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Bhatnagar, Rohit; Lin, Bing

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The Joint Transshipment and Production Control Policies for Multi-Location Production/Inventory Systems

Rohit Bhatnagar^a Bing Lin^{b*}

^a Division of Information Technology and Operations Management

Nanyang Business School, Nanyang Technological University

Nanyang Avenue, 639798, Singapore

^b Business School, Jiangsu Normal University, 221116, China

Abstract

In this paper we study the joint transshipment and production control policies for multi-location production/inventory systems in which items are manufactured and stocked at each location to meet incoming demand. We formulate the problem as a make-to-stock queue to gain insight into the following questions: (1) how much demand at a location should be covered by transshipment from other locations, and when to produce or stop production? (2) is there a simple structure associated with the optimal policy, and whether a simple decision rule can be implemented for transshipment control? (3) can effective heuristic policies be developed to solve the multi-location problems? For the two-location problem, we characterize the optimal policy as monotone switching-curve policy. To address the multi-location problem, we develop two heuristic policies. One is obtained from the one-step policy improvement based on policy iteration and the other from the one-step lookahead method based on the approximation of the optimal cost function. Numerical examples are used to illustrate the optimal and heuristic policies and compare their performance for various cases.

Keywords: *Inventory; Transshipment; Make-To-Stock; Dynamic Programming/Optimal Control; Heuristic Policy*

* Corresponding Author: Email: linbing@pmail.ntu.edu.sg; Tel: 86-516-83867883; Fax: 86-516-83536936

1. Introduction

With today's state of the art information systems, firms can utilize data flow among remotely located production facilities to drive efficient communication, coordination and control in operations. Such "networked manufacturing" enables firms to optimize the production planning and inventory management functions of the entire business and has gained considerable attention in industry as well as academia. In recent years, Adidas, the world's second largest sportswear company, has envisioned a faster, leaner, and more consumer-centric future and has initiated the "Speedfactory" project to cope with the increasing trend of low volume and customized production. The firm plans to shift production closer to the customers in its end market by starting up local manufacturing in several "mini-factories". Localized production has also been implemented in other industries. For instance, Nissan, the Japanese auto maker, has plants both in Tennessee, USA, and in Canton, China to satisfy local demand. Similarly, firms such as Caterpillar and Tesla have established localized production in China to satisfy rising demand in Asia and to counter increasing logistics costs. This trend towards localized production will likely intensify in future and this will also increase the complexity of networked manufacturing. One way to deal with this complexity is by utilizing the available data flow to implement a more responsive *transshipment* policy. Transshipment is a traditional inventory management method involving cross-shipping of inventory across locations. This will make networked manufacturing more flexible and efficient.

This research is motivated by the question of how to achieve flexible and efficient production planning and inventory control in a networked production system with many localized mini-factories. A traditional supply chain typically has multiple levels. For instance, a manufacturer supplies distributors and distributors in turn supply retailers. Inventory control at all three levels often involves the pooling ("reduced inventory") versus proximity ("responsive service") tradeoff. If the firm could cross-ship stocks from one location to another, the localized production system can achieve the benefits of pooling as well as quick response to customers without the physical centralization of stocks. For this purpose, we consider the *inventory transshipment* method which has been widely used in automotive, machine tool, and retailing industries. Transshipment has

proven to be an effective managerial tool in reducing inventory costs by *virtual pooling* of inventories at different locations.

Various applications of inventory transshipment have been studied in previous literature. These papers can be roughly categorized into single-period problems and multi-period problems and model formulations for the multi-period problems can be further classified into the periodic-review models and the continuous-review models. The single-period problems are relatively simple and optimal solutions can be derived for the two-location as well as the general N location problems. These attracted some attention in the earlier days of research on transshipment, as reported for instance in, Gross (1963), Krishnan and Rao (1965), Karmarkar and Patel (1977), Herer and Rashit (1999). For the periodic-review problems, two-location problems are the main focus of researchers because the optimal policy usually can be characterized and readily derived in computation. In this line of research, Das (1975) considered a two-location stochastic inventory system with joint inventory and transshipment controls. They established both stock transfer and storage rules under certain regularity conditions. Tagaras (1989) studied a two-location inventory system with zero replenishment lead time. The system with non-negligible replenishment lead time was further analyzed by Tagaras and Cohen (1992). Archibald et al. (1997) studied a two-location inventory system in which stockouts can be satisfied by either transshipments or emergency orders. Yang and Qin (2007) considered a two-location capacitated production/inventory system and introduced a new concept of *virtual transshipment* into their model. Virtual transshipment allows demand emerging from one location to be allocated to the other plant and satisfied directly by the stock therein without a physical lateral transshipment between the two plants. If the other plant has negative inventory level it would backorder the allocated demand. Hu et al. (2008) considered a lost-sales model with uncertain production capacities and characterized the optimal production and transshipment policies for a two-location production/inventory system. Chen et al. (2015) employed the concept of L^q -convexity/-concavity, a variable transformation/inversion technique, to prove the structural properties of the optimal value function in Hu et al. (2008). With this new method, the analysis of the model in Hu et al. (2008) was significantly simplified. More recently, Abouee-Mehrzi et al. (2015) considered a finite-horizon lost-sales inventory

system for two retailers. Each of the retailers can replenish inventory either from a supplier or via transshipment from the other retailer. They characterized the optimal joint replenishment and transshipment policies as switching curves. Ramakrishna et al. (2015) studied a two-item two-warehouse inventory control problem that allowed transshipment between warehouses and emergency orders. They proposed a heuristic approach to address the replenishment and transshipment control decisions. The general N location problems are notoriously difficult to analyze due to the high-dimensional state space of the intended problems. Herein, we outline a few studies in this area. Karmarkar (1981) characterized the optimal policies for the multi-location multi-period problems with identical costs. Robinson (1990) considered both optimal and heuristic policies for inventory ordering in the context of multiple retailer outlets with transshipments among these outlets. The optimal solution can be derived analytically either for the two-outlet case or the case with identical costs at all outlets. Liu et al. (2016) used virtual transshipment to enable virtual inventory pooling in a multi-location inventory problem. Their study differed from other transshipment literature in that they did not consider physical transshipment but virtual stock transfer. With no transshipment costs associated with the problem, they characterized the optimal policy and provided simple algorithms to compute the policy. Recently, Meissner and Senicheva (2018) applied approximate dynamic programming to study the multi-location lost-sales inventory system with lateral transshipment and derived a near-optimal transshipment policy.

For the continuous-review models, previous research centered around the N -location inventory systems and the predetermined $(S, S-1)$ and/or (R, Q) policies are often employed and evaluated. Lee (1987), Axsäter (1990), Sherbrooke (1992), Alfredsson and Verrijdt (1999) considered the inventory system with one-for-one stock replenishment policy where transshipments are triggered by stockouts. Further, Grahovac and Chakravarty (2001) analyzed a similar inventory system in which transshipments take place as soon as the inventory level is below a certain level. Axsäter (2003) considered a number of parallel warehouses facing compound Poisson demand and developed a simple performance-guaranteed decision rule for lateral transshipment based on the improvement from the no-transshipping policy. Moreover, Paterson et al. (2012) enhanced Axsäter (2003)'s decision rule with a policy which further allowed additional

stock redistribution in response to stockout after an initial improvement from the no-transshipping policy. Seidscher and Minner (2013) studied both proactive and reactive transshipments in multi-location problems and compared the performance of various transshipment rules. Alvarez et al. (2014) took an approximation approach to the multi-item, multi-warehouse problem with emergency shipment from the upstream supplier and lateral transshipment. They found significant cost savings from lateral transshipment compared with the option of using only emergency shipment. In contrast, our model formulation is in the continuous-time production/inventory setting that has been rarely studied previously. In particular, we concentrate on characterizing the structural properties of the optimal policy for the two-location problem. To address the general N -location problem, we develop new heuristic policies distinct from the existing policies.

The readers are referred to Paterson et al. (2011) which provides a comprehensive literature review, classification, and future research directions for the inventory transshipment problem in various contexts.

While some papers are closely related to our research, there are many differences in our model which makes it unique. Regarding the difference in model formulation, Yang and Qin (2007), Hu et al. (2008) and Abouee-Mehrzi et al. (2015) are periodic-review models while Zhao et al. (2008) and this paper are continuous-time models. Further, for the periodic-review models, Yang and Qin (2007) and Hu et al. (2008) are single-echelon two-plant models where Yang and Qin (2007) study the backorder case while Hu et al. (2008) consider the lost-sales case. Abouee-Mehrzi et al. (2015) is a finite-horizon two-echelon lost-sales model with two-retailers and one supplier. For the continuous-time models, Zhao et al. (2008) and this paper are both single-echelon models. Zhao et al. (2008) consider only the backorder case while this paper studied both the backorder and lost-sales cases. Moreover, the introduction of batch demand into our model makes the proof technique in this paper significantly different from Zhao et al. (2008) for the unit demand case. Specifically, an important motivation for writing this paper is that we seek to extend the work of Zhao et al. (2008) who put a very restrictive condition on the cost parameters (yielding nearly identical unit backorder costs at the two locations) in order to characterize the optimal policy. Finally, only this paper studies the general multi-location case, i.e. models including the case of more than two locations. In

summary, the model formulation and the analysis of this paper is significantly different from the above related papers. This paper adds insights to the literature on the two-location production/inventory systems with transshipment by bridging the missing link of the continuous-time model with lost sales, as well as removing the restrictive condition on cost parameters in Zhao et al. (2008) for characterizing the optimal policy.

More generally, the main contributions of this paper are as follows:

- (1) We characterize the optimal joint transshipment and production policies of the two-location problem for both backorder and lost-sales cases. Monotone properties for the switching curves and optimal cost function are established.
- (2) Two heuristic policies are developed for the multi-location problem: one is the one-step improved policy based on the policy improvement method; the other is the one-step lookahead policy derived from the approximation of the optimal cost function. We further characterize the heuristic policies as the type of switching-curve (surface) policies.
- (3) A simple decision rule associated with the transshipment control is derived under certain restrictions on the cost parameters.

The rest of the paper is organized as follows: In Section 2, we formulate the models for the two-location problems and characterize the optimal policies. In Section 3, we study the multi-location problem and develop two heuristic policies. Finally, in Section 4, we offer concluding comments.

2. The Two-Location Problem

2.1 Model Formulation

Suppose a firm manufactures and stocks identical products at two locations to meet customer orders. If demand at a location cannot be met by local stock, it is possible to cover part or all of the demand by transferring stocks from the other location. To minimize the total inventory and transshipment costs across all locations, we model the joint transshipment and production controls in the context of make-to-stock queues. Customer orders arrive in accordance with a Poisson process with rate λ_i , $i = 1, 2$. The quantities $\{D_{i,t}\}$ demanded at the arrival epochs $t = 1, 2, \dots$ are discrete, independent and identically distributed (i.i.d.) random variables which follow a probability distribution

$P_i(D_i = d_i) = p_i(d_i)$, $d_i = 1, 2, \dots, m_i$, $\sum_{d_i=1}^{m_i} p_i(d_i) = 1$, $i = 1, 2$, for all t . Moreover, the quantities demanded at two locations are independent of each other and also independent of the arrival processes. Here, the assumption of independent demand at the two locations makes sense for several cases, especially when the product is a convenience good (e.g. bottled water) and consumer behavior at one location will not affect that at other locations. The production time at each location follows an exponential distribution with parameter $1/\mu_i$, $i = 1, 2$. Further, assume $\lambda_1 E(D_1) + \lambda_2 E(D_2) < \mu_1 + \mu_2$, i.e., the total production capacities are greater than the total demands of the two locations. This is a typical assumption for the make-to-stock queue models which guarantees the stability of the queueing systems in the long run. The problem is illustrated in Figure 1.

Figure 1: The Two-Location Transshipment Problem

In this paper, we regard the virtual transshipment cost as the nonnegative difference of two delivery costs, i.e., the delivery costs from the production facilities to the place where a demand is generated. Suppose that production facility at location 1 mainly serves region 1 and production facility at location 2 mainly serves region 2. Further, the delivery cost from location i , $i = 1, 2$, to any place within region j , $j = 1, 2$, is a constant c_{ij} . Define the transshipment costs $r_{12} = c_{12} - c_{22} \geq 0$ and $r_{21} = c_{21} - c_{11} \geq 0$. The costs r_{21} and r_{12} are, respectively, the unit transshipment cost incurred by transferring stocks from the production facility at location 2 to any place in region 1, and from the production facility at location 1 to any place in region 2. Note that, intuitively, when the transshipment costs are significantly larger in comparison to the backorder costs, transferring stocks from the other location to meet the local demand is not cost-efficient and is less likely to happen. Moreover, as in many previous studies, we assume that the

transshipments take no time. It is often the case that transshipment lead time is significantly shorter when compared with the replenishment/production lead time.

The state $x = (x_1, x_2)^T$, indicating the inventory levels at two locations, lies in the state space $X = Z^2$. Let $\pi(x) = h_1x_1^+ + b_1x_1^- + h_2x_2^+ + b_2x_2^-$ be the inventory cost rate where $x_1^+ = \max\{x_1, 0\}$, $x_1^- = \max\{-x_1, 0\}$, $x_2^+ = \max\{x_2, 0\}$, $x_2^- = \max\{-x_2, 0\}$. Here, h_i and b_i , $i=1, 2$, are the holding cost and backorder cost per unit time, respectively. The evolution of the system is influenced by the control $a = (a_1^1, a_1^2, a_2^1, a_2^2, \tilde{a}_1, \tilde{a}_2)$, where a_1^1 and a_2^2 represent the local stock used to fill local demand, also referred to as the action of satisfying local demand with local stock, while a_1^2 and a_2^1 represent the stock transshipped from location 2 to 1 and from location 1 to 2, respectively, also referred to as the action of meeting demand by transshipments. These actions are constrained by the demand, i.e. $a_1^1 + a_1^2 = d_1$ and $a_2^1 + a_2^2 = d_2$. The production control action $\tilde{a}_i$, $i = 1, 2$, takes two possible values: $\tilde{a}_i = 0$ (no production) and $\tilde{a}_i = 1$ (production). Thus, control a is constrained by a finite set $A(x)$, i.e. $a \in A(x)$. The admissible policy u consists of a sequence of functions $u = \{u_0, u_1, u_2, \dots\} \in U$, where each function u_k maps state x into the control $a = u_k(x) \in A(x)$ for each x in X , and U is the set of all admissible policies.

Given the initial state x_0 , we try to find an admissible policy $u = \{u_0, u_1, u_2, \dots\}$ that minimizes the total expected cost with discount rate $\alpha > 0$ over an infinite horizon:

$$V_u(x_0) = E_u^x \left\{ \int_0^\infty e^{-\alpha t} \pi(x_t) dt + \sum_{k=1}^\infty e^{-\alpha \tau_{1,k}} a_1^2(u) r_{21} + \sum_{k=1}^\infty e^{-\alpha \tau_{2,k}} a_2^1(u) r_{12} \right\}.$$

where $\tau_{1,k}$ and $\tau_{2,k}$, $k = 1, 2, \dots$ are the respective customer arrival times at two locations. The actions $a_1^2(u)$ and $a_2^1(u)$ are associated with the transshipment controls in policy u and $a_1^2(u)r_{21}$ and $a_2^1(u)r_{12}$ are the transshipment costs incurred at $\tau_{1,k}$ and $\tau_{2,k}$ respectively. Given that the initial state is x_0 and policy u is employed, the random sequence $x_t = \{x_{n(t)}, t \geq 0\}$ forms a controlled Markov chain. The expectation is relative to x_t . Among all admissible u , we seek an optimal one u^* to minimize $V_u(x_0)$. Then the optimal cost function, denoted by $f(x)$, is

$$f(x_0) \equiv V_{u^*}(x_0) = \min_{u \in U} V_u(x_0).$$

Let $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$ and define the operators T_{1,d_1} , T_{2,d_2} , T_1 and T_2 as follows:

$$T_{1,d_1}f(x) = \min_{\substack{a_1^1 + a_1^2 = d_1 \\ a_1^1 \geq 0, a_1^2 \geq 0}} \left\{ a_1^2 r_{21} + f(x - a_1^1 e_1 - a_1^2 e_2) \right\},$$

$$T_{2,d_2}f(x) = \min_{\substack{a_2^1 + a_2^2 = d_2 \\ a_2^1 \geq 0, a_2^2 \geq 0}} \left\{ a_2^1 r_{12} + f(x - a_2^1 e_1 - a_2^2 e_2) \right\},$$

$$T_1 f(x) = \min \{ f(x + e_1), f(x) \},$$

$$T_2 f(x) = \min \{ f(x + e_2), f(x) \}.$$

The operators T_1 and T_2 are associated with the production decisions at two locations. For the construction of the transshipment control operators T_{1,d_1} and T_{2,d_2} , let us give a relatively detailed illustration. For instance, for T_{1,d_1} , it is a minimization operator that can be applied to $f(x)$. Here, a customer order arrives at location 1 with demand size d_1 , which is a finite random variable; the nonnegative decisions a_1^1 and a_1^2 then are to determine the quantity to be met with local stock at production facility 1 and how much to be filled by stock from production facility 2, respectively. Hence, $a_1^1 + a_1^2 = d_1$. After applying T_{1,d_1} to $f(x)$, the transshipment cost $a_1^2 r_{21}$ is incurred by shipping a_1^2 units from location 2 to location 1, leading to the change of state from (x_1, x_2) to $(x_1 - a_1^1, x_2 - a_1^2)$. For a more detailed construction of various operators, readers are referred to Koole (1998).

Similar to Yang and Qin (2007), we also allow virtual transshipments, that is, demand generated at one location can be switched to and backordered by the production facility at the other location. This mechanism is more flexible and cost efficient than physical transshipment where demand can be only backordered by the local plant. Moreover, we can characterize the optimal policy as a switching-curve type without the restrictive condition on cost parameters in Zhao et al. (2008). If virtual transshipment is not allowed, the optimal policy cannot be characterized as a simple switching-curve type for the general two-location problem and thereby the transshipment control is not easily implemented in practice.

Here, we assume that transshipments take no time. Then, let the uniformized transition rate be $\Lambda = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$. Following the uniformization technique in

Lippman (1975), it can be shown that the optimal cost function f satisfies the Bellman equation

$$f = Tf, \quad (1)$$

where the operator T is defined by

$$Tf(x) = \frac{1}{\alpha + \Lambda} \left[\pi(x) + \sum_{i=1}^2 \sum_{d_i=1}^{m_i} \lambda_i p_i(d_i) T_{i,d_i} f(x) + \sum_{i=1}^2 \mu_i T_i f(x) \right]. \quad (2)$$

In the right-hand-side (RHS) of (2), at each transition epoch, a customer order arrives at location i , $i = 1, 2$, with probability λ_i/Λ and order d_i units with probability $p_i(d_i)$, or the production of an item is completed with probability μ_i/Λ at location i , $i = 1, 2$. A more detailed construction of the RHS of (2) can be found in Bertsekas (1995).

Remark: In the following discussion, we denote T^n as the composition of T with itself n times and T_u as the operator associated with a stationary policy $u \in U$. And we use $f \leq g$ to indicate the point-wise inequality $f(x) \leq g(x)$, for any $x \in X$.

2.2 Characterization of the Optimal Policies

In this subsection, we will characterize the optimal policies for the joint transshipment and production controls. It will be shown that the structure of the optimal policies can be characterized as a set of switching curves. Let Ω be the set of functions on Z^2 and if $f \in \Omega$, then

- (a) $f(x+e_1) - f(x) \uparrow x_1 \uparrow x_2$,
- (b) $f(x+e_2) - f(x) \uparrow x_2 \uparrow x_1$,
- (c) $f(x+e_1) - f(x+e_2) \uparrow x_1 \downarrow x_2$,
- (d) $f(x+e_2) - f(x+e_1) \uparrow x_2 \downarrow x_1$.

Notations $\uparrow$ and $\downarrow$ refer to nondecreasing and nonincreasing, respectively. The $f(x+e_1) - f(x) \uparrow x_1$ in (a) implies the discrete convexity of $f(x)$ in x_1 and $f(x+e_2) - f(x) \uparrow x_2$ in (b) denotes the discrete convexity of $f(x)$ in x_2 . Both $f(x+e_1) - f(x) \uparrow x_2$ in (a) and $f(x+e_2) - f(x) \uparrow x_1$ in (b) refer to the supermodularity

of $f(x)$. Here, (c) and (d) are identical and referred to as superconvexity. Furthermore, from properties (a)-(d), we can readily deduce the following properties:

$$(a') f(x + \gamma e_1) - f(x) \uparrow x_1 \uparrow x_2,$$

$$(b') f(x + \beta e_2) - f(x) \uparrow x_1 \uparrow x_2,$$

$$(c') f(x + \gamma e_1) - f(x + \beta e_2) \uparrow x_1 \downarrow x_2 \text{ when } \gamma \geq \beta,$$

$$(d') f(x + \beta e_2) - f(x + \gamma e_1) \uparrow x_2 \downarrow x_1 \text{ when } \beta \geq \gamma,$$

where γ and β are both positive integers. The properties (a')-(d') will be employed in the proof of Lemma 1 for convenience.

Lemma 1. $T_{1,d_1}f, T_{2,d_2}f, T_1f, T_2f$ and $Tf \in \Omega$, if $f \in \Omega$.

Proof: Please see the online appendix B for the proof. ■

Lemma 1 states that structural properties (a)-(d) are preserved by $T_{1,d_1}, T_{2,d_2}, T_1, T_2$ and T . Then, we need to characterize the structural properties of the optimal cost function f . Based on Lemma 1, we can prove that the f in (1) retains the structural properties (a)-(d), leading to the following Lemma 2.

Lemma 2. The optimal cost function $f \in \Omega$.

Proof: Please see the appendix A for the proof. ■

To characterize the optimal policy, we need to define some switching functions. For location 1, noting $f(x + e_2) - f(x + e_1) \uparrow x_2$ in property (d), we define the switching function associated with the decision of satisfying demand with local stock as

$$S_{11}(x_1, a_1^1, d_1) = \min\{x_2 \mid f(x - a_1^1 e_1 - a_1^2 e_2) - f(x - (a_1^1 - 1)e_1 - (a_1^2 + 1)e_2) - r_{21} \geq 0, \\ \text{given } x_1, a_1^1 = 1, \dots, d_1, a_1^1 + a_1^2 = d_1\}$$

And the switching function associated with the decision of transshipping stock from location 2 to location 1 is

$$S_{21}(x_1, a_1^2, d_1) = \min\{x_2 \mid f(x - a_1^1 e_1 - a_1^2 e_2) - f(x - (a_1^1 - 1)e_1 - (a_1^2 + 1)e_2) - r_{21} \geq 0, \\ \text{given } x_1, a_1^2 = 0, 1, \dots, d_1 - 1, a_1^1 + a_1^2 = d_1\}$$

Noting that $f(x + e_1) - f(x) \uparrow x_1$ in property (a), we can define the switching function associated with the production decision as

$$S_3(x_2) = \min\{x_1 \mid f(x + e_1) - f(x) \geq 0, \text{ given } x_2\}.$$

For location 2, we have similar definitions as

$$S_{12}(x_2, a_2^1, d_2) = \min\{x_1 \mid f(x - a_2^1 e_1 - a_2^2 e_2) - f(x - (a_2^1 + 1)e_1 - (a_2^2 - 1)e_2) - r_{12} \geq 0, \\ \text{given } x_2, a_2^1 = 0, 1, \dots, d_2 - 1, a_2^1 + a_2^2 = d_2\}$$

$$S_{22}(x_2, a_2^2, d_2) = \min\{x_1 \mid f(x - a_2^1 e_1 - a_2^2 e_2) - f(x - (a_2^1 + 1)e_1 - (a_2^2 - 1)e_2) - r_{12} \geq 0, \\ \text{given } x_2, a_2^2 = 1, \dots, d_2, a_2^1 + a_2^2 = d_2\}$$

$$S_4(x_1) = \min\{x_2 \mid f(x + e_2) - f(x) \geq 0, \text{ given } x_1\}.$$

For the above switching functions, we always set the switching function value to be ∞ if its corresponding set is empty. To take a closer look, note that each of $S_{11}(x_1, a_1^1, d_1)$ and $S_{22}(x_2, a_2^2, d_2)$ is differentiated by the decision of satisfying demand with local stock and demand size. That is, for each value of a_1^1 and d_1 , there exists a switching function $S_{11}(x_1)$ with respect to x_1 ; and for each value of a_2^2 and d_2 , there is a switching function $S_{22}(x_2)$. The same interpretation can be applied to $S_{21}(x_1, a_1^2, d_1)$ and $S_{12}(x_2, a_2^1, d_2)$. The $S_3(x_2)$ and $S_4(x_1)$ are associated with the production decisions. The switching functions are the so-called switching curves which are shown to be monotone in actions.

Lemma 3. For the switching curves associated with the decisions of filling demand at two locations, $S_{11}(x_1, a_1^1, d_1)$ is strictly decreasing in a_1^1 ; $S_{21}(x_1, a_1^2, d_1)$ is strictly increasing in a_1^2 ; $S_{12}(x_2, a_2^1, d_2)$ is strictly increasing in a_2^1 ; $S_{22}(x_2, a_2^2, d_2)$ is strictly decreasing in a_2^2 .

Proof: Please see the appendix A for the proof. ■

Lemma 3 ensures that the series of switching curves associated with the action of filling demand will never meet and cross each other. Otherwise, it will cause a contradiction in decision-making. For instance, if $S_{12}(x_2, a_2^1, d_2)$ and $S_{12}(x_2, a_2^1 + 1, d_2)$ meet and cross each other, then for those states below $S_{12}(x_2, a_2^1 + 1, d_2)$ and above $S_{12}(x_2, a_2^1, d_2)$, the corresponding optimal decisions are to transship two units based on the decision rule associated with $S_{12}(x_2, a_2^1 + 1, d_2)$ while it is also optimal to transship nothing based on the decision rule associated with $S_{12}(x_2, a_2^1, d_2)$. Lemma 3 also implies the existence of states between $S_{21}(x_1, a_1^2, d_1)$ and $S_{21}(x_1, a_1^2 + 1, d_1)$ (including on $S_{21}(x_1, a_1^2, d_1)$) and between $S_{12}(x_2, a_2^1, d_2)$ and $S_{12}(x_2, a_2^1 + 1, d_2)$ (including on

$S_{12}(x_2, a_2^1, d_2)$). As a result, we should discuss two cases (i)-b and (ii)-b in Theorem 1 to identify the decisions for these states. In the meanwhile, we show the existence of an optimal stationary policy for (1).

Lemma 4. There exists an optimal stationary policy.

Proof: Please see the appendix A for the proof. ■

Then we characterize the optimal policy as shown in the following theorem.

Theorem 1. The optimal actions are prescribed by the switching curves; the optimal stationary policy is characterized by the switching curves.

- (i). For $S_{21}(x_1, a_1^2, d_1)$, there are three cases:
 - (a) There is no transshipment to region 1 if $x_2 < S_{21}(x_1, 0, d_1)$;
 - (b) The $(a_1^2 + 1)$ units of demand at region 1 are satisfied by the transshipment from location 2 if inventory level x_2 satisfies $S_{21}(x_1, a_1^2, d_1) \leq x_2 < S_{21}(x_1, a_1^2 + 1, d_1)$, $a_1^2 = 0, 1, \dots, d_1 - 2$;
 - (c) The d_1 units at location 2 are transshipped to region 1 if $x_2 \geq S_{21}(x_1, d_1 - 1, d_1)$.
- (ii). For $S_{12}(x_2, a_2^1, d_2)$, there are three cases:
 - (a) There is no transshipment to region 2 if $x_1 < S_{12}(x_2, 0, d_2)$;
 - (b) The $(a_2^1 + 1)$ units of demand at region 2 are satisfied by the transshipment from location 1 if inventory level x_1 satisfies $S_{12}(x_2, a_2^1, d_2) \leq x_1 < S_{12}(x_2, a_2^1 + 1, d_2)$, $a_2^1 = 0, 1, \dots, d_2 - 2$;
 - (c) The d_2 units at location 1 are transshipped to region 2 if $x_1 \geq S_{12}(x_2, d_2 - 1, d_2)$.
- (iii). At location 1, produce when $x_1 < S_3(x_2)$; otherwise, stop production.
- (iv). At location 2, produce when $x_2 < S_4(x_1)$; otherwise, stop production.

Proof: Please see the appendix A for the proof. ■

Optimal policies for filling demand with local stock are associated with $S_{11}(x_1, a_1^1, d_1)$ and $S_{22}(x_2, a_2^2, d_2)$. Decision rules can be developed similarly to (i) and (ii). However, it is also convenient to use (i) and (ii) to compute a_1^1 and a_2^2 based on the

equations $a_1^1 + a_1^2 = d_1$ and $a_2^1 + a_2^2 = d_2$. Hence, we do not include the decision rules of $S_{11}(x_1, a_1^1, d_1)$ and $S_{22}(x_2, a_2^2, d_2)$ in Theorem 1.

Theorem 1 gives us only the decision rules for guiding transshipment and production. To gain insight into a comprehensive graphic delineation of the switching curves, we give a more detailed characterization of the switching curve by presenting some monotone properties in the following proposition.

Proposition 1.

- (a) $S_{11}(x_1, a_1^1, d_1) \uparrow x_1 \uparrow d_1 \uparrow r_{21}$; $S_{21}(x_1, a_1^2, d_1) \uparrow x_1 \downarrow d_1 \uparrow r_{21}$; $S_{12}(x_2, a_2^1, d_2) \uparrow x_2 \downarrow d_2 \uparrow r_{12}$;
 $S_{22}(x_2, a_2^2, d_2) \uparrow x_2 \uparrow d_2 \uparrow r_{12}$;
 (b) $S_3(x_2) \downarrow x_2$; $S_4(x_1) \downarrow x_1$; $S_3(x_2) \geq 0$; $S_4(x_1) \geq 0$; $\lim_{x_2 \rightarrow +\infty} S_3(x_2)$ and $\lim_{x_1 \rightarrow +\infty} S_4(x_1)$ exist.

Proof: Please see the appendix A for the proof. ■

In part (a) of the above proposition, $S_{21}(\cdot)$ is associated with the decision for transshipment from location 2 to region 1. When stock level x_1 at location 1 increases, it becomes less likely to transfer stocks from location 2 to region 1, which is reflected in the increasing of $S_{21}(\cdot)$. As the demand value d_1 increases, it is more likely to make stock transshipment from location 2, which is reflected in the decreasing of $S_{21}(\cdot)$. With the increasing transshipment costs, the switching curves associated with transshipment decisions are increasing accordingly. Intuitively, it becomes less beneficial to transship due to the higher transshipment costs. In part (b), the switching curves for production are monotone and associated with nonnegative inventories. Intuitively, when the inventory at the other location is increasing, it is less likely to increase local inventory. This is reflected by a decreasing production switching curve with respect to the inventory at the other location. Further, when the inventory at the other location is increasing significantly, the local production switching curve gradually becomes a threshold level.

For the multi-location production/inventory systems with transshipment, cost parameters and/or other parameters often play a pivotal role in charactering the optimal policy and establishing some decision rules. Next, before presenting a simple decision rule based upon the restrictive condition on some cost parameters, we first investigate the monotone properties related to the question of how cost parameters and other problem parameters affect the optimal cost.

Proposition 2. $f(x) \uparrow r_{21} \uparrow r_{12} \uparrow h_i \uparrow b_i \downarrow d_i \downarrow \mu_i$, for $i = 1, 2$, and all $x \in X$.

Proof: The results are readily proved by the value iteration method and hence the proof is omitted. ■

Moreover, if some parameters satisfy certain conditions, we can derive a simple decision rule.

Condition (I): $h_j - h_i \leq \alpha r_{ji}$, $i, j = 1, 2$.

Then we have the following decision rule.

Proposition 3. If Condition (I) is satisfied, the inequality $f(x - ke_i) - f(x - (k-1)e_i - e_j) \leq r_{ji}$, $i, j = 1, 2$, holds when $x_i \geq k$.

Proof: Please see the online appendix B for the proof. ■

Proposition 3 implies that, under Condition (I), it is optimal to satisfy the demand with local stock until it is depleted. Intuitively, when the unit holding cost rates are not much different and stocks are still available at two locations, it is not optimal for one location to borrow stocks from the other which will incur additional transshipment costs, noting that, the savings from holding cost cannot compensate for the transshipment cost.

In Zhao et al. (2008), to establish the structural properties (a)-(d), the cost parameters need to satisfy

Condition (II): $b_i - b_j \leq \alpha r_{ji}$, $i, j = 1, 2$ and $i \neq j$.

Based on Condition (II), it can be proved that the following inequality holds for the optimal cost function: $f(x - e_i) - f(x - e_j) \leq r_{ji}$, $i, j = 1, 2$, holds when $x_j \leq 0$. These inequalities are required to prove that the operators associated with transshipment control preserve the structural properties (a)-(d). Since α usually takes a very small value, this condition is quite restrictive. Furthermore, in Zhao et al.'s model, there is an additional option of transshipment controls at the production completion epochs which is also essential to establish the structural properties of the optimal cost function.

2.3 A Numerical Example

Consider a two-location problem with $\lambda_1 = 1$, $\lambda_2 = 1.2$ and $\mu_1 = 4$, $\mu_2 = 4$. Demand sizes at two locations are assumed to follow the probability distribution $P_i(D_i = d_i) = p_i(d_i)$, $d_i = 1, 2, 3$ with $P_i(D_i = 1) = 0.5$, $P_i(D_i = 2) = 0.3$, $P_i(D_i = 3) = 0.2$, $i = 1, 2$. The transshipment

costs are $r_{12} = 5$ and $r_{21} = 5$. The holding cost rate and backorder cost rate are $h_1 = 1$, $h_2 = 1$, $b_1 = 10$, $b_2 = 9$. We derive the optimal decisions by value iteration from $f_n = Tf_{n-1}$. The continuous-time discount rate α is set to be 0.1 to achieve a fast convergence in iterations.

To deal with the infinite state space, Ha (1997) truncated the state space by linear approximation of the optimal cost function along the boundaries. But their method cannot be applied to our case of compound Poisson demand. Instead, we compute the optimal cost function by directly truncating the state space. If the final optimal costs associated with the state space $[\xi_1, \eta_1] \times [\xi_2, \eta_2]$ are desired, then we apply n iterations from the initial truncated state space $[\xi_1 - nd_{1,\max}, \eta_1 + n] \times [\xi_2 - nd_{2,\max}, \eta_2 + n]$. Here, $d_{i,\max}$, $i = 1, 2$, is the maximum amount of the demand.

For the above example, after 597 iterations from $f_0(x) = 0$, we obtain $f_{597}(0,0) = 82.767998$ and $f_{596}(0,0) = 82.765656$, yielding $f_{197}(0,0) - f_{196}(0,0) = 0.0023422$. Hence, the optimal cost for zero initial stocks at two locations is about 82.77. Correspondingly, for instance, the obtained optimal transshipment decisions (for $d_i = 3$, $i = 1, 2$) and the production decisions are listed in Figure 2 and Figure 3, respectively.

Figure 2: The Switching Curves for Optimal Transshipment Decisions

$(\lambda_1 = 1, \lambda_2 = 1.2, \mu_1 = 4, \mu_2 = 4, P_i(D_i = 1) = 0.5, P_i(D_i = 2) = 0.3, P_i(D_i = 3) = 0.2, i = 1, 2,$
 $r_{12} = 5, r_{21} = 5, h_1 = 1, h_2 = 1, b_1 = 10, b_2 = 9, \alpha = 0.1)$

Figure 3: The Switching Curves for Optimal Production Decisions

$(\lambda_1 = 1, \lambda_2 = 1.2, \mu_1 = 4, \mu_2 = 4, P_i(D_i = 1) = 0.5, P_i(D_i = 2) = 0.3, P_i(D_i = 3) = 0.2, i = 1, 2,$
 $r_{12} = 5, r_{21} = 5, h_1 = 1, h_2 = 1, b_1 = 10, b_2 = 9, \alpha = 0.1)$

2.4 A Lost-Sales Model

Here, we consider a lost-sales model for the two-location transshipment problem with both discounted and long-run average cost criteria. The orders arrive according to the Poisson processes and assume $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$. Let R_1 and R_2 be the unit revenue for accepting orders at two locations (or equivalently, the unit penalty costs for lost sales). Define the operators U_1, U_2, T_1 and T_2 on Z_+^2 by

$$U_1 V(x) = \begin{cases} \min \{V(x - e_1) - R_1, r_{21} + V(x - e_2) - R_1\} & x_1 > 0, x_2 > 0 \\ V(x - e_1) - R_1 & x_1 > 0, x_2 = 0 \\ \min \{V(x), r_{21} + V(x - e_2) - R_1\} & x_1 = 0, x_2 > 0 \\ V(x) & x_1 = 0, x_2 = 0 \end{cases},$$

$$U_2 V(x) = \begin{cases} \min \{V(x - e_2) - R_2, r_{12} + V(x - e_1) - R_2\} & x_1 > 0, x_2 > 0 \\ V(x - e_2) - R_2 & x_1 = 0, x_2 > 0 \\ \min \{V(x), r_{12} + V(x - e_1) - R_2\} & x_1 > 0, x_2 = 0 \\ V(x) & x_1 = 0, x_2 = 0 \end{cases},$$

$$T_1 V(x) = \min \{V(x + e_1), V(x)\},$$

$$T_2 V(x) = \min \{V(x + e_2), V(x)\}.$$

Here, U_1 and U_2 are associated with transshipment controls while T_1 and T_2 are for the production decisions. Then uniformize the transition rate as $\Lambda = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$ and follow the uniformization technique in Lippman (1975), we derive the Bellman equation

$$V_\alpha = TV_\alpha, \quad (3)$$

where

$$TV_\alpha(x) = \frac{1}{\alpha + \Lambda} \left[h(x) + \sum_{i=1}^2 \lambda_i U_i V_\alpha(x) + \sum_{i=1}^2 \mu_i T_i V_\alpha(x) \right]. \quad (4)$$

In (3), we append a subscript α to the optimal cost function V , indicating that V_α is associated with the Markov decision process with a continuous-time discount rate of α . Such a notation will also appear in the following Theorem 3. Analogously, the construction of (4) is parallel to that of (2).

To establish the structural properties and characterize the optimal policy, we need the assumptions $R_1 - R_2 \leq r_{21}$ and $R_2 - R_1 \leq r_{12}$ and the following properties in addition to the Properties (a) – (d) in Section 2.2:

$$(e) \ R_1 + V(x + e_1) \geq V(x),$$

$$(f) \ R_2 + V(x + e_2) \geq V(x).$$

The assumptions $R_1 - R_2 \leq r_{21}$ (i.e. $R_1 - r_{21} \leq R_2$) and $R_2 - R_1 \leq r_{12}$ (i.e. $R_2 - r_{12} \leq R_1$) imply that the marginal profit earned by satisfying local demand with local stock is always higher than that obtained by transshipping stock to the other location to meet its demand. Properties (e) and (f) imply that the expected cost brought down by producing one more unit is limited by the unit revenue R_1 (equivalent to the unit penalty cost for lost sales). Analogously, by the approach in proof of Lemma 1, we can show that the

structural properties (a)-(f) are preserved by U_1 , U_2 , T_1 and T_2 . Similar to Lemma 2 we can establish the structural properties (a)-(f) for $V_\alpha(x)$.

Then we define some switching functions to characterize the optimal policies. For decisions at location 1:

$$\tilde{S}_1(x_1) = \min \{x_2 \mid V_\alpha(x - e_1) - V_\alpha(x - e_2) - r_{21} \geq 0, \text{ for } x_1 > 0, x_2 > 0\},$$

$$\tilde{S}_1(x_1 = 0) = \min \{x_2 \mid V_\alpha(x) - V_\alpha(x - e_2) - r_{21} + R_1 \geq 0\},$$

$$\tilde{S}_3(x_2) = \min \{x_1 \mid V_\alpha(x + e_1) - V_\alpha(x) \geq 0, \text{ given } x_2\}.$$

The $\tilde{S}_2(x_2)$, $\tilde{S}_2(x_2 = 0)$, and $\tilde{S}_4(x_1)$ for decisions at location 2 can be defined analogously. Then we characterize the optimal policy in the following theorem.

Theorem 2. (i) There exists an optimal stationary policy;

(ii) The optimal decisions at location 1 are:

- (a) For $x_1 > 0$ and $x_2 > 0$, satisfy the demand with local stock if $x_2 < \tilde{S}_1(x_1)$, otherwise, satisfy the demand with stock from location 2;
- (b) For $x_1 > 0$ and $x_2 = 0$, satisfy the demand with local stock;
- (c) For $x_1 = 0$ and $x_2 > 0$, satisfy the demand with stock from location 2 if $x_2 \geq \tilde{S}_1(x_1 = 0)$, otherwise, the demand is lost;
- (d) For $x_1 = 0$ and $x_2 = 0$, the demand is lost.
- (e) Produce when $x_1 < \tilde{S}_3(x_2)$; otherwise, stop production;

(iii) The optimal decisions at location 2 are:

- (a) For $x_1 > 0$ and $x_2 > 0$, satisfy the demand with local stock if $x_1 < \tilde{S}_2(x_2)$, otherwise, satisfy the demand with stock from location 1;
- (b) For $x_1 = 0$ and $x_2 > 0$, satisfy the demand with local stock;
- (c) For $x_1 > 0$ and $x_2 = 0$, satisfy the demand with stock from location 1 if $x_1 \geq \tilde{S}_2(x_2 = 0)$, otherwise, the demand is lost;
- (d) For $x_1 = 0$ and $x_2 = 0$, the demand is lost.
- (e) Produce when $x_2 < \tilde{S}_4(x_1)$; otherwise, stop production.

Proof: The proof parallels that of the preceding backorder case and hence it is omitted. ■

We now consider the long-run average cost criterion. Without loss of optimality, we add to the original problem two constraints: no production at location 1 when $R_1 < h_1 x_1 / \Lambda$, and no production at location 2 when $R_2 < h_2 x_2 / \Lambda$. The constraints state that if the unit revenue (equivalently, the unit penalty cost for lost sales) is less than the expected holding cost until the next transition epoch, it is better to stop production. Then the original problem is converted into a problem of finite state space and action set.

Theorem 3. (i) The relative cost function $v(x) = \lim_{\alpha \rightarrow 0} (V_\alpha(x) - V_\alpha(0))$ exists and retains the structural properties (a)-(f); the optimal long-run average cost $g = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\alpha + \Lambda} V_\alpha(0)$ exists; (ii) $v(x)$ and g satisfy the optimality equation $g/\Lambda + v(x) = Tv(x)$. The stationary policy associated with those decisions of (ii) and (iii) in Theorem 2 (with $V_\alpha(x)$ replaced by $v(x)$ in the switching functions) is optimal and attains the minimum in the right hand side (RHS) of the above optimality equation.

Proof: Please see the appendix A for the proof. ■

To address the lost-sales problem, we develop a simple heuristic policy as follows which performs very well. Under this heuristic policy, the transshipment controls for the case $x_1 > 0$ and $x_2 > 0$ of the operators U_1 and U_2 are redefined as:

$$U_1 v(x) = \begin{cases} v(x - e_1) - R_1 & x_1 > 0, x_2 \geq 0 \\ \min \{v(x), r_{21} + v(x - e_2) - R_1\} & x_1 = 0, x_2 > 0, \\ v(x) & x_1 = 0, x_2 = 0 \end{cases}$$

$$U_2 v(x) = \begin{cases} v(x - e_2) - R_2 & x_1 \geq 0, x_2 > 0 \\ \min \{v(x), r_{12} + v(x - e_1) - R_2\} & x_1 > 0, x_2 = 0 \\ v(x) & x_1 = 0, x_2 = 0 \end{cases}$$

For the redefined operators, demand is always filled with local stock and possible transshipment could happen only when local inventory is exhausted.

For the production control, we apply the base-stock policy to find two integer thresholds k_1 and k_2 which are referred to as the base stock levels at two locations. Then, according to the base-stock policy, the operators for production control can be written as:

$$T_1 v(x) = v(x + e_1) I \{x_1 < k_1\} + v(x) I \{x_1 \geq k_1\},$$

$$T_2 v(x) = v(x + e_2) I \{x_2 < k_2\} + v(x) I \{x_2 \geq k_2\}.$$

To reduce the workload of a complete two-dimensional search, we first apply the value iteration method to find the optimal switching curves $\tilde{S}_3(x_2)$ and $\tilde{S}_4(x_1)$ and then conduct a search for k_1 from $1 + \max_{x_2} \tilde{S}_3(x_2)$ to 1 and for k_2 from $1 + \max_{x_1} \tilde{S}_4(x_1)$ to 1 in such a sequence. Typically, k_1 and k_2 can be obtained after a few steps of search in our numerical examples. We compare the performance of the optimal and heuristic policies for 16 cases with different parameters. The results are shown in Table 1. In Table 1, we use “optimal” to represent the long-run average cost associated with optimal policy and “heuristic” to represent the long-run average cost associated with the heuristic policy. Obviously, the optimal cost and the cost of heuristic policy are not much different for all cases. Especially, for the cases 6 and 2 in which the transshipment costs and unit revenues are close in number, the two long-run average costs are almost identical.

Table 1: Comparison of Optimal and Heuristic Polices for the Lost-Sales Models

example	r_{21}	r_{12}	h_1	h_2	R_1	R_2	λ_1	λ_2	μ_1	μ_2	Optimal	Heuristic	k_1	k_2	Difference (%)
1	5	5	1	1	10	10	1	1.2	1.5	1.5	-15.66	-15.62	3	3	-0.26
2	8	8	1	1	10	10	1	1.2	1.5	1.5	-15.1	-15.09	3	3	-0.07
3	3	8	1	1	10	10	1	1.2	1.5	1.5	-15.59	-15.56	2	4	-0.19
4	5	5	1	2	10	10	1	1.2	1.5	1.5	-14.28	-14.24	3	2	-0.28
5	5	5	3	1	10	10	1	1.2	1.5	1.5	-13.51	-13.39	1	4	-0.89
6	5	5	1	1	6	6	1	1.2	1.5	1.5	-7.74	-7.74	2	3	0.00
7	5	5	1	1	10	15	1	1.2	1.5	1.5	-21.36	-21.29	3	4	-0.33
8	5	5	1	1	10	10	1.4	1.2	1.5	1.5	-18.36	-18.31	4	4	-0.27
9	5	5	3	1	10	10	1.4	1.2	1.5	1.5	-15.77	-15.73	2	4	-0.25
10	5	5	1	2	10	10	1	1.4	1.5	1.5	-15.47	-15.36	3	3	-0.71
11	5	5	1	2	10	10	1.4	1.4	1.5	1.5	-18.06	-17.97	5	3	-0.50
12	5	5	1	1	10	10	1	1.2	1.5	2	-16.19	-16.14	3	3	-0.31
13	5	5	1	1	10	10	1	1.2	2	2	-16.59	-16.57	2	3	-0.12
14	5	5	1	2	10	10	1	1.2	2	2	-15.21	-15.14	2	2	-0.46
15	5	5	2	1	10	10	1	1.2	2	2	-15.33	-15.19	2	3	-0.91
16	5	5	2	1	10	10	1	1.8	2	1.5	-18.46	-18.36	2	6	-0.54

3. The Multi-Location Transshipment Problem

3.1 Model Formulation

It seems very difficult to characterize the optimal policy for the general k -location problems,. Alternatively, we can develop heuristic policies. Consider a multi-location problem, demand arrival rate and production rate are assumed to be λ_i and μ_i , $i = 1, 2, \dots$,

k , respectively. The probability distribution of the demand size is $P_i(D_i = d_i) = p_i(d_i)$, $i = 1, 2, \dots, k$. Assume that transshipment is possible between any two locations. Here, our only task is to develop the heuristic policy. Hence, we focus on the case that demand originating in one region cannot be backordered by the plant of the other location. Then, define the operators by

$$T_{i,d_i}J(x) = \min_{\substack{a_i^j + \sum_{j \neq i} a_i^j = d_i \\ a_i^i \geq 0 \\ x_j \geq a_i^j \geq 0, j \neq i \\ a_i^j = 0, \text{ if } x_j < 0}} \left\{ \sum_{j \neq i} r_{ji} a_i^j + J(x - a_i^i e_i - \sum_{j \neq i} a_i^j e_j) \right\},$$

$$T_i J(x) = \min \{J(x + e_i), J(x)\},$$

where r_{ji} is the unit transshipment cost from location j to i , $i \neq j$, and $i, j = 1, \dots, k$. The a_i^i denotes the decision of meeting demand with local stock at location i , and a_i^j the transshipment decision from location j to i , $i \neq j$, $i, j = 1, \dots, k$. Following the uniformization technique in Lippman (1975), by rescaling the time to achieve $\alpha + \bar{\Lambda} = 1$, we can derive the Bellman equation

$$J = TJ,$$

where $TJ(x) = \pi(x) + \sum_{i=1}^k \sum_{d_i=1}^{m_i} \lambda_i p_i(d_i) T_{i,d_i} J(x) + \sum_{i=1}^k \mu_i T_i J(x)$ can be constructed analogously to that of (2).

3.2 The One-Step Improved Policy

It is hard to characterize the optimal policy of the above multi-location problem. Note that the policy-iteration algorithm of Markov decision process usually achieves the largest cost improvements in the first several iterations. This suggests that we can develop a heuristic policy based on the one-step policy iteration from any admissible policy. For instance, the decision rule in Axsäter (2003) can be regarded as a one-step improved policy from an initial policy of no-transshipment, i.e. a policy that transshipment is forbidden. Readers are referred to Tijms (2003) for a more detailed discussion on this topic.

We follow the procedure below to develop and characterize a heuristic policy, referred to as the *one-step improved policy*. First, we decompose the k -location problem into k independent single-location problems without transshipments among them. Under the no-transshipment policy, we derive the optimal cost function associated with the multi-location problem based on the sum of the optimal cost functions associated with the single-location problems. Second, we apply one-step policy iteration from the no-transshipment policy to compute the heuristic policy for the k -location problem. Moreover, we characterize the heuristic policy as a monotone switching-curve (or surface) policy.

We first formulate the model for the k -location problem with no transshipments. Let $x = (x_1, x_2, \dots, x_k)^T \in X$ be the vector of inventory levels, where the state space $X = \mathbb{Z}^k$.

Define the penalty cost function $\pi(x) = \sum_{i=1}^k \pi_i(x)$, $\pi_i(x_i) = h_i x_i^+ + b_i x_i^-$, $i = 1, 2, \dots, k$. Let

$\bar{\Lambda} = \sum_{i=1}^k \lambda_i + \sum_{i=1}^k \mu_i$. Following the uniformization technique in Lippman (1975), we rescale

the time to achieve $\alpha + \bar{\Lambda} = 1$. Then following the construction process in Bertsekas (1995) we can derive that the optimal cost function $\bar{J}$ satisfies the Bellman equation

$$\bar{J} = \bar{T} \bar{J},$$

where $\bar{T} \bar{J}(x) = \pi(x) + \sum_{i=1}^k \sum_{d_i=1}^{m_i} \lambda_i p_i(d_i) \bar{J}(x - d_i e_i) + \sum_{i=1}^k \mu_i \min \{ \bar{J}(x + e_i), \bar{J}(x) \}$.

Here, assume $\lambda_i E(D_i) < \mu_i$, $i = 1, 2, \dots, k$. Since there are no transshipments among these locations, production and inventory control at each location can be regarded as an independent single-location problem. For each single-location problem, we obtain

$$\bar{J}_i(x_i) = \pi_i(x_i) + \sum_{d_i=1}^{m_i} \lambda_i p_i(d_i) \bar{J}_i(x_i - d_i) + \mu_i \min \{ \bar{J}_i(x_i + 1), \bar{J}_i(x_i) \} + (\bar{\Lambda} - \lambda_i - \mu_i) \bar{J}_i(x_i),$$

for $i = 1, 2, \dots, k$. Note that for each single-location problem we use the identical uniformized transition rate $\alpha + \bar{\Lambda}$. Then we can show that $\bar{J}$ can be derived by adding up $\bar{J}_i(x_i)$.

Proposition 4. $\bar{J}(x) = \sum_{i=1}^k \bar{J}_i(x_i)$.

Proof: Please see the online appendix B for the proof. ■

It is well known that policy iteration algorithm usually achieves the largest cost improvements in first several iterations. This suggests we can develop a heuristic policy based on the one-step policy iteration from any admissible policy. Then $\bar{J}$ can be regarded as the cost function associated with no-transshipment policy and derived based on Proposition 4. Then we can compute $T\bar{J}(x)$ to obtain an improved policy u , yielding

$$T_u \bar{J}(x) = T\bar{J}(x).$$

Proposition 5. $J_u \leq \bar{J}$.

Proof: the proof is simple and hence it is omitted. ■

Proposition 5 asserts that the policy u for the original multi-location problem always performs no worse than the no-transshipment policy.

Next, we characterize the policy u . Based on Proposition 4, we can derive the following structural properties:

- (g) $\bar{J}(x + e_i) - \bar{J}(x) \uparrow x_i$,
- (h) $\bar{J}(x + e_i) - \bar{J}(x) = \bar{J}(x + e_i + e_j) - \bar{J}(x + e_j), i \neq j$,
- (i) $\bar{J}(x + e_i) - \bar{J}(x + e_j) \uparrow x_i \downarrow x_j, i \neq j$.

In (h), the equal sign implies that $\bar{J}(x)$ can be regarded as both submodular and supmodular in the direction $(\dots, i, \dots, j, \dots)$. By the properties (g) – (i), each location can be separately characterized since $\bar{J}$ is the sum of $\bar{J}_i$ as indicated in Proposition 4. In the following discussion, the indices are set to $i, j, m = 1, 2, \dots, k$. Then for location i , define the following switching function associated with the transshipment decision from location j to i .

$$\bar{S}_{ji}(x_i, a_i^j, d_i) = \min\{x_j \mid \bar{J}(x - a_i^i e_i - a_i^j e_j) - \bar{J}(x - (a_i^i - 1)e_i - (a_i^j + 1)e_j) - r_{ji} \geq 0, \text{ given } x_i, \text{ other } x_m, m \neq i, j, \text{ are arbitrarily given, } a_i^j = 0, 1, \dots, d_i - 1, a_i^i + a_i^j = d_i\}$$

If $j = i$, $\bar{S}_{ji}(\cdot)$ is associated with the decision of filling demand with local stock. Then define the following switching function for the production decision at location i .

$$\bar{S}_i = \min\{x_i \mid \bar{J}(x + e_i) - \bar{J}(x) \geq 0\}.$$

By the same arguments as those in Section 2, we can derive the monotone properties for these switching functions. Then we can characterize policy u in the following proposition.

Proposition 6. The one-step improved policy for transshipment control is characterized by switching curves (surfaces) $\bar{S}_{ji}(x_i, a_i^j, d_i)$. At location i , given the state $x = (x_1, x_2, \dots, x_k)$, the transshipment decisions are

- (1) There is no transshipment to location i if $x_j < \bar{S}_{ji}(x_i, 0, d_i)$;
- (2) The a_i^j units of demand at location i are covered by the transshipment from location j if x_j satisfies $\bar{S}_{ji}(x_i, a_i^j, d_i) \leq x_j < \bar{S}_{ji}(x_i, a_i^j + 1, d_i)$, $a_i^j = 0, 1, \dots, d_i - 1$;
- (3) The d_i units at location j are transshipped to location i if $x_j \geq \bar{S}_{ji}(x_i, d_i, d_i)$.

Proof: The proof parallels that of Theorem 1 and hence it is omitted. ■

The one-step improved policy for production control at each location is a base-stock policy. At location i , produce when $x_i < \bar{S}_i$, and do not produce when $x_i \geq \bar{S}_i$.

Next, we illustrate the above results with a three-location example which has the arrival rates $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\lambda_3 = 0.8$ and the production rates $\mu_1 = 6$, $\mu_2 = 7$, $\mu_3 = 5$. Demand sizes are assumed to follow $P_i\{D_i = d_i\} = 0.2$, $d_i = 1, 2, \dots, 5$, $i = 1, 2$. $r_{21} = 5$ and $r_{31} = 6$ denote the transshipment costs from location 2 to 1 and from location 3 to 1, respectively. All the rest transshipment costs equal 5. The holding cost rates are $h_1 = h_2 = h_3 = 1$ and the backorder cost rates are $b_1 = b_2 = b_3 = 10$. The discount rate is $\alpha = 0.05$. For a demand of 5 units at location 1, we compute $T_{1,d_1} \bar{J}(x)$ to obtain the decisions for the selected states (x_1 and x_2 are taken from -8 to 8, given $x_3 = 4$) as listed in Tables 2 and 3. The digits in the tables, for instance, the bold “5” in Table 2 indicates that given the inventory state ($x_1 = 6$, $x_2 = -3$, $x_3 = 4$), the heuristic policy is to fill the demand with 5 units at location 1. The bold “3” in Table 3 indicates that given the inventory state ($x_1 = 0$, $x_2 = 1$, $x_3 = 4$), the heuristic policy requires 3 units to be transferred from location 3 to 1 for the demand at location 1.

Table 2: Demand Filling Decisions for the Three-Location Problem

	x_2	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
x_1	demand filling decisions at location 1 ($x_3=4$)																	
-8	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-7	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-6	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-5	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-4	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-3	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
2	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2
3	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
8	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 3: Transshipment Decisions for the Three-Location Problem

	x_2	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	
x_1	transshipments from location 2 to location 1 ($x_3=4$)																		transshipment from location 3 to location 1 ($x_3=4$)																	
-8		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-7		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-6		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-5		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-4		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-3		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-2		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
-1		0	0	0	0	0	0	0	0	0	1	2	2	3	3	4	5	5	4	4	4	4	4	4	4	4	4	4	3	3	2	2	1	0	0	
0		0	0	0	0	0	0	0	0	0	1	1	2	2	3	4	5	5	4	4	4	4	4	4	4	4	4	3	2	2	1	1	0	0		
1		0	0	0	0	0	0	0	0	0	0	1	1	2	3	4	4	4	3	3	3	3	3	3	3	3	3	2	2	1	1	0	0	0		
2		0	0	0	0	0	0	0	0	0	0	0	1	2	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	0	0	0	0	
3		0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	
4		0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	
5		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
6		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
7		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
8		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

3.3 The One-Step Lookahead Policy

To address the multi-location problem, we intend to develop another heuristic policy, referred to as *one-step lookahead policy*, which is based on the method of approximation of the optimal cost function. Readers are referred to Bertsekas (1995) for a more detailed discussion on the topic of limited lookahead policy.

Here, we obtain the one-step lookahead policy by computing $T\tilde{J}$, where $\tilde{J}$ is an approximation of the optimal cost function J . Here, for an approximation of J , we

consider the linear combination: $\tilde{J} = \theta J^u + (1 - \theta)J^L$, where $0 < \theta \leq 1$, J^u and J^L are the upper bound and lower bound of J . It is obvious that $\tilde{J}$ is an upper bound of J . Then we just need to find a lower bound of J . The most obvious lower bound of J is 0. Immediately, we have the following approximation.

Approximation 1: $\tilde{J} = 0.5\bar{J} + 0.5 \times 0$ with $\theta = 0.5$.

Alternatively, we can obtain other lower bounds by the method of *state aggregation* in dynamic programming. Let $h = \min_i \{h_i\}$, $b = \min_i \{b_i\}$, and $y = \sum_i x_i$, $i = 1, 2, \dots, k$. Here, we derive the lower bounds for two cases which differ in the probability distribution of demand size.

In the first case, the probability distributions of demand size at all locations are identical with $P(D_i = d) = p(d)$, $d = 1, 2, \dots, m$. In the second case, the demand size of different locations are not identical in probability distribution and $P(D_i = d_i) = p_i(d_i)$, $d_i = 1, 2, \dots, m_i$, $i = 1, 2, \dots, k$.

Then we use the following Bellman equation without the transshipment controls to obtain a lower bound of J .

$$J^L = T^L J^L \quad (5)$$

The dynamic programming operator for the first case is as follows:

$$T^L J^L(y) = hy^+ + by^- + \sum_{i=1}^k \sum_{d=1}^m \lambda_i p(d) J^L(y-d) + \sum_{i=1}^k \mu_i \min \{J^L(y+1), J^L(y)\}. \quad (6)$$

For the second case, to formulate the dynamic programming operator, we first homogenize the demand size with $\bar{m} = \max_i \{m_i\}$ and then choose μ'_i with $\mu'_i \geq \mu_i$ for all i , to achieve $\bar{m} \sum_i \lambda_i < \sum_i \mu'_i$. Then the dynamic programming operator reads

$$T^L J^L(y) = hy^+ + by^- + \sum_{i=1}^k \lambda_i J^L(y - \bar{m}) + \sum_{i=1}^k \mu'_i \min \{J^L(y+1), J^L(y)\}. \quad (7)$$

Note that in (6) and (7), we should rescale time to achieve $\alpha + \sum_i \lambda_i + \sum_i \mu_i = 1$ and $\alpha + \sum_i \lambda_i + \sum_i \mu'_i = 1$. Next we show that J^L in (5) is indeed a lower bound of J .

Proposition 7. $J^L(y) \leq J(x)$ for all $x \in X$.

Proof: Please see the online appendix B for the proof. ■

Based on the upper bound $\bar{J}(x)$ and the lower bound $J^L(y)$ of $J(x)$, we can construct another approximation.

Approximation 2: $\tilde{J}(x) = 0.5J^u(x) + 0.5J^L(y)$ by selecting $\theta = 0.5$.

Finally, we can derive a stationary policy $\tilde{u}$ by computing $T\tilde{J}$ from

$$T_{\tilde{u}}\tilde{J} = T\tilde{J}.$$

It is convenient to show that $\tilde{J}(x)$ retains the structural properties (g)-(i). Hence, the switching curve (surface) policy for transshipment control as well as the base-stock policy for production control can be also applied to $\tilde{u}$.

3.4 Comparison of Different Policies

Table 4: Cost Savings from the Transshipment Associated with Various Policies

state (x_1, x_2)	$r_{12}=r_{21}=2$						$r_{12}=r_{21}=4$					
	(-3,3)	(-2,2)	(-1,1)	(1,-1)	(2,-2)	(3,-3)	(-3,3)	(-2,2)	(-1,1)	(1,-1)	(2,-2)	(3,-3)
no transshipment	122.7	109.9	101.9	105.4	116.7	132.8	122.7	109.9	101.9	105.4	116.7	132.8
with transshipment	83	75.79	71.2	71.52	76.37	83.8	92.78	84.97	79.82	80.45	86.1	94.33
cost savings	32.35%	31.06%	30.15%	32.13%	34.57%	36.89%	24.37%	22.71%	21.69%	23.65%	26.24%	28.96%
improved policy	84.9	77.59	73.02	73.39	78.47	86.72	94.92	87.05	81.94	82.65	88.44	97.32
cost savings	30.80%	29.42%	28.36%	30.35%	32.77%	34.69%	22.62%	20.81%	19.61%	21.56%	24.23%	26.71%
approximation 1	84.89	77.58	73.01	73.39	78.46	86.71	93.98	86.11	80.97	81.55	87.37	96.29
cost savings	30.81%	29.41%	28.35%	30.37%	32.77%	34.71%	23.39%	21.67%	20.56%	22.61%	25.15%	27.49%
approximation 2	83.27	76.06	71.48	71.81	76.67	84.12	94.02	86.26	81.11	81.56	87.17	95.37
cost savings	32.14%	30.79%	29.85%	31.87%	34.30%	36.66%	23.37%	21.51%	20.40%	22.62%	25.30%	28.19%
state (x_1, x_2)	$r_{12}=r_{21}=6$						$r_{12}=r_{21}=8$					
	(-3,3)	(-2,2)	(-1,1)	(1,-1)	(2,-2)	(3,-3)	(-3,3)	(-2,2)	(-1,1)	(1,-1)	(2,-2)	(3,-3)
no transshipment	122.7	109.9	101.9	105.4	116.7	132.8	122.7	109.9	101.9	105.4	116.7	132.8
with transshipment	99.86	91.38	85.61	86.59	93.05	102.1	105.2	96.04	89.67	91.01	98.36	108.3
cost savings	18.60%	16.87%	16.01%	17.82%	20.28%	23.10%	14.22%	12.63%	12.03%	13.63%	15.73%	18.42%
improved policy	102.5	94	88.31	89.4	96.02	105.6	106.5	97.28	90.88	92.44	99.84	110.2
cost savings	16.45%	14.49%	13.36%	15.15%	17.73%	20.46%	13.16%	11.51%	10.84%	12.27%	14.46%	16.98%
approximation 1	101.2	92.72	86.75	87.65	94.26	103.9	107.41	97.97	91.25	92.59	100.25	110.59
cost savings	17.51%	15.65%	14.90%	16.82%	19.24%	21.75%	12.46%	10.86%	10.45%	12.15%	14.10%	16.72%
approximation 2	102.23	93.87	87.88	88.59	94.99	103.99	109.16	99.81	93.09	94.16	101.54	111.34
cost savings	16.68%	14.59%	13.76%	15.95%	18.60%	21.69%	11.04%	9.18%	8.65%	10.66%	12.99%	16.16%

In this subsection, we compare cost savings of different policies by studying the two-location problems. Here, we consider four cases which differ only in the transshipment costs: $r_{12} = r_{21} = 2, 4, 6, 8$. Other parameters are the same as those of the example in Section 2.3. The value iterations are conducted for 300 stages for all cases since the cost differences between the two stages 300 and 299 at state $(0, 0)$ are all less than 0.1. The results are presented in Table 4 (Here, improved policy refers to the one-step improve policy; approximations 1 and 2 are used to derive the one-step lookahead policy). And we have the following observations: first, the cost savings from possible transshipments are nonincreasing with respect to the transshipping costs in consistent with $f(x) \uparrow r_{21} \uparrow r_{12}$ in Propositions 2 and 5; second, The cost savings from approximation 1 are no better than those from approximation 2 in the cases of lower transshipping costs but approximation 1 performs better than approximation 2 in the cases of higher transshipping costs; third, for a large number of cases, the cost savings from approximation 1 exceed those from the improved policy. In a comprehensive assessment, the approximation 1 of the one-step lookahead policy not only performs very well but also has a simpler form for calculation. As a result, we then concentrate on the performance of approximation 1 by studying more cases. The results are shown in Table 5.

To obtain the approximation 1 of the one-step lookahead policy, we need to select a function from a parametric class of functions $\bar{f}(x, \theta) = \theta \bar{f}(x)$, where $0 < \theta \leq 1$, to approximate the optimal cost function. Note that approximation 1 is equivalent to the one-step improved policy when $\theta = 1$. For the cases in Table 4, we use $\theta = 0.5$. Here, we propose a bisection method to find a better θ . For example, for $r_{21} = r_{12} = 6$, we first compute the case of $\theta = 1$. Secondly, we compute the case of $\theta = 0.5$. Thirdly, we compute two cases: $\theta = 0.25$ and $\theta = 0.75$, of which the one-step lookahead policy with $\theta = 0.75$ achieves a better result. Hence, in the fourth step, we compute two cases: $\theta = (1+0.75)/2 = 0.875$ and $\theta = (0.5+0.75)/2 = 0.625$. Among the three cases: $\theta = 0.625$, $\theta = 0.75$ and $\theta = 0.875$, the one-step lookahead policy with $\theta = 0.75$ achieves the best result. We may further compute two cases: $\theta = (0.625+0.75)/2 = 0.6875$ and $\theta = (0.75+0.875)/2 = 0.8125$. But we find the results of $\theta = 0.75$ are good enough compared with those of other cases. For the case of $r_{21} = r_{12} = 4$, either $\theta = 0.5$ or $\theta = 0.625$ makes a good approximation. Here, note that the results of the case $\theta = 0.75$ are identical with those of

the case $\theta = 0.875$. This implies the approximation 1 with $\theta = 0.75$ and that with $\theta = 0.875$ may yield the same one-step lookahead policy.

Table 5: Cost Savings from the Transshipment Associated with the Approximation 1 of the One-Step Lookahead Policy

state (x_1, x_2)	$r_{12}=r_{21}=4$						$r_{12}=r_{21}=6$					
	(-3,3)	(-2,2)	(-1,1)	(1,-1)	(2,-2)	(3,-3)	(-3,3)	(-2,2)	(-1,1)	(1,-1)	(2,-2)	(3,-3)
no transshipment	122.7	109.9	101.9	105.4	116.7	132.8	122.7	109.9	101.9	105.4	116.7	132.8
full transshipment	92.78	84.97	79.82	80.45	86.1	94.33	99.86	91.38	85.61	86.59	93.05	102.1
cost savings(%)	24.37%	22.71%	21.69%	23.65%	26.24%	28.96%	18.60%	16.87%	16.01%	17.82%	20.28%	23.10%
approx. 1 ($\theta=0.25$)	101.6	93.44	87.25	87.89	94.28	102.9	115.2	103.7	96.06	97.9	107	119.1
cost savings(%)	17.17%	15.00%	14.40%	16.59%	19.23%	22.53%	6.14%	5.69%	5.76%	7.09%	8.33%	10.28%
approx. 1 ($\theta=0.5$)	93.98	86.11	80.97	81.55	87.37	96.29	101.2	92.72	86.75	87.65	94.26	103.9
cost savings(%)	23.39%	21.67%	20.56%	22.61%	25.15%	27.49%	17.51%	15.65%	14.90%	16.82%	19.24%	21.75%
approx. 1 ($\theta=0.625$)	93.88	85.99	80.89	81.58	87.42	96.33	100.4	91.8	86.02	87.06	93.67	103.3
cost savings(%)	23.47%	21.77%	20.65%	22.58%	25.10%	27.45%	18.19%	16.49%	15.61%	17.37%	19.74%	22.18%
approx. 1 ($\theta=0.75$)	94.55	86.67	81.55	82.32	88.12	97.01	100.3	91.69	85.92	86.99	93.6	103.3
cost savings(%)	22.93%	21.16%	19.99%	21.88%	24.50%	26.94%	18.28%	16.59%	15.70%	17.44%	19.80%	22.23%
approx. 1 ($\theta=0.875$)	94.55	86.67	81.55	82.32	88.12	97.01	101.4	92.92	87.21	88.5	95.15	104.8
cost savings(%)	22.93%	21.16%	19.99%	21.88%	24.50%	26.94%	17.32%	15.47%	14.45%	16.01%	18.48%	21.10%
approx. 1 ($\theta=1$)	94.92	87.05	81.94	82.65	88.44	97.32	102.5	94	88.31	89.4	96.02	105.6
cost savings(%)	22.62%	20.81%	19.61%	21.56%	24.23%	26.71%	16.45%	14.49%	13.36%	15.15%	17.73%	20.46%

4. Conclusions

Characterizing the transshipment policy in the general multi-location inventory system has been a challenging task for many years and has attracted the attention of many researchers. In this paper, we studied joint production and transshipment controls in multi-location, make-to-stock systems. Through virtual transshipment, an effective managerial tool for inventory pooling, we obtained the optimal joint transshipment and production control policies for both the backorder and lost-sales cases of two-location problems, as well as two heuristic policies for the general multi-location problem.

For the two-location problem, we characterized the structure of the optimal policies as monotone switching curves without restrictive conditions on cost parameters as done in Zhao et al. (2008). Our numerical examples verified the structure and monotone properties of the optimal policy. Further, under certain regularity conditions which require nearly identical unit holding cost rates among different locations, we

derived a simple decision rule for guiding transshipment. The decision rule states that there is no transshipment if one has stock on hand. In other words, it is always optimal to meet demand with local stock until it is depleted.

For the multi-location problem, we developed two heuristic policies. The first heuristic was obtained by the method of one-step policy improvement which first decomposed the original complex multi-location problem into multiple easily handled single-location problems. Next, the one-step policy iteration was applied to obtain the heuristic result, based on the observation that policy iteration typically achieves the largest improvement in the first few steps of iteration. The second heuristic we developed was the one-step lookahead policy. This was computed from the Bellman equation with an approximate optimal cost function derived from a linear combination of the upper and lower bounds of the original optimal cost function. Finally, in a comprehensive numerical assessment of the heuristic policies, we found that the one-step lookahead policy computed from half of an upper bound (i.e. the cost function associated with no transshipment) of the optimal cost function performed very well.

From the practitioner perspective, the structural properties of the optimal policy and the two heuristic policies help managers gain insights into the operation of the multi-location production/inventory system with transshipment and provide them with more guidance for designing efficient and effective decision rules. Finally, some interesting issues remain for future studies. For instance, we may consider the case where transshipment time is non zero. For such a case, it would be valuable to explore the structural properties and characterize the optimal policies.

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Appendix A

Proof of Lemma 2

From Lemma 1, note that for any bounded $f_0 \in \Omega$, $T^n f_0 \in \Omega$ for all n . Since $T^n f_0(x)$ takes the point-wise convergence to $f(x)$ for all x as $n \rightarrow \infty$, we obtain $f \in \Omega$. ■

Proof of Lemma 3

We prove that $S_{21}(x_1, a_1^2, d_1)$ is increasing in a_1^2 . By the definition,

$$f(x_1 - a_1^1, S_{21}(x_1, a_1^2, d_1) - a_1^2) - f(x_1 - a_1^1 + 1, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 1) - r_{21} \geq 0.$$

Since $S_{21}(x_1, a_1^2, d_1)$ is the smallest value to satisfy the above inequality, we have

$$f(x_1 - a_1^1, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 1) - f(x_1 - a_1^1 + 1, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 2) - r_{21} < 0.$$

From property (d), we have

$$f(x_1 - a_1^1 + 1, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 1) - f(x_1 - a_1^1 + 2, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 2) - r_{21} < 0. \quad (A1)$$

From the definition of $S_{21}(x_1, a_1^2 + 1, d_1)$,

$$f(x_1 - a_1^1 + 1, S_{21}(x_1, a_1^2 + 1, d_1) - a_1^2 - 1) - f(x_1 - a_1^1 + 2, S_{21}(x_1, a_1^2 + 1, d_1) - a_1^2 - 2) - r_{21} \geq 0. \quad (A2)$$

Since $S_{21}(x_1, a_1^2 + 1, d_1)$ is the smallest value to satisfy the above inequality, then $S_{21}(x_1, a_1^2 + 1, d_1) > S_{21}(x_1, a_1^2, d_1)$ after comparing (A1) with (A2). Thus, $S_{21}(x_1, a_1^2, d_1)$ is strictly increasing in a_1^2 . Similarly, we prove the results for other switching curves. ■

Proof of Lemma 4

The finite control set $A(x)$ suffices to guarantee the existence of an admissible stationary policy u which attains the minimum in the RHS of (1), i.e. $Tf = T_{\text{uf}}$. Noting the cost per stage is always nonnegative, by the result in Bertsekas (1995), u is optimal. ■

Proof of Theorem 1

In (i), for (a), from the definition, $S_{21}(x_1, 0, d_1)$ is the smallest value to satisfy $f(x - d_1 e_1) \geq f(x - (d_1 - 1)e_1 - e_2) + r_{21}$. Then for any $x_2 < S_{21}(x_1, 0, d_1)$, we have $f(x - d_1 e_1) < f(x - (d_1 - 1)e_1 - e_2) + r_{21}$, that is, it is optimal to have no transshipment.

In (b), by the definition of $S_{21}(x_1, a_1^2, d_1)$, $f(x - a_1^1 e_1 - a_1^2 e_2) \geq f(x - (a_1^1 - 1)e_1 - (a_1^2 + 1)e_2) + r_{21}$, for any $x_2 \geq S_{21}(x_1, a_1^2, d_1)$, i.e., it is better to transfer $(a_1^2 + 1)$ units instead of a_1^2 units. On the other hand, by the definition, $S_{21}(x_1, a_1^2 + 1, d_1)$ is the smallest value to satisfy

$$f(x - (a_1^1 - 1)e_1 - (a_1^2 + 1)e_2) \geq f(x - (a_1^1 - 2)e_1 - (a_1^2 + 2)e_2) + r_{21}.$$

From $f(x + e_2) - f(x + e_1) \uparrow x_2$, for any $x_2 < S_{21}(x_1, a_1^2 + 1, d_1)$,

$f(x - (a_1^1 - 1)e_1 - (a_1^2 + 1)e_2) < f(x - (a_1^1 - 2)e_1 - (a_1^2 + 2)e_2) + r_{21}$, i.e., it is better to transfer $(a_1^2 + 1)$ units instead of $(a_1^2 + 2)$ units. Hence, combining the above two cases, we have that if $S_{21}(x_1, a_1^2, d_1) \leq x_2 < S_{21}(x_1, a_1^2 + 1, d_1)$, it is optimal to transfer $(a_1^2 + 1)$ units.

In (c), by the definition of $S_{21}(x_1, d_1 - 1, d_1)$ and Property (d), for $x_2 \geq S_{21}(x_1, d_1 - 1, d_1)$, $f(x - e_1 - (d_1 - 1)e_2) \geq f(x - d_1 e_2) + r_{21}$, i.e., it is optimal to have all demand at location 1 filled by the transshipment from location 2.

For (ii), the proof is similar to that of (i).

For (iii), if $x_1 < S_3(x_2)$, we have $f(x + e_1) < f(x)$ by the definition of $S_3(x_2)$ and property (a). Hence, it is optimal to produce at the inventory level x_1 ; otherwise, if $x_1 \geq S_3(x_2)$, we have $f(x + e_1) \geq f(x)$, i.e., it is optimal to have no production.

For (iv), the proof is the same as that of (iii).

The actions prescribed by the switching curves minimize the RHS of (1). Hence, the optimal stationary policy is characterized as switching curves. ■

Proof of Proposition 1

In part (a), we only prove the case for $S_{21}(x_1, a_1^2, d_1)$. The other cases can be proved analogously. For $S_{21}(x_1, a_1^2, d_1) \uparrow x_1$, by the definition,

$$f(x_1 - a_1^1, S_{21}(x_1, a_1^2, d_1) - a_1^2) - f(x_1 - a_1^1 + 1, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 1) - r_{21} \geq 0. \quad (\text{A3})$$

From (A3) and $f(x + e_2) - f(x + e_1) \downarrow x_1$ in property (d),

$$f(x_1 - a_1^1 - 1, S_{21}(x_1, a_1^2, d_1) - a_1^2) - f(x_1 - a_1^1, S_{21}(x_1, a_1^2, d_1) - a_1^2 - 1) - r_{21} \geq 0. \quad (\text{A4})$$

By the definition of $S_{21}(x_1 - 1, a_1^2, d_1)$,

$$f(x_1 - a_1^1 - 1, S_{21}(x_1 - 1, a_1^2, d_1) - a_1^2) - f(x_1 - a_1^1, S_{21}(x_1 - 1, a_1^2, d_1) - a_1^2 - 1) - r_{21} \geq 0. \quad (\text{A5})$$

Since $S_{21}(x_1 - 1, a_1^2, d_1)$ is the least value to satisfy (A5), we obtain $S_{21}(x_1, a_1^2, d_1) \geq S_{21}(x_1 - 1, a_1^2, d_1)$ after comparing (A4) with (A5). Thus, $S_{21}(x_1, a_1^2, d_1)$ is nondecreasing in x_1 .

For $S_{21}(x_1, a_1^2, d_1) \downarrow d_1$, we need to prove $S_{21}(x_1, a_1^2, d_1) \geq S_{21}(x_1, a_1^2, d_1 + 1)$. By the definition $S_{21}(x_1, a_1^2, d_1)$ and property (d),

$$f(x_1 - (d_1 + 1 - a_1^2), S_{21}(x_1, a_1^2, d_1) - a_1^2) - f(x_1 - (d_1 - a_1^2), S_{21}(x_1, a_1^2, d_1) - (a_1^2 + 1)) - r_{21} \geq$$

$$f(x_1 - (d_1 - a_1^2), S_{21}(x_1, a_1^2, d_1) - a_1^2) - f(x_1 - (d_1 - a_1^2 - 1), S_{21}(x_1, a_1^2, d_1) - (a_1^2 + 1)) - r_{21} \geq 0.$$

Since $S_{21}(x_1, a_1^2, d_1 + 1)$ is the least value of x_2 to satisfy

$$f(x_1 - (d_1 + 1 - a_1^2), x_2 - a_1^2) - f(x_1 - (d_1 - a_1^2), x_2 - (a_1^2 + 1)) - r_{21} \geq 0.$$

Hence, for fixed x_1 and a_1^2 , $S_{21}(x_1, a_1^2, d_1) \geq S_{21}(x_1, a_1^2, d_1 + 1)$.

For $S_{21}(x_1, a_1^2, d_1) \uparrow r_{21}$, suppose $r'_{21} \geq r''_{21}$ and $S'_{21}(x_1, a_1^2, d_1)$ and $S''_{21}(x_1, a_1^2, d_1)$ are the switching curves associated with r'_{21} and r''_{21} , respectively. By the definition,

$$f(x_1 - a_1^1, S'_{21}(x_1, a_1^2, d_1) - a_1^2) - f(x_1 - a_1^1 + 1, S'_{21}(x_1, a_1^2, d_1) - a_1^2 - 1) \geq r'_{21} \geq r''_{21}. \quad (\text{A6})$$

Since $S''_{21}(x_1, a_1^2, d_1)$ is the least value of x_2 to satisfy

$$f(x_1 - a_1^1, x_2 - a_1^2) - f(x_1 - a_1^1 + 1, x_2 - a_1^2 - 1) \geq r''_{21}, \quad (\text{A7})$$

we have $S'_{21}(x_1, a_1^2, d_1) \geq S''_{21}(x_1, a_1^2, d_1)$ by comparing (A6) with (A7). Hence, $S_{21}(x_1, a_1^2, d_1) \uparrow r_{21}$.

In part (b), the monotone properties of $S_3(x_2)$ and $S_4(x_1)$ can be proved similarly as those in part (a).

$S_3(x_2) \geq 0$ and $S_4(x_1) \geq 0$ directly follow from the inequality: $f(x + e_i) \leq f(x)$, $i = 1, 2$, when $x_i < 0$. It can be readily verified that the inequality is preserved by T_{1,d_1} , T_{2,d_2} , T_1 , T_2 and thus T . Then, applying the value iteration method, we have that the inequality holds when $x_i < 0$.

Finally, the existence of two limitations is evident. ■

Proof of Theorem 3

In (i), $v(x)$ inherits the structural properties (a)-(f) from $V_\alpha(x)$, the optimal discounted cost function. The existence of $v(x)$, g and the results in (ii) follows from Propositions 2.1 and 2.6 of Chapter 4 in volume II of Bertsekas (1995). ■