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A Matrix-based IRLS Algorithm for the Least l_p -norm Design of 2-D FIR Filters

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Abstract Fast design of two-dimensional (2-D) FIR filters in the least l_p -norm sense is investigated in this brief. The design problem is first formulated in a matrix form and then solved by a matrix-based iterative reweighted least squares (IRLS) algorithm. The proposed algorithm includes two loops: one for updating the weighting function and the other for solving the weighted least squares (WLS) subproblems. These WLS subproblems are solved using an efficient matrix-based WLS algorithm, which is an iterative procedure with its initial iterative matrix being the solution matrix in the last iteration, resulting in a considerable CPU-time saving. Through analysis, the new algorithm is shown to have a lower complexity than existing methods. Three design examples are provided to illustrate the high computational efficiency and design precision of the proposed algorithm.

Keywords 2-D FIR filter · Least l_p -norm design · Matrix-based algorithm · Iterative reweighted least squares algorithm

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1 Introduction

In the past several decades, many methods have been reported for the optimal design of 2-D FIR filters due to their widespread applications in 2-D digital signal processing. Among them, one idea is to extend design algorithms for 1-D filters to 2-D cases by compacting the 2-D filter coefficients into a vector (Algazi et al (1986); Burrus et al (1994); Barreto and Burrus (1994); Kok et al (2008); Lai (2008); Lu (2002); Lu and Hinamoto (2011); Tseng and Lee (2013); Aggarwal et al (2016)). We refer to such methods as vectorized methods. A disadvantage of these methods is that the required amount of computations is rather large, since the number of coefficients of an $(N \times N)$ -th-order 2-D filter is the square of that of an N -th-order 1-D filter.

On the other hand, several matrix-based design algorithms (Hsieh et al (1997); Aravena and Gu (1996); Zhu et al (1999); Zhao and Lai (2011, 2013); Hong et al (2013, 2016); Zhao et al (2016)) have been proposed for designing 2-D filters. They formulate the design problem and solve for the coefficients of 2-D filters in their natural matrix forms. Many numerical simulations have verified that these matrix-based algorithms can greatly improve the design of 2-D filters compared with vectorized methods, especially in terms of the computational efficiency.

The most commonly used criteria in filter design are the least squares (LS) (or weighted least squares (WLS)) and the minimax criteria. In practice, there are many situations where both LS (or WLS) and minimax criteria are not suitable (Adams et al (1998); Diniz and Netto (1999); Savin et al (1999); Mousa (2012); Hamamoto et al (2015)). To obtain a better filter performance, one should use a mixture criterion or make a compromise between the two criteria. The least l_p -norm design criterion considered in this brief can give a good trade-off between the LS (or WLS) design and the minimax design in terms of the computational complexity and approximation error. It is a more general criterion, including LS ($p = 2$) as its special case and approaching minimax as p goes to ∞ . Moreover, it allows assigning different values of p in different frequency bands (Burrus et al (1994)).

For the least l_p -norm approximation problem, because there are no analytical methods to obtain the optimal solution, numerical iterative methods have to be used. One of the most efficient methods for the linear least l_p -norm approximation is the iterative reweighted least squares (IRLS) technique, which breaks the complicated original problem into a series of WLS subproblems. The IRLS technique was first developed by Lawson (Lawson (1961)) to solve the minimax approximation and then extended to the least l_p -norm approximation by Rice and Usow (Rice and Usow (1968)). Its extensions and variants include Karlovitz (1970); Chalmers et al (1983); Watson (1988) and so on. The IRLS technique and its modified versions have been applied to filter design problems in the least l_p -norm or minimax senses by many authors (Algazi et al (1986); Burrus et al (1994); Lim et al (1992); Barreto and Burrus (1994); Diniz and Netto (1999); Vargas and Burrus (2009); Vargas (2012)). Although it has been verified in practice that the IRLS technique offers a tailored and efficient route to the least l_p -norm design of 1-D filters, the computational complexity is still very high when directly extending this approach to the 2-D case using vectorized methods (Algazi et al (1986); Barreto and Burrus (1994); Burrus et al (1994)) due to the large number of variables and iterations.

In this brief, we extend the matrix-based WLS algorithm recently presented in Zhao et al (2016) to the design of 2-D least l_p -norm filters by formulating the least l_p -norm design of 2-D FIR filters in a matrix form and incorporating the algorithm with the IRLS technique in Rice and Usow (1968). Due to the high efficiency of the WLS algorithm in Zhao et al (2016), the proposed matrix-based IRLS algorithm can efficiently design the least l_p -norm

and minimax 2-D FIR filters with a high design precision. Three design examples are given to demonstrate its effectiveness and usefulness.

This brief is organized as follows. In Section 2, the least l_p -norm design problem of 2-D FIR filters is formulated as an optimization problem in matrix forms. Then, an efficient matrix-based IRLS algorithm for such design problem is developed and a complexity analysis is given in Section 3. In Section 4, three design examples are provided to demonstrate the effectiveness and usefulness of the presented algorithm and comparisons are made with existing algorithms. Finally, the conclusions are drawn in Section 5.

2 Least l_p -norm design of 2-D FIR filters

The frequency response of a general $N_1 \times N_2$ 2-D FIR filter can be formulated as

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) e^{-j(n_1\omega_1 + n_2\omega_2)}, \quad (1)$$

where $\{h(n_1, n_2), n_1 = 0, 1, \dots, N_1 - 1; n_2 = 0, 1, \dots, N_2 - 1\}$ are the impulse responses, ω_1 and ω_2 are the horizontal and vertical frequencies, respectively.

In this brief, we consider the quadrantly symmetric 2-D FIR filter, whose impulse responses $h(n_1, n_2)$'s are real numbers and satisfy the restrictions:

$$h(n_1, n_2) = h(n_1, N_2 - 1 - n_2) = h(N_1 - 1 - n_1, n_2). \quad (2)$$

As a result, the 2-D filter has a linear phase response and its magnitude response, denoted by $M(\omega_1, \omega_2)$, is even symmetric with respect to ω_1 and ω_2 , i.e.,

$$M(\omega_1, \omega_2) = M(-\omega_1, \omega_2) = M(\omega_1, -\omega_2). \quad (3)$$

Using the restrictions (2), the magnitude response can be written in a matrix-vector form as

$$M(\omega_1, \omega_2) = \psi^T(\omega_1, N_1) \mathbf{A} \psi(\omega_2, N_2), \quad (4)$$

where $\psi(\omega, N)$ is a vector given by

$$\psi(\omega, N) = \begin{cases} [\frac{1}{\sqrt{2}}, \cos(\omega), \cos(2\omega), \dots, \cos(\frac{N-1}{2}\omega)]^T, & \text{for odd } N, \\ [\cos(\frac{\omega}{2}), \cos(\frac{3\omega}{2}), \dots, \cos(\frac{N-1}{2}\omega)]^T, & \text{for even } N, \end{cases}$$

and $\mathbf{A}$ is an $L_1 \times L_2$ real coefficient matrix of the filter with $L_k = N_k/2$ for even N_k or $L_k = (N_k + 1)/2$ for odd N_k ($k=1, 2$). The ij -th entry in $\mathbf{A}$, denoted by a_{ij} , is a real coefficient related to $\{h(n_1, n_2)\}$ by certain rules (Zhao et al (2016)).

The design of a 2-D FIR filter is to determine the coefficient matrix $\mathbf{A}$ such that $M(\omega_1, \omega_2)$ can best approximate a desired magnitude response $D(\omega_1, \omega_2)$ in some senses. In view of the symmetry (3), we only need to consider the approximation on the region $\Omega = \{(\omega_1, \omega_2) | 0 \leq \omega_1 \leq \pi; 0 \leq \omega_2 \leq \pi\}$.

Firstly, we define an error function

$$E(\omega_1, \omega_2) = M(\omega_1, \omega_2) - D(\omega_1, \omega_2),$$

and discretize the region Ω by an $M_1 \times M_2$ rectangular frequency grid defined as $\hat{\Omega} = \{(\omega_{1i}, \omega_{2j}) | i = 1, 2, \dots, M_1; j = 1, 2, \dots, M_2\}$, where $0 \leq \omega_{11} < \omega_{12} < \dots < \omega_{1, M_1} \leq \pi$, and $0 \leq \omega_{21} < \omega_{22} < \dots < \omega_{2, M_2} \leq \pi$.

The least l_p -norm design of linear-phase 2-D FIR filters can be formulated as

$$\min_{\mathbf{A}} E_p(\mathbf{A}), \quad (5)$$

where $p \geq 2$ and the cost function $E_p(\mathbf{A})$ is given by

$$E_p(\mathbf{A}) = \left(\sum_{i=1}^{M_1} \sum_{j=1}^{M_2} (\bar{w}_{ij} |E(\omega_{1i}, \omega_{2j})|)^p \right)^{1/p}. \quad (6)$$

Here, $\bar{w}_{ij}$'s are nonnegative weights imposed on frequency points $(\omega_{1i}, \omega_{2j})$'s. For the frequency points $(\omega_{1i}, \omega_{2j})$'s in the transition band, $\bar{w}_{ij}$'s are set to be 0.

To express $E_p(\mathbf{A})$ in matrix forms, in this brief we define the following l_p norm ($p \geq 2$), denoted by $\|\mathbf{M}\|_p$, for a matrix $\mathbf{M} = (m_{ij})$,

$$\|\mathbf{M}\|_p = \left(\sum_{i,j} |m_{ij}|^p \right)^{1/p}. \quad (7)$$

To avoid confusion, it should be pointed out that the l_p norm defined in (7) is the l_p vector norm on a matrix but not the matrix norm induced by the l_p vector norm (Horn and Johnson (1994)). Obviously, when $p = 2$, the norm (7) reduces to the matrix Frobenius norm. When $p = \infty$, $\|\mathbf{M}\|_\infty = \max_{i,j} \{|m_{ij}|\}$.

Using the l_p norm defined in (7), the cost function (6) can be rewritten as

$$E_p(\mathbf{A}) = \|(\mathbf{P}^T \mathbf{A} \mathbf{Q} - \mathbf{D}) \circ \bar{\mathbf{W}}\|_p, \quad (8)$$

where $\mathbf{D}$ and $\bar{\mathbf{W}}$ are two $M_1 \times M_2$ matrices with their ij -th entries being $D(\omega_{1i}, \omega_{2j})$ and $\bar{w}_{ij}$, respectively, and $\mathbf{P}$ and $\mathbf{Q}$ are also matrices given by

$$\begin{aligned} \mathbf{P} &= [\psi(\omega_{11}, N_1), \psi(\omega_{12}, N_1), \dots, \psi(\omega_{1, M_1}, N_1)] \in \mathbf{R}^{L_1 \times M_1}, \\ \mathbf{Q} &= [\psi(\omega_{21}, N_2), \psi(\omega_{22}, N_2), \dots, \psi(\omega_{2, M_2}, N_2)] \in \mathbf{R}^{L_2 \times M_2}. \end{aligned}$$

In (8), “ $\circ$ ” denotes the Hadamard matrix product, i.e., the entry-wise product (Horn and Johnson (1994)).

Thus, we know that when $p = 2$, the design problem (5) is just the least squares design. When $p = \infty$, the cost function (6) becomes

$$E_\infty(\mathbf{A}) = \max_{i,j} \{\bar{w}_{ij} |E(\omega_{1i}, \omega_{2j})|\}, \quad (9)$$

and thereby the design problem (5) is the minimax design

$$\min_{\mathbf{A}} \max_{i,j} \{\bar{w}_{ij} |E(\omega_{1i}, \omega_{2j})|\}. \quad (10)$$

3 Matrix-based IRLS algorithm

3.1 Description of the algorithm

Firstly, it is easy to know that minimizing $E_p(\mathbf{A})$ is equivalent to minimizing $(E_p(\mathbf{A}))^p$. By definition (7), we obtain

$$(E_p(\mathbf{A}))^p = \|(\mathbf{P}^T \mathbf{A} \mathbf{Q} - \mathbf{D})^{(\frac{p}{2})} \circ \bar{\mathbf{W}}^{(\frac{p}{2})}\|_2^2,$$

where the superscript $(\cdot)$ defines the entry-wise power operation on a matrix, i.e., for a matrix $\mathbf{M}$ with its ij -th entry being m_{ij} , $\mathbf{M}^{(b)}$ denotes the matrix whose ij -th entry is $|m_{ij}|^b$,

$$\mathbf{M}^{(b)} = (|m_{ij}|^b). \quad (11)$$

Further, $(E_p(\mathbf{A}))^p$ can be written as

$$(E_p(\mathbf{A}))^p = \|(\mathbf{P}^T \mathbf{A} \mathbf{Q} - \mathbf{D}) \circ (\mathbf{P}^T \mathbf{A} \mathbf{Q} - \mathbf{D})^{(\frac{p-2}{2})} \circ \bar{\mathbf{W}}^{(\frac{p}{2})}\|_2^2. \quad (12)$$

Equation (12) suggests us to solve the design problem (5) iteratively by

$$\mathbf{A}_{n+1} = \arg\{\min_{\mathbf{A}} \|(\mathbf{P}^T \mathbf{A} \mathbf{Q} - \mathbf{D}) \circ \mathbf{W}_n^{(\frac{1}{2})}\|_2^2\}, \quad (13)$$

where

$$\mathbf{W}_n = (\mathbf{P}^T \mathbf{A}_n \mathbf{Q} - \mathbf{D})^{(p-2)} \circ \bar{\mathbf{W}}^{(p)} \quad (14)$$

and $\mathbf{A}_n$ is the solution matrix obtained in the n -th iteration. Equations (13) and (14) constitute a basic IRLS algorithm for the least l_p -norm design of 2-D FIR filters. The problem (13) is a WLS design problem, which can be solved efficiently using the matrix-based WLS algorithm recently proposed in Zhao et al (2016).

However, the basic IRLS algorithm has numerical problems and does not always converge when p is large. Some modified algorithms (Rice and Usow (1968); Karlovitz (1970); Chalmers et al (1983); Watson (1988); Barreto and Burrus (1994); Burrus et al (1994); Lim et al (1992); Vargas and Burrus (2009)) had been proposed to address the convergence issue. The algorithm by Rice and Usow (Rice and Usow (1968)) is a simple but efficient one. It will be used to update the weight matrix (14) in this brief.

Following Rice and Usow (1968), we rewrite (14) as

$$\begin{aligned} \mathbf{W}_n &= [(\mathbf{P}^T \mathbf{A}_n \mathbf{Q} - \mathbf{D})^{(p-2)} \circ \bar{\mathbf{W}}^{(p)}]^{(\alpha)} \circ (\mathbf{P}^T \mathbf{A}_n \mathbf{Q} - \mathbf{D})^{(\beta)} \circ \bar{\mathbf{W}}^{(p-\alpha p)} \\ &= \mathbf{W}_n^{(\alpha)} \circ (\mathbf{P}^T \mathbf{A}_n \mathbf{Q} - \mathbf{D})^{(\beta)} \circ \bar{\mathbf{W}}^{(p-\alpha p)} \end{aligned} \quad (15)$$

where α and β are two positive real numbers satisfying

$$\alpha > 0, \beta > 0 \text{ and } \alpha + \beta/(p-2) = 1.$$

Equation (15) suggests the following updating formula for the weighting matrix $\mathbf{W}_n$,

$$\mathbf{W}_n = \mathbf{W}_{n-1}^{(\alpha)} \circ (\mathbf{P}^T \mathbf{A}_n \mathbf{Q} - \mathbf{D})^{(\beta)} \circ \bar{\mathbf{W}}^{(p-\alpha p)}, \quad (16)$$

where $\mathbf{W}_{n-1}$ is the weighting matrix in the $(n-1)$ -th iteration.

By introducing a parameter $\gamma = \alpha/\beta$, the entry-wise powers of $\mathbf{W}_{n-1}$, $(\mathbf{P}^T \mathbf{A}_n \mathbf{Q} - \mathbf{D})$ and $\bar{\mathbf{W}}$ in (16) can be represented by p and γ as,

$$\alpha = \frac{\gamma(p-2)}{\gamma(p-2)+1}, \quad \beta = \frac{p-2}{\gamma(p-2)+1}, \quad \text{and} \quad p - \alpha p = \frac{p}{\gamma(p-2)+1}. \quad (17)$$

Note that for the special case of $p = \infty$ (corresponding to the minimax design (10)), using the expressions in (17), we have

$$\lim_{p \rightarrow \infty} \alpha = 1, \quad \lim_{p \rightarrow \infty} \beta = \frac{1}{\gamma}, \quad \text{and} \quad \lim_{p \rightarrow \infty} (p - \alpha p) = \frac{1}{\gamma}. \quad (18)$$

Based on the above description, we propose a matrix-based IRLS algorithm for the least l_p -norm design of 2-D FIR filters as follows:

Matrix-based IRLS Algorithm

Step 1: Given filter lengths N_1 and N_2 , choose two proper design tolerances $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, a small number $\tau > 0$, and a discretized frequency grid $\hat{\Omega}$. Construct the desired magnitude matrix $\mathbf{D}$, and the matrices $\mathbf{P}$, $\mathbf{Q}$ and $\bar{\mathbf{W}}$. Choosing a proper value for parameter γ , for $2 \leq p < \infty$, obtain α , β and $p - \alpha p$ using (17) and for $p = \infty$, obtain α , β and $p - \alpha p$ using (18). Initialize the weighting matrix $\mathbf{W}_0 = \bar{\mathbf{W}}^{(2)}$ and the initial iterative matrix $\mathbf{A}_0$ properly. Let $n = 0$.

Step 2: Solve the following WLS subproblem for $\mathbf{A}_{n+1}$,

$$\mathbf{A}_{n+1} = \arg\{\min_{\mathbf{A}} \| (\mathbf{P}^T \mathbf{A} \mathbf{Q} - \mathbf{D}) \circ \mathbf{W}_n^{(\frac{1}{2})} \|_2^2\}$$

using the matrix-based WLS algorithm (Zhao et al (2016)). In the matrix-based WLS algorithm which is an iterative one, $\mathbf{A}_n$ and ε_1 are used as its initial iterative matrix and design tolerance, respectively.

Step 3: Update the weighting matrix $\mathbf{W}_n$. First let

$$\mathbf{W}_{n+1} = \mathbf{W}_n^{(\alpha)} \circ (\mathbf{P}^T \mathbf{A}_{n+1} \mathbf{Q} - \mathbf{D})^{(\beta)} \circ \bar{\mathbf{W}}^{(p-\alpha p)},$$

and then normalize it by

$$\mathbf{W}_{n+1} = \mathbf{W}_{n+1} / \|\mathbf{W}_{n+1}\|_{\infty}.$$

Finally, amend the weighting matrix as

$$\mathbf{W}_{n+1} = (\mathbf{W}_{n+1} + \tau) \circ \text{sign}(\bar{\mathbf{W}}),$$

where $\text{sign}(\cdot)$ is the signum function.

Step 4: If $\|\mathbf{A}_{n+1} - \mathbf{A}_n\|_F < \varepsilon_2$, terminate the algorithm. Otherwise, let $n = n + 1$ and go back to Step 2. $\square$

In the above algorithm, the small positive number τ is used to avoid the weighting matrix entries on the passband and stopband accidentally becoming zero and thus to speed the convergence of the algorithm. Generally, $\tau = 10^{-5}$ is a good choice. The parameter γ can be used to adjust the convergence rate, and a good empirical value is 0.65. For the initial iterative matrix $\mathbf{A}_0$ in Step 1, $\mathbf{A}_0 = (\mathbf{P}\mathbf{P}^T)^{-1} \mathbf{P} \mathbf{D} \mathbf{Q}^T (\mathbf{Q} \mathbf{Q}^T)^{-1}$ is a good choice.

Because the matrix-based WLS algorithm in Zhao et al (2016) is an iterative procedure, the proposed matrix-based IRLS algorithm includes two loops: one for solving the WLS subproblem in Step 2, and the other for updating the weighting matrix. To avoid confusion, we call the former the WLS iteration, and the later the IRLS iteration.

In addition, for the case of $p = 2$, by (17) we have $\alpha = 0$, $\beta = 0$ and $p - \alpha p = 2$. Hence, no IRLS iteration is required for this case.

3.2 Analysis and Comparison of Computational Complexity

For IRLS algorithms, the computational burden lies in solving the WLS subproblems. The vectorized WLS algorithm using matrix inversion as described in Zhao et al (2016) has the complexity of about $M_1 M_2 L_1^2 L_2^2 + O(L_1^3 L_2^3)$. While, the matrix-based WLS algorithm used in the proposed IRLS algorithm only needs about $L_1 L_2 (M_1 + M_2) + M_1 M_2 (L_1 + L_2)$ multipliers in one iteration (i.e., in one WLS iteration). Moreover, the iteration number required by the matrix-based WLS algorithm is generally much less than its maximum value $L_1 L_2$. When the IRLS iteration numbers required by the vectorized and the matrix-based IRLS algorithms are close to each other, the matrix-based algorithm is obviously more efficient than the vectorized one.

As another comparison, we give in the following the complexity of the sequential constrained least squares (SCLS) and the mixed l_2/l_∞ algorithms in Hong et al (2013) for minimax design of 2-D FIR filters. Both of them employ the matrix-based constrained least squares (CLS) algorithm, which is also an iterative one. The complexity of the CLS algorithm in one iteration (referred to as CLS iteration) is about $O(M_1 L_2 (M_2 + L_1) + L_1 L_2 (t + L_2))$, where t is the number of Lagrangian multipliers. When the magnitude error upper-bound constraint is not tight, t is small, and the complexity per CLS iteration is about the same order as that per iteration of the matrix-based WLS algorithm. But when the upper bound is close to the minimax error, t will approximate $L_1 L_2$. Counting the overall WLS iteration and CLS iteration numbers, the matrix-based IRLS algorithm is generally much more efficient for the design of high-order filters than the SCLS and the mixed l_2/l_∞ algorithms.

4 Design examples and Comparisons

Three design examples, one for least l_p -norm design and two for minimax design, are presented in this section to illustrate the good performance of the matrix-based IRLS algorithm. All algorithms are implemented in MATLAB programs, and all designs are done on a Dell vostro-5470 laptop with an Intel(R) Core(TM) i5-4210U CPU (2.4-GHz) and an 8-Gb memory. The frequency grid is taken as

$$\hat{\Omega} = \{(\frac{(i-1)\pi}{M_1-1}, \frac{(j-1)\pi}{M_2-1}) | i = 1, 2, \dots, M_1; j = 1, 2, \dots, M_2\}$$

with $M_1 = 5N_1$ and $M_2 = 5N_2$, the same as that in Hong et al (2013).

Example 1: Least l_p -norm design of $N \times N$ fan filters with passband Ω_p and stopband Ω_s in the first quadrant described as

$$\begin{aligned} \Omega_p &= \{(\omega_1, \omega_2) \mid 0 \leq \omega_1 \leq \omega_2 \tan(\frac{\pi}{6}), 0 \leq \omega_2 \leq \pi\}, \\ \Omega_s &= \{(\omega_1, \omega_2) \mid \omega_2 \tan(\frac{\pi}{6}) + 0.08\pi \leq \omega_1 \leq \pi, 0 \leq \omega_2 \leq \pi\}. \end{aligned}$$

The weights are set to be 1 on the passband and stopband and 0 on the transition band.

Table 1 Comparison among the CVX, the vectorized IRLS algorithm and the matrix-based IRLS algorithm in Example 1

N	p	CVX with SeDuMi			Vectorized IRLS algorithm				Matrix-based IRLS algorithm			
		$T_d(s)$	E_∞	E_p	$T_d(s)$	N_i	E_∞	E_p	$T_d(s)$	N_i	E_∞	E_p
11	4	5.28	4.0502×10^{-1}	8.9854×10^{-1}	0.11	10	4.0503×10^{-1}	8.9854×10^{-1}	0.015	9	4.0502×10^{-1}	8.9854×10^{-1}
	15	8.60	3.2987×10^{-1}	3.6313×10^{-1}	0.12	43	3.2991×10^{-1}	3.6314×10^{-1}	0.031	18	3.2980×10^{-1}	3.6314×10^{-1}
	25.5	13.47	3.1982×10^{-1}	3.3538×10^{-1}	0.23	89	3.1981×10^{-1}	3.3543×10^{-1}	0.044	24	3.1965×10^{-1}	3.3539×10^{-1}
	32	7.83	3.1719×10^{-1}	3.2887×10^{-1}	0.27	101	3.1718×10^{-1}	3.2891×10^{-1}	0.051	28	3.1719×10^{-1}	3.2889×10^{-1}
21	4	71.7	1.9913×10^{-1}	5.0959×10^{-1}	0.47	9	1.9913×10^{-1}	5.0959×10^{-1}	0.035	9	1.9914×10^{-1}	5.0959×10^{-1}
	15	—*	—*	—*	1.57	36	1.3390×10^{-1}	1.5798×10^{-1}	0.10	19	1.3387×10^{-1}	1.5798×10^{-1}
	25.5	—*	—*	—*	2.56	57	1.2450×10^{-1}	1.3730×10^{-1}	0.15	28	1.2458×10^{-1}	1.3730×10^{-1}
	32	—*	—*	—*	2.94	67	1.2226×10^{-1}	1.3226×10^{-1}	0.16	28	1.2212×10^{-1}	1.3226×10^{-1}
35	4	—**	—**	—**	3.66	8	7.0201×10^{-2}	2.1864×10^{-1}	0.10	8	7.0217×10^{-2}	2.1864×10^{-1}
	15	—**	—**	—**	13.6	32	4.8850×10^{-2}	5.7115×10^{-2}	0.19	15	4.8795×10^{-2}	5.7115×10^{-2}
	25.5	—**	—**	—**	24.3	56	4.5514×10^{-2}	4.9545×10^{-2}	0.21	13	4.5691×10^{-2}	4.9635×10^{-2}
	32	—**	—**	—**	27.8	65	4.4683×10^{-2}	4.7834×10^{-2}	0.22	14	4.4871×10^{-2}	4.7962×10^{-2}

* Program fails.

** Out of memory.

Fig. 1 Maximum magnitude error E_∞ vs p for the 21×21 filters

We use the popular convex optimization software CVX (with solver SeDuMi), the vectorized IRLS algorithm and the proposed matrix-based IRLS algorithm to design such filters of sizes 11×11 , 21×21 and 35×35 for different values of p . The design parameters in the IRLS algorithms are taken to be $\epsilon_1 = \epsilon_2 = \tau = 10^{-5}$, and $\gamma = 0.65$, respectively. The IRLS iteration number (N_i), the design time (T_d) in seconds (s), the maximum magnitude error ($E_\infty = E_\infty(\mathbf{A})$) and the l_p norm error ($E_p = E_p(\mathbf{A})$), are listed in Table 1.

In the CVX method, the function `minimize()` is used to minimize the l_p norm error E_p . By comparing the values of E_p (The smaller E_p is, the better the solution is in the least l_p -norm design), we see that these algorithms have a similar design precision. Obviously, the IRLS algorithms are more efficient than CVX and are capable of designing high-order filters with large values of p . When comparing the two IRLS algorithms, as expected, the matrix-based IRLS algorithm needs much less design time than the vectorized IRLS algorithm. In addition, we can see that the maximum magnitude error E_∞ will decrease gradually as the

Fig. 2 WLS iteration numbers vs IRLS iteration for the 21×21 filters

Fig. 3 Maximum magnitude error E_∞ and l_p norm error E_p vs IRLS iteration for the 21×21 filter with $p = 32$

value of p increases. This is depicted in Fig. 1, which gives the change of E_∞ with p for the 21×21 filters.

Since the solution matrix $\mathbf{A}_n$ of the n -th IRLS iteration is used as the initial iterative matrix for the WLS algorithm in the $(n+1)$ -th IRLS iteration, the WLS iterations consumed by the WLS algorithm will decrease as the IRLS algorithm proceeds, resulting in considerable CPU-time savings. This can be seen from the WLS iteration numbers in different IRLS iterations for the 21×21 filters shown in Fig. 2. In addition, the average values of WLS iteration numbers are 6.7, 20.2 and 23.5 for $p=4$, 15 and 32, respectively, which are much less than its maximum value $L_1 L_2 = 121$.

Further, we also find that as the IRLS algorithm proceeds, the errors E_p and E_∞ of the intermediate filters in different IRLS iterations will decrease gradually, which can be seen

Table 2 Comparison among the algorithms in Hsieh et al (1997); Hong et al (2013) and the matrix-based IRLS algorithm in Example 2

N	Iterative WLS algorithm (Hsieh et al (1997))			SCLS algorithm (Hong et al (2013))			Mixed l_2/l_∞ algorithm (Hong et al (2013))			Matrix-based IRLS algorithm		
	$T_d(s)$	N_i	E_∞	$T_d(s)$	N_c	E_∞	$T_d(s)$	N_c	E_∞	$T_d(s)$	N_w	E_∞
19	1.5	1470	7.9470×10^{-2}	0.56	1310	7.8485×10^{-2}	0.58	1063	7.8415×10^{-2}	2.1	3932	7.8392×10^{-2}
31	3.1	1147	3.0797×10^{-2}	5.1	4597	2.2368×10^{-2}	6.6	3916	2.2369×10^{-2}	5.1	3422	2.2362×10^{-2}
43	4.1	970	2.7265×10^{-2}	82	20000	7.4177×10^{-3}	51	10832	7.4166×10^{-3}	5.5	3990	7.4170×10^{-3}
51	8.6	1362	2.2775×10^{-2}	886	75050	3.9441×10^{-3}	174	15273	3.9516×10^{-3}	7.3	1897	3.9478×10^{-3}
61	—*	—*	—*	6446	253331	1.5193×10^{-3}	528	21669	1.5264×10^{-3}	5.6	2055	1.5273×10^{-3}

* Fails to converge

Fig. 4 Magnitude response of the 35×35 least l_{32} -norm fan filter

from Fig. 3 for the 21×21 filter with $p = 32$. Finally, Fig. 4 gives the magnitude response of the 35×35 fan filter designed by the proposed algorithm for $p = 32$.

Example 2: Minimax (least l_∞ -norm) design of $N \times N$ 2-D circular FIR filters with the passband Ω_p and stopband Ω_s described as

$$\begin{aligned}\Omega_p &= \{(\omega_1, \omega_2) | \omega_1^2 + \omega_2^2 \leq (0.4\pi)^2\}, \\ \Omega_s &= \{(\omega_1, \omega_2) | \omega_1^2 + \omega_2^2 \geq (0.5\pi)^2, |\omega_1| \leq \pi, |\omega_2| \leq \pi\}.\end{aligned}$$

The weights are set to be 1 on the passband and stopband and 0 on the transition band.

Here, we compare the proposed matrix-based IRLS algorithm with some other matrix-based algorithms, i.e., the iterative WLS algorithm in Hsieh et al (1997), and the SCLS and the mixed l_2/l_∞ algorithms in Hong et al (2013). The design parameters ε_1 , τ and γ are the same as that in Example 1, i.e., $\varepsilon_1 = \tau = 10^{-5}$, and $\gamma = 0.65$, but let $\varepsilon_2 = 10^{-6}$. Filters of various sizes are designed and the results are listed in Table 2. As in Table 1, N_i represents the IRLS iteration number. Considering that the complexity of one CLS iteration is similar to that of one WLS iteration as mentioned in Section 3.2, we account the total CLS and

Fig. 5 Magnitude response of the 31×31 minimax circular filter

WLS iteration numbers, denoted by N_c and N_w , respectively, in this example to compare these algorithms.

From the table, we can see that when compared with the iterative WLS algorithm (Hsieh et al (1997)) which also has a high computational efficiency but sometimes does not converge, our algorithm always converges, and obtains obviously better solutions, i.e., smaller E_∞ 's, consuming nearly equal design time.

Compared with the algorithms in Hong et al (2013), the proposed matrix-based IRLS algorithm needs slightly longer design time for low-order filters, but consumes much less design time with an acceptable precision for high-order filters. What is more, we find the average values of WLS iteration numbers of the proposed algorithm are extremely small in this example. They are 2.6, 2.4, 5.9, 2.4 and 9.0 for the 19×19 , 31×31 , 43×43 , 51×51 and 61×61 filters, respectively.

Moreover, in the matrix-based IRLS algorithm, a tradeoff can be made between the design precision and design time by tuning the value of ϵ_2 . For example, if let $\epsilon_2 = 5 \times 10^{-6}$ for the 19×19 filter, we obtain $T_d = 0.54s$, $N_w = 2472$ and $E_\infty = 7.8453 \times 10^{-2}$. If let $\epsilon_2 = 0.2 \times 10^{-6}$ for the 61×61 filter, we obtain $T_d = 28s$, $N_w = 4093$ and $E_\infty = 1.5192 \times 10^{-3}$. Fig. 5 shows the magnitude response of the 31×31 filter designed by the matrix-based IRLS algorithm.

In the above two examples, the weights on the passband and stopband are uniform. In fact, by (6), the proposed algorithm allows the use of different weights on the passband and stopband. The following example illustrates this flexibility of our proposed design method.

Example 3: Minimax (least l_∞ -norm) design of 25×25 2-D FIR diamond filter with the passband Ω_p and stopband Ω_s described as

$$\begin{aligned}\Omega_p &= \{(\omega_1, \omega_2) \mid |\omega_1| + |\omega_2| \leq 0.5\pi\}, \\ \Omega_s &= \{(\omega_1, \omega_2) \mid |\omega_1| + |\omega_2| \geq 0.7\pi, |\omega_1| \leq \pi, |\omega_2| \leq \pi\}.\end{aligned}$$

In addition, the maximum magnitude error in the stopband is required not greater than 0.01.

Fig. 6 Magnitude response of the 25×25 diamond filter

Firstly, we design such 2-D diamond filter using the semi-definite programming (SDP) method (Lu (2002)) by SDPA-M (Fujisawa et al (2005)). It consumes 1236 seconds and the maximum magnitude errors of the resulting filter in passband and stopband are 2.0648×10^{-2} and 0.01, respectively. Then we design the filter using the proposed algorithm. The weights on the passband, stopband and transition band are set to be 1, w_p and 0, respectively. It is easy to know by trial and error that w_p would locate in the interval $[1, 3]$ for satisfying the error requirement in stopband. Therefore, we use the bisection method to find the appropriate w_p . After calling the matrix-based IRLS algorithm 15 times (including $w_p = 1$ and 3) and consuming 75 seconds, we obtain $w_p = 2.0676$ and the corresponding filter with its maximum magnitude errors in the passband and stopband being 2.0672×10^{-2} and 0.01, respectively. Compared with the SDP method, the proposed one needs much less design time with a slightly larger passband magnitude error. Fig. 6 shows the magnitude response of the designed 25×25 diamond filter.

As a final comment, from Tables 1 and 2, we can see that the design time of the proposed matrix-based IRLS algorithm does not increase much with the filter order. This is mainly because the overall WLS iteration number of the proposed algorithm increases just slightly or even decreases with the filter order. For the convergence of the IRLS algorithm, there have been many in-depth discussions in the literature (Lawson (1961); Rice and Usow (1968); Karlovitz (1970); Watson (1988); Vargas (2012)), and the proposed matrix-based IRLS algorithm always converges in practical 2-D filter designs.

5 Conclusion

A matrix-based IRLS algorithm has been proposed for the design of 2-D FIR filters in the least l_p -norm sense. The algorithm is derived from the matrix formulation of the design problem and uses a highly efficient matrix-based WLS algorithm as its iteration core. Design examples and comparisons with existing methods show that the proposed algorithm has achieved an obvious improvement in the design efficiency and solution precision.

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