

4.4 Multiple Coexistence of Steady States in Sparse Networks

The observation that the initial majority opinion tends to completely dominate the whole network in the final state does not hold in sufficiently sparse networks. As illustrated in Fig. 4.3, the state of minority may persist to exist in the steady state where the coexistence of the two states maintains stable.

Figure 4.2: Temporal fraction of 0-state nodes under the influence majority rule on Erdos-Renyi (ER) networks with a size of $N = 20000$, $k_{\min} = 5$, and $k_{\text{cutoff}} = 10$. The simulation results are averaged over 50 independent realizations with error bars representing the standard deviation.

The study in [129] shows that for 4-regular infinite graphs with $k_{\min} \geq 5$, the network state under the classic majority rule shall converge to unanimity. We observe that when $k_{\min} < 5$ and the networks are sufficiently sparse, the coexistence of two states may sustain under both rules. As to why there could be a coexistence of different states, we argue that the main reasons are different for different rules. For CMR, it is because that the interconnected low-degree nodes could form up separating boundaries between communities favoring opposite opinions.

Figure 4.3 shows that the fraction of the minority state decreases over time and finally remains unchanged where the state coexistence persists. This occurs only on sparse networks where low-degree nodes constitute a reasonably large portion of the population. They may thus spread widely across the network with a good chance to be connected to each other, forming up inter-community separating boundaries between communities holding the opposite opinions. In a densely connected network, on the other hand, the network nodes are with higher degrees, giving them a good

Figure 4.3: The fractions of nodes with state 0 in time on a scale-free network with a size of $N = 20000$ and an average degree of $\langle k \rangle = 4.5$ under CMR and IMR, respectively. Each curve corresponds to simulation results on the same network with the same parameters but a different randomly-generated initial set of 0-state nodes.

chance to be connected to many nodes of different states and consequently, quickly adopt the state of majority. A separating boundary formed up by low-degree nodes holding a minority state can hardly sustain under such case.

We show an example of the maintaining of low-degree nodes in a minority group while the group size decreases in time until they form a stable structure in Fig. 4.4. The decline of the average degree, as well as the slope of the best-fitting line, results from the low-degree nodes constituting a larger and larger proportion of the minority community during the dynamic process. This process ends when the population reaches an equilibrium where the two states coexist. Figure 4.5 illustrates such a

coexistence of opposite states in a sparse network. Due to the limited size of the network, only a single community of majority-opinion holders is observed. Nevertheless, we could see that minority-opinion holders are connected to each other and form up their own sustained community.

Figure 4.4: Temporal degree distribution of minority-state nodes from the initial state to the final steady state. A scale-free network of a size of $N = 20000$ and an average degree of $\langle k \rangle = 4.5$ is employed. The fraction of the minority in the initial and final state is 0.45 and 0.2 , respectively. The best-fitting lines (black solid lines) and the 95% confidence intervals for their slopes (CI) are obtained by using the linear least-squares regression.

Under the regime of IMR, as illustrated in Fig. 4.2, global consensus would be achieved in dense networks. In sparse networks, as can be observed in Fig. 4.3 - IMR, majority and minority states coexist. Note that, though the two subfigures of Fig. 4.3 may appear to be similar to each other at the first sight, the coexistences of opposite opinions under the two different majority rules are due to different reasons. Specifically, under IMR, highly connected hub nodes play a critical role. The communities of minority-opinion holders center around some hub nodes while most of the other nodes of the communities are directly linked to those hubs (see Fig. 4.6). These hubs keep the state of their neighbors difficult to be changed according to the rule: In the community of nodes holding the minority state, a low-degree node has a

Figure 4.5: A snapshot of a 500-node network in the steady state governed by CMR. Peripheral nodes of the minority (blue) are mostly of low degrees and connected with each other (via black links), forming up a sustained minority group while being exposed to the majority (green) (via white links). Size of each node reflects its nodal degree.

small number of links. At least one of these links connects to one of the hubs of the minority state while most of the other links possibly connect to nodes in majority communities but not their hubs. Under such case, nodes holding the minority state may persist to exist in sparse networks. In dense networks, on the other hand, a minority-state node even having connection with a hub of its state may still have

Figure 4.6: A snapshot of a 500-node network in the steady state governed by IMR. The two state communities are formed, each containing its own hubs that sustain the persistence of the two communities. Size of each node reflects its nodal degree.

a good chance to connect to one or more high-degree nodes of the majority-state communities, giving it a good chance to change its state. This may lead to a series of events of nodes leaving the minority communities, ultimately causing the vanishing of the minority state.

From the above discussions, we may also expect that, in sparse networks under IMR, a minority-state holder not linked to a minority-state hub may tend to change its state. The communities of the two states shall thus become more and more disassortative until they get stabilized. We verify such is indeed the case by measuring the assortativity of the minority-state communities. Specifically, we remove nodes holding the majority state and then compute the assortativity coefficient of the remaining subgraph using the following equation [160]:

$$r = \frac{\sum_{xy} xy(e_{xy} - a_x b_y)}{\sigma_a \sigma_b}, \quad (4.9)$$

where e_{xy} is the fraction of edges in the graph that connect a node of degree x to another one of degree y , $a_x = \sum_y e_{xy}$, $b_y = \sum_x e_{xy}$, σ_a and σ_b are the standard deviations of the distributions a_x and b_y , respectively. Figure 4.7 shows an example where the assortativity coefficient decreases as opinion evolution goes on.

Figure 4.8 further shows the steady-state fraction of nodes with state 0 in respect of its portion at the beginning. It reveals that the larger proportion the minority opinion initially takes, the wider range the coexistence may finally have. More noticeably, if the initial minority and majority are sufficiently close to each other (including the special case where each of them occupies 50% of the population at the beginning), the initial minority state may indeed have a chance to become the majority state in the end. Our numerical simulation experiences show that this is most likely to happen when the largest hubs hold the minority state at the beginning

Figure 4.7: Temporal assortativity coefficient of 0-state community on a network with a size of $N = 30000$, $k_{\min} = 3$, $k_{\text{cutoff}} = 70$ and $\langle k \rangle = 5$. $F_0(0) = 0.45$. The simulation results are averaged over 50 independent realizations, 10 on each of 5 randomly generated networks, with error bars representing the standard deviation.

of the evolution. Note that the relatively large standard deviations in Fig. 4.7 and the relatively wide ranges of the box plots in Fig. 4.8 mainly come from the existence of multi-steady states, as will be discussed below. Also note that in the ER networks, since there are not obvious hub nodes with very high degrees, the evolution dynamics under IMR appears to be similar to, though not exactly the same as, that under CMR.

Lastly, it is worth pointing out that Fig. 4.3 reveals an important phenomenon that can be observed in sparse networks governed by either of the two majority rules. That is, both rules may drive the systems to *multiple* steady states. Specifically, with the same initial percentage of the minority state with independent random initial state allocation, on the same underlying network, the evolution may end up with different final states capturing various portions of the population. This is mainly due to the randomness of the decline of the minority under the majority effects.

Figure 4.8: Box plots showing the statistical fraction of 0-state nodes in the initial state versus that in the steady state. Simulations are carried out on networks with a size of $N = 100000$, $k_{\min} = 3$, $k_{\text{cutoff}} = 70$ and an average degree of $\langle k \rangle = 4.5$. The results for each $F_0(0)$ are obtained from 50 independent realizations: we generate 5 random scale-free networks and carry out 10 realizations of simulations on each of them, each realization has an independent random allocation of initial state on network nodes.

To be specific, “majority-oriented” individuals facilitates the dominant growth of the initial majority opinion in dense networks, generally resulting in the vanishing of the minority. In a sparse network, groups formed by interconnected minority-opinion holders are more likely to form up and sustain. Such minority-opinion groups, however, could be vulnerable to small fluctuations of the system state, allowing them to be “swallowed” by the majority-opinion communities. For example, under CMR, when a small number of minority-opinion holders change their mind, say, when slightly more than half of their neighbors are majority-opinion holders, cascading effects may cause more minority-opinion holders to change their mind, leading to a significant decline or even total vanishment of a minority community. The vulnerability hence results in the multi-steady states at the end of the system

evolution.

Such vulnerability of minority-opinion communities also exists under IMR, but to a less extent, as it typically takes changing the opinion of a hub to eliminate a minority-opinion community under IMR. This may explain why in sparse networks, minority state tends to occupy a relatively larger portion of nodes at final steady state under IMR than that under CMR, as can be observed in Fig. 4.3.

4.5 Conclusion

We studied the dynamics of social complex networks under the classic majority rule and the influence majority rule, respectively. It is found that under both rules, in dense networks, global consensus could be steadily achieved, while in sparse networks, multi-steady states of opinion coexistence may be observed. Closer observations, however, showed that the coexistence of opposite opinions under the two rules is caused by different reasons: interconnected low-degree minority-state nodes form into their own clusters under the classic majority rule, while hub nodes may play a critical role in forming up different communities under the influence majority rule.

Many future studies could be carried out. As mentioned earlier, we let the influence of each node in a social network be measured by its degree. Other cases, say, where influence of a node is positively correlated to its nodal degree but there is not a simple linear relationship between them, may be worth considering. Further, different nodes may have different resistances to state change. For example, a high-degree hub may be more reluctant to change its state to follow the majority. Effects of such factors shall be of our future research interest.

Chapter 5

Biased Consensus Making and Emergence of Extremism

5.1 Introduction

Emergence of extremism can be seen as one of the most appealing properties of social systems. The coincidence, to a certain extent, between existing mathematical model-based observations and social phenomena triggers more studies to uncover the key factors leading to the formation, the persistence, and even the proliferation of extreme attitudes on social networks. In processes leading to such extremism, the notion of *extreme agents* who intrinsically maintain a low susceptibility to persuasion and high persistence of opinion is typically presented. Such agents are also labeled as zealots, extremists, or inflexibles interchangeably. A few existing studies demonstrate that the presence of extremists may swap a large part or even a whole of the population, causing a social network, with an open conflict, to end up being separated into different or even opposing opinion communities [101, 161–165]. Other studies devote to finding other possible explanations to the emergence of extremism, e.g., by introducing tolerance threshold as a function of opinion, such as [166, 167], or basing mathematical models on well-known social psychology theories such as biased assimilation (see [168]) in studies [169, 170] or group polarization (see [171]) in [172].

Motivated by observations that (i) assuming the pre-existence of extreme opinions does not explain how they emerge at the first place; and (ii) the existence of hatred may not necessarily be the force driving the emergence of extremism, we propose a new approach that allows extremism to emerge from largely “normal” actions that are less drastic than hatred or violence. Specifically, a model is proposed, which may appear to be quite similar to consensus making in the classic bounded-tolerance models, with the only difference that the consensus making could be slightly or significantly biased towards the two ends of the opinion spectrum. That is, instead of agreeing on a central value between two opinions in consensus making as that in the classical Deffuant model, we let right-wing opinion holders agree on an opinion that is to a certain extent biased towards right, while left-wing opinion holders’ consensus tends to be somewhat biased towards left. It is shown that under such case, the normal consensus making, with a certain level of *bias* which may arguably be a part of human nature, may allow the emergence or even prevalence of extremism. This may help explain why extremism ideas can be observed in almost any human societies.

Our studies then further consider the effects of two factors which arguably may be observed in many, though definitely not all, social systems: (i) communities holding opinions closer to the two ends (far-left and far-right) may tend to be more biased towards their respective ends (extremist communities tend to be self-stimulated to be more extreme [168, 169, 173, 174]); and (ii) extremists tend to be less tolerant of different opinions [175, 176]. By slightly extending the basic biased consensus making model to take these two factors into account, we show that the former factor, in fact, offers moderate opinions a better chance to survive compared to that of the case where bias extent is universal across the opinion space. As to the latter factor, generally speaking, it contributes to significantly dwarf the size of the

extremist communities rather than helping them to prevail. Such observations, to a certain extent, may help explain why, though extremism widely exists in most social systems, extreme opinions seldom prevail to become the mainstream opinions of human societies. Brief discussions are also given on why the latter factor, though it hinders the growth of extremism communities, can nevertheless be observed in so many social systems.

The rest of this chapter is organized as follows. The biased consensus making model will be proposed in Section 5.2, followed by some brief theoretical analyses. Main simulation results and discussions will be presented in Sections 5.3, where we shall demonstrate how the simple model under different settings of bias may allow extremism to easily emerge. Also, we will evaluate and have some brief discussions on the effects of the relatively stronger bias and lower tolerance of extreme opinion holders in Section 5.4. Section 5.5 presents results on real-life networks. Finally, Section 5.6 shall conclude the chapter.

5.2 Biased Consensus Making (BCM) Model

5.2.1 Model Formulation

Let $G = (V, E)$ denote a connected undirected graph (network) with a network size N composed of a set of vertices (nodes) $V = \{1, \dots, N\}$ and a set of edges (links) $E \subseteq V \times V$. The sets V and E represent individuals and the social connections between them, respectively. Let $\langle k \rangle = \frac{2|E|}{|V|}$ denote the average nodal degree. At time step $t = 0$, each node $u \in V$ holds a continuous-valued opinion limited in $[0, 1]$, denoted by $o_u(t)$, where values 0 and 1 correspond to the most opposing and most agreeable opinions, respectively, and values in between represent other relatively moderate, including neutral, opinions. At each time step t , a node randomly chosen from V , say u , interacts with one of its neighbors, say v . They communicate and as a result,

may change their current opinions following the updating rule: if their opinions differ an amount larger than a certain tolerance range $d \in [0, 1]$, i.e., $|o_u(t) - o_v(t)| > d$, they maintain their current opinions; otherwise, they adopt a new opinion as follows:

$$\begin{aligned} o_u(t+1) &= o_v(t+1) \\ &= \bar{o}_{u,v}(t) + \delta \frac{|o_u(t) - o_v(t)|}{2} \text{sgn}(\bar{o}_{u,v}(t) - 0.5), \end{aligned} \quad (5.1)$$

where $\bar{o}_{u,v}(t) = \frac{o_u(t) + o_v(t)}{2}$, $\delta \in [0, 1]$ is a function quantifying the relative extent of bias in local consensus making, termed as *bias parameter*. A larger value of δ implies the existence of a stronger bias. In the present study, we investigate two cases of δ :

- $\delta = \alpha$ is a scalar, $\alpha \in [0, 1]$, referred to as *opinion-independent bias* case, where there exists an identical extent of bias over the entire opinion range.
- $\delta = \delta(o_u(t), o_v(t)) = 2\beta |\bar{o}_{u,v}(t) - 0.5|$, $\beta \in [0, 1]$, referred to as *opinion-dependent bias* case, where consensus making interactions on regions closer to the two ends of the opinion axis are more biased, and vice versa.

Note that other models certainly can also be proposed to reflect the non-uniform distribution of bias at different opinion ranges. As the main focus of this chapter is to evaluate the effects of bias in consensus making and such bias may allow extremism to easily emerge and persist, rather than proposing a general framework, studies on more general bias models and their corresponding effects are largely out of the scope of this chapter.

Definition 5.2.1. (Transition matrix) Let $\mathbf{o}(t) = (o_1(t), \dots, o_N(t))^T$ denote the opinion profile at time step t . We call $A(\mathbf{o}(t), t) \in [0, 1]^{N \times N}$ the transition matrix at time step t of the process (5.1) if it is row-stochastic and (5.1) can be rewritten as

$$\mathbf{o}(t+1) = A(\mathbf{o}(t), t)\mathbf{o}(t). \quad (5.2)$$

Opinion profile at time step t can be obtained by recursive iteration

$$\begin{aligned}
\mathbf{o}(t) &= A(\mathbf{o}(t-1), t-1)A(\mathbf{o}(t-2), t-2)\dots A(\mathbf{o}(0), 0)\mathbf{o}(0) \\
&= A_C(0, t)\mathbf{o}(0),
\end{aligned} \tag{5.3}$$

where $A_C(t_0, t_1) := A(\mathbf{o}(t_1-1), t_1-1)A(\mathbf{o}(t_1-2), t_1-2)\dots A(\mathbf{o}(t_0), t_0)$ is defined as the *accumulation matrix* from time step t_0 to t_1 .

5.2.2 Convergence of the Biased Consensus Making Model

In this section, we consider the convergence of the model (5.1) in complete graphs.

We first present the following lemma proved in [102].

Lemma 5.2.2. *Define $A(t) := A(\mathbf{o}(t), t)$ and let $(A(t) \in [0, 1]^{N \times N})_{t \in \mathbb{N}_0}$ be the series of transition matrices. If each matrix of the series satisfies the three following properties:*

1. *The diagonal of $A(t)$ is positive,*
2. *Zero-entries of $A(t)$ are symmetric,*
3. *The positive entries of $A(t)$ do not converge to zero,*

then, there exists a time step t_0 and pairwise disjoint clusters of agents $C_1 \cup \dots \cup C_p = V$ such that

$$\lim_{t \rightarrow \infty} A_C(0, t) = \text{diag}(K_1, \dots, K_p)A_C(0, t_0), \tag{5.4}$$

where $K_1, \dots, K_p$ are consensus matrices with the sizes of $C_1, \dots, C_p$, respectively, and the multiplication of K_i with an arbitrary vector gives a vector of equal entries. (Note that to have the block diagonal form, the matrix indices are sorted according to $C_1, \dots, C_p$.)

Theorem 5.2.3. (Convergence of the BCM model) *Consider the biased consensus making model (5.1). For any initial state $\mathbf{o}(0) \in [0, 1]^N$, $\mathbf{o}(t)$ converges to a constant opinion profile $\mathbf{o}^\infty \in [0, 1]^N$ as $t \rightarrow \infty$ where $\mathbf{o}_i^\infty = \mathbf{o}_j^\infty$ or $|\mathbf{o}_i^\infty - \mathbf{o}_j^\infty| > d$ for all $i \neq j$.*

Proof. Following the rule (5.1), a pair of agents u and v is chosen at random at time step t . We assume, without loss of generality, $o_u(t) \leq o_v(t)$. Then, Eq. (5.1) can be rewritten as follows:

$$\begin{aligned}
o_u(t+1) &= o_v(t+1) \\
&= \left(\frac{1}{2} - \frac{\delta}{2} \text{sgn}(\bar{o}_{u,v}(t) - \frac{1}{2})\right)o_u(t) + \left(\frac{1}{2} + \frac{\delta}{2} \text{sgn}(\bar{o}_{u,v}(t) - \frac{1}{2})\right)o_v(t)
\end{aligned} \tag{5.5}$$

The transition matrix $A(\mathbf{o}(t), t)$ of process (5.1) can then be constructed:

$$A(\mathbf{o}(t), t) = \begin{cases} I_N & \text{if } |o_u(t) - o_v(t)| > d \\ A^*(\mathbf{o}(t), t) & \text{if } |o_u(t) - o_v(t)| \leq d \end{cases}, \quad (5.6)$$

where I_N is an identity matrix of size N and the matrix $A^*(\mathbf{o}(t), t) \in [0, 1]^{N \times N}$ is defined by:

$$A_{ij}^*(\mathbf{o}(t), t) = \begin{cases} \frac{1}{2} - \frac{\delta}{2} \text{sgn}(\bar{o}_{u,v}(t) - \frac{1}{2}) & \text{if } j = u, i \in \{u, v\} \\ \frac{1}{2} + \frac{\delta}{2} \text{sgn}(\bar{o}_{u,v}(t) - \frac{1}{2}) & \text{if } j = v, i \in \{u, v\} \\ 1 & \text{if } i = j \notin \{u, v\} \\ 0 & \text{if } i \neq j, \{i, j\} \neq \{u, v\} \end{cases}. \quad (5.7)$$

For $\delta \in [0, 1]$, $A(\mathbf{o}(t), t)$ has all the three properties presented in Lemma 5.2.2. It follows that multiplication of K_i with an arbitrary vector gives a vector of equal entries. This result implies that the agents' opinions evolve until a certain time step t_0 when the agents group into independent clusters, each converging to the same limiting opinion.

In addition, off-block diagonal entries of $\text{diag}(K_1, \dots, K_k)$ are zeros, implying that agents in each cluster C_i do not exchange opinions with any agents of other clusters. This state persists when all the clusters C_i pairwise have opinions differing by an amount larger than d . This completes the proof. $\square$

5.2.3 Dynamics Equation of Opinion Evolution

Denote the probability density of opinion o at time step t as $P_o(t)$. Adopting the mean-field method, we derive the dynamics equation of $P_o(t)$ as follows:

$$\begin{aligned} & \frac{dP_o(t)}{dt} \\ &= \iint_{|o_1 - o_2| \leq d} P_{o_1}(t) P_{o_2}(t) [\sigma(o - \frac{o_1 + o_2}{2} - \delta \frac{|o_1 - o_2|}{2} \text{sgn}(\frac{o_1 + o_2}{2} - 0.5)) - \sigma(o - o_1)] do_1 do_2, \end{aligned} \quad (5.8)$$

where $\sigma(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$. The equation is not amenable of simple solutions.

By numerical calculations of the equation, we could produce the temporal opinion evolution and obtain the quasi-steady state as the time is sufficiently long. We show a well matching between simulation and theoretical results in Sec. 5.3.

5.3 Effects of Bias in Opinion Cluster Formation

We start by showing the temporal evolution of opinion distributions under different values of α in Fig. 5.1. The results for $\alpha = 0$ is equivalent to those of the classical Deffuant model with a convergence rate – the rate of two agents moving towards each other in an opinion space in a local consensus making – of 0.5 [177].

In the case, all nodes converge to 5 major opinion clusters (which is roughly equal to $\frac{1}{2d}$ [177,178]) interpreted as communities of agents holding similar opinions.

We can observe that with an increasing α value, the major clusters tend to shift to the two ends of the opinion axis. We hereafter refer to node bearing these opinions right at, or relatively nearby, the two ends as extremists. The bias in local interactions leads to the existence of these opinions, concurrently the decline in size of clusters of moderate opinions (intermediate opinions).

Figure 5.1: Illustration of temporal distributions of opinions of 20000 agents for $d = 0.1$, $\alpha = 0$ (a), 0.5 (b), and 1 (c). Each time step, at which the temporary opinion distribution is recorded, corresponds to 20000 pair-wise interactions.

Figure 5.2 depicts, for the case with opinion-independent bias, the opinion distributions at equilibrium where the opinion evolution has come to an end. As can be observed, the opinions in the steady state diverge into several discontinuous regions holding different opinions. These regions are generally pushed towards the two ends of the opinion axis due to the effects of the bias. Figure 5.2 – *middle row* shows the co-existence of the two highest peaks corresponding to the co-existence of two major communities acquiring the majority of the population. Both of these communities hold moderate opinions until the bias in local interactions causes the

polarized peaks to shift to the two ends. It is of interest to observe that such microscopic/local bias may eventually cause the extremism in a macro/global scope. Noticeably, even when $d = 0.3$, which is sufficiently large for the majority of the population to converge to the single cluster of neutral opinion where there is no bias in opinion formation [178], the existence of bias, when it is strong enough, may separate the peak into two shifting to the two ends, as can be observed in Fig. 5.2 – *bottom row*. Besides the general trend of agents shifting to the extremities under relatively high extents of bias, another common observation for the three cases of different d values is the gradual vanishment of intermediate opinions, most noticeably the disappearance of the neutral opinion around 0.5. Note that Fig. 5.2 shows the results for both the ER and SF networks. As stated earlier, all main conclusions remain the same in the two networks.

We shall then study the effects of the function δ in opinion formation with opinion-dependent bias, where different system dynamics emerge. Specifically, it is found that while the unbalance in local consensus making still pushes the clusters towards the two ends, having an opinion-dependent bias may lower the chance that extremist clusters can dominate the society. Observing Fig. 5.2 and Fig. 5.3, we could see that, when extremists are keeping their bias level largely unchanged, lowering the bias level among the rest of the population gives moderate opinion clusters a much higher chance to survive or even prevail in the final steady state.

This can be explained. For the case where extremists have a stronger bias in the opinion-dependent bias model, the term $|\bar{o} - 0.5|$ lets opinion formation closer to the two ends of the opinion axis have a faster speed shifting towards their respective ends than the other opinion formations in between. This difference in shifting speed may quickly enlarge the distance, measured on the opinion axis, between extremists and other moderate opinion holders. Once this distance is larger than the toler-

ance range, extremists can no longer drag more moderate opinion holders to change their opinions. Such speed asynchronization and the consequent quick isolation of extremist clusters give moderate opinion clusters a better chance to sustain, or even prevail in some cases.

An interesting observation is since the quick isolation of extremist clusters may become a weaker effect when the tolerance range d is relatively large, in a population that is relatively more *open-minded*, if the so-called open-mindedness is revealed as having a larger tolerance range rather than a weaker bias, extremist clusters may actually have a better chance to prevail when the bias extent is sufficiently large. Figure 5.3 – *bottom row* illustrates such a phenomenon where two polarized peaks dominate a system with a large tolerance range of $d = 0.3$. Note that theoretical analysis results are also presented in Fig. 5.3, which have a reasonably good match with simulation results.

Figure 5.2: Steady-state opinion distribution for $\alpha = 0, 0.5$ and 0.8 (left, middle, and right column panels, respectively) and $d = 0.1, 0.2$, and 0.3 (top, middle, and bottom row panels, respectively) in ER (blue) and SF (red) networks with a size $N = 2 \times 10^4$ and an average degree $\langle k \rangle = 20$.

Figure 5.3: Steady-state opinion distribution for $\beta = 0, 0.5$, and 0.8 (left, middle, and right column panels, respectively) and ranges $d = 0.1, 0.2$, and 0.3 (top row, middle row, and bottom row panels, respectively) in ER networks with a size $N = 2 \times 10^4$ and an average degree $\langle k \rangle = 20$. Both simulation (symbols) and theoretical (solid lines) results are presented.

Figure 5.3 also reveals that the local bias makes it more difficult for a social network to reach a complete consensus. We demonstrate in Fig. 5.4 that the local bias prolongs the global polarization state (with respect to d) in which the two largest clusters holding roughly symmetric opposing and favoring opinions coexisting in the steady state. As can be seen, the polarization-to-consensus transition point shifts to the right in respect of d -axis with an increasing value of β , meaning that in a system with a stronger bias, it requires a larger tolerance range to achieve system-wide complete consensus.

5.4 Heterogeneous Tolerance Range

Intuitively, due to the intrinsic characteristic of the attitude resistance of extremists, it is relatively more difficult for them to have *effective* contacts that may lead to a change in their current attitude. Inferentially, a person holding a stronger stance

Figure 5.4: Sizes of largest (circles) and second-largest (triangles) steady-state clusters in ER networks with a size $N = 2 \times 10^4$ and an average degree $\langle k \rangle = 20$.

tends to have a lower tolerance level. Such observations have been supported by quite a few studies on social psychology, e.g., [175, 176, 179]. It is of interest to investigate the effects of the extremists' lower tolerance on the growth of their clusters. We hereby adopt an idea proposed by [167] to quantify the tolerance range of an individual according to his/her current state. Specifically, we let an agent holding an extreme opinion have a narrower range of tolerance, and vice versa. As such, two agents would undergo an effective interaction only if the difference between their current opinions is within each other's tolerance ranges, or in other words, the difference is not bigger than the narrower one of the two agents' tolerance ranges. The heterogeneous tolerance is based on the assumption that people holding extreme opinions tend to be more convinced while those who have moderate initial opinion express the uncertainty [167]. It is worth noting that in [167], the heterogeneous tolerance is taken as a key force driving the emergence of extremism while in our model, we do not hypothesize it as a factor causing extremism but adopt it as a possible explanation of why extremist groups are not so popular in most cases. The results will be detailed in the following parts of this chapter.

A simple formula of heterogeneous tolerance is introduced as shown below, where d is a symmetric function of o with respect to $o = 0.5$ with the minimum and maximum values of d being $d_{\min}$ and $d_{\max}$, respectively:

$$d(o) = 2(d_{\min} - d_{\max})|o - 0.5| + d_{\max}. \quad (5.9)$$

Figure 5.5: Steady-state opinion distribution versus β in a network of 2×10^4 agents with an average degree $\langle k \rangle = 20$. The tolerance range is a symmetric function of opinion with respect to $o = 0.5$ with $d_{\min}, d_{\max} = 0.03, 0.3$, respectively.

Figure 5.5 shows that, when extremists have a narrower tolerance range, the size of their clusters may become significantly smaller than those of the moderate-opinion holders. This reveals that the extremists' lower tolerance towards different opinions may be one of the main reasons why, though we frequently witness the existence of extremist groups, we seldom see them prevail.

It is interesting to observe that having a lower tolerance level dwarfs extremist clusters so significantly yet most, if not all, extremist groups do have relatively small tolerance ranges in opinion formation. Why don't these groups adopt a different strategy to become more tolerant, in which case they may have a much better chance to grow bigger and stronger? The answer may be that though having a larger

tolerance range definitely helps cluster growth in opinion formation by consensus making, a more tolerant cluster may however unavoidably also be more tolerant towards other types of opinion changes (also known as mutation) [113, 180, 181]. The effects of such changes could be significant, especially when the cluster requests extensive devotion from its members, emotionally, physically and/or financially. Tolerating such opinion changes may allow the cluster to be quickly diminished under certain circumstances [182]. The best strategy for an extremist cluster to sustain, therefore, may be maintaining a low tolerance against any “unfavorable” opinion changes, even imposing penalties and punishments on such changes when necessary. Further studies are certainly needed to verify the above hypothesis.

5.5 Numerical Simulations on Real-life Networks

In this section, we examine the opinion dynamics on networks retrieved from the real world. Ego networks are of a typical topology in real-life social networks [183, 184], each consisting of a focal node (ego) connected to its neighbors (alters) and other links among those alters. We carry out simulations on a network of 10 connected ego networks extracted from Facebook [185]. The network has 4039 nodes and 88234 links in total.

Figure 5.6 shows the temporal evolution of opinions in the network under the influences of opinion-dependent bias and heterogeneous tolerance. As that in Sec. 5.4, we set $d_{\min} = 0.03$ and $d_{\max} = 0.3$, respectively. Two different cases with different β values reveal similar observations: the peaks closer to the two ends of the opinion axis quickly form up, largely due to their smaller tolerance ranges, stop adopting newcomers, and maintain their sizes while centrist opinions keep evolving.

Most of these moderates finally end up gathering together in the relatively large cluster holding opinions around 0.5. As $d_{\max} = 0.3$ is a relatively large value, only a

single big cluster emerges in the moderate opinion region. With a smaller $d_{\max}$ value, multiple moderate clusters would be formed up. More noticeably, the presence of the bias causes a significant increase in the sizes of the extremist clusters compared to that in the classical case ($\beta = 0$). These observations resonate with those in Sec. 5.4. Other cases with different parameter values have also been tested. Basically, all conclusions holding for synthetic networks remain valid in real-life networks.

Note that the network used in this experiment has a strong community (modularity) structure, where each constituent ego network has a relatively denser internal connectivity than that among different ego networks. This causes a higher frequency of intra-ego network interactions, resulting in emergence of extremism in each component ego network. We demonstrate such an observation in Fig. 5.7.

Figure 5.6: Time evolution of opinions in ego network for $\beta = 0$ (top panels) and $\beta = 0.5$ (bottom panels). The setting of d is as that in Sec. 5.4.

Figure 5.7: Steady-state opinion distributions in the three largest ego networks for $\beta = 0.5$.

5.6 Conclusion

We examined the effects of bias in interactions and emergence of extremism in complex networks. When the bias parameter is a scalar identical across the opinion spectrum, all nodes tend to move toward the two opinion extremities at a relatively similar pace, resulting in them shifting to the utmost opinions together and finally form separate clusters then. The degree of extremism of these clusters depends on the bias extent in pair-wise consensus making. The agents are under the effects of both consensus making and biasing and therefore, they do not form up into two clusters close to the two opinion extremities completely but rather diverge to clusters that distance from each other an amount larger than d . When the bias is formulated as a function of opinion, i.e., the further from the neutral viewpoint where the interaction happens, the more biased toward the corresponding extremity the consensus making is. The opinion-dependent bias parameter, in one way, leads to the emergence of the extremism, and in another way, isolates these extreme communities from the rest of the population holding moderate opinions. Also, due to the co-existence of moderate and extreme views, it makes the social networks harder to reach the state of global consensus.

We then set the tolerance range as a function of opinion based on the idea that

a node with an extreme view has a narrow tolerance range for social contacts, and conversely. It causes the descending popularity of extremity at equilibrium. This functional range may provide one more explanation to the question of why bias in local compromise may lead to the emergence of extremism, yet its proliferation.

Chapter 6

Opinion Formation in Multicultural Social Networks

6.1 Introduction

In Chapters 3, 4 and 5, we have mainly studied two different models and their extensions, namely the consensus making rule and majority game rule, respectively. We may argue that such different rules reflect different tendencies or, to a certain extent, even different cultures in social opinion formation. Specifically, in a majority game, individuals may strongly feel the existence and the impacts of the local majority opinion and tend to stay with the majority as much as such is possible, while in a consensus making interaction, individuals largely ignore the influences of majority opinions and focus on each specific social interaction. A question may naturally arise: what if some or all individuals are under strong influences of the local majority opinion while achieving consensus making in each specific opinion evolution in social interactions?

To investigate the answer to such a problem, for the first time to the best of our knowledge, we propose a model of opinion formation model under interplay of consensus making and majority game (termed as Majority Oriented Social inTeraction or MOST rule hereafter for convenience of discussion). In this model, an individual A may achieve consensus making with an individual B if and only if the opinion

of individual B is within the tolerance ranges of most of individual A 's neighbors; in other words, consensus making between A and B would happen only when individual B is "tolerated-able" for most of individual A 's social connections, while the significance of the difference between their opinions is largely ignored. We found that under such case, when agents in a population have a relatively small tolerance threshold, the influence of majority may drive the opinion system to a state of severe fragmentation where agents tend to only slightly change or largely maintain their initial opinions. Conversely, when the tolerance threshold becomes reasonably larger, the majority rule may promote the population to gain a global consensus.

We then move forward to consider a hybrid model where a certain percentage of individuals adopt MOST rule in their opinion changing, while the other nodes follow the standard consensus making rule. It is found that similar to that under the MOST rule model, the system behaves in a non-monotonic fashion: the presence of majority-oriented agents promotes a universal consent only if the population is relatively open-minded with a relatively large tolerance range; otherwise, even consensus making in small local groups may be prevented. Noticeably, even when agents gather into diverse opinion clusters (opinion communities), there may be a significant proportion of clusters with coexistence of members following the two different rules.

The rest part of this chapter is organized as follows. Section 6.2 will examine MOST model. Section 6.3 studies the hybrid model. Finally, Section 6.4 concludes the chapter and discusses a few future works.

6.2 Majority Oriented Social Interaction Model

6.2.1 Model Definition

Nodes interact on an undirected graph $G = (V, E)$. At time t , a randomly chosen link between two nodes u and v becomes active and the two end nodes interact. Independently, each node, say u , considers the opinions of its immediate neighbors and changes its opinion to be closer to that of the other end node, say v , if and only if for the majority of u 's neighbors, v 's opinion is within their tolerance ranges. The mathematical description of u 's opinion updating rule is as follows:

$$o_u(t+1) = \begin{cases} o_u(t) + \mu(o_v(t) - o_u(t)), & \text{if } |\{w | |o_w(t) - o_v(t)| \leq d, w \in \eta(u)\}| > \frac{|\eta(u)|}{2} \\ o_u(t), & \text{otherwise.} \end{cases} \quad (6.1)$$

It has been confirmed that the value of μ does not affect the final configuration of the opinions; hence we fix μ at 0.5.

6.2.2 Simulation Results

We here present the model dynamics based on numerical simulations. We start by showing the time evolution of the opinion distribution on ER networks. The initial opinions follow a uniform distribution.

Figure 6.1 depicts the gradual convergence of opinions to the central value for relative large values of d . When the tolerance range is sufficiently narrow, as we could see in Fig. 6.1(a), the severe opinion diversity is retained for a massively long evolution. It is interesting to observe that in a society with strong impacts of majority rules and rather limited tolerance, formation of opinion communities holding their respective community consensus, which can be steadily observed when the system is under the consensus making rule, may be seriously depressed. It can

Figure 6.1: Time evolution of opinions in an ER networks with $N = 20000$ and $\langle k \rangle = 20$.

be explained that in such less open-minded networks, a node may find it hard to achieve consensus making with a connected node as that node's opinion may not be within the tolerance ranges of the majority of its neighbors. Conversely, if the population is sufficiently open-minded, they are easy to move closer to each other. The majority rule may eventually drive almost the whole system to adopt a single consensus.

We show the tipping point in the d -domain at which the network turns from fragmentation to complete consensus, denoted by d_T , in Fig. 6.2. Our intensive simulations confirm that the value is around $d_T = 0.13$, approximately equaling the half of the transition point in the classical Deffuant model [178]. As the analysis to obtain an accurate value of d_T remains challenging, we shall leave it to our future studies.

Figure 6.2: Critical tolerance threshold in a continuous-valued opinion network under majority rule. The ER networks with various sizes and $\langle k \rangle = 20$ are used.

6.3 Mixed Rules Model

6.3.1 Model Description

We partition the network population into two sets, named as *majority-oriented group* and *consensus making-oriented group* denoted by $V_M \subset V$ and $V_C \subset V$, respectively. The relative density of the two types of nodes is quantified by $\nu = \frac{|V_M|}{|V_M| + |V_C|}$.

To briefly recall the rule of consensus making in a single network following the Deffuant rule only: at time step t , if a randomly selected edge $e = \{u, v\}$ is active, the opinions held by u and v at time step $t + 1$ are determined by the following rule: if $|o(u, t) - o(v, t)| \leq d$, then the two nodes make consensus:

$$\begin{aligned} o(u, t + 1) &= o(u, t) + \mu(o(v, t) - o(u, t)), \\ o(v, t + 1) &= o(v, t) + \mu(o(u, t) - o(v, t)). \end{aligned} \quad (6.2)$$

In a pairwise interaction, a node follows its own rule depending on its pre-assigned characteristic. Specifically, each node with its consistent intrinsic personality/characteristic either gain a consensus with a social partner who shares a sufficiently similar opinion as described in the rule above or follow MOST rule, as defined in Eq. (6.1). Note that two interacting nodes behave independently. For

example, when node $u \in V_M$ interacts with node $v \in V_C$, it is possible that node v changes its opinion to be closer to that of node u , while node u maintains its opinion unchanged following MOST rule.

6.3.2 Simulation Results

Figure 6.3: Steady-state opinion distribution in ER networks with $N = 20000$ and $\langle k \rangle = 20$ in various mixing ratio.

We let the opinions start with a uniform distribution and evolve until the steady state is reached. Figure 6.3 shows the complex effects of ν in opinion formation. As discussed earlier, for $d < d_T$, e.g., $d = 0.1$, the majority rule tends to diversify the opinions. This effect still holds in networks sustaining the mixing rules. As can be seen in Fig. 6.3–($d = 0.1$), when $\nu = 0$ (classical Deffuant model), there exists 5 major steady opinion clusters. With an increasing value of ν , however, the opinion distribution is flattened.

In contrast, in a population that is relatively open-minded, i.e., when $d > d_T$, the majority effect promotes the emergence of a global consensus at opinion around 0.5 (the mean of the initial opinion distribution). For instance, for $d = 0.2$ and $\nu = 0$, all individuals have compromise preference and the network finally diverges to two

major clusters. With an increasing proportion of majority-oriented agents, these clusters eventually merge into a single major cluster around opinion 0.5. In short, in a network with relatively large tolerance, all individuals tend to adopt universal consent, and more importantly, the presence of majority-oriented entities enhances such effects.

We further evaluate the extent of convergence to unanimity versus various mixing proportions of the two rules. Specifically, we count those nodes holding a state of $0.5 \pm \epsilon$ in the final steady state as being already converged to unanimity. In this chapter, ϵ is set at 10^{-3} . Also, in the scope of this chapter, we define the critical tolerance threshold as the threshold value at which 99% of the population converges to unanimity. Figure 6.4 reveals that when the majority nodes constitute more than half of the network size, d_c starts to decline. In other words, having a larger portion of majority-oriented individual eases the convergence to unanimity. Such observations support the earlier argument. It is worth noting that though the correlation of d_c and ν as shown in Fig. 6.4 basically remains consistent for different network sizes, the critical value of ν at which d_c starts to decrease fluctuates in different rounds of simulation. Obtaining an equation for estimating critical value of ν and achieving in-depth understanding of system dynamics when ν is larger than the critical value remains as challenges. Such topics will be tackled in our future studies.

Figure 6.4: Critical tolerance threshold versus ν .

Finally, we take a closer look at the structure of each opinion cluster. An opinion cluster is a connected subgraph formed by nodes holding states differing by only a relatively small value. We fix this value at 0.01 and obtain opinion clusters by removing links between nodes holding opinions with a difference larger than that. Values of local ν (i.e., the value of ν for each separate cluster) for the separate opinion clusters are computed and the histograms of their distributions are presented in Fig. 6.5.

Figure 6.5: Histogram of local ν in separate opinion clusters under different global values of ν and $d = 0.1$.

It shows that each opinion cluster may accommodate individuals favoring different opinion formation rules, which may imply the feasibility of coexistence of people with different cultural backgrounds following different rules of opinion formation, as long as these individuals are holding similar opinions. Our model may help better understand how people of diverse cultures live together to form up relatively stable social systems.

6.4 Conclusions

In this chapter, our studies show that when two effects, namely consensus making between interacting individuals and majority opinions in neighborhood, come in together to have an interplay on opinion formation, some interesting new dynamics may emerge. Further, if different portions of a population follow different rules in their opinion formation, say, due to the different cultural background they have, system dynamics may become more complex, and arguably, more interesting as well.

Future studies may be carried out on analyzing the critical value d_T , and for evaluating the cases where people following different rules are not randomly distributed at the first place. Besides, there are many different ways to take into account the effects of both the majority opinion and the bounded confidence. It would be very interesting to investigate a few different such models and see what different results we may get, and which ones of these results better resemble some real-life observations. What may be even more interesting is to investigate whether a general framework could be proposed for a whole class of different models with general rules and guidelines revealing how opinions would evolve in such systems. Such works would be of our future research interest.

Chapter 7

Conclusions and Future Work

7.1 Major Contributions

In this thesis, we carried out studies on dynamics of opinion formation on complex social networks. The main observations made in the studies can be summarized as follows:

- i. We firstly investigated the opinion dynamics on multiplex networks of two identical layers. It is found that the presence of multiplexity in the underlying networks either enhances or diminishes the consensus formation compared to the case of single, non-interacting networks. The specific effects depend on the relative magnitudes of the tolerance ranges on each layer. A global consensus state where the entire population share the same opinion is reached if the sum of two tolerance ranges is equal to 1, a double of the critical range in opinion systems on single networks.
- ii. The majority rules in binary opinion systems are analyzed. For the first time, to the best of our knowledge, the social influence of agents is taken into account. Both the classical and variant rules allow achieving a global consensus in relatively dense networks while in sparse networks, the two different rules sustain different mechanisms to maintain the multiple co-existences of states at equilibrium.

- iii. Emergence of extremism is studied. A simple factor of bias in pairwise social interactions is examined on the top of the bounded-confidence model. It is revealed that the proliferation of extremism may be rooted from microscopic bias in interactions. Furthermore, extremism-oriented entities' low tolerance level could, in one way, ensure the persistence and in the other way, dwarf the dominance of extremist groups.
- iv. Dynamics of opinion formation under the interplay of consensus formation and majority rule is examined. The network is partitioned into two groups following either of the rules. It is shown that if the tolerance range is relatively large, majority rule promotes a complete consensus across the population; otherwise, counter-intuitively to a certain extent, majority rule may drive public opinions to the state of severe fragmentation.

7.2 Future Research Direction

Besides the extensions we have discussed in Chapters 3-6, our future research may extend into several directions as follows:

- Individuals with a different number of connections may have different social influences. Therefore, the degree heterogeneity may have significant effects on opinion formation ruled by pair-wise consensus making. Further investigations of the influences of degree heterogeneity may be of importance.
- Opinion dynamics on adaptive multiplex networks. It is based on the fact that real-life networks and systems are not always static during the evolution of dynamic processes. Co-evolution of multiplex networks and opinion systems is therefore of interest.
- Opinion formation in multiplex social systems where opinion differences on

different layers play certain more sophisticated roles in consensus making and opinion polarization. For example, we may consider the case where opinion differences on different layers are of different importance to involved individuals.

Author's Publications

Journal Publications

1. Vu Xuan Nguyen, Gaoxi Xiao, Xin-Jian Xu, Guoqi Li, and Zhen Wang. Opinion formation on multiplex scale-free networks. *EPL (Europhysics Letters)*, 121(2):26002, 2018.
2. Vu Xuan Nguyen, Gaoxi Xiao, Xin-Jian Xu, Qingchu Wu, and Cheng-Yi Xia. Dynamics of opinion formation under majority rules on complex social networks. *Scientific Reports*, 10(1):1-9, 2020.

Conference Publications

1. Vu Xuan Nguyen and Gaoxi Xiao. Global consensus making on multiplex scale-free networks. In *2017 6th International Conference on Computer Science and Network Technology (ICCSNT)*, pages 347–351. IEEE, 2017.
2. Yi Yu, Vu Xuan Nguyen, and Gaoxi Xiao. Effects of initial state on opinion formation in complex social networks with noises. *Proc. ICCSCSS'2018*, Oct. 2018.

Papers Currently under Review

1. Vu Xuan Nguyen, Gaoxi Xiao, Jie Zhou, Guoqi Li, and Beibei Li. Bias in social interactions and emergence of extremism in complex social networks.

manuscript submitted for publication, 2019.

Papers Currently under Preparation

1. Vu Xuan Nguyen, Lock Yue Chew, and Gaoxi Xiao. Opinion dynamics under the interplay of consensus making and majority rule. manuscript under preparation.

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