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An Alternating Variable Technique for the Constrained Minimax Design of Frequency-Response-Masking Filters

Ruijie Zhao^{1,2} · Xiaoping Lai³ ·
David B. H. Tay⁴ · Zhiping Lin²

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Abstract The Frequency-Response-Masking (FRM) technique is one of the most efficient approaches to the design of narrow transition band FIR filters. The constrained minimax design of linear-phase FRM FIR filters in the frequency domain is considered in this paper. The corresponding optimization problem is a nonconvex one. To deal with the nonconvex design problem and improve the FRM filter performance, we propose an algorithm to alternately optimize different subsets of the design variables by fixing the remaining ones. In this way, the nonconvex optimization problem is converted into a series of linear programming subproblems defined on different frequency bands, which are then solved alternately and iteratively. Moreover, the new algorithm converges to a better FRM filter than those obtained by several competitive methods and is flexible to incorporate linear constraints in the design. Several design examples are provided to demonstrate the advantages of the proposed algorithm.

Ruijie Zhao
E-mail: zhao_rj@sdu.edu.cn

Xiaoping Lai
E-mail: laixp@hdu.edu.cn

David B. H. Tay
E-mail: david.tay@deakin.edu.au

Zhiping Lin
E-mail: ezplin@ntu.edu.sg

¹ School of Mechanical, Electrical and Information Engineering, Shandong University, Weihai, 264209, China

² School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, 639798

³ Institute of Information and Control, Hangzhou Dianzi University, Hangzhou, 310018, China

⁴ School of Information Technology, Deakin University, Waurn Ponds, Victoria 3216, Australia.

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1 Introduction

Narrow transition band finite impulse response (FIR) digital filters have found many applications in digital signal processing, such as wireless communication systems, multimedia systems, and so on [3, 4, 16, 22]. One of the most efficient approaches to synthesizing narrow transition band filters is the frequency-response-masking (FRM) technique, which was originally proposed by Lim in [17] and then improved by many researchers, e.g., Lim and Lian in [11, 13, 18, 19], Lee *et al.* in [8–10], Saramäki *et al.* in [29–32, 37] and Lu *et al.* in [24–26]. The FRM technique produces filters with very sparse coefficients that require fewer number of multipliers and adders in their implementation than those yielded by the conventional direct design methods. The basic structure of a filter synthesized using the FRM technique is shown in Fig. 1.

The FRM FIR filters have been investigated extensively in the past thirty years and many design methods have been reported. These methods include the one-stage techniques [1, 4, 8–15, 17–20, 24–26, 29–33, 35, 37–39, 41], multistage techniques [7, 17–19, 21, 23–25, 29, 33, 34, 36, 37, 40] and their variants [12, 15, 19, 20, 30, 34–36]. This paper focuses on the one-stage FRM techniques.

The main difficulty in the optimal design of FRM FIR filters is that the frequency response of the FRM filter is nonlinear with respect to the filter coefficients. This nonlinearity leads to a highly nonconvex optimization problem [28]. Thus, in most of the existing methods, the involved subfilters, i.e., $H_a(z^M)$, $H_{ma}(z)$ and $H_{mc}(z)$ shown in Fig. 1, are either designed separately, or optimized jointly using nonlinear optimization techniques. The methods in [13, 15, 17, 18, 29], for example, designed the subfilters separately. Though the subfilters can be efficiently designed using the Remez exchange algorithm [27], the resultant overall FRM filters synthesized by the separately designed subfilters in these methods are generally not good enough for very crucial applications.

On the other hand, many researchers have developed design methods that optimize the subfilters jointly [1, 8–10, 24–26, 32, 33, 37–39]. In most of these methods, all the subfilter coefficients are treated as a whole design vector and optimized simultaneously. In [38], the sequential quadratic programming (SQP) algorithm was employed to jointly optimize the subfilters. In [39], the authors presented a weighted least square (WLS) algorithm to optimize the subfilters simultaneously. In [33], a neural network based algorithm was applied to the optimization of FRM filters. In [24–26], Lu and Hinamoto developed FRM filter design algorithms respectively based on the second-order cone programming [24], the semidefinite programming (SDP) [25] and the convex-concave procedure (CCP) [26]. However, all methods mentioned above require the use of complicated nonlinear optimization techniques.

In another class of joint optimization techniques, the subfilters are optimized alternately [1, 8–10, 32, 37]. We call them alternating variable approaches, the name of which firstly appeared in [9]. The method in [9] focuses on the FRM filter design with power-of-two coefficients, where the subfilters are improved iteratively by fixing the others and the optimization of a subfilter is done via a branch and bound method over a subregion of the search space. In [32, 37], the methods proposed by Saramäki *et al.* first find an initial FRM filter by alternately optimizing the subfilters and then further optimize these obtained subfilters simultaneously using a nonlinear optimization technique. The minimax design of FRM filters is considered in [1, 8, 10], where the nonlinear minimax design problem is first transformed into a series of nonlinear WLS problems using the iterative reweighted least-squares (IRLS) technique similar to [39]. Each resultant nonlinear WLS problem is then alternately decomposed into some linear WLS subproblems by fixing some subfilters. In [8, 10], $H_a(z)$ and the set of $\{H_{ma}(z), H_{mc}(z)\}$ are alternately optimized by fixing $\{H_{ma}(z), H_{mc}(z)\}$ and $H_a(z)$, respectively. In [1], $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are successively optimized by fixing $\{H_{mc}(z), H_{ma}(z)\}$, $\{H_a(z), H_{mc}(z)\}$ and $\{H_a(z), H_{ma}(z)\}$, respectively. Though the methods in [1, 8, 10] do not need complicated nonlinear optimization, due to the analytic solutions to the linear WLS problems, the IRLS technique generally results in near-optimal solution to the minimax problem. Moreover, all the above methods based on the alternating variable approach do not incorporate constraints in the design procedure.

In this paper, we consider the minimax design of linear-phase FRM FIR filters, and incorporate in the design with some possible practical design constraints [5, 6], say the magnitude error constraints in the frequency domain. Considering the high nonconvexity of this design problem and inspired by the methods in [1, 8–10, 32, 37], we propose a new alternating variable algorithm that converts the original nonlinear approximation problem into a series of linear approximation subproblems [28]. Compared to the existing design methods, the new algorithm has the following advantages:

(1) The subfilter pairs $\{H_{ma}(z), H_{mc}(z)\}$ and $\{H_a(z), H_{mc}(z)\}$ are alternately optimized. That is, the first pair is optimized by keeping $H_a(z)$ invariant and the second pair is optimized by keeping the difference of $H_{ma}(z)$ and $H_{mc}(z)$, i.e., $H_{ac}(z) = H_{ma}(z) - H_{mc}(z)$, invariant. Thus, more coefficients are optimized in each iteration with the new algorithm compared to previous alternating variable based methods.

(2) The new algorithm includes two phases. The solution filter obtained in the first phase serves as the initial filter of the second phase. The subproblems for optimizing $H_a(z), H_{mc}(z)$ in both phases and for optimizing $H_{ma}(z), H_{mc}(z)$ in the second phase are defined on the same frequency band as the original problem, but that for optimizing $H_{ma}(z), H_{mc}(z)$ in the first phase is defined on a different frequency band. Practical designs show that the two-phase method has achieved better solutions than several existing methods that optimize all subfilters always based on the frequency band of interest of the original problem.

Fig. 1 Structure of FRM filters (a) $N_{ma} \geq N_{mc}$; (b) $N_{ma} < N_{mc}$

(3) The optimization subproblems in both phases of the new algorithm are linear programming problems. No complicated nonlinear optimization techniques are needed and any linear constraints with respect to the subfilter coefficients can be incorporated into the algorithm easily.

Note that the proposed algorithm always converges, though not necessarily to the globally optimal solution. Moreover, a large number of simulation examples show that the new algorithm gets better FRM filters than several competitive design methods.

The rest of the paper is organized as follows. Section 2 gives a brief overview of the FRM FIR filter and expresses its frequency response in the vector form. Section 3 formulates the constrained minimax design problem of FRM FIR filters and describes the proposed alternating variable algorithm. Design examples and comparison with existing methods are presented in Section 4. Finally, the conclusion is drawn in Section 5.

2 FRM FIR filter and its frequency response

Consider the FRM filter structure as shown in Fig. 1. Here, $H_a(z^M)$ with M being a given interpolation factor is the band-edge shaping filter, which is obtained by replacing z in the prototype filter $H_a(z)$ with z^M . $H_{ma}(z)$ and $H_{mc}(z)$ are two masking filters.

Assume that the subfilters $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are FIR filters of lengths N_a , N_{ma} and N_{mc} , respectively. We can write the transfer functions

of these subfilters as

$$\begin{aligned} H_a(z) &= \psi(z, N_a)^T \mathbf{h}_a, \\ H_{ma}(z) &= \psi(z, N_{ma})^T \mathbf{h}_{ma}, \\ H_{mc}(z) &= \psi(z, N_{mc})^T \mathbf{h}_{mc}, \end{aligned}$$

where $\mathbf{h}_a$, $\mathbf{h}_{ma}$ and $\mathbf{h}_{mc}$ represent the real impulse response vectors of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$, respectively, as given by

$$\begin{aligned} \mathbf{h}_a &= [h_a(0), h_a(1), \dots, h_a(N_a - 1)]^T, \\ \mathbf{h}_{ma} &= [h_{ma}(0), h_{ma}(1), \dots, h_{ma}(N_{ma} - 1)]^T, \\ \mathbf{h}_{mc} &= [h_{mc}(0), h_{mc}(1), \dots, h_{mc}(N_{mc} - 1)]^T, \end{aligned}$$

and $\psi(z, N)$ is a vector defined by

$$\psi(z, N) = [1, z^{-1}, z^{-2}, \dots, z^{-(N-1)}]^T.$$

Then, the transfer function of the overall FRM filter, denoted by $H(z)$, is given as follows, i.e., for $N_{ma} \geq N_{mc}$,

$$H(z) = H_a(z^M)H_{ma}(z) + z^{-D_2}[z^{-D_1} - H_a(z^M)]H_{mc}(z), \quad (1)$$

and for $N_{ma} < N_{mc}$,

$$H(z) = z^{-D_2}H_a(z^M)H_{ma}(z) + [z^{-D_1} - H_a(z^M)]H_{mc}(z). \quad (2)$$

We further assume that $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are linear-phase filters, N_{ma} and N_{mc} are either both even or both odd, $M(N_a - 1)$ is even [25], and D_1 and D_2 in Fig. 1 are set to [36]

$$\begin{aligned} D_1 &= \frac{M(N_a - 1)}{2}, \\ D_2 &= \frac{|N_{ma} - N_{mc}|}{2}, \end{aligned}$$

then, $H(z)$ will have a linear phase response.

For simplicity, in the following, we let N_a , N_{ma} and N_{mc} be all odd. The algorithm proposed in the paper can be applied to other cases satisfying above assumptions with only minimal changes.

Let $H_a(\omega)$, $H_{ma}(\omega)$ and $H_{mc}(\omega)$ be the zero-phase frequency responses of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$, respectively. By defining a vector $\varphi(\omega, K)$ of the form,

$$\varphi(\omega, K) = [1, \cos(\omega), \cos(2\omega), \dots, \cos((K-1)\omega)]^T,$$

these zero-phase frequency responses can be written in vector forms as

$$H_a(\omega) = \varphi(\omega, K_a)^T \mathbf{b}_a, \quad (3)$$

$$H_{ma}(\omega) = \varphi(\omega, K_{ma})^T \mathbf{b}_{ma}, \quad (4)$$

$$H_{mc}(\omega) = \varphi(\omega, K_{mc})^T \mathbf{b}_{mc}, \quad (5)$$

where $\mathbf{b}_a$, $\mathbf{b}_{ma}$ and $\mathbf{b}_{mc}$ are the coefficient vectors related to $\mathbf{h}_a$, $\mathbf{h}_{ma}$ and $\mathbf{h}_{mc}$, respectively, and can be denoted by

$$\begin{aligned}\mathbf{b}_a &= [b_a(0), b_a(1), \dots, b_a(K_a - 1)]^T, \\ \mathbf{b}_{ma} &= [b_{ma}(0), b_{ma}(1), \dots, b_{ma}(K_{ma} - 1)]^T, \\ \mathbf{b}_{mc} &= [b_{mc}(0), b_{mc}(1), \dots, b_{mc}(K_{mc} - 1)]^T.\end{aligned}$$

In (3-5), K_a , K_{ma} and K_{mc} are given, respectively, by

$$\begin{aligned}K_a &= (N_a + 1)/2, \\ K_{ma} &= (N_{ma} + 1)/2, \\ K_{mc} &= (N_{mc} + 1)/2.\end{aligned}$$

Using the above notation, the frequency response of the overall filter $H(z)$ in (1) can be written as

$$\begin{aligned}H(e^{j\omega}) &= e^{-j\omega \frac{M(N_a-1)}{2}} H_a(M\omega) e^{-j\omega \frac{N_{ma}-1}{2}} H_{ma}(\omega) \\ &+ e^{-j\omega \frac{N_{ma}-N_{mc}}{2}} [e^{-j\omega \frac{M(N_a-1)}{2}} - e^{-j\omega \frac{M(N_a-1)}{2}} H_a(M\omega)] e^{-j\omega \frac{N_{mc}-1}{2}} H_{mc}(\omega),\end{aligned}$$

where $H_a(M\omega)$ is obtained by replacing ω in $H_a(\omega)$ with $M\omega$. By extracting the factor $e^{-j\omega \frac{M(N_a-1)+N_{ma}-1}{2}}$, we obtain,

$$H(e^{j\omega}) = e^{-j\omega \frac{M(N_a-1)+N_{ma}-1}{2}} [H_a(M\omega)H_{ma}(\omega) + (1 - H_a(M\omega))H_{mc}(\omega)].$$

Therefore, the zero-phase frequency response of $H(z)$ in (1) is

$$H(\omega) = H_a(M\omega)H_{ma}(\omega) + [1 - H_a(M\omega)]H_{mc}(\omega). \quad (6)$$

In a similar way, it can be shown that the zero-phase frequency response for (2) has the same expression as (6). Then, using (3)-(5), for either (1) or (2), we have

$$\begin{aligned}H(\omega) &= \varphi(M\omega, K_a)^T \mathbf{b}_a \varphi(\omega, K_{ma})^T \mathbf{b}_{ma} \\ &+ [1 - \varphi(M\omega, K_a)^T \mathbf{b}_a] \varphi(\omega, K_{mc})^T \mathbf{b}_{mc}. \quad (7)\end{aligned}$$

3 Constrained minimax design of FRM FIR filters

As an example, let us consider the design of a low-pass FRM filter with passband and stopband edge frequencies ω_p and ω_s , respectively. Then, all of the subfilters $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ are low-pass filters as well [17]. We denote the passband and stopband edges of $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ by (θ, ϕ) , (θ_a, ϕ_a) , and (θ_c, ϕ_c) , respectively.

For any given $0 < \omega_p < \omega_s < \pi$ and positive integer M , we have the following two cases for the edge frequencies (θ, ϕ) , (θ_a, ϕ_a) , and (θ_c, ϕ_c) to satisfy [17],

Case A:

$$\begin{aligned} m &= \lfloor M\omega_p/(2\pi) \rfloor, \\ \begin{cases} \theta &= M\omega_p - 2m\pi, \\ \phi &= M\omega_s - 2m\pi, \end{cases} \\ \begin{cases} \theta_a &= \omega_p, \\ \phi_a &= [2(m+1)\pi - \phi]/M, \end{cases} \\ \begin{cases} \theta_c &= (2m\pi - \theta)/M, \\ \phi_c &= \omega_s, \end{cases} \end{aligned}$$

where $\lfloor M\omega_p/(2\pi) \rfloor$ denotes the largest integer less than or equal to $M\omega_p/(2\pi)$; and

Case B:

$$\begin{aligned} m &= \lceil M\omega_p/(2\pi) \rceil, \\ \begin{cases} \theta &= 2m\pi - M\omega_s, \\ \phi &= 2m\pi - M\omega_p, \end{cases} \\ \begin{cases} \theta_a &= [2(m-1)\pi + \phi]/M, \\ \phi_a &= \omega_s, \end{cases} \\ \begin{cases} \theta_c &= \omega_p, \\ \phi_c &= (2m\pi + \theta)/M, \end{cases} \end{aligned}$$

where $\lceil M\omega_p/(2\pi) \rceil$ denotes the smallest integer larger than or equal to $M\omega_p/(2\pi)$.

In our design method to be described later, we need to know which of the above two sets of subfilter edge frequencies should be chosen for given design specifications. The only set with θ and ϕ satisfying $0 < \theta < \phi < \pi$ is the proper one.

Now, we assume that the passband and stopband edge frequencies of $H(z)$, i.e., ω_p and ω_s , the interpolation factor M , and the subfilter lengths, i.e., N_a , N_{ma} and N_{mc} , are given. Let $D(\omega)$ be the desired zero-phase frequency response, i.e., $D(\omega) = 1$ for $\omega \in [0, \omega_p]$ and $D(\omega) = 0$ for $\omega \in [\omega_s, \pi]$.

Then, the constrained minimax design of FRM FIR filters can be formulated as follows:

$$\min_{\mathbf{b}_a, \mathbf{b}_{ma}, \mathbf{b}_{mc}} \max_{\omega \in \Omega} \{W(\omega)|H(\omega) - D(\omega)|\}, \quad (8a)$$

$$s.t. \quad |H(\omega) - D(\omega)| \leq \rho(\omega), \quad \omega \in \Omega_c, \quad (8b)$$

where Ω and Ω_c are two disjoint frequency subsets of the frequency band of interest, Ω_0 , which is usually the union of the passband and stopband, i.e., $\Omega_0 = ([0, \omega_p] \cup [\omega_s, \pi])$, $W(\omega)$ is a nonnegative weighting function on Ω , and $\rho(\omega) > 0$ is an error-bound function used to constrain the approximation error on Ω_c .

To solve problem (8) for the subfilter coefficient vectors $\mathbf{b}_a$, $\mathbf{b}_{ma}$ and $\mathbf{b}_{mc}$, we rewrite (6) as

$$\begin{aligned} H(\omega) &= H_a(M\omega)[H_{ma}(\omega) - H_{mc}(\omega)] + H_{mc}(\omega) \\ &= H_a(M\omega)H_{ac}(\omega) + H_{mc}(\omega), \end{aligned} \quad (9)$$

where $H_{ac}(\omega) = H_{ma}(\omega) - H_{mc}(\omega)$. Using (3)-(5), we get

$$H(\omega) = \varphi(M\omega, K_a)^T \mathbf{b}_a [\varphi(\omega, K_{ma})^T \mathbf{b}_{ma} - \varphi(\omega, K_{mc})^T \mathbf{b}_{mc}] + \varphi(\omega, K_{mc})^T \mathbf{b}_{mc}. \quad (10)$$

To further formulate the design problem, we have two cases for the lengths of subfilters $H_{ma}(z)$ and $H_{mc}(z)$ to consider in the following: One is $N_{ma} \geq N_{mc}$ and the other is $N_{ma} < N_{mc}$.

(i) When $N_{ma} \geq N_{mc}$, we have

$$\varphi(\omega, K_{mc})^T \mathbf{b}_{mc} = \varphi(\omega, K_{ma})^T \bar{\mathbf{b}}_{mc}, \quad (11)$$

where the K_{ma} -dimensional vector $\bar{\mathbf{b}}_{mc}$ is a lengthened version of $\mathbf{b}_{mc}$ with padded zeros at the end. Substituting (11) into (10) yields

$$H(\omega) = \varphi(M\omega, K_a)^T \mathbf{b}_a \varphi(\omega, K_{ma})^T \mathbf{b}_{ac} + \varphi(\omega, K_{mc})^T \mathbf{b}_{mc}, \quad (12)$$

where $\mathbf{b}_{ac} = \mathbf{b}_{ma} - \bar{\mathbf{b}}_{mc}$. Obviously, we have

$$H_{ac}(\omega) = \varphi(\omega, K_{ma})^T \mathbf{b}_{ac}.$$

From (7) or (10), we can see that $H(\omega)$ is nonlinear with respect to the set of coefficient vectors $\{\mathbf{b}_a, \mathbf{b}_{ma}, \mathbf{b}_{mc}\}$. If one wants to optimize $\{\mathbf{b}_a, \mathbf{b}_{ma}, \mathbf{b}_{mc}\}$ jointly, one approach is to apply nonlinear optimization techniques. By looking into (9) and (12), however, we find that, if $\mathbf{b}_a$ is fixed, $H(\omega)$ is linear with respect to $\{\mathbf{b}_{ac}, \mathbf{b}_{mc}\}$, and if $\mathbf{b}_{ac}$ is fixed, $H(\omega)$ is linear with respect to $\{\mathbf{b}_a, \mathbf{b}_{mc}\}$. Considering this, we transform the design problem into some linear approximation subproblems using the alternating variable method and then solve these subproblems based on a set of dense grid points $\{\omega_j\}_{j=1}^J$ in the frequency bands of interest.

When the coefficient vector $\mathbf{b}_{ac}$ is fixed, by using (12) and introducing a variable δ , the problem (8) can be converted into a linear programming problem as follows,

$$\min_{\mathbf{b}_a, \mathbf{b}_{mc}, \delta} \delta \quad (13a)$$

$$s.t. \quad |\varphi(M\omega_j, K_a)^T \mathbf{b}_a \varphi(\omega_j, K_{ma})^T \mathbf{b}_{ac} + \varphi(\omega_j, K_{mc})^T \mathbf{b}_{mc} - D(\omega_j)| \leq \delta / W(\omega_j), \quad \omega_j \in \Omega \subset \Omega_0, \quad (13b)$$

$$|\varphi(M\omega_j, K_a)^T \mathbf{b}_a \varphi(\omega_j, K_{ma})^T \mathbf{b}_{ac} + \varphi(\omega_j, K_{mc})^T \mathbf{b}_{mc} - D(\omega_j)| \leq \rho(\omega_j), \quad \omega_j \in \Omega_c \subset \Omega_0. \quad (13c)$$

Similarly, when the coefficient vector $\mathbf{b}_a$ is fixed, by using (7) (instead of (12) in this case) and introducing a variable δ , the problem (8) can be

converted into another linear programming problem as follows,

$$\min_{\mathbf{b}_{ma}, \mathbf{b}_{mc}, \delta} \delta \quad (14a)$$

$$s.t. \quad |\varphi(M\omega_j, K_a)^T \mathbf{b}_a \varphi(\omega_j, K_{ma})^T \mathbf{b}_{ma} + [1 - \varphi(M\omega_j, K_a)^T \mathbf{b}_a] \\ \cdot \varphi(\omega_j, K_{mc})^T \mathbf{b}_{mc} - D(\omega_j)| \leq \delta/W(\omega_j), \quad \omega_j \in \Omega \subset \Omega_0, \quad (14b)$$

$$|\varphi(M\omega_j, K_a)^T \mathbf{b}_a \varphi(\omega_j, K_{ma})^T \mathbf{b}_{ma} + [1 - \varphi(M\omega_j, K_a)^T \mathbf{b}_a] \\ \cdot \varphi(\omega_j, K_{mc})^T \mathbf{b}_{mc} - D(\omega_j)| \leq \rho(\omega_j), \quad \omega_j \in \Omega_c \subset \Omega_0. \quad (14c)$$

(ii) When $N_{ma} < N_{mc}$, we have

$$\varphi(\omega, K_{ma})^T \mathbf{b}_{ma} = \varphi(\omega, K_{mc})^T \bar{\mathbf{b}}_{ma}, \quad (15)$$

where the K_{mc} -dimensional vector $\bar{\mathbf{b}}_{ma}$ is a lengthened version of $\mathbf{b}_{ma}$ with padded zeros at the end. Thus, $H_{ac}(\omega)$ in (9) becomes

$$H_{ac}(\omega) = \varphi(\omega, K_{mc})^T \mathbf{b}_{ac},$$

with $\mathbf{b}_{ac} = \bar{\mathbf{b}}_{ma} - \mathbf{b}_{mc}$. Substituting (15) into (10) yields

$$H(\omega) = \varphi(M\omega, K_a)^T \mathbf{b}_a \varphi(\omega, K_{mc})^T \mathbf{b}_{ac} + \varphi(\omega, K_{mc})^T \mathbf{b}_{mc}. \quad (16)$$

Equation (16) looks very similar to (12). However, for this case we cannot directly solve a linear program similar to (13) for $\mathbf{b}_a$ and $\mathbf{b}_{mc}$ by fixing $\mathbf{b}_{ac}$. This is because if $\mathbf{b}_{ac}$ is fixed, the last $K_{mc} - K_{ma}$ components of $\mathbf{b}_{mc}$ are fixed as well. Thus, we partition $\mathbf{b}_{mc}$ as follows

$$\mathbf{b}_{mc}^T = [\mathbf{b}_{mc1}^T, \mathbf{b}_{mc2}^T],$$

where $\mathbf{b}_{mc1}$ includes the first K_{ma} components of $\mathbf{b}_{mc}$ and $\mathbf{b}_{mc2}$ includes the last $K_{mc} - K_{ma}$ components. Correspondingly, $\varphi(\omega, K_{mc})$ is partitioned as follows

$$\varphi(\omega, K_{mc})^T = [\varphi_1(\omega, K_{mc})^T, \varphi_2(\omega, K_{mc})^T],$$

and

$$\varphi(\omega, K_{mc})^T \mathbf{b}_{mc} = \varphi_1(\omega, K_{mc})^T \mathbf{b}_{mc1} + \varphi_2(\omega, K_{mc})^T \mathbf{b}_{mc2}.$$

Then, $H(\omega)$ is written as

$$H(\omega) = \varphi(M\omega, K_a)^T \mathbf{b}_a \varphi(\omega, K_{mc})^T \mathbf{b}_{ac} \\ + \varphi_1(\omega, K_{mc})^T \mathbf{b}_{mc1} + \varphi_2(\omega, K_{mc})^T \mathbf{b}_{mc2}. \quad (17)$$

If the coefficient vector $\mathbf{b}_{ac}$ is fixed (then, $\mathbf{b}_{mc2}$ is fixed as well), by using (17) and introducing a variable δ , the problem (8) can be converted into a linear programming problem as follows,

$$\min_{\mathbf{b}_a, \mathbf{b}_{mc1}, \delta} \delta \quad (18a)$$

$$s.t. \quad |\varphi(M\omega_j, K_a)^T \mathbf{b}_a \varphi(\omega_j, K_{mc})^T \mathbf{b}_{ac} + \varphi_1(\omega_j, K_{mc})^T \mathbf{b}_{mc1} + \varphi_2(\omega_j, K_{mc})^T \mathbf{b}_{mc2} - D(\omega_j)| \leq \delta/W(\omega_j), \quad \omega_j \in \Omega \subset \Omega_0, \quad (18b)$$

$$|\varphi(M\omega_j, K_a)^T \mathbf{b}_a \varphi(\omega_j, K_{mc})^T \mathbf{b}_{ac} + \varphi_1(\omega_j, K_{mc})^T \mathbf{b}_{mc1} + \varphi_2(\omega_j, K_{mc})^T \mathbf{b}_{mc2} - D(\omega_j)| \leq \rho(\omega_j), \quad \omega_j \in \Omega_c \subset \Omega_0. \quad (18c)$$

Similarly, if the coefficient vector $\mathbf{b}_a$ is fixed, by using (7) and introducing a variable δ , the problem for this case, as shown in (8), can be converted into the same linear programming problem as (14).

It is found that if we design the FRM filter by alternately solving (13) (or (18)) and (14) both on the frequency band $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$, the solutions are generally not good. As an alternative, we consider a two-phase approach. In the first phase, (13) (or (18)) is solved on $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$, but (14) on a different $\Omega_0 = [0, \varpi_p] \cup [\varpi_s, \pi]$ where

$$\begin{aligned} \varpi_p &= 2m\pi/M, \quad \varpi_s = (2m+1)\pi/M, \quad \text{for Case A;} \\ \varpi_p &= (2m-1)\pi/M, \quad \varpi_s = 2m\pi/M, \quad \text{for Case B.} \end{aligned}$$

Then, in the second phase, both problems are solved on the same $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$ to refine the solution. Simulation results show that this two-phase approach works very well.

The proposed alternating variable algorithm incorporating the two-phase approach is summarized as follows.

Phase 1

Step 1: Given the initial subfilters $H_a(z)$, $H_{ma}(z)$ and $H_{mc}(z)$ specified respectively on the frequency bands $\Omega_0 = [0, \theta] \cup [\phi, \pi]$, $[0, \theta_{ma}] \cup [\phi_{ma}, \pi]$ and $[0, \theta_{mc}] \cup [\phi_{mc}, \pi]$, whose coefficient vectors are denoted as $\mathbf{b}_a^{(0)}$, $\mathbf{b}_{ma}^{(0)}$ and $\mathbf{b}_{mc}^{(0)}$. Synthesize the FRM filter using these subfilters and evaluate its maximum approximation error denoted by $\delta_1^{(0)}$. Set $n = 0$.

Step 2: If $N_{ma} \geq N_{mc}$, get $\bar{\mathbf{b}}_{mc}^{(n)}$ by padding $\mathbf{b}_{mc}^{(n)}$ with zeros and let $\mathbf{b}_{ac}^{(n)} = \mathbf{b}_{ma}^{(n)} - \bar{\mathbf{b}}_{mc}^{(n)}$. Fix $\mathbf{b}_{ac}$ at $\mathbf{b}_{ac}^{(n)}$, and then solve the linear programming problem (13) on $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$. Denote the solution as $\{\mathbf{b}_a^{(n+1)}, \mathbf{b}_{mc}^{(n+1)}, \delta_1^{(n+1)}\}$.

Step 3: If $N_{ma} < N_{mc}$, get $\bar{\mathbf{b}}_{ma}^{(n)}$ by padding $\mathbf{b}_{ma}^{(n)}$ with zeros and let $\mathbf{b}_{ac}^{(n)} = \bar{\mathbf{b}}_{ma}^{(n)} - \mathbf{b}_{mc}^{(n)}$. Fix $\mathbf{b}_{ac}$ at $\mathbf{b}_{ac}^{(n)}$, and then solve the linear programming problem (18) on $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$. Denote the solution as $\{\mathbf{b}_a^{(n+1)}, \mathbf{b}_{mc1}^{(n+1)}, \delta_1^{(n+1)}\}$.

Step 4: If $\delta_1^{(n+1)} < \delta_1^{(n)}$, let $\hat{\delta}_1 = \delta_1^{(n+1)}$, $\hat{\mathbf{b}}_a = \mathbf{b}_a^{(n+1)}$.

Step 5: If $|\delta_1^{(n+1)} - \delta_1^{(n)}| < \varepsilon_1$ or $n > N_{max}$, terminate Phase 1, where ε_1 is a given error tolerance and N_{max} is a given maximum allowed iteration number.

Step 6: Fix $\mathbf{b}_a$ at $\mathbf{b}_a^{(n+1)}$, and then solve the linear programming problem (14) on $\Omega_0 = [0, \varpi_p] \cup [\varpi_s, \pi]$. Denote the solution as $\{\mathbf{b}_{ma}^{(n+1)}, \mathbf{b}_{mc}^{(n+1)}, \tilde{\delta}_1^{(n+1)}\}$. Let $n = n + 1$ and go back to Step 2.

The solution in the first phase corresponding to the minimum $\delta_1^{(n)} = \hat{\delta}_1$, is used as the initial solution for the second phase.

Phase 2

Step 1: Let $\mathbf{b}_a^{(0)} = \hat{\mathbf{b}}_a$, $\delta_2^{(0)} = \hat{\delta}_1$ and set $n = 0$.

Step 2: Fix $\mathbf{b}_a$ at $\mathbf{b}_a^{(n)}$, and then solve the linear programming problem (14) on $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$. Denote the solution as $\{\mathbf{b}_{ma}^{(n+1)}, \mathbf{b}_{mc}^{(n+1)}, \tilde{\delta}_2^{(n+1)}\}$.

Step 3: If $N_{ma} \geq N_{mc}$, get $\bar{\mathbf{b}}_{mc}^{(n+1)}$ by padding $\mathbf{b}_{mc}^{(n+1)}$ with zeros and let $\mathbf{b}_{ac}^{(n+1)} = \mathbf{b}_{ma}^{(n+1)} - \bar{\mathbf{b}}_{mc}^{(n+1)}$. Fix $\mathbf{b}_{ac}$ at $\mathbf{b}_{ac}^{(n+1)}$, and then solve the linear programming problem (13) on $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$. Denote the solution as $\{\mathbf{b}_a^{(n+1)}, \mathbf{b}_{mc}^{(n+1)}, \delta_2^{(n+1)}\}$.

Step 4: If $N_{ma} < N_{mc}$, get $\bar{\mathbf{b}}_{ma}^{(n+1)}$ by padding $\mathbf{b}_{ma}^{(n+1)}$ with zeros and let $\mathbf{b}_{ac}^{(n+1)} = \bar{\mathbf{b}}_{ma}^{(n+1)} - \mathbf{b}_{mc}^{(n+1)}$. Fix $\mathbf{b}_{ac}$ at $\mathbf{b}_{ac}^{(n+1)}$, and then solve the linear programming problem (18) on $\Omega_0 = [0, \omega_p] \cup [\omega_s, \pi]$. Denote the solution as $\{\mathbf{b}_a^{(n+1)}, \mathbf{b}_{mc1}^{(n+1)}, \delta_2^{(n+1)}\}$. Update $\mathbf{b}_{mc}^{(n+1)}$ by using $\mathbf{b}_{mc1}^{(n+1)}$ and $\mathbf{b}_{ac}^{(n+1)}$.

Step 5: If $|\delta_2^{(n+1)} - \delta_2^{(n)}| < \varepsilon_2$, terminate the algorithm, where ε_2 is a given error tolerance. $\mathbf{b}_a^{(n+1)}$, $\mathbf{b}_{mc}^{(n+1)}$ and $\mathbf{b}_{ma}^{(n+1)}$ are the optimized solution, where $\mathbf{b}_{ma}^{(n+1)}$ is obtained by using $\mathbf{b}_{mc}^{(n+1)}$ and $\mathbf{b}_{ac}^{(n+1)}$. Otherwise, let $n = n + 1$ and go back to Step 2.

Remark 1: For the initial subfilters $H_a(\omega)$, $H_{ma}(\omega)$ and $H_{mc}(\omega)$ in Step 1 of Phase 1, if there are no magnitude error constraints, we design them in the (weighted) minimax sense using the Remez algorithm [27]. Otherwise, if there are magnitude error constraints, we design these initial subfilters in the constrained minimax sense using the improved Remez algorithm in [5]. But to give better optimized solutions according to the ripple analysis in [17], we use an error-bound function $\beta\rho(\omega)$, i.e., $\rho(\omega)$ multiplied by a factor β , in the improved Remez algorithm for designing these initial subfilters. Generally, we take $0 < \beta \leq 1$.

Remark 2: It should be pointed out that because the design problems in Step 2 (or 3) and Step 6 in Phase 1 are not defined on the same frequency band Ω_0 , $\delta_1^{(n)}$ in Phase 1 may not decrease monotonically, which will be seen in the next section. However, it is easy to show that $\delta_2^{(n)}$ in Phase 2 should be monotonically decreasing. In Step 2, we fix $\mathbf{b}_a$ at $\mathbf{b}_a^{(n)}$ to solve (14). This leads to $\tilde{\delta}_2^{(n+1)} \leq \delta_2^{(n)}$. In Step 3 (or 4) of Phase 2, we fix $\mathbf{b}_{ac}$ at $\mathbf{b}_{ac}^{(n+1)}$ to solve (13)

(or (18)). This leads to $\delta_2^{(n+1)} \leq \tilde{\delta}_2^{(n+1)}$. Thus, we have $\delta_2^{(0)} \geq \tilde{\delta}_2^{(1)} \geq \delta_2^{(1)} \geq \tilde{\delta}_2^{(2)} \geq \delta_2^{(2)} \geq \dots$.

Noting that there is a maximum allowed iteration number in Phase 1, an immediate consequence of **Remark 2** is the following convergence theorem:

Theorem 1 *The proposed alternating variable algorithm always converges.*

Remark 3: It is easily seen that problems (13) (or (18)) and (14) solved in our algorithm steps have $\frac{N_a+N_{ma}}{2} + 2$ and $\frac{N_{ma}+N_{mc}}{2} + 2$ decision variables, respectively. Moreover, a linear programming with n variables and m linear inequality constraints has a complexity of order mn^2 when $m > n$ [2]. Then, the complexity of our alternating variable algorithm is order $4PJ \max\{N_1^2, N_2^2\}$, where P denotes the number of total iterations in Phases 1 and 2, J is the number of frequency grid points on Ω_0 , $N_1 = \frac{N_a+N_{ma}}{2} + 2$, and $N_2 = \frac{N_{ma}+N_{mc}}{2} + 2$.

4 Design Examples

In this section, two design examples are provided to illustrate the performance of the proposed alternating variable algorithm. The simulation is performed in MATLAB 7.11 on a Dell vostro-5470 laptop with an Intel(R) Core(TM) i5-4210U CPU (2.4-GHz) and an 8-Gb memory.

Example 1: Design of linear-phase lowpass FRM FIR filter with passband and stopband edge frequencies: $\omega_p = 0.60\pi$, $\omega_s = 0.61\pi$. The allowed maximum passband and stopband errors are both 0.01.

This example was considered by several researchers in [17, 25, 26, 33, 38]. For comparison, we first design such FRM filters without any constraints, i.e., $\Omega = [0, 0.60\pi] \cup [0.61\pi, \pi]$ and $\Omega_c = \emptyset$. In the proposed algorithm, the error tolerances and the maximum allowed iteration numbers are taken to be $\varepsilon_1 = 10^{-5}$, $N_{max} = 20$ for Phase 1 and $\varepsilon_2 = 10^{-8}$ for Phase 2, respectively. The interpolation factor is always taken as $M = 9$ in this example.

Firstly, as in [17, 25, 26], the following filter order specifications are used: $N_a = 45$, $N_{ma} = 41$ and $N_{mc} = 33$. The weighting function is set to $W(\omega) \equiv 1$. At the same time, as in [26], a set of $J = 1100$ uniformly distributed grid points in $[0, \omega_p] \cup [\omega_s, \pi]$, including the frequency band edges, are used. The proposed algorithm converges after 20 iterations in Phase 1 and 17 iterations in Phase 2, consuming 247 CPU seconds. The design results along with a comparison with the methods in [17, 25, 26] are listed in Table 1, where A_p and A_s denote the peak-to-peak passband ripple and the minimum stopband attenuation defined by [26]

$$A_p = 20 \log_{10} \left(\frac{1 + \delta_p}{1 - \delta_p} \right), \quad A_s = -20 \log_{10}(\delta_s),$$

with δ_p and δ_s denoting the maximum passband and stopband errors, respectively. P_1 and P_2 represent the iteration numbers required in Phase 1 and Phase 2 in our algorithm, respectively.

Table 1 Comparison with the methods in [17, 25, 26, 38]

Methods	J	(N_a, N_{ma}, N_{mc})	(P_1, P_2)	A_p (dB)	A_s (dB)
Method in [17]	—*	(45, 41, 33)	—	0.1792	40.96
Method in [25]	900	(45, 41, 33)	—	0.1348	42.25
Method in [26]	1100	(45, 41, 33)	—	0.1320	42.49
Proposed algorithm	1100	(45, 41, 33)	(20, 17)	0.1277	42.67
Method in [38]	—*	(45, 19, 27)	—	0.1737	40.00
Proposed algorithm	2500	(45, 19, 27)	(6, 9)	0.1641	40.49

* Not given

From Table 1, we can see that the proposed alternating variable algorithm obtains an FRM filter with better magnitude response behaviour in both the passband and stopband than those reported in [17, 25, 26]. Figure 2 shows the magnitude responses of the FRM filters obtained by the proposed algorithm and the recent method in [26] in the entire frequency band. Figures 3 and 4 give the passband and stopband details. Figures 5 and 6 depict the changes of $\delta_1^{(n)}$ and $\delta_2^{(n)}$ versus the iteration number, respectively. It is seen that in Phase 1 of the algorithm, $\delta_1^{(n)}$ decreases quickly in the first several iterations, and then decreases gradually though not monotonically. In Phase 2, $\delta_2^{(n)}$ decreases monotonically.

In [38], the lengths of the subfilters were further reduced to $N_a = 45$, $N_{ma} = 19$, $N_{mc} = 27$, respectively. The SQP method was used to obtain an FRM filter with $A_p = 0.1737$ dB ($\delta_p = 0.01$) and $A_s = 40$ dB ($\delta_s = 0.01$). With the same specifications as in [38], we design another filter using the proposed algorithm. It is pointed out that from our design experience and the suggestion in [37], the total number of grid points should be about 3 to 7 times the order of the overall filter for accurate designs. Therefore, we use $J = 2500$ points in our algorithm in this design for a fair comparison. The design results are also listed in Table 1. They are slightly better than that obtained by the SQP algorithm.

Next, we consider the constrained designs. In [26, 33], the filter order specifications with $N_a = 43$, $N_{ma} = 37$ and $N_{mc} = 29$ were used in the design. Their results are listed in Table 2. For comparison, we design the filter under the magnitude constraint with a constant $\rho(\omega)$ corresponding to $A_p = 0.2600$ dB, where the number of grid points, and the passband and stopband are taken as those in [26, 33], i.e., $J = 1000$, $\Omega = [0.61\pi, \pi]$, and $\Omega_c = [0, 0.60\pi]$. When using the improved Remez algorithm in [5] to get the initial subfilters for Phase 1 of our algorithm, the parameter β mentioned in **Remark 1** is set to 0.5 in this constrained design. The design results shown in Table 2 are better than those in [26, 33].

Our experience tells us that the initial subfilters will affect the final design results in the constrained designs. Specifically, when using different β values, the design results may be different. To show this, we design a few more filters under the constraint $A_p = 0.1737$ dB ($\Omega = [0.61\pi, \pi]$, $\Omega_c = [0, 0.60\pi]$, $\rho(\omega) = 0.01$) or the constraint $A_s = 40$ dB ($\Omega = [0, 0.60\pi]$, $\Omega_c = [0.61\pi, \pi]$, $\rho(\omega) =$

Fig. 2 Magnitude response of the unconstrained FRM filters by the proposed algorithm (solid line) and the method in [26] (dashed line) in entire frequency band

Fig. 3 Magnitude response of the unconstrained FRM filter by the proposed algorithm (solid line) and the method in [26] (dashed line) in passband

Fig. 4 Magnitude response of the unconstrained FRM filter by the proposed algorithm (solid line) and the method in [26] (dashed line) in stopband

Fig. 5 Change of $\delta_1^{(n)}$ in Phase 1

Fig. 6 Change of $\delta_2^{(n)}$ in Phase 2

0.01), all with $N_a = 45$, $N_{ma} = 41$ and $N_{mc} = 33$. Here, $J = 2500$ frequency points are used and β varies from 0.1 to 1 with a step of 0.05. The values of δ_p for the constrained designs with $A_s = 40$ dB and the values of δ_s for the constrained designs with $A_p = 0.1737$ dB for different values of β are shown in Fig. 7. It is observed from the figure that in general a small β is preferable. Figure 8 shows the magnitude response of the filter obtained by our algorithm with $A_s = 40$ dB and $\beta = 0.45$.

We also design other four filters under the constraint $A_s = 40$ dB or $A_p = 0.1737$ dB with $N_a = 45$, $N_{ma} = 41$, $N_{mc} = 33$ as in [17, 25, 26] and $N_a = 45$, $N_{ma} = 19$, $N_{mc} = 27$ as in [38]. The number of frequency grid points is also taken as $J = 2500$. The parameter β is set to 0.45 for $A_s = 40$ dB and 0.65 for $A_p = 0.1737$ dB for the former filter order specifications, and 0.3 for $A_s = 40$ dB and 0.5 for $A_p = 0.1737$ dB in the later case. Results of the four designs are tabulated in Table 3. After comparing the constrained design results in Table 3 with those unconstrained design results in Table 1, it can be seen that the proposed algorithm allows the designers to control the maximum passband and stopband magnitude errors more accurately by imposing explicit constraints on the magnitude.

Example 2: Design of linear-phase lowpass FRM FIR filter with an extremely narrow transition band, i.e., $\omega_p = 0.4\pi$, $\omega_s = 0.402\pi$. The allowed maximum passband and stopband errors are $\delta_p = 0.01$ and $\delta_s = 0.001$, respectively.

This example is taken from [10, 25, 32, 37], where the best feasible filters were obtained at $\{M = 21, N_a = 123, N_{ma} = 56, N_{mc} = 78\}$ and at $\{M =$

Table 2 Comparison with the methods in [26,33]

Methods	J	β	(N_a, N_{ma}, N_{mc})	(P_1, P_2)	A_p (dB)	A_s (dB)
Method in [33]	1000	—	(43,37,29)	—	0.2652	42.42
Method in [26]	1000	—	(43,37,29)	—	0.2600	43.01
Proposed algorithm	1000	0.5	(43,37,29)	(20,7)	0.2600	43.82

Table 3 Results of constrained designs with different specifications

β	(N_a, N_{ma}, N_{mc})	(P_1, P_2)	A_p (dB)	A_s (dB)
0.45	(45,41,33)	(20,6)	0.1065	40.00
0.65	(45,41,33)	(13,4)	0.1737	43.86
0.3	(45,19,27)	(19,4)	0.1596	40.00
0.5	(45,19,27)	(20,81)	0.1737	41.29

Fig. 7 (a) The change of δ_p versus β for the constrained design of FRM filter with $A_s = 40$ dB; (b) The change of δ_s versus β for the constrained design of FRM filter with $A_p = 0.1737$ dB

Fig. 8 Magnitude response of the constrained FRM filter with $A_s = 40$ dB and $\beta = 0.45$

21, $N_a = 125$, $N_{ma} = 56$, $N_{mc} = 76$ }. In the design by our alternating variable algorithm, the weighting function $W(\omega)$ is taken as $w_p = 1$ in passband and $w_s = 10$ in stopband, respectively.

This design example is challenging due to the very narrow transition band and very high orders of the subfilters. The aim here is to show the versatility and reliability of the proposed algorithm. To achieve highly accurate results, the number J of frequency grid points in the design should be sufficiently large, too. This will lead to a slow convergence. We speed up our algorithm using the approach described in Section 3.4 of [37] in Phase 1 of our algorithm. Firstly, we solve the linear programming problems on a sparse subset of the overall set of frequency grid points Ω_0 . Then in Step 4 of Phase 1, we evaluate the weighted error function on Ω_0 and add its extremal points to update the sparse subset. If the sparse subset does not change any more or the iteration number reaches the maximum allowed value (here, we set $N_{max} = 80$ in this example), Phase 1 terminates.

The design results are listed in Table 4. The proposed algorithm has achieved an FRM filter satisfying the design specifications in both passband and stopband. The design results are slightly better than those in [10] but slightly worse than those in [37]. In addition, the possible reason that the method in [25] gets the smaller values of δ_p and δ_s is the insufficient frequency grid points, as mentioned in [37]. Through the design, we see that the proposed algorithm is capable of designing FRM filters with very narrow transition band

Table 4 Comparison with the methods in [10,25,32,37]

Methods	J	(N_a, N_{ma}, N_{mc})	(P_1, P_2)	δ_p	δ_s
Method in [25]	1000	(123,56,78)	–	0.00989	0.000898
Method in [10]	10001	(123,56,78)	–	0.00986	0.000986
Method in [37,32]	20000	(123,56,78)	–	0.01001	0.001001
Method in [37,32]	20000	(125,56,76)	–	0.00988	0.000988
Proposed algorithm	10001	(123,56,78)	(80,3)	0.00984	0.000984
Proposed algorithm	20000	(125,56,76)	(73,11)	0.00996	0.000996

and very high order. Moreover, no complicated nonlinear optimization techniques are employed in our algorithm.

5 Conclusion

An alternating variable algorithm is proposed in this paper for the constrained minimax design of linear-phase FRM FIR filters. The algorithm transforms the original design problem into a series of linear programming problems by alternately fixing different subsets of design coefficients. Then by iteratively solving these linear programming problems, an optimized solution can be obtained without using nonlinear optimization techniques. Simulation results show that the new algorithm can achieve better filters than several existing methods in general. Moreover, because it is based on linear programming, constraints which are linear with respect to the subfilter coefficients can be incorporated into the proposed algorithm easily.

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