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Slow viscous flow of two porous spherical particles translating along the axis of a cylinder

Xin Yao, Chyi Huey Ng, Jia Rui Amanda Teo, Marcos[†] and Teck Neng Wong

School of Mechanical and Aerospace Engineering, Nanyang Technological University,
Singapore 639798, Singapore

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This article describes the motion of two freely moving porous spherical particles located along the axis of a cylindrical tube with background Poiseuille flow at low Reynolds number. The stream function and a framework based on cylindrical harmonics are adopted to solve the flow field around the particles and the flow within the tube, respectively. The two solutions are employed in an iterated framework using the method of reflections. We first consider the case of two identical particles, followed by two particles with different dimensions. In both cases, the drag force coefficients of the particles are solved as functions of the separation distance between the particles and the permeability of the particles. The detailed flow field in the vicinity of the two particles is investigated by plotting the streamlines and velocity contours. We find that the particle-particle interaction is dependent on the separation distance, particle sizes and permeability of the particles. Our analysis reveals that when the permeability of the particles is large, the streamlines are more parallel and the particle-particle interaction has less effect on the particle motion. We further show that a smaller permeability and bigger particle size generally tend to squeeze the streamlines and velocity contour towards the wall.

Key words: Particle/fluid flow, Stokesian dynamics, Porous media

1. Introduction

The systematic investigation of particles at low Reynolds number involving both analytical and numerical studies traces back to the work of Brenner (1963, 1964) and O'Neill (1968, 1969), which have provided a comprehensive review of the literature and various techniques for viscous flow containing solid particles. Stokes flow with more particles was analyzed by Mazur & van Saarloos (1982) and Kim (1987) analytically, and Wilson (2013) numerically. The presence of a boundary significantly affects the flow behavior (Shen *et al.* 2017). A large body of literature has considered flow behavior with boundary effect, including the general solutions under different background flows (Lamb & Caflisch 1993) and extra forces acting on the fluid (Marcos *et al.* 2005*a,b*; Feng *et al.* 2016; Huang *et al.* 2016, 2017). One typical example of a commonly encountered boundary is a cylindrical tube. The motion of small particles in a cylindrical tube at low Reynolds number is of importance to many fields, such as drug delivery through blood capillaries, lubrication theory, microfluidic and nanofluidic applications in biotechnology

[†] Email address for correspondence: marcos@ntu.edu.sg

and chemical reaction engineering (Ho & Tai 1998; Bhattacharya *et al.* 2010; Marcos *et al.* 2014; Koh & Marcos 2014). The motion of a single particle traveling in a cylindrical tube had been investigated by Brenner & Happel (1958), Hasimoto (1976), Greenstein & Happel (1968) and Bhattacharya *et al.* (2010). When there are two or more particles traveling in the tube, the flow field is more complex due to particle-particle interaction (Zhong & Duan 2016; Zhong *et al.* 2017). The motion of two or more spherical particles translating along the axis of the tube has been investigated by Sonshine & Brenner (1966), Wang & Skalak (1969), Leichtberg *et al.* (1976) and spheroidal particles by Chen & Skalak (1970). Navardi & Bhattacharya (2013) calculated the hydrodynamic forces on two particles traveling in a cylindrical tube. The drag force and torque tensors were obtained. A related result using the same methodology was presented by Navardi *et al.* (2015), who gave the Stokesian simulation of two particles freely moving in a cylinder.

The flow field around a moving porous particle depends on its permeability. When the particle has zero permeability, the fluid has to travel around the particle. When the permeability is nonzero, the fluid can pass through the particle. Many materials, such as agarose or hydrogel, are highly permeable. They are typically used as a membrane for separation devices (Kapur *et al.* 1996; Mika & Childs 2001; Buehler & Anderson 2002). Agarose has also been fabricated in nanoscale or microscale for bioengineering and energy storage applications (Chudasama *et al.* 2016; Kim *et al.* 2017). Permeable particles could be modeled as porous materials. The flow within the porous region can be described using the Darcy equation or the Brinkman equation while the flow of the fluid in the cylinder is governed by the steady-state Stokes equation. Darcy Law states that the fluid flow is linearly proportional to the pressure gradient and the gravitational force (Mokadam 1961). However, the Darcy equation is not compatible with the existence of a boundary layer region (Beavers & Joseph 1967; Neale & Nader 1974). The Brinkman equation, however, has been found to be consistent with the predictions made by applying Darcy Law without the need to resort to empirical boundary conditions. The validity of the Brinkman equation has been extensively treated in literature (Nield & Bejan 2006). Though there are several limitations, the Brinkman equation with continuity of velocity and normal stress at the porous interface is believed to be a successful model of steady Stokes flow past highly permeable particles (Prakash & Sekhar 2012). Higdon & Kojima (1981) derived asymptotic results for Stokes flow passing a porous particle of large and small permeability with arbitrary geometry using the Brinkman equation. Padmavathi *et al.* (1993) derived a solution for non-axisymmetric Stokes flow passing through a stationary porous sphere. Neale *et al.* (1973) discussed various possible solutions to the problem of unbounded creeping flow passing through a porous sphere and generalized the solution to cover a swarm of spheres. Vainshtein *et al.* (2002) examined the flow around and within an isolated porous spheroid. Mo & Sangani (1994) proposed a method to compute Stokes flow interactions among spherical objects, including the case for N porous particles within a periodic cubic unit cell.

Studies on porous particles translating in a cylindrical channel have been more limited. Saad & Faltas (2014) solved the flow of a single porous particle in a cylindrical channel using an analytical-numerical method, constructing a general solution from the superposition of the fundamental solutions in both cylindrical and spherical coordinate systems and solving the boundary conditions using a collocation technique. To the best of our knowledge, there is no literature on the solution to the problem of creeping flow passing and within two or more porous spherical particles translating in a cylinder.

This paper details an semi-analytical approach to solve for the flow of two freely translating porous particles in a viscous fluid along the axis of a cylindrical tube. The flow inside and outside the porous particles are governed by the Brinkman equation

FIGURE 1. Schematics of two porous spheres translating along the axis of a cylindrical tube with radius R_0 . Two spheres, A and B with radius a and b respectively, are placed along the axis of the tube. The background flow field of the cylinder has a parabolic profile with the maximum velocity U_0 at the axis. L is the separation distance between the centers of spheres A and B .

and the Stokes equation, respectively. The fluid passing through the porous material will induce drag forces acting on the two particles. The calculation is performed in an iterated framework using the method of reflections. We analyzed the hydrodynamic forces and velocities of the two particles, which are dependent on the distance between the two particles and particle sizes. This information can give a better understanding of the interaction between porous particles.

2. Problem statement

We consider two porous spherical particles translating freely along the axis of a cylindrical tube of radius R_0 with a background Poiseuille flow at low Reynolds number (figure 1). In this model, (R, Φ, Z) , giving the radial distance, azimuthal angle, axial distance, represents the cylindrical coordinates with the Z axis coinciding with the tube axis while $(r_{a(b)}, \theta_{a(b)}, \varphi_{a(b)})$, giving the radial distance, polar angle, azimuthal angle, are the spherical coordinates distinguished by appending the subscripts a and b with their corresponding origins at the centers of particle A and particle B , respectively. L is the separation distance between the centers of particle A and B along the axis of the cylinder. Due to the relative positions where the two spheres are along the axis (axisymmetric configuration) and the uniform permeability, each particle only has a translational velocity $\mathbf{U}$ along the Z direction with subscript a and b corresponding to particle A and particle B , respectively. In this section, we outline the governing equations and boundary conditions involved, and the procedures to approach this problem.

2.1. Governing equations and boundary conditions

The flow field around the moving porous particles depends on their permeability. When the two particles are impermeable (zero permeability), the fluid has to travel around the particles. If the permeability is nonzero, the fluid can pass through the two particles (Region marked in figure 1 with dashed rectangle). In such case, at low Reynolds number, the governing equations for the model are: the Stokes equation for the external steady Newtonian fluid flow outside the particles ($r_a > a$, $r_b > b$), and the Brinkman equation for the internal porous region ($0 < r_a < a$, $0 < r_b < b$):

$$\nabla^2 \mathbf{v}^e = \frac{1}{\mu} \nabla p^e, \quad \nabla \cdot \mathbf{v}^e = 0, \quad (2.1)$$

$$\nabla^2 \mathbf{v}^i = \frac{1}{\mu} \nabla p^i + \frac{1}{k} \mathbf{v}^i, \quad \nabla \cdot \mathbf{v}^i = 0, \quad (2.2)$$

where μ is the viscosity of the fluid and k is the permeability of the porous material. The superscripts "e" and "i" represent the external and internal flow of the porous particles, respectively. The flow field $\mathbf{v}$ and induced stress tensors components t_{rr} and $t_{r\theta}$ are continuous at the fluid-particle interface. The boundary conditions are:

$$(\mathbf{v}^e - \mathbf{U}) \cdot \mathbf{n} = \mathbf{v}^i \cdot \mathbf{n}, \quad (2.3)$$

$$(\mathbf{v}^e - \mathbf{U}) \cdot \mathbf{s} = \mathbf{v}^i \cdot \mathbf{s}, \quad (2.4)$$

$$t_{rr}^e = t_{rr}^i, \quad (2.5)$$

$$t_{r\theta}^e = t_{r\theta}^i, \quad (2.6)$$

where $\mathbf{n}$ is the unit vector normal to the porous-fluid interface and pointing to the external fluid and $\mathbf{s}$ is the tangential vector at the interface. It should be noted that $\mathbf{v}^i$ represents the relative velocity of the fluid inside the porous particle with respect to the origin of the particle and $\mathbf{v}^e$ refers to the velocity in an inertial reference frame. Therefore, the particle velocity must be subtracted from $\mathbf{v}^e$ to construct the boundary conditions (2.3)-(2.4). On the wall of the cylindrical tube, the no-slip boundary condition applies:

$$\mathbf{v}^e = 0. \quad (2.7)$$

2.2. Semi-analytical implementation

2.2.1. Calculation procedure based on the method of reflections

Since the governing equations and boundary conditions are linear, the velocity and pressure fields can be expressed as a summation of fields:

$$\mathbf{v} = \mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1)} + \mathbf{v}_b^{(2)} + \mathbf{v}_c^{(3)} + \mathbf{v}_a^{(4)} + \mathbf{v}_b^{(5)} + \dots, \quad (2.8)$$

$$p = p_c^{(0)} + p_a^{(1)} + p_b^{(2)} + p_c^{(3)} + p_a^{(4)} + p_b^{(5)} + \dots, \quad (2.9)$$

where the superscript (j) refers to the j th reflection ($j = 1, 2, 3, \dots$) and subscripts a , b and c refer to the boundary condition satisfying particle A , particle B and the cylinder, respectively. Each term of $\mathbf{v}$ and p , separately satisfies the governing equations (2.1)-(2.2) and the boundary conditions resulting from the previous terms. For example, if the subscript of the last term is b , the summation of the terms satisfies the boundary condition of particle B . It should be noted that the reflections representing the flow fields due to the porous particles (reflections with subscripts "a" and "b") consist of two components: the flow inside and outside of the porous particles with superscripts "i" and "e", respectively. Therefore, (2.8) becomes:

$$\mathbf{v}^e = \mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1,e)} + \mathbf{v}_b^{(2,e)} + \mathbf{v}_c^{(3)} + \mathbf{v}_a^{(4,e)} + \mathbf{v}_b^{(5,e)} + \dots, \quad (2.10)$$

$$\mathbf{v}_a^i = \mathbf{v}_a^{(1,i)} + \mathbf{v}_a^{(4,i)} + \mathbf{v}_a^{(7,i)} + \dots, \quad (2.11)$$

$$\mathbf{v}_b^i = \mathbf{v}_b^{(2,i)} + \mathbf{v}_b^{(5,i)} + \mathbf{v}_b^{(8,i)} + \dots. \quad (2.12)$$

Based on the boundary conditions (2.3)-(2.6), the following relations can be obtained:

$$\mathbf{v}_c^{(0)} = [U_0(1 - R^2/R_0^2)]\mathbf{Z}, \quad (2.13)$$

$$\mathbf{v}_a^{(1)} = \begin{cases} (\mathbf{v}_a^{(1,e)} + \mathbf{v}_c^{(0)} - \mathbf{U}_a) \cdot \mathbf{n} = \mathbf{v}_a^{(1,i)} \cdot \mathbf{n}, & \text{at } r_a = a \\ (\mathbf{v}_a^{(1,e)} + \mathbf{v}_c^{(0)} - \mathbf{U}_a) \cdot \mathbf{s} = \mathbf{v}_a^{(1,i)} \cdot \mathbf{s}, & \text{at } r_a = a \\ t_{a,rr}^{(1,e)} = t_{a,rr}^{(1,i)}, & \text{at } r_a = a \\ t_{a,r\theta}^{(1,e)} = t_{a,r\theta}^{(1,i)}, & \text{at } r_a = a \\ 0, & \text{at } Z = \pm\infty, \end{cases} \quad (2.14)$$

$$\mathbf{v}_b^{(2)} = \begin{cases} (\mathbf{v}_b^{(2,e)} + \mathbf{v}_a^{(1,e)} + \mathbf{v}_c^{(0)} - \mathbf{U}_b) \cdot \mathbf{n} = \mathbf{v}_b^{(2,i)} \cdot \mathbf{n}, & \text{at } r_b = b \\ (\mathbf{v}_b^{(2,e)} + \mathbf{v}_a^{(1,e)} + \mathbf{v}_c^{(0)} - \mathbf{U}_b) \cdot \mathbf{s} = \mathbf{v}_b^{(2,i)} \cdot \mathbf{s}, & \text{at } r_b = b \\ t_{b,rr}^{(2,e)} = t_{b,rr}^{(2,i)}, & \text{at } r_b = b \\ t_{b,r\theta}^{(2,e)} = t_{b,r\theta}^{(2,i)}, & \text{at } r_b = b \\ 0, & \text{at } Z = \pm\infty, \end{cases} \quad (2.15)$$

$$\mathbf{v}_c^{(3)} = \begin{cases} -\mathbf{v}_a^{(1,e)} - \mathbf{v}_b^{(2,e)}, & \text{at } R = R_0 \\ 0, & \text{at } Z = \pm\infty, \end{cases} \quad (2.16)$$

$$\mathbf{v}_a^{(4)} = \begin{cases} (\mathbf{v}_a^{(4,e)} + \mathbf{v}_b^{(2,e)} + \mathbf{v}_c^{(3)}) \cdot \mathbf{n} = \mathbf{v}_a^{(4,i)} \cdot \mathbf{n}, & \text{at } r_a = a \\ (\mathbf{v}_a^{(4,e)} + \mathbf{v}_b^{(2,e)} + \mathbf{v}_c^{(3)}) \cdot \mathbf{s} = \mathbf{v}_a^{(4,i)} \cdot \mathbf{s}, & \text{at } r_a = a \\ t_{a,rr}^{(4,e)} = t_{a,rr}^{(4,i)}, & \text{at } r_a = a \\ t_{a,r\theta}^{(4,e)} = t_{a,r\theta}^{(4,i)}, & \text{at } r_a = a \\ 0, & \text{at } Z = \pm\infty, \end{cases} \quad (2.17)$$

$$\mathbf{v}_b^{(5)} = \begin{cases} (\mathbf{v}_b^{(5,e)} + \mathbf{v}_a^{(4,e)} + \mathbf{v}_c^{(3)}) \cdot \mathbf{n} = \mathbf{v}_b^{(5,i)} \cdot \mathbf{n}, & \text{at } r_b = b \\ (\mathbf{v}_b^{(5,e)} + \mathbf{v}_a^{(4,e)} + \mathbf{v}_c^{(3)}) \cdot \mathbf{s} = \mathbf{v}_b^{(5,i)} \cdot \mathbf{s}, & \text{at } r_b = b \\ t_{b,rr}^{(5,e)} = t_{b,rr}^{(5,i)}, & \text{at } r_b = b \\ t_{b,r\theta}^{(5,e)} = t_{b,r\theta}^{(5,i)}, & \text{at } r_b = b \\ 0, & \text{at } Z = \pm\infty, \end{cases} \quad (2.18)$$

⋮

For illustration, the calculation is conducted from the background flow having a parabolic profile, $\mathbf{v}_c^{(0)}$. The following reflection, $\mathbf{v}_a^{(1)}$ due to the presence of particle A, is required to make the summation $\mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1)}$ satisfy the boundary condition of particle A (as shown in (2.3)-(2.6)). To acquire the boundary condition of $\mathbf{v}_a^{(1)}$, we transform the expression of $\mathbf{v}_c^{(0)}$ in the cylindrical coordinates into the spherical coordinates (details on the coordinate transformation are provided in the appendix A). The expression of $\mathbf{v}_a^{(1)}$ can be calculated by means of stream function which will be described in §2.2.2. Similarly, the following reflection $\mathbf{v}_b^{(2)}$ is included to make the summation $\mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1)} + \mathbf{v}_b^{(2)}$ satisfy the boundary condition of particle B. The expression can be solved using similar procedure in the spherical coordinates of particle B. The added reflection $\mathbf{v}_c^{(3)}$ is needed to make the summation satisfy the boundary condition of the tube. The expressions derived in spherical coordinates of particle A, $\mathbf{v}_a^{(1)}$, and particle B, $\mathbf{v}_b^{(2)}$, should be transformed into cylindrical coordinates, such that $\mathbf{v}_c^{(3)}$ can be solved by utilizing cylindrical harmonics shown in §2.2.3.

In general, the reflection $\mathbf{v}_a^{(1)}$ arising from $\mathbf{v}_c^{(0)}$ brings in the representative geometry dimension a (the radius of particle A) and R_0 (the radius of the tube) included in the form a/R_0 by using stream function. This indicates that the error at least scales as a/R_0 (Kim & Karrila 2005). The next reflection $\mathbf{v}_b^{(2)}$ is calculated from $\mathbf{v}_a^{(1)}$ and $\mathbf{v}_c^{(0)}$. The velocity field introduces the term b/R_0 and b/L , where L is the separation distance between the two particles. This indicates that the error at least scales as the maximum value of b/R_0 and b/L . The following reflection $\mathbf{v}_c^{(3)}$ is required to satisfy the boundary condition: $\mathbf{v}_c^{(3)} = -\mathbf{v}_a^{(1)} - \mathbf{v}_b^{(2)}$ at the cylinder wall. The magnitude of $\mathbf{v}_c^{(3)}$ scales as the same order of $\mathbf{v}_a^{(1)}$ and $\mathbf{v}_b^{(2)}$ at the wall, which is smaller than that near $\mathbf{v}_a^{(1)}$ and $\mathbf{v}_b^{(2)}$. This indicates the error of $\mathbf{v}_c^{(3)}$ is smaller than the error of $\mathbf{v}_a^{(1)}$ and $\mathbf{v}_b^{(2)}$. Therefore, the subsequent reflection can correct the error of the flow field to higher order. By repeating the procedures and adding more reflections, the results are expected to converge. This convergence is verified in all our calculations (see Appendix B).

2.2.2. Solution based on stream function

The solution of the external and internal flow of the porous particle in spherical coordinates can be expressed using the stream functions (Saad & Faltas 2014):

$$\Phi^e = \sum_{n=2}^{\infty} (A_n r^{-n+1} + B_n r^{-n+3}) C_n(\cos \theta), \quad (2.19)$$

$$\Phi^i = \sum_{n=2}^{\infty} (\tilde{A}_n r^n + \sqrt{r} \tilde{B}_n I_{n-\frac{1}{2}}(\alpha r)) C_n(\cos \theta), \quad (2.20)$$

where $\alpha = \sqrt{\frac{1}{k}}$; $I_{n-\frac{1}{2}}$ is the modified Bessel function of the first kind; C_n is the Gegenbauer polynomial of the first kind of order n and degree $-1/2$; A_n , B_n , $\tilde{A}_n$ and $\tilde{B}_n$ are unknown coefficients which can be determined from the boundary conditions as shown in (2.3)-(2.6).

The velocity components in spherical coordinates can be derived from the stream functions:

$$v_r^{e,i} = -\frac{1}{r^2 \sin \theta} \frac{\partial \Phi^{e,i}}{\partial \theta}, \quad (2.21)$$

$$v_\theta^{e,i} = \frac{1}{r \sin \theta} \frac{\partial \Phi^{e,i}}{\partial r}, \quad (2.22)$$

By substituting (2.19)-(2.20) into (2.21)-(2.22), the velocity components can be expressed as:

$$v_r^e = \sum_{n=2}^{\infty} -r^{-n-1} (A_n + B_n r^2) P_{n-1}(\cos \theta), \quad (2.23)$$

$$v_r^i = \sum_{n=2}^{\infty} -\frac{1}{r^2} \left[r^n \tilde{A}_n + \sqrt{r} \tilde{B}_n I_{n-\frac{1}{2}}(\alpha r) \right] P_{n-1}(\cos \theta), \quad (2.24)$$

$$v_\theta^e = \sum_{n=2}^{\infty} -r^{-n-1} [(n-1)A_n + (n-3)B_n r^2] \csc \theta C_n(\cos \theta), \quad (2.25)$$

$$v_\theta^i = \sum_{n=2}^{\infty} -\frac{1}{r^2} \left\{ n r^n \tilde{A}_n + \sqrt{r} \left[n I_{n-\frac{1}{2}}(\alpha r) + \alpha r I_{n+\frac{1}{2}}(\alpha r) \right] \tilde{B}_n \right\} \csc \theta C_n(\cos \theta), \quad (2.26)$$

where P_{n-1} is the Legendre Polynomial of order $n - 1$. The components of the stress tensor can be derived from the velocity components:

$$t_{rr}^{e,i} = -p^{e,i} + 2\mu \frac{\partial v_r^{e,i}}{\partial r} \quad (2.27)$$

$$t_{r\theta}^{e,i} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta^{e,i}}{r} \right) + \frac{1}{r} \frac{\partial v_r^{e,i}}{\partial \theta} \right]. \quad (2.28)$$

The pressure can be obtained from the governing equations:

$$p^e = -2\mu \sum_{n=2}^{\infty} \frac{2n-3}{n} B_n r^{-n} P_{n-1}(\cos \theta), \quad (2.29)$$

$$p^i = \mu \sum_{n=2}^{\infty} \frac{\alpha^2}{n-1} \tilde{A}_n r^{n-1} P_{n-1}(\cos \theta). \quad (2.30)$$

By substituting these expansions into the boundary conditions (2.3)-(2.6), the coefficients in the stream functions can be obtained. It should be reminded that our implementation in this work is based on the method of reflections. The stream function solutions developed here are applicable for the reflections involving particles A and B . For example, (2.14) is used to satisfy the boundary condition of particle A . To construct the algebraic equations for solving the unknown coefficients A_n , B_n , $\tilde{A}_n$ and $\tilde{B}_n$ for this reflection $\mathbf{v}_a^{(1)}$, we transform the expression of $\mathbf{v}_c^{(0)}$ from the cylindrical coordinates into the spherical coordinates (details on the coordinate transformation are provided in the appendix A). It should be noted that this procedure is not the re-expansion of the solution. It is a direct transformation of the unit vector and variables from cylindrical coordinates to spherical coordinates. Since (2.14) is a boundary condition at the surface of particle A ($r_a = a$), the transformed $\mathbf{v}_c^{(0)}$ is only a function of θ_a . An arbitrary function of θ can be expanded into series using either Gegenbauer Polynomial or Legendre Polynomial (Happel & Brenner 1983):

$$f(\cos \theta) = \sum_{n=2}^{\infty} \tilde{a}_n C_n(\cos \theta) \quad (2.31)$$

where

$$\tilde{a}_n = \frac{1}{2} n(n-1)(2n-1) \int_{-1}^1 \frac{f(\cos \theta) C_n(\cos \theta)}{\sin^2 \theta} d(\cos \theta) \quad (2.32)$$

$$g(\cos \theta) = \sum_{n=2}^{\infty} \tilde{b}_{n-1} P_{n-1}(\cos \theta) \quad (2.33)$$

where

$$\tilde{b}_{n-1} = \frac{1}{2} (2n-1) \int_{-1}^1 g(\cos \theta) P_{n-1}(\cos \theta) d(\cos \theta) \quad (2.34)$$

The $\mathbf{v}_c^{(0)}$, which is now in spherical coordinates, can be expressed into either Gegenbauer or Legendre polynomial, and substituted into (2.14). The first boundary condition of (2.14) can then be transformed into:

$$\sum_{n=2}^{\infty} -a^{-n-1}(A_n + B_n a^2)P_{n-1}(\cos \theta_a) + \sum_{n=2}^{\infty} \tilde{b}_{n-1}P_{n-1}(\cos \theta_a) = \sum_{n=2}^{\infty} -\frac{1}{a^2} \left[a^n \tilde{A}_n + \sqrt{a} \tilde{B}_n I_{n-\frac{1}{2}}(\alpha a) \right] P_{n-1}(\cos \theta_a). \quad (2.35)$$

The first summation term on the left hand side of (2.35) is the general solution of the external flow of particle A, $\mathbf{v}_a^{(1,e)}$, given by (2.23). The second summation term on the left hand side is $\mathbf{v}_c^{(0)}$ expressed in Legendre polynomial. The right hand side represents the general solution of the internal flow of particle A, $\mathbf{v}_a^{(1,i)}$, given by (2.24). Equating term by term under the summation sign, an algebraic equation can be constructed. Similarly, the other three algebraic equations can be constructed from (2.14). The coefficients A_n , B_n , $\tilde{A}_n$ and $\tilde{B}_n$ in the stream functions for this reflection $\mathbf{v}_a^{(1)}$ can then be obtained. Similarly, other reflections with subscripts "a" and "b" can be solved by using the corresponding boundary conditions satisfying the surface of the two particles.

2.2.3. Solution based on cylindrical harmonics

In cylindrical coordinates, the velocity components are shown in terms of (R, Φ, Z) . Without loss of generality, the general solution to solve the flow field inside the cylindrical tube is given by Happel & Brenner (1983):

$$\mathbf{v} = \sum_{k=-\infty}^{\infty} \left[\nabla \times (\Omega_k \mathbf{Z}) + \nabla \Psi_k + R \frac{\partial}{\partial R} \nabla \Pi_k + \frac{\partial \Pi_k}{\partial Z} \mathbf{Z} \right], \quad (2.36)$$

$$p = -2\mu \sum_{k=-\infty}^{\infty} \frac{\partial^2 \Pi_k}{\partial Z^2}, \quad (2.37)$$

where Π_k , Ψ_k and Ω_k are cylindrical harmonic functions. In our previous work (Yao *et al.* 2017), we have developed a framework to obtain such cylindrical harmonic functions with the following standard forms:

$$M_k = \frac{1}{\pi} \begin{pmatrix} \cos k\Phi \int_0^{\infty} \frac{1}{\lambda} m_{ak} I_k(\lambda R) \sin \lambda Z \, d\lambda \\ + \sin k\Phi \int_0^{\infty} \frac{1}{\lambda} m_{bk} I_k(\lambda R) \sin \lambda Z \, d\lambda \\ + \cos k\Phi \int_0^{\infty} \frac{1}{\lambda} m_{ck} I_k(\lambda R) \cos \lambda Z \, d\lambda \\ + \sin k\Phi \int_0^{\infty} \frac{1}{\lambda} m_{dk} I_k(\lambda R) \cos \lambda Z \, d\lambda \end{pmatrix}, \quad (2.38)$$

where M refers to Π , Ψ or Ω and m refers to π , ψ or ω . π_{ik} , ψ_{ik} and ω_{ik} are unknown functions of λ , where $i = a, b, c, d$. Since the flow is axisymmetric and the boundary conditions are homogeneous in our study, the velocity expression will not be a function of Φ which requires $k = 0$ and only π_{a0} , π_{c0} , ψ_{a0} and ψ_{c0} contribute to the velocity expression. Hence, we have the following simplified forms:

$$\Pi_0 = \frac{1}{\pi} \left(\int_0^\infty \frac{1}{\lambda} \pi_{a0} I_0(\lambda R) \sin \lambda Z \, d\lambda + \int_0^\infty \frac{1}{\lambda} \pi_{c0} I_0(\lambda R) \cos \lambda Z \, d\lambda \right), \quad (2.39)$$

$$\Psi_0 = \frac{1}{\pi} \left(\int_0^\infty \frac{1}{\lambda} \psi_{a0} I_0(\lambda R) \sin \lambda Z \, d\lambda + \int_0^\infty \frac{1}{\lambda} \psi_{c0} I_0(\lambda R) \cos \lambda Z \, d\lambda \right). \quad (2.40)$$

By expanding (2.36), the flow field can be described by:

$$\mathbf{v} = \left(\frac{\partial \Psi_0}{\partial R} + R \frac{\partial^2 \Pi_0}{\partial R^2} \right) \mathbf{R} + \left(\frac{\partial \Psi_0}{\partial Z} + R \frac{\partial^2 \Pi_0}{\partial R \partial Z} + \frac{\partial \Pi_0}{\partial Z} \right) \mathbf{Z} \quad (2.41)$$

We note the boundary condition at the wall of the tube ($R = R_0$) has the following form:

$$\mathbf{V}(R_0, Z) = V_R \mathbf{R} + V_Z \mathbf{Z}, \quad (2.42)$$

where

$$V_R = \frac{\partial \Psi_0}{\partial R} + R_0 \frac{\partial^2 \Pi_0}{\partial R^2}, \quad (2.43)$$

$$V_Z = \frac{\partial \Psi_0}{\partial Z} + R_0 \frac{\partial^2 \Pi_0}{\partial R \partial Z} + \frac{\partial \Pi_0}{\partial Z}. \quad (2.44)$$

We now proceed to construct equations using the boundary conditions to solve the unknown coefficients in (2.39)-(2.40). By substituting (2.39)-(2.40), (2.43) can be written as

$$V_R = \frac{\partial \Psi_0}{\partial R} + R_0 \frac{\partial^2 \Pi_0}{\partial R^2} = \frac{1}{\pi} \left\{ \int_0^\infty [\psi_{c0} I_0'(\lambda R_0) + \pi_{c0} \lambda R_0 I_0''(\lambda R_0)] \cos \lambda Z \, d\lambda + \int_0^\infty [\psi_{a0} I_0'(\lambda R_0) + \pi_{a0} \lambda R_0 I_0''(\lambda R_0)] \sin \lambda Z \, d\lambda \right\}. \quad (2.45)$$

By utilizing Fourier integral, we can eliminate variable Z and (2.45) can be written as:

$$\int_{-\infty}^\infty V_R \cos \lambda Z \, dZ = \psi_{c0} I_0'(\lambda R_0) + \pi_{c0} \lambda R_0 I_0''(\lambda R_0), \quad (2.46)$$

$$\int_{-\infty}^\infty V_R \sin \lambda Z \, dZ = \psi_{a0} I_0'(\lambda R_0) + \pi_{a0} \lambda R_0 I_0''(\lambda R_0). \quad (2.47)$$

Similarly, the following equations can be obtained from the boundary condition in the Z direction as given by (2.44):

$$\int_{-\infty}^\infty V_Z \cos \lambda Z \, dZ = \psi_{a0} I_0(\lambda R_0) + \pi_{a0} \lambda R_0 I_0'(\lambda R_0) + \pi_{a0} I_0(\lambda R_0), \quad (2.48)$$

$$\int_{-\infty}^\infty V_Z \sin \lambda Z \, dZ = -\psi_{c0} I_0(\lambda R_0) - \pi_{c0} \lambda R_0 I_0'(\lambda R_0) - \pi_{c0} I_0(\lambda R_0). \quad (2.49)$$

By solving (2.46)-(2.49) simultaneously, π_{a0} , π_{c0} , ψ_{a0} and ψ_{c0} can be obtained with

the following forms and finally the flow field in cylindrical harmonics can be obtained using (2.41).

$$\psi_{c0} = \frac{[I_0(\lambda R_0) + \lambda R_0 I_0'(\lambda R_0)] \int_{-\infty}^{\infty} V_R \cos \lambda Z \, dZ + \lambda R_0 I_0''(\lambda R_0) \int_{-\infty}^{\infty} V_Z \sin \lambda Z \, dZ}{\lambda R_0 I_0'''(\lambda R_0) + I_0(\lambda R_0)[I_0'(\lambda R_0) - \lambda R_0 I_0''(\lambda R_0)]} \quad (2.50)$$

$$\psi_{a0} = \frac{[I_0(\lambda R_0) + \lambda R_0 I_0'(\lambda R_0)] \int_{-\infty}^{\infty} V_R \sin \lambda Z \, dZ - \lambda R_0 I_0''(\lambda R_0) \int_{-\infty}^{\infty} V_Z \cos \lambda Z \, dZ}{\lambda R_0 I_0'''(\lambda R_0) + I_0(\lambda R_0)[I_0'(\lambda R_0) - \lambda R_0 I_0''(\lambda R_0)]} \quad (2.51)$$

$$\pi_{c0} = \frac{I_0(\lambda R_0) \int_{-\infty}^{\infty} V_R \cos \lambda Z \, dZ + I_0''(\lambda R_0) \int_{-\infty}^{\infty} V_Z \sin \lambda Z \, dZ}{-\lambda R_0 I_0'''(\lambda R_0) + I_0(\lambda R_0)[-I_0'(\lambda R_0) + \lambda R_0 I_0''(\lambda R_0)]} \quad (2.52)$$

$$\pi_{a0} = \frac{I_0(\lambda R_0) \int_{-\infty}^{\infty} V_R \sin \lambda Z \, dZ - I_0''(\lambda R_0) \int_{-\infty}^{\infty} V_Z \cos \lambda Z \, dZ}{-\lambda R_0 I_0'''(\lambda R_0) + I_0(\lambda R_0)[-I_0'(\lambda R_0) + \lambda R_0 I_0''(\lambda R_0)]} \quad (2.53)$$

Therefore, the unknown coefficients can be solved based on the boundary conditions. For example, $\mathbf{v}_c^{(3)}$ is required to make the summation $\mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1)} + \mathbf{v}_b^{(2)} + \mathbf{v}_c^{(3)}$ satisfy the boundary condition of the tube (as shown in (2.16)). We firstly transform the expression of $\mathbf{v}_a^{(1)}$ and $\mathbf{v}_b^{(2)}$ from the spherical coordinates into the cylindrical coordinates (details on the coordinate transformation are provided in the appendix A) and set $R = R_0$ to acquire the boundary condition of $\mathbf{v}_c^{(3)}$. This boundary condition (2.16) is substituted into (2.50)-(2.53) and the unknown coefficients π_{a0} , π_{c0} , ψ_{a0} and ψ_{c0} can be solved. Similarly, other reflections with subscript "c" can be solved by using the corresponding boundary conditions satisfying the wall of the tube.

2.2.4. Drag force and velocities of particles

After obtaining the velocity fields using both stream functions and cylindrical harmonics for the particles and the cylinder, respectively, the total velocity flow field can then be obtained using (2.10)-(2.12). However, only the flow fields derived from the stream functions (the reflection terms with subscripts a and b) contribute to the drag forces acting on the two particles (Happel & Brenner 1983). Therefore, the drag force on the two particles follows:

$$F_a = F_a^{(1)} + F_a^{(4)} + F_a^{(7)} + F_a^{(10)} + \dots, \quad (2.54)$$

$$F_b = F_b^{(2)} + F_b^{(5)} + F_b^{(8)} + F_b^{(11)} + \dots. \quad (2.55)$$

On the other hand, the drag force acting on the surface of a porous particle by the external fluid can be obtained from the stream function (Happel & Brenner 1983):

$$F = \mu\pi \int_0^\pi r^3 \sin^3 \theta \frac{\partial}{\partial r} \left(\frac{E^2 \Phi^e}{r^2 \sin^2 \theta} \right) d\theta \quad (2.56)$$

where E^2 is the axisymmetric Stokesian operator:

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \quad (2.57)$$

By substituting the stream function given by (2.19) into (2.56), we can obtain the simple relation (Saad & Faltas 2014):

$$F = -4\mu\pi B_2, \quad (2.58)$$

where B_2 is the only remaining coefficient in the stream function contributing to the drag force due to orthogonality of Gegenbauer polynomial.

We can now apply (2.56) to each term in (2.54)-(2.55) and solve the corresponding coefficient B_2 using the boundary conditions as presented in (2.13)-(2.18). Since the boundary conditions given by (2.13)-(2.18) are functions of particle velocities U_a , U_b and background flow U_0 , the drag force will also be functions of U_a , U_b and U_0 . Therefore, the drag force (2.54)-(2.55) can also be expressed as:

$$\mathbf{F}_a = 6\pi\mu a(D_a U_0 + K_a U_a + K_{ab} U_b)\mathbf{Z}, \quad \mathbf{F}_b = 6\pi\mu a(D_b U_0 + K_{ba} U_a + K_b U_b)\mathbf{Z}, \quad (2.59)$$

where $\mathbf{F}_a$ and $\mathbf{F}_b$ are the drag forces on particle A and B ; D_a , D_b and K_a , K_b are the coefficients related to the background flow and particle velocities, respectively; K_{ab} and K_{ba} are the coefficients referring to hydrodynamic interaction between the two particles.

At low Reynolds number, the inertial forces are negligible, thus the total drag force on the particle is zero and using (2.58), the velocities of particles U_a , U_b can be obtained. Subsequently, the total flow field can be obtained by substituting U_a , U_b into (2.10)-(2.12). The practical implementation is numerically conducted for given geometrical parameters, including particle radii, permeability, cylinder radius and separation distance. We can calculate any arbitrary number of reflections in our solution when needed for convergence. As described earlier, the calculation procedure for the method of reflections is conducted based on an iterative process by adding more terms. We terminate the calculation when the solution reaches a convergence, i.e. when the magnitude of the last term is three orders smaller than the sum of the previous terms (details on the convergence are provided in the appendix B). We provide a detailed example on how to conduct the calculation for the first four reflections in appendix C.

3. Results and discussion

3.1. Approximate formula for drag force

Since the drag force given by (2.58) only includes coefficient B_2 , we approximate the stream function outside the porous particle (2.19) as:

$$\Phi^e = r B_2 C_2(\cos \theta). \quad (3.1)$$

By substituting (3.1) into (2.56) and using (2.14), the first reflection of drag force acting on particle A (i.e. the first term in (2.54)) can be obtained:

$$\mathbf{F}_a^{(1)} = 6\pi\mu a(D_a^{(1)} U_0 + K_a^{(1)} U_a)\mathbf{Z} \quad (3.2)$$

where

$$D_a^{(1)} = \frac{2a^2 [a\alpha(3R_0^2\alpha^2 - 2a^2\alpha^2 - 12) \cosh a\alpha + (6a^2\alpha^2 + 12 - 3R_0^2\alpha^2) \sinh a\alpha]}{3R_0^2 [a\alpha(3 + 2a^2\alpha^2) \cosh a\alpha - 3 \sinh a\alpha]}, \quad (3.3)$$

$$K_a^{(1)} = \frac{2a^2\alpha^2(\sinh a\alpha - a\alpha \cosh a\alpha)}{a\alpha(3 + 2a^2\alpha^2) \cosh a\alpha - 3 \sinh a\alpha}. \quad (3.4)$$

Here, $K_{ab}^{(1)}$ is zero, since (2.14) has not included the boundary condition of particle B .

Similarly, by using (2.15), the first reflection of drag force on particle B (i.e. the first term in (2.55)) is given by:

$$\mathbf{F}_b^{(2)} = 6\pi\mu a(D_b^{(2)}U_0 + K_{ba}^{(2)}U_a + K_b^{(2)}U_b)\mathbf{Z} \quad (3.5)$$

where

$$D_b^{(2)} = \frac{\left\{ \begin{aligned} & \left[\begin{aligned} & 3a^4b^4\alpha^4(6 + b^2\alpha^2 - 3L^2\alpha^2) \\ & (12 + 8a^2\alpha^2 - 3R_0^2\alpha^2 + 2a^4\alpha^4 - a^2R_0^2\alpha^4) \\ & + 3ab^4L^3\alpha^4(3 + 2a^2\alpha^2)(12 + 2b^2\alpha^2 - 3R_0^2\alpha^2) \end{aligned} \right] \cosh a\alpha \cosh b\alpha \sinh a\alpha \\ & + \left[\begin{aligned} & -4.5a^3b^4\alpha^3(6 + b^2\alpha^2 - 3L^2\alpha^2)(4 + 2a^2\alpha^2 - R_0^2\alpha^2) \\ & + L^3(-9b^6\alpha^5 - 54b^4\alpha^3 + 13.5b^4R_0^2\alpha^5) \end{aligned} \right] \cosh b\alpha \sinh^2 a\alpha \\ & + \left[\begin{aligned} & 13.5a^3b^3\alpha^2(2 + b^2\alpha^2 - L^2\alpha^2)(4 + 2a^2\alpha^2 - R_0^2\alpha^2) \\ & + L^3(27b^5\alpha^4 + 54b^3\alpha^2 - 13.5b^3R_0^2\alpha^4) \end{aligned} \right] \sinh b\alpha \sinh^2 a\alpha \\ & - \left[\begin{aligned} & 4.5a^4b^3\alpha^3(2 + b^2\alpha^2 - L^2\alpha^2) \\ & (12 - 3R_0^2\alpha^2 + 2a^4\alpha^4 + 8a^2\alpha^2 - a^2R_0^2\alpha^4) \\ & + 4.5ab^3L^3\alpha^3(3 + 2a^2\alpha^2)(4 + 2b^2\alpha^2 - R_0^2\alpha^2) \end{aligned} \right] \sinh b\alpha \sinh 2a\alpha \\ & - \left[\begin{aligned} & 0.5a^5b^4\alpha^5(3 + 2a^2\alpha^2)(6 + b^2\alpha^2 - 3L^2\alpha^2) \\ & (12 + 2a^2\alpha^2 - 3R_0^2\alpha^2) \\ & + 0.5a^2b^4L^3\alpha^5(3 + 2a^2\alpha^2)^2(12 + 2b^2\alpha^2 - 3R_0^2\alpha^2) \end{aligned} \right] \cosh b\alpha \cosh^2 a\alpha \\ & + \left[\begin{aligned} & 1.5a^5b^3\alpha^4(3 + 2a^2\alpha^2)(2 + b^2\alpha^2 - L^2\alpha^2) \\ & (12 + 2a^2\alpha^2 - 3R_0^2\alpha^2) \\ & + 1.5a^2b^3L^3\alpha^4(3 + 2a^2\alpha^2)^2(4 + 2b^2\alpha^2 - R_0^2\alpha^2) \end{aligned} \right] \sinh b\alpha \cosh^2 a\alpha \end{aligned} \right\}}{3a\alpha^2L^3R_0^2[(3b\alpha + 2b^3\alpha^3) \cosh b\alpha - 3 \sinh b\alpha][(1.5a\alpha + a^3\alpha^3) \cosh a\alpha - 1.5 \sinh a\alpha]^2}, \quad (3.6)$$

$$K_{ba}^{(2)} = \frac{2a^3\alpha^2b^3 \left(\begin{aligned} & 3a^2\alpha^2 \cosh^2 a\alpha + 2a^4\alpha^4 \cosh^2 a\alpha \\ & + 3 \sinh^2 a\alpha - 3a\alpha \sinh 2a\alpha \\ & - a^3\alpha^3 \sinh 2a\alpha \end{aligned} \right) \left(\begin{aligned} & 6b\alpha \cosh b\alpha - 5b^3\alpha^3 \cosh b\alpha \\ & + 15bL^2\alpha^3 \cosh b\alpha - 6 \sinh b\alpha \\ & + 3b^2\alpha^2 \sinh b\alpha - 15L^2\alpha^2 \sinh b\alpha \end{aligned} \right)}{5aL^3[(3a\alpha + 2a^3\alpha^3) \cosh a\alpha - 3 \sinh a\alpha]^2[b\alpha(3 + 2b^2\alpha^2) \cosh b\alpha - 3 \sinh b\alpha]} \quad (3.7)$$

$$K_b^{(2)} = \frac{2b^3\alpha^2(\sinh b\alpha - b\alpha \cosh b\alpha)}{(3ab\alpha + 2ab^3\alpha^3) \cosh b\alpha - 3a \sinh b\alpha}. \quad (3.8)$$

Thus far, we have derived the first term of (2.55), given by (3.5), as the approximate drag force to the first leading order acting on particle B , in which the boundary conditions of particle A , particle B and the cylindrical tube have been taken into account. Following the same procedure as above, the approximate drag acting on particle A is simply (3.5) by exchanging the letters b and a .

3.2. Model validation

In this section, we validate our semi-analytical solution with previous works, where the motion of two solid particles has been studied. As α goes to infinity, the permeability approaches zero and the particle becomes a solid sphere. We firstly validate our full model

FIGURE 2. Drag force coefficients (a) D_a or D_b and (b) K_a or K_b when $a/R_0 = 0.1$, $b/R_0 = 0.2$. (c) $|K_a + K_{ab}|$ when $a/R_0 = b/R_0 = 0.1$. (d) $|K_a - K_{ab}|$ when $a/R_0 = b/R_0 = 0.01$. All the drag force coefficients are plotted as a function of the normalized distance L/R_0 .

solution with our previous work (Yao *et al.* 2017), which is also based on the method of reflections. We consider the case $a/R_0 = 0.1$, $b/R_0 = 0.2$ and $\alpha R_0 = 50000$ by using the procedures introduced in §2. At such a large α , the fluid going through the internal region is extremely limited and can be approximated as the solid particle case. Figure 2(a,b) shows that the drag force coefficients due to the background flow (D_a and D_b) and particle velocities (K_a and K_b) evaluated at $\alpha R_0 = 50000$ are in good agreement with our previous work (Yao *et al.* 2017). This has confirmed the consistency of the two frameworks. Moreover, Sonshine & Brenner (1966) theoretically investigated the motion of two or more solid particles travelling along the axis of a cylindrical tube without the background flow by developing the point force formula based on the stream function. The particles are equally spaced along the axis and translate with a constant velocity. We compare our solution with their work for the case that ($a/R_0 = b/R_0 = 0.1$, $\alpha R_0 = 50000$) (figure 2c). We can clearly see that the drag force coefficients calculated from our full semi-analytical results are in good agreement with the previous work except when the two particles are very close ($L/R_0 < 0.3$). In addition, the drag force of two identical particles approaching each other with a constant velocity in an unbounded fluid based on the lubrication theory has been previously solved (Jeffrey 1982). We then consider the case $a/R_0 = b/R_0 = 0.01$ and $\alpha R_0 = 50000$ by using the procedures introduced in §2 to compare with the lubrication theory solution. In such case of a small particle size, the wall effect in our solution is negligible. It should be noted that when the two particles are very close, our solution cannot match with the lubrication theory solution (figure 2d). This could be due to the fact that the method of reflections is not accurate to solve

FIGURE 3. (a) Drag force coefficient D_b as a function of normalized distance L/R_0 for $\alpha R_0 = 5, 10, 20$. (b) Relative error as a function of αR_0 .

cases when the two solid particles ($\alpha R_0 = 50000$) are very close, consistent with what reported in the literature (Bhattacharya *et al.* 2010).

Here we present the validation of the approximate solution derived in §3.1. In figure 3(a), we compare our full semi-analytical results with the approximate solution given by (3.6) when $a/R_0 = 0.1$, $b/R_0 = 0.2$ and $\alpha R_0 = 5, 10, 20$. As αR_0 increases, D_b increases which indicates the porous particle experiences larger drag force due to the background flow. Since (3.6) is the approximate expression of the drag force, it is not unexpected that the magnitude is smaller than that of the full semi-analytical solution. As αR_0 decreases, the difference between (3.6) and the semi-analytical result becomes smaller (figure 3a), which implies that the particles' contribution to the drag force is not as important. To better highlight this, we investigate the effect of αR_0 on the relative error between the full semi-analytical solution and the approximate solution for $a/R_0 = 0.1$, $b/R_0 = 0.2$ and $L/R_0 = 0.301$. Figure 3(b) demonstrates that the decrease in αR_0 leads to the decrease in the relative error. When αR_0 is smaller than 5, the relative error is less than 10 percent which indicates that (3.6) is a good approximate solution of the drag force acting on the porous particle. We repeated our calculations for (3.7)-(3.8) and found the same conclusion that these approximate solutions are valid when αR_0 is smaller than 5. We can also infer that the particle-particle interaction decreases as αR_0 decreases.

3.3. Two identical particles moving along the axis of the tube

Previous works have investigated the permeability of porous media theoretically and experimentally. Kapur *et al.* (1996) measured the permeability of polymer gels when they are stabilized within a microporous support. Mika & Childs (2001) developed a model in which gel networks are considered as an assemblage of spheres to calculate the hydrodynamic permeability. They have provided the equation to calculate the radius a_p of such sphere. Theoretical works have provided the equations to calculate the hydrodynamic permeability based on a_p and porosity ε . One of the most widely used equations to calculate the permeability is from Brinkman (1949)

$$k = \frac{a_p^2}{18} \left(3 + \frac{4}{1 - \varepsilon} - 3\sqrt{\frac{8}{1 - \varepsilon} - 3} \right). \quad (3.9)$$

Model	$a_p(\text{m})$	$\alpha(\text{m}^{-1})$	$R_0(\text{m})$
Brinkman (1949)	2.18×10^{-9}	2.49×10^8	1.6×10^{-7}
Happel & Brenner (1983)	2.18×10^{-9}	2.77×10^8	1.5×10^{-7}
Carman-Kozeny(Krainov <i>et al.</i> 2014)	2.18×10^{-9}	1.31×10^8	3.2×10^{-7}
Experiment(Kapur <i>et al.</i> 1996)	Not applicable	2.26×10^8	1.8×10^{-7}

 TABLE 1. Variables corresponding to $\alpha R_0 = \sqrt{1/k} R_0 = 40$

Happel & Brenner (1983) have also derived the relationship based on the cell model method (Mika & Childs 2001)

$$k = \frac{2a_p^2}{9(1-\varepsilon)} \left[\frac{3 - 4.5(1-\varepsilon)^{1/3} + 4.5(1-\varepsilon)^{5/3} - 3(1-\varepsilon)^2}{3 + 2(1-\varepsilon)^{5/3}} \right]. \quad (3.10)$$

Mika & Childs (2001) have shown the comparison between the theoretical work and experimental data conducted by Kapur *et al.* (1996) and both equations (3.9)-(3.10) give a good prediction of the permeability of the gel. In addition, the permeability can be predicted with Carman-Kozeny formula (Krainov *et al.* 2014)

$$k = \frac{\varepsilon^3 a_p^2}{45(1-\varepsilon)^2}. \quad (3.11)$$

In our study, we consider material with large porosity, such as a polymer gel with porosity $\varepsilon = 0.96$. Based on the equation and parameters from Mika & Childs (2001), we can calculate $a_p = 2.66 \times 10^{-10} / \sqrt{0.372(1-\varepsilon)}$ m. Using (3.9)-(3.11), $\alpha = \sqrt{1/k}$ can be obtained. In our main discussion, we choose $\alpha R_0 = 40$ and $\alpha R_0 = 100$. Using these values, we summarized the calculated values of a_p , α and R_0 in tables 1-2. The experiments of Kapur *et al.* (1996) reported a value of $\alpha = 2.26 \times 10^8 \text{ m}^{-1}$. Using this value, we also calculated the corresponding R_0 in tables 1-2.

It can be seen that R_0 is at least one order larger than a_p . In our study, the minimum size of the porous particle is $a/R_0 = 0.1$ and a is at least six times larger than a_p . Although the Brinkman equation is only fully accurate for $a_p/a \ll 1$, it can be used as a first-order approximation in a_p/a provided that $a_p/a < 1$ (Auriault 2009). It should be noted that R_0 is in the nanometer scale based on our calculation result. This nano scale of fluidic structures is of interest with a great variety of applications, ranging from drug delivery (Sinha *et al.* 2004), energy transfer (Daiguji *et al.* 2004; Van Der Heyden *et al.* 2006) and biochemical analysis (Foquet *et al.* 2002). Previous works have shown the procedure to fabricate the nanochannel. Lee *et al.* (2003) have developed a nanochannel fabrication technique by using chemical-mechanical polishing and thermal oxidation. They have successfully fabricated narrow channels (i.e., 10 nm long, 25 nm wide and 100 nm deep) with smooth inner surfaces. Song *et al.* (2012) have fabricated nanochannel by utilizing the crystallinity-dependent anisotropic wet etching of silicon. The dimension of the nanochannel is down to 16nm in depth and 175nm in width and up to 8cm in length. Even at such scale, the Stokes equation and Brinkman equation are still valid (Urbakh *et al.* 1995; Bocquet & Charlaix 2010).

We first consider two identical porous particles moving along the axis of the tube ($a/R_0 = b/R_0 = 0.1$ and $a/R_0 = b/R_0 = 0.3$) with $\alpha R_0 = 40$ and $\alpha R_0 = 100$. The semi-analytical results such as drag force coefficients and particle velocities are plotted against the normalized separation distance between the two particles L/R_0 . Due to kinematic reversibility at low Reynolds number, the coefficients of the two identical particles are

Model	$a_p(\text{m})$	$\alpha(\text{m}^{-1})$	$R_0(\text{m})$
Brinkman (1949)	2.18×10^{-9}	2.49×10^8	4.0×10^{-7}
Happel & Brenner (1983)	2.18×10^{-9}	2.77×10^8	3.8×10^{-7}
Carman-Kozeny(Krainov <i>et al.</i> 2014)	2.18×10^{-9}	1.31×10^8	8.0×10^{-7}
Experiment(Kapur <i>et al.</i> 1996)	Not applicable	2.26×10^8	4.5×10^{-7}

TABLE 2. Variables corresponding to $\alpha R_0 = \sqrt{1/k} R_0 = 100$

FIGURE 4. Drag force coefficients and normalized velocity of two identical porous particles. (a) drag force coefficient D ; (b) drag force coefficient K ; (c) drag force coefficient K^* and (d) normalized particle velocities $U = U_a = U_b$. Two particle sizes and two αR_0 are considered: $a/R_0 = b/R_0 = 0.1, 0.3$; $\alpha R_0 = 40, 100$.

expected to be identical. Our semi-analytical results have also verified that this is the case. For convenience, the new variables D ($D = D_a = D_b$), K ($K = K_a = K_b$) and K^* ($K^* = K_{ab} = K_{ba}$) are utilized to represent the coefficients (figure 4(a,b,c)).

The coefficient D (figure 4a), describing the drag force due to the background flow on the particles when they are at rest, increases monotonically from a minimum to a constant value. At small separation distance L/R_0 , the effect of particle interaction is strong inducing small velocity gradient between them and thus small drag force. Details on the streamlines and velocity contour will be discussed below. As L/R_0 increases, the effect of particle interaction diminishes and each particle behaves as a single particle case, indicated by the constant D . In addition, bigger particles have larger drag force coefficients due to the stronger wall effect. When αR_0 is larger ($\alpha R_0 = 100$), the particle has a smaller permeability hence it is harder for the fluid to go through the internal region

FIGURE 5. Streamlines (left) and normalized velocity contour in the Z -direction v_z/U_0 (right) around two identical particles $a/R_0 = b/R_0 = 0.1$. Since it is an axisymmetric model, we only plot half region of the flow field: $\alpha R_0 = 40$ (top half) and $\alpha R_0 = 100$ (bottom half). Two representative separation distances are considered: $L/R_0 = 0.3$ (a) and $L/R_0 = 0.7$ (b).

of the particle. The porous particle experiences a larger drag force than that with smaller αR_0 ($\alpha R_0 = 40$). Thus, as αR_0 increases, the coefficient D has a similar relationship with L/R_0 but with larger magnitude.

Unlike the coefficient D , K represents the drag force contribution due to the movement of one particle while the other is at rest in the absence of the background flow (figure 4b). This drag force is against the direction of the particle motion which explains the negative sign of K . When the particle is moving in the fluid, the presence of the neighbouring resting particle increases the velocity gradient in the vicinity of the moving particle and thus also increases the drag force acting on the moving particle. This effect is more pronounced as the two particles get closer due to stronger particle-particle interaction. As the separation distance L/R_0 increases, K approaches the single particle case. In addition, smaller αR_0 allows the fluid to smoothly pass through the particle, represented by the smaller coefficient K magnitude. Unsurprisingly, we also found that bigger particles have a larger drag force magnitude due to the stronger wall effect.

The drag coefficient K^* represents the particle-particle interaction in the absence of background flow (figure 4c). It describes the drag force acting on a resting particle due to the movement of the neighbouring particle. The coefficient K^* is positive since the drag force is in the same direction as the moving particle. When the neighboring particle is moving, it drags the surrounding fluid inducing drag force on the resting particle. As the two particles get very close, the coefficient K^* reaches its maximum value which arises from a large shear rate in the gap between the two particles. It is not unexpected that as L/R_0 increases, K^* decays to zero since the hydrodynamic interaction becomes negligible.

As mentioned earlier, the velocities of particles can be calculated by applying zero total drag force due to the negligible inertia effect at low Reynolds number. The velocity of particle A is the same as particle B since the two particles are identical ($U = U_a = U_b$), which implies the two particles will not attract nor repel. This is consistent with the kinematic reversibility argument. Figure 4(d) shows that the particle velocity is almost unaffected by the separation distance L/R_0 , with the smallest magnitude when the two particles are almost in contact. We also find that smaller αR_0 gives rise to larger particle velocity, as it is easier for the fluid to go through the particle. When the particle is bigger, it possesses a smaller velocity due to the non uniformity of the background parabolic flow as well as the interaction with the cylinder wall.

FIGURE 6. Streamlines (left) and normalized velocity contour in the Z -direction v_z/U_0 (right) around two identical particles $a/R_0 = b/R_0 = 0.3$. Since it is an axisymmetric model, we only plot half region of the flow field: $\alpha R_0 = 40$ (top half) and $\alpha R_0 = 100$ (bottom half). Two representative separation distances are considered: $L/R_0 = 0.7$ (a) and $L/R_0 = 1.5$ (b).

To better highlight the hydrodynamic interaction between the two porous particles, we choose two representative separation distance ($L/R_0 = 0.3, 0.7$) with $a/R_0 = b/R_0 = 0.1$ and $\alpha R_0 = 40, 100$, to investigate the corresponding streamlines and velocity contour in the Z -direction in the vicinity of the two particles (figure 5) when they are fixed. The pattern of the streamlines is related to the permeability of the particle. Large permeability (small αR_0) allows fluid to go through the internal region of the particle easily and thus results in streamlines that are almost parallel to the axis of the tube (top half of figure 5(a,b)). Due to higher fluid resistance to move through the porous particle, the Z -direction velocity of the fluid inside the particle is smaller than that outside. When $\alpha R_0 = 100$, the particle is less permeable inducing evident distortion of the streamlines in the vicinity of the two particles (bottom half of figure 5(a,b)). The streamlines and velocity contour are squeezed towards the wall of the tube. The Z -direction velocity of the fluid in the vicinity of the two particles is smaller than that of $\alpha R_0 = 40$ because of the higher fluid resistance inside the particle. The velocity contour is denser when αR_0 is larger indicating larger shear rates, which explains why the particle with larger αR_0 experiences a larger drag force than that with smaller αR_0 . The effect of separation distance L/R_0 on the velocity field can also be seen from figure 5. At small L/R_0 , the boundary layer of the two particles interact and thus as the streamline exits the first particle, it does not recover to its upstream condition before entering the second particle (figure 5a). Such distortion is not visible when the two particles are far apart (figure 5b). The fluid in the gap between the two particles experiences smaller shear rate when they are close to each other.

Figure 6 shows the streamlines and velocity contour in the Z -direction around two identical particles when $a/R_0 = b/R_0 = 0.3$. We can see that the distortion of the streamlines is more significant since the two particles are bigger than those presented in figure 5. The velocity contour further reinforces this phenomenon. Here, the particles squeeze the streamlines towards the wall and the velocity contour in the vicinity around the boundary of the particle are denser than that in figure 5. As mentioned earlier, the denser velocity contour indicates a larger velocity gradient, which results in a larger drag force. The effect of αR_0 on the velocity field for the larger particles here is similar to that in figure 5. However, it is interesting to note that even though the background velocity

FIGURE 7. Drag force coefficient D_a of particle size ratio $b/a = 2, 5, 7$ for $\alpha R_0 = 40$ (a) and $\alpha R_0 = 100$ (b). The size of particle A, $a/R_0 = 0.1$. The horizontal line refers to the single particle case.

is parabolic and the local fluid velocity moving at the axis of the tube is maximum, we find that the velocity of the fluid inside of the particle is smaller than the velocity of the fluid outside the particle nearer to the wall, due to the fluid resistance inside the particle. This effect is more evident when the particle is bigger with sufficiently large αR_0 (i.e. sufficiently large fluid resistance). In the limit of αR_0 approaching zero, the particle is completely permeable (simply becomes a control volume), the velocity field will therefore be unaffected regardless of the size of the particle.

3.4. Two particles with unequal sizes moving along the axis of the tube

We now consider the case when the two particles are unequal. We keep the size of particle A as $a/R_0 = 0.1$ and vary the radius of particle B with $b/a = 2, 5, 7$. Figure 7 shows the normalized drag force coefficient D_a . As mentioned previously, when the two particles are close to each other, a larger region where the velocity gradient of the fluid is small is induced in between them that causes the smaller D_a . As the separation distance L/R_0 increases, D_a approaches to the single particle result. Such effect is more pronounced with the bigger sizes of particle B ($b/a = 2, 5, 7$) due to the stronger particle-particle interaction. Similar to figure 4(a), the permeability influences the drag force. When the particle's permeability is small ($\alpha R_0 = 100$, figure 7b), it is harder for the fluid to go through the internal region of the particle resulting in a larger drag force on particle A. At larger permeability ($\alpha R_0 = 40$, figure 7a), the flow field is more parallel and the particle-particle interaction has relatively less effect on the particle's motion which causes smaller D_a magnitude as well as smaller variation of D_a against L/R_0 .

Figure 8 shows the variation of the coefficient D_b against separation distance L/R_0 . Interestingly, D_b does not vary monotonously when particle B is bigger than particle A except for $a/R_0 = 0.1, b/R_0 = 0.2$ and $\alpha R_0 = 40$ (figure 8a). As L/R_0 increases, the coefficient D_b decreases first to a minimum value and then increases to a constant representing the single particle case. Similar to D_a , smaller permeability ($\alpha R_0 = 100$) causes the porous particle to experience a larger D_b (figure 8 (b,d,f)).

To investigate the detailed flow field around the two particles, we choose three representative separation distances ($L/R_0 = 0.301, 0.35$ and 0.6) with $b/a = 2$ when $\alpha R_0 = 40, 100$ and plot the corresponding streamlines and velocity contour in the Z -direction in the vicinity of the two particles when they are fixed along the axis (figure 9). The permeability of the particle and separation distance L/R_0 can significantly affect the flow field. At $\alpha R_0 = 40$ (top half), it is easier for the fluid to pass through the two particles and

FIGURE 8. Drag force coefficient D_b of particle size ratio $b/a = 2$ (a, b), $b/a = 5$ (c, d), $b/a = 7$ (e, f). The size of particle A, $a/R_0 = 0.1$; $\alpha R_0 = 40$ (left column) and $\alpha R_0 = 100$ (right column). The horizontal line refers to the single particle case.

the streamlines are almost parallel to the axis of the tube with smaller distortion. In addition, bigger particle can induce larger distortion and the streamlines are squeezed towards the wall when the fluid passes through particle B. As discussed earlier, due to the resistance inside the porous particle, the fluid has a smaller Z-direction velocity inside the particle than that outside. This phenomenon is more evident for larger particle B. At $\alpha R_0 = 100$ (bottom half), the particle has smaller permeability and the fluid tends to move around the particle. Especially in the vicinity of particle B, the streamlines are extremely squeezed and more fluid flows outside the particle. Furthermore, the velocity contour at $\alpha R_0 = 100$ is stretched outwards as compared with $\alpha R_0 = 40$ due to the larger distortion. Such effect causes the velocity contour to become denser around the

FIGURE 9. Streamlines (left) and normalized velocity contour in the Z -direction v_z/U_0 (right) around two particles $a/R_0 = 0.1$ and $b/a = 2$. Since it is an axisymmetric model, we only plot half region of the flow field: $\alpha R_0 = 40$ (top half) and $\alpha R_0 = 100$ (bottom half). Three representative separation distances are considered: $L/R_0 = 0.301$ (a), $L/R_0 = 0.35$ (b), and $L/R_0 = 0.6$ (c). The three representative L/R_0 have been marked in figure 8(a) and (b).

two particles, which is more apparent around particle B . Since dense velocity contour indicates large velocity gradient, the two particles experience large drag force which agrees with that previously discussed (figures 5 and 6). Taking a closer look at the velocity contour of $\alpha R_0 = 40$, when the two particles are almost in contact (figure 9a top half) and when the two particles are slightly apart (figure 9b top half), the contour looks almost identical, except that the contour is slightly denser in figure 9 (b). When the separation distance L/R_0 is even larger (figure 9c top half), the velocity contour is even denser. This explains the monotonically increase in D_b in figure 8 (a). However, for small permeability ($\alpha R_0 = 100$), the velocity contour when the two particles are almost in contact is denser than that when the two particles are slightly apart (figure 9(a,b) bottom half). When the separation distance L/R_0 is large (figure 9c bottom half), the velocity contour is denser than that in figures 9(a,b) (bottom half). The corresponding drag coefficient D_b is therefore observed to decrease at small L/R_0 and increase at large L/R_0 (figure 8b).

The drag coefficients K_a and K_b show the drag force due to the movement of one particle while the other is at rest in the absence of the background flow. Figure 10 shows the variation of K_a which is due to the movement of particle A against L/R_0 . The magnitude of K_a decreases monotonously to the value of the single particle case. Furthermore, the bigger the resting particle B is, the larger the magnitude of K_a due to larger velocity gradient between the two particles. Also, when αR_0 is small, the flow field is more parallel and the particle-particle interaction has relatively less effect on the particle particle motion. Therefore, both the magnitude and the variation of K_a against

FIGURE 10. Drag force coefficient K_a of various particle size ratio $b/a = 2, 5, 7$ with $\alpha R_0 = 40$ (a) and $\alpha R_0 = 100$ (b). The size of particle A, $a/R_0 = 0.1$.

L/R_0 is significantly smaller than that of larger αR_0 (figure 10). Similar behaviour is also observed in the coefficient K_b (figure 11).

The coefficients K_{ab} and K_{ba} represent the particle-particle interaction. The drag force has the same direction as the moving neighboring particle which makes coefficients positive (figure 12). Our semi-analytical results have validated that the two coefficients are identical, consistent with what reported in the literature (Purcel 1977; Reuland *et al.* 1978). When two particles travel at low Reynolds number, the resistance matrix is symmetric based on Lorentz reciprocal theorem (Reuland *et al.* 1978; Higdon & Kojima 1981; Happel & Brenner 1983). Thus, the two coefficients K_{ab} and K_{ba} (off diagonal components of the matrix) are identical. When the neighboring particle is moving, it drags the surrounding fluid inducing drag force on the resting particle. In both cases, when the two particles are very close, the coefficient K_{ab} and K_{ba} have the maximum value which arises from a large shear rate in the gap between the moving and the resting particle. As the separation distance increases, the coefficient decays to zero which indicates that the hydrodynamic interaction can be neglected. When αR_0 is larger (figure 12b), the particle is less permeable resulting in a larger magnitude of K_{ab} and K_{ba} .

We investigated the effect of the separation distance L/R_0 and radii ratios b/a on the velocities of particles U_a and U_b (figure 13). The two particles are moving freely along the axis of the tube. We can see that the particle interaction only has significantly effect on U_a (figure 13(a,c)) and not so much on U_b (b,d)). The effect is more significant when b/a is larger. However, the effect of the particle interaction on particle B is weak, which can be seen from the near constant U_b (figure 13(b,d)). In addition, the permeability can influence the particle motion. When $\alpha R_0 = 40$, the particle is more permeable and has larger velocity than that for the case when $\alpha R_0 = 100$. Furthermore, U_a is always larger than U_b for any given αR_0 . This implies the two particles are approaching when the smaller particle A is placed behind the bigger particle B. Using the kinematic reversibility, we can deduce that the two particles will repel from each other if we reverse the direction of the Poiseuille flow, or when the smaller particle is in front of the bigger particle.

3.5. Effect of permeability on the particle-particle interaction

Our previous work on solid spheres (Yao *et al.* 2017) has found that the characteristic distance L^* to neglect particle-particle interaction is three times the sum of the particles radii. This characteristic distance is defined as the distance when the velocity of each particle is within 1 percent difference than that of the single particle case. However,

FIGURE 11. Drag force coefficient K_b of various particle size ratio $b/a = 2$ (a, b), $b/a = 5$ (c, d), $b/a = 7$ (e, f) with $\alpha R_0 = 40$ (left column) and $\alpha R_0 = 100$ (right column). The size of particle A, $a/R_0 = 0.1$.

this characteristic distance is expected to change, since particle's permeability changes the flow field and correspondingly the particles' velocities. As previously discussed, the particle-particle interaction becomes stronger as αR_0 increases. This indicates the characteristic distance should also increase with αR_0 . Moreover, if αR_0 is small enough, the particle-particle interaction can be completely neglected even when the two particles are almost in touch, i.e. in this case $L^*/R_0 = L^*/R_0$. We can then define this particular αR_0 as the characteristic αR_0 . Figure 14(a) shows the relationship between the normalized characteristic distance L^*/R_0 and αR_0 . When αR_0 is smaller than 100, L^*/R_0 increases significantly followed by a gentler linear increase at larger αR_0 . When $a/R_0 = 0.1$ and $b/a = 2$, L^*/R_0 has a linear relationship with αR_0 . Increase in αR_0 , i.e. a less permeable

FIGURE 12. Drag force coefficient K_{ab} (lines) or K_{ba} (symbols) of various particle size ratio $b/a = 2, 5, 7$ with $\alpha R_0 = 40$ (a) and $\alpha R_0 = 100$ (b). The size of particle A, $a/R_0 = 0.1$.

FIGURE 13. (a and c) Normalized velocity of particle A. The horizontal line refers to the velocity of the single particle case. ; (b and d) Normalized velocity of particle B. $\alpha R_0 = 40$ (a, b) and $\alpha R_0 = 100$ (c, d).

porous particle, leads to an increase in the overall internal resistance, which finally results in stronger particle-particle interaction together with larger characteristic distance. Since the particle-particle interaction becomes stronger when b/a is larger, the characteristic $\alpha^* R_0$ decreases as b/a increases (figure 14b).

FIGURE 14. (a) Effect of permeability parameter αR_0 on characteristic separation distance L^*/R_0 . The size of particle A, $a/R_0 = 0.1$ and particle size ratio $b/a = 2, 3, 4$. (b) Effect of particle size ratio on the characteristic permeability parameter $\alpha^* R_0$.

FIGURE 15. Streamlines (left) and normalized velocity contour in the Z-direction v_z/U_0 (right) around two particles $a/R_0 = 0.1$, $b/a = 2$ and $L/R_0 = 0.35$ with different permeabilities. Since it is an axisymmetric model, we only plot half region of the flow field. Two combinations of different permeabilities are considered: $\alpha R_0 = 40$ for particle A (smaller particle), $\alpha R_0 = 100$ for particle B (bigger particle)(top half) and $\alpha R_0 = 100$ for particle A (smaller particle), $\alpha R_0 = 40$ for particle B (bigger particle)(bottom half).

3.6. Two particles with different permeabilities

In this section, we study the case when the two porous particles with different permeabilities and plot the streamlines and velocity contours in the Z-direction in the vicinity of the two particles when they are fixed along the axis (figure 15). When the permeability of particle A ($\alpha R_0 = 40$) is smaller than that of particle B ($\alpha R_0 = 100$) (top half), it is easier for the fluid to pass through particle A and the streamlines are almost parallel to the axis of the tube. The streamlines around particle B have been pushed towards the wall. This phenomenon is more evident in the velocity contours, which have more uniform distribution for particle A as compared with particle B. These features are in accord with our previous discussion. Similarly, if we switch the permeability of the two particle (bottom half), the distribution of the streamlines and velocity contours will be changed accordingly.

4. Conclusion

In this paper, a systematic procedure to calculate the slow viscous flow of two porous spherical particles travelling along the axis of a cylindrical tube is presented. This procedure utilizes the method of reflections which treats the boundary conditions of particles and the tube separately. The stream function and a framework based on cylindrical harmonics are employed to solve the flow field around the particles and the

FIGURE 16. Axonometric drawing of the coordinate system of two particles along the axis of a cylindrical tube.

flow within the tube, respectively. By utilizing this procedure, we consider two cases: two identical particles and then unequal sizes. In both cases, the drag force coefficients and velocities of the particles are obtained as functions of the separation distance between the particles and the permeability of the porous particle. In addition, we analyzed the detailed flow field in the vicinity of the two particles by presenting the streamlines and velocity contour. When the particles are porous, the permeability can significantly alter the flow behavior. We found that the particle-particle interaction, which influences the velocity gradient around the particles, is dependent on the separation distance, particle sizes and permeability of the porous particles. As the permeability of the particles increases, the streamlines are more parallel and the particle-particle interaction has less effect on the particle motion. This is particularly prominent when $\alpha R_0 < 10$. We further showed that smaller permeability and bigger particle size generally tend to squeeze the streamlines and velocity contour towards the wall. It is also found that the characteristic distance to neglect the particle-particle interaction increases as particle's permeability decreases and the characteristic $\alpha^* R_0$ decreases with particle size ratio.

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Appendix A. Coordinate setup and transformation

The expression of the flow fields are represented based on the unit vectors and variables in the current system. We conduct the transformation by converting the unit vectors in the current coordinate system to a Cartesian coordinates as a transitional step before further converting them into the new coordinate system. We then replace the variables in the current coordinate system with that in the next system.

Consider an arbitrary point P in space (figure 16); the position can be described with various coordinates. $P_a(P_b)$ is the projection of P on the surface perpendicular to the axis of the tube and passes through the center of particle $A(B)$. For each particle, there are Cartesian coordinates $(X_{a(b)}, Y_{a(b)}, Z_{a(b)})$; spherical coordinates $(r_{a(b)}, \theta_{a(b)}, \varphi_{a(b)})$, which are distinguished by appending the subscripts a and b in relation to particle A and B with the common origin at the sphere center, respectively. The origin of the cylindrical

coordinates with variables (R, Φ, Z) is set at the center of particle A , so that $\varphi_a = \Phi = \varphi_b$ and $Z_a = Z$. The unit vectors in spherical coordinates with respect to particle A ($\mathbf{i}_a^r, \mathbf{i}_a^\theta, \mathbf{i}_a^\varphi$) can be transformed to the corresponding Cartesian coordinates ($\mathbf{i}_a^X, \mathbf{i}_a^Y, \mathbf{i}_a^Z$) using the following relation:

$$\begin{bmatrix} \cos \varphi_a \sin \theta_a & \cos \varphi_a \cos \theta_a & -\sin \varphi_a \\ \sin \varphi_a \sin \theta_a & \sin \varphi_a \cos \theta_a & \cos \varphi_a \\ \cos \theta_a & -\sin \theta_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a^r \\ \mathbf{i}_a^\theta \\ \mathbf{i}_a^\varphi \end{bmatrix} = \begin{bmatrix} \mathbf{i}_a^X \\ \mathbf{i}_a^Y \\ \mathbf{i}_a^Z \end{bmatrix}. \quad (\text{A } 1)$$

The unit vectors in the Cartesian coordinates with respect to particle A is identical to that of particle B and we have:

$$\begin{bmatrix} \mathbf{i}_a^X \\ \mathbf{i}_a^Y \\ \mathbf{i}_a^Z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_b^X \\ \mathbf{i}_b^Y \\ \mathbf{i}_b^Z \end{bmatrix}. \quad (\text{A } 2)$$

These unit vectors can be transformed from Cartesian coordinates to spherical coordinates with respect to particle B using:

$$\begin{bmatrix} \cos \varphi_b \sin \theta_b & \sin \varphi_b \sin \theta_b & \cos \theta_b \\ \cos \varphi_b \cos \theta_b & \sin \varphi_b \cos \theta_b & -\sin \theta_b \\ -\sin \varphi_b & \cos \varphi_b & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_b^X \\ \mathbf{i}_b^Y \\ \mathbf{i}_b^Z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_b^r \\ \mathbf{i}_b^\theta \\ \mathbf{i}_b^\varphi \end{bmatrix}. \quad (\text{A } 3)$$

Using equations(A1)-(A3) and noting that $\varphi_a = \varphi_b$, we have:

$$\begin{bmatrix} \cos \theta_a \cos \theta_b + \sin \theta_a \sin \theta_b & \cos \theta_a \sin \theta_b - \cos \theta_b \sin \theta_a & 0 \\ \cos \theta_b \sin \theta_a - \cos \theta_a \sin \theta_b & \cos \theta_a \cos \theta_b + \sin \theta_a \sin \theta_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a^r \\ \mathbf{i}_a^\theta \\ \mathbf{i}_a^\varphi \end{bmatrix} = \begin{bmatrix} \mathbf{i}_b^r \\ \mathbf{i}_b^\theta \\ \mathbf{i}_b^\varphi \end{bmatrix}. \quad (\text{A } 4)$$

In addition, the variables $(r_a, \theta_a, \varphi_a)$ can be expressed as a function of $(r_b, \theta_b, \varphi_b)$ by solving the following equations due to the geometry configuration:

$$\left. \begin{aligned} r_a \cos \theta_a - r_b \cos \theta_b &= L, \\ -r_a \sin \theta_a \cos \varphi_a + r_b \sin \theta_b \cos \varphi_a &= 0, \\ r_a \sin \theta_a \sin \varphi_a - r_b \sin \theta_b \sin \varphi_a &= 0 \\ \cos^2 \theta_a + \sin^2 \theta_a &= 1, \\ \cos^2 \varphi_a + \sin^2 \varphi_a &= 1. \end{aligned} \right\} \quad (\text{A } 5)$$

These give:

$$\left. \begin{aligned} r_a &= (r_b^2 + 2Lr_b \cos \theta_b + L^2)^{1/2}, \\ \cos \theta_a &= \frac{L + r_b \cos \theta_b}{(r_b^2 + 2Lr_b \cos \theta_b + L^2)^{1/2}}, \\ \sin \theta_a &= \frac{r_b \sin \theta_b}{(r_b^2 + 2Lr_b \cos \theta_b + L^2)^{1/2}}, \\ \cos \varphi_a &= \cos \varphi_b, \\ \sin \varphi_a &= \sin \varphi_b. \end{aligned} \right\} \quad (\text{A } 6)$$

The expression can then be transformed from the spherical coordinates with respect to particle A to that with respect to particle B . Similarly, the flow field in the spherical coordinates with respect to particle A can be transformed into the cylindrical coordinates $(\mathbf{i}^R, \mathbf{i}^\Phi, \mathbf{i}^Z)$ using the following relations:

$$\begin{bmatrix} \sin \theta_a & \cos \theta_a & 0 \\ 0 & 0 & 1 \\ \cos \theta_a & -\sin \theta_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_a^r \\ \mathbf{i}_a^\theta \\ \mathbf{i}_a^\varphi \end{bmatrix} = \begin{bmatrix} \mathbf{i}^R \\ \mathbf{i}^\Phi \\ \mathbf{i}^Z \end{bmatrix}, \quad (\text{A } 7)$$

$$\left. \begin{aligned} r_a &= (R^2 + Z^2)^{1/2}, \\ \cos \theta_a &= \frac{Z}{(R^2 + Z^2)^{1/2}}, \\ \sin \theta_a &= \frac{R}{(R^2 + Z^2)^{1/2}}, \\ \cos \varphi_a &= \cos \Phi, \\ \sin \varphi_a &= \sin \Phi. \end{aligned} \right\} \quad (\text{A } 8)$$

And the expression can be transformed from the spherical coordinates with respect to particle B to the cylindrical coordinates using the following relations:

$$\begin{bmatrix} \sin \theta_b & \cos \theta_b & 0 \\ 0 & 0 & 1 \\ \cos \theta_b & -\sin \theta_b & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}_b^r \\ \mathbf{i}_b^\theta \\ \mathbf{i}_b^\varphi \end{bmatrix} = \begin{bmatrix} \mathbf{i}^R \\ \mathbf{i}^\Phi \\ \mathbf{i}^Z \end{bmatrix}. \quad (\text{A } 9)$$

$$\left. \begin{aligned} r_b &= [R^2 + (Z - L)^2]^{1/2}, \\ \cos \theta_b &= \frac{Z - L}{[R^2 + (Z - L)^2]^{1/2}}, \\ \sin \theta_b &= \frac{R}{[R^2 + (Z - L)^2]^{1/2}}, \\ \cos \varphi_b &= \cos \Phi, \\ \sin \varphi_b &= \sin \Phi. \end{aligned} \right\} \quad (\text{A } 10)$$

It should be noted that all the above transformations are reversible.

FIGURE 17. Drag force coefficients D_a (a,c) and D_b (b,d) as the iteration proceeds when $\alpha R_0 = 100$. Two representative cases are considered: $a/R_0 = 0.1, b/a = 2, L/R_0 = 0.301$ (a,b) and $a/R_0 = 0.1, b/a = 7, L/R_0 = 0.801$ (c,d).

Appendix B. Convergence test

Each reflection in (2.8) can be solved by using both stream functions and cylindrical harmonics for the particles and the cylinder, respectively. As presented in §2.2.4, only the flow fields derived from the stream functions (the reflection terms with subscripts a and b) contribute to the drag forces acting on the two particles. Due to the method of reflections, the magnitude of the velocity field will be smaller as elaborated in §2.2.1. Since the drag force is derived based on the stream function (see §2.2.4), convergence of drag force coefficients also implies convergence of the stream function and thus the velocity fields. Figure 17 shows the drag force coefficients D_a and D_b which are the cumulative summation of the previous reflections as the iteration proceeds. We consider two cases: $a/R_0 = 0.1, b/R_0 = 0.2, L/R_0 = 0.301$ (a, b) and $a/R_0 = 0.1, b/R_0 = 0.7, L/R_0 = 0.801$ (c, d). By using the method of reflections, the magnitude of the last term of the drag force will decrease as the iteration proceeds and leads to convergence. When the magnitude of the last term is three orders smaller than the sum of the previous terms, the calculation is terminated. We calculate the rate of convergence C_r using the following relation (Schatzman & Schatzman 2002):

$$\lim_{k \rightarrow \infty} \frac{|x_{k+1} - N|}{|x_k - N|} = C_r, \quad (\text{B1})$$

where x_k is the sequence at the k th term of D_a or D_b in figure 17; N is the convergent sequence of x_k . In our calculation, N is chosen to be the value of D_a or D_b when the

iteration terminates to calculate C_r using (B1). As the iteration proceeds, C_r approaches zero in all the cases which indicates the iteration converges superlinearly. In addition, the flow field around the porous particles and inside the cylindrical tube can be solved in the corresponding stream functions and cylindrical harmonics. Since the solution based on stream functions is an infinite series, we can only include a finite number of terms in each reflection. We equally chose 40 discrete points on the particle surface and calculated the velocity value. We found that the first three terms in the stream function series can satisfy the boundary conditions with the error less than 0.1%. Therefore, we only need to include these terms as the leading expression in the calculation.

Appendix C. Details of the calculation for the first four reflections

In this section, we provide a detailed example on how to conduct the calculation for the first four reflections ($\mathbf{v}_a^{(1)}$, $\mathbf{v}_b^{(2)}$, $\mathbf{v}_c^{(3)}$, $\mathbf{v}_a^{(4)}$). Other reflections follow the same procedure. The practical implementation is numerically conducted for given geometrical parameters, including particle radii, permeability, cylinder radius and separation distance. It should be noted that the reflections representing the flow fields due to the porous particles (reflections with subscripts a and b) consist of two components: the flow inside and outside of the porous particles with superscripts i and e , respectively. $\mathbf{v}_c^{(0)}$, representing the background flow, has a parabolic profile given by (2.13). The next reflection $\mathbf{v}_a^{(1)}$ in (2.8), representing the reflection due to the boundary condition of particle A , is required to make the summation $\mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1)}$ satisfy the boundary condition of particle A (see (2.3)-(2.6)). $\mathbf{v}_a^{(1)}$ can be solved using the stream function presented in §2.2.2. To solve the unknown coefficients for $\mathbf{v}_a^{(1)}$ (A_n , B_n , $\tilde{A}_n$ and $\tilde{B}_n$ as shown in (2.23)-(2.26)), we need to construct a set of algebraic equations. We first transform expression of $\mathbf{v}_c^{(0)}$ in the cylindrical coordinates into the spherical coordinates with respect to particle A to calculate $\mathbf{v}_a^{(1)}$ (i.e. the geometrical relations added in the appendix A). The transformed $\mathbf{v}_c^{(0)}$ has the following expression:

$$\begin{aligned} \mathbf{v}_c^{(0)} = & (\cos \theta_a U_0 - \cos \theta_a U_a - \cos \theta_a \sin^2 \theta_a r_a^2 U_0) \mathbf{i}_a^r \\ & - (\sin \theta_a U_0 - \sin \theta_a U_a - \sin^3 \theta_a r_a^2 U_0) \mathbf{i}_a^\theta, \end{aligned} \quad (\text{C } 1)$$

where (r_a, θ_a) giving the radial distance, polar angle are the spherical coordinates with the origins at the center of particle A . ($\mathbf{i}_a^r$, $\mathbf{i}_a^\theta$) are the unit vectors in the spherical coordinates with respect to particle A . We set $r_a = a$ in order to obtain the boundary condition at the surface of particle A and the transformed $\mathbf{v}_c^{(0)}$ is now only a function of θ_a . An arbitrary function of θ can be expanded using either Gegenbauer Polynomial or Legendre Polynomial ((2.31)-(2.34)). The $\mathbf{v}_c^{(0)}$, which is now in spherical coordinates, can be expressed into either Gegenbauer or Legendre polynomial, and substitute them into the boundary condition of particle A given by (2.14). For example, the first boundary condition of (2.14) can be written as:

$$\begin{aligned} \sum_{n=2}^{\infty} -a^{-n-1} (A_n + B_n a^2) P_{n-1}(\cos \theta_a) + \sum_{n=2}^{\infty} \tilde{b}_{n-1} P_{n-1}(\cos \theta_a) = \\ \sum_{n=2}^{\infty} -\frac{1}{a^2} \left[a^n \tilde{A}_n + \sqrt{a} \tilde{B}_n I_{n-\frac{1}{2}}(\alpha a) \right] P_{n-1}(\cos \theta_a) \end{aligned} \quad (\text{C } 2)$$

The first summation term on the left hand side of (C 2) is the general solution outside particle A , $\mathbf{v}_a^{(1,e)}$, given by (2.23). The second summation term on the left hand side is $\mathbf{v}_c^{(0)}$ expressed into Legendre polynomial. The right hand side represents the general solution inside particle A , $\mathbf{v}_a^{(1,i)}$, given by (2.24). By equating term by term under the summation sign, an algebraic equation can be constructed.

$$-a^{-n-1}(A_n + B_n a^2) + \tilde{b}_{n-1} = -\frac{1}{a^2} \left[a^n \tilde{A}_n + \sqrt{a} \tilde{B}_n I_{n-\frac{1}{2}}(\alpha a) \right] \quad (\text{C } 3)$$

Similarly, three other algebraic equations can be constructed from the remaining three boundary conditions in (2.14), using the same procedure.

$$-a^{-n-1} [(-1+n)A_n + (-3+n)a^2 B_n] + \hat{b}_{n-1} = -\frac{1}{a^2} \left[na^n \tilde{A}_n + \sqrt{a}(nI_{n-\frac{1}{2}}(\alpha a) + \alpha a I_{n+\frac{1}{2}}(\alpha a)) \tilde{B}_n \right], \quad (\text{C } 4)$$

$$-2a^{-n-2} [(-1+n)A_n + (-1+n)a^2 B_n] + 2(2-3/n)a^{-n} B_n + \check{b}_{n-1} = \frac{1}{a^3} \left[2(-2+n)a^n \tilde{A}_n + 2\sqrt{a}((-2+n)I_{n-\frac{1}{2}}(\alpha a) + \alpha a I_{n+\frac{1}{2}}(\alpha a)) \tilde{B}_n + \frac{\alpha^2}{n-1} a^{n-1} \tilde{A}_n \right], \quad (\text{C } 5)$$

$$2a^{-n-2} [(-1+n^2)A_n + (-2+n)na^2 B_n] + \breve{b}_{n-1} = \frac{1}{a^{7/2}\alpha} \left[\begin{aligned} &2(n-2)na^{n+1/2}\alpha \tilde{A}_n + 2n(-2+n)(1+2n)I_{n+\frac{1}{2}}(\alpha a) \tilde{B}_n \\ &+ (2n-1)\alpha^2 a^2 I_{n+\frac{1}{2}}(\alpha a) \tilde{B}_n + (2n\alpha a(n-2) + \alpha^3 a^3) I_{n+\frac{3}{2}}(\alpha a) \tilde{B}_n \end{aligned} \right]. \quad (\text{C } 6)$$

The coefficients A_n , B_n , $\tilde{A}_n$ and $\tilde{B}_n$ in the stream functions for this reflection $\mathbf{v}_a^{(1)}$ can then be obtained by solving these four algebraic equations simultaneously. Moving on to the next reflection, similarly, the reflection $\mathbf{v}_b^{(2)}$ in (2.8) is included to make the summation $\mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1)} + \mathbf{v}_b^{(2)}$ satisfy the boundary condition of particle B . The boundary condition is given by (2.15). Thus, $\mathbf{v}_b^{(2)}$ can be solved using similar procedure in the spherical coordinates of particle B . We first transform expression of $\mathbf{v}_c^{(0)}$ in the cylindrical coordinates and $\mathbf{v}_a^{(1,e)}$ in the spherical coordinates with respect to particle A into the spherical coordinates with respect to particle B . The first boundary condition of (2.15) can be written as

$$\sum_{n=2}^{\infty} -b^{-n-1}(A_n + B_n b^2)P_{n-1}(\cos \theta_b) + \sum_{n=2}^{\infty} \tilde{b}_{n-1}P_{n-1}(\cos \theta_b) = \sum_{n=2}^{\infty} -\frac{1}{b^2} \left[b^n \tilde{A}_n + \sqrt{b} \tilde{B}_n I_{n-\frac{1}{2}}(\alpha b) \right] P_{n-1}(\cos \theta_b). \quad (\text{C } 7)$$

The first summation term on the left hand side is the general solution outside particle B , $\mathbf{v}_b^{(2)}$, given by (2.23). The second summation term on the left hand side is $\mathbf{v}_c^{(0)} + \mathbf{v}_a^{(1,e)}$ expressed into Legendre polynomial. The right hand side represents the general solution inside particle B , $\mathbf{v}_b^{(2,i)}$, given by (2.24). Similarly, three other algebraic equations can be constructed from the remaining three boundary conditions in (2.15), using the same procedure. Up to this point, $\mathbf{v}_b^{(2)}$ has been obtained. The next reflection, $\mathbf{v}_c^{(3)}$ in (2.8), is needed to make the summation satisfy the boundary condition of the tube, given

by (2.16). The expressions $\mathbf{v}_a^{(1,e)}$ and $\mathbf{v}_b^{(2,e)}$ have been obtained previously referring to the spherical coordinates of particle A and B , respectively (see (2.23) and (2.25)). We then transform $\mathbf{v}_a^{(1,e)}$ and $\mathbf{v}_b^{(2,e)}$ into cylindrical coordinates (details on the coordinate transformation are provided in the appendix A) and set $R = R_0$ to obtain the boundary condition (2.16), such that $\mathbf{v}_a^{(1)}$ and $\mathbf{v}_b^{(2)}$ now have the following forms:

$$V_R = \sum_{n=2}^{\infty} F_R(Z, n), \quad (\text{C } 8)$$

$$V_Z = \sum_{n=2}^{\infty} F_Z(Z, n). \quad (\text{C } 9)$$

$\mathbf{v}_a^{(1)}$ and $\mathbf{v}_b^{(2)}$ (now expressed as V_R and V_Z) are substituted into (2.50)-(2.53) to solve the unknown coefficients π_{a0} , π_{c0} , ψ_{a0} and ψ_{c0} . Finally, $\mathbf{v}_c^{(3)}$ can be obtained by substituting these coefficients into the general solution (2.36). The next reflection, $\mathbf{v}_a^{(4)}$ in (2.8), is needed to make the summation satisfy the boundary condition of particle A , given by (2.17). The expressions $\mathbf{v}_b^{(2)}$ and $\mathbf{v}_c^{(3)}$ have been solved referring to the spherical coordinates of particle B and cylindrical coordinates of the tube, respectively. We transform $\mathbf{v}_b^{(2)}$ and $\mathbf{v}_c^{(3)}$ into the spherical coordinates of particle A and set $r_a = a$ in order to obtain the boundary condition at the surface of particle A . The calculation procedure of $\mathbf{v}_a^{(4)}$ is the same as $\mathbf{v}_a^{(1)}$. The transformed $\mathbf{v}_b^{(2)}$ and $\mathbf{v}_c^{(3)}$ will be expanded using either Gegenbauer Polynomial or Legendre Polynomial ((2.31)-(2.34)) to construct algebraic equations. It should be noted that we need to numerically conduct the two dimensional integration for $\mathbf{v}_c^{(3)}$. For example, the velocity expression $\mathbf{v}_c^{(3)}$ will not be a function of Φ and we can assume that the velocity components have the following forms in the cylindrical coordinates:

$$v_R = \int_0^{\infty} f_R(\lambda, R, Z) d\lambda, \quad (\text{C } 10)$$

$$v_Z = \int_0^{\infty} f_Z(\lambda, R, Z) d\lambda. \quad (\text{C } 11)$$

v_R and v_Z are only functions of R and Z , together with the dummy variable λ . We conduct the coordinate transformation by replacing the variable and unit vector in the spherical coordinates with respect to particle A and set $r_a = a$ in order to obtain the boundary condition at the surface of particle A . The transformed velocity components are now only a function of θ_a :

$$v_r = \int_0^{\infty} f_r(\lambda, \theta_a) d\lambda, \quad (\text{C } 12)$$

$$v_{\theta} = \int_0^{\infty} f_{\theta}(\lambda, \theta_a) d\lambda, \quad (\text{C } 13)$$

As mentioned earlier, we need to expand the velocity components and the components of the stress tensor using either Gegenbauer Polynomial or Legendre Polynomial to solve the unknown coefficients in the stream functions (A_n , B_n , $\hat{A}_n$ and $\hat{B}_n$ as shown in (2.23)-(2.26)). For example, we can expand v_r using Legendre Polynomial to get the first boundary condition of (2.14). The expansion of an arbitrary function of θ using Legendre Polynomial is shown in (2.33)-(2.34). We substitute v_r into (2.34) to obtain the

expansion coefficients $\tilde{b}_{n-1}$:

$$\tilde{b}_{n-1} = \frac{1}{2}(2n-1) \int_{-1}^1 \int_0^\infty f_r(\lambda, \theta_a) d\lambda P_{n-1}(\cos \theta_a) d(\cos \theta_a) \quad (\text{C } 14)$$

Due the complexity of the problem, we numerically conduct the two dimensional integration with respect to λ and θ_a . It should be noted that even it is numerical integration, we can calculate an arbitrary number of terms when needed for convergence without loss of generality. This indicates we can calculate an arbitrary number of the coefficients ($A_n, B_n, \tilde{A}_n$ and $\tilde{B}_n$ as shown in (2.23)-(2.26)) when needed for convergence. The remaining calculation procedure of $\mathbf{v}_a^{(4)}$ is the same as $\mathbf{v}_a^{(1)}$. Here, we completed the calculation for the first four reflections ($\mathbf{v}_a^{(1)}, \mathbf{v}_b^{(2)}, \mathbf{v}_c^{(3)}, \mathbf{v}_a^{(4)}$).

REFERENCES

- AURIAULT, J.-L. 2009 On the domain of validity of brinkman's equation. *Transport Porous Med.* **79** (2), 215–223.
- BEAVERS, G.S. & JOSEPH, D.D. 1967 Boundary conditions at a naturally permeable wall. *J. Fluid Mech.* **30** (1), 197–207.
- BHATTACHARYA, S., MISHRA, C. & BHATTACHARYA, S. 2010 Analysis of general creeping motion of a sphere inside a cylinder. *J. Fluid Mech.* **642**, 295–328.
- BOCQUET, L. & CHARLAIX, E. 2010 Nanofluidics, from bulk to interfaces. *Chem. Soc. Rev.* **39** (3), 1073–1095.
- BRENNER, H. 1963 The stokes resistance of an arbitrary particle. *Chem. Eng. Sci.* **18** (1), 1–25.
- BRENNER, H. 1964 The stokes resistance of an arbitrary particle-ii. an extension. *Chem. Eng. Sci.* **19** (9), 599–629.
- BRENNER, H. & HAPPEL, J. 1958 Slow viscous flow past a sphere in a cylindrical tube. *J. Fluid Mech.* **4** (2), 195–213.
- BRINKMAN, H.C. 1949 A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Appl. Sci. Res.* **1** (1), 27–34.
- BUEHLER, K.L. & ANDERSON, J.L. 2002 Solvent effects on the permeability of membrane-supported gels. *Ind. Eng. Chem. Res.* **41** (3), 464–472.
- CHEN, T.C. & SKALAK, R. 1970 Stokes flow in a cylindrical tube containing a line of spheroidal particles. *Appl. Sci. Res.* **22** (1), 403–441.
- CHUDASAMA, N.A., PRASAD, K. & SIDDHANTA, A.K. 2016 Agarose functionalization: Synthesis of peg-agarose amino acid nano-conjugate - its structural ramifications and interactions with bsa in a varying ph regime. *Carbohydr. Polym.* **151**, 735–742.
- DAIGUJI, H., YANG, P., SZERI, A.J. & MAJUMDAR, A. 2004 Electrochemomechanical energy conversion in nanofluidic channels. *Nano Lett.* **4** (12), 2315–2321.
- FENG, H., WONG, T.N., CHE, Z. & MARCOS 2016 Chaotic micromixer utilizing electro-osmosis and induced charge electro-osmosis in eccentric annulus. *Phys. Fluids* **28** (6).
- FOQUET, M., KORLACH, J., ZIPFEL, W., WEBB, W.W. & CRAIGHEAD, H.G. 2002 Dna fragment sizing by single molecule detection in submicrometer-sized closed fluidic channels. *Anal. Chem.* **74** (6), 1415–1422.
- GREENSTEIN, T. & HAPPEL, J. 1968 Theoretical study of the slow motion of a sphere and a fluid in a cylindrical tube. *J. Fluid Mech.* **34** (4), 705–710.
- HAPPEL, J. & BRENNER, H. 1983 *Low Reynolds Number Hydrodynamics: With Special Applications to Particulate Media*. Springer Netherlands.
- HASIMOTO, H. 1976 Slow motion of a small sphere in a cylindrical domain. *J. Phys. Soc. Jpn.* **41** (6), 2143–2144.
- HIGDON, J.J.L. & KOJIMA, M. 1981 On the calculation of stokes' flow past porous particles. *Int. J. Multiphas Flow* **7** (6), 719–727.
- HO, C.-M. & TAI, Y.-C. 1998 Micro-electro-mechanical-systems (mems) and fluid flows. *Annu. Rev. Fluid Mech.* **30**, 579–612.
- HUANG, Y., CHEN, J., WONG, T. & LIOW, J.-L. 2016 Experimental and theoretical

- investigations of non-newtonian electro-osmotic driven flow in rectangular microchannels. *Soft Matter* **12** (29), 6206–6213.
- HUANG, Y., WANG, Y.L. & WONG, T.N. 2017 Ac electric field controlled non-newtonian filament thinning and droplet formation on the microscale. *Lab Chip* **17** (17), 2969–2981.
- JEFFREY, D.J. 1982 Low-reynolds-number flow between converging spheres. *Mathematika* **29** (1), 58–66.
- KAPUR, V., CHARKOUDIAN, J.C., KESSLER, S.B. & ANDERSON, J.L. 1996 Hydrodynamic permeability of hydrogels stabilized within porous membranes. *Ind. Eng. Chem. Res.* **35** (9), 3179–3185.
- KIM, S. 1987 Stokes flow past three spheres: An analytic solution. *Phys. Fluids* **30** (8 , Aug. 1987), 2309–2314.
- KIM, S. & KARRILA, S.J. 2005 *Microhydrodynamics: Principles and Selected Applications*. Dover Publications.
- KIM, S.-K., KOO, H.-J., LIU, J. & BRAUN, P.V. 2017 Flexible and wearable fiber microsupercapacitors based on carbon nanotube-agarose gel composite electrodes. *Acs Appl. Mater. Inter.* **9** (23), 19925–19933.
- KOH, J.B.Y. & MARCOS 2014 Effect of dielectrophoresis on spermatozoa. *Microfluid Nanofluid* **17** (4), 613–622.
- KRAINOV, V.P., SMIRNOV, B.M. & TERESHONOK, D.V. 2014 Permeability of porous materials for liquid and gases. *Europhys. Lett.* **108** (3).
- LAMB, H. & CAFLISCH, R. 1993 *Hydrodynamics*. Cambridge University Press.
- LEE, C., YANG, E.-H., MYUNG, N.V. & GEORGE, T. 2003 A nanochannel fabrication technique without nanolithography. *Nano Lett.* **3** (10), 1339–1340.
- LEICHTBERG, S., PFEFFER, R. & WEINBAUM, S. 1976 Stokes flow past finite coaxial clusters of spheres in a circular cylinder. *Int. J. Multiphas Flow* **3** (2), 147–169.
- MARCOS, KANG, Y.J., OOI, K.T., YANG, C. & WONG, T.N. 2005a Frequency-dependent velocity and vorticity fields of electro-osmotic flow in a closed-end cylindrical microchannel. *J. Micromech. Microeng.* **15** (2), 301–312.
- MARCOS, OOI, K.T., YANG, C., CHAI, J.C. & WONG, T.N. 2005b Developing electro-osmotic flow in closed-end micro-channels. *Int. J. Eng. Sci.* **43** (17-18), 1349–1362.
- MARCOS, TRAN, N.P., SAINI, A.R., ONG, K.C.H. & CHIA, W.J. 2014 Analysis of a swimming sperm in a shear flow. *Microfluid Nanofluid* **17** (5), 809–819.
- MAZUR, P. & VAN SAARLOOS, W. 1982 Many-sphere hydrodynamic interactions and mobilities in a suspension. *Physica A* **115** (1-2), 21–57.
- MIKA, A.M. & CHILDS, R.F. 2001 Calculation of the hydrodynamic permeability of gels and gel-filled microporous membranes. *Ind. Eng. Chem. Res.* **40** (7), 1694–1705.
- MO, G. & SANGANI, A.S. 1994 A method for computing stokes flow interactions among spherical objects and its application to suspensions of drops and porous particles. *Phys. Fluids* **6** (5), 1637–1652.
- MOKADAM, R. G. 1961 Thermodynamic analysis of the darcy law. *J. Appl. Mech.* **28** (2), 208–212.
- NAVARDI, S. & BHATTACHARYA, S. 2013 General methodology to evaluate two-particle hydrodynamic friction inside cylinder-bound viscous fluid. *Comput. Fluids* **76**, 149–169.
- NAVARDI, S., BHATTACHARYA, S. & WU, H. 2015 Stokesian simulation of two unequal spheres in a pressure-driven creeping flow through a cylinder. *Comput. Fluids* **121**, 145–163.
- NEALE, G., EPSTEIN, N. & NADER, W. 1973 Creeping flow relative to permeable spheres. *Chem. Eng. Sci.* **28** (10), 1865–1874.
- NEALE, G. & NADER, W. 1974 Practical significance of brinkman’s extension of darcy’s law: Coupled parallel flows within a channel and a bounding porous medium. *Can. J. Chem. Eng.* **52** (4), 475–478.
- NIELD, D.A. & BEJAN, A. 2006 *Convection in Porous Media*. Springer New York.
- O’NEILL, M.E. 1968 A sphere in contact with a plane wall in a slow linear shear flow. *Chem. Eng. Sci.* **23** (11), 1293–1298.
- O’NEILL, M.E. 1969 Exact solutions of the equations of slow viscous flow generated by the asymmetrical motion of two equal spheres. *Appl. Sci. Res.* **21** (1), 452–466.
- PADMAYATHI, B.S., AMARANATH, T. & NIGAM, S.D. 1993 Stokes flow past a porous sphere using brinkman’s model. *Z. Angew. Math. Phys.* **44** (5), 929–939.

- PRAKASH, J. & SEKHAR, G.P.R. 2012 Arbitrary oscillatory stokes flow past a porous sphere using brinkman model. *Meccanica* **47** (5), 1079–1095.
- PURCEL, E.A. 1977 Life at low reynolds number. *Am. J. Phys.* **45** (1), 3–11.
- REULAND, P., FELDERHOF, B.U. & JONES, R.B. 1978 Hydrodynamic interaction of two spherically symmetric polymers. *Physica A* **93** (3-4), 465–475.
- SAAD, E.I. & FALTAS, M.S. 2014 Slow motion of a porous sphere translating along the axis of a circular cylindrical pore subject to a stress jump condition. *Transport Porous Med.* **102** (1), 91–109.
- SCHATZMAN, M. & SCHATZMAN, M. 2002 *Numerical Analysis: A Mathematical Introduction*. Clarendon Press.
- SHEN, X., MARCOS & FU, H.C. 2017 Traction reveals mechanisms of wall effects for microswimmers near boundaries. *Phys Rev E* **95** (3).
- SINHA, P.M., VALCO, G., SHARMA, S., LIU, X. & FERRARI, M. 2004 Nanoengineered device for drug delivery application. *Nanotechnology* **15** (10), S585–S589.
- SONG, Y., KIM, M.-O., KWON, D.-S., KIM, Y.-J. & KIM, J. 2012 Facile fabrication of sub-20-nm nanochannels based on crystallinity- dependent anisotropic etching of silicon. *Microelectron. Eng.* **98**, 309–312.
- SONSHINE, R.M. & BRENNER, H. 1966 The stokes translation of two or more particles along the axis of an infinitely long circular cylinder. *Appl. Sci. Res.* **16** (1), 425–454.
- URBAKH, M., DAIKHIN, L. & KLAFTER, J. 1995 Velocity profiles and the brinkman equation in nanoscale confined liquids. *Europhys. Lett.* **32** (2), 125–130.
- VAINSHTAIN, P., SHAPIRO, M. & GUTFINGER, C. 2002 Creeping flow past and within a permeable spheroid. *Int. J. Multiphas Flow* **28** (12), 1945–1963.
- VAN DER HEYDEN, F.H.J., BONTHUIS, D.J., STEIN, D., MEYER, C. & DEKKER, C. 2006 Electrokinetic energy conversion efficiency in nanofluidic channels. *Nano Lett.* **6** (10), 2232–2237.
- WANG, H. & SKALAK, R. 1969 Viscous flow in a cylindrical tube containing a line of spherical particles. *J. Fluid Mech.* **38** (1), 75–96.
- WILSON, H.J. 2013 Stokes flow past three spheres. *J. Comput. Phys.* **245**, 302–316.
- YAO, X., MARCOS & WONG, T.N. 2017 Slow viscous flow around two particles in a cylinder. *Microfluid Nanofluid* **21** (10).
- ZHONG, X. & DUAN, F. 2016 Flow regime and deposition pattern of evaporating binary mixture droplet suspended with particles. *Eur. Phys. J. E* **39** (2), 1–6.
- ZHONG, X., XIE, H. & DUAN, F. 2017 Deposition patterns from evaporating sessile droplets with suspended mixtures of multi-sized and multi-species hydrophilic and non-adsorbing nanoparticles. *Appl. Therm. Eng.* **111**, 1565–1572.