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Design of Pipelined IIR Filters Using Two-Stage Frequency-Response Masking Technique

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Abstract—A pipelined, high-speed IIR filter based on the frequency-response masking (FRM) technique has been proposed by Johansson and Wanhammar. In their method, the bandedge shaping filters are a pair of IIR power complementary filters whose z -transform functions are functions of z^M and the masking filters are linear phase finite impulse response (FIR) filters. The IIR bandedge shaping filters have M delays in the feedback loop and can be pipelined by M stages. In this paper, we present a two-stage FRM method for reducing the number of multipliers of the pipelined IIR filter proposed by Johansson and Wanhammar for a given magnitude response specification and a given number of pipeline stages in the feedback loop. In our design, all the subfilters are optimized jointly. We show that our two-stage masking design requires fewer multipliers if the number of pipeline stages in the feedback loop is larger than four.

Index Terms—Infinite impulse response (IIR) filter, pipelined implementation, frequency-response masking (FRM), sharp filter, optimization.

I. INTRODUCTION

IT has been reported in [1]–[5] that the frequency-response masking (FRM) technique [6]–[13] can be extended to the synthesis of infinite impulse response (IIR) filters. Fig. 1 shows the structure of the IIR FRM filter proposed by Johansson and Wanhammar [1]. The masking filters are linear phase finite impulse response (FIR) filters whose z -transform transfer functions are $F_{ma}(z)$ and $F_{mc}(z)$. The bandedge shaping filters are a pair of power complementary IIR filters realized as a parallel connection of two allpass filters whose z -transform transfer functions are $A_1(z^M)$ and $A_2(z^M)$, where M is a positive integer. When implemented using pipelined arithmetic, the IIR bandedge shaping filters can be pipelined by M stages since the feedback loops of $A_1(z^M)$ and $A_2(z^M)$ contain M delays.

The filter in Fig. 1 is a one-stage masking design. For given passband and stopband ripples, the number of multipliers of the masking filters increases when the number of pipeline stages in the feedback loop (M) increases. This is because the transition width of the masking filter decreases when M increases. The number of multipliers of the IIR bandedge shaping filter is not sensitive to the change of M . When M is large, the total number of multipliers of the filter in Fig. 1 is very large since the number of multipliers of the masking filters is

Fig. 1. The structure of the pipelined IIR filter proposed in [1]. The filter is synthesized using the FRM technique.

very large. It is shown in [1] that the number of multipliers of the masking filters can be reduced by using the constrained masking filters, i.e., $F_{mc}(z) = z^{-K} - (-1)^K F_{ma}(-z)$ where integer K is the delay of the masking filters. However, this technique is only effective when the filter bandedges are between $(1 - 1/M)\pi/2$ and $(1 + 1/M)\pi/2$. This range is very restrictive when M is very large. Thus, the usage of the constrained masking filters is very limited.

In this paper, we present a novel two-stage FRM approach for the synthesis of pipelined IIR filters. The proposed two-stage masking design requires much fewer multipliers than the one-stage masking design for the same number of pipeline stages in the feedback loop and for the same set of magnitude response specifications. The proposed technique is independent of the positions of the bandedges. The structure of the proposed two-stage FRM filter is presented in Section II. The two-stage FRM filter is designed by optimizing all the subfilters jointly, which is described in Section III. Section IV shows that our two-stage masking design requires fewer multipliers than the one-stage masking design when the number of pipeline stages in the feedback loop is larger than four. A design example is given in Section V to demonstrate the effectiveness of our method. The proposed method can be extended to multi-stage masking design. An example of three-stage design is presented in Section VI.

Throughout this paper, we assume that all the FIR subfilters have zero phase. Causality can be restored by inserting appropriate number of delays into the FIR subfilters.

II. FILTER STRUCTURE

Fig. 2 shows the structure of the pipelined IIR filter synthesized using the two-stage FRM technique where M_1 and M_2 are positive integers. The feedback loop can be pipelined by $M_1 M_2$ stages. Let $H(z)$ be the z -transform transfer function of the overall filter. $H(z)$ is given by

$$H(z) = H_a^{(1)}(z^{M_1})F_{ma}^{(1)}(z) + H_c^{(1)}(z^{M_1})F_{mc}^{(1)}(z) \quad (1)$$

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Fig. 2. The structure of pipelined IIR filter synthesized using the two-stage FRM technique.

where

$$H_a^{(1)}(z) = H_a^{(2)}(z^{M_2})F_{ma}^{(2)}(z) + H_c^{(2)}(z^{M_2})F_{mc}^{(2)}(z) \quad (2a)$$

$$H_c^{(1)}(z) = H_a^{(2)}(z^{M_2})(1 - F_{ma}^{(2)}(z)) + H_c^{(2)}(z^{M_2})(1 - F_{mc}^{(2)}(z)). \quad (2b)$$

$\{H_a^{(1)}(z), H_c^{(1)}(z)\}$ and $\{H_a^{(2)}(z), H_c^{(2)}(z)\}$ are the prototype bandedge shaping filters in the first and second stage of the two-stage masking design, respectively. $\{F_{ma}^{(1)}(z), F_{mc}^{(1)}(z)\}$ and $\{F_{ma}^{(2)}(z), F_{mc}^{(2)}(z)\}$ are the masking filters in the first and second stage of the two-stage masking design, respectively. They are linear phase FIR filters with symmetrical impulse responses. The synthesis of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ is illustrated in Fig. 3 where $H_a^{(2)}(e^{jM_2\omega})$, $H_c^{(2)}(e^{jM_2\omega})$, $F_{ma}^{(2)}(\omega)$, $F_{mc}^{(2)}(\omega)$, $H_a^{(1)}(e^{j\omega})$ and $H_c^{(1)}(e^{j\omega})$ are the frequency responses of $H_a^{(2)}(z^{M_2})$, $H_c^{(2)}(z^{M_2})$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$, respectively.

The prototype bandedge shaping filters in the second stage, $H_a^{(2)}(z)$ and $H_c^{(2)}(z)$, are composed of two allpass filters connected in parallel. Their transfer functions are given by

$$H_a^{(2)}(z) = \frac{A_1(z) + A_2(z)}{2} \quad (3a)$$

$$H_c^{(2)}(z) = \frac{A_1(z) - A_2(z)}{2} \quad (3b)$$

where $A_x(z)$ ($x = 1, 2$) is an allpass filter whose frequency response magnitude $|A_x(e^{j\omega})|$ equals unity. $H_a^{(2)}(z)$ and $H_c^{(2)}(z)$ are a pair of power complementary filters. From (2) and (3), we have

$$H_a^{(1)}(z) + H_c^{(1)}(z) = A_1(z^{M_2}). \quad (4)$$

Since the sum of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ is an allpass filter, $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ are allpass complementary [14].

III. JOINT OPTIMIZATION OF SUBFILTERS

To obtain an optimum design for $H(z)$, the subfilters are jointly optimized. We shall first formulate the optimization problem.

Let the order of $A_x(z)$ ($x = 1, 2$) be N_x . When $H_a^{(2)}(z)$ and $H_c^{(2)}(z)$ are lowpass and highpass filters, respectively, the orders of $A_1(z)$ and $A_2(z)$ differ by one. In the following discussion, we use $N_2 = N_1 + 1$. The frequency response of $A_x(z)$ can be written as

$$A_x(e^{j\omega}) = e^{-jN_x\omega} \tilde{A}_x(e^{j\omega}) \quad (5)$$

Fig. 3. The synthesis of $H_a^{(1)}(z)$ and $H_c^{(1)}(z)$ using the FRM technique.

where

$$\tilde{A}_x(e^{j\omega}) = \prod_{i=1}^{\lceil N_x/2 \rceil} \frac{1 + a_{xi}(1)e^{j\omega} + a_{xi}(2)e^{j2\omega}}{1 + a_{xi}(1)e^{-j\omega} + a_{xi}(2)e^{-j2\omega}}. \quad (6)$$

Denote the coefficient vector of $A_x(z)$ as

$$\mathbf{a}_x = [a_{x1}(1), a_{x1}(2), a_{x2}(1), a_{x2}(2), \dots, a_{x\lceil N_x/2 \rceil}(1), a_{x\lceil N_x/2 \rceil}(2)]^T \quad (7)$$

where $\lceil x \rceil$ denotes the smallest integer larger than or equal to x . Let the impulse response of $F_{mr}^{(i)}(z)$ ($i = 1, 2$ and $r = a, c$) be $h_{mr}^{(i)}(n)$, $n = 0, \dots, L_{imr} - 1$ where L_{imr} is the length of $F_{mr}^{(i)}(z)$. For $\{F_{ma}^{(1)}(z), F_{mc}^{(1)}(z)\}$, L_{1ma} and L_{1mc} must be both odd or both even. For $\{F_{ma}^{(2)}(z), F_{mc}^{(2)}(z)\}$, $(L_{2ma} - 1)M_1$ and $(L_{2mc} - 1)M_1$ must be even. The frequency response of $F_{mr}^{(i)}(z)$ can be written as

$$F_{mr}^{(i)}(\omega) = h_{mr}^{(i)}\left(\frac{L_{imr} - 1}{2}\right) + \sum_{n=1}^{\frac{L_{imr}-1}{2}} 2h_{mr}^{(i)}\left(\frac{L_{imr} - 1}{2} - n\right) \cos(\omega n) \quad (8)$$

for L_{imr} odd, and written as

$$F_{mr}^{(i)}(\omega) = \sum_{n=1}^{\frac{L_{imr}}{2}} 2h_{mr}^{(i)}\left(\frac{L_{imr}}{2} - n\right) \cos\left(\omega\left(n - \frac{1}{2}\right)\right) \quad (9)$$

for L_{imr} even. Denote the coefficient vector of $F_{mr}^{(i)}(z)$ as

$$\mathbf{f}_{imr} = \left[h_{mr}^{(i)}\left(\frac{L_{imr} - 1}{2}\right), 2h_{mr}^{(i)}\left(\frac{L_{imr} - 1}{2} - 1\right), \dots, 2h_{mr}^{(i)}(0) \right]^T. \quad (10)$$

for L_{imr} odd, and as

$$\mathbf{f}_{imr} = \left[h_{mr}^{(i)}\left(\frac{L_{imr}}{2} - 1\right), 2h_{mr}^{(i)}\left(\frac{L_{imr}}{2} - 2\right), \dots, 2h_{mr}^{(i)}(0) \right]^T. \quad (11)$$

for L_{imr} even.

Let the coefficient vector of the overall filter be $\mathbf{p} = [\mathbf{a}_1^T, \mathbf{a}_2^T, \mathbf{f}_{2ma}^T, \mathbf{f}_{2mc}^T, \mathbf{f}_{1ma}^T, \mathbf{f}_{1mc}^T]^T$. Let $\tilde{H}(e^{j\omega}, \mathbf{p})$ denote the frequency response of the overall filter without the linear phase term $e^{-jM_1M_2N_1\omega}$.

$$\tilde{H}(e^{j\omega}, \mathbf{p}) = \tilde{H}_a^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc})F_{ma}^{(1)}(\omega, \mathbf{f}_{1ma}) + \tilde{H}_c^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc})F_{mc}^{(1)}(\omega, \mathbf{f}_{1mc}) \quad (12)$$

where

$$\begin{aligned} & \tilde{H}_a^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc}) \\ &= \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)F_{ma}^{(2)}(M_1\omega, \mathbf{f}_{2ma}) \\ &+ \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)F_{mc}^{(2)}(M_1\omega, \mathbf{f}_{2mc}) \\ & \tilde{H}_c^{(1)}(e^{jM_1\omega}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{f}_{2ma}, \mathbf{f}_{2mc}) \\ &= \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)[1 - F_{ma}^{(2)}(M_1\omega, \mathbf{f}_{2ma})] \\ &+ \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2)[1 - F_{mc}^{(2)}(M_1\omega, \mathbf{f}_{2mc})] \\ & \tilde{H}_a^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2) \\ &= \frac{1}{2}\{\tilde{A}_1(e^{jM_1M_2\omega}, \mathbf{a}_1) + e^{-jM_1M_2\omega}\tilde{A}_2(e^{jM_1M_2\omega}, \mathbf{a}_2)\} \\ & \tilde{H}_c^{(2)}(e^{jM_1M_2\omega}, \mathbf{a}_1, \mathbf{a}_2) \\ &= \frac{1}{2}\{\tilde{A}_1(e^{jM_1M_2\omega}, \mathbf{a}_1) - e^{-jM_1M_2\omega}\tilde{A}_2(e^{jM_1M_2\omega}, \mathbf{a}_2)\} \end{aligned}$$

The joint optimization of subfilters can be formulated as the following optimization problem

$$\underset{\delta, \mathbf{p}}{\text{minimize}} \quad \delta \quad (14a)$$

$$\text{subject to} \quad |\tilde{H}(e^{j\omega}, \mathbf{p})|^2 - 1 \leq W(\omega)\delta, \quad \forall \omega \in \Omega_p \quad (14b)$$

$$|\tilde{H}(e^{j\omega}, \mathbf{p})| \leq W(\omega)\delta, \quad \forall \omega \in \Omega_s \quad (14c)$$

$$a_{xi}(1) - a_{xi}(2) < 1, \quad (14d)$$

$$-a_{xi}(1) - a_{xi}(2) < 1, \quad (14e)$$

$$a_{xi}(2) < 1, \quad i = 1, 2, \dots, \lceil N_x/2 \rceil \quad (14f)$$

where $W(\omega)$ is a prescribed weight function, and Ω_p and Ω_s are the sets of passband and stopband grid points, respectively. Constraints (14d)-(14f) ensure the stability of the allpass filters $A_1(z)$ and $A_2(z)$.

Problem (14) can be solved using sequential convex programming [15]. The solution may be a local optimum. It is important to find a good initial solution such that the optimization procedure converges to a useful optimum. Given $M_1, M_2, N_1, N_2, L_{2ma}, L_{2mc}, L_{1ma}, L_{1mc}$ and the bandedges of the overall filter, a good initialization technique is as follows.

- Step 1:* Compute the bandedges of $H_a^{(2)}(z)[H_c^{(2)}(z)]$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ using the formulas in [6].
- Step 2:* Initialize $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ and $F_{mc}^{(1)}(z)$ as linear phase FIR filters with appropriate bandedges.
- Step 3:* Initialize $H_a^{(2)}(z)$ or $H_c^{(2)}(z)$ as an odd-order digital elliptic filter. Then, $A_1(z)$ and $A_2(z)$ are obtained based on the poles of the elliptic filter.

IV. MULTIPLIER SAVING

For a given magnitude response specification and a given pipeline stages in the feedback loop, the number of multipliers

of the IIR bandedge shaping filter will not have a significant difference in the one-stage and two-stage masking designs. However, the number of multipliers of the FIR masking filters may be significantly different.

The length L of a lowpass linear phase FIR filter with transition width $\Delta \cdot f_s$, where f_s is the sampling frequency, passband ripple magnitude δ_1 and stopband ripple magnitude δ_2 , is given by [16]

$$L \approx \frac{\Phi_1(\delta_1, \delta_2)}{\Delta} - \Phi_2(\delta_1, \delta_2)\Delta + 1 \quad (15)$$

where

$$\begin{aligned} \Phi_1(\delta_1, \delta_2) &= [0.005309(\log_{10} \delta_1)^2 + 0.07114 \log_{10} \delta_1 - 0.4761] \\ &\times \log_{10} \delta_2 - 0.00266(\log_{10} \delta_1)^2 \\ &- 0.5941 \log_{10} \delta_1 - 0.4278 \end{aligned} \quad (16a)$$

$$\begin{aligned} \Phi_2(\delta_1, \delta_2) &= 11.01217 + 0.51244(\log_{10} \delta_1 - \log_{10} \delta_2). \end{aligned} \quad (16b)$$

The first term in (15) is the dominant term when $\Delta \leq 0.2$. Thus, we have

$$L \approx \frac{\Phi_1(\delta_1, \delta_2)}{\Delta}, \quad \text{for } \Delta \leq 0.2. \quad (17)$$

Let the maximum allowed passband and stopband ripple magnitudes of $H(z)$ be δ_p and δ_s , respectively. When estimating the lengths of the masking filters, the passband and stopband ripple magnitudes of $F_{mr}^{(i)}(z)$ ($i = 1, 2$ and $r = a, c$) are chosen as δ_p and δ_s , respectively. The sum of the normalized transition widths of $F_{ma}^{(i)}(z)$ and $F_{mc}^{(i)}(z)$ is $1/M_i$. It has been shown in [7] that $L_{ima} + L_{imc}$ is minimum when $\Delta_{ima} = \Delta_{imc} = 1/(2M_i)$ where Δ_{ima} and Δ_{imc} are the normalized transition widths of $F_{ma}^{(i)}(z)$ and $F_{mc}^{(i)}(z)$, respectively. Thus, for the two-stage masking design, the total number of coefficients of the masking filters is given by

$$L_{T2S} \approx 4\Phi_1(\delta_p, \delta_s)(M_1 + M_2). \quad (18)$$

If the number of pipeline stages in the feedback loop of the one-stage masking design is the same as that of the two-stage masking design, i.e., $M = M_1M_2$, the total number of coefficients of the masking filters for one-stage masking design is given by

$$L_{T1S} \approx 4\Phi_1(\delta_p, \delta_s)M_1M_2. \quad (19)$$

In the FRM technique, the value of M is larger than one. From (18) and (19), we have $L_{T2S} \leq L_{T1S}$. The equality holds when $M_1 = M_2 = 2$. Thus, the two-stage masking design requires fewer multipliers than the one-stage masking design when the number of pipeline stages in the feedback loop is larger than four.

There may exist a few possible values of (M_1, M_2) satisfying $M_1M_2 = M_{des}$ where M_{des} is the desired number of pipeline stages in the feedback loop. The number of multipliers is minimum when $M_1 = M_2 = \sqrt{M_{des}}$. Thus, M_1 and M_2 should be chosen such that $M_1 \approx M_2$.

TABLE I
MULTIPLIER SAVINGS IN PERCENTAGE FOR DIFFERENT NUMBER OF
PIPELINE STAGES IN THE FEEDBACK LOOP

Pipeline stages	M	Multipliers	Saving (%)
9	Two-stage: $M_1 = 3, M_2 = 3$	35	27.1
	One-stage: $M = 9$	48	
15	Two-stage: $M_1 = 5, M_2 = 3$	40	42.9
	One-stage: $M = 15$	70	
21	Two-stage: $M_1 = 7, M_2 = 3$	47	50.0
	One-stage: $M = 21$	94	
25	Two-stage: $M_1 = 5, M_2 = 5$	47	57.3
	One-stage: $M = 25$	110	
27	Two-stage: $M_1 = 9, M_2 = 3$	52	54.8
	One-stage: $M = 27$	115	
35	Two-stage: $M_1 = 7, M_2 = 5$	52	63.6
	One-stage: $M = 35$	143	
45	Two-stage: $M_1 = 9, M_2 = 5$	60	67.0
	One-stage: $M = 45$	182	

V. EXAMPLE

We choose the design specifications given in Example 1 of [1] as an example to illustrate our method. The design specifications are: passband edge = 0.495π , stopband edge = 0.5π , maximum passband peak-to-peak ripple = 0.2 dB, minimum stopband attenuation = 50 dB. Our design algorithm is implemented using MATLAB and CVX [17]. For the purpose of comparison, one-stage FRM filter [1] is also designed using the joint optimization method similar to that described in Section III. Filters with different number of pipeline stages in the feedback loop are designed. The actual lengths of the FIR masking filters are often less than the estimated lengths obtained using (15). The total number of multipliers required by the two-stage and one-stage masking designs is listed in Table I. As can be seen from Table I, our two-stage masking design produces significant savings in the total number of multipliers compared to the one-stage design. The magnitude responses of subfilters and overall filter in the two-stage design with $M_1 = 5$ and $M_2 = 3$ are shown in Fig. 4. The coefficient values are shown in Table II.

For the two-stage masking design, the number of multipliers increases with $M_1 + M_2$ as shown in Section IV. In general, for the two-stage design, the number of multipliers increases when the number of pipeline stages increases, i.e., $M_1 M_2$ increases. For the cases where $M_1 M_2$ increases but $M_1 + M_2$ remains unchanged, the number of multipliers may be unchanged or approximately the same. This can be seen in Table I when $M_1 M_2$ increases from 21 to 25 and when $M_1 M_2$ increases from 27 to 35.

Compared to the one-stage design, the two-stage design requires two extra adders for obtaining $1 - F_{ma}^{(2)}(z^{M_1})$ and $1 - F_{mc}^{(2)}(z^{M_1})$ and requires two extra adders for summing the outputs of $F_{ma}^{(2)}(z^{M_1})$ and $F_{mc}^{(2)}(z^{M_1})$ and the outputs of $1 - F_{ma}^{(2)}(z^{M_1})$ and $1 - F_{mc}^{(2)}(z^{M_1})$. However, the total number of adders required by the FIR masking filters in the two-stage design is significantly less than that in the one-stage design. Thus, the two-stage design requires much fewer adders than the one-stage design. For the example where the number of

TABLE II
COEFFICIENT VALUES OF $A_1(z)$, $A_2(z)$, $F_{ma}^{(2)}(z)$, $F_{mc}^{(2)}(z)$, $F_{ma}^{(1)}(z)$ AND $F_{mc}^{(1)}(z)$ FOR THE TWO-STAGE DESIGN WITH $M_1 = 5$ AND $M_2 = 3$

$a_{11}(1) = 0.392178919257735$	$a_{21}(1) = 0.526338754929022$
$a_{11}(2) = 0.000000000000000$	$a_{21}(2) = 0.344477735714563$
$a_{12}(1) = 0.403321799782520$	$a_{22}(1) = 0.367715987627043$
$a_{12}(2) = 0.622962482782130$	$a_{22}(2) = 0.897874990280713$
$h_{ma}^{(2)}(0) = -0.023971821827727$	$h_{mc}^{(2)}(0) = -0.009750471728404$
$h_{ma}^{(2)}(1) = -0.053383192172249$	$h_{mc}^{(2)}(1) = 0.054317748630854$
$h_{ma}^{(2)}(2) = -0.053593374608275$	$h_{mc}^{(2)}(2) = -0.002088312263685$
$h_{ma}^{(2)}(3) = 0.044741642245504$	$h_{mc}^{(2)}(3) = -0.118512917293624$
$h_{ma}^{(2)}(4) = 0.195738377622483$	$h_{mc}^{(2)}(4) = 0.241692140679120$
$h_{ma}^{(2)}(5) = 0.190838915003943$	$h_{mc}^{(2)}(5) = 0.763263289408957$
$h_{ma}^{(2)}(6) = 0.261304368450267$	
$h_{ma}^{(1)}(0) = -0.006174845786805$	$h_{mc}^{(1)}(0) = 0.008928644269163$
$h_{ma}^{(1)}(1) = 0.004774264078534$	$h_{mc}^{(1)}(1) = -0.006976160125584$
$h_{ma}^{(1)}(2) = 0.012157356331036$	$h_{mc}^{(1)}(2) = -0.029564738978457$
$h_{ma}^{(1)}(3) = -0.016360153407917$	$h_{mc}^{(1)}(3) = 0.001778538625737$
$h_{ma}^{(1)}(4) = -0.012369129145799$	$h_{mc}^{(1)}(4) = 0.043799863590131$
$h_{ma}^{(1)}(5) = 0.051563500442353$	$h_{mc}^{(1)}(5) = 0.004010001455358$
$h_{ma}^{(1)}(6) = -0.001801809142000$	$h_{mc}^{(1)}(6) = -0.087819828790875$
$h_{ma}^{(1)}(7) = -0.114365872463247$	$h_{mc}^{(1)}(7) = -0.051913118330990$
$h_{ma}^{(1)}(8) = 0.086890263327473$	$h_{mc}^{(1)}(8) = 0.181032342211771$
$h_{ma}^{(1)}(9) = 0.494460856174286$	$h_{mc}^{(1)}(9) = 0.404739512763396$

pipeline stages is 15, 80 adders is saved by using the two-stage design.

VI. MULTI-STAGE MASKING DESIGN

Our proposed technique may be extended to multi-stage masking design. We shall illustrate this by using an example of three-stage masking design. Fig. 5 shows the structure of the pipelined IIR filter synthesized using the three-stage FRM technique. The feedback loop is pipelined by $M_1 M_2 M_3$ stages. Consider the design in Table I whose number of pipeline stages is 45. If three-stage design is used, we may choose $M_1 = 5$, $M_2 = 3$ and $M_3 = 3$ and the total number of multipliers is 53. Compared to the two-stage design, the three-stage design produces a 11.7% saving in total number of multipliers.

VII. CONCLUSION

In this paper, we present a two-stage FRM technique for designing pipelined IIR filter. In our design, all the subfilters are optimized jointly. It is shown that, when the number of pipeline stages in the feedback loop is larger than four, our two-stage masking design requires fewer multipliers than the one-stage masking design [1] meeting the same magnitude response specifications. The superiority of the two-stage design over the one-stage design is also demonstrated by an example. The proposed method can be used to design any piecewise constant frequency response such as a bandpass filter.

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Fig. 4. Magnitude responses of (a) Bandedge shaping filters $H_a^{(2)}(z^{15})$ (solid line) and $H_c^{(2)}(z^{15})$ (dashed line), (b) masking filters $F_{ma}^{(2)}(z^5)$ (solid line) and $F_{mc}^{(2)}(z^5)$ (dashed line), (c) bandedge shaping filters $H_a^{(1)}(z^5)$ (solid line) and $H_c^{(1)}(z^5)$ (dashed line), (d) masking filters $F_{ma}^{(1)}(z)$ (solid line) and $F_{mc}^{(1)}(z)$ (dashed line), (e) overall filter $H(z)$, and (f) $H(z)$ in the passband.

Fig. 5. The structure of pipelined IIR filter synthesized using the three-stage FRM technique.

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