Stochastic orderings by nonlinear expectations

Ly, Sel

2020

Ly, S. (2020). Stochastic Orderings by Nonlinear Expectations. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/145282>

<https://doi.org/10.32657/10356/145282>

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**STOCHASTIC ORDERINGS BY NONLINEAR
EXPECTATIONS**

**LY SEL
SCHOOL OF PHYSICAL AND MATHEMATICAL
SCIENCES
2020**

**STOCHASTIC ORDERINGS BY NONLINEAR
EXPECTATIONS**

LY SEL

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2020

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Abstract

We study the theory of stochastic order under the nonlinear expectations framework, including g - and G -expectations, which leads to more general concepts of orderings in comparison with the standard linear expectation setting. The g -expectation $\mathcal{E}_g$ is generated by a solution of Backward Stochastic Differential Equation (BSDE) and it provides a new robust tool for option pricing in finance. In the same manner, we denote by $\mathcal{E}_G$ a nonlinear expectation generated by a solution of G -Backward Stochastic Differential Equation (G -BSDE), which is defined on G -expectation spaces. In this dissertation, our main contributions consist of four parts as follows:

(i) We extend several properties of monotonicity, convexity and continuous dependence for the solutions of Forward-Backward Stochastic Differential Equations (FBSDEs) and associated semilinear parabolic Partial Differential Equations (PDEs). Also, we obtain analogous results for the solutions of G -Forward-Backward Stochastic Differential Equations (G -FBSDEs) and associated G -PDEs (Hamilton-Jacobi-Bellman-type equations). These properties are very crucial to constitute comparison results for the monotonic, convex, increasing convex g - and G -stochastic orderings.

(ii) We give some characterizations for a better understanding of g -stochastic orderings which are defined via the g -evaluations and g -expectations $\mathcal{E}_g$. We then derive several sufficient conditions for the convex, increasing convex and monotonic g -stochastic orderings of diffusion processes at the terminal time T . Analogous comparison results for g -risk measures have been proposed as consequences, in terms of concave g -stochastic orderings. Applications of the g -stochastic orderings to contingent claim price comparison under different hedging portfolio constraints are also provided.

(iii) In order to deal with financial problems under uncertain volatility, we proceed to work on G -expectation spaces and propose analogously sufficient conditions for the convex, increasing convex and monotonic G -stochastic orderings and G -risk measures of G -diffusion processes at the terminal time T as in the case of g -stochastic orderings setting. Also, we provide applications of the G -stochastic orderings to contingent claim superhedging price comparison under ambiguous coefficients.

(iv) We have derived comparisons results between linear, sublinear and nonlinear expectations, including the linear expectation $\mathbb{E}$, g -expectation $\mathcal{E}_g$, sublinear expectation $\widehat{\mathbb{E}}$, G -expectation $\mathbb{E}_G$ and the nonlinear expectation $\mathcal{E}_G$ in the sense of convex

and increasing convex orders. Specifically, we derive sufficient conditions for comparing between $\mathcal{E}_g[\varphi(X_T)]$ and $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$, $\mathbb{E}_G[\varphi(X_T^G)]$ and $\mathbb{E}_G[\varphi(\widehat{X})]$, $\mathcal{E}_g[\varphi(X_T)]$ and $\mathcal{E}_G[\varphi(X_T^G)]$, as well as between $\mathbb{E}[\varphi(X_T)]$ and $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$ as consequences, where φ are convex (or non-decreasing convex) functions, $\widehat{X}$ denotes a G -normal distributed random variable, X_T and X_T^G are terminal values of diffusion stochastic processes driven by the classical Brownian motion and G -Brownian motion, respectively.

In option pricing setting, inequalities in the g - and G -stochastic orderings can be interpreted as comparisons of portfolio values, option prices, and risk measures. In stochastic control problems, they can be used for comparing the cost or value functions. Furthermore, the inequalities in the g - and G -stochastic orderings can also offer us lower and upper bounds information and the concentration of measure inequalities for random variables with uncertain distributions. As such, they can be applied in decision making under volatility uncertainty. Besides, our results can serve as alternative comparison theorems for the solutions of FBSDEs and G -FBSDEs.

Acknowledgements

I have come to the final stage of my PhD program at Nanyang Technological University, Singapore. This thesis would not have been completed without the support and inspiration of a number of people. So, I would like to render my sincere thanks to the people who made this day possible.

First and foremost, it is my pleasure to express the warmest appreciation and deepest thanks to my supervisor, Professor Nicolas Privault, for his wholehearted support, constant encouragement and invaluable guidance throughout all stages of my research work. From the bottom of my heart, I am eternally grateful to him.

I am also grateful to the members of my thesis advisory committee and staffs of the School of Physical and Mathematical Sciences for their continuous support during my PhD journey. Especially, I truly appreciate and sincerely thank the qualifying examination committee, for their valuable feedback and suggestions.

I am so thankful for all help and assistance from my brother, sister, extended family members and all true friends. I am so glad to have all of you in my life.

Lastly, and most importantly, I convey my heartfelt thanks to my dearest parents and my beloved wife, the mother of my daughter, for all your love and the sacrifices made for me. You are my everything today and tomorrow. I may not always say how much I love you but I have the biggest place in my heart only for you all.

Table of Contents

Abstract	1
Acknowledgements	3
Table of Contents	4
List of Abbreviations and Symbols	8
1 Introduction	11
2 Preliminaries	17
2.1 Stochastic Orderings	18
2.2 Stochastic Differential Equations	20
2.3 g -Expectations	24
2.3.1 Backward Stochastic Differential Equations	24
2.3.2 g -Evaluation and g -Expectation	25
2.3.3 Relations between g -Expectations and Classical Expectations .	27
2.4 Risk Measures	31
2.4.1 Coherent and Convex Risk Measures	31
2.4.2 Risk Measures by g -Expectations	35
2.5 Sublinear Expectations	36
2.5.1 Definitions and Properties	36
2.5.2 G -Expectation and G -Brownian Motion	40
2.5.3 G -Itô Integrals	43
2.5.4 Stochastic Differential Equations Driven by G -Brownian Motion	45

3	Monotonicity, Convexity and Continuous Dependence of FBSDEs and PDEs	47
3.1	Forward-Backward SDEs and Associated Parabolic PDEs	48
3.1.1	Forward-Backward SDEs	48
3.1.2	Associated Parabolic PDEs	50
3.2	Monotonicity, Convexity and Continuous Dependence Results on FB-SDEs and PDEs	51
3.2.1	Monotonicity of FBSDE Solutions	51
3.2.2	Continuous Dependence of FBSDE Solutions	54
3.2.3	Convexity of Semilinear Parabolic PDE Solutions	56
3.3	G -FBSDEs and Associated G -PDEs	63
3.3.1	G -FBSDEs	63
3.3.2	Associated G -PDEs (HJB-type equations)	66
3.4	Monotonicity, Convexity and Continuous Dependence Results on G -FBSDEs and G -PDEs	68
3.4.1	Monotonicity of G -FBSDE Solutions	68
3.4.2	Continuous Dependence of G -FBSDE Solutions	71
3.4.3	Convexity of Nonlinear G -PDE Solutions	77
4	Stochastic Orderings by g-Expectations	87
4.1	Introduction	88
4.2	g -Stochastic Orderings	91
4.3	Some Characterizations of g -Stochastic Orderings	95
4.4	Comparison Results for g -Stochastic Orderings	104
4.4.1	g -Stochastic Orderings with Convex Drifts	104
4.4.2	g -Stochastic Orderings with Partially Convex Drifts	110
4.5	Comparison Results for g -Risk Measures	115
4.6	Comparisons of Option Prices under Different Constraints	117
5	Stochastic Orderings under G-Expectation Framework	127
5.1	Introduction	128
5.2	G -Stochastic Orderings	131
5.3	Comparison Results for G -Stochastic Orderings	134
5.3.1	G -Stochastic Orderings with Convex Drifts	135
5.3.2	G -Stochastic Orderings with Partially Convex Drifts	141
5.4	Comparison Results for G -Risk Measures	147

5.5	Comparisons of Option Superhedging Prices under Ambiguous Coefficients	151
6	Stochastic Orderings in Linear, Sublinear and Nonlinear expectations	155
6.1	A Representation of G -Expectation	156
6.2	Monotonicity, Convexity and Concavity of G -heat PDEs	157
6.3	Comparisons of g -Expectations with G -Normal Distribution	160
6.4	Comparisons of G -Expectations with G -Normal Distribution	174
6.5	Comparisons of Nonlinear Expectations $\mathcal{E}_g$ and $\mathcal{E}_G$	178
7	Conclusion and Future Work	187
	References	189

List of Abbreviations

a.s.	Almost surely
q.s.	Quasi surely
PDEs	Partial Differential Equations
SDEs	Stochastic Differential Equations
FSDEs	Forward Stochastic Differential Equations
BSDEs	Backward Stochastic Differential Equations
FBSDEs	Forward-Backward Stochastic Differential Equations

List of Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of non-negative real numbers
$\mathbb{R}_-$	Set of non-positive real numbers
$\mathbb{E}$	Linear expectation
$\widehat{\mathbb{E}}$	Sublinear expectation
$\mathbb{E}_G$	G -expectation
$\mathcal{E}_g$	g -expectations generated by BSDEs
$\mathcal{E}_G$	nonlinear expectations generated by G -BSDEs
$(W_t)_{t \geq 0}$	Standard Brownian motion
$(B_t)_{t \geq 0}$	G -Brownian motion

Chapter 1

Introduction

Suppose that given two random variables X and Y , how can we compare them? One of the simplest ways is we make a comparison by using their expectations which are known as centers of their distributions. In case the two means are equal, then one may be interested in comparing their variances used as dispersion measures for the distributions. The use of expectation and variance is often referred to as *mean-variance analysis*. However, in many applications, mean and variance values are not enough to make reasonable decisions. Moreover, there are a number of distributions in the literature that they may not exist expectations and variances as well. For the purpose of comparing random variables or probability distributions, the theory of stochastic orders has been developed and extensively studied over the past decades. One of the most popular approaches to define stochastic orderings is utilizing expectation of functions of random variables, i.e., comparison between $\mathbb{E}[f(X)]$ and $\mathbb{E}[f(Y)]$, for a certain class of functions f . For example in utility theory, we denote by X and Y the returns of two certain portfolios, if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all non-decreasing concave utility functions f such that expectations exist, then we say that Y dominates X in the increasing concave order. In this case, the second portfolio (Y) would be preferred over the first portfolio (X) by all risk-averse investors. A similar conclusion would follow for risk-seeking investors if such an inequality holds for all non-decreasing convex utility functions f . In finance setting, the terminology of *stochastic dominance* is more frequently used rather than stochastic ordering. In this case, the increasing concave order is equivalent to the Second-order Stochastic Dominance (SSD). More generally, the theory of stochastic orders has found a wide range of applications in various fields such as reliability theory, economics, actuarial sciences, operation research, risk management, biology,

option evaluation, etc., see e.g. [68, 22, 85, 90, 56, 7, 83] for more detailed information about different types of stochastic orderings and their potential applications.

As can be seen, the stochastic orders above are defined under the linear expectation framework. In this study, we aim to extend the theory of stochastic order under the nonlinear expectations framework. In [74], Peng introduced a nonlinear expectation called g -expectation $\mathcal{E}_g$ which is generated by a solution of the Backward Stochastic Differential Equation (BSDE). The g -expectation $\mathcal{E}_g$ preserves almost properties of the classical expectations $\mathbb{E}$, except linearity. Also, the g -expectation $\mathcal{E}_g$ has been applied to contingent claim pricing in financial markets, as well as to stochastic control theory and the study of dynamic risk measures, see e.g. [72, 74, 27, 65, 75, 32, 5, 79, 81, 70], and it generalizes the classical notion of expectation which corresponds to the choice $g := 0$, that is, $\mathcal{E}_0 \equiv \mathbb{E}$. Later, with the aim of dealing with financial problems under uncertain volatility, Peng [77] also introduced a new family of sublinear expectations $\hat{\mathbb{E}}$ that can be represented as supremum of linear expectations. Since then, a new form of stochastic calculus has been developed on sublinear expectation spaces which generalize usual probability spaces. In particular, Peng [77, 80] constructed a typical sublinear expectation called G -expectation $\mathbb{E}_G$ and the corresponding G -Brownian motion with related problems such as G -Itô formula, G -Itô integrals and stochastic differential equations driven by G -Brownian motion. In addition, the general sublinear expectation $\hat{\mathbb{E}}$ and a particular G -expectation have been used to establish a new central limit theorem for G -normal distributions [78], and for the study dynamic risk measures [76, 82]. Those results have been applied to contingent claim pricing in financial markets with uncertain volatility [41, 95], as well as to stochastic control theory [97, 44] and to the robust mean-variance hedging [10]. Recently, Song [87, 88] has extended Stein's method to G -normal approximation under sublinear expectations.

Motivated by both theoretical results and potential applications of the stochastic orders, g -expectations, G -expectations, and Forward-Backward Stochastic Differential Equations (FBSDEs), we go beyond the linear setting of the standard expectation operator $\mathbb{E}$ to study stochastic orderings from the nonlinear expectations point of view, such as $\mathcal{E}_g$ generated by BSDEs and $\mathcal{E}_G$ generated by G -BSDEs, which are called g -stochastic orderings and G -stochastic orderings, respectively, see Chapter 4 and Chapter 5 for more details. More generally, inequalities in these new types of stochastic orderings can be interpreted as comparisons of portfolio values, option prices, and risk measures. The g - and G -stochastic orderings can be also used for the comparison of expected utilities and as such, they are applicable to risk manage-

ment, statistical mechanics, potential theory, decision theory, and actuarial sciences. Our results regarding g - and G -stochastic orderings can be also viewed as alternative comparison theorems for the solutions of usual FBSDEs and G -FBSDEs. In option pricing problems, they can also offer us lower and upper bounds of the robust prices under uncertain volatility. In the literature under the linear expectation framework, comparison bounds in convex ordering have been established in [26] for option prices with convex payoff functions in the continuous diffusion case, via a martingale approach and based on the classical Kolmogorov equation as well as the propagation of convexity property for Markov semigroups. This approach has been generalized to semimartingales in [38], [8], [9], see also [51], [3], [66].

The rest of the dissertation is organized as follows:

In Chapter 2, we basically review to readers some backgrounds and notations on the theory of classical stochastic order, stochastic differential equations, nonlinear g -expectations theory, risk measures, theory of sublinear expectations and related problems. Firstly, in Section 2.1, we start to recall the fundamental stochastic orders of monotonic, convex, increasing convex, concave and increasing concave orderings types which are defined via linear expectations. They are partial orders used for comparing random variables that can be applied to make reasonable decisions in certain senses. Next, in Section 2.2, we remind three main tools including Itô's formula, Girsanov's theorem and Feynman-Kac's formula which are frequently utilized in the studies of stochastic differential equations. Section 2.3 is one of the main contents of this chapter, where we present the theory of g -expectation, a type of nonlinear expectation which is defined by a solution of the backward stochastic differential equation. The g -expectations preserve almost properties of the classical expectation, except linearity. In order to have a comparison, we also give some relationships between the g -expectation and linear expectation in Subsection 2.3.3. In Section 2.4, axiomatic systems of static and dynamic risk measures are briefly recalled. Here, we also introduce coherent and convex risk measures induced by the g -expectations. Section 2.5 is very important in which we devote to focus on the sublinear expectation theory, including definitions, properties and related results, and they will be used through Chapter 3, Chapter 5 and Chapter 6. Constructed on sublinear expectation spaces, new concepts of G -expectations, G -Brownian motion, G -Itô integrals, G -Stochastic differential equations and G -Itô formula have been established and extensively studied.

Chapter 3 is devoted to proving some necessary results on monotonicity, convexity and continuous dependence property of solutions of FBSDEs, G -FBSDEs and cor-

responding solutions of associated semilinear parabolic PDEs and Hamilton-Jacobi-Bellman (HJB) equations, respectively. These properties are very crucial to constitute comparison results for the monotonic, convex, increasing convex g - and G -stochastic orderings presented in Chapter 4, Chapter 5 and Chapter 6 later. We divide Chapter 3 into two parts. The first part consists of Section 3.1-3.2 in which we study the FBSDEs driven by a standard Brownian motion and corresponding semilinear parabolic PDEs. Firstly, in Section 3.1.1, we provide an explicit solution for the FBSDE with a linear generator, which is generalized from the linear BSDE. This result is needed for proving monotonicity of solutions of FBSDEs as shown in Section 3.2.1. Next, in Section 3.1.2, we state some theorems under smoothness and boundedness assumptions on coefficients to ensure that semilinear parabolic PDEs admit classically smooth solutions. Later on, those theorems together with the continuous dependence property given in Section 3.2.2, are applied to obtain convexity of solutions of the mentioned PDEs as presented in Section 3.2.3. In the second part of Chapter 3, we move on to study FBSDEs driven by G -Brownian motion (G -FBSDEs) and corresponding G -PDEs (HJB-type equations), which are introduced in Section 3.3. As the same first part, we establish very important results on monotonicity, convexity and continuous dependence property of solutions of the G -FBSDEs and associated HJB equations presented in the last Section 3.4.

In Chapter 4, we define nonlinear g -evaluations and g -expectations generated by solutions of the backward stochastic differential equations. Then in Section 4.2, by using these nonlinear expectations, we introduce new concepts of monotonic, convex and increasing convex g -stochastic orderings, which leads to more general concepts of orderings in comparison with the standard linear expectation setting. In order to have a better understanding of the g -stochastic orderings, in Section 4.3, we first give some characterizations for the cases of sublinear and convex generators. Section 4.4 provides the main results of this chapter for the g -stochastic orderings with two approaches. The first approach presents in Section 4.4.1, which is relied on comparison results of solutions of the semilinear parabolic PDEs, we derive several sufficient conditions for the convex, increasing convex and monotonic g -stochastic orderings of diffusion processes under the nonlinear g -expectations and g -evaluations. This first approach needs quite strong assumptions since it requires convexity of the function

$$(x, y, z) \mapsto f_i(t, x, y, z) := z\mu_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)),$$

for both $i = 1$ and $i = 2$. In the second approach, given in Section 4.4.2, we use stochastic arguments approach to get those g -stochastic orderings results that only

requires convexity of the function $f_i(t, x, y, z)$ for $i = 1$ or $i = 2$. As consequences, in Section 4.5, we derive results for comparing g -risk measures which are induced by the g -expectations. They can be also viewed as concave g -stochastic orderings. We note that our approaches also rely on comparison results for forward-backward stochastic differential equations and on several extensions of convexity, monotonicity and continuous dependence property of the solutions of associated semilinear parabolic partial differential equations as presented in Section 3.2 of Chapter 3. In the last Section 4.6, applications of g -stochastic orderings to contingent claim price comparison under different hedging portfolio constraints are provided.

Chapter 5 is developed by motivations from the results of the g -stochastic orderings in Chapter 4. Here, we study stochastic orderings by a nonlinear expectation denoted by $\mathcal{E}_G$, which is generated by a solution of the G -Backward Stochastic Differential Equation (G -BSDE). Firstly, in Section 5.2, we define the so-called G -stochastic orderings by using the nonlinear expectations $\mathcal{E}_G$, including monotonic, convex and increasing convex orderings. We then derive several sufficient conditions with two different approaches for the convex, increasing convex and monotonic G -stochastic orderings of G -diffusion stochastic processes driven by G -Brownian motion. In the first approach, given in Section 5.3.1, we simply apply comparison results of solutions of the associated HJB-type equations, also called G -PDEs. This first approach requires convexity of the functions

$$(x, y, z) \mapsto M_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)),$$

and

$$(x, y, z) \mapsto N_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x)),$$

for both $i = 1$ and $i = 2$. Next, in the second approach presented in Section 5.3.2, we use stochastic analysis arguments under the G -expectation framework to obtain comparison results for the G -stochastic orderings that only needs convexity of the functions $M_i(t, x, y, z)$ and $N_i(t, x, y, z)$, for $i = 1$ or $i = 2$. As consequences, in Section 5.4 we derive results for comparing G -risk measures which are induced by the $\mathcal{E}_G$ -expectations. Here, we also note that our approaches rely on comparison results for the G -FBSDEs and on several extensions of convexity, monotonicity and continuous dependence property for the solutions of associated HJB equations, which have been shown in Section 3.4 of Chapter 3. In the last Section 5.5, we study the effects of ambiguous drift and volatility coefficients of G -FSDEs to associated contingent claim superhedging prices and portfolio values by applying G -stochastic ordering results.

In Chapter 6, we derive comparison results in the sense of convex and increasing convex orders between the linear expectation $\mathbb{E}$, sublinear expectation $\widehat{\mathbb{E}}$, g -expectation $\mathcal{E}_g$, G -expectation $\mathbb{E}_G$ and nonlinear expectation $\mathcal{E}_G$, including both kinds of lower means $-\mathcal{E}[-\varphi(\cdot)]$ and upper means $\mathcal{E}[\varphi(\cdot)]$, where $\mathcal{E}$ represents for one of the expectations $\mathbb{E}$, $\mathbb{E}_G$, $\widehat{\mathbb{E}}$, $\mathcal{E}_g$ and $\mathcal{E}_G$ above. In Section 6.1, we first start by recalling a concrete representation for the G -expectations $\mathbb{E}_G$, which shows a connection to linear expectations. Next, in Section 6.2, we show the monotonicity, convexity and concavity of solutions of G -heat PDEs, which are important properties to constitute comparison results in later sections. Then in Section 6.3, we derive sufficient conditions for several comparisons between diffusion stochastic processes $(X_t)_{t \in [0, T]}$ at terminal time and G -normal distributed random variables $\widehat{X}$, i.e., between $\mathcal{E}_g[\varphi(X_T)]$ and $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$, a consequence between $\mathbb{E}[\varphi(X_T)]$ and $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$, as well as between $-\mathcal{E}_g[-\varphi(X_T)]$ and $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})]$, and between $\mathbb{E}[\varphi(X_T)]$ and $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})]$. Such inequalities can offer us lower and upper bounds information and the concentration of measure inequalities for the g -expectations of terminal values of the diffusion SDEs. Moving to Section 6.4, we show analogous results for comparing between terminal values of G -diffusion stochastic processes $(X_t^G)_{t \in [0, T]}$ and G -normal distributed random variables, i.e., between $\mathbb{E}_G[\varphi(X_T^G)]$ and $\mathbb{E}_G[\varphi(\widehat{X})]$, and between $-\mathbb{E}_G[-\varphi(X_T^G)]$ and $-\mathbb{E}_G[-\varphi(\widehat{X})]$. In the last Section 6.5, we devote to present some comparisons between the nonlinear expectations $\mathcal{E}_g$ and $\mathcal{E}_G$ that enable us to have some orderings of $(X_t)_{t \in [0, T]}$ and $(X_t^G)_{t \in [0, T]}$, specifically between $\mathcal{E}_g[\varphi(X_T)]$ and $\mathcal{E}_G[\varphi(X_T^G)]$. This chapter also reuses comparison results for forward, for forward-backward, and for G -forward-backward stochastic differential equations, together with the monotonicity, convexity and continuous dependence property of solutions of the associated semilinear parabolic PDEs and HJB equations (G -PDEs).

Chapter 2

Preliminaries

In this chapter, we basically review to the readers some backgrounds and notations on the theory of classical stochastic orders, stochastic differential equations, nonlinear g -expectations theory, risk measures, theory of sublinear expectations and related problems. Firstly, in Section 2.1, we start to recall the fundamental stochastic orders of monotonic, convex, increasing convex, concave and increasing concave orderings types which are defined via linear expectations. They are partial orders used for comparing random variables that can be applied to make reasonable decisions in certain senses. Next, in Section 2.2, we remind three main tools including Itô's formula, Girsanov's theorem and Feynman-Kac's formula which are frequently utilized in the studies of stochastic differential equations. Section 2.3 is one of the main contents of this chapter, where we present the theory of g -expectation, a type of nonlinear expectation which is defined by a solution of the backward stochastic differential equation. The g -expectations preserve almost properties of the classical expectation, except linearity. In order to have a comparison, we also give some relationships between the g -expectation and linear expectation in Subsection 2.3.3. In Section 2.4, the axiomatic systems of static and dynamic risk measures are briefly recalled. Here, we also introduce coherent and convex risk measures induced by the g -expectations. Section 2.5 is very important in which we devote to focus on the sublinear expectation theory, including definitions, properties and related results, and they will be used through Chapter 3, Chapter 5 and Chapter 6. Constructed on sublinear expectation spaces, new concepts of G -expectations, G -Brownian motion, G -Itô integrals, G -Stochastic differential equations and G -Itô formula have been established and extensively studied.

2.1 Stochastic Orderings

Stochastic orderings are concepts of partial orders for comparing random variables. In the decision theory, stochastic ordering is formally referred as *stochastic dominance*. Normally, they are defined in terms of expectations of a certain function class of random variables provided that the expectations exist, which will also be assumed throughout this dissertation. Below we briefly review some of their definitions and results, and we refer to books, e.g. [68, 85] for other types of stochastic orders and their applications.

Definition 2.1.1. *For given two random variables X and Y , we define stochastic orders*

- (i) $X \leq_{st} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, for all non-decreasing functions ϕ .
- (ii) $X \leq_{cx} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, for all convex functions ϕ .
- (iii) $X \leq_{icx} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, for all non-decreasing convex functions ϕ .
- (iv) $X \leq_{cv} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, for all concave functions ϕ .
- (v) $X \leq_{icv} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, for all non-decreasing concave functions ϕ .

The ordering $\leq_{st}$ is usually called the *usual stochastic order* and in case the function ϕ represents for a class of non-decreasing utility functions, it is often referred as the *First-order Stochastic Dominance (FSD)*. For example, given two portfolios with returns denoted by X and Y , respectively, then any rational decision-maker with non-decreasing utility functions would prefer the return Y over the return X if $X \leq_{st} Y$. The stochastic ordering $\leq_{st}$ can be characterized by cumulative distributions, survival functions and quantile functions, see e.g. [68, 85]. Specifically, we have the following results.

Theorem 2.1.2. *For two random variables X and Y , the following statements are equivalent:*

- (1) $X \leq_{st} Y$.
- (2) $F_X(t) := \mathbb{P}(X \leq t) \geq \mathbb{P}(Y \leq t) =: F_Y(t)$, for all $t \in \mathbb{R}$.
- (3) $\bar{F}_X(t) := \mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t) =: \bar{F}_Y(t)$, for all $t \in \mathbb{R}$.

(4) $F_X^{-1}(p) \leq F_Y^{-1}(p)$, for all $p \in (0, 1)$, where the quantile function F^{-1} is defined by

$$F^{-1}(p) := \inf\{t \in \mathbb{R} : F(t) \geq p\}, \quad p \in (0, 1).$$

(5) There exist two random variables X' and Y' , defined on the same probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, such that $X' \sim F_X$ and $Y' \sim F_Y$ and $X'(\omega') \leq Y'(\omega')$, for all $\omega' \in \Omega'$.

The *increasing concave order* $\leq_{icv}$ is known as the *Second-order Stochastic Dominance (SSD)*. In the financial setting, the non-decreasing concave (resp. convex) utility functions are used to represent the attitude of *risk-averse* (resp. *risk-seeking*) investors. This means that for the two risky returns X and Y , if $X \leq_{icv} Y$, then any risk-averse decision-maker would prefer Y over X .

The *increasing convex order* $\leq_{icx}$ is also known as *stop-loss order* used in actuarial sciences since this ordering is applied to rank losses (risks) instead of returns. Here, one shown that $X \leq_{icx} Y$, if and only if the stop-loss transformation is ordered:

$$\mathbb{E}[\max\{X - t, 0\}] \leq \mathbb{E}[\max\{Y - t, 0\}], \quad t \in \mathbb{R}.$$

Hence, given the two risks X and Y , any risk-averse decision-maker would prefer the risk X over the risk Y if $X \leq_{icx} Y$. We also see that $X \leq_{icv} Y$ if and only if $-Y \leq_{icx} -X$, so it is sufficient to consider the ordering $\leq_{icx}$. Besides, the *increasing convex order* $\leq_{icx}$ can be described by the orderings $\leq_{st}$ and $\leq_{cx}$ as in the following theorem.

Theorem 2.1.3. *Let two random variables X and Y define on the same probability space $(\Omega, \mathcal{F}, P)$ with $X \leq_{icx} Y$. Then there exists a random variable Z defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$X \leq_{st} Z \leq_{cx} Y.$$

It is similar to the Theorem 2.1.2, we have characterizations for the convex ordering as follows:

Theorem 2.1.4. *For two random variables X and Y , the following statements are equivalent:*

- (1) $X \leq_{cx} Y$.
- (2) There exist two random variables X' and Y' , defined on the same probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, such that $X' \sim F_X$, $Y' \sim F_Y$ and $\mathbb{E}[Y'|X'] = X'$.

(3) $\mathbb{E}[X] = \mathbb{E}[Y]$ and for all $\alpha \in (0, 1)$, we have

$$\int_{\alpha}^1 F_X^{-1}(p) dp \leq \int_{\alpha}^1 F_Y^{-1}(p) dp.$$

2.2 Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(W_t)_{t \geq 0}$ be a standard one dimensional Brownian motion on this space. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the augmented filtration such that $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\} \cup \mathcal{N}$, for any $t \geq 0$, where $\mathcal{N}$ is the collection of all $\mathbb{P}$ -null sets.

Consider a Stochastic Differential Equation (SDE) of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad 0 \leq t \leq T. \quad (2.1)$$

where drift coefficient $\mu(t, x)$ and volatility coefficient $\sigma(t, x)$ are assumed throughout this thesis to be satisfied the following condition:

(A.2.1) For every $t \in [0, T]$, the functions $x \mapsto \mu(t, x)$ and $x \mapsto \sigma(t, x)$ are globally Lipschitz, i.e.,

$$|\mu(t, x) - \mu(t, y)| \leq C|x - y| \quad \text{and} \quad |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|, \quad x, y \in \mathbb{R},$$

In particular, $x \mapsto \mu(t, x)$ and $x \mapsto \sigma(t, x)$ satisfy the linear growth conditions:

$$|\mu(t, x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma(t, x)| \leq C(1 + |x|), \quad x \in \mathbb{R},$$

for some positive constant $C > 0$.

Under the condition (A.2.1), the equation (2.1) admits a unique *adapted solution* $(X_t)_{t \in [0, T]}$, i.e., for each $t \in [0, T]$, X_t is a $\mathcal{F}_t$ -measurable random variable.

Itô Formula

In order to solve a SDE as well as to derive dynamic equations of transformations of a stochastic process, we normally apply the Itô formula as below.

Lemma 2.2.1 (The Itô Formula). *Let $(X_t)_{t \geq 0}$ be a stochastic process satisfying the equation (2.1). Let $u(t, x)$ be a continuously differentiable function with respect*

to time t and twice differentiable with respect to space x . Then for any $t \geq 0$, we have

$$\begin{aligned} du(t, X_t) = & \left(\frac{\partial u}{\partial t}(t, X_t) + \mu(t, X_t) \frac{\partial u}{\partial x}(t, X_t) + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2 u}{\partial x^2}(t, X_t) \right) dt \\ & + \sigma(t, X_t) \frac{\partial u}{\partial x}(t, X_t) dW_t. \end{aligned}$$

Girsanov's Theorem

In the theory of stochastic process, **Girsanov's theorem** is one of the key results which enables us to describe how the dynamic equation of a stochastic process is changed when we change the original measure to an equivalent probability measure. Below we state a version of Girsanov's theorem, which is frequently used in the thesis.

Theorem 2.2.2 (Girsanov's Theorem). *Let $(W_t)_{t \in [0, T]}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\theta_t)_{t \in [0, T]}$ be an adapted process to natural filtration of $(W_t)_{t \in [0, T]}$ and satisfies the following condition:*

$$\mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{1}{2} \int_0^T \theta_s^2 ds \right) \right] < \infty. \quad (2.2)$$

Defining the following stochastic processes:

$$M_t := \exp \left(- \int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right),$$

and

$$\widetilde{W}_t := W_t + \int_0^t \theta_s ds.$$

Then $(\widetilde{W}_t)_{t \in [0, T]}$ is a Brownian motion under the new measure $\mathbb{Q}$ defined by:

$$Q(A) := \int_A M_T(\omega) d\mathbb{P}(\omega), \quad \text{for all } A \in \mathcal{F}.$$

Note that (2.2) is known as the Novikov condition and we then have $\mathbb{E}_{\mathbb{P}}[M_t] = 1$, for all $t \geq 0$. Moreover, the process $(M_t)_{t \in [0, T]}$ is a positive martingale under that condition, i.e., $\mathbb{E}_{\mathbb{P}}[M_t | \mathcal{F}_s] = M_s$, for $0 \leq s \leq t \leq T$. In addition, the measure $\mathbb{Q}$ is called a *risk-neutral measure* which is usually applied to fairly price contingent claims in finance.

Feynman-Kac Formula

Stochastic Differential Equations (SDEs) can be linked to parabolic Partial Differential Equations (PDEs) via the so-called Feynman-Kac formula. In this framework, solutions of PDEs are solved by simulating random paths of a stochastic process and then taking expectation. Conversely, solutions of some SDEs can also be derived by the help of PDEs solutions. The latter fact will be clearer in Chapter 3 when we consider a connection between forward and backward SDEs.

Theorem 2.2.3 (Feynman-Kac Formula). *Let $(X_t)_{t \geq 0}$ be a diffusion process satisfying the SDEs*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t^{\mathbb{Q}}, \quad 0 \leq t \leq T, \quad (2.3)$$

where $(W_t^{\mathbb{Q}})_{t \in [0, T]}$ denotes a Brownian motion under the measure $\mathbb{Q}$. Let $u(t, x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ satisfy the following PDEs

$$\frac{\partial u}{\partial t}(t, x) + \mu(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 u}{\partial x^2}(t, x) - \alpha(t, x)u(t, x) + \beta(t, x) = 0, \quad (2.4)$$

for all $(t, x) \in [0, T) \times \mathbb{R}$ with final condition $u(T, x) = \phi(x)$, where $\mu(t, x)$ and $\sigma(t, x)$ satisfy (A.2.1), and $\alpha : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ and $\beta : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ are continuous functions. Then

$$u(t, x) = \mathbb{E}_{\mathbb{Q}} \left[\phi(X_T) e^{-\int_t^T \alpha(s, X_s) ds} + \int_t^T \beta(r, X_r) e^{-\int_t^r \alpha(s, X_s) ds} dr \mid X_t = x \right],$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

In the next sequel, we provide a simple example for demonstrating the use of the Itô formula, Girsanov's theorem and the Feynman-Kac formula and the most important thing is to lead to definitions of Backward Stochastic Differential Equations (BSDEs) and corresponding the Peng's g -expectations which will be given in the next section.

Example 2.2.4. Black-Scholes Model

Consider the problem of pricing a European call option with underlying risky stock obeying the following SDEs:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu \in \mathbb{R}, \sigma > 0$. Also, we suppose that there exists a risk-free asset, whose price is $E_t = E_0 \exp(rt)$, where r is a risk-free rate. Its dynamic equation is defined by

$$dE_t = rE_t dt.$$

Let p_t and q_t be portions of wealth invested in the risky stock S_t and in the risk-free asset E_t at the time t , respectively. Let $(V_t)_{t \in [0, T]}$ be the wealth process of the portfolio. If we consider only the self-financing strategy, then the value of the portfolio will be given by

$$dV_t = q_t dE_t + p_t dS_t = q_t r E_t dt + p_t \mu S_t dt + p_t \sigma S_t dW_t.$$

As we have $q_t E_t = V_t - p_t S_t$, we get that

$$dV_t = \left(rV_t + \frac{\mu - r}{\sigma} \sigma p_t S_t \right) dt + p_t \sigma S_t dW_t.$$

In a complete market, we know that a contingent claim with payoff $\phi(S_T)$ at the terminal time T , can be fairly priced by the value of V_t with $V_T = \phi(S_T)$. Setting

$$\theta = \frac{\mu - r}{\sigma}, \quad Z_t = \sigma p_t S_t,$$

then we have that

$$V_t = \phi(S_T) - \int_t^T (rV_s + \theta Z_s) ds - \int_t^T Z_s dW_s. \quad (2.5)$$

Discounting $\tilde{V}_t = e^{-rt} V_t$, $\tilde{Z}_t = e^{-rt} Z_t$. Then, by applying the Itô formula to $f(t, V_t) = e^{-rt} V_t$, we obtain that

$$\tilde{V}_t = e^{-rT} \phi(S_T) - \int_t^T \theta \tilde{Z}_s ds - \int_t^T \tilde{Z}_s dW_s.$$

Defining

$$\widetilde{W}_t := W_t + \int_0^t \theta ds = W_t + \theta t \text{ and } \frac{d\mathbb{Q}}{d\mathbb{P}} := \exp \left(- \int_0^t \theta dW_s - \frac{1}{2} \int_0^t \theta^2 ds \right).$$

Then, by the Girsanov theorem, we have $(\widetilde{W}_t)_{t \in [0, T]}$ is a Brownian motion under the measure $\mathbb{Q}$, and hence

$$\tilde{V}_t = e^{-rT} \phi(S_T) - \int_t^T \tilde{Z}_s d\widetilde{W}_s.$$

Therefore, we get $\tilde{V}_t = e^{-rT} \mathbb{E}_{\mathbb{Q}}[\phi(S_T) | \mathcal{F}_t]$, i.e., $V_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[\phi(S_T) | \mathcal{F}_t]$. In particular, at the time $t = 0$, the fair option price is defined by $V_0 = e^{-rT} \mathbb{E}_{\mathbb{Q}}[\phi(S_T)]$. Note that by using the Brownian motion $(\tilde{W}_t)_{t \in [0, T]}$ under the probability measure $\mathbb{Q}$, we can write SDEs of the risky stock S_t above as:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.$$

Thus, if we define

$$u(t, x) := \mathbb{E}_{\mathbb{Q}} \left[\phi(X_T) e^{-r(T-t)} \mid X_t = x \right],$$

then, by the Feynman-Kac formula, $u(t, x)$ satisfies the equation (2.4) with $\mu(t, x) = \alpha(t, x) = r$ and $\beta(t, x) = 0$, for all $(t, x) \in [0, T] \times \mathbb{R}$.

Remark 2.2.5. *As can be seen, the equation (2.5) is a linear BSDE with the generator function $g(t, y, z) := -(ry + \theta z)$ and its solution $(V_t)_{t \in [0, T]}$ can be written in term of the conditional linear expectation. From this point of view, Peng [74] generalized the linear BSDEs to nonlinear BSDEs and hence to define conditional nonlinear expectations called g -expectations and nonlinear Feynman-Kac formulas.*

2.3 g -Expectations

2.3.1 Backward Stochastic Differential Equations

Let $L^2(\Omega, \mathcal{F}_t) = L^2(\Omega, \mathcal{F}_t, \mathbb{P})$, with $t \in [0, T]$, denote the space of all real-valued, $\mathcal{F}_t$ -measurable and square integrable random variables applying with the L^2 -norm. Let $L^2(\Omega \times [0, T]) = L^2(\Omega \times [0, T], \mathbb{R})$ denote the space of all real-valued, $\mathcal{F}_T$ -adapted processes $(V_t)_{t \in [0, T]}$ with $\mathbb{E} \left[\int_0^T V_t^2 dt \right] < \infty$.

The backward stochastic differential equation is defined as

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad 0 \leq t \leq T, \quad (2.6)$$

with terminal condition $Y_T = \xi \in L^2(\Omega, \mathcal{F}_T)$, or it may write in term of the integration

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (2.7)$$

where g is called the generator of the BSDE satisfying the following condition:

(A.2.2) g is uniformly Lipschitz continuous in (y, z) , i.e. there exists a constant $C > 0$ such that for all $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \text{ a.s.} \quad (2.8)$$

The existence and uniqueness solution of the BSDE is ensured by a theorem below, see Proposition 2.2 in [72] and also Theorem 2.1 in [27] for the proof.

Theorem 2.3.1 (Existence and Uniqueness). *Let g satisfy the assumption (A.2.2) and $\int_0^T g(\cdot, 0, 0)ds \in L^2(\Omega \times [0, T])$. Then there exists a unique pair of processes $(Y_t, Z_t)_{t \in [0, T]} \in L^2(\Omega \times [0, T]) \times L^2(\Omega \times [0, T])$ that solves the BSDE (2.6) with terminal condition $Y_T = \xi$.*

For a comparison theorem of solutions of BSDEs, we refer to Theorem 2.2 in [27].

Theorem 2.3.2 (Comparison Theorem). *Let $g_i, i = 1, 2$ satisfy the assumption (A.2.2) and $\int_0^T g_i(\cdot, 0, 0)ds \in L^2(\Omega \times [0, T])$. Let $(Y_t^{(i)}, Z_t^{(i)})_{t \in [0, T]}$ for $i = 1, 2$ be solutions of the following BSDEs*

$$-dY_t^{(i)} = g_i(t, Y_t^{(i)}, Z_t^{(i)})dt - Z_t^{(i)}dW_t, \quad 0 \leq t \leq T, \quad (2.9)$$

with terminal condition $Y_T^{(i)} = \xi_i$. If we suppose that

$$(i) \quad \xi_1 \leq \xi_2, \text{ a.s.}$$

$$(ii) \quad g_1(t, Y_t^{(2)}, Z_t^{(2)}) \leq g_2(t, Y_t^{(2)}, Z_t^{(2)}), \text{ a.s.}$$

then we have $Y_t^{(1)} \leq Y_t^{(2)}$ almost surely for any time $t \in [0, T]$. Moreover, if $Y_0^{(1)} = Y_0^{(2)}$ then $\xi_1 = \xi_2$ and $g_1(t, Y_t^{(2)}, Z_t^{(2)}) = g_2(t, Y_t^{(2)}, Z_t^{(2)}), \text{ a.s.}$

2.3.2 g -Evaluation and g -Expectation

First of all, we introduce g -evaluation and outline some properties. However, we need an additional assumption on g as follows:

(A.2.3) $g(\cdot, 0, 0) = 0, \text{ a.s.}$

Definition 2.3.3 (g -Evaluation). *Let g satisfy the Assumption (A.2.2)-(A.2.3). For any $\xi \in L^2(\Omega, \mathcal{F}_T)$, let $(Y_t, Z_t)_{t \in [0, T]} \in L^2(\Omega \times [0, T]) \times L^2(\Omega \times [0, T])$ be the solution of the BSDE (2.6) with terminal condition ξ . The g -evaluation of ξ at the time t , denoted by $\mathcal{E}_g^{t, T}[\xi]$, is defined by*

$$\mathcal{E}_g^{t, T}[\xi] := Y_t. \quad (2.10)$$

In [75], Peng proved that $\mathcal{E}_g^{t,T}$ defined by (2.10) satisfies the following axiomatic assumptions of a $\mathcal{F}_t$ -consistent nonlinear evaluation:

Proposition 2.3.4 ($\mathcal{F}_t$ -consistent nonlinear evaluation). *Let g satisfy the Assumption (A.2.2)-(A.2.3). The functional $\mathcal{E}_g^{t,T}$ defined by (2.10) satisfies the following properties:*

- (i) **Monotonicity:** $\mathcal{E}_g^{t,T}[\xi_1] \leq \mathcal{E}_g^{t,T}[\xi_2]$, a.s. if $\xi_1 \leq \xi_2$, a.s.
- (ii) **Constant-preserving:** $\mathcal{E}_g^{t,T}[\xi] = \xi$, if $\xi \in L^2(\Omega, \mathcal{F}_t)$. In particular, $\mathcal{E}_g^{0,0}[c] = c$, where $c \in \mathbb{R}$ is a constant.
- (iii) **Zero-one law:** For each $t \leq T$, $\mathcal{E}_g^{t,T}[\mathbb{1}_A \xi] = \mathbb{1}_A \mathcal{E}_g^{t,T}[\xi]$, for all $A \in \mathcal{F}_t$.
- (iv) **Time consistent:** $\mathcal{E}_g^{s,t}[\mathcal{E}_g^{t,T}[\xi]] = \mathcal{E}_g^{s,T}[\xi]$, if $s \leq t \leq T$.

In general, we have that $\mathcal{E}_g^{t,T}[c] \neq c$ for a constant $c \in \mathbb{R}$, if $t \neq T$. However, under a stronger assumption than (A.2.3), say,

(A.2.3') $g(\cdot, y, 0) = 0$, a.s. for each $y \in \mathbb{R}$.

Then, it is surely that $\mathcal{E}_g^{t,T}[c] = c$ for all constants $c \in \mathbb{R}$ and $t \leq T$, see [74]. In this case, the g -evaluation $\mathcal{E}_g^{t,T}[\cdot]$ becomes a **conditional g -expectation**. From now on, we will simply write it as $\mathcal{E}_g[\cdot | \mathcal{F}_t]$.

Also, we note that the g -expectation $\mathcal{E}_g[\xi | \mathcal{F}_0] = \mathcal{E}_g[\xi]$ is definitely deterministic. Even though, the g -expectation is not linear in general, it almost preserves basic properties of the classical expectation.

Proposition 2.3.5 (Properties of the conditional g -expectations). *Let g satisfy the Assumption (A.2.2)-(A.2.3'). The g -expectation $\mathcal{E}_g$ defined as in (2.10) satisfies the following properties*

- (i) Let ξ be an $\mathcal{F}_t$ -measurable random variable. Then $\mathcal{E}_g[\xi | \mathcal{F}_t] = \xi$. In particular, we have $\mathcal{E}_g[0 | \mathcal{F}_t] = 0$ and $\mathcal{E}_g[1 | \mathcal{F}_t] = 1$.
- (ii) For all $\xi \in L^2(\Omega, \mathcal{F}_T)$ and $s, t \in [0, T]$, we have $\mathcal{E}_g[\mathcal{E}_g[\xi | \mathcal{F}_s] | \mathcal{F}_t] = \mathcal{E}_g[\xi | \mathcal{F}_{s \wedge t}]$, where $s \wedge t$ denotes the minimum of s and t . In particular, we have

$$\mathcal{E}_g[\mathcal{E}_g[\xi | \mathcal{F}_t]] = \mathcal{E}_g[\xi].$$

- (iii) If $\xi_1 \geq \xi_2$, then $\mathcal{E}_g[\xi_1 | \mathcal{F}_t] \geq \mathcal{E}_g[\xi_2 | \mathcal{F}_t]$. In this case $\mathcal{E}_g[\xi_1] = \mathcal{E}_g[\xi_2]$ if and only if $\xi_1 = \xi_2$, a.s.

(iv) For all $A \in \mathcal{F}_t$, we have $\mathcal{E}_g[\mathbb{1}_A \xi | \mathcal{F}_t] = \mathbb{1}_A \mathcal{E}_g[\xi | \mathcal{F}_t]$.

(v) If ξ is independent of $\mathcal{F}_t$, then $\mathcal{E}_g[\xi | \mathcal{F}_t] = \mathcal{E}_g[\xi]$.

(vi) Let $g(t, y, z)$ be independent of $y \in \mathbb{R}$ and sub-additive in $z \in \mathbb{R}$, i.e.,

$$g(t, z_1 + z_2) \leq g(t, z_1) + g(t, z_2), \quad z_1, z_2 \in \mathbb{R}.$$

Then for all $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T)$, $t \in [0; T]$, we have

$$\mathcal{E}_g[\xi_1 + \xi_2 | \mathcal{F}_t] \leq \mathcal{E}_g[\xi_1 | \mathcal{F}_t] + \mathcal{E}_g[\xi_2 | \mathcal{F}_t].$$

(vii) Similarly, let $g(t, y, z)$ be independent of $y \in \mathbb{R}$ and super-additive in $z \in \mathbb{R}$, i.e.,

$$g(t, z_1 + z_2) \geq g(t, z_1) + g(t, z_2), \quad z_1, z_2 \in \mathbb{R}.$$

Then for all $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T)$, $t \in [0; T]$, we have

$$\mathcal{E}_g[\xi_1 + \xi_2 | \mathcal{F}_t] \geq \mathcal{E}_g[\xi_1 | \mathcal{F}_t] + \mathcal{E}_g[\xi_2 | \mathcal{F}_t].$$

For proofs of the Proposition 2.3.5 above, we refer to [74, 48, 70].

2.3.3 Relations between g -Expectations and Classical Expectations

First, the following proposition shows a relationship between g -expectation and the usual expectation.

Proposition 2.3.6. *Let $(Y_t, Z_t)_{t \in [0; T]} \in L^2(\Omega \times [0, T]) \times L^2(\Omega \times [0, T])$ be the solution of the BSDE (2.6) with terminal condition $Y_T = \xi$. We have that*

$$\mathcal{E}_g[\xi | \mathcal{F}_t] = \mathbb{E}[\xi | \mathcal{F}_t] + \mathbb{E}\left[\int_t^T g(s, Y_s, Z_s) ds \mid \mathcal{F}_t\right]. \quad (2.11)$$

Proof. Since $Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s$, and by definition of the conditional g -expectation with noting that Y_t is $\mathcal{F}_t$ -measurable, we get

$$\mathcal{E}_g[\xi | \mathcal{F}_t] = Y_t = \mathbb{E}[Y_t | \mathcal{F}_t] = \mathbb{E}[\xi | \mathcal{F}_t] + \mathbb{E}\left[\int_t^T g(s, Y_s, Z_s) ds \mid \mathcal{F}_t\right].$$

□

As a consequence, we have the following properties for the g -expectations.

Corollary 2.3.7. *Let $(Y_t, Z_t)_{t \in [0;T]} \in L^2(\Omega \times [0, T]) \times L^2(\Omega \times [0, T])$ be the solution of the BSDE (2.6) with terminal condition $Y_T = \xi$.*

(i) *If $g(\cdot, x, y) = 0$, a.s. for all $x, y \in \mathbb{R}$, then $\mathcal{E}_0[\xi|\mathcal{F}_t] = \mathbb{E}[\xi|\mathcal{F}_t]$, a.s. Furthermore, we have that*

$$\mathbb{E}[\xi|\mathcal{F}_t] = \xi - \int_t^T Z_s dW_s, \quad \text{Var}[\xi|\mathcal{F}_t] = \mathbb{E} \left[\int_t^T Z_s^2 ds \mid \mathcal{F}_t \right].$$

(ii) *If $g(\cdot, x, y) \leq 0$, a.s. for all $x, y \in \mathbb{R}$, then $\mathcal{E}_g[\xi|\mathcal{F}_t] \leq \mathbb{E}[\xi|\mathcal{F}_t]$, a.s.*

(iii) *If $g(\cdot, x, y) \geq 0$, a.s. for all $x, y \in \mathbb{R}$, then $\mathcal{E}_g[\xi|\mathcal{F}_t] \geq \mathbb{E}[\xi|\mathcal{F}_t]$, a.s.*

Proof. (i) It is obvious from (2.11), we have $\mathcal{E}_0[\xi|\mathcal{F}_t] = \mathbb{E}[\xi|\mathcal{F}_t]$, a.s. in the case of $g(\cdot, x, y) = 0$, a.s. for all $x, y \in \mathbb{R}$. Since $\mathcal{E}_0[\xi|\mathcal{F}_t] = Y_t = \xi - \int_t^T Z_s dW_s$, one has that

$$\mathbb{E}[\xi|\mathcal{F}_t] = \xi - \int_t^T Z_s dW_s.$$

By the definition of variance and applying the equality above, one obtains that

$$\text{Var}[\xi|\mathcal{F}_t] = \mathbb{E} \left[\left(\xi - \mathbb{E}[\xi|\mathcal{F}_t] \right)^2 \right] = \mathbb{E} \left[\left(\int_t^T Z_s dW_s \right)^2 \right] = \mathbb{E} \left[\int_t^T Z_s^2 ds \right].$$

Part (ii) and (iii) are also straightforward from the relation (2.11). $\square$

Next, we are interested in calculating the g -expectation of the second and the third term of BDSE (2.7).

Proposition 2.3.8. *Let $(Y_t, Z_t)_{t \in [0;T]}$, $(Y_t^*, Z_t^*)_{t \in [0;T]}$, and $(Y_t^{**}, Z_t^{**})_{t \in [0;T]} \in L^2(\Omega \times [0, T]) \times L^2(\Omega \times [0, T])$ be the solutions of the BSDE (2.6) with terminal condition $Y_T = \xi$, $Y_T^* = \int_0^T Z_s dW_s$, and $Y_T^{**} = \int_0^T g(s, Y_s, Z_s) ds$, respectively. Then we have*

$$(i) \quad \mathcal{E}_g \left[\int_0^T Z_s dW_s \right] = \mathbb{E} \left[\int_0^T g(s, Y_s^*, Z_s^*) ds \right].$$

$$\text{If } g(\cdot, x, y) = 0, \text{ a.s. for all } x, y \in \mathbb{R}, \text{ then } \mathcal{E}_g \left[\int_0^T Z_s dW_s \right] = 0.$$

$$(ii) \quad \mathcal{E}_g \left[\int_0^T g(s, Y_s, Z_s) ds \right] = \mathbb{E} \left[\int_0^T g(s, Y_s, Z_s) ds \right] + \mathbb{E} \left[\int_0^T g(s, Y_s^{**}, Z_s^{**}) ds \right].$$

Proof. (i) By the assumptions, we have

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s,$$

and

$$Y_t^* = \int_0^T Z_s dW_s + \int_t^T g(s, Y_s^*, Z_s^*) ds - \int_t^T Z_s^* dW_s.$$

Hence,

$$\mathcal{E}_g \left[\int_0^T Z_s dW_s \right] = Y_0^* = \int_0^T Z_s dW_s + \int_0^T g(s, Y_s^*, Z_s^*) ds - \int_0^T Z_s^* dW_s.$$

By taking expectation on both sides of the equation above and using the fact that the expectation of Itô's integrals are equal to zero, we get

$$\mathbb{E} \left[\mathcal{E}_g \left[\int_0^T Z_s dW_s \right] \right] = \mathcal{E}_g \left[\int_0^T Z_s dW_s \right] = \mathbb{E} \left[\int_0^T g(s, Y_s^*, Z_s^*) ds \right].$$

(ii) Similarly, since $(Y_t^{**}, Z_t^{**})_{t \in [0, T]}$ is the solution of the BDSE

$$Y_t^{**} = \int_0^T g(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s^{**}, Z_s^{**}) ds - \int_t^T Z_s^{**} dW_s.$$

Then, we also obtain

$$\mathcal{E}_g \left[\int_0^T g(s, Y_s, Z_s) ds \right] = \mathbb{E} \left[\int_0^T g(s, Y_s, Z_s) ds \right] + \mathbb{E} \left[\int_0^T g(s, Y_s^{**}, Z_s^{**}) ds \right].$$

□

For the classical expectation $\mathbb{E}$, we have that $\mathbb{E}[\xi + c] = \mathbb{E}[\xi] + c$ and $\mathbb{E}[c\xi] = c \mathbb{E}[\xi]$ for all constants $c \in \mathbb{R}$. For the g -expectation $\mathcal{E}_g$, however, that translation invariance and homogeneity do not hold in general.

Proposition 2.3.9. *Let $(Y_t, Z_t)_{t \in [0, T]} \in L^2(\Omega \times [0, T]) \times L^2(\Omega \times [0, T])$ be the solution of the BSDE (2.6) with terminal condition $Y_T = \xi$. Then we have*

$$(i) \quad \mathcal{E}_g[\xi + c] = \mathcal{E}_{\bar{g}^{(c)}}[\xi] + c, \text{ where } \bar{g}^{(c)}(t, y, z) = g(t, y + c, z), \quad c \in \mathbb{R}.$$

$$(ii) \quad \mathcal{E}_g[\xi + c] = \mathcal{E}_g[\xi] + c + \mathbb{E} \left[\int_0^T (g(s, \bar{Y}_s + c, Z_s) - g(s, Y_s, Z_s)) ds \right],$$

where $\bar{Y}_s = \mathcal{E}_{\bar{g}^{(c)}}[\xi | \mathcal{F}_s]$.

- (iii) $\mathcal{E}_g[c\xi] = c\mathcal{E}_{\tilde{g}^{(c)}}[\xi]$, where $\tilde{g}^{(c)}(t, y, z) = \frac{1}{c}g(t, cy, cz)$, $c \in \mathbb{R} \setminus \{0\}$.
- (iv) $\mathcal{E}_g[c\xi] = c\mathcal{E}_g[\xi] + \mathbb{E} \left[\int_0^T (g(s, c\tilde{Y}_s, c\tilde{Z}_s) - cg(s, Y_s, Z_s)) ds \right]$,
 where $(\tilde{Y}_t, \tilde{Z}_t)_{t \in [0, T]}$ is the solutions of the BSDE with generator $\tilde{g}^{(c)}(t, y, z) = \frac{1}{c}g(t, cy, cz)$, $c \neq 0$ and terminal condition $\tilde{Y}_T = \xi$.

Proof. For the part (i), denote by $(Y_t^*, Z_t^*)_{t \in [0, T]}$ the solution of the backward SDE

$$\begin{aligned} Y_t^* &= \xi + c + \int_t^T g(s, Y_s^*, Z_s^*) ds - \int_t^T Z_s^* dW_s, \quad t \in [0, T], \\ Y_t^* - c &= \xi + \int_t^T \bar{g}^{(c)}(s, Y_s^* - c, Z_s^*) ds - \int_t^T Z_s^* dW_s. \end{aligned}$$

Let $(\bar{Y}_t, \bar{Z}_t)_{t \in [0, T]}$ be the solution of the BSDE (2.6) with the generator $\bar{g}^{(c)}(t, y, z) = g(t, y + c, z)$ and the terminal condition $Y_T = \xi$, i.e.,

$$\bar{Y}_t = \xi + \int_t^T \bar{g}^{(c)}(s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dW_s, \quad (2.12)$$

By uniqueness of the solution $(\bar{Y}_t, \bar{Z}_t)_{t \in [0, T]}$ of the backward SDE (2.12), we get that

$$Y_t^* - c = \bar{Y}_t, \quad Z_t^* = \bar{Z}_t.$$

Thus, $\mathcal{E}_g[\xi + c] = \mathcal{E}_{\bar{g}^{(c)}}[\xi] + c$.

The assertion (ii) is a consequence of the assertion (i) and Proposition 2.3.6. In fact, we have

$$\begin{aligned} \mathcal{E}_g[\xi + c] &= c + \mathcal{E}_{\bar{g}^{(c)}}[\xi] \\ &= c + \mathbb{E}[\xi] + \mathbb{E} \left[\int_0^T g(s, \bar{Y}_s + c, Z_s) ds \right] \\ &= c + \mathcal{E}_g[\xi] - \mathbb{E} \left[\int_0^T g(s, Y_s, Z_s) ds \right] + \mathbb{E} \left[\int_0^T g(s, \bar{Y}_s + c, Z_s) ds \right]. \end{aligned}$$

The assertion (iii) is proved as the same fashion of (i). For any $c \neq 0$, we denote by $(Y_t^{**}, Z_t^{**})_{t \in [0, T]}$ the solution of the backward SDE

$$Y_t^{**} = c\xi + \int_t^T g(s, Y_s^{**}, Z_s^{**}) ds - \int_t^T Z_s^{**} dW_s, \quad t \in [0, T],$$

$$\frac{Y_t^{**}}{c} = \xi + \int_t^T \tilde{g}^{(c)} \left(s, \frac{Y_s^{**}}{c}, \frac{Z_s^{**}}{c} \right) ds - \int_t^T \frac{Z_s^{**}}{c} dW_s.$$

By uniqueness of the solution $(\tilde{Y}_t, \tilde{Z}_t)_{t \in [0, T]}$ of the backward SDE

$$\tilde{Y}_t = \xi + \int_t^T \tilde{g}^{(c)} \left(s, \tilde{Y}_s, \tilde{Z}_s \right) ds - \int_t^T \tilde{Z}_s dW_s, \quad t \in [0, T],$$

we get that $Y_t^{**}/c = \tilde{Y}_t$ and $Z_t^{**}/c = \tilde{Z}_t$, $t \in [0, T]$. Thus, at $t = 0$ we have

$$\mathcal{E}_g[c\xi] = c\mathcal{E}_{\tilde{g}^{(c)}}[\xi].$$

It is similar to (ii), the assertion (iv) is a consequence of the assertion (iii) and Proposition 2.3.6. $\square$

Remark 2.3.10. In [48], the authors proved equivalences that

- (i) $\mathcal{E}_g[\xi + c] = \mathcal{E}_g[\xi] + c$ if and only if g is independent of y , i.e., $g(t, y, z) = g(t, z)$ for all $t \in [0, T]$, $y, z \in \mathbb{R}$.
- (ii) $\mathcal{E}_g[c\xi] = c\mathcal{E}_g[\xi]$, $c > 0$ if and only if g is positively homogeneous in (y, x) , i.e., $g(t, \alpha y, \alpha z) = \alpha g(t, y, z)$ for all $\alpha > 0$, $t \in [0, T]$, $y, z \in \mathbb{R}$.

2.4 Risk Measures

2.4.1 Coherent and Convex Risk Measures

Let $\mathcal{X}$ be space all of financial positions in which investors are interested in. Here, we consider $\mathcal{X} = L^2(\Omega, \mathcal{F}, \mathbb{P})$. A *risk measure* ρ is a mapping $\rho : \xi \mapsto \mathbb{R}$, used to quantify riskiness of financial positions. There are different families of risk measures which have been proposed and studied such as coherent risk measures and convex risk measures based on some axiomatic assumptions as follows.

Axioms for static risk measure ρ :

- (a) **Positivity:** if $\xi \geq 0$, then $\rho(\xi) \leq \rho(0)$, for all $\xi \in \mathcal{X}$.
- (b) **Monotonicity:** if $\xi_1 \leq \xi_2$, then $\rho(\xi_1) \geq \rho(\xi_2)$, for all $\xi_1, \xi_2 \in \mathcal{X}$.
- (c) **Constancy:** $\rho(c) = -c$, for all constant $c \in \mathbb{R}$.

- (d) **Translatability:** $\rho(\xi + c) = \rho(\xi) - c$, for all constant $c \in \mathbb{R}$ and $\xi \in \mathcal{X}$.
- (e) **Sublinearity:**
 - (e1) **Positive homogeneity:** $\rho(c\xi) = c\rho(\xi)$, for all $c \geq 0$ and $\xi \in \mathcal{X}$.
 - (e2) **Sub-additivity:** $\rho(\xi_1 + \xi_2) \leq \rho(\xi_1) + \rho(\xi_2)$, for all $\xi_1, \xi_2 \in \mathcal{X}$.
- (f) **Convexity:** $\rho(\lambda\xi_1 + (1 - \lambda)\xi_2) \leq \lambda\rho(\xi_1) + (1 - \lambda)\rho(\xi_2)$, for all $\xi_1, \xi_2 \in \mathcal{X}$ and $\lambda \in (0, 1)$.
- (g) **Lower semi-continuity:** $\{\xi : \rho(\xi) \leq \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$.

Note that we represent a loss of the position ξ in term of positive amount, and by convention, ξ is called *acceptable* when $\rho(\xi) \leq 0$ and *unacceptable* otherwise. Thus, if the risk measure $\rho(\xi)$ is negative, then it is like a gain or no loss and the investors can withdraw the capital of an amount $-\rho(\xi)$, while the position still remain *acceptable*. For the axiom of **(a) Positivity**, it means that if the position ξ always generates gain, then its riskiness is less than the riskiness of the null position. **(b) Monotonicity** can be interpreted in the same manner, if the position ξ_1 has almost surely worse outcome than the position ξ_2 , then ξ_1 should have a greater riskiness than ξ_2 .

The **(c) Constancy** is clear that the riskiness of positions with constant net worth is simply opposite of their net worth. For the **(d) Translatability**, it tells us that if adding a constant amount to the position, then its riskiness will be reduced by that amount. Also, it implies that $\rho(\xi + \rho(\xi)) = 0$, so $\rho(\xi)$ is the minimum capital that we need to add to the original position such that it is neutrally acceptable.

The risk measure is required to satisfy the sublinearity. In particular, by the **(e1) Positive homogeneity**, we see that the riskiness of the position ξ will be increased by the factor λ if the investors increase the size of the position by the factor λ . Meanwhile, the **(e2) Sub-additivity** ensures that diversification will produce a smaller riskiness than a sum of individual investments. In a comparison, the **(f) Convexity** encourages only diversification through portfolios composed of a convex combination of single positions.

We have briefly given interpretations of the axioms above. Based on this system, one can define different families of risk measures.

Definition 2.4.1 (Static coherent risk measure). A functional $\rho : \mathcal{X} \mapsto \mathbb{R}$ is called a static coherent risk measure if it satisfies **(b) Monotonicity**, **(d) Translatability**, and **(e) Sublinearity**.

We note that **(b) Monotonicity** implies **(a) Positivity**. The condition **(e1) Positive homogeneity** implies that $\rho(0) = 0$ and together with **(d) Translatability**, it implies **(c) Constancy**. In [20], the authors give a representation theorem for coherent risk measures.

Theorem 2.4.2. *Let $\rho : \xi \mapsto \mathbb{R}$ be a risk measure. Then it is a coherent risk measure if and only if there exists a closed convex set $\mathcal{P}$ of $\mathbb{P}$ -continuous probability measures such that*

$$\rho(\xi) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-\xi]. \quad (2.13)$$

Definition 2.4.3 (Static convex risk measure). *A functional $\rho : \xi \mapsto \mathbb{R}$ is called a static convex risk measure if it satisfies **(b) Monotonicity**, **(d) Translatability**, **(f) convexity**, **(g) Lower semi-continuity** and $\rho(0) = 0$.*

The convex risk measure is an extension of the coherent risk measure. Analogously, there also exists a representation for convex risk measures.

Theorem 2.4.4. *Let $\rho : \xi \mapsto \mathbb{R}$ be a risk measure. Then it is a convex risk measure if and only if there exists a closed convex set $\mathcal{P}$ of $\mathbb{P}$ -continuous probability measures such that*

$$\rho(\xi) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-\xi - F(X)], \quad (2.14)$$

where F is a convex conjugate of $\rho(\cdot)$, which is a penalty term and depends on the scenarios, see e.g. [31].

Moving on, we consider dynamic risk measures. Let $L^0(\Omega, \mathcal{F}_t)$ be a space of all finite valued $\mathcal{F}_t$ -measurable random variables. Denote by $(\rho_t)_{t \in [0, T]}$ a dynamic risk measure defined as follows:

Definition 2.4.5 (Dynamic risk measures). *A dynamic risk measure $(\rho_t)_{t \in [0, T]}$ is any map satisfying:*

- (i) $\rho_t : \xi \mapsto L^0(\Omega, \mathcal{F}_t)$, for all $t \in [0, T]$.
- (ii) ρ_0 is a static risk measure.
- (iii) $\rho_T(\xi) = -\xi$, for all $\xi \in \mathcal{X}$.

Similarly, we provide the axioms for which families of dynamic risk measures to be satisfied.

Axioms for dynamic risk measure ρ :

- (a) **Dynamic positivity:** if $\xi \geq 0$, then $\rho_t(\xi) \leq \rho_t(0)$, for all $t \in [0, T]$ and $\xi \in \mathcal{X}$.
- (b) **Dynamic monotonicity:** if $\xi_1 \leq \xi_2$, then $\rho_t(\xi_1) \geq \rho_t(\xi_2)$, for all $t \in [0, T]$ and $\xi_1, \xi_2 \in \mathcal{X}$.
- (c) **Dynamic constancy:** $\rho_t(c) = -c$, for all $t \in [0, T]$ and constant $c \in \mathbb{R}$.
- (d) **Dynamic translatability:** $\rho_t(\xi + \alpha) = \rho_t(\xi) - \alpha$, for all $t \in [0, T]$, α is $\mathcal{F}_t$ -measurable in ξ and $\xi \in \mathcal{X}$.
- (e) **Dynamic sublinearity:**
 - (e1) **Dynamic positive homogeneity:** $\rho_t(c\xi) = c\rho_t(\xi)$, for all $t \in [0, T]$ and $c \geq 0$ and $\xi \in \mathcal{X}$.
 - (e2) **Dynamic sub-additivity:** $\rho_t(\xi_1 + \xi_2) \leq \rho_t(\xi_1) + \rho_t(\xi_2)$, for all $t \in [0, T]$ and $\xi_1, \xi_2 \in \mathcal{X}$.
- (f) **Dynamic convexity:** $\rho_t(\lambda\xi_1 + (1 - \lambda)\xi_2) \leq \lambda\rho_t(\xi_1) + (1 - \lambda)\rho_t(\xi_2)$, for all $t \in [0, T]$ and $\xi_1, \xi_2 \in \mathcal{X}$ and $\lambda \in (0, 1)$.
- (g) **Dynamic lower semi-continuity:** $\{\xi : \rho_t(\xi) \leq \gamma\}$ is closed in $\mathcal{X}$ for all $t \in [0, T]$ and any $\gamma \in \mathbb{R}$.

Definition 2.4.6. A dynamic risk measure $(\rho_t)_{t \in [0, T]}$ is called:

- (i) **coherent** if it satisfies (b) *Dynamic monotonicity*, (d) *Dynamic translatability*, and (e) *Dynamic sublinearity*.
- (ii) **convex** if it satisfies (b) *Dynamic monotonicity*, (d) *Dynamic translatability*, (e) *Dynamic convexity*, (g) *Dynamic lower semi-continuity* and $\rho_t(0) = 0$.
- (iii) **time-consistent** if for all $t \in [0, T]$, $\xi \in \mathcal{X}$, $A \in \mathcal{F}_t$, we have

$$\rho_t(\xi \mathbb{1}_A) = \rho_0(-\rho_t(\xi) \mathbb{1}_A).$$

2.4.2 Risk Measures by g -Expectations

Let g satisfy the Conditions (A.2.2)-(A.2.3'). We define a static risk measure $\rho^g : L^2(\Omega, \mathcal{F}_T) \mapsto \mathbb{R}$ and a dynamic risk measure $\rho_t^g : L^2(\Omega, \mathcal{F}_T) \mapsto L^2(\Omega, \mathcal{F}_t)$ by using the g -expectations as follows:

$$\rho^g(\xi) := \mathcal{E}_g[-\xi], \quad \rho_t^g(\xi) := \mathcal{E}_g[-\xi | \mathcal{F}_t], \quad t \in [0, T], \xi \in L^2(\Omega, \mathcal{F}_T). \quad (2.15)$$

From now on, we call ρ^g and $(\rho_t^g)_{t \in [0, T]}$ are the **static g -risk measure** and the **dynamic g -risk measure**, respectively. Their properties rely on properties of the function g .

Proposition 2.4.7 (Proposition 11 of [32]).

- (i) If g is sublinearity in (y, z) , then ρ^g and $(\rho_t^g)_{t \in [0, T]}$ are a static coherent and a dynamic coherent risk measure, respectively.
- (ii) If g is convex in (y, z) , then ρ^g and $(\rho_t^g)_{t \in [0, T]}$ are a static convex and a dynamic convex risk measure, respectively.

As a consequence, we have representation results for g -risk measures by representation theorems of the coherent and the convex risk measures.

Corollary 2.4.8 (Corollary 12 of [32]).

- (i) If g is convex in (y, z) then ρ^g can be represented as

$$\rho_g(\xi) = \sup_{Q \in \mathcal{P}_g} \left\{ \mathbb{E}_Q[-\xi] - F_g\left(\frac{dQ}{dP}\right) \right\}, \quad \xi \in L^2(\Omega, \mathcal{F}_T), \quad (2.16)$$

where $F_g : L^2(\Omega, \mathcal{F}_T) \mapsto \mathbb{R} \cup \{+\infty\}$ is the convex functional defined by

$$F_g(\xi') := \sup_{\xi \in L^2(\Omega, \mathcal{F}_T)} \{ \mathbb{E}[\xi' \xi] - \mathcal{E}_g[\xi] \},$$

and $\mathcal{P}_g$ is the non-empty convex set defined by

$$\mathcal{P}_g := \{Q \in \mathcal{M} : \frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}_T) \text{ and } F_g\left(\frac{dQ}{dP}\right) < \infty\}.$$

- (ii) If g is sublinear in (y, z) then the representation of ρ^g in (2.16) becomes

$$\rho_g(\xi) = \sup_{Q \in \mathcal{P}_g} \mathbb{E}_Q[-\xi], \quad \xi \in L^2(\Omega, \mathcal{F}_T). \quad (2.17)$$

In the same fashion, we also have representations for the dynamic g -risk measure $(\rho_t^g)_{t \in [0, T]}$ as the Corollary above by simply replacing expectations by conditional expectations.

2.5 Sublinear Expectations

2.5.1 Definitions and Properties

Let Ω be a given set (known as sample space) and let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω such that $c \in \mathcal{H}$ for all constants c and $|\xi| \in \mathcal{H}$ if $\xi \in \mathcal{H}$. Furthermore, we assume that if $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}$, then $\varphi(\xi_1, \dots, \xi_n) \in \mathcal{H}$, for each $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R}^n)$, where $\mathcal{C}_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of locally Lipschitz functions satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad (2.18)$$

for some constant C and m depending only on φ . We regard $\mathcal{H}$ as a space of random variables and $\mathcal{C}_{l.Lip}(\mathbb{R}^n)$ as a test function space. In general, $\mathcal{C}_{l.Lip}(\mathbb{R}^n)$ can be replaced by one of the following spaces:

- $L^\infty(\mathbb{R}^n)$: the space of bounded Borel-measurable functions;
- $\mathcal{C}_B(\mathbb{R}^n)$: the space of bounded and continuous functions;
- $\mathcal{C}_b^k(\mathbb{R}^n)$: the space of continuously differentiable functions with bounded derivatives of all orders less than or equal to k ;
- $\mathcal{C}_B^k(\mathbb{R}^n)$: the space of *bounded* and continuously differentiable functions with bounded derivatives of all orders less than or equal to k ;
- $\mathcal{C}_{B.Unif}(\mathbb{R}^n)$: the space of bounded and uniformly continuous functions;
- $\mathcal{C}_{Lip}(\mathbb{R}^n)$: the space of Lipschitz continuous functions;
- $\mathcal{C}_{B.Lip}(\mathbb{R}^n)$: the space of bounded and Lipschitz continuous functions.

Definition 2.5.1 (Sublinear expectation). *A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$, satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

- (i) *Monotonicity: If $X \leq Y$, then $\widehat{\mathbb{E}}[X] \leq \widehat{\mathbb{E}}[Y]$.*
- (ii) *Constant preserving: $\widehat{\mathbb{E}}[c] = c$, for $c \in \mathbb{R}$.*
- (iii) *Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$.*
- (iv) *Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda \geq 0$.*

The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space.

In case $\widehat{\mathbb{E}}$ satisfies only the monotonicity and the constant preserving property, $\widehat{\mathbb{E}}$ is called a *nonlinear expectation*. For instance, the g -expectation $\mathcal{E}_g$ is a nonlinear expectation in general. The sublinear expectation $\widehat{\mathbb{E}}$ is also called an upper expectation since it can be represented by supremum of a family of linear expectations as in the following theorem, see Theorem 2.1 in Chapter I, § 2 of [80].

Theorem 2.5.2 (Representation of sublinear expectations). *Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sublinear expectation space. Then there exists a family of linear expectation $\{\mathbb{E}_\theta : \theta \in \Theta\}$ defined on $\mathcal{H}$ such that*

$$\widehat{\mathbb{E}}[X] = \sup_{\theta \in \Theta} \mathbb{E}_\theta[X], \quad X \in \mathcal{H},$$

and for each $X \in \mathcal{H}$, there exist $\theta_X \in \Theta$ such that $\widehat{\mathbb{E}}[X] = \mathbb{E}_{\theta_X}[X]$.

Denote by $\mathcal{P}_f$ the collection of all finitely additive probability measures on $(\Omega, \mathcal{F})$. In Theorem 2.4, Chapter I, § 2 of [80], Peng also provides another version of the representation that for each sublinear expectation $\widehat{\mathbb{E}}$, there exists a subset $\mathcal{P}$ of $\mathcal{P}_f$ such that

$$\widehat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X], \quad X \in \mathcal{H}. \quad (2.19)$$

In this case, $\mathcal{P}$ is called a set that represents the sublinear expectation $\widehat{\mathbb{E}}$.

Definition 2.5.3 (Distributions on sublinear expectation spaces). *Given $X \in \mathcal{H}$, define the function $\mathbb{F}_X$ by*

$$\mathbb{F}_X[\varphi] := \widehat{\mathbb{E}}[\varphi(X)], \quad \varphi \in \mathcal{C}_{l.Lip}(\mathbb{R}). \quad (2.20)$$

The function $\mathbb{F}_X$ is called the distribution of X under $\widehat{\mathbb{E}}$.

It is easy to see that $(\mathbb{R}, \mathcal{C}_{l.Lip}(\mathbb{R}), \mathbb{F}_X)$ also forms a sublinear expectation space and by Theorem 2.5.2, the distribution of X is known to admit the representation

$$\mathbb{F}_X[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}} \varphi(x) F_X(\theta, dx). \quad (2.21)$$

for a family $(F_X(\theta, \cdot))_{\theta \in \Theta}$ of usual probability distributions defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The distribution of $X \in \mathcal{H}$ defines the following parameters:

$$\underline{\mu} := -\widehat{\mathbb{E}}[-X], \quad \bar{\mu} := \widehat{\mathbb{E}}[X], \quad \text{and} \quad \underline{\sigma}^2 := -\widehat{\mathbb{E}}[-X^2], \quad \bar{\sigma}^2 := \widehat{\mathbb{E}}[X^2], \quad (2.22)$$

and the intervals $[\underline{\mu}, \bar{\mu}]$ and $[\underline{\sigma}^2, \bar{\sigma}^2]$ characterize the mean-uncertainty and the variance-uncertainty of X . In the theory of sublinear expectation, the following property is very useful and frequently applied, see Proposition 3.6 in Chapter I, § 2 of [80].

Proposition 2.5.4. *Let X and Y be two random variables defined on sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. If $\widehat{\mathbb{E}}[Y] = -\widehat{\mathbb{E}}[-Y]$, i.e., Y has no mean-uncertainty, then we have*

$$\widehat{\mathbb{E}}[X + \alpha Y] = \widehat{\mathbb{E}}[X] + \alpha \widehat{\mathbb{E}}[Y], \quad \alpha \in \mathbb{R}.$$

In particular, if $\widehat{\mathbb{E}}[Y] = \widehat{\mathbb{E}}[-Y] = 0$, then $\widehat{\mathbb{E}}[X + \alpha Y] = \widehat{\mathbb{E}}[X]$ for all $\alpha \in \mathbb{R}$.

Definition 2.5.5 (Identical distribution). *Let X_1 and X_2 be two random variables defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\widehat{\mathbb{E}}_1[\varphi(X_1)] = \widehat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.*

If in addition, $\widehat{\mathbb{E}}_1[\varphi(X_1)] \leq \widehat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$, then we say that the distribution of X_2 is *stronger* than the distribution of X_1 . In this case, we have

$$\{F_{X_1}(\theta_1, \cdot) : \theta_1 \in \Theta_1\} \subset \{F_{X_2}(\theta_2, \cdot) : \theta_2 \in \Theta_2\}.$$

Definition 2.5.6 (Independence). *Given two random variables X and Y , the random variable Y is said to be independent from X under $\widehat{\mathbb{E}}$ if for every test function $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R}^2)$, we have*

$$\widehat{\mathbb{E}}[\varphi(X, Y)] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}(\varphi(x, Y))_{|x=X}]. \quad (2.23)$$

In addition, Y is said to be weakly independent from X under $\widehat{\mathbb{E}}$ if (2.23) holds for all test functions of the form

$$\varphi(x, y) := \psi_0(y) + \psi_1(y)|x| + \psi_2(y)|x|^2, \quad \psi_0, \psi_1, \psi_2 \in \mathcal{C}_{l.Lip}(\mathbb{R}).$$

In addition, a random variable $\overline{X}$ is called an *independent copy* of X if $X \stackrel{d}{=} \overline{X}$ and $\overline{X}$ is independent from X .

Remark 2.5.7.

- (i) *Y is independent from X means that after realization $X = x$, the uncertainty of distributions $\{F_Y(\theta, \cdot) : \theta \in \Theta\}$ of Y does not change.*
- (ii) ***Y is independent from X** does not imply that **X is independent from Y** , see Example 3.13 in [80] at page 10.*

Definition 2.5.8 (G-normal distribution). A random variable X in the sublinear expectation $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called G -normally distributed if

$$aX + b\overline{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X, \quad a, b \geq 0,$$

where $\overline{X}$ is an independent copy of X and, $G : \mathbb{R} \rightarrow \mathbb{R}$ is the function defined by $G(a) := \frac{1}{2}\widehat{\mathbb{E}}[aX^2]$, $a \in \mathbb{R}$.

By [80, 77, 78], letting $\underline{\sigma}^2 := -\widehat{\mathbb{E}}[-\widehat{X}^2]$ and $\overline{\sigma}^2 := \widehat{\mathbb{E}}[\widehat{X}^2]$, $\widehat{X}$ is G -normally distributed if and only if

$$G(a) := \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \overline{\sigma}^2} (\sigma^2 a) = \frac{1}{2} (\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \quad a \in \mathbb{R}, \quad (2.24)$$

with $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$, and for each $\varphi \in \mathcal{C}_{\text{Lip}}(\mathbb{R})$, the function $u_\varphi(t, x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}\widehat{X})]$ solves the G -heat equation

$$\frac{\partial u_\varphi}{\partial t}(t, x) - G\left(\frac{\partial^2 u_\varphi}{\partial x^2}(t, x)\right) = 0, \quad u_\varphi(0, x) = \varphi(x). \quad (2.25)$$

In particular, we have $\widehat{\mathbb{E}}[\varphi(\widehat{X})] = P_1^G(\varphi) := u_\varphi(1, 0)$, and the function $G : \mathbb{R} \rightarrow \mathbb{R}$ in (2.24) is sublinear, monotonic and bounded, with $-\underline{\sigma}^2|a|/2 \leq G(a) \leq \overline{\sigma}^2|a|/2$, $a \in \mathbb{R}$.

In the sequel, we will only deal with the non-degenerate G -normal distribution $\mathcal{N}(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2])$ for which there exists $\underline{\sigma}^2 > 0$ such that G is non-degenerate, i.e.

$$G(b) - G(a) \geq \frac{\underline{\sigma}^2}{2}(b - a), \quad a \leq b.$$

In this case, the equation (2.25) admits a unique classical solution $u(t, x) \in C^{1,2}([0, T) \times \mathbb{R})$, see e.g. [80, 77].

Remark 2.5.9. Given $\widehat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2])$, we can calculate $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$ for some cases as follows:

(i) For each φ convex, we have

$$\widehat{\mathbb{E}}[\varphi(\widehat{X})] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\overline{\sigma}^2 y) \exp\left(-\frac{y^2}{2}\right) dy.$$

(ii) For each φ concave, we have

$$\widehat{\mathbb{E}}[\varphi(\widehat{X})] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\underline{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy.$$

In particular, we have

$$\widehat{\mathbb{E}}[\widehat{X}] = \widehat{\mathbb{E}}[-\widehat{X}] = 0, \quad \widehat{\mathbb{E}}[\widehat{X}^2] = \bar{\sigma}^2, \quad -\widehat{\mathbb{E}}[-\widehat{X}^2] = \underline{\sigma}^2,$$

and

$$\widehat{\mathbb{E}}[\widehat{X}^4] = 3\bar{\sigma}^4, \quad -\widehat{\mathbb{E}}[-\widehat{X}^4] = 3\underline{\sigma}^4.$$

2.5.2 G -Expectation and G -Brownian Motion

In this section, we recall some notations and backgrounds on the theory of G -expectation, G -Brownian motion and related stochastic calculus, more details see [80, 77]. First, we will give a general definition of G -Brownian motion on a given sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Later on, we construct a G -expectation based on the sublinear expectation $\widehat{\mathbb{E}}$ and corresponding a G -Brownian motion on a certain G -expectation space.

Definition 2.5.10. Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sublinear expectation space. A stochastic process $(B_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called one dimensional G -Brownian motion under $\widehat{\mathbb{E}}$ if it satisfies the following conditions:

(i) $B_0(\omega) = 0$.

(ii) For each $s, t \geq 0$, the increment $B_{t+s} - B_t$ follows G -normal distribution $N(\{0\}, [s\underline{\sigma}^2, s\bar{\sigma}^2])$ and it is independent from $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$, for each $n \in \mathbb{N}$ and $t_1 \leq t_2 \leq \dots \leq t_n \leq t$.

We can see that the definition of G -Brownian motion is quite similar to definition of the usual Brownian motion, but here its increments are assumed to be G -normal distributed with uncertain variances. The G -Brownian motion can be characterized by the following theorem.

Theorem 2.5.11 (Theorem 1.6, Chapter III, § 1 of [80]). Let $(B_t)_{t \geq 0}$ be a stochastic process defined on the sublinear expectation $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ such that

(i) $B_0(\omega) = 0$.

- (ii) For each $s, t \geq 0$, $B_{t+s} - B_t$ and B_s are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$, for each $n \in \mathbb{N}$ and $t_1 \leq \dots \leq t_n \leq t$.
- (iii) For each $t \geq 0$, $(B_t)_{t \geq 0}$ has no mean-uncertainty, i.e., $\widehat{\mathbb{E}}[B_t] = \widehat{\mathbb{E}}[-B_t] = 0$.
- (iv) $\lim_{t \rightarrow 0} \widehat{\mathbb{E}}[B_t^3]t^{-1} = 0$.

Then $(B_t)_{t \geq 0}$ is a G -Brownian motion on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with the function $G(\cdot)$ defined by $G(a) := \frac{1}{2} \widehat{\mathbb{E}}[aB_1^2]$, $a \in \mathbb{R}$.

A construction of G -expectation and G -Brownian motion

Let $\mathcal{C}_{l.Lip}(\mathbb{R})$ be the space of all local Lipschitz real functions on $\mathbb{R}$. Also, let $\Omega := C_0(\mathbb{R}^+)$ denote the space of $\mathbb{R}$ -valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$, and equipped with the distance

$$d(\omega^1, \omega^2) := \sum_{i=1}^{\infty} \frac{1}{2^i} \max \left\{ \left(\max_{t \in [0, i]} |\omega_t^1 - \omega_t^2| \right), 1 \right\}.$$

Following [77], for each $t \in [0, \infty)$, we set

$$\Omega_t := \{\omega_{\cdot \wedge t} : \omega \in \Omega\}, \quad \mathcal{F}_t := \mathcal{B}_t(\Omega) = \mathcal{B}(\Omega_t), \quad \mathcal{F}_{t+} := \mathcal{B}_{t+}(\Omega) = \bigcap_{s>t} \mathcal{B}_s(\Omega),$$

and $\mathcal{F} := \bigvee_{s>t} \mathcal{F}_s$, where $(\Omega, \mathcal{F})$ is the canonical space equipped with the natural filtration and $B_t(\omega) = \omega_t$, for $t \in [0, \infty)$, $\omega \in \Omega$, is the corresponding canonical process. For each fixed $T \geq 0$, we let

$$L_{ip}(\Omega, \mathcal{F}_T) := \left\{ \varphi(B_{t_1 \wedge T}, \dots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \dots, t_n \in [0, \infty), \varphi \in \mathcal{C}_{l.Lip}(\mathbb{R}^n) \right\},$$

$$L_{ip}(\Omega, \mathcal{F}) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega, \mathcal{F}_n).$$

We now construct a sublinear expectation on $L_{ip}(\Omega, \mathcal{F})$ called G -expectation.

Definition 2.5.12 (G -expectation). *i) The G -expectation is the sublinear expectation defined on $L_{ip}(\Omega, \mathcal{F})$ by*

$$\mathbb{E}_G[X] := \widehat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \dots, \sqrt{t_n - t_{n-1}}\xi_n)], \quad (2.26)$$

for X of the form $X := \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega, \mathcal{F})$, where $\xi_1, \dots, \xi_n$ are identically G -normally distributed random variables in a

sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ such that ξ_{i+1} is independent from $(\xi_1, \dots, \xi_i)$, for $i = 1, 2, \dots, n-1$. The corresponding canonical process $B_t(\omega) = \omega_t$, $\omega \in \Omega$, $t \geq 0$, is called a ***G-Brownian motion***.

ii) The ***conditional G-expectation*** of $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ under $\mathcal{F}_{t_j}$ is defined by

$$\begin{aligned}\mathbb{E}_G[X | \mathcal{F}_{t_j}] &= \widehat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}}) | \mathcal{F}_{t_j}] \\ &= \widetilde{\varphi}(B_{t_1} - B_{t_0}, \dots, B_{t_j} - B_{t_{j-1}}),\end{aligned}\quad (2.27)$$

where $\widetilde{\varphi}(x_1, \dots, x_{t_j}) = \widehat{\mathbb{E}}[\varphi(x_1, \dots, x_{t_j}, \sqrt{t_{j+1} - t_j}\xi_j, \dots, \sqrt{t_n - t_{n-1}}\xi_n)]$.

Let $\mathcal{P}$ is a weakly compact set that represents the G -expectation $\mathbb{E}_G$ as in (2.19). We define a capacity $c(\cdot)$ by

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{F}_T. \quad (2.28)$$

In this case, if $c(A) = 0$, then the set A is called a polar set. A property that holds outside a polar set is referred to hold *quasi-surely* (*q.s.*).

For $0 \leq s \leq t$, we have $\mathbb{E}_G[B_t - B_s | \mathcal{F}_s] = -\mathbb{E}_G[-(B_t - B_s) | \mathcal{F}_s] = 0$, and

$$\mathbb{E}_G[(B_t - B_s)^2 | \mathcal{F}_s] = \overline{\sigma}^2(t - s) \text{ and } -\mathbb{E}_G[-(B_t - B_s)^2 | \mathcal{F}_s] = \underline{\sigma}^2(t - s).$$

In [80, 77], the author proved that $\mathbb{E}_G[\cdot]$ consistently defines a nonlinear expectation on the vector lattice $L_{ip}(\Omega, \mathcal{F}_T)$ as well as on $L_{ip}(\Omega, \mathcal{F})$. These spaces then can be made completion which are denoted by $L_G^1(\Omega, \mathcal{F}_T)$ and $L_G^1(\Omega, \mathcal{F})$ under norm $\|X\|_1 := \mathbb{E}_G[|X|]$, $X \in L_{ip}^0(\Omega, \mathcal{F}_T)$, resp. $X \in L_{ip}^0(\Omega, \mathcal{F})$. The conditional G -expectation $\mathbb{E}_G[\cdot | \mathcal{F}_t]$ is a continuous map on $L_{ip}(\Omega, \mathcal{F}_t)$ (resp. $L_{ip}(\Omega, \mathcal{F})$) under the norm $\|\cdot\|_1$, hence it can be continuously extended to define on $L_G^1(\Omega, \mathcal{F}_t)$ (resp. $L_G^1(\Omega, \mathcal{F})$) under that norm.

Below we list some properties of the conditional G -expectation $\mathbb{E}_G[\cdot | \mathcal{F}_t]$ that hold for random variables in $L_{ip}(\Omega, \mathcal{F}_T)$ and still hold in $L_G^1(\Omega, \mathcal{F})$.

Proposition 2.5.13 (Proposition 2.3, Chapter III, § 2 of [80]).

- (i) $\mathbb{E}_G[X | \mathcal{F}_t] = X$, for $X \in L_G^1(\Omega, \mathcal{F}_t)$, $t \leq T$.
- (ii) If $X \leq Y$, then $\mathbb{E}_G[X | \mathcal{F}_t] \leq \mathbb{E}_G[Y | \mathcal{F}_t]$.
- (iii) $\mathbb{E}_G[X | \mathcal{F}_t] - \mathbb{E}_G[Y | \mathcal{F}_t] \leq \mathbb{E}_G[X - Y | \mathcal{F}_t]$.

(iv) $\mathbb{E}_G [\mathbb{E}_G[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}_G[X|\mathcal{F}_{t \wedge s}]$. In particular, $\mathbb{E}_G [\mathbb{E}_G[X|\mathcal{F}_t]] = \mathbb{E}_G[X]$.

(v) $\mathbb{E}_G[X + \eta|\mathcal{F}_t] = \mathbb{E}_G[X|\mathcal{F}_t] + \eta$, for $\eta \in L_G^1(\Omega, \mathcal{F}_t)$.

(vi) $\mathbb{E}_G[\eta X|\mathcal{F}_t] = \eta^+ \mathbb{E}_G[X|\mathcal{F}_t] + \eta^- \mathbb{E}_G[-X|\mathcal{F}_t]$, for each bounded $\eta \in L_G^1(\Omega, \mathcal{F}_t)$.

2.5.3 G -Itô Integrals

For $p > 1$, let $L_G^p(\Omega, \mathcal{F}) := \{X \in L_G^1(\Omega, \mathcal{F}) : |X|^p \in L_G^1(\Omega, \mathcal{F})\}$, which is a Banach space under the norm $\|X\|_p := (\mathbb{E}_G[|X|^p])^{1/p}$. Let now $p \geq 1$ be fixed, and let $\pi_T^N := \{t_0, \dots, t_N\}$ be a subdivision of the interval $[0, T]$. We consider the following type of simple processes defined by

$$\eta_t(\omega) = \sum_{j=0}^N \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t)$$

where $\xi_j \in L_G^p(\Omega, \mathcal{F}_{t_i})$, $i = 0, 1, 2, \dots, N-1$ are given. The collection of these type of processes is denoted by $M_G^{p,0}(0, T)$. The G -Itô integral is defined as

$$I(\eta) = \int_0^T \eta_s dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

For $p \geq 1$, we denote by $M_G^p(0, T)$, $H_G^p(0, T)$ and $S_G^p(0, T)$ the respective completion of $M_G^{p,0}(0, T)$ under the norms

$$\|\eta_t\|_{M^p} := \left(\mathbb{E}_G \left[\int_0^T |\eta_t|^p dt \right] \right)^{1/p}, \quad \|\eta_t\|_{H^p} := \left(\mathbb{E}_G \left[\left(\int_0^T |\eta_t|^p dt \right)^{p/2} \right] \right)^{1/p},$$

and

$$\|\eta_t\|_{S^p} := \left(\mathbb{E}_G \left[\sup_{t \in [0, T]} |\eta_t|^p \right] \right)^{1/p}.$$

Proposition 2.5.14. (Lemma 3.5, Exercise 3.8, Chapter III, § 3 of [80]) The mapping $I : M_G^{2,0}(0, T) \rightarrow L_G^2(\Omega, \mathcal{F}_T)$ is a linear continuous mapping and thus can be continuously extended to $M_G^2(0, T) \rightarrow L_G^2(\Omega, \mathcal{F}_T)$. In fact, we have $\mathbb{E}_G \left[\int_0^T \eta_s dB_s \right] = 0$, and

$$\underline{\sigma}^2 \mathbb{E}_G \left[\int_0^T \eta_s^2 ds \right] \leq \mathbb{E}_G \left[\left(\int_0^T \eta_s dB_s \right)^2 \right] \leq \bar{\sigma}^2 \mathbb{E}_G \left[\int_0^T \eta_s^2 ds \right].$$

The following property of the G -Itô integrals will be useful:

$$\mathbb{E}_G[X + \int_r^T \eta_t dB_t | \mathcal{F}_s] = \mathbb{E}_G[X | \mathcal{F}_s], \quad \forall X \in L_G^1(\Omega, \mathcal{F}), 0 \leq s \leq r \leq t \leq T.$$

Given $\pi_t^N = \{t_0^N = 0, t_1^N, \dots, t_N^N = t\}$ be a family of subdivisions of the interval of $[0, t]$, we consider

$$\begin{aligned} B_t^2 &= \sum_{j=0}^{N-1} (B_{t_{j+1}^N}^2 - B_{t_j^N}^2) \\ &= \sum_{j=0}^{N-1} B_{t_j^N} (B_{t_{j+1}^N} - B_{t_j^N}) + \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2. \end{aligned}$$

where the first term tends to $\int_0^t B_s dB_s$ as $\mu(\pi_t^N) \rightarrow 0$. The limit of the second term is denoted by $\langle B \rangle_t$, i.e. we have

$$\langle B \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2 \int_0^t B_s dB_s,$$

and $(\langle B \rangle_t)_{t \geq 0}$ is called the quadratic variation process of the G -Brownian motion $(B_t)_{t \geq 0}$. It is an increasing stochastic process and perfectly characterize the uncertainty of the G -Brownian motion $(B_t)_{t \geq 0}$. In addition, the quadratic variation process $(\langle B \rangle_t)_{t \geq 0}$ is just like the G -Brownian motion $(B_t)_{t \geq 0}$ its self since its increment $\langle B \rangle_{t+s} - \langle B \rangle_t$ is independent of $\mathcal{F}_t$ and identically distributed with $\langle B \rangle_s$. We have the following proposition.

Proposition 2.5.15 (Lemma 4.1, Chapter III, § 4 of [80]). *For each $0 \leq s \leq t < \infty$, we have*

$$\begin{aligned} \mathbb{E}_G[\langle B \rangle_t - \langle B \rangle_s | \mathcal{F}_s] &= \bar{\sigma}^2(t - s), \\ -\mathbb{E}_G[-(\langle B \rangle_t - \langle B \rangle_s) | \mathcal{F}_s] &= \underline{\sigma}^2(t - s), \end{aligned}$$

where $\underline{\sigma}^2 = -\mathbb{E}_G[-B_1^2]$ and $\bar{\sigma}^2 = \mathbb{E}_G[B_1^2]$.

For each $\eta \in M_G^{1,0}(0, T)$, we define an integral with respect to $d\langle B \rangle_t$ as follows:

$$Q_{0,T}(\eta) = \int_0^T \eta_s d\langle B \rangle_s := \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}).$$

Then, $Q_{0,T}(\eta)$ is continuous, so it can be continuously extended to $M_G^1(0, T)$ and it holds that, see Exercise 4.8 in Chapter III, § 4 of [80],

$$\underline{\sigma}^2 \mathbb{E}_G \left[\int_0^T |\eta_s| ds \right] \leq \mathbb{E}_G \left[\int_0^T |\eta_s| d\langle B \rangle_s \right] \leq \bar{\sigma}^2 \mathbb{E}_G \left[\int_0^T |\eta_s| ds \right], \quad \eta_t \in M_G^1(0, T). \quad (2.29)$$

Besides, we have the following isometry property by Proposition 4.5 in Chapter III, § 4 of [80]:

$$\mathbb{E}_G \left[\left(\int_0^T \eta_s dB_s \right)^2 \right] = \mathbb{E}_G \left[\int_0^T \eta_s^2 d\langle B \rangle_s \right], \quad \eta_t \in M_G^2(0, T). \quad (2.30)$$

Moreover, by Corollary 5.4 in Chapter III, § 5 of [80], the process $(\langle B \rangle_t)_{t \geq 0}$ satisfies the bounds:

$$\underline{\sigma}^2(T-t) \leq \langle B \rangle_T - \langle B \rangle_t \leq \bar{\sigma}^2(T-t),$$

where $\underline{\sigma}^2 = -\mathbb{E}_G[-B_1^2]$ and $\bar{\sigma}^2 = \mathbb{E}_G[B_1^2]$, which can be used to show that

$$\underline{\sigma}^2 \int_t^T \eta_s ds \leq \int_t^T \eta_s d\langle B \rangle_s \leq \bar{\sigma}^2 \int_t^T \eta_s ds, \quad (2.31)$$

for any $\eta_s \in M_G^1(0, T)$ such that $\eta_s \geq 0$, *q.s.* (quasi-surely). We will use this property in Chapter 3. Moreover, by Proposition 1.4 in Chapter IV, § 1 of [80] page 72, we have

$$M_t := \int_0^t \eta_s d\langle B \rangle_s - 2 \int_0^t G(\eta_s) ds, \quad \eta_t \in M_G^1(0, T), \quad (2.32)$$

is a G -martingale, i.e., $\mathbb{E}_G[M_t | \mathcal{F}_s] = M_s$ for $0 \leq s \leq t$, which will be used in the proof of Theorem 5.3.7 in Chapter 5.

2.5.4 Stochastic Differential Equations Driven by G -Brownian Motion

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a fixed sublinear expectation space. We consider a SDE driven by one-dimension G -Brownian motion $(B_t)_{t \in [0, T]}$:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t h(s, X_s) d\langle B \rangle_s + \int_0^t \sigma(t, x) dB_t, \quad (2.33)$$

where $b(\cdot, x)$, $h(\cdot, x)$, $\sigma(\cdot, x) \in M_G^2(0, T)$ for each $x \in \mathbb{R}$ and they are assumed to be satisfied the Lipschitz condition, i.e., $|f(t, x) - f(t, y)| \leq C|x - y|$ for any $t \in$

$[0, T]$, $x, y \in \mathbb{R}$ and $f = b, h$ and σ , respectively. Under this condition, see [80], there exists a unique solution $(X_t)_{t \in [0, T]} \in M_G^2(0, T)$ satisfying the above SDE (2.33).

In the sequel, we introduce the G -Itô formula which plays the most important role in stochastic calculus related to G -Brownian motion, see [57, 98, 80, 77].

Theorem 2.5.16 (G -Itô formula). *Let $u(t, x) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ and $(X_t)_{t \in [0, T]}$ be the process defined as in (2.33). Then for each $t \in [0, T]$, we have*

$$\begin{aligned} u(t, X_t) - u(0, X_0) &= \int_0^t \left(\frac{\partial u}{\partial t}(s, X_s) + b(s, X_s) \frac{\partial u}{\partial x}(s, X_s) \right) ds \\ &\quad + \int_0^t \left(h(s, X_s) \frac{\partial u}{\partial x}(s, X_s) + \frac{1}{2} \sigma^2(s, X_s) \frac{\partial^2 u}{\partial x^2}(s, X_s) \right) d\langle B \rangle_s \\ &\quad + \int_0^t \sigma(s, X_s) \frac{\partial u}{\partial x}(s, X_s) dB_s. \end{aligned} \quad (2.34)$$

In fact, the G -Itô formula was first proved for the function $u(t, x) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ with $\frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x) \in \mathcal{C}_{B.Lip}([0, T] \times \mathbb{R})$, see e.g. [80, 77]. Later on, in [98], the authors give an extension that applies for the function $u(t, x) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$, and only requires the derivatives $\frac{\partial u}{\partial x}(t, x)$ to be Lipschitz continuous in $x \in \mathbb{R}$ with $\frac{\partial^2 u}{\partial x^2}(t, x)$ is uniformly continuous in $x \in \mathbb{R}$. Eventually, in [57], Li and Peng relaxed all Lipschitz and uniform continuity conditions to provide a general G -Itô formula stated in Theorem 2.5.16 as above.

Chapter 3

Monotonicity, Convexity and Continuous Dependence of FBSDEs and PDEs

This chapter is devoted to proving some necessary results on monotonicity, convexity and continuous dependence property of solutions of the FBSDEs, G -FBSDEs and corresponding solutions of the associated semilinear parabolic PDEs and Hamilton-Jacobi-Bellman (HJB) equations, respectively. These properties are very crucial to constitute comparison results for the monotonic, convex, increasing convex g - and G -stochastic orderings presented in Chapter 4, Chapter 5 and Chapter 6 later. We divide Chapter 3 into two parts. The first part consists of Section 3.1-3.2 in which we study FBSDEs driven by a standard Brownian motion and corresponding semilinear parabolic PDEs. Firstly, in Section 3.1.1, we provide an explicit solution for the FBSDE with a linear generator, which is generalized from the linear BSDE. This result is needed for proving monotonicity of solutions of FBSDEs as shown in Section 3.2.1. Next, in Section 3.1.2, we state some theorems under smoothness and boundedness assumptions on coefficients to ensure that semilinear parabolic PDEs admit classically smooth solutions. Later on, those theorems together with the continuous dependence property given in Section 3.2.2, are applied to obtain convexity of solutions of the mentioned PDEs as presented in Section 3.2.3. In the second part of Chapter 3, we move on to study FBSDEs driven by G -Brownian motion (G -FBSDEs) and corresponding G -PDEs (HJB-type equations), which are introduced in Section 3.3. As the same first part, we establish very important results on monotonicity, convexity and continuous dependence property of solutions of G -FBSDEs and associated HJB equations presented in the last Section 3.4.

3.1 Forward-Backward SDEs and Associated Parabolic PDEs

3.1.1 Forward-Backward SDEs

Consider a forward SDE of the form

$$dX_s^{t,x} = \mu(s, X_s^{t,x})dt + \sigma(s, X_s^{t,x})dW_s, \quad 0 \leq t \leq s \leq T, \quad (3.1)$$

with initial condition $X_t^{t,x} = x$, and whose coefficients are assumed throughout this dissertation to satisfy the condition (A.3.1) as follows:

(A.3.1) For every $t \in [0, T]$, the functions $x \mapsto \mu(t, x)$ and $x \mapsto \sigma(t, x)$ are globally Lipschitz, i.e.

$$|\mu(t, x) - \mu(t, y)| \leq C|x - y| \quad \text{and} \quad |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|, \quad x, y \in \mathbb{R},$$

In particular, $x \mapsto \mu(t, x)$ and $x \mapsto \sigma(t, x)$ satisfy the linear growth conditions

$$|\mu(t, x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma(t, x)| \leq C(1 + |x|), \quad x \in \mathbb{R},$$

for some positive constant $C > 0$.

The associated backward SDE is defined by

$$Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T g(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})d\tau - \int_s^T Z_\tau^{t,x}dW_\tau, \quad 0 \leq t \leq s \leq T, \quad (3.2)$$

with terminal condition $Y_T^{t,x} = \phi(X_T^{t,x}) \in L^2(\Omega, \mathcal{F}_T)$, where the generator $g(\cdot, x, y, z)$ of (3.2) is an $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process in $L^2(\Omega \times [0, T])$ for all $x, y, z \in \mathbb{R}$, which satisfies the following conditions denoted by (A.3.2)-(A.3.3).

(A.3.2) The function $g(t, x, y, z)$ is uniformly Lipschitz in (x, y, z) , i.e., there exists $C > 0$ such that

$$|g(t, x_2, y_2, z_2) - g(t, x_1, y_1, z_1)| \leq C(|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|)$$

$$\text{a.s., } x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}, t \in [0, T],$$

(A.3.3) $g(\cdot, x, 0, 0) = 0$ a.s. for all $x \in \mathbb{R}$.

By Theorem 2.1 of [27], see also Proposition 2.2 of [72], under (A.3.2)-(A.3.3) there exists a unique pair $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$ of adapted processes in $L^2(\Omega \times [0, T])$ that solves the BSDE (3.2). Note that the Condition (A.3.3) is just a particular case and a need for defining g -evaluation in Chapter 4 later. In general, it is enough to assume that $g(\cdot, x, 0, 0) \in L^2(\Omega \times [0, T])$ for all $x \in \mathbb{R}$.

In the sequel, we will also consider the following condition for the function ϕ :

(A.3.4) The function ϕ is continuous on $\mathbb{R}$ and has the polynomial growth

$$|\phi(x)| \leq C(1 + |x|^p), \quad x \in \mathbb{R}, \text{ for some } p \geq 1/2 \text{ and } C > 0. \quad (3.3)$$

Linear FBSDEs

In this subsection, we consider a forward SDE which is associated with a backward SDE of linear generator. For a simplicity, we drop the superscript x of $(X_t^x)_{t \in [0, T]}$. Here we let $(X_t)_{t \in [0, T]}$ satisfy the forward diffusion equation

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad (3.4)$$

where μ and σ satisfy (A.3.1), with an associated linear backward SDE

$$dY_t = -(a_t X_t + b_t Y_t + c_t Z_t + k_t) dt + Z_t dW_t, \quad (3.5)$$

with terminal condition $Y_T = \phi(X_T)$, where $(a_t)_{t \in [0, T]}$, $(b_t)_{t \in [0, T]}$ and $(c_t)_{t \in [0, T]}$ are real-valued, $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted bounded processes, and $(k_t)_{t \in [0, T]}$ is a real-valued $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process such that $\mathbb{E} \left[\int_0^T k_t^2 dt \right] < \infty$.

The following Lemma 3.1.1, which will be used in the proof of Proposition 3.2.2 later, extends a classical result from linear BSDEs to linear FBSDEs.

Lemma 3.1.1. *Let $(X_t)_{t \in [0, T]}$ be the solution of (3.4), and let $(Y_t, Z_t)_{t \in [0, T]}$ be the solution of (3.5). Then the process $(Y_t)_{t \in [0, T]}$ is given in an explicit form as*

$$Y_t = \frac{1}{\Gamma_t} \mathbb{E} \left[\Gamma_T \phi(X_T) + \int_t^T (a_s X_s + k_s) \Gamma_s ds \mid \mathcal{F}_t \right],$$

where $(\Gamma_t)_{t \in [0, T]}$ is the geometric Brownian motion

$$\Gamma_t := \exp \left(\int_0^t b_s ds + \int_0^t c_s dW_s - \frac{1}{2} \int_0^t c_s^2 ds \right), \quad t \in [0, T].$$

Proof. We have

$$\begin{aligned}
d(\Gamma_s Y_s) &= \Gamma_s dY_s + Y_s d\Gamma_s + d\langle \Gamma_s, Y_s \rangle \\
&= \Gamma_s (- (a_s X_s + b_s Y_s + c_s Z_s + k_s) ds + Z_s dW_s) + Y_s \Gamma_s (b_s ds + c_s dW_s) \\
&\quad + c_s Z_s \Gamma_s ds \\
&= - (a_s X_s \Gamma_s + k_s \Gamma_s) ds + (c_s Y_s \Gamma_s + Z_s \Gamma_s) dW_s.
\end{aligned}$$

Hence,

$$\Gamma_T Y_T - \Gamma_t Y_t = - \int_t^T (a_s X_s \Gamma_s + k_s \Gamma_s) ds + \int_t^T (c_s Y_s \Gamma_s + Z_s \Gamma_s) dB_s, \quad (3.6)$$

and by taking conditional expectation on both sides of (3.6), we find

$$\begin{aligned}
Y_t &= \mathbb{E}[Y_t | \mathcal{F}_t] \\
&= \frac{1}{\Gamma_t} \mathbb{E}[\Gamma_T Y_T | \mathcal{F}_t] + \frac{1}{\Gamma_t} \mathbb{E} \left[\int_t^T (a_s X_s \Gamma_s + k_s \Gamma_s) ds \mid \mathcal{F}_t \right] \\
&= \frac{1}{\Gamma_t} \mathbb{E} \left[\Gamma_T \phi(X_T) + \int_t^T (a_s X_s + k_s) \Gamma_s ds \mid \mathcal{F}_t \right], \quad t \in [0, T].
\end{aligned}$$

□

Remark 3.1.2. We note that Lemma 3.1.1 also holds for the case of the BSDE (3.5) which is nonlinear in $(X_t)_{t \geq 0}$.

3.1.2 Associated Parabolic PDEs

Throughout the remaining of this chapter, we assume that $g(t, x, y, z)$ is a deterministic function, in addition to (A.3.1)-(A.3.3). The function $u(t, x) := Y_t^{t, x}$ can be shown to be a viscosity solution of the backward PDE

$$\frac{\partial u}{\partial t}(t, x) + \mu(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 u}{\partial x^2}(t, x) + g\left(t, x, u(t, x), \sigma(t, x) \frac{\partial u}{\partial x}(t, x)\right) = 0, \quad (3.7)$$

$x \in \mathbb{R}$, $t \in [0, T]$, with a terminal condition $u(T, x) = \phi(x)$ satisfying (A.3.4), see Theorem 2.2 in [71], Theorem 4.3 of [73] and Theorem 4.2 of [27].

In the sequel, we let $\mathcal{C}^{p, q}([0, T] \times \mathbb{R})$ denote the space of functions $f(t, x)$ which are p times continuously differentiable in $t \in [0, T]$, $p \geq 1$, and q times differentiable in $x \in \mathbb{R}$, $q \geq 1$. We also let $\mathcal{C}_b^k(\mathbb{R}^n)$ denote the space of continuously differentiable

functions whose partial derivatives of orders one to k are uniformly bounded on $\mathbb{R}^n$. In Theorem 3.1.3 below, we state an existence result for classical solutions under stronger smoothness assumptions on BSDE coefficients, see Theorem 3.2 of [73], Theorem 8.1 in § V.8 page 495, and Theorem 7.1 in § VII.7 page 596 of [53].

Theorem 3.1.3. *Assume (A.3.3) and in addition that $\mu(t, \cdot)$, $\sigma(t, \cdot)$, $\phi \in \mathcal{C}_b^3(\mathbb{R})$, and that $g(t, \cdot, \cdot, \cdot) \in \mathcal{C}_b^3(\mathbb{R}^3)$ for any $t \in [0, T]$. Then the function $u(t, x) := Y_t^{t,x}$ is a classical solution in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ of the backward PDE (3.7) with terminal condition $u(T, \cdot) = \phi$.*

Under the conditions of Theorem 3.1.3, by Proposition 4.3 of [27] the solution $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$ of (3.2) satisfies $Y_s^{t,x} = u(s, X_s^{t,x})$ and $Z_s^{t,x} = \sigma(s, X_s^{t,x}) \frac{\partial u}{\partial x}(s, X_s^{t,x})$, $0 \leq t \leq s \leq T$. In addition, by Theorem 2.2 in [65] or Proposition 3.3 in [64], we have the following result.

Theorem 3.1.4. *Under the assumptions of Theorem 3.1.3, suppose additionally that $\sigma(t, x)$ is bounded above and below by strictly positive constants. Then the first derivative in $t \in [0, T]$ and the first and second derivatives in $x \in \mathbb{R}$ of $u(t, x)$ are bounded for all $(t, x) \in [0, T] \times \mathbb{R}$.*

As in [23], we denote by $\mathcal{C}^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$, $\eta \in (0, 1)$, the space of functions $f(t, x)$ which are differentiable in $t \in [0, T]$ and twice differentiable in $x \in \mathbb{R}$ with $\frac{\partial f}{\partial t}(t, x)$ and $\frac{\partial^2 f}{\partial x^2}(t, x)$ being respectively $\eta/2$ -Hölder continuous and η -Hölder continuous in $(t, x) \in [0, T] \times \mathbb{R}$, and define the space $\mathcal{C}^{k+\eta}(\mathbb{R})$ analogously for $k \geq 1$. By Theorem 2.3 in [23], see also page 236 of [65], we have the following result.

Theorem 3.1.5. *In addition to the assumptions of Theorem 3.1.4, suppose that for some $\eta \in (0, 1)$ the functions $\mu(\cdot, \cdot)$, $\sigma(\cdot, \cdot)$ and $g(\cdot, \cdot, y, z)$ are in $\mathcal{C}^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$ for all $y, z \in \mathbb{R}$, and that $\phi \in \mathcal{C}^{4+\eta}(\mathbb{R})$. Then the function $u(t, x)$ is a classical solution in $\mathcal{C}^{2+\eta/2, 4+\eta}([0, T] \times \mathbb{R})$ of the backward PDE (3.7) with terminal condition $u(T, \cdot) = \phi$.*

3.2 Monotonicity, Convexity and Continuous Dependence Results on FBSDEs and PDEs

3.2.1 Monotonicity of FBSDE Solutions

In Proposition 3.2.1 and Proposition 3.2.2, we prove the monotonicity results needed in the proofs of Theorem 4.4.2, Corollaries 4.4.3-4.4.4, 4.4.6 and Theorem 4.4.8 in

Chapter 4. We apply Lemma 3.1.1 to derive monotonicity results for FBSDE flows of the form

$$dX_s^{t,x} = \mu(t, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \quad X_t^{t,x} = x, \quad (3.8a)$$

$$dY_s^{t,x} = -g(s, X_s^{s,x}, Y_s^{t,x}, Z_s^{t,x})ds + Z_s^{t,x}dW_s, \quad Y_T^{t,x} = \phi(X_T^{t,x}), \quad (3.8b)$$

$0 \leq t \leq s \leq T$. We first prove a monotonicity result for the solution $(X_s^{t,x})_{s \in [t, T]}$ of the SDE (3.8a), which will be used to prove non-decreasing property of $Y_s^{t,x}$ and $u(t, x)$ in Proposition 3.2.2.

Proposition 3.2.1. *Under the assumption (A.3.1), the solution $(X_s^{t,x})_{s \in [t, T]}$ of (3.8a) is a.s. non-decreasing in x for all $t \in [0, T]$ and $s \in [t, T]$.*

Proof. Let $\hat{X}_s^{t,x,y} := X_s^{t,y} - X_s^{t,x}$ for $x \leq y$, $s \in [t, T]$, and consider the processes

$$\hat{\mu}_u := \frac{\mu(u, X_u^{t,y}) - \mu(u, X_u^{t,x})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}} \quad \text{and} \quad \hat{\sigma}_u := \frac{\sigma(u, X_u^{t,y}) - \sigma(u, X_u^{t,x})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}},$$

$u \in [t, T]$. We note that the processes $(\hat{\mu}_u)_{u \in [t, T]}$ and $(\hat{\sigma}_u)_{u \in [t, T]}$ are bounded since $\mu(t, x)$ and $\sigma(t, x)$ are Lipschitz in x , and $(\hat{X}_s^{t,x,y})_{s \in [t, T]}$ satisfies the equation

$$\hat{X}_s^{t,x,y} = y - x + \int_t^s \hat{\mu}_u \hat{X}_u^{t,x,y} du + \int_t^s \hat{\sigma}_u \hat{X}_u^{t,x,y} dW_u, \quad s \in [t, T],$$

which yields

$$\hat{X}_s^{t,x,y} = (y - x) \exp \left(\int_t^s \hat{\mu}_u du + \int_t^s \hat{\sigma}_u dW_u - \frac{1}{2} \int_t^s \hat{\sigma}_u^2 du \right) \geq 0, \quad 0 \leq t \leq s \leq T.$$

□

The next monotonicity result is used for the proofs of Theorem 4.4.2, Corollaries 4.4.3-4.4.4, 4.4.6 and Theorem 4.4.8 in Chapter 4.

Proposition 3.2.2. *Assume that the coefficients μ , σ , g and ϕ satisfy (A₁)-(A.3.4). If $\phi(x)$ and $g(t, x, y, z)$ are non-decreasing in $x \in \mathbb{R}$ for all $t \in [0, T]$ and $y, z \in \mathbb{R}$, then the solution $(Y_s^{t,x})_{s \in [t, T]}$ of (3.8b) is a.s. non-decreasing in x for all $s \in [t, T]$. As a consequence, if $u(t, x)$ is solution of the backward PDEs (3.7), then $u(t, x)$ is also a non-decreasing function of $x \in \mathbb{R}$ for all $t \in [0, T]$.*

Proof. Letting $\widehat{X}_s = X_s^{t,y} - X_s^{t,x}$, $\widehat{Y}_s = Y_s^{t,y} - Y_s^{t,x}$, $\widehat{Z}_s = Z_s^{t,y} - Z_s^{t,x}$ and $\widehat{Y}_T = \phi_2(X_T^{t,y}) - \phi_1(X_T^{t,x})$, we have

$$\widehat{Y}_s = \widehat{Y}_T + \int_s^T (g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})) du - \int_s^T \widehat{Z}_u dW_u.$$

Defining the processes $(a_u)_{u \in [0, T]}$, $(b_u)_{u \in [0, T]}$, and $(c_u)_{u \in [0, T]}$ as

$$\begin{aligned} a_u &:= \frac{g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}}, \\ b_u &:= \frac{g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y})}{Y_u^{t,y} - Y_u^{t,x}} \mathbf{1}_{\{Y_u^{t,y} \neq Y_u^{t,x}\}}, \\ c_u &:= \frac{g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})}{Z_u^{t,y} - Z_u^{t,x}} \mathbf{1}_{\{Z_u^{t,y} \neq Z_u^{t,x}\}}, \quad u \in [0, T], \end{aligned}$$

which are $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted and bounded since $g(u, x, y, z)$ is Lipschitz, and using the decomposition

$$\begin{aligned} g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) \\ = g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) \\ + g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) \\ + g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}), \end{aligned}$$

we have

$$\widehat{Y}_s = \widehat{Y}_T + \int_s^T (a_u \widehat{X}_u + b_u \widehat{Y}_u + c_u \widehat{Z}_u) du - \int_s^T \widehat{Z}_u dW_u, \quad s \in [0, T].$$

Hence, by Lemma 3.1.1, we get

$$\widehat{Y}_s = \frac{1}{\Gamma_s} \mathbb{E} \left[\Gamma_T \widehat{Y}_T + \int_s^T a_u \widehat{X}_u \Gamma_u du \mid \mathcal{F}_s \right], \quad s \in [t, T], \quad (3.9)$$

where

$$\Gamma_s := \exp \left(\int_t^s c_u dW_u - \frac{1}{2} \int_t^s c_u^2 du + \int_0^s b_u du \right), \quad s \in [t, T].$$

By Proposition 3.2.1 the solution $(X_s^{t,x})_{s \in [t, T]}$ of the forward SDE (3.8a) satisfies $\widehat{X}_s = X_s^{t,y} - X_s^{t,x} \geq 0$ for all $s \in [t, T]$ if $x \leq y$, and since $g(s, x, y, z)$ is non-decreasing in x we have $a_s \geq 0$ a.s., $s \in [t, T]$. Since $\phi(x)$ is non-decreasing, we have $\widehat{Y}_T = \phi(X_T^{t,y}) - \phi(X_T^{t,x}) \geq 0$, a.s.. Hence, by (3.9) we have $\widehat{Y}_s = Y_s^{t,x} - Y_s^{t,y} \geq 0$, $s \in [t, T]$, if $x \leq y$, which implies the monotonicity of $(Y_s^{t,x})_{s \in [t, T]}$ with respect to $x \in \mathbb{R}$. Therefore, we also get $u(t, x) \leq u(t, y)$, $x \leq y$, $t \in [0, T]$, since $u(t, x) = Y_t^{t,x}$. $\square$

3.2.2 Continuous Dependence of FBSDE Solutions

The next Proposition 3.2.3 result extends the argument of Theorem 9.7 in [67] to the setting of FBSDEs. Other continuous dependence results are available in the literature such as Theorem 3.3 of [47], which however requires uniform estimates on coefficients.

Proposition 3.2.3. *Consider the family of forward-backward stochastic differential equations*

$$\begin{cases} X_{n,t} = x_{n,0} + \int_0^t \mu_n(s, X_{n,s})ds + \int_0^t \sigma_n(s, X_{n,s})dW_s, \\ Y_{n,t} = \phi_n(X_{n,T}) + \int_t^T g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s})ds - \int_t^T Z_{n,s}dW_s, \end{cases}$$

where, for every $n \geq 1$, $X_{n,0} = x_{n,0} \in \mathbb{R}$, and the coefficients μ_n , σ_n , g_n and ϕ_n satisfy (A₁)-(A.3.4) for a same $C > 0$. Assume the convergence $x_{n,0} \rightarrow x_0$ and pointwise convergences

$$\mu_n(t, x) \rightarrow \mu(t, x), \quad \sigma_n(t, x) \rightarrow \sigma(t, x), \quad g_n(t, x, y, z) \rightarrow g(t, x, y, z),$$

for all $t \in [0, T]$, and $x, y, z \in \mathbb{R}$ as $n \rightarrow \infty$, and the strong convergence $\phi_n(x_n) \rightarrow \phi(x)$ whenever $x_n \rightarrow x \in \mathbb{R}$, where μ , σ , g and ϕ satisfy (A.3.1)-(A.3.4) for a same constant $C > 0$. Then for all $t \in [0, T]$, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} [|Y_{n,t} - Y_t|^2] = 0,$$

where $(Y_t)_{t \in \mathbb{R}_+}$ is solution of the FBSDE

$$\begin{aligned} X_t &= x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\ Y_t &= \phi(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{aligned}$$

Proof. For $n \geq 1$, let

$$\hat{Y}_{n,t} := Y_{n,t} - Y_t, \quad \hat{X}_{n,t} := X_{n,t} - X_t, \quad \hat{Z}_{n,t} := Z_{n,t} - Z_t, \quad \text{and} \quad \hat{\phi}_n(x) := \phi_n(x) - \phi(x),$$

$t \in [0, T]$, $x \in \mathbb{R}$. Applying the Itô formula to $|\hat{Y}_{n,t}|^2$ and taking expectation on both sides yields

$$\mathbb{E} [|\hat{Y}_{n,t}|^2] = \mathbb{E} [(\phi_n(X_{n,T}) - \phi(X_T))^2] - \mathbb{E} \left[\int_t^T |\hat{Z}_{n,s}|^2 ds \right]$$

$$+2 \mathbb{E} \left[\int_t^T \widehat{Y}_{n,s} (g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s)) ds \right]$$

By the inequality $2ab \leq (6C^2)a^2 + b^2/(6C^2)$, we have

$$\begin{aligned} 2\widehat{Y}_{n,s}(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s)) &\leq 6C^2|\widehat{Y}_{n,s}|^2 \\ &\quad + \frac{1}{6C^2}(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))^2. \end{aligned}$$

Next, letting

$$\widehat{g}_n(t, x, y, z) := g_n(t, x, y, z) - g(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

we have

$$\begin{aligned} &(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))^2 \\ &\leq 2(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g_n(s, X_s, Y_s, Z_s))^2 + 2(\widehat{g}_n(s, X_s, Y_s, Z_s))^2 \\ &\leq 2C^2(|X_{n,s} - X_s| + |Y_{n,s} - Y_s| + |Z_{n,s} - Z_s|)^2 + 2(\widehat{g}_n(s, X_s, Y_s, Z_s))^2 \\ &\leq 6C^2(|\widehat{X}_{n,s}|^2 + |\widehat{Y}_{n,s}|^2 + |\widehat{Z}_{n,s}|^2) + 2(\widehat{g}_n(s, X_s, Y_s, Z_s))^2, \end{aligned}$$

by the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$. Combining the above estimates, we find

$$\begin{aligned} \mathbb{E} [|\widehat{Y}_{n,t}|^2] &\leq \mathbb{E} [(\phi_n(X_{n,T}) - \phi(X_T))^2] \\ &\quad + (6C^2 + 1) \mathbb{E} \left[\int_t^T |\widehat{Y}_{n,s}|^2 ds \right] + \mathbb{E} \left[\int_t^T |\widehat{X}_{n,s}|^2 ds \right] \\ &\quad + \frac{1}{3C^2} \mathbb{E} \left[\int_t^T (\widehat{g}_n(s, X_s, Y_s, Z_s))^2 ds \right], \end{aligned}$$

which yields

$$\begin{aligned} \mathbb{E} [|\widehat{Y}_{n,t}|^2] &\leq C'' \left(\mathbb{E} [(\phi_n(X_{n,T}) - \phi(X_T))^2] + \int_t^T \mathbb{E} [|\widehat{X}_{n,s}|^2] ds \right. \\ &\quad \left. + \mathbb{E} \left[\int_t^T (\widehat{g}_n(s, X_s, Y_s, Z_s))^2 ds \right] \right) \end{aligned}$$

by Gronwall's inequality, and therefore

$$\begin{aligned} \mathbb{E} [|\widehat{Y}_{n,t}|^2] &\leq C'' \mathbb{E} [(\phi_n(X_{n,T}) - \phi(X_T))^2] \\ &\quad + C''(T - t) \sup_{s \in [t, T]} \mathbb{E} [|\widehat{X}_{n,s}|^2] + C'' \int_t^T \mathbb{E} [(\widehat{g}_n(s, X_s, Y_s, Z_s))^2] ds. \end{aligned}$$

We note that since $x_{n,0} \rightarrow x_0$, and $\mu_n(t, x) \rightarrow \mu$, $\sigma_n(t, x) \rightarrow \sigma(t, x)$ pointwise when $n \rightarrow \infty$, by Theorem 9.7 of [67] we have $\lim_{n \rightarrow \infty} \mathbb{E} [\sup_{t \in [0, T]} |\hat{X}_{n,t}|^2] = 0$. Hence, by the condition $|\hat{g}_n(t, x, y, z)| \leq 2C(|x| + |y| + |z|)$ and the pointwise limit $\lim_{n \rightarrow \infty} \hat{g}_n(t, x, y, z) = 0$, by the Lebesgue dominated convergence we find

$$\lim_{n \rightarrow \infty} \int_t^T \mathbb{E} [(\hat{g}_n(s, X_s, Y_s, Z_s))^2] ds = 0, \quad t \in [0, T].$$

Finally, by the strong convergence of $(\phi_n)_{n \geq 1}$ to ϕ and the uniform integrability

$$\sup_{n \geq 1} \mathbb{E} \left[\sup_{t \in [0, T]} |X_{n,t}|^{2p} \right] < \infty, \quad p \geq 1,$$

see Theorem 9.2 in [67], we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E} [(\phi_n(X_{n,T}) - \phi(X_T))^2] = 0,$$

and we conclude to

$$\lim_{n \rightarrow \infty} \mathbb{E} [|Y_{n,t} - Y_t|^2] = 0, \quad t \in [0, T].$$

The proof is complete. □

Remark 3.2.4. *From the point of view of Proposition 3.2.3, $Y_{n,t}$ converges to Y_t in mean square for each $t \in [0, T]$. This implies that $Y_{n,t}$ converges to Y_t in probability for each $t \in [0, T]$. Hence, there exists a subsequence $Y_{n_k,t}$ of $Y_{n,t}$ which almost surely converges to Y_t for each $t \in [0, T]$. We frequently apply this argument for approximations in later chapters.*

3.2.3 Convexity of Semilinear Parabolic PDE Solutions

In this section, we extend the convexity result Theorem 1.1 in [11] for semilinear PDEs under a weaker convexity condition on the nonlinear drift $(x, y, z) \mapsto f(t, x, y, z)$ in the one-dimensional case, as required by applications in finance, see Examples 4.6.3-4.6.5. For this we remark that, in our one-dimensional setting, the constant rank Theorem 2.3 in [11], see also Theorem 1.2 in [12], only requires convexity of the nonlinear drift $f(t, x, y, z)$ in $(x, y) \in \mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$, instead of global convexity in (x, y, z) . Precisely, we note that Condition (2.6) in Theorem 2.3 of [11] reduces to (3.12) below.

Theorem 3.2.5. (Theorem 2.3, [11]). Assume that $u(t, x)$ is a $\mathcal{C}^{2,4}([0, T] \times \mathbb{R})$ convex solution of the PDE

$$\frac{\partial u}{\partial t}(t, x) + F\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\right) = 0, \quad (3.11)$$

and that $F(t, x, y, z, w)$ is a $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^4)$ function that satisfies the elliptic condition

$$\frac{\partial F}{\partial w}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\right) > 0, \quad t \in [0, T], \quad x \in \mathbb{R},$$

and

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\right) + 2b \frac{\partial^2 F}{\partial x \partial y}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\right) \\ + b^2 \frac{\partial^2 F}{\partial y^2}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\right) \geq 0, \end{aligned} \quad (3.12)$$

for all $t \in [0, T]$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. Then the sign $\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right)$ of $\frac{\partial^2 u}{\partial x^2}(t, x)$ is constant in $x \in \mathbb{R}$ for any $t \in (0, T)$, and we have

$$\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(s, x)\right) \geq \operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right), \quad 0 \leq s \leq t < T.$$

We note that in our one dimensional setting, the rank of $\frac{\partial^2 u}{\partial x^2}(t, x)$ is $\{0, 1\}$ -valued, with the relation $\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right) \equiv \operatorname{rank}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right)$ provided that $x \mapsto u(t, x)$ convex, $t \in [0, T]$, $x \in \mathbb{R}$. By adapting arguments of [11], using the constant rank Theorem 3.2.5 and a new Lemma 3.2.7, we will prove the following Theorem 3.2.6 which will be used in the proofs of Theorems 4.4.1-4.4.2, Corollaries 4.4.4-4.4.5, and Theorems 4.4.7-4.4.8 and Corollaries 4.5.2-4.5.3.

We note that by Theorem 2.3 and Condition (2.6) in [11], convexity of the solution $u(t, x)$ of the PDE (3.7) is ensured by the joint convexity of $(x, y, z) \mapsto f(t, x, y, z) := F(t, x, y, z, 0)$, where $F(t, x, y, z, w)$ is the semilinear function

$$F(t, x, y, z, w) := z\mu(t, x) + g(t, x, y, z\sigma(t, x)) + \frac{w}{2}\sigma^2(t, x).$$

However, this joint convexity condition is too strong for our applications in mathematical finance, and in Theorem 3.2.6 we show that it can be relaxed into the partial convexity of $F(t, x, y, z, 0)$ in (x, y) and in (y, z) . Namely, Theorem 3.2.6 extends Theorem 1.1 of [11] on the convexity of the solution $u(t, x)$ of the PDE (3.7) by

only assuming convexity in (x, y) and in (y, z) of the function $f(t, x, y, z)$ in (3.13) below, instead of joint convexity in $(x, y, z) \mapsto f(t, x, y, z)$. For this, we start by assuming additional regularity conditions on coefficients, and then use approximation arguments with help of the continuous dependence (stability) Theorem 3.2.3, see Lemma 3.2.7.

Theorem 3.2.6. *Assume that the coefficients μ, σ, g and ϕ satisfy (A.3.1)-(A.3.4). Suppose that $u(t, x)$ is a $C^{1,2}([0, T] \times \mathbb{R})$ solution of (3.7) with terminal condition $u(T, x) = \phi(x)$, and that the function*

$$f(t, x, y, z) := z\mu(t, x) + g(t, x, y, z\sigma(t, x)) \quad (3.13)$$

satisfies the following conditions:

(H₁) $(x, y) \mapsto f(t, x, y, z)$ is convex on $\mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$,

(H₂) $(y, z) \mapsto f(t, x, y, z)$ is convex on $\mathbb{R}^2$ for every $(t, x) \in [0, T] \times \mathbb{R}$.

Then the function $x \mapsto u(t, x)$ is convex on $\mathbb{R}$ for all $t \in [0, T]$, provided that $u(T, x) = \phi(x)$ convex in $x \in \mathbb{R}$.

Proof. We proceed by extending the proof argument of Theorem 1.1 in [11] by using Theorem 3.2.5 and an approximation argument. We start by assuming that the following conditions, which are stronger than (A.3.1)-(A.3.4), hold for some $\eta \in (0, 1)$.

(H₃) $\mu(\cdot, \cdot), \sigma(\cdot, \cdot), g(\cdot, \cdot, y, z) \in C_b^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$ for all $y, z \in \mathbb{R}$, and $\phi(\cdot) \in C_b^{4+\eta}(\mathbb{R})$,

(H₄) $\sigma(\cdot, \cdot)$ satisfies the bound

$$0 < c \leq \sigma(t, x) \leq C, \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (3.14)$$

for some constants $c, C > 0$.

(H₅) For some $C > 0$ and $\alpha > 0$ we have

$$\left| \frac{\partial^2 g}{\partial z^2}(t, x, y, z) \right| \leq C(1 + x^2)^{-\alpha}, \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^3.$$

Under Conditions (H₃)-(H₄) the function $F(t, y, z, w)$ defined as

$$F(t, x, y, z, w) := f(t, x, y, z) + \frac{w}{2}\sigma^2(t, x), \quad t \in [0, T], \quad x, y, z, w \in \mathbb{R},$$

is in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^4)$ and the solution $u(t, x)$ of (3.11) is in $\mathcal{C}^{2,4}([0, T] \times \mathbb{R})$ by Theorem 3.1.3. Besides, we note that by Theorem 3.1.5, the first and the second partial derivatives of $u(t, x)$ with respect to x are bounded uniformly in $(t, x) \in [0, T] \times \mathbb{R}$. Setting $h(x) := (1 + x^2)^{\alpha+1}$, for any $K \in \mathbb{R}$ and $\varepsilon > 0$, we define

$$v_K(t, x) := e^{-Kt}h(x) \quad \text{and} \quad u_\varepsilon(t, x) := u(t, x) + \varepsilon v_K(t, x), \quad x \in \mathbb{R}, \quad t \in [0, T].$$

Next, we let

$$E_\varepsilon := \left\{ (t, x) \in [0, T] \times \mathbb{R} : \frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) \leq 0 \right\}$$

and suppose that $E_\varepsilon \neq \emptyset$. From the relation $h''(x) \geq (1 + x^2)^\alpha$ and the bound $\left| \frac{\partial^2 u}{\partial x^2}(t, x) \right| \leq C$ we get

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) \geq \varepsilon e^{-Kt}(1 + x^2)^\alpha - C,$$

therefore there exists $R_\varepsilon > 0$ such that $\frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) > 0$ for all $|x| \geq R_\varepsilon$, and we have $E_\varepsilon \subseteq [0, T] \times B(R_\varepsilon)$, where $B(R_\varepsilon)$ is the centered open ball with radius R_ε , so that E_ε is compact. Consequently, the supremum

$$\tau_0 := \sup\{t \in [0, T] : (t, x) \in E_\varepsilon \text{ for some } x \in \mathbb{R}\}$$

is attained at some $(\tau_0, x_0) \in E_\varepsilon$ with $x_0 \in B(R_\varepsilon)$, such that $\frac{\partial^2 u_\varepsilon}{\partial x^2}(\tau_0, x_0) \leq 0$. In addition, by the convexity assumption on $x \mapsto u(T, x)$ we have

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}(T, x) = \frac{\partial^2 u}{\partial x^2}(T, x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(T, x) \geq \varepsilon e^{-KT}h''(x) > 0, \quad x \in \mathbb{R},$$

hence $\tau_0 < T$ and by the continuity of u_ε we have $\frac{\partial^2 u_\varepsilon}{\partial x^2}(\tau_0, x) \geq 0$, $x \in \mathbb{R}$, since $\frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) > 0$ for all $t \in (\tau_0, T)$ and $x \in \mathbb{R}$. Consequently, the function $u_\varepsilon(t, x)$ is convex in x on $[\tau_0, T] \times B(R_\varepsilon)$.

On the other hand, we note that $\frac{\partial^2 u_\varepsilon}{\partial x^2}(\tau_0, x_0) = 0$ for $x_0 \in B(R_\varepsilon)$, and that $u_\varepsilon(t, x)$ satisfies the equation

$$\frac{\partial u_\varepsilon}{\partial t}(t, x) + F_{K, \varepsilon} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), \frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) \right) = 0,$$

where

$$F_{K,\varepsilon}(t, x, y, z, w) := -\varepsilon \frac{\partial v_K}{\partial t}(t, x) + \frac{1}{2} \sigma^2(t, x) \left(w - \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \quad (3.15) \\ + f \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x) \right).$$

By the constant rank Theorem 3.2.5 and Lemma 3.2.7 below, we deduce that $\frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x)$ vanishes on $[\tau_0, T) \times B(R_\varepsilon)$ hence $\tau_0 = T$, which is a contradiction showing that $E_\varepsilon = \emptyset$. Therefore, we have

$$\frac{\partial^2 u}{\partial x^2}(t, x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) = \frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) > 0, \quad (t, x) \in [0, T] \times \mathbb{R},$$

and after letting ε tend to 0, we conclude that

$$\frac{\partial^2 u}{\partial x^2}(t, x) \geq 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Finally, we relax the above conditions (H_3) – (H_5) under the hypotheses (A.3.1)–(A.3.4) by applying the above argument to sequences $(\mu_n)_{n \geq 1}$, $(\sigma_n)_{n \geq 1}$, $(g_n)_{n \geq 1}$ of $\mathcal{C}_b^{2,3}$ functions and $(\phi_n)_{n \geq 1}$ of $\mathcal{C}_b^5$ functions satisfying (H_3) and (A.3.1)–(A.3.4), and converging pointwise respectively to μ , σ , g , and ϕ while preserving the convexity of the approximations $(\phi_n)_{n \geq 1}$ and $(f_n)_{n \geq 1}$ defined in (3.13), as well as Condition (3.14), see e.g. [4], Lemma 1 of [55], and Problem 1.4.14 in [99]. In order to satisfy (H_5) , we replace g_n with $\tilde{g}_n$ obtained by smoothing out the piecewise $\mathcal{C}^1$ function

$$x \mapsto \left(g_n(t, -n, y, z) + (x + n) \frac{\partial g_n}{\partial x}(t, -n, y, z) \right) \mathbf{1}_{(-\infty, -n)}(x) \\ + g_n(t, x, y, z) \mathbf{1}_{[-n, n]}(x) + \left(g_n(t, n, y, z) + (x - n) \frac{\partial g_n}{\partial x}(t, n, y, z) \right) \mathbf{1}_{(n, \infty)}(x),$$

by convolution in x with the Gaussian kernel $e^{-nx^2/2}/\sqrt{2\pi/n}$, and we conclude by the continuous dependence Proposition 3.2.3. $\square$

The proof of Theorem 3.2.6 relies on the following lemma.

Lemma 3.2.7. *Under conditions (H_1) – (H_5) above we can choose $K \in \mathbb{R}$ such that the function $(x, y) \mapsto F_{K,\varepsilon}(t, x, y, z, 0)$ in (3.15) satisfies Condition (3.12).*

Proof. We need to show that

$$S(b) := \frac{\partial^2 F_{K,\varepsilon}}{\partial x^2} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), 0 \right) + 2b \frac{\partial^2 F_{K,\varepsilon}}{\partial x \partial y} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), 0 \right) \quad (3.16)$$

$$\begin{aligned}
 & + b^2 \frac{\partial^2 F_{K,\varepsilon}}{\partial y^2} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), 0 \right) \\
 & \geq 0,
 \end{aligned}$$

for all $b \in \mathbb{R}$. Using (3.15) we find

$$\begin{aligned}
 S(b) &= 2 \frac{\partial^2 f}{\partial x^2} - 2 \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right) \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2} \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right)^2 \frac{\partial^2 f}{\partial y^2} \\
 &+ \frac{1}{2} \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right)^2 \frac{\partial^2 f}{\partial y^2} + 2 \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right) \frac{\partial^2 f}{\partial y \partial z} + 2 \left(\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right)^2 \frac{\partial^2 f}{\partial z^2} \\
 &- \frac{\partial^2 f}{\partial x^2} - \left(\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right)^2 \frac{\partial^2 f}{\partial z^2} - \varepsilon \left(\frac{1}{2} \frac{\partial^2 \sigma^2}{\partial x^2}(t, x) + \frac{\partial f}{\partial y} + 2 \frac{\partial^2 f}{\partial x \partial z} \right) \frac{\partial^2 v_K}{\partial x^2}(t, x) \\
 &- \varepsilon \frac{\partial^3 v_K}{\partial x^2 \partial t}(t, x) - \varepsilon \left(\frac{\partial f}{\partial z} + 2 \sigma(t, x) \frac{\partial \sigma}{\partial x}(t, x) \right) \frac{\partial^3 v_K}{\partial x^3}(t, x) - \frac{\varepsilon}{2} \sigma^2(t, x) \frac{\partial^4 v_K}{\partial x^4}(t, x),
 \end{aligned}$$

for all $b \in \mathbb{R}$, where the derivatives of f are evaluated at the point $(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x))$. Since $f(t, x, y, z)$ is convex in (x, y) , and in (y, z) , we have

$$2 \frac{\partial^2 f}{\partial x^2} - 2 \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right) \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2} \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right)^2 \frac{\partial^2 f}{\partial y^2} \geq 0,$$

and

$$\frac{1}{2} \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right)^2 \frac{\partial^2 f}{\partial y^2} + 2 \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \left(\varepsilon \frac{\partial v_K}{\partial x}(t, x) - b \right) \frac{\partial^2 f}{\partial y \partial z} + 2 \left(\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right)^2 \frac{\partial^2 f}{\partial z^2} \geq 0,$$

for all $b \in \mathbb{R}$. To conclude it suffices to show the inequality

$$\begin{aligned}
 S' &:= - \frac{\partial^2 f}{\partial x^2} - \varepsilon \left(\frac{\partial^2 v_K}{\partial x^2}(t, x) \right)^2 \frac{\partial^2 f}{\partial z^2} - \left(\frac{1}{2} \frac{\partial^2 \sigma^2}{\partial x^2}(t, x) + \frac{\partial f}{\partial y} + 2 \frac{\partial^2 f}{\partial x \partial z} \right) \frac{\partial^2 v_K}{\partial x^2}(t, x) \\
 &- \frac{\partial^3 v_K}{\partial x^2 \partial t}(t, x) - \left(\frac{\partial f}{\partial z} + 2 \sigma(t, x) \frac{\partial \sigma}{\partial x}(t, x) \right) \frac{\partial^3 v_K}{\partial x^3}(t, x) - \frac{1}{2} \sigma^2(t, x) \frac{\partial^4 v_K}{\partial x^4}(t, x) \\
 &\geq 0.
 \end{aligned}$$

Thanks to Conditions (H₃)-(H₄), we find that there exists $C > 0$ such that

$$\begin{aligned}
 \left| \frac{\partial^2 f}{\partial x^2}(t, x, y, z) \right| &\leq C(1 + z^2), & \left| \frac{\partial f}{\partial y}(t, x, y, z) \right| &\leq C, \\
 \left| \frac{\partial f}{\partial x \partial z}(t, x, y, z) \right| &\leq C(1 + |z|), & \left| \frac{\partial f}{\partial z}(t, x, y, z) \right| &\leq C(1 + |x|)
 \end{aligned}$$

for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^3$, hence

$$\left| \frac{\partial^2 f}{\partial x^2} \left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x) \right) \right| \leq C \left(1 + \left(\frac{\partial u}{\partial x}(t, x) \right)^2 \right) \leq C' \quad (3.17)$$

and

$$\left| \frac{\partial^2 f}{\partial x \partial z} \left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x) \right) \right| \leq C \left(1 + \left| \frac{\partial u}{\partial x}(t, x) \right| \right) \leq C', \quad (3.18)$$

$(t, x) \in [0, T] \times \mathbb{R}$, for some $C' > 0$. By the relation $v_K(t, x) := e^{-Kt}(1+x^2)^{\alpha+1}$ and Condition (H_5) , we find

$$\left| \left(\frac{\partial^2 v_K}{\partial x^2}(t, x) \right)^2 \frac{\partial^2 f}{\partial z^2} \left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x) \right) \right| \leq C(\alpha) e^{-2Kt} (1+x^2)^\alpha, \quad (3.19)$$

for some constant $C(\alpha) > 0$ depending on $\alpha > 0$. Next, we note that for some $C'(\alpha) > 0$ we have

$$\left| \frac{\partial^2 v_k}{\partial x^2}(t, x) \right| \leq C'(\alpha) e^{-Kt} (1+x^2)^\alpha, \quad \left| \frac{\partial^3 v_k}{\partial x^3}(t, x) \right| \leq C'(\alpha) e^{-Kt} (1+|x|)(1+x^2)^{\alpha-1},$$

and

$$\left| \frac{\partial^4 v_k}{\partial x^4}(t, x) \right| \leq C'(\alpha) e^{-Kt} (1+x^2)^{\alpha-1},$$

hence from (H_3) , (3.14), (3.18) we check that

$$\left| \frac{1}{2} \frac{\partial^2 \sigma^2}{\partial x^2} + \frac{\partial f}{\partial y} + 2 \frac{\partial^2 f}{\partial x \partial z} \right| \left| \frac{\partial^2 v_K}{\partial x^2}(t, x) \right| \leq C'(\alpha) e^{-Kt} (1+x^2)^\alpha.$$

Similarly, thanks to the conditions (H_3) and (3.14)

$$\left| \frac{\partial f}{\partial z} + 2\sigma(t, x) \frac{\partial \sigma}{\partial x}(t, x) \right| \left| \frac{\partial^3 v_K}{\partial x^3}(t, x) \right| \leq C'(\alpha) e^{-Kt} (1+x^2)^\alpha,$$

and

$$\frac{1}{2} \sigma^2(t, x) \left| \frac{\partial^4 v_k}{\partial x^4}(t, x) \right| \leq C'(\alpha) e^{-Kt} (1+x^2)^\alpha. \quad (3.20)$$

Combining (3.19)-(3.20), the inequality $\frac{\partial^3 v_k}{\partial x^2 \partial t}(t, x) \leq -KC''(\alpha) e^{-Kt} (1+x^2)^\alpha$ for some constant $C''(\alpha) > 0$, and Condition (3.17), we obtain

$$\begin{aligned} S' &\geq e^{-Kt} (KC''(\alpha) - 3C'(\alpha)) (1+x^2)^\alpha - \varepsilon C(\alpha) e^{-2Kt} (1+x^2)^\alpha - C' \\ &\geq 0, \end{aligned}$$

for $K = K(\alpha, \varepsilon)$ large enough, $t \in [0, T]$, $x \in \mathbb{R}$. $\square$

Remark 3.2.8. One can follow the procedure above to obtain a concave version of Theorem 3.2.6 for the solution $u(t, x)$ of the PDE (3.7), i.e., if we change Conditions (H_1) -(H_2) by the following conditions:

(H'_1) $(x, y) \mapsto f(t, x, y, z)$ is concave on $\mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$,

(H'_2) $(y, z) \mapsto f(t, x, y, z)$ is concave on $\mathbb{R}^2$ for every $(t, x) \in [0, T] \times \mathbb{R}$.

Then the function $x \mapsto u(t, x)$ is concave on $\mathbb{R}$ for all $t \in [0, T]$, provided that $u(T, x) = \phi(x)$ concave in $x \in \mathbb{R}$.

3.3 G -FBSDEs and Associated G -PDEs

In this section, we move to work on the G -expectation space $(\Omega, L_G^1(\Omega, \mathcal{F}), \mathbb{E}_G)$. Let $(B_t)_{t \geq 0}$ be G -Brownian motion with $-\mathbb{E}_G[-B_1^2] = \underline{\sigma}^2$ and $\mathbb{E}_G[B_1^2] = \bar{\sigma}^2$, where in one dimension, the function G has the following form

$$G(a) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \bar{\sigma}^2} (\sigma^2 a) = \frac{1}{2} (\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \quad a \in \mathbb{R}. \quad (3.21)$$

3.3.1 G -FBSDEs

We consider a G -FSDE defined on $M_G^2(0, T)$ as

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + h(s, X_s^{t,x})d\langle B \rangle_s + \sigma(s, X_s^{t,x})dB_s, \quad \text{with } X_t^{t,x} = x, \quad 0 \leq t \leq s \leq T, \quad (3.22)$$

and an associated G -BSDE

$$\begin{aligned} Y_s^{t,x} &= \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\langle B \rangle_r \\ &\quad - \int_s^T Z_r^{t,x} dB_r - (K_T^{t,x} - K_s^{t,x}), \end{aligned} \quad (3.23)$$

where $\varphi(X_T^{t,x}) \in L_G^2(\Omega, \mathcal{F}_T)$, and $K_s^{t,x}$ is a decreasing G -martingale, $b, h, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $f, g : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are deterministic functions and satisfy the following conditions:

(AA.3.1) The functions $b(t, x)$, $h(t, x)$ and $\sigma(t, x)$ are continuous in $t \in [0, T]$ and Lipschitz continuous in x , i.e., there is a constant $C > 0$ such that for all $x, y \in \mathbb{R}$,

- (i) $|b(t, x) - b(t, y)| \leq C|x - y|$,
- (ii) $|h(t, x) - h(t, y)| \leq C|x - y|$,
- (iii) $|\sigma(t, x) - \sigma(t, y)| \leq C|x - y|$.

In particular, we work under the linear growth conditions

$$b(t, x) \leq C(1 + |x|), \quad \sigma(t, x) \leq C(1 + |x|), \quad h(t, x) \leq C(1 + |x|),$$

on $b(t, x)$, $\sigma(t, x)$ and $h(t, x)$.

(AA.3.2) The functions $\varphi(x)$, $f(t, x, y, z)$, and $g(t, x, y, z)$ are also continuous in $t \in [0, T]$ and are locally Lipschitz in x and uniformly Lipschitz in (y, z) , i.e., there exist a positive integer m and constant $C > 0$ such that for $x, x', y, y', z, z' \in \mathbb{R}$,

(i)

$$|f(t, x, y, z) - f(t, x', y', z')| \leq C [(1 + |x|^m + |x'|^m) |x - x'| + |y - y'| + |z - z'|],$$

(ii)

$$|g(t, x, y, z) - g(t, x', y', z')| \leq C [(1 + |x|^m + |x'|^m) |x - x'| + |y - y'| + |z - z'|],$$

$$(iii) \quad |\varphi(x) - \varphi(x')| \leq C (1 + |x|^m + |x'|^m) |x - x'|.$$

(AA.3.3) Furthermore, we assume that

$$f(t, x, 0, 0) = g(t, x, 0, 0) = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Under Condition (AA.3.1), the G -FBSE (3.22) admits a unique solution $(X_t)_{t \in [0, T]} \in M_G^2(0, T)$ by Theorem 2.2 in Chapter V of [80], while under the Conditions (AA.3.2)-(AA.3.3), and by Theorem 4.1 in [42], the G -BSDE (3.23) admit a unique solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s \in [t, T]} \in (S_G^2(0, T) \times M_G^2(0, T) \times L_G^2(\Omega, \mathcal{F}_T))$, for $\varphi(X_T^{t,x}) \in L_G^2(\Omega, \mathcal{F}_T)$. The Condition (AA.3.3) is needed for the later definition of G -evaluation in Chapter 5. However in general, it is sufficient to assume that $f(\cdot, x, 0, 0), g(\cdot, x, 0, 0)$ are in $M_G^2(0, T)$, for all $x \in \mathbb{R}$.

Linear G -FBSDE

Let $(X_t)_{t \in [0, T]}$ satisfy the G -forward diffusion equation

$$dX_t = b(t, X_t) dt + h(t, X_t) d\langle B \rangle_t + \sigma(t, X_t) dB_t, \quad (3.24)$$

where b, h, σ satisfy (AA.3.1), with an associated linear G -backward SDE

$$dY_t = -(p_t X_t + a_t Y_t + b_t Z_t + m_t) dt - (q_t X_t + c_t Y_t + d_t Z_t + n_t) d\langle B \rangle_t + Z_t dB_t + dK_t, \quad (3.25)$$

with terminal condition $Y_T = \varphi(X_T)$, where $(a_t)_{t \in [0, T]}$, $(b_t)_{t \in [0, T]}$, $(c_t)_{t \in [0, T]}$, and $(d_t)_{t \in [0, T]}$ are real-valued and $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted bounded processes in $M_G^p(0, T)$, $p > 1$, and $(p_t)_{t \in [0, T]}$, $(q_t)_{t \in [0, T]}$, $(m_t)_{t \in [0, T]}$ and $(n_t)_{t \in [0, T]}$ are a real-valued $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process in $M_G^p(0, T)$, $p > 1$.

Following [43], we construct an auxiliary extended sublinear $\tilde{G}$ -expectation space $(\tilde{\Omega}_T, L^1_{\tilde{G}}(\tilde{\Omega}, \mathcal{F}_T), \tilde{\mathbb{E}}_G)$ with $\tilde{\Omega}_T := C_0([0, T], \mathbb{R}^2)$ and

$$\tilde{G}(A) := \frac{1}{2} \sup_{\sigma^2 \leq \nu \leq \bar{\sigma}^2} \text{tr} \left[A \begin{pmatrix} \nu & 1 \\ 1 & \nu^{-1} \end{pmatrix} \right], \quad A \in \mathbb{S}_2,$$

where $\mathbb{S}_2$ is a space of 2×2 symmetric matrices.

Let $(B_t)_{t \in [0, T]}$ and $(\tilde{B}_t)_{t \in [0, T]}$ be the canonical process in the extended space, and we check that $\langle B, \tilde{B} \rangle_t = t$. The following Lemma 3.3.1, which will be used in the proof of Theorem 3.4.2, extends a classical result from linear G -BSDEs to linear G -FBSDEs. This result is quite similar to Lemma 3.1.1, but here we apply the G -Itô formula with noting that $dB_t^2 = d\langle B \rangle_t$, see e.g. Section 5.4 of [77].

Lemma 3.3.1. *Let $(X_t)_{t \in [0, T]}$ be the solution of (3.24), and let $(Y_t, Z_t, K_t)_{t \in [0, T]}$ be the solution of (3.25). Then the process $(Y_t)_{t \in [0, T]}$ is given in an explicit form as*

$$Y_t = \frac{1}{\Gamma_t} \tilde{\mathbb{E}}_G \left[\Gamma_T Y_T + \int_t^T \Gamma_s (p_s X_s + m_s) ds + \int_t^T \Gamma_s (q_s X_s + n_s) d\langle B \rangle_s \middle| \mathcal{F}_t \right], \quad (3.26)$$

where $(\Gamma_t)_{t \in [0, T]}$ is the solution of $\tilde{G}$ -SDE:

$$\Gamma_t = 1 + \int_0^t a_s \Gamma_s ds + \int_0^t c_s \Gamma_s d\langle B \rangle_s + \int_0^t d_s \Gamma_s dB_s + \int_0^t b_s \Gamma_s d\tilde{B}_s.$$

Proof. By applying the G -Itô formula to $Y_t \Gamma_t$, we have

$$\begin{aligned} d(\Gamma_s Y_s) &= \Gamma_s dY_s + Y_s d\Gamma_s + d\langle \Gamma_s, Y_s \rangle \\ &= -\Gamma_s (p_s X_s + a_s Y_s + b_s Z_s + m_s) ds - \Gamma_s (q_s X_s + c_s Y_s + d_s Z_s + n_s) d\langle B \rangle_s \\ &\quad + \Gamma_s Z_s dB_s + \Gamma_s dK_s + Y_s \Gamma_s \left(a_s ds + c_s d\langle B \rangle_s + d_s dB_s + b_s d\tilde{B}_s \right) \\ &\quad + d_s Z_s \Gamma_s d\langle B \rangle_s + b_s Z_s \Gamma_s ds \\ &= -(p_s X_s \Gamma_s + m_s \Gamma_s) ds - (q_s X_s \Gamma_s + n_s \Gamma_s) d\langle B \rangle_s + (Z_s \Gamma_s + d_s Y_s \Gamma_s) dB_s \\ &\quad + b_s Y_s \Gamma_s d\tilde{B}_s + \Gamma_s dK_s. \end{aligned}$$

By taking conditional expectation under $\tilde{\mathbb{E}}_G$ on both sides and noting that $\int_0^t \Gamma_s dK_s$ is a $\tilde{G}$ -martingale, we find

$$Y_t = \tilde{\mathbb{E}}_G [Y_T | \mathcal{F}_t]$$

$$= \frac{1}{\Gamma_t} \tilde{\mathbb{E}}_G \left[\Gamma_T Y_T + \int_t^T (p_s X_s \Gamma_s + m_s \Gamma_s) ds + \int_t^T (q_s X_s \Gamma_s + n_s \Gamma_s) d\langle B \rangle_s \middle| \mathcal{F}_t \right],$$

Note that by the formula (3.3) in [43], the process $(\Gamma_t)_{t \in [0, T]}$ has the form

$$\Gamma_t = \exp \left(\int_0^t (a_s - b_s d_s) ds + \int_0^t c_s d\langle B \rangle_s \right) \mathcal{E}_t^B \mathcal{E}_t^{\tilde{B}},$$

where

$$\mathcal{E}_t^B := \exp \left(\int_0^t d_s dB_s - \frac{1}{2} \int_0^t d_s^2 d\langle B \rangle_s \right) \quad \text{and} \quad \mathcal{E}_t^{\tilde{B}} := \exp \left(\int_0^t b_s d\tilde{B}_s - \frac{1}{2} \int_0^t b_s^2 d\langle \tilde{B} \rangle_s \right),$$

$t \in [0, T]$. $\square$

Remark 3.3.2. We note that Lemma 3.3.1 also holds for the case of the G -BSDE (3.25) which is nonlinear in $(X_t)_{t \geq 0}$.

3.3.2 Associated G -PDEs (HJB-type equations)

As in the case of the g -expectation and the usual FBSDEs setting, the G -backward SDEs can also be linked to the G -forwards SDEs via a type of Hamilton-Jacobi-Bellman equation which will be also called a G -PDE, where the function G is defined as in (3.21). In fact, in [43], they proved the following nonlinear Feynman-Kac formula under (A4.3.1)-(A4.3.2).

Theorem 3.3.3 (Theorem 4.5 of [43]). *Under (A4.3.1)-(A4.3.2). Let $u(t, x) := Y_t^{t, x}$ for $(t, x) \in [0, T] \times \mathbb{R}$. Then $u(t, x)$ is the unique viscosity solution of the HJB equation*

$$\frac{\partial u}{\partial t}(t, x) + F \left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x) \right) = 0, \quad (3.27)$$

with $u(T, x) = \varphi(x)$,

where

$$F(t, x, y, z, w) := 2G(H(t, x, y, z, w)) + zb(t, x) + f(t, x, y, z\sigma(t, x)), \quad (3.28)$$

and

$$H(t, x, y, z, w) := \frac{w}{2} \sigma^2(t, x) + zh(t, x) + g(t, x, y, z\sigma(t, x)). \quad (3.29)$$

The next definition of $\mathcal{C}^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$, $\eta \in (0, 1)$ comes from [23].

Definition 3.3.4. *i) Let $\mathcal{C}^{p,q}([0, T] \times \mathbb{R})$ denote the space of functions $v(t, x)$ which are p times continuously differentiable in $t \in [0, T]$ and q times differentiable in $x \in \mathbb{R}$, for $p, q \geq 1$.*

ii) Let $\mathcal{C}_b^{p,q}([0, T] \times \mathbb{R}^n)$ denote the subspace of $\mathcal{C}^{p,q}([0, T] \times \mathbb{R})$ whose partial derivatives with respect to x (resp. t) of orders 1 to q (resp. 1 to p) are uniformly bounded on $\mathbb{R}^n$ (resp. on $[0, T]$).

iii) Let $\mathcal{C}^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$, $\eta \in (0, 1)$, denote the space of functions $v(t, x)$ which are differentiable in $t \in [0, T]$ and twice differentiable in $x \in \mathbb{R}$, such that $\frac{\partial v}{\partial t}(t, x)$ and $\frac{\partial^2 v}{\partial x^2}(t, x)$ are respectively $\eta/2$ -Hölder continuous and η -Hölder continuous in $(t, x) \in [0, T] \times \mathbb{R}$. The space $\mathcal{C}^{k+\eta}(\mathbb{R})$ is defined analogously for $k \geq 1$.

We also define the norm

$$\begin{aligned} \|v\|_{1,2,\eta} := & \|v\|_{\infty} + \left\| \frac{\partial v}{\partial t} \right\|_{\infty} + \left\| \frac{\partial v}{\partial x} \right\|_{\infty} + \left\| \frac{\partial^2 v}{\partial x^2} \right\|_{\infty} \\ & + \sup_{(t,x) \neq (t',x')} \frac{|\partial v(t,x)/\partial t - \partial v(t',x')/\partial t| + |\partial^2 v(t,x)/\partial x^2 - \partial^2 v(t',x')/\partial x^2|}{|t - t'|^{\eta/2} + |x - x'|^{\eta}}, \end{aligned} \quad (3.30)$$

on $\mathcal{C}^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$, where $\|\cdot\|_{\infty}$ denotes the supremum norm

$$\|v\|_{\infty} := \sup_{(t,x) \in [0,T] \times \mathbb{R}} |v(t, x)|.$$

The next Theorem 3.3.5 states an existence result for classical solutions of the HJB equation (3.27) under stronger additional boundedness conditions on coefficients in comparison with Theorems 3.1.3-3.1.4 in Section 3.1.2.

Theorem 3.3.5. *In addition to (AA.3.1)-(AA.3.2), assume that $b(t, \cdot)$, $h(t, \cdot)$, $\sigma(t, \cdot)$, φ are bounded functions in $\mathcal{C}_b^3(\mathbb{R})$, and that $f(t, \cdot, \cdot, \cdot)$, $g(t, \cdot, \cdot, \cdot)$ are bounded functions in $\mathcal{C}_b^3(\mathbb{R}^3)$ for any $t \in [0, T]$. In addition, suppose that $\sigma(t, x)$ is bounded above and below on $[0, T] \times \mathbb{R}$ by strictly positive constants. Then the function $u(t, x) := Y_t^{t,x}$ is a bounded classical solution in $\mathcal{C}_b^{1,2}([0, T] \times \mathbb{R})$ of the HJB equation (3.27) with terminal condition $u(T, \cdot) = \varphi$.*

Proof. The argument is similar to the proof of Theorem 4.5 in Appendix C of [80]. We note that the function $F(t, x, y, z, w)$ in (3.28) satisfies Definition 6.1.1 page 271 of [52] as in Example 6.1.8 page 279 therein. Hence, by Theorem 6.4.3 page 301 of [52] there exists $\eta \in (0, 1)$ such that for each $\kappa > 0$ we have $\|u\|_{1,2,\eta} < \infty$ on $[0, T - \kappa] \times \mathbb{R}$, therefore $u(t, x)$ is bounded and belongs to $\mathcal{C}_b^{1,2}([0, T] \times \mathbb{R})$. $\square$

Other variations of Theorem 3.3.5 above include Theorem 4.2 of [30] and Theorem 1.5 of [93] when $(y, z) \mapsto f(t, x, y, z)$ and $(y, z) \mapsto g(t, x, y, z)$ are linear. In the case of semilinear parabolic PDEs, we refer to Theorem 3.2 of [73], Proposition 3.3 in [64], Theorem 2.2 in [65], see also Theorem 8.1 in § V.8 page 495, and Theorem 7.1 in § VII.7 page 596 of [53].

The following theorem will be used to prove convexity of the solution $u(t, x)$ of the HJB equation (3.27). It shows that $u(t, x) \in \mathcal{C}^{2+\eta/2, 4+\eta}([0, T] \times \mathbb{R})$ under additional conditions, such as the terminal condition $\varphi \in \mathcal{C}^{4+\eta}(\mathbb{R})$.

Theorem 3.3.6. *Suppose, in addition to the conditions of Theorem 3.3.5, that for some $\eta \in (0, 1)$, the functions $b(\cdot, \cdot)$, $h(\cdot, \cdot)$, $\sigma(\cdot, \cdot)$ and $f(\cdot, \cdot, y, z)$, $g(\cdot, \cdot, y, z)$ are in $\mathcal{C}^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$ for all $y, z \in \mathbb{R}$, and that $\varphi \in \mathcal{C}^{4+\eta}(\mathbb{R})$. Then the solution $u(t, x)$ of the HJB equation (3.27) is a classical solution in $\mathcal{C}^{2+\eta/2, 4+\eta}([0, T] \times \mathbb{R})$.*

Proof. As in the proof of Theorem 3.3.5 we have $\|u\|_{1,2,\eta} < \infty$ on $[0, T - \kappa] \times \mathbb{R}$ for some $\eta \in (0, 1)$. In order to show that $\|u\|_{2,4,\eta} < \infty$, we successively apply Schauder interior estimates to the difference quotients which are used to approximate the derivatives of $u(t, x)$ as in Theorem 2.2 of [23], see the inequality (4.15) in Theorem 4.5 of [94] and Theorem 6.17 of [34]. $\square$

Under the conditions of Theorem 3.3.5, by applying the G -Itô formula to $u(s, X_s^{t,x})$, $s \in [t, T]$, one can verify that $Y_s^{t,x} := u(s, X_s^{t,x})$, $Z_s^{t,x} := \sigma(s, X_s^{t,x}) \frac{\partial u}{\partial x}(s, X_s^{t,x})$, and

$$\begin{aligned} K_s^{t,x} := & \int_0^s H \left(\tau, X_\tau^{t,x}, u(\tau, X_\tau^{t,x}), \frac{\partial u}{\partial x}(\tau, X_\tau^{t,x}), \frac{\partial^2 u}{\partial x^2}(\tau, X_\tau^{t,x}) \right) d\langle B \rangle_\tau \\ & - 2 \int_0^s G \left(H \left(\tau, X_\tau^{t,x}, u(\tau, X_\tau^{t,x}), \frac{\partial u}{\partial x}(\tau, X_\tau^{t,x}), \frac{\partial^2 u}{\partial x^2}(\tau, X_\tau^{t,x}) \right) \right) d\tau \end{aligned}$$

solve the G -BSDEs (3.23).

3.4 Monotonicity, Convexity and Continuous Dependence Results on G -FBSDEs and G -PDEs

3.4.1 Monotonicity of G -FBSDE Solutions

Proposition 3.4.1. *Assume that (A4.3.1) holds. The solution $(X_s^{t,x})_{t \in [0, T]}$ of (3.22) is q.s. non-decreasing in x for all $s \in [t, T]$ and $t \in [0, T]$.*

Proof. Letting $\widehat{X}_s^{t,x,y} := X_s^{t,y} - X_s^{t,x}$ for $y \geq x$, $s \in [t, T]$, and consider the processes,

$$\widehat{b}_s := \frac{b(s, X_s^{t,y}) - b(s, X_s^{t,x})}{X_s^{t,y} - X_s^{t,x}} \mathbf{1}_{\{X_s^{t,y} \neq X_s^{t,x}\}}, \quad \widehat{h}_s := \frac{h(s, X_s^{t,y}) - h(s, X_s^{t,x})}{X_s^{t,y} - X_s^{t,x}} \mathbf{1}_{\{X_s^{t,y} \neq X_s^{t,x}\}},$$

and $\widehat{\sigma}_s := \frac{\sigma(s, X_s^{t,y}) - \sigma(s, X_s^{t,x})}{X_s^{t,y} - X_s^{t,x}} \mathbf{1}_{\{X_s^{t,y} \neq X_s^{t,x}\}}$. Then the processes $(\widehat{b}_s)_{s \in [t, T]}$, $(\widehat{h}_s)_{s \in [t, T]}$ and $(\widehat{\sigma}_s)_{s \in [t, T]}$ are bounded in $M_G^2(0, T)$ and $\widehat{X}_s^{t,x,y}$ satisfies the equation

$$\widehat{X}_s^{t,x,y} = (y - x) + \int_t^s \widehat{b}_r \widehat{X}_r dr + \int_t^s \widehat{h}_r \widehat{X}_r d\langle B \rangle_r + \int_t^s \widehat{\sigma}_r \widehat{X}_r dB_r.$$

which yields

$$\widehat{X}_s^{t,x,y} = (y - x) \exp \left(\int_t^s \widehat{b}_r dr + \int_t^s \widehat{h}_r d\langle B \rangle_r + \int_t^s \widehat{\sigma}_r dB_r - \frac{1}{2} \int_t^s \widehat{\sigma}_r^2 d\langle B \rangle_r \right) \geq 0,$$

according to the formula (3.3) in [43]. $\square$

Next, we show a non-decreasing property with respect to x of the solution $(Y_s^{t,x})_{s \in [t, T]}$ of the G-BSDE (3.23) as we did for the solution of the usual BSDE. This result is frequently utilized to derive G-monotonic and increasing convex orderings presented in Chapter 5.

Proposition 3.4.2. *Assume that (AA.3.1)-(AA.3.3) hold. If $f(t, x, y, z)$, $g(t, x, y, z)$ and $\varphi(x)$ are non-decreasing in $x \in \mathbb{R}$ for all $t \in [0, T]$ and $y, z \in \mathbb{R}$, then the solution $(Y_s^{t,x})_{s \in [t, T]}$ of (3.23) is q.s. non-decreasing in x for all $s \in [t, T]$. As a consequence, if $u(t, x)$ is solution of the backward PDEs (3.27), then $u(t, x)$ is also a non-decreasing function of $x \in \mathbb{R}$ for all $t \in [0, T]$.*

Proof. Letting $\widehat{X}_s = X_s^{t,y} - X_s^{t,x}$, $\widehat{Y}_s = Y_s^{t,y} - Y_s^{t,x}$, $\widehat{Z}_s = Z_s^{t,y} - Z_s^{t,x}$, $\widehat{K}_s = K_s^{t,y} - K_s^{t,x}$ and $\widehat{Y}_T = \varphi(X_T^{t,y}) - \varphi(X_T^{t,x})$ for $y \geq x$, $s \in [t, T]$, we have

$$\begin{aligned} \widehat{Y}_s &= \widehat{Y}_T + \int_s^T (f(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})) du - \int_s^T \widehat{Z}_u dB_u \\ &\quad + \int_s^T (g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})) d\langle B \rangle_u - \int_s^T d\widehat{K}_u. \end{aligned}$$

Defining the processes $(p_u)_{u \in [0, T]}$, $(a_u)_{u \in [0, T]}$, and $(b_u)_{u \in [0, T]}$ as

$$p_u := \frac{f(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}},$$

$$\begin{aligned}
a_u &:= \frac{f(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y})}{Y_u^{t,y} - Y_u^{t,x}} \mathbf{1}_{\{Y_u^{t,y} \neq Y_u^{t,x}\}}, \\
b_u &:= \frac{f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})}{Z_u^{t,y} - Z_u^{t,x}} \mathbf{1}_{\{Z_u^{t,y} \neq Z_u^{t,x}\}}, \quad u \in [0, T],
\end{aligned}$$

and similarly defining the processes $(q_u)_{u \in [0, T]}$, $(c_u)_{u \in [0, T]}$, and $(d_u)_{u \in [0, T]}$ as

$$\begin{aligned}
q_u &:= \frac{g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}}, \\
c_u &:= \frac{g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y})}{Y_u^{t,y} - Y_u^{t,x}} \mathbf{1}_{\{Y_u^{t,y} \neq Y_u^{t,x}\}}, \\
d_u &:= \frac{g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})}{Z_u^{t,y} - Z_u^{t,x}} \mathbf{1}_{\{Z_u^{t,y} \neq Z_u^{t,x}\}}, \quad u \in [0, T],
\end{aligned}$$

which are processes defined in $M_G^2(0, T)$ and using the decomposition

$$\begin{aligned}
&f(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) \\
&= f(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) \\
&\quad + f(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) \\
&\quad + f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}),
\end{aligned}$$

and

$$\begin{aligned}
&g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) \\
&= g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) \\
&\quad + g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) \\
&\quad + g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}),
\end{aligned}$$

we have

$$\begin{aligned}
\widehat{Y}_s &= \widehat{Y}_T + \int_s^T (p_u \widehat{X}_u + a_u \widehat{Y}_u + b_u \widehat{Z}_u) du - \int_s^T \widehat{Z}_u dB_u \\
&\quad + \int_s^T (q_u \widehat{X}_u + c_u \widehat{Y}_u + d_u \widehat{Z}_u) d\langle B \rangle_u - \int_s^T d\widehat{K}_u, \quad s \in [0, T].
\end{aligned}$$

Hence by Lemma 3.3.1, we get that

$$\widehat{Y}_s = \frac{1}{\Gamma_t} \widetilde{\mathbb{E}}_G \left[\Gamma_T \widehat{Y}_T + \int_t^T p_u \widehat{X}_u \Gamma_u du + \int_t^T q_u \widehat{X}_u \Gamma_u d\langle B \rangle_u \middle| \Omega_t \right], \quad (3.31)$$

where

$$\Gamma_s = \exp \left(\int_t^s (a_u - b_u d_u) du + \int_t^s c_u d\langle B \rangle_u \right) \mathcal{E}_s^B \mathcal{E}_s^{\tilde{B}},$$

and

$$\mathcal{E}_s^B := \exp \left(\int_t^s d_u dB_u - \frac{1}{2} \int_t^s d_u^2 d\langle B \rangle_u \right), \quad \mathcal{E}_s^{\tilde{B}} := \exp \left(\int_t^s b_u d\tilde{B}_u - \frac{1}{2} \int_t^s b_u^2 d\langle \tilde{B} \rangle_u \right),$$

$t \in [0, T]$. By Proposition 3.4.1 the solution $(X_s^{t,x})_{s \in [t, T]}$ of the forward SDE (3.22) satisfies $\hat{X}_s = X_s^{t,y} - X_s^{t,x} \geq 0$ for all $s \in [t, T]$ if $x \leq y$, and since $f(s, x, y, z)$ and $g(s, x, y, z)$ are non-decreasing in x , we have $p_s \geq 0$ and $q_s \geq 0$ q.s., $s \in [t, T]$. Since $\varphi(x)$ is non-decreasing, we have $\hat{Y}_T = \varphi(X_T^{t,x}) - \varphi(X_T^{t,y}) \geq 0$, q.s. Hence, by (3.31) we have $\hat{Y}_s = Y_s^{t,x} - Y_s^{t,y} \geq 0$, $s \in [t, T]$, if $x \leq y$, which implies monotonicity of $(Y_s^{t,x})_{s \in [t, T]}$ with respect to $x \in \mathbb{R}$. Thus, we also get $u(t, x) \leq u(t, y)$, $x \leq y$, $t \in [0, T]$, since $u(t, x) = Y_t^{t,x}$. $\square$

3.4.2 Continuous Dependence of G -FBSDE Solutions

In this section, we prove stability results for the solutions of G -FBSDEs. Letting $L^0(\Omega)$ denote the space of all $\mathcal{F}$ -measurable real functions, as in e.g. Exercise VI.3.3 in [80] we let

$$L_G^1(\Omega, \mathcal{F}) = \{ \xi \in L^0(\Omega) : \xi \text{ has a quasi-continuous version and } \lim_{n \rightarrow \infty} \mathbb{E}_G[|\xi| \mathbf{1}_{\{\xi > n\}}] = 0 \},$$

denote the space of uniformly integrable random variables under $\mathbb{E}_G$. By e.g. Theorem VI.2.5 in [80], the G -expectation $\mathbb{E}_G$ can be represented as

$$\mathbb{E}_G[\xi] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\xi], \quad \xi \in L_G^1(\Omega, \mathcal{F}),$$

where $\mathcal{P}$ is a weakly compact family of probability measures on $(\Omega, \mathcal{F})$, and by Corollary VI.1.33 in [80] it satisfies the monotone continuity (regularity) property

$$\mathbb{E}_G[\xi] \downarrow 0 \quad \text{for any sequence } (\xi_n)_{n \geq 1} \subset L_G^1(\Omega, \mathcal{F}) \quad \text{such that } \xi_n \downarrow 0, \text{ q.s.},$$

Therefore, the dominated convergence Theorem 3.4 in [46], or Theorem 10.1.10 in [99] applies to the G -expectation $\mathbb{E}_G$, and it will be applied in the following proofs although it may not hold under general sublinear expectations. Theorems 3.4.3 and 3.4.4 are stated using convergence in L_G^2 , and we note that if $\lim_{n \rightarrow \infty} \mathbb{E}_G[|Y_{n,t} - Y_t|^2] = 0$ then by Proposition VI.1.17 in [80] there exists a subsequence $(Y_{n_k, t})_{k \geq 1}$ converging to Y_t quasi-surely, $t \in [0, T]$.

Theorem 3.4.3. *Consider the family of G -forward-backward stochastic differential equations*

$$X_{n,t} = x_{n,0} + \int_0^t b_n(s, X_{n,s})ds + \int_0^t h_n(s, X_{n,s})d\langle B \rangle_s + \int_0^t \sigma_n(s, X_{n,s})dB_s, \quad (3.32a)$$

$$\begin{aligned} Y_{n,t} = & \varphi_n(X_{n,T}) + \int_t^T f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s})ds + \int_t^T g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s})d\langle B \rangle_s \\ & - \int_t^T Z_{n,s}dB_s - \int_t^T dK_{n,s}, \end{aligned} \quad (3.32b)$$

where for every $n \geq 1$, $X_{n,0} = x_{n,0} \in \mathbb{R}$, and the coefficients $b_n, h_n, \sigma_n, f_n, g_n$ and φ_n satisfy (AA.3.1)-(AA.3.3) for a same $C > 0$. Assume the convergence of the sequence $x_{n,0} \rightarrow x_0$ and pointwise convergences $b_n(t, x) \rightarrow b(t, x)$, $h_n(t, x) \rightarrow h(t, x)$, $\sigma_n(t, x) \rightarrow \sigma(t, x)$, and

$$f_n(t, x, y, z) \rightarrow f(t, x, y, z), \quad g_n(t, x, y, z) \rightarrow g(t, x, y, z), \quad \varphi_n(x) \rightarrow \varphi(x),$$

for all $t \in [0, T]$, and $x, y, z \in \mathbb{R}$ as $n \rightarrow \infty$, where b, h, σ, φ and f, g satisfy (AA.3.1)-(AA.3.3) for the same $C > 0$. Then for all $t \in [0, T]$ we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_G [|Y_{n,t} - Y_t|^2] = 0,$$

where $(Y_t)_{t \in \mathbb{R}_+}$ is solution of the G -FBSDE

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t h(s, X_s)d\langle B \rangle_s + \int_0^t \sigma(s, X_s)dB_s, \quad (3.33a)$$

$$\begin{aligned} Y_t = & \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds + \int_t^T g(s, X_s, Y_s, Z_s)d\langle B \rangle_s \\ & - \int_t^T Z_sdB_s - \int_t^T dK_s. \end{aligned} \quad (3.33b)$$

Proof. For $n \geq 1$, let

$$\widehat{Y}_{n,t} := Y_{n,t} - Y_t, \quad \widehat{X}_{n,t} := X_{n,t} - X_t, \quad \widehat{Z}_{n,t} := Z_{n,t} - Z_t, \quad \text{and} \quad \widehat{\varphi}_n(x) := \varphi_n(x) - \varphi(x),$$

$t \in [0, T]$, $x \in \mathbb{R}$. Applying the G -Itô formula to $|\widehat{Y}_{n,t}|^2$, taking G -expectation on both sides and noting that $\int_t^T 2\widehat{Y}_{n,s}dK_s$ is a martingale under $\mathbb{E}_G$, this yields

$$\mathbb{E}_G [|\widehat{Y}_{n,t}|^2] \leq \mathbb{E}_G [(\varphi_n(X_{n,T}) - \varphi(X_T))^2]$$

$$\begin{aligned}
 & + \mathbb{E}_G \left[\int_t^T 2\widehat{Y}_{n,s} (f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - f(s, X_s, Y_s, Z_s)) ds \right. \\
 & + \int_t^T 2\widehat{Y}_{n,s} (g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s)) d\langle B \rangle_s \\
 & \left. - \int_t^T |\widehat{Z}_{n,s}|^2 d\langle B \rangle_s \right].
 \end{aligned}$$

By the inequality $2ab \leq (12C^2)a^2 + b^2/(12C^2)$, we have

$$\begin{aligned}
 & 2\widehat{Y}_{n,s} (g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s)) \\
 & \leq 12C^2 |\widehat{Y}_{n,s}|^2 + \frac{1}{12C^2} (g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))^2.
 \end{aligned}$$

Letting

$$\hat{g}_n(t, x, y, z) := g_n(t, x, y, z) - g(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

we have

$$\begin{aligned}
 & (g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))^2 \\
 & \leq 2(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g_n(s, X_s, Y_s, Z_s))^2 + 2(\hat{g}_n(s, X_s, Y_s, Z_s))^2 \\
 & \leq 2C^2 ((1 + X_{n,s}^m + X_s^m) |X_{n,s} - X_s| + |Y_{n,s} - Y_s| + |Z_{n,s} - Z_s|)^2 \\
 & \quad + 2(\hat{g}_n(s, X_s, Y_s, Z_s))^2 \\
 & \leq 6C^2 ((1 + X_{n,s}^m + X_s^m)^2 |X_{n,s} - X_s|^2 + |Y_{n,s} - Y_s|^2 + |Z_{n,s} - Z_s|^2) \\
 & \quad + 2(\hat{g}_n(s, X_s, Y_s, Z_s))^2,
 \end{aligned}$$

by the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$. Similarly, by the inequality $2ab \leq (12C^2/\underline{\sigma}^2)a^2 + b^2/(12C^2/\underline{\sigma}^2)$, we have

$$\begin{aligned}
 & 2\widehat{Y}_{n,s} (f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - f(s, X_s, Y_s, Z_s)) \\
 & \leq \frac{12C^2}{\underline{\sigma}^2} |\widehat{Y}_{n,s}|^2 + \frac{\underline{\sigma}^2}{12C^2} (f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - f(s, X_s, Y_s, Z_s))^2.
 \end{aligned}$$

Letting

$$\hat{f}_n(t, x, y, z) := f_n(t, x, y, z) - f(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

we then have

$$(f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - f(s, X_s, Y_s, Z_s))^2$$

$$\begin{aligned} &\leq 6C^2((1 + X_{n,s}^m + X_s^m)^2 |X_{n,s} - X_s|^2 + |Y_{n,s} - Y_s|^2 + |Z_{n,s} - Z_s|^2) \\ &\quad + 2(\hat{f}_n(s, X_s, Y_s, Z_s))^2. \end{aligned}$$

In addition, by the inequality $(a+b)^2 \leq 2(a^2+b^2)$ and the locally Lipschitz property of φ_n , we have

$$\begin{aligned} (\varphi_n(X_{n,T}) - \varphi(X_T))^2 &= ((\varphi_n(X_{n,T}) - \varphi_n(X_T)) + \hat{\varphi}_n(X_T))^2 \\ &\leq 2(\varphi_n(X_{n,T}) - \varphi_n(X_T))^2 + 2(\hat{\varphi}_n(X_T))^2 \\ &\leq 2C'(1 + X_{n,T}^m + X_T^m)^2 |X_{n,T} - X_T|^2 + 2(\hat{\varphi}_n(X_T))^2. \end{aligned}$$

Combining the above estimates and noting that $\underline{\sigma}^2 \int_t^T |\hat{Z}_{n,s}|^2 ds \leq \int_t^T |\hat{Z}_{n,s}|^2 d\langle B \rangle_s \leq \bar{\sigma}^2 \int_t^T |\hat{Z}_{n,s}|^2 ds$, by the point of view of Corollary 5.4 in [80] page 54, see also Remark 3.10, p.775 in [42], we then find

$$\begin{aligned} \mathbb{E}_G [|\hat{Y}_{n,t}|^2] &\leq 2C' \mathbb{E}_G [(1 + X_{n,T}^m + X_T^m)^2 |\hat{X}_{n,T}|^2] + 2 \mathbb{E}_G [(\hat{\varphi}_n(X_T))^2] \\ &\quad + \left(\frac{12C^2}{\underline{\sigma}^2} + \frac{\underline{\sigma}^2}{2} + 12C^2\bar{\sigma}^2 + \frac{\bar{\sigma}^2}{2} \right) \int_t^T \mathbb{E}_G [|\hat{Y}_{n,s}|^2] ds \\ &\quad + \frac{\underline{\sigma}^2 + \bar{\sigma}^2}{2} \int_t^T \mathbb{E}_G [(1 + X_{n,s}^m + X_s^m)^2 |\hat{X}_{n,s}|^2] ds \\ &\quad + \frac{\underline{\sigma}^2}{6C^2} \int_t^T \mathbb{E}_G [(\hat{f}_n(s, X_s, Y_s, Z_s))^2] ds \\ &\quad + \frac{\bar{\sigma}^2}{6C^2} \int_t^T \mathbb{E}_G [(\hat{g}_n(s, X_s, Y_s, Z_s))^2] ds, \end{aligned}$$

By the Hölder inequality, and then by Gronwall's inequality, we get

$$\begin{aligned} \mathbb{E}_G [|\hat{Y}_{n,t}|^2] &\leq C'' \left((1 + x_{n,0}^{2m} + x_0^{2m}) (\mathbb{E}_G [|\hat{X}_{n,T}|^4])^{1/2} + \mathbb{E}_G [(\hat{\varphi}_n(X_T))^2] \right. \\ &\quad + (1 + x_{n,0}^{2m} + x_0^{2m}) \int_t^T (\mathbb{E}_G [|\hat{X}_{n,s}|^4])^{1/2} ds \\ &\quad \left. + \int_t^T \mathbb{E}_G [(\hat{f}_n(s, X_s, Y_s, Z_s))^2] ds + \int_t^T \mathbb{E}_G [(\hat{g}_n(s, X_s, Y_s, Z_s))^2] ds \right), \end{aligned}$$

where we used $\mathbb{E}_G [|X_s|^p] \leq C^*(1 + |X_0|^p)$ for any $s \in [0, T]$ and $p \geq 2$, where constant $C^* > 0$ depends on p, C, T, G , see e.g. Proposition 4.1 in [43]. Therefore, we have

$$\mathbb{E}_G [|\hat{Y}_{n,t}|^2] \leq C'' (1 + x_{n,0}^{2m} + x_0^{2m}) \left((\mathbb{E}_G [|\hat{X}_{n,T}|^4])^{1/2} + (T-t) \left(\sup_{s \in [t, T]} \mathbb{E}_G [|\hat{X}_{n,s}|^4] \right)^{1/2} \right)$$

$$\begin{aligned}
 & + C'' \left(\mathbb{E}_G [(\hat{\varphi}_n(X_T))^2] + \int_t^T \mathbb{E}_G [(\hat{f}_n(s, X_s, Y_s, Z_s))^2] ds \right. \\
 & \left. + \int_t^T \mathbb{E}_G [(\hat{g}_n(s, X_s, Y_s, Z_s))^2] ds \right).
 \end{aligned}$$

We note that since $x_{n,0} \rightarrow x_0$, and $\mu_n(t, x) \rightarrow \mu$ with $\sigma_n(t, x) \rightarrow \sigma(t, x)$ pointwise when $n \rightarrow \infty$, by Theorem 3.4.4 below we have $\lim_{n \rightarrow \infty} \mathbb{E}_G [\sup_{t \in [0, T]} |\hat{X}_{n,t}|^4] = 0$. Hence, by the conditions $|\hat{f}_n(t, x, y, z)| \leq 2C(|x|^{m+1} + |y| + |z|)$, $|\hat{g}_n(t, x, y, z)| \leq 2C(|x|^{m+1} + |y| + |z|)$ and $|\hat{\varphi}_n(x)| \leq 2C'|x|^{m+1}$ and the pointwise limits $\lim_{n \rightarrow \infty} \hat{\varphi}_n(x) = 0$, $\lim_{n \rightarrow \infty} \hat{f}_n(t, x, y, z) = 0$, $\lim_{n \rightarrow \infty} \hat{g}_n(t, x, y, z) = 0$, and by the dominated convergence Theorem 3.4 in [46] under the G -expectation $\mathbb{E}_G$, see also Theorem 10.1.10 in [99], we conclude to

$$\lim_{n \rightarrow \infty} \mathbb{E}_G [|Y_{n,t} - Y_t|^2] = 0, \quad t \in [0, T].$$

□

The following theorem has been used in the proof of Theorem 3.4.3 above. It is an extension of Theorem 9.7 in [67] from SDEs under the linear expectation framework to G -SDEs under the G -expectation framework.

Theorem 3.4.4. *Assume the coefficients b_n, h_n, σ_n , satisfy (AA₁)-(AA.3.3) for a same $C > 0$. Assume the convergence of the sequence $x_{n,0} \rightarrow x_0$, and pointwise convergences $b_n(t, x) \rightarrow b(t, x)$, $h_n(t, x) \rightarrow h(t, x)$ and $\sigma_n(t, x) \rightarrow \sigma(t, x)$, for all $t \in [0, T]$, and $x, y, z \in \mathbb{R}$ as $n \rightarrow \infty$, where b, h , and σ satisfy (AA.3.1)-(AA.3.3) for the same $C > 0$. Then the solutions of equations (3.32a) converge to solutions of (3.33a). Moreover, for all $t \in [0, T]$ and $p \geq 1$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{E}_G \left[\sup_{t \in [0, T]} |X_{n,t} - X_t|^{2p} \right] = 0.$$

Proof. Letting $\Delta_n(t) := \mathbb{E}_G \left[\sup_{s \in [0, t]} |X_{n,s} - X_s|^{2p} \right]$, we estimate

$$\begin{aligned}
 \Delta_n(t) & \leq 2^{2p-1} \left(|x_{n,0} - x_0|^{2p} + \mathbb{E}_G \left[\sup_{s \in [0, t]} \left| \int_0^s (b_n(u, X_{n,u}) - b(u, X_u)) du \right|^{2p} \right] \right. \\
 & \quad + \mathbb{E}_G \left[\sup_{s \in [0, t]} \left| \int_0^s (h_n(u, X_{n,u}) - h(u, X_u)) d\langle B \rangle_u \right|^{2p} \right] \\
 & \quad \left. + \mathbb{E}_G \left[\sup_{s \in [0, t]} \left| \int_0^s (\sigma_n(u, X_{n,u}) - \sigma(u, X_u)) dB_u \right|^{2p} \right] \right)
 \end{aligned}$$

By Lemma 30 in [77] and the Hölder inequality, we find

$$\begin{aligned}
& \mathbb{E}_G \left[\sup_{s \in [0, t]} \left| \int_0^s (\sigma_n(u, X_{n,u}) - \sigma(u, X_u)) dB_u \right|^{2p} \right] \\
& \leq \mathbb{E}_G \left[\left| \int_0^t (\sigma_n(u, X_{n,u}) - \sigma(u, X_u))^2 du \right|^p \right] \\
& \leq 2^{2p-1} t^{p-1} \left(\int_0^t \mathbb{E}_G \left[|\sigma_n(u, X_{n,u}) - \sigma_n(u, X_u)|^{2p} \right] du \right. \\
& \quad \left. + \int_0^t \mathbb{E}_G \left[|\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p} \right] du \right) \\
& \leq 2^{2p-1} t^{p-1} \left(C^{2p} \int_0^t \mathbb{E}_G \left[|X_{n,u} - X_u|^{2p} \right] du + \int_0^t \mathbb{E}_G \left[|\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p} \right] du \right) \\
& \leq 2^{2p-1} t^{p-1} \left(C^{2p} \int_0^t \Delta_n(u) du + \int_0^t \mathbb{E}_G \left[|\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p} \right] du \right).
\end{aligned}$$

Similarly, by the Hölder inequality, we get

$$\begin{aligned}
& \mathbb{E}_G \left[\sup_{s \in [0, t]} \left| \int_0^s (b_n(u, X_{n,u}) - b(u, X_u)) du \right|^{2p} \right] \\
& \leq 2^{2p-1} t^{2p-1} \left(C^{2p} \int_0^t \Delta_n(u) du + \int_0^t \mathbb{E}_G \left[|b_n(u, X_u) - b(u, X_u)|^{2p} \right] du \right),
\end{aligned}$$

and by the Hölder inequality and Lemma 34 in [77], we find

$$\begin{aligned}
& \mathbb{E}_G \left[\sup_{s \in [0, t]} \left| \int_0^s (h_n(u, X_{n,u}) - h(u, X_u)) d\langle B \rangle_u \right|^{2p} \right] \\
& \leq \mathbb{E}_G \left[\left| \int_0^t (h_n(u, X_{n,u}) - h(u, X_u)) d\langle B \rangle_u \right|^{2p} \right] \\
& \leq 2^{2p-1} t^{2p-1} \left(\int_0^t \mathbb{E}_G \left[|h_n(u, X_{n,u}) - h_n(u, X_u)|^{2p} \right] du \right. \\
& \quad \left. + \int_0^t \mathbb{E}_G \left[|\sigma_n(u, X_u) - h(u, X_u)|^{2p} \right] du \right) \\
& \leq 2^{2p-1} t^{p-1} \left(C^{2p} \int_0^t \mathbb{E}_G \left[|X_{n,u} - X_u|^{2p} \right] du + \int_0^t \mathbb{E}_G \left[|h_n(u, X_u) - h(u, X_u)|^{2p} \right] du \right) \\
& \leq 2^{2p-1} t^{p-1} \left(C^{2p} \int_0^t \Delta_n(u) du + \int_0^t \mathbb{E}_G \left[|h_n(u, X_u) - h(u, X_u)|^{2p} \right] du \right).
\end{aligned}$$

Summing up, we obtain

$$\Delta_n(T) \leq C' \left(|x_{n,0} - x_0|^{2p} + \int_0^T \Delta_n(u) du + \int_0^T \mathbb{E}_G \left[|b_n(u, X_u) - b(u, X_u)|^{2p} \right] du \right)$$

$$\begin{aligned}
 & + \int_0^T \mathbb{E}_G \left[\left| h_n(u, X_u) - h(u, X_u) \right|^{2p} \right] du \\
 & + \int_0^T \mathbb{E}_G \left[\left| \sigma_n(u, X_u) - \sigma(u, X_u) \right|^{2p} \right] du \Big),
 \end{aligned}$$

and hence by the Gronwall inequality, we have

$$\begin{aligned}
 \Delta_n(T) & \leq C'' \left(|x_{n,0} - x_0|^{2p} + \int_0^T \mathbb{E}_G \left[\left| b_n(u, X_u) - b(u, X_u) \right|^{2p} \right] du \right. \\
 & \quad + \int_0^T \mathbb{E}_G \left[\left| h_n(u, X_u) - h(u, X_u) \right|^{2p} \right] du \\
 & \quad \left. + \int_0^T \mathbb{E}_G \left[\left| \sigma_n(u, X_u) - \sigma(u, X_u) \right|^{2p} \right] du \right)
 \end{aligned}$$

Noting that

$$\begin{aligned}
 \mathbb{E}_G \left[\left| b_n(u, X_u) - b(u, X_u) \right|^{2p} \right] & \leq 2^{2p-1} \mathbb{E}_G \left[\left(\left| b_n(u, X_u) \right|^{2p} + \left| b(u, X_u) \right|^{2p} \right) \right] \\
 & \leq 2^{2p-1} C^{2p} \mathbb{E}_G \left[1 + \left(\left| X_u \right|^{2p} + \left| X_u \right|^{2p} \right) \right] \\
 & \leq 2^{2p-1} C^{2p} C''' (1 + 2|x_0|^{2p}) < \infty.
 \end{aligned}$$

by Proposition 4.1 in [43], which implies boundedness by a constant and integrability, i.e., $b_n(u, X_u) - b(u, X_u) \in L_G^{2p}(\Omega, \mathcal{F}) \subset L_G^1(\Omega, \mathcal{F})$. Similarly, one can obtain bounds for $\mathbb{E}_G \left[\left| h_n(u, X_u) - h(u, X_u) \right|^{2p} \right]$ and $\mathbb{E}_G \left[\left| \sigma_n(u, X_u) - \sigma(u, X_u) \right|^{2p} \right]$. Therefore, we can derive $\Delta_n(T) \rightarrow 0$ as $n \rightarrow \infty$ by the pointwise limits $\lim_{n \rightarrow \infty} (b_n(t, x) - b(t, x)) = 0$, $\lim_{n \rightarrow \infty} (h_n(t, x) - h(t, x)) = 0$, $\lim_{n \rightarrow \infty} (\sigma_n(t, x) - \sigma(t, x)) = 0$ and $\lim_{n \rightarrow \infty} (x_{n,0} - x_0) = 0$ and the dominated convergence Theorem 3.4 in [46] under the G -expectation $\mathbb{E}_G$, see also Theorem 10.1.10 in [99]. $\square$

3.4.3 Convexity of Nonlinear G -PDE Solutions

Consider the G -PDEs (HJB equations)

$$\frac{\partial u}{\partial t}(t, x) + F \left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x) \right) = 0, \quad (3.34)$$

with $u(T, x) = \varphi(x)$, where

$$F(t, x, y, z, w) = N(t, x, y, z) + 2G \left(\frac{1}{2} \sigma^2(t, x) w + M(t, x, y, z) \right), \quad (3.35)$$

$$G(a) = \frac{1}{2} \sup_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \{\sigma^2 a\} = \frac{1}{2} \max\{a\underline{\sigma}^2, a\bar{\sigma}^2\}, \quad (3.36)$$

$$M(t, x, y, z, w) = h(t, x)z + g(t, x, y, z\sigma(t, x)), \quad (3.37)$$

and

$$N(t, x, y, z, w) = b(t, x)z + f(t, x, y, z\sigma(t, x)). \quad (3.38)$$

In this section, we extend the convexity result Theorem 4.1 in [11] to the nonlinear G-PDEs (3.34) under a weaker convexity condition on the nonlinear drift $(x, y, z) \mapsto M(t, x, y, z)$ and $(x, y, z) \mapsto N(t, x, y, z)$ in the one-dimensional case, as required by applications in finance. For this we remark that, in our one-dimensional setting, the constant rank Theorem 2.3 in [11], see also Theorem 1.2 in [12], only requires convexity of the nonlinear drift $M(t, x, y, z)$ and $N(t, x, y, z)$ in $(x, y) \in \mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$, instead of global convexity in (x, y, z) . Precisely, we note that Condition (2.6) in Theorem 2.3 of [11] reduces to (3.40) below.

Theorem 3.4.5. (Theorem 2.3, [11]). Assume that $u(t, x)$ is a $\mathcal{C}^{2,4}([0, T] \times \mathbb{R})$ convex solution of the PDE

$$\frac{\partial u}{\partial t}(t, x) + F\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\right) = 0, \quad (3.39)$$

and that $F(t, x, y, z, w)$ is a $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^4)$ function that satisfies the elliptic condition

$$\frac{\partial F}{\partial w}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\right) > 0, \quad t \in [0, T], \quad x \in \mathbb{R},$$

and

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\right) + 2b \frac{\partial^2 F}{\partial x \partial y}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\right) \\ + b^2 \frac{\partial^2 F}{\partial y^2}\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\right) \geq 0, \end{aligned} \quad (3.40)$$

for all $t \in [0, T]$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. Then the sign $\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right)$ of $\frac{\partial^2 u}{\partial x^2}(t, x)$ is constant in $x \in \mathbb{R}$ for any $t \in (0, T)$, and we have

$$\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(s, x)\right) \geq \operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right), \quad 0 \leq s \leq t < T.$$

We note that in our one dimensional setting, the rank of $\frac{\partial^2 u}{\partial x^2}(t, x)$ is $\{0, 1\}$ -valued, with the relation $\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right) \equiv \operatorname{rank}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right)$ provided that $x \mapsto u(t, x)$ convex, $t \in [0, T]$, $x \in \mathbb{R}$. By adapting arguments of [11], using the constant rank

Theorem 3.4.5 and a new Lemma 3.4.7, we will prove the following Theorem 3.4.6 which will be used in the proofs of Theorems 5.3.1-5.3.2, Corollaries 5.3.4-5.3.5, and Theorems 5.3.7-5.3.8. Theorem 3.4.6 extends Theorem 4.1 of [11] on the convexity of the solution $u(t, x)$ of the PDE (3.34) by only assuming convexity in (x, y) and in (y, z) of the function $M(t, x, y, z)$ and $N(t, x, y, z)$ defined in (3.41) and (3.38) above, respectively, instead of joint convexity of $(x, y, z) \mapsto M(t, x, y, z)$ and $(x, y, z) \mapsto N(t, x, y, z)$.

In comparison with Theorem 3.2.6, in Theorem 3.4.6 below we need to deal with the lack of smoothness of the function $G(a)$. For this, we use a family of smooth approximations G_δ of G , which admit derivatives up to the desired orders and still yield the convexity of $u(t, x)$ after taking limits of the corresponding solutions $u_\delta(t, x)$.

Theorem 3.4.6. *Assume that Conditions (AA.3.1)-(AA.3.3) hold. Suppose that $u(t, x)$ is a $C^{1,2}([0, T] \times \mathbb{R})$ solution of (3.34) with terminal condition $u(T, x) = \varphi(x)$, and that the function*

$$M(t, x, y, z) := zh(t, x) + g(t, x, y, z\sigma(t, x)) \quad (3.41)$$

and

$$N(t, x, y, z) := zb(t, x) + f(t, x, y, z\sigma(t, x)) \quad (3.42)$$

satisfy the following conditions:

(HH₁) $(x, y) \mapsto M(t, x, y, z)$ and $(x, y) \mapsto N(t, x, y, z)$ are convex on $\mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$,

(HH₂) $(y, z) \mapsto M(t, x, y, z)$ and $(y, z) \mapsto N(t, x, y, z)$ are convex on $\mathbb{R}^2$ for every $(t, x) \in [0, T] \times \mathbb{R}$.

Then the function $x \mapsto u(t, x)$ is convex on $\mathbb{R}$ for all $t \in [0, T]$, provided that $u(T, x) = \varphi(x)$ convex in $x \in \mathbb{R}$.

Proof.

We proceed by extending the proof argument of Theorem 4.1 in [11] by using Theorem 3.4.5 on modified equations. Fixing $\delta > 0$, let

$$G_\delta(a) := \frac{1}{\delta} \int_{\mathbb{R}} \rho\left(\frac{a-b}{\delta}\right) G(b) db, \quad a \in \mathbb{R},$$

where $\rho(x)$ is a mollifying function on $\mathbb{R}$. Denote by $u_\delta(t, x)$ the solution of the PDE

$$\frac{\partial u_\delta}{\partial t}(t, x) + F_\delta\left(t, x, u_\delta(t, x), \frac{\partial u_\delta}{\partial x}(t, x), \frac{\partial^2 u_\delta}{\partial x^2}(t, x)\right) = 0, \quad (3.43)$$

with terminal condition $u_\delta(T, x) = \varphi(x)$, where

$$F_\delta(t, x, y, z, w) = N(t, x, y, z) + 2G_\delta(H(t, x, y, z, w)), \quad (3.44)$$

and

$$H(t, x, y, z, w) := \frac{w}{2}\sigma^2(t, x) + M(t, x, y, z).$$

We first prove the convexity of $u_\delta(t, x)$. We start by assuming that the following conditions, which are stronger than (AA.3.1)-(AA.3.3), hold for some $\eta \in (0, 1)$.

(HH₃) $b(\cdot, \cdot), h(\cdot, \cdot), \sigma(\cdot, \cdot), f(\cdot, \cdot, y, z), g(\cdot, \cdot, y, z)$ are bounded functions in the space $\mathcal{C}_b^{1+\eta/2, 2+\eta}([0, T] \times \mathbb{R})$ for all $y, z \in \mathbb{R}$, and $\varphi(\cdot) \in C_b^{4+\eta}(\mathbb{R})$ is bounded.

(HH₄) $\sigma(\cdot, \cdot)$ satisfies the bound

$$0 < c \leq \sigma(t, x) \leq C, \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (3.45)$$

for some constants $c, C > 0$.

(HH₅) For some $C > 0$ and $\alpha > 0$ we have

$$\begin{aligned} \left| \frac{\partial^2 f}{\partial z^2}(t, x, y, z) \right| &\leq \frac{C}{(1+x^2)^\alpha}, & (t, x, y, z) \in [0, T] \times \mathbb{R}^3, \\ \left| \frac{\partial^2 g}{\partial z^2}(t, x, y, z) \right| &\leq \frac{C}{(1+x^2)^\alpha}, & (t, x, y, z) \in [0, T] \times \mathbb{R}^3. \end{aligned}$$

Under Conditions (HH₃)-(HH₄), the function $F_\delta(t, y, z, w)$ defined by (3.44) is in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^4)$ and the solution $u_\delta(t, x)$ of (3.43) is in $\mathcal{C}^{2,4}([0, T] \times \mathbb{R})$ by Theorem 3.3.6. Besides, we note that by Theorem 3.3.5, the first and the second partial derivatives of $u_{\delta,k}(t, x)$ with respect to x are bounded uniformly in $(t, x) \in [0, T] \times \mathbb{R}$.

Setting $h(x) := (1+x^2)^{\alpha+1}$, for any $K \in \mathbb{R}$ and $\varepsilon > 0$, we define

$$v_K(t, x) := e^{-Kt}h(x) \quad \text{and} \quad u_{\delta,\varepsilon}(t, x) := u_\delta(t, x) + \varepsilon v_K(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Next, we let

$$E_{\delta,\varepsilon} := \left\{ (t, x) \in [0, T] \times \mathbb{R} : \frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x) \leq 0 \right\}$$

and suppose that $E_{\delta,\varepsilon} \neq \emptyset$. From the relation $h''(x) \geq (1+x^2)^\alpha$ and the bound $\left| \frac{\partial^2 u_\delta}{\partial x^2}(t, x) \right| \leq C$, we get

$$\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x) \geq \varepsilon e^{-Kt}(1+x^2)^\alpha - C,$$

therefore there exists $R_\varepsilon > 0$ such that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x) > 0$ for all $|x| \geq R_\varepsilon$, and we have $E_{\delta,\varepsilon} \subseteq [0, T] \times B(R_\varepsilon)$, where $B(R_\varepsilon)$ is the centered open ball with radius R_ε , so that $E_{\delta,\varepsilon}$ is compact. Consequently, the supremum

$$\tau_0 := \sup \{t \in [0, T] : (t, x) \in E_{\delta,\varepsilon} \text{ for some } x \in \mathbb{R}\}$$

is attained at some $(\tau_0, x_0) \in E_{\delta,\varepsilon}$ with $x_0 \in B(R_\varepsilon)$, such that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(\tau_0, x_0) \leq 0$. In addition, by the convexity assumption on $x \mapsto u_\delta(T, x)$ we have

$$\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(T, x) = \frac{\partial^2 u_\delta}{\partial x^2}(T, x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(T, x) \geq \varepsilon e^{-KT} h''(x) > 0, \quad x \in \mathbb{R},$$

hence $\tau_0 < T$ and by the continuity of $u_{\delta,\varepsilon}$ we have $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(\tau_0, x) \geq 0$, $x \in \mathbb{R}$, since $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x) > 0$ for all $t \in (\tau_0, T)$ and $x \in \mathbb{R}$. Consequently, the function $u_{\delta,\varepsilon}(t, x)$ is convex in x on $[\tau_0, T] \times B(R_\varepsilon)$.

On the other hand, we note that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(\tau_0, x_0) = 0$ for $x_0 \in B(R_\varepsilon)$, and that $u_{\delta,\varepsilon}(t, x)$ satisfies the equation

$$\frac{\partial u_{\delta,\varepsilon}}{\partial t}(t, x) + F_{K,\delta,\varepsilon}\left(t, x, u_{\delta,\varepsilon}(t, x), \frac{\partial u_{\delta,\varepsilon}}{\partial x}(t, x), \frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x)\right) = 0,$$

where

$$\begin{aligned} F_{K,\delta,\varepsilon}(t, x, y, z, w) := & -\varepsilon \frac{\partial v_K}{\partial t}(t, x) + N\left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x)\right) \\ & + 2G_\delta\left(H\left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), w - \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x)\right)\right). \end{aligned} \quad (3.46)$$

By the constant rank Theorem 3.4.5 and Lemma 3.4.7 below we deduce that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x)$ vanishes on $[\tau_0, T] \times B(R_\varepsilon)$ hence $\tau_0 = T$, which is a contradiction showing that $E_{\delta,\varepsilon} = \emptyset$. Therefore, we have

$$\frac{\partial^2 u_\delta}{\partial x^2}(t, x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) = \frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t, x) > 0, \quad (t, x) \in [0, T] \times \mathbb{R},$$

and after letting ε tend to 0, we conclude that

$$\frac{\partial^2 u_\delta}{\partial x^2}(t, x) \geq 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Next, we relax the above Conditions (HH_3) – (HH_5) under the $(AA.3.1)$ – $(AA.3.3)$ by applying the above argument to sequences $(b_n)_{n \geq 1}$, $(h_n)_{n \geq 1}$, $(\sigma_n)_{n \geq 1}$, $(f_n)_{n \geq 1}$, $(g_n)_{n \geq 1}$ of $\mathcal{C}_b^{2,3}$ bounded functions and $(\varphi_n)_{n \geq 1}$ of $\mathcal{C}_b^5$ bounded functions satisfying (HH_3) and $(AA.3.1)$ – $(AA.3.3)$, and converging pointwise respectively to b, h, σ, f, g , and φ , while preserving the convexity of the approximations $(\varphi_n)_{n \geq 1}$ and $(M_n)_{n \geq 1}, (N_n)_{n \geq 1}$ defined in (3.41) and (3.42) , as well as Condition (3.45) . In order to satisfy (HH_5) , we replace f_n, g_n with $\tilde{f}_n, \tilde{g}_n$ obtained by smoothing out the piecewise $\mathcal{C}^1$ functions

$$\begin{aligned} x \mapsto & \left(f_n(t, -n, y, z) + (x+n) \frac{\partial f_n}{\partial x}(t, -n, y, z) \right) \mathbf{1}_{(-\infty, -n)}(x) \\ & + f_n(t, x, y, z) \mathbf{1}_{[-n, n]}(x) + \left(f_n(t, n, y, z) + (x-n) \frac{\partial f_n}{\partial x}(t, n, y, z) \right) \mathbf{1}_{(n, \infty)}(x) \end{aligned}$$

and

$$\begin{aligned} x \mapsto & \left(g_n(t, -n, y, z) + (x+n) \frac{\partial g_n}{\partial x}(t, -n, y, z) \right) \mathbf{1}_{(-\infty, -n)}(x) \\ & + g_n(t, x, y, z) \mathbf{1}_{[-n, n]}(x) + \left(g_n(t, n, y, z) + (x-n) \frac{\partial g_n}{\partial x}(t, n, y, z) \right) \mathbf{1}_{(n, \infty)}(x) \end{aligned}$$

by convolution in x with the Gaussian kernel $e^{-nx^2/2}/\sqrt{2\pi/n}$, and we apply the continuous dependence Theorem 3.4.3 by letting n tend to infinity. Finally, the convexity of the original solution $u(t, x)$ then follows by taking the limit as $\delta \rightarrow 0$, as on page 49 of [11]. $\square$

The proof of Theorem 3.4.6 relies on the following lemma.

Lemma 3.4.7. *Under conditions (HH_1) – (HH_5) above we can choose $K \in \mathbb{R}$ such that the function $(x, y) \mapsto F_{K, \delta, \varepsilon}(t, x, y, z, 0)$ in (3.46) satisfies the Condition (3.40).*

Proof. We need to show that

$$\begin{aligned} S(b) &:= \frac{\partial^2 F_{K, \delta, \varepsilon}}{\partial x^2} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), 0 \right) + 2b \frac{\partial^2 F_{K, \delta, \varepsilon}}{\partial x \partial y} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), 0 \right) \\ &\quad + b^2 \frac{\partial^2 F_{K, \delta, \varepsilon}}{\partial y^2} \left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), 0 \right) \\ &\geq 0, \end{aligned} \tag{3.47}$$

for all $b \in \mathbb{R}$. Using (3.46), we find

$$\frac{\partial F_{K, \delta, \varepsilon}}{\partial x}(t, x, y, z, 0) = -\varepsilon \frac{\partial^2 v_K}{\partial x \partial t}(t, x) + \frac{\partial N}{\partial x} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x) \right)$$

$$+ 2 \frac{\partial H}{\partial x} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \frac{\partial G_\delta}{\partial a},$$

and

$$\begin{aligned} \frac{\partial F_{K,\delta,\varepsilon}}{\partial y}(t, x, y, z, 0) &= \frac{\partial N}{\partial y} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x) \right) \\ &+ 2 \frac{\partial H}{\partial y} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \frac{\partial G_\delta}{\partial a}, \end{aligned}$$

where we write for short

$$\frac{\partial G_\delta}{\partial a} := \frac{\partial G_\delta}{\partial a} \left(H(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x)) \right).$$

Similarly, we find second derivatives for $F_{K,\delta,\varepsilon}$

$$\begin{aligned} \frac{\partial^2 F_{K,\delta,\varepsilon}}{\partial x^2}(t, x, y, z, 0) &= -\varepsilon \frac{\partial^3 v_K}{\partial x^2 \partial t}(t, x) + \frac{\partial^2 N}{\partial x^2} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x) \right) \\ &+ 2 \frac{\partial^2 H}{\partial x^2} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \frac{\partial G_\delta}{\partial a} \\ &+ 2 \left(\frac{\partial H}{\partial x} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \right)^2 \frac{\partial^2 G_\delta}{\partial a^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 F_{K,\delta,\varepsilon}}{\partial x \partial y}(t, x, y, z, 0) &= \frac{\partial^2 N}{\partial x \partial y} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x) \right) \\ &+ 2 \frac{\partial H}{\partial x} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \\ &\times \frac{\partial H}{\partial y} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \frac{\partial^2 G_\delta}{\partial a^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 F_{K,\delta,\varepsilon}}{\partial y^2}(t, x, y, z, 0) &= \frac{\partial^2 N}{\partial y^2} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x) \right) \\ &+ 2 \frac{\partial^2 H}{\partial y^2} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \frac{\partial G_\delta}{\partial a} \\ &+ 2 \left(\frac{\partial H}{\partial y} \left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \right)^2 \frac{\partial^2 G_\delta}{\partial a^2}. \end{aligned}$$

Therefore, we can write

$$S(b) = -\varepsilon \frac{\partial^3 v_K}{\partial x^2 \partial t}(t, x) + S_1(b) + 2S_2(b) \frac{\partial G_\delta}{\partial a} + 2S_3(b) \frac{\partial^2 G_\delta}{\partial a^2},$$

where

$$\begin{aligned} S_1(b) &= \frac{\partial^2 N}{\partial x^2} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x) \right) + 2b \frac{\partial^2 N}{\partial x \partial y} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x) \right) \\ &\quad + b^2 \frac{\partial^2 N}{\partial y^2} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x) \right), \end{aligned}$$

and

$$\begin{aligned} S_2(b) &= \frac{\partial^2 H}{\partial x^2} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \\ &\quad + 2b \frac{\partial^2 H}{\partial x \partial y} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \\ &\quad + b^2 \frac{\partial^2 H}{\partial y^2} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right), \end{aligned}$$

and

$$\begin{aligned} S_3(b) &= \left(\frac{\partial H}{\partial x} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \right. \\ &\quad \left. + b \frac{\partial H}{\partial y} \left(t, x, u_{\delta, \varepsilon}(t, x), \frac{\partial u_{\delta, \varepsilon}}{\partial x}(t, x), -\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) \right) \right)^2, \end{aligned}$$

for all $b \in \mathbb{R}$. Noting that since $G(a)$ is a convex function on $\mathbb{R}$, so is $G_\delta(a)$. Thus, we have $S_3(b) \partial^2 G_\delta / \partial a^2 \geq 0$, $b \in \mathbb{R}$. Besides, we have $0 \leq \partial G_\delta(a) / \partial a \leq \underline{\sigma}^2 + \bar{\sigma}^2$, $a \in \mathbb{R}$. Hence, following the proof arguments of the Lemma 3.2.7 in Section 3.2.3, under the hypotheses (HH_1) – (HH_5) , we can choose positive constant $K_1(\varepsilon, \alpha)$ and $K_2(\varepsilon, \alpha, \underline{\sigma}^2, \bar{\sigma}^2)$ large enough such that

$$S_1(b) - \frac{\varepsilon}{2} \frac{\partial^3 v_{K_1}}{\partial x^2 \partial t}(t, x) \geq 0, \quad 2S_2(b) \frac{\partial G_\delta}{\partial a} - \frac{\varepsilon}{2} \frac{\partial^3 v_{K_2}}{\partial x^2 \partial t}(t, x) \geq 0, \quad b \in \mathbb{R}.$$

Taking $K(\varepsilon, \alpha) = \max \{K_1(\varepsilon, \alpha), K_2(\varepsilon, \alpha, \underline{\sigma}^2, \bar{\sigma}^2)\}$, we conclude to $S(b) \geq 0$, $b \in \mathbb{R}$. $\square$

Remark 3.4.8. One can follow the procedure above to obtain a concave version of Theorem 3.4.6 for the solution $u(t, x)$ of the PDE (3.34), i.e., if we change Conditions (HH_1) – (HH_2) by the following conditions:

(HH'_1) $(x, y) \mapsto M(t, x, y, z)$ and $(x, y) \mapsto N(t, x, y, z)$ are concave on $\mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$,

(HH'_2) $(y, z) \mapsto M(t, x, y, z)$ and $(y, z) \mapsto N(t, x, y, z)$ are concave on $\mathbb{R}^2$ for every $(t, x) \in [0, T] \times \mathbb{R}$.

Then the function $x \mapsto u(t, x)$ is concave on $\mathbb{R}$ for all $t \in [0, T]$, provided that $u(T, x) = \varphi(x)$ concave in $x \in \mathbb{R}$.

Chapter 4

Stochastic Orderings by g -Expectations

In this chapter, we define nonlinear g -evaluations and g -expectations generated by solutions of the backward stochastic differential equations. Then in Section 4.2, by using these nonlinear expectations, we introduce new concepts of monotonic, convex and increasing convex g -stochastic orderings, which leads to more general concepts of orderings in comparison with the standard linear expectation setting. In order to have a better understanding of the g -stochastic orderings, in Section 4.3, we first give some characterizations for the cases of sublinear and convex generators. Section 4.4 provides the main results of this chapter for the g -stochastic orderings with two approaches. The first approach presents in Section 4.4.1, which is relied on comparison results of solutions of the semilinear parabolic PDEs, we derive several sufficient conditions for the convex, increasing convex and monotonic g -stochastic orderings of diffusion processes under the nonlinear g -expectations and g -evaluations. This first approach needs quite strong assumptions since it requires convexity of the function

$$(x, y, z) \mapsto f_i(t, x, y, z) := z\mu_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)),$$

for both $i = 1$ and $i = 2$. In the second approach, given in Section 4.4.2, we use stochastic arguments approach to get those g -stochastic orderings results that only requires convexity of the function $f_i(t, x, y, z)$ for $i = 1$ or $i = 2$. As consequences, in Section 4.5, we derive results for comparing g -risk measures which are induced by the g -expectations. They can be also viewed as concave g -stochastic orderings. We note that our approaches also rely on comparison results for forward-backward stochastic differential equations and on several extensions of convexity, monotonicity and continuous dependence property of the solutions of associated semilinear parabolic partial differential equations as presented in Section 3.2 of Chapter 3. In

the last Section 4.6, applications of g -stochastic orderings to contingent claim price comparison under different hedging portfolio constraints are provided.

4.1 Introduction

In comparison with standard mean-variance analysis, partial orderings of probability distributions provide additional information which can be used in applications to risk management. In this framework, a random variable X is said to be dominated by another random variable Y if

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)], \quad (4.1)$$

for all $\phi : \mathbb{R} \rightarrow \mathbb{R}$ in a certain class of functions, where $\mathbb{E}[X]$ denotes the usual expectation of the random variable X . For example, if X and Y represent the lifetimes of two devices A and B then the stochastic ordering (4.1) for all non-decreasing and bounded functions ϕ , tells that the device B will be likely survive longer than the device A . Stochastic ordering has found a wide range of applications in various fields such as reliability theory, economics, actuarial sciences, operation research, risk management, biology, option evaluation, etc., see e.g. [68, 22, 85, 90, 56, 7, 83]. In the von Neumann-Morgenstern expected utility theory, a portfolio with return $X^{(2)}$ dominates a portfolio with return $X^{(1)}$ in the increasing concave order if (4.1) holds for all non-decreasing concave utility functions ϕ , in which case, the second portfolio would be preferred over the first portfolio by risk-averse investors, see Theorem 1.35 in [90]. Similarly, if (4.1) holds for all non-decreasing convex utility functions ϕ , the second portfolio would be preferred over the first portfolio by risk-seeking investors, see Theorem 1.37 and the notion of stochastic dominance in Theorem 2.4 in [90].

Comparison bounds in convex ordering have been established in [26] for option prices with convex payoff functions in the continuous diffusion case, via a martingale approach based on the classical Kolmogorov equation and the propagation of convexity property for Markov semigroups. This approach has been generalized to semimartingales in [38], [8], [9], see also [51], [3], [66].

On the other hand, empirical experiments have shown that many uncertain phenomena cannot be fully modeled using the linear expectation operator $\mathbb{E}[\cdot]$, as in e.g. the Allais and Ellsberg paradoxes. Choquet's expectation has been proposed as an nonlinear alternative that relies on capacities instead of probability measures, see

[37, 36] for the construction of monotonic and increasing convex stochastic orders and application to financial optimization.

In this chapter, we go beyond the linear setting of the standard expectation operator $\mathbb{E}$, and we study stochastic orderings from the point of view of nonlinear expectations, which leads to more general concepts of stochastic ordering in nonadditive models. Uncertainty orders have been constructed in [92] on the sublinear expectation space. Here, we extend this approach to the settings of g -expectations and g -evaluations, which are not sublinear in general. Namely, we use the g -expectation and g -evaluation $\mathcal{E}_g[X]$ of a random variable X , defined by [74], [75] as the initial value Y_0 for a pair $(Y_t, Z_t)_{t \in [0, T]}$ of adapted processes which solves a Backward Stochastic Differential Equation (BSDE) of the form

$$-dY_t = g(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \quad 0 \leq t \leq T, \quad (4.2)$$

with terminal condition $Y_T = X$, where the function $g(t, x, y, z)$ is called the BSDE generator, $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion, and $(X_t)_{t \in \mathbb{R}_+}$ is diffusion process driven by $(W_t)_{t \in \mathbb{R}_+}$.

BSDEs were first introduced by [13] in the linear case, and then extended by [72] to the nonlinear case. BSDEs and the corresponding g -expectations $\mathcal{E}_g$ have been applied to contingent claim pricing, stochastic control theory, utility maximization and dynamic risk measures, see e.g. [72, 74, 27, 65, 75, 75, 32, 28, 29, 49].

In this chapter, we study stochastic orderings from the point of view of nonlinear g -expectations. Consider two risky assets with positive prices $(X_t^{(i)})_{t \in [0, T]}$, $i = 1, 2$, given by

$$dX_t^{(i)} = X_t^{(i)} a_i(t, X_t^{(i)})dt + X_t^{(i)} b_i(t, X_t^{(i)})dW_t, \quad i = 1, 2,$$

a risk-free asset $E_t := E_0 e^{rt}$, where r is an interest rate, and two portfolios with prices $Y_t^{(i)} = p_t^{(i)} E_t + q_t^{(i)} X_t^{(i)}$, under the self-financing conditions

$$\begin{aligned} dY_t^{(i)} &= q_t^{(i)} dE_t + p_t^{(i)} dX_t^{(i)} \\ &= (rY_t^{(i)} + (a_i(t, X_t^{(i)}) - r)p_t^{(i)} X_t^{(i)})dt + p_t^{(i)} X_t^{(i)} b_i(t, X_t^{(i)})dW_t, \quad i = 1, 2, \end{aligned} \quad (4.3)$$

which lead to the BSDEs

$$Y_t^{(i)} = Y_T^{(i)} + \int_t^T g_i(s, X_s^{(i)}, Y_s^{(i)}, Z_s^{(i)})ds - \int_t^T Z_s^{(i)} dW_s,$$

where $Z_t^{(i)} := p_t^{(i)} X_t^{(i)} b_i(t, X_t^{(i)})$ and

$$g_i(t, x, y, z) := -ry - z\theta_i(t, x), \quad \text{with} \quad \theta_i(t, x) = \frac{a_i(t, x) - r}{b_i(t, x)}, \quad i = 1, 2.$$

We say that $X_T^{(1)}$ is dominated by $X_T^{(2)}$ in the convex g_1, g_2 -stochastic ordering, i.e. $X_T^{(1)} \leq_{g_1, g_2}^{conv} X_T^{(2)}$ if

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})]$$

for all convex functions ϕ , where $Y_0^{(i)} := \mathcal{E}_{g_i}[\phi(X_T^{(i)})]$ represent the fair prices at time $t = 0$ of the options with convex payoffs $Y_T^{(i)} := \phi(X_T^{(i)})$, $i = 1, 2$.

Here, the use of distinct generators $g_i(t, x, y, z)$ is motivated by the comparison of different contingent claims under different hedging strategies, for example in the case of misspecified volatility coefficients or for hedging under constraints, see the examples presented in Section 4.6. In case $g(t, x, y, z) = \beta(t)z$, the increasing convex g -stochastic ordering $\leq_g^{conv}$ is equivalent to the (classical) increasing convex ordering with respect to the capacities μ_g , where $\mu_g[A] := \mathcal{E}_g[\mathbf{1}_A]$, $A \in \mathcal{F}$, see [16, 37].

Main results

In Theorem 4.4.1, we derive sufficient conditions on two BSDE generators $g_1(t, x, y, z)$, $g_2(t, x, y, z)$ for the convex ordering

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

in nonlinear expectations $\mathcal{E}_{g_1}$, $\mathcal{E}_{g_2}$, for all convex functions $\phi(x)$ with polynomial growth, where $X_T^{(1)}$ and $X_T^{(2)}$ are the terminal values of the solutions of two forward Stochastic Differential Equations (SDEs)

$$dX_t^{(1)} = \mu_1(t, X_t^{(1)})dt + \sigma_1(t, X_t^{(1)})dW_t,$$

$$dX_t^{(2)} = \mu_2(t, X_t^{(2)})dt + \sigma_2(t, X_t^{(2)})dW_t,$$

with $X_0^{(1)} = X_0^{(2)}$, under the bound

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}.$$

The proof of Theorem 4.4.1 is based on the comparison Theorem 2.4 in Appendix C of [80], provided that

$$z\mu_1(t, x) + g_1(t, x, y, z\sigma_1(t, x)) \leq z\mu_2(t, x) + g_2(t, x, y, z\sigma_2(t, x)), \quad x, y, z \in \mathbb{R}, \quad t \in [0, T],$$

and both functions $(x, y, z) \mapsto f_i(t, x, y, z) := z\mu_i(t, x) + g_i(t, x, y, z\sigma_i(t, x))$ are convex in (x, y) and in (y, z) on $\mathbb{R}^2$ for $i = 1, 2$ and $t \in [0, T]$. Several extensions are considered on increasing convex and monotonic orderings in Theorem 4.4.2 and Corollaries 4.4.3-4.4.4, with the particular cases of equal drifts and equal volatilities treated in Corollaries 4.4.5 and 4.4.6.

This approach requires convexity of the function $(x, y, z) \mapsto f_i(t, x, y, z)$ for both $i = 1$ and $i = 2$. In Section 4.4.2 we relax those conditions using a stochastic calculus approach, by only requiring the convexity of the function $f_i(t, x, y, z)$ for $i = 1$ or $i = 2$ in Theorems 4.4.7 and 4.4.8, which respectively deal with the convex and increasing convex orders.

Related comparison results for g -risk measures are presented in Corollaries 4.5.2-4.5.3, using the quantity $\mathcal{E}_{g_i}[-\phi(X_T^{(i)}) | \mathcal{F}_t] = -\mathcal{E}_{g_i^{(-1)}}[\phi(X_T^{(i)}) | \mathcal{F}_t]$ which makes sense as a dynamic g -risk measure, where $g_i^{(-1)}(t, x, y, z) := -g_i(t, x, -y, -z)$. Here, the choice of generator function g_i determines the investor's portfolio strategy and the corresponding risk measures, see Section 4.6 for examples.

The proofs of Theorems 4.4.1-4.5.3 rely on an extension of convexity properties of the solutions of nonlinear parabolic Partial Differential Equations (PDEs) which is proved in Theorem 3.2.6. The convexity properties of solutions of nonlinear PDEs have been studied by several authors, see e.g. Theorem 3.1 in [60], Theorem 2.1 in [33], and Theorem 1.1 in [11], see also Theorem 1 in [1] in the elliptic case. Those works typically require global convexity of the nonlinear drifts $f(t, x, y, z)$ in all state variables (x, y, z) , a condition which is too strong for our applications to finance in Examples 4.6.1-4.6.5 below. For this reason, in Theorem 3.2.6 we extend Theorem 1.1 of [11] in dimension one, by replacing the global convexity of the nonlinear drift $f_i(t, x, y, z)$ in (x, y, z) with its convexity in (x, y) and (y, z) , $i = 1, 2$.

4.2 g -Stochastic Orderings

In this section, we recall some notations on FBSDEs which have been considered in Chapter 3 and we also introduce g -expectations, g -evaluations and g -stochastic orderings generated by solutions of the BSDEs. For a given $T > 0$, let $(W_t)_{t \in [0, T]}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{F}_t)_{t \in [0, T]}$ the augmented filtration such that $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t) \vee \mathcal{N}$, $t \in [0, T]$, where $\mathcal{N}$ is the collection of all $\mathbb{P}$ -null sets. We also let $L^2(\Omega, \mathcal{F}_t) := L^2(\Omega, \mathcal{F}_t, \mathbb{P})$, $t \in [0, T]$.

Forward-Backward SDEs

Consider a forward SDE of the form

$$dX_s^{t,x} = \mu(s, X_s^{t,x})dt + \sigma(s, X_s^{t,x})dW_s, \quad 0 \leq t \leq s \leq T. \quad (4.5)$$

with initial condition $X_t^{t,x} = x$. The associated backward SDE is defined by

$$Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T g(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})d\tau - \int_s^T Z_\tau^{t,x}dW_\tau, \quad 0 \leq t \leq s \leq T, \quad (4.6)$$

with terminal condition $Y_T^{t,x} = \phi(X_T^{t,x}) \in L^2(\Omega, \mathcal{F}_T)$, where coefficients of (4.5) are assumed throughout this chapter to satisfy Condition (A.4.1):

(A.4.1) For every $t \in [0, T]$, the functions $x \mapsto \mu(t, x)$ and $x \mapsto \sigma(t, x)$ are globally Lipschitz, i.e.

$$|\mu(t, x) - \mu(t, y)| \leq C|x - y| \quad \text{and} \quad |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|, \quad x, y \in \mathbb{R},$$

In particular, $x \mapsto \mu(t, x)$ and $x \mapsto \sigma(t, x)$ satisfy the linear growth conditions

$$|\mu(t, x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma(t, x)| \leq C(1 + |x|), \quad x \in \mathbb{R},$$

for some positive constant $C > 0$.

The generator $g(t, x, y, z)$ of (4.6) for all $x, y, z \in \mathbb{R}$, satisfies Conditions (A.4.2)-(A.4.3) throughout the thesis:

(A.4.2) The function $g(t, x, y, z)$ is uniformly Lipschitz in (x, y, z) , i.e., there exists $C > 0$ such that

$$|g(t, x_2, y_2, z_2) - g(t, x_1, y_1, z_1)| \leq C(|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|)$$

for $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, $t \in [0, T]$,

(A.4.3) $g(t, x, 0, 0) = 0$, for all $t \in [0, T]$, $x \in \mathbb{R}$.

The function ϕ satisfy the Condition (A.4.4):

(A.4.4) The function ϕ is continuous on $\mathbb{R}$ and has the polynomial growth

$$|\phi(x)| \leq C(1 + |x|^p), \quad x \in \mathbb{R}, \text{ for some } p \geq 1/2 \text{ and } C > 0. \quad (4.7)$$

By Theorem 2.1 of [27], see Proposition 2.2 of [72], under (A.4.2)-(A.4.4) there exists a unique pair $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$ of adapted processes in $L^2(\Omega \times [0, T])$ that solves the BSDE (4.6).

Next, we state the definition of the g -evaluation and g -expectation.

g -evaluation and g -expectation

Definition 4.2.1. Given $\xi \in L^2(\Omega, \mathcal{F}_T)$ and the backward SDE

$$\begin{cases} dY_t^{0,x} = -g(t, X_t^{0,x}, Y_t^{0,x}, Z_t^{0,x})dt + Z_t^{0,x}dW_t, & 0 \leq t \leq T, \\ Y_T^{0,x} = \xi, \end{cases} \quad (4.8)$$

we respectively call

$$\mathcal{E}_g[\xi] := Y_0^{0,x} \quad \text{and} \quad \mathcal{E}_g[\xi \mid \mathcal{F}_t] := Y_t^{0,x}$$

the g -evaluation and the $\mathcal{F}_t$ -conditional g -evaluation of ξ , $t \in [0, T]$.

Under (A.4.3), one can show in addition that the map $\xi \mapsto \mathcal{E}_g[\xi]$ preserves all properties of the classical expectation $\mathbb{E}$, except for linearity and the property $\mathcal{E}_g[c] = c$ for constant $c \in \mathbb{R}$, see Relation (34) and Theorem 3.4 in [75].

In the sequel we make the (stronger than (A.4.3)) assumption

$$(A.4.3') \quad g(\cdot, x, y, 0) = 0 \text{ a.s. for all } x, y \in \mathbb{R},$$

the g -evaluation $\mathcal{E}_g$ becomes the g -expectation, which satisfies the property $\mathcal{E}_g[c] = c$ for constant $c \in \mathbb{R}$, see Relation (36.2) and Lemma 36.3 in [74]. We note that the results of Sections 4.4 and 4.5 remain valid for g -expectations if we assume (A.4.3') instead of (A.4.3).

In the sequel, we will state definitions of orderings under the g -evaluation $\mathcal{E}_g[\cdot]$, and more generally under the conditional g -evaluation and the conditional g -expectation $\mathcal{E}_g[\cdot \mid \mathcal{F}_t]$, $t \in [0, T]$, under (A.4.3') instead of (A.4.3).

 g -stochastic orderings

Stochastic orderings with respect to capacity have been studied in [37, 36] by using Choquet's expectation and uncertainty orders have been constructed in [92] on the sublinear expectation space. Here, we extend this approach to the comparison of random variables $X^{(1)}, X^{(2)}$ in the settings of g -expectations and g -evaluations, which are not sublinear in general, via the condition

$$\mathcal{E}_{g_1}[\phi(X^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X^{(2)})], \quad (4.9)$$

in nonlinear expectations $\mathcal{E}_{g_1}, \mathcal{E}_{g_2}$, for all $\phi(x)$ in a certain class of functions having polynomial growth. In general, different portfolios or hedging strategies may corresponding to different generators g_1, g_2 as can be seen in Examples 4.6.3 and 4.6.5.

Definition 4.2.2. Let g_1, g_2 satisfy (A.4.2)-(A.4.3). For any $X^{(1)}, X^{(2)} \in L^2(\Omega, \mathcal{F}_T)$, we say that

- 1) $X^{(1)}$ is dominated by $X^{(2)}$ in the monotonic g_1, g_2 -ordering, i.e. $X^{(1)} \leq_{g_1, g_2}^{\text{mon}} X^{(2)}$, if (4.9) holds for all non-decreasing functions $\phi(x)$ satisfying (4.7).
- 2) $X^{(1)}$ is dominated by $X^{(2)}$ in the convex g_1, g_2 -ordering, i.e. $X^{(1)} \leq_{g_1, g_2}^{\text{conv}} X^{(2)}$, if (4.9) holds for all convex functions $\phi(x)$ satisfying (4.7).
- 3) $X^{(1)}$ is dominated by $X^{(2)}$ in the increasing convex g_1, g_2 -ordering, i.e. $X^{(1)} \leq_{g_1, g_2}^{\text{iconv}} X^{(2)}$, if (4.9) holds for all non-decreasing convex functions $\phi(x)$ satisfying (4.7).

We note that

$$(i) \quad X^{(1)} \leq_{g_1, g_2}^{\text{mon}} X^{(2)} \implies X^{(1)} \leq_{g_1, g_2}^{\text{iconv}} X^{(2)}, \text{ and}$$

$$(ii) \quad X^{(1)} \leq_{g_1, g_2}^{\text{conv}} X^{(2)} \implies X^{(1)} \leq_{g_1, g_2}^{\text{iconv}} X^{(2)},$$

and we simply write $\leq_g^{\text{mon}}, \leq_g^{\text{conv}}, \leq_g^{\text{iconv}}$ if $g_1 = g_2 := g$.

In Example 4.6.1 below, we have $g_1(t, x, y, z) = g_2(t, x, y, z)$ if two processes $(X_t^{(1)})_{t \in [0, T]}$ and $(X_t^{(2)})_{t \in [0, T]}$ have the same market risk premiums, i.e., $\frac{a_1(t, x) - r}{b_1(t, x)} = \frac{a_2(t, x) - r}{b_2(t, x)}$, for all $(t, x) \in [0, T] \times \mathbb{R}$. Also, we note for the g -expectation $\mathcal{E}_g[\xi]$ that the generator function g does not affect the original probability distribution of ξ , hence it is possibly reasonable to define the g -stochastic orderings under different generator functions $g_1 \neq g_2$.

Replacing convex functions with concave functions yields the corresponding notions of concave and increasing concave g_1, g_2 -orderings denoted by $X^{(1)} \leq_{g_1, g_2}^{\text{conc}} X^{(2)}$ and $X^{(1)} \leq_{g_1, g_2}^{\text{iconc}} X^{(2)}$, respectively, which are characterized in the next proposition.

Proposition 4.2.3. Given two random variables $X^{(i)} \in L^2(\Omega, \mathcal{F}_T)$, $i = 1, 2$ and the generator $g^{(-1)}(t, x, y, z) := -g(t, x, -y, -z)$, for $(t, x, y, z) \in [0, T] \times \mathbb{R}^3$, we have that

$$(i) \quad X^{(1)} \leq_g^{\text{conc}} X^{(2)} \text{ if and only if } -X^{(2)} \leq_{g^{(-1)}}^{\text{conv}} -X^{(1)}.$$

$$(ii) \quad X^{(1)} \leq_g^{\text{iconc}} X^{(2)} \text{ if and only if } -X^{(2)} \leq_{g^{(-1)}}^{\text{iconv}} -X^{(1)}.$$

Proposition 4.2.3 is a direct consequence of the following lemma.

Lemma 4.2.4. Letting $g^{(a)}(t, x, y, z) := ag(t, x, y/a, z/a)$, $x, y, z \in \mathbb{R}$, $t \in [0, T]$, we have

$$\mathcal{E}_{g^{(a)}}[a\xi] = a\mathcal{E}_{g^{(1)}}[\xi], \quad a \neq 0,$$

Proof. Given $(X_t)_{t \in [0, T]}$ an $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process, let $(Y_t, Z_t)_{t \in [0, T]}$ denote the solution of (4.8) with terminal condition $Y_T = \xi \in L^2(\Omega, \mathcal{F}_T)$, and let $(\bar{Y}_t, \bar{Z}_t)_{t \in [0, T]}$ denote the solution of the backward SDE

$$\bar{Y}_t = a\xi + \int_t^T g^{(a)}(s, X_s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T],$$

with generator $g^{(a)}(t, x, y, z)$ and terminal condition $\bar{Y}_T = a\xi$, i.e.

$$\frac{\bar{Y}_t}{a} = \xi + \int_t^T g\left(s, X_s, \frac{\bar{Y}_s}{a}, \frac{\bar{Z}_s}{a}\right) ds - \frac{1}{a} \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T].$$

The uniqueness of the solution (Y_t, Z_t) of the backward SDEs

$$Y_t = \xi + \int_t^T g(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

yields $\bar{Z}_t/a = Z_t$ and $\bar{Y}_t/a = Y_t$, $t \in [0, T]$, and we conclude by taking $t = 0$. $\square$

4.3 Some Characterizations of g -Stochastic Orderings

For a better understanding of our stochastic orderings, we provide some characterizations and transfer them to probability distributions setting. First, we consider the case when the generator functions are convex functions. Denote by $\mathcal{M}$ the set of all absolutely continuous probability measures $\mathbb{Q}$ defined on $(\Omega, \mathcal{F}_T)$.

First, we have some representations of g -evaluations and g -expectations as below:

Remark 4.3.1. • If $g(t, x, y, z) = \alpha(t, x)y + \beta(t, x)z$ is linear in $(y, z) \in \mathbb{R}^2$ for all $(t, x) \in [0, T] \times \mathbb{R}$, where $\alpha(t, x)$ and $\beta(t, x)$ are bounded functions, see Example 4.6.1, then the g -expectation $\mathcal{E}_g[\cdot]$ satisfies

$$\mathcal{E}_g[\xi \mid \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}} \left[\xi \int_t^T \alpha(s, X_s) ds \mid \mathcal{F}_t \right],$$

where $\mathbb{Q}$ is the probability measure defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp \left(\int_0^T \beta(s, X_s) dW_s - \frac{1}{2} \int_0^T \beta^2(s, X_s) ds \right), \quad (4.10)$$

as follows by applying the Itô formula to $Y_t \exp \left(\int_0^t \alpha(s, X_s) ds \right)$ using the Brownian motion $\widetilde{W}_t := W_t + \int_0^t \beta(s, X_s) ds$ under $\mathbb{Q}$.

- If $g(t, x, y, z) = \alpha(t)|z| + \beta(t)z$, $t \in [0, T]$, $z \in \mathbb{R}$, where $\alpha(t)$ and $\beta(t)$ are two continuous function on $[0, T]$, then $\mathcal{E}_g[\xi]$ coincides with Choquet's expectation

$$\mathcal{E}_g[\xi] = \mathbb{E}_{\mu_g}[\xi] := \int_{-\infty}^0 (\mu_g(\xi > x) - 1)dx + \int_0^{\infty} \mu_g(\xi > x)dx$$

for ξ of the form $\xi = y + zW_T$, where $\mu_g[A] := \mathcal{E}_g[\mathbf{1}_A]$, $A \in \mathcal{F}$, is the corresponding capacity, see [16].

Moreover, $\mathcal{E}_g[\cdot]$ coincides with the linear expectation $\mathcal{E}_{\mu_g}[\cdot]$ if and only if $\alpha(t) = 0$ a.s., $t \in [0, T]$, see Theorem 1 in [16], and in case $\beta(t) = 0$, $t \in [0, T]$, we have

$$\mathcal{E}_g[\xi] = \sup_{\mathbb{Q} \in \mathcal{P}_g} \mathbb{E}_{\mathbb{Q}}[\xi], \text{ if } \alpha(t) > 0, t \in [0, T], \quad (4.11a)$$

$$\mathcal{E}_g[\xi] = \inf_{\mathbb{Q} \in \mathcal{P}_g} \mathbb{E}_{\mathbb{Q}}[\xi], \text{ if } \alpha(t) < 0, t \in [0, T], \quad (4.11b)$$

where $\mathcal{P}_g$ is the set of probability measures $\mathbb{Q} \in \mathcal{M}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp \left(- \int_0^T v_t dW_t - \frac{1}{2} \int_0^T v_t^2 dt \right),$$

where $(v_t)_{t \in [0, T]}$ is $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted and $|v_t| \leq |\alpha(t)|$, $t \in [0, T]$, see Theorem 2.2 in [17] and also Example 1 in [16].

- When $g(t, x, y, z)$ is convex in $(y, z) \in \mathbb{R}^2$ for all $(t, x) \in [0, T] \times \mathbb{R}$, which is the case in Theorems 4.4.1-4.4.2, Corollaries 4.4.4-4.4.5, and Theorems 4.4.7-4.4.8, we have the representation

$$\mathcal{E}_g[\xi] = \sup_{\mathbb{Q} \in \mathcal{P}_g} \left(\mathbb{E}_{\mathbb{Q}}[\xi] - F_g \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right), \quad (4.12)$$

where $F_g : L^2(\Omega, \mathcal{F}_T) \mapsto \mathbb{R} \cup \{+\infty\}$ is the convex functional defined by

$$F_g(X) := \sup_{\xi \in L^2(\Omega, \mathcal{F}_T)} (\mathbb{E}[\xi X] - \mathcal{E}_g[\xi]), \quad X \in L^2(\Omega, \mathcal{F}_T), \quad (4.13)$$

and $\mathcal{P}_g$ is the non-empty convex set of prior probability measures representing model uncertainty and defined by

$$\mathcal{P}_g := \left\{ \mathbb{Q} \in \mathcal{M} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\Omega, \mathcal{F}_T) \text{ and } F_g \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) < \infty \right\}, \quad (4.14)$$

where $\mathcal{M}$ is the set of probability measures on $(\Omega, \mathcal{F}_T)$ which are absolutely continuous with respect to $\mathbb{P}$, see Corollary 12 of [32].

- If $g(t, x, y, z)$ is sublinear in $(y, z) \in \mathbb{R}^2$ for all $(t, x) \in [0, T] \times \mathbb{R}$, which is the case in Examples 4.6.3-4.6.5, then (4.12) becomes

$$\mathcal{E}_g[\xi] = \sup_{\mathbb{Q} \in \mathcal{P}_g} \mathbb{E}_{\mathbb{Q}}[\xi], \quad (4.15)$$

see Corollary 12 in [32] and also [19], [18]. We also note that if $g_1 \leq g_2$, then we have $F_{g_1} \geq F_{g_2}$ and $\mathcal{P}_{g_1} \subseteq \mathcal{P}_{g_2}$. Indeed, $g_1 \leq g_2$ will imply that $\mathcal{E}_{g_1}[\xi] \leq \mathcal{E}_{g_2}[\xi]$, $\xi \in L^2(\Omega, \mathcal{F}_T)$, hence by the definition of F_{g_i} , we deduce that $F_{g_1}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \geq F_{g_2}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)$, which implies that $\mathcal{P}_{g_1} \subseteq \mathcal{P}_{g_2}$.

From the above facts, we have the following two propositions.

Proposition 4.3.2. Assume that $g_i(t, x, y, z), i = 1, 2$ is convex in $(y, z) \in \mathbb{R}^2$, for all $(t, x) \in [0, T] \times \mathbb{R}$. Then $\mathcal{E}_{g_1}[\phi(X^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X^{(2)})]$ can be represented as

$$\sup_{\mathbb{Q} \in \mathcal{P}_{g_1}} \left\{ \mathbb{E}_{\mathbb{Q}}[\phi(X^{(1)})] - F_{g_1}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \right\} \leq \sup_{\mathbb{Q} \in \mathcal{P}_{g_2}} \left\{ \mathbb{E}_{\mathbb{Q}}[\phi(X^{(2)})] - F_{g_2}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \right\}, \quad (4.16)$$

where $F_{g_i} : L^2(\Omega, \mathcal{F}_T) \mapsto \mathbb{R} \cup \{+\infty\}$ is the convex functional defined as in (4.13), and $\mathcal{P}_{g_i}$ is the non-empty convex set defined as in (4.14). Moreover, if $g_1 \leq g_2$, then we have $F_{g_1} \geq F_{g_2}$ and $\mathcal{P}_{g_1} \subseteq \mathcal{P}_{g_2}$.

Proof. This is a direct consequence of Corollary 12 in [32]. $\square$

Proposition 4.3.3. Assume that $g_i(t, x, y, z), i = 1, 2$ is sublinear in $(y, z) \in \mathbb{R}^2$ for all $(t, x) \in [0, T] \times \mathbb{R}$. Then $\mathcal{E}_{g_1}[\phi(X^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X^{(2)})]$ can be represented as

$$\sup_{\mathbb{Q} \in \mathcal{P}_{g_1}} \mathbb{E}_{\mathbb{Q}}[\phi(X^{(1)})] \leq \sup_{\mathbb{Q} \in \mathcal{P}_{g_2}} \mathbb{E}_{\mathbb{Q}}[\phi(X^{(2)})], \quad (4.17)$$

where in this case $\mathcal{P}_{g_i}$ is the non-empty convex set defined by

$$\mathcal{P}_{g_i} := \left\{ \mathbb{Q} \in \mathcal{M} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\Omega, \mathcal{F}_T) \text{ and } F_{g_i}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = 0 \right\}, \quad i = 1, 2.$$

Proof. Note that sublinearity is equivalent to convexity and plus positive homogeneity. In this case, $\mathcal{E}_{g_i}$ is a sublinear expectation. Again, the comparison above is a sequence of the second part of Corollary 12 in [32]. $\square$

Furthermore, in case of sublinear generator functions we have some more characterizations of the monotonic g -stochastic ordering.

Proposition 4.3.4. *Assume that $g_i(t, x, y, z), i = 1, 2$ is sublinear in $(y, z) \in \mathbb{R}^2$, for $(t, x) \in [0, T] \times \mathbb{R}$. Let $X^{(1)}$ and $X^{(2)}$ be two random variables in $L^2(\Omega, \mathcal{F}_T)$. Then the following conditions are equivalent.*

- (i) $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$.
- (ii) $\inf_{\mathbb{Q} \in \mathcal{P}_{g_1}} \mathbb{Q}(X^{(1)} \leq t) \geq \inf_{\mathbb{Q} \in \mathcal{P}_{g_2}} \mathbb{Q}(X^{(2)} \leq t)$, for all $t \in \mathbb{R}$, where the two non-empty convex sets $\mathcal{P}_{g_1}$ and $\mathcal{P}_{g_2}$ of absolutely continuous probability measures $\mathbb{Q}$ defined as in Proposition 4.3.3.
- (iii) $(F_1^*)^{-1}(u) \leq (F_2^*)^{-1}(u)$, for $u \in (0, 1)$, where

$$(F_i^*)^{-1}(u) := \inf\{t \in \mathbb{R} : F_i^*(t) \geq u\},$$
 and $F_i^*(t) := \mathbb{Q}_i^*(X^{(i)} \leq t) := \inf_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{Q}(X^{(i)} \leq t)$, $\mathbb{Q}_i^* \in \mathcal{P}_{g_i}$, $i = 1, 2$.
- (iv) There exist two random variables $\hat{X}^{(1)}$ and $\hat{X}^{(2)}$, defined on the same probability space, such that $\hat{X}^{(1)} \sim F_1^*$ and $\hat{X}^{(2)} \sim F_2^*$ and $\mathbb{P}(\hat{X}^{(1)} \leq \hat{X}^{(2)}) = 1$.
- (v) $\psi(X^{(1)}) \leq_{g_1, g_2}^{mon} \psi(X^{(2)})$ for all non-decreasing function ψ .

Proof. **(i) $\Rightarrow$ (ii):** Since $g_i(t, x, y, z), i = 1, 2$ is sublinear in $(y, z) \in \mathbb{R}^2$, for $(t, x) \in [0, T] \times \mathbb{R}$, then by Proposition 4.3.3 it is clear that there exist two non-empty convex sets $\mathcal{P}_{g_1}$ and $\mathcal{P}_{g_2}$ of absolutely continuous probability measures $\mathbb{Q}$ such that $\mathcal{E}_{g_i}[\phi(X^{(i)})] := \sup_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[\phi(X^{(i)})]$, $i = 1, 2$. Suppose that $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$, i.e., $\mathcal{E}_{g_1}[\phi(X^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X^{(2)})]$ for all non-decreasing function ϕ . Now we take $\phi(x) := \mathbf{1}_{\{x > t\}}$, $t \in \mathbb{R}$, which is a non-decreasing function in x for each fixed $t \in \mathbb{R}$. Then, we have

$$\begin{aligned} \sup_{\mathbb{Q} \in \mathcal{P}_{g_1}} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{X^{(1)} > t\}}] &\leq \sup_{\mathbb{Q} \in \mathcal{P}_{g_2}} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{X^{(2)} > t\}}], \\ \sup_{\mathbb{Q} \in \mathcal{P}_{g_1}} \mathbb{Q}(X^{(1)} > t) &\leq \sup_{\mathbb{Q} \in \mathcal{P}_{g_2}} \mathbb{Q}(X^{(2)} > t), \\ \inf_{\mathbb{Q} \in \mathcal{P}_{g_1}} \mathbb{Q}(X^{(1)} \leq t) &\geq \inf_{\mathbb{Q} \in \mathcal{P}_{g_2}} \mathbb{Q}(X^{(2)} \leq t). \end{aligned}$$

(ii) $\Leftrightarrow$ (iii): For each $u \in (0, 1)$, we note that

$$\{t \in \mathbb{R} : F_i^*(t) \geq u\} = [(F_i^*)^{-1}(u), +\infty), \quad i = 1, 2.$$

We have $F_1^*(t) \geq F_2^*(t)$ is equivalent to

$$\begin{aligned} [(F_1^*)^{-1}(u), +\infty) &= \{t \in \mathbb{R} | F_1^*(t) \geq u\} \\ &\supseteq \{t \in \mathbb{R} | F_2^*(t) \geq u\} \\ &= [(F_2^*)^{-1}(u), +\infty). \end{aligned}$$

Hence, it is equivalent to $(F_1^*)^{-1}(u) \leq (F_2^*)^{-1}(u)$, for all $u \in (0, 1)$.

(iii) $\Rightarrow$ (i): Suppose that the assertion (iii) hold true. Taking any non-decreasing function ϕ , we have

$$\phi((F_1^*)^{-1}(u)) \leq \phi((F_2^*)^{-1}(u)), \quad u \in (0, 1),$$

and

$$\int_0^1 \phi((F_1^*)^{-1}(u)) du \leq \int_0^1 \phi((F_2^*)^{-1}(u)) du = \int_0^1 \phi((F_2^*)^{-1}(v)) dv.$$

By changing variables $u = F_1^*(x_1)$ and $v = F_2^*(x_2)$, we get

$$\int_{\mathbb{R}} \phi(x_1) dF_1^*(x_1) \leq \int_{\mathbb{R}} \phi(x_2) dF_2^*(x_2),$$

and

$$\mathbb{E}_{\mathbb{Q}_1^*}[\phi(X^{(1)})] \leq \mathbb{E}_{\mathbb{Q}_2^*}[\phi(X^{(2)})], \quad \phi \text{ is non-decreasing,}$$

which allow us to conclude to $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$, where $(y, z) \mapsto g_i(t, x, y, z)$ is sublinear on $\mathbb{R}^2$, for $(t, x) \in [0, T] \times \mathbb{R}$ and $i = 1, 2$.

(iii) $\Rightarrow$ (iv): Let U be a uniformly distributed random variable on the interval $(0, 1)$. We define $\widehat{X}^{(i)} := (F_i^*)^{-1}(U)$, $i = 1, 2$ then it is clear that $\widehat{X}^{(1)} \sim F_1^*$ and $\widehat{X}^{(2)} \sim F_2^*$ are defined on the same probability space. Suppose that $(F_1^*)^{-1}(u) \leq (F_2^*)^{-1}(u)$, for all $u \in (0, 1)$, then it follows that $\mathbb{P}[(F_1^*)^{-1}(U) \leq (F_2^*)^{-1}(U)] = 1$.

(iv) $\Rightarrow$ (iii): Consider $\widehat{X}^{(i)}$, $i = 1, 2$ is defined as above. Defining

$$\overline{F}_i^*(t) := \mathbb{P}(\widehat{X}^{(i)} > t), \quad i = 1, 2, \quad t \in \mathbb{R}.$$

Since $\mathbb{P}(\widehat{X}^{(1)} \leq \widehat{X}^{(2)}) = 1$, we have $\{\omega \in \Omega : \widehat{X}^{(1)}(\omega) > t\} \subseteq \{\omega \in \Omega : \widehat{X}^{(2)}(\omega) > t\}$. Hence, $\overline{F}_1^*(t) \leq \overline{F}_2^*(t)$. It implies that $F_1^*(t) \geq F_2^*(t)$ and $(F_1^*)^{-1}(u) \leq (F_2^*)^{-1}(u)$, for $u \in (0, 1)$.

(v) $\Rightarrow$ (i): Suppose that $\psi(X^{(1)}) \leq_{g_1, g_2}^{mon} \psi(X^{(2)})$ for all non-decreasing function ψ . Then it is obvious that $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$ by taking $\psi(x) = x$.

(i) $\Rightarrow$ (v): We now suppose that $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$. For two non-decreasing functions ϕ and ψ , we define $h(x) := \phi \circ \psi(x)$, then $h(x)$ is also non-decreasing. Checking that

$$\mathcal{E}_{g_1} [h(X^{(1)})] \leq \mathcal{E}_{g_2} [h(X^{(2)})],$$

and

$$\mathcal{E}_{g_1} [\phi(\psi(X^{(1)}))] \leq \mathcal{E}_{g_2} [\phi(\psi(X^{(2)}))],$$

by definition of the monotonic g -stochastic ordering, we obtain

$$\psi(X^{(1)}) \leq_{g_1, g_2}^{mon} \psi(X^{(2)}).$$

□

Next, we consider characterization of the convex g -stochastic orderings. In the case of $\phi(x)$ convex, one may be interested in when does the g -expectation satisfy the Jensen inequality? That is,

$$\mathcal{E}_g[\phi(X)] \geq \phi(\mathcal{E}_g[X]), \quad \phi \text{ is convex.}$$

This problem has been studied by some authors, see e.g. [18]. They shown that the g -expectation satisfy Jensen's inequality if only if the function $g(t, x, y, z)$ does not depend on y and it is super-homogeneous, i.e.,

$$g(t, x, \lambda z) \geq \lambda g(t, x, z), \quad \lambda \in \mathbb{R}, (x, z) \in \mathbb{R}^2, t \in [0, T].$$

In one dimension, such the function g has exactly the following form

$$g(t, x, z) := \alpha(t)|z| + \beta(t)z, \quad (x, z) \in \mathbb{R}^2, t \in [0, T], \quad (4.18)$$

where $\alpha(t)$ and $\beta(t)$ are continuous functions on $[0, T]$.

Proposition 4.3.5. *Given the generator functions $g_i(t, x, y, z) = \alpha^{(i)}(t)|z| + \beta^{(i)}(t)z$, for $i = 1, 2$ and $(x, y, z) \in \mathbb{R}^3$, $t \in [0, T]$, where $\alpha^{(i)}(t)$ and $\beta^{(i)}(t)$ are bounded functions on $[0, T]$ such that $\alpha^{(i)}(t) \geq 0$, $t \in [0, T]$. Then $X^{(1)} \leq_{g_1, g_2}^{conv} X^{(2)}$ can be represented as*

$$\sup_{\mathbb{Q} \in \mathcal{P}_{g_1}} \mathbb{E}_{\mathbb{Q}}[\phi(X^{(1)})] \leq \sup_{\mathbb{Q} \in \mathcal{P}_{g_2}} \mathbb{E}_{\mathbb{Q}}[\phi(X^{(2)})], \quad (4.19)$$

where in this case $\mathcal{P}_{g_i}$ is the non-empty convex set defined by

$$\mathcal{P}_{g_i} := \left\{ \mathbb{Q} \in \mathcal{M} : \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_T \right] := \exp \left(-\frac{1}{2} \int_0^T (v_s^{(i)} + \beta^{(i)}(s))^2 ds - \int_0^T (v_s^{(i)} + \beta^{(i)}(s)) dW_s \right) \right\},$$

where $|v_s^{(i)}| \leq \alpha^{(i)}(s)$, a.s., for $i = 1, 2$, and $s \in [0, T]$. Moreover, we have

$$\phi(\mathcal{E}_{g_1}[X^{(1)}]) \leq \mathcal{E}_{g_1}[\phi(X^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X^{(2)})],$$

for all convex function ϕ such that $\phi(X^{(i)}) \in L^2(\Omega, \mathcal{F}_T)$, $i = 1, 2$.

Proof. We follow arguments in the proof of Lemma 3 in [19]. Let $(Y_t^{(i)}, Z_t^{(i)})_{t \in [0, T]}$, be the solution of the following BSDEs

$$Y_t^{(i)} = \phi(X_T^{(i)}) + \int_t^T \left(\alpha^{(i)}(s) |Z_s^{(i)}| + \beta^{(i)}(s) Z_s^{(i)} \right) ds - \int_t^T Z_s^{(i)} dW_s, \quad 0 \leq t \leq T, \quad i = 1, 2. \quad (4.20)$$

Defining $\overline{W}_t := W_t + \int_0^t (c_s^{(i)} + \beta^{(i)}(s)) dW_s$, where $c_s^{(i)} := \alpha^{(i)}(s) \text{sgn}(Z_s^{(i)})$ and $\text{sgn}(\cdot)$ denotes the sign function:

$$\text{sgn}(x) := \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Let $\mathbb{Q}^c$ denote a probability measure such that

$$\mathbb{E} \left[\frac{d\mathbb{Q}^c}{d\mathbb{P}} \mid \mathcal{F}_T \right] := \exp \left(-\frac{1}{2} \int_0^T (c_s^{(i)} + \beta^{(i)}(s))^2 ds - \int_0^T (c_s^{(i)} + \beta^{(i)}(s)) dW_s \right).$$

Then $(\overline{W}_t)_{t \in [0, T]}$ is a Brownian motion under $\mathbb{Q}^c$ and the equation (4.20) becomes

$$Y_t^{(i)} = \phi(X_T^{(i)}) - \int_t^T Z_s^{(i)} d\overline{W}_s, \quad 0 \leq t \leq T, \quad i = 1, 2.$$

Thus, we have

$$Y_t^{(i)} = \mathbb{E}_{\mathbb{Q}^c}[\phi(X_T^{(i)}) \mid \mathcal{F}_t] \leq \text{ess. sup}_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[\phi(X_T^{(i)}) \mid \mathcal{F}_t]. \quad (4.21)$$

On the other hand, let $(d_s^{(i)})_{s \in [0, T]}$ be a bounded process by $\alpha^{(i)}(s)$, $s \in [0, T]$, $i = 1, 2$. i.e. $|d_s^{(i)}| \leq \alpha^{(i)}(s)$, a.s. for any $s \in [0, T]$, and denote by $(\tilde{Y}_t^{(i)}, \tilde{Z}_t^{(i)})$ the unique solution of the following BSDE

$$\tilde{Y}_t^{(i)} = \phi(X_T^{(i)}) + \int_t^T (d_s^{(i)} + \beta^{(i)}(s)) \tilde{Z}_s^{(i)} ds - \int_t^T \tilde{Z}_s^{(i)} dW_s, \quad 0 \leq t \leq T, \quad i = 1, 2. \quad (4.22)$$

Also, we define $\widetilde{W}_t := W_t + \int_0^t (d_s^{(i)} + \beta^{(i)}(s))dW_s$, where $|d_s^{(i)}| \leq \alpha^{(i)}(s)$, a.s. and defining the probability measure $\mathbb{Q}^d$ such that

$$\mathbb{E} \left[\frac{d\mathbb{Q}^d}{d\mathbb{P}} \mid \mathcal{F}_T \right] := \exp \left(-\frac{1}{2} \int_0^T (d_s^{(i)} + \beta^{(i)}(s))^2 ds - \int_0^T (d_s^{(i)} + \beta^{(i)}(s))dW_s \right).$$

Then $(\widetilde{W}_t)_{t \in [0, T]}$ is a Brownian motion under $\mathbb{Q}^d$ and the equation (4.22) becomes

$$\widetilde{Y}_t^{(i)} = \phi(X_T^{(i)}) - \int_t^T \widetilde{Z}_s^{(i)} d\widetilde{W}_s, \quad 0 \leq t \leq T, \quad i = 1, 2.$$

Noting that $(d_t + \beta(t))z \leq \alpha(t)|z| + \beta(t)z$, for all $(t, z) \in [0, T] \times \mathbb{R}$. By the comparison theorem for the backward SDE, see e.g. [74], we get $\mathbb{E}_{\mathbb{Q}^d}[\phi(X_T^{(i)}) \mid \mathcal{F}_t] = \widetilde{Y}_t^{(i)} \leq Y_t^{(i)}$. Hence, we have

$$\text{ess. sup}_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[\phi(X_T^{(i)}) \mid \mathcal{F}_t] \leq \widetilde{Y}_t^{(i)}. \quad (4.23)$$

From (4.21) and (4.23), we obtain $\widetilde{Y}_t^{(i)} = \text{ess. sup}_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[\phi(X_T^{(i)}) \mid \mathcal{F}_t]$. In particular, at the time $t = 0$, we have

$$\mathcal{E}_{g_i}[\phi(X_T^{(i)})] = \widetilde{Y}_0^{(i)} = \sup_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[\phi(X_T^{(i)})].$$

Therefore, we have the desired representation. In addition, when $g_i(t, x, y, z) = \alpha^{(i)}(t)|z| + \beta^{(i)}(t)z$, for $i = 1, 2$ and $(x, y, z) \in \mathbb{R}^3$, $t \in [0, T]$, then the g -expectations satisfy Jensen's inequality, see [18]. $\square$

From the characterizations above, we see that comparison under g -stochastic orderings can be viewed as the comparison under uncertain distributions. This will allow us to define some uncertain parameters such as uncertain means and uncertain variances. Denote by $\underline{\mu}_i$, $\bar{\mu}_i$, $\underline{\sigma}_i^2$, and $\bar{\sigma}_i^2$ the lower mean, upper mean, lower variance and upper variance of the random variable $X^{(i)}$ under the g_i -expectation $\mathcal{E}_{g_i}$, $i = 1, 2$, respectively, i.e.,

$$\underline{\mu}_i := -\mathcal{E}_{g_i}[-X^{(i)}], \quad \bar{\mu}_i := \mathcal{E}_{g_i}[X^{(i)}], \quad \underline{\sigma}_i^2 := -\mathcal{E}_{g_i}[-(X^{(i)})^2], \quad \bar{\sigma}_i^2 := \mathcal{E}_{g_i}[(X^{(i)})^2].$$

For example, if we take g_i as in Proposition 4.3.5, then it is easy to verify that

$$\underline{\mu}_i = \inf_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[X^{(i)}], \quad \bar{\mu}_i = \sup_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[X^{(i)}], \quad (4.24)$$

$$\underline{\sigma}_i^2 = \inf_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[(X^{(i)})^2], \quad \bar{\sigma}_i^2 = \sup_{\mathbb{Q} \in \mathcal{P}_{g_i}} \mathbb{E}_{\mathbb{Q}}[(X^{(i)})^2]. \quad (4.25)$$

Remark 4.3.6. We recall that Lemma 4.2.4, which will be used in the proofs of Proposition 4.3.7, Corollaries 4.5.2-4.5.3 below, shows in particular that

$$-\mathcal{E}_g[-\phi(X_T)] = \mathcal{E}_{g^{(-1)}}[\phi(X_T)],$$

where $g^{(-1)}(t, x, y, z) := -g(t, x, -y, -z)$, for $(t, x, y, z) \in [0, T] \times \mathbb{R}^3$.

We have the following properties.

Proposition 4.3.7. Let $X^{(i)} \in L^2(\Omega, \mathcal{F}_T)$, and $g_i^{(-1)}(t, x, y, z) := -g_i(t, x, -y, -z)$, for $i = 1, 2$ and $(t, x, y, z) \in [0, T] \times \mathbb{R}^3$. We have

(i) If $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$, then $\bar{\mu}_1 \leq \bar{\mu}_2$.

(ii) If $X^{(1)} \leq_{g_1^{(-1)}, g_2^{(-1)}}^{mon} X^{(2)}$, then $\underline{\mu}_1 \leq \underline{\mu}_2$.

(iii) If $X^{(1)} \leq_{g_1, g_2}^{conv} X^{(2)}$, then $[\underline{\mu}_1, \bar{\mu}_1] \subseteq [\underline{\mu}_2, \bar{\mu}_2]$ and $\bar{\sigma}_1^2 \leq \bar{\sigma}_2^2$.

(iv) If $X^{(1)} \leq_{g_1^{(-1)}, g_2^{(-1)}}^{conv} X^{(2)}$, then $[\underline{\mu}_2, \bar{\mu}_2] \subseteq [\underline{\mu}_1, \bar{\mu}_1]$ and $\underline{\sigma}_1^2 \leq \underline{\sigma}_2^2$.

Proof. **(i):** It is straightforward by taking $\phi(x) = x$, which is an increasing function, then $X^{(1)} \leq_{g_1, g_2}^{mon} X^{(2)}$ implies that

$$\bar{\mu}_1 := \mathcal{E}_{g_1}[X^{(1)}] \leq \mathcal{E}_{g_2}[X^{(2)}] := \bar{\mu}_2.$$

Part **(ii):** Note that $\mathcal{E}_{g_i^{(-1)}}[X^{(i)}] = -\mathcal{E}_{g_i}[-X^{(i)}]$, by Lemma 4.2.4. Thus, by using $\phi(x) = x$, we immediately have from the definition of $X^{(1)} \leq_{g_1^{(-1)}, g_2^{(-1)}}^{mon} X^{(2)}$,

$$\underline{\mu}_1 := \mathcal{E}_{g_1^{(-1)}}[X^{(1)}] \leq \mathcal{E}_{g_2^{(-1)}}[X^{(2)}] := \underline{\mu}_2.$$

Assertion (iii) is proved similarly by taking $\phi(x) = x$ to get $\underline{\mu}_1 \leq \underline{\mu}_2$ and then using $\phi(x) = -x$ to obtain $\mathcal{E}_{g_1}[-X^{(1)}] \leq \mathcal{E}_{g_2}[-X^{(2)}]$. It is equivalent to $-\mathcal{E}_{g_1}[-X^{(1)}] \geq -\mathcal{E}_{g_2}[-X^{(2)}]$. Thus, we have that $\underline{\mu}_2 \leq \underline{\mu}_1 \leq \bar{\mu}_1 \leq \bar{\mu}_2$, it means that $X^{(2)}$ has a larger range of uncertain mean than $X^{(1)}$. Finally, we take $\phi(x) = x^2$ to derive that $\bar{\sigma}_1^2 \leq \bar{\sigma}_2^2$. Similarly, one can prove the assertion (iv). $\square$

4.4 Comparison Results for g -Stochastic Orderings

4.4.1 g -Stochastic Orderings with Convex Drifts

Consider the forward SDEs

$$dX_t^{(1)} = \mu_1(t, X_t^{(1)})dt + \sigma_1(t, X_t^{(1)})dW_t, \quad X_0^{(1)} = x_0^{(1)}, \quad (4.26a)$$

$$dX_t^{(2)} = \mu_2(t, X_t^{(2)})dt + \sigma_2(t, X_t^{(2)})dW_t, \quad X_0^{(2)} = x_0^{(2)}, \quad (4.26b)$$

and the associated BSDEs

$$\begin{cases} dY_t^{(1)} = -g_1(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)})dt + Z_t^{(1)}dW_t, & Y_T^{(1)} = \phi(X_T^{(1)}), \\ dY_t^{(2)} = -g_2(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)})dt + Z_t^{(2)}dW_t, & Y_T^{(2)} = \phi(X_T^{(2)}), \end{cases}$$

and let

$$f_i(t, x, y, z) := z\mu_i(t, x) + g_i(t, x, y, z\sigma_i), \quad t \in [0, T], \quad x, y, z \in \mathbb{R}, \quad i = 1, 2. \quad (4.27)$$

In all following theorems, the convexity of

$$(x, y) \mapsto f_i(t, x, y, z), \quad \text{resp.} \quad (y, z) \mapsto f_i(t, x, y, z)$$

on $\mathbb{R}^2$ is understood to hold for all $(t, z) \in [0, T] \times \mathbb{R}$, resp. for all $(t, x) \in [0, T] \times \mathbb{R}$. All coefficients μ_i , σ_i , g_i , and ϕ are assumed throughout this chapter to satisfy Conditions (A.4.1)-(A.4.4). The next result is a consequence of the comparison Theorem 2.4 in Appendix C of [80]. We note that Condition (B₁) can be shown to be necessary for convex ordering by taking $\phi(x) = x$ as in Theorem 3.2 of [15].

Theorem 4.4.1. (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$, and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(B_1) \quad f_1(t, x, y, z) \leq f_2(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(B_2) \quad (x, y) \mapsto f_i(t, x, y, z) \text{ and } (y, z) \mapsto f_i(t, x, y, z) \text{ are convex on } \mathbb{R}^2 \text{ for } i = 1, 2.$$

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})], \quad (4.28)$$

for all convex functions $\phi(x)$ satisfying (4.7).

Proof. We start by assuming that the function ϕ and the coefficients $\mu_i(t, \cdot)$, $\sigma_i(t, \cdot)$ and $g_i(t, \cdot, \cdot, \cdot)$ are $\mathcal{C}_b^3$ functions for all $t \in [0, T]$. By Theorem 3.1.3, the functions $u_1(t, x) := Y_t^{(1), t, x}$ and $u_2(t, x) := Y_t^{(2), t, x}$ are solutions of the backward PDEs (4.29) which are continuous in t and x :

$$\frac{\partial u_i}{\partial t}(t, x) + \mu_i(t, x) \frac{\partial u_i}{\partial x}(t, x) + \frac{1}{2} \sigma_i^2(t, x) \frac{\partial^2 u_i}{\partial x^2}(t, x) + g_i\left(t, x, u_i(t, x), \sigma_i(t, x) \frac{\partial u_i}{\partial x}(t, x)\right) = 0, \quad (4.29)$$

for $i = 1, 2$, $x \in \mathbb{R}$, $t \in [0, T]$, with a terminal condition $u(T, x) = \phi(x)$. Letting

$$h_i(t, x, y, z, w) := f_i(t, x, y, z) + \frac{w}{2} \sigma_i^2(t, x), \quad i = 1, 2, \quad (4.30)$$

we rewrite (4.29) as

$$\frac{\partial u_i}{\partial \tau}(\tau, x) = h_i\left(\tau, x, u_i(\tau, x), \frac{\partial u_i}{\partial x}(\tau, x), \frac{\partial^2 u_i}{\partial x^2}(\tau, x)\right) \text{ with } u_i(0, x) = \phi(x), \quad i = 1, 2, \quad (4.31)$$

by setting $\tau := T - t$. We also assume that there exists constants $c, C' > 0$ such that

$$0 < c \leq \sigma_1(t, x) \leq \sigma_2(t, x) \leq C', \quad t \in [0, T], \quad x \in \mathbb{R}. \quad (4.32)$$

In this case, by Theorem 3.1.4 the second derivative $\left| \frac{\partial^2 u_i}{\partial x^2}(t, x) \right|$ is bounded by $C'' > 0$. In addition, under (B_2) , both solutions $u_1(t, x)$ and $u_2(t, x)$ of (4.31) are convex functions of x by Theorem 3.2.6 in Chapter 3, hence we have $\frac{\partial^2 u_i}{\partial x^2}(\tau, x) \geq 0$, $\tau \in [0, T]$, $x \in \mathbb{R}$. Therefore, in (4.31) we can replace $h_i(t, x, y, z, w)$ in (4.30) with

$$\tilde{h}_i(t, x, y, z, w) := f_i(t, x, y, z) + \frac{(\min(w, C''))^+}{2} \sigma_i^2(t, x), \quad i = 1, 2, \quad (4.33)$$

where $w^+ = \max(w, 0)$, and rewrite the backward PDEs (4.31) as

$$\frac{\partial u_i}{\partial \tau}(\tau, x) = \tilde{h}_i\left(\tau, x, u_i(\tau, x), \frac{\partial u_i}{\partial x}(\tau, x), \frac{\partial^2 u_i}{\partial x^2}(\tau, x)\right) \text{ with } u_i(0, x) = \phi(x), \quad i = 1, 2.$$

Next, for all $\tau \in [0, T]$ and $x_1, x_2, y, z \in \mathbb{R}$ we have

$$|f_i(\tau, x_2, y, z) - f_i(\tau, x_1, y, z)|$$

$$\begin{aligned}
&\leq |z| |\mu_i(\tau, x_2) - \mu_i(\tau, x_1)| + |g_i(\tau, x_2, y, z\sigma_i(\tau, x_2)) - g_i(\tau, x_1, y, z\sigma_i(\tau, x_1))| \\
&\leq C|z||x_2 - x_1| + C(|x_2 - x_1| + |z| |\sigma_i(\tau, x_2) - \sigma_i(\tau, x_1)|) \\
&\leq C|z||x_2 - x_1| + C(1 + |z|)|x_2 - x_1|, \quad i = 1, 2,
\end{aligned}$$

hence

$$\begin{aligned}
&|\tilde{h}_i(\tau, x_2, y, z, w) - \tilde{h}_i(\tau, x_1, y, z, w)| \\
&\leq |f_i(\tau, x_2, y, z) - f_i(\tau, x_1, y, z)| + \frac{(\min(w, C''))^+}{2} |\sigma_i^2(\tau, x_2) - \sigma_i^2(\tau, x_1)| \\
&\leq C|z||x_2 - x_1| + C(1 + |z|)|x_2 - x_1| + \frac{C''}{2} |\sigma_i(\tau, x_2) - \sigma_i(\tau, x_1)| (\sigma_i(\tau, x_1) + \sigma_i(\tau, x_2)) \\
&\leq C|z||x_2 - x_1| + C(1 + |z|)|x_2 - x_1| + CC'C''|x_2 - x_1| \\
&\leq (C + CC'C'')(1 + |x_1| + |x_2| + |y|)(1 + |z|)|x_2 - x_1|, \quad i = 1, 2,
\end{aligned}$$

which shows that Condition (G) of Theorem 2.4 in Appendix C of [80] is satisfied with $\omega(x) = \bar{\omega}(x) := Cx$. In addition, by the conditions (4.32) and (B_1) we have

$$\begin{aligned}
&\tilde{h}_2(\tau, x, y, z, w) - \tilde{h}_1(\tau, x, y, z, w) \\
&= f_2(\tau, x, y, z) - f_1(\tau, x, y, z) + \frac{(\min(w, C''))^+}{2} (\sigma_2^2(\tau, x) - \sigma_1^2(\tau, x)) \\
&\geq 0, \quad x, y, z, w \in \mathbb{R}, \quad \tau \in [0, T].
\end{aligned}$$

Besides, we have $\tilde{h}_2(\tau, x, y, z, w_1) \leq \tilde{h}_2(\tau, x, y, z, w_2)$ when $w_1 \leq w_2$, and

$$\begin{aligned}
&|\tilde{h}_2(\tau, x, y_1, z, w_1) - \tilde{h}_2(\tau, x, y_2, z, w_2)| \leq \frac{1}{2} \sigma_2^2(\tau, x) |w_1 - w_2| \\
&\quad + |f_2(\tau, x, y_1, z) - f_2(\tau, x, y_2, z)| \\
&\leq C(|y_1 - y_2| + |w_1 - w_2|),
\end{aligned}$$

$(\tau, x) \in [0, T] \times \mathbb{R}$, $(y_1, z, w_1), (y_2, z, w_2) \in \mathbb{R}^3$, hence $\tilde{h}_2(t, x, y, z, w)$ is Lipschitz in y and w . Therefore, by the comparison Theorem 2.4 in Appendix C of [80] it follows that $u_1(t, x) \leq u_2(t, x)$ for all $t \in [0, T]$, from which we conclude to

$$Y_0^{(1)} = u_1(0, X_0^{(1)}) \leq Y_0^{(2)} = u_2(0, X_0^{(1)}) = u_2(0, X_0^{(2)}),$$

hence

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all convex functions ϕ in $\mathcal{C}_b^3(\mathbb{R})$. In order to extend (4.28) to coefficients satisfying (A.4.1)-(A.4.4) without assuming the bound (4.32), we apply the above argument to sequences $(\mu_{n,i})_{n \geq 1}$, $(\sigma_{n,i})_{n \geq 1}$, $(g_{n,i})_{n \geq 1}$, $(\phi_n)_{n \geq 1}$ of $\mathcal{C}_b^3$ functions as in Theorem 3.1.3, converging respectively pointwise μ_i , σ_i , g_i and strongly to ϕ i.e. $\phi_n(x_n) \rightarrow \phi(x)$ whenever

$x_n \rightarrow x \in \mathbb{R}$, satisfying (A.4.1)-(A.4.4) with a same constant $C > 0$ for all $n \geq 1$, and such that

$$0 < c_n \leq \sigma_{n,1}(t, x) \leq \sigma_{n,2}(t, x) \leq C_n, \quad t \in [0, T], \quad x \in \mathbb{R}, \quad n \geq 1, \quad (4.34)$$

for some constants $c_n, C_n > 0$. We note that the approximations $(\phi_n)_{n \geq 1}$ and $(f_{n,i})_{n \geq 1}$ defined by (4.27), can be chosen to be convex, see e.g. [4], Lemma 1 of [55], and Problem 1.4.14 in [99]. The continuous dependence Theorem 3.2.3 then yields the convergence of the corresponding sequences $(Y_{n,0}^{(i)})_{n \geq 1}$ of BSDE solutions, concluding the proof. $\square$

By similar arguments, we derive the following Theorem 4.4.2 for the increasing convex ordering. The proof of Theorem 4.4.2 is first stated for $\mathcal{C}_b^3$ coefficients ϕ , $\mu_i(t, x)$, $\sigma_i(t, x)$ and $g_i(t, x, y, z)$ under (4.34), and then extended to coefficients satisfying (A.4.1)-(A.4.4) by applying the continuous dependence Theorem 3.2.3 as in the proof of Theorem 4.4.1.

Theorem 4.4.2. (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(B'_1) \quad f_1(t, x, y, z) \leq f_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(B'_2) \quad (x, y) \mapsto f_i(t, x, y, z) \text{ and } (y, z) \mapsto f_i(t, x, y, z) \text{ are both convex respectively on } \mathbb{R}^2 \text{ and } \mathbb{R} \times \mathbb{R}_+, \text{ for } i = 1, 2, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T],$$

$$(B'_3) \quad x \mapsto g_i(t, x, y, z) \text{ is non-decreasing on } \mathbb{R} \text{ for } i = 1, 2, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{iconv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all non-decreasing convex functions $\phi(x)$ satisfying (4.7).

Proof. Under (B'_3) , when $\phi(x)$ and $g_i(t, x, y, z)$, $i = 1, 2$, are non-decreasing in x , Proposition 3.2.2 tells us that the PDE solutions $u_1(t, x)$ and $u_2(t, x)$ satisfy

$$\frac{\partial u_1}{\partial x}(t, x) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t, x) \geq 0, \quad t \in [0, T],$$

hence Conditions (B_1) -(B_2) only need to hold for $z \geq 0$, and the conclusion follows by repeating the arguments in the proof of Theorem 4.4.1. $\square$

We note that in case $\sigma_1(t, x) = \sigma_2(t, x)$ the convexity of $u_i(t, x)$, $i = 1, 2$, is no longer required in the proofs of Theorems 4.4.1-4.4.2, and one can then remove Condition (B'_2) to obtain a result for the monotonic order.

Corollary 4.4.3. (Monotonic order with equal volatilities). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma(t, x) := \sigma_1(t, x) = \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(B''_1) \quad f_1(t, x, y, z) \leq f_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(B''_2) \quad x \mapsto g_i(t, x, y, z) \text{ is non-decreasing on } \mathbb{R} \text{ for } i = 1, 2, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{mon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all non-decreasing functions $\phi(x)$ satisfying (4.7).

Proof. When $\sigma_1(t, x) = \sigma_2(t, x)$ we can repeat the proof of Theorem 4.4.1 by using h_i in (4.30), without defining $\tilde{h}_i$ in (4.33) and without assuming (B_2) , and then follow the proof argument of Theorem 4.4.2 without requiring the convexity of $u_i(t, x)$, $i = 1, 2$. $\square$

Ordered drifts

Theorem 4.4.2 also admits the following version in the case of ordered drifts.

Corollary 4.4.4. (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$\mu_1(t, x) \leq \mu_2(t, x) \quad \text{and} \quad 0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the following conditions:

$$(C_1) \quad g_1(t, x, y, z) \leq g_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(C_2) \quad g_i(t, x, y, z) \text{ is non-decreasing in } z \text{ for } i = 1 \text{ or } i = 2, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T],$$

$$(C_3) \quad g_i(t, x, y, z) \text{ is non-decreasing in } x \text{ for } i = 1, 2, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T],$$

(C₄) $(x, y) \mapsto f_i(t, x, y, z)$ and $(y, z) \mapsto f_i(t, x, y, z)$ are both convex respectively on $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R}_+$ for $i = 1, 2$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0, T]$.

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{iconv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all non-decreasing convex functions $\phi(x)$ satisfying (4.7).

Proof. Under (C₃), since $\phi(x)$ and $g_i(t, x, y, z)$, $i = 1, 2$, are non-decreasing in x , by Proposition 3.2.2 the solutions $u_1(t, x)$ and $u_2(t, x)$ of (4.31) are non-decreasing in x and, as in the proof of Theorem 4.4.2, one can take $z \geq 0$ since $\frac{\partial u_i}{\partial x}(t, x) \geq 0$. Assuming that e.g. $g_1(t, x, y, z)$ is non-decreasing in z under (C₂), then by $z\sigma_1(t, x) \leq z\sigma_2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, and (C₁), we have

$$g_1(t, x, y, z\sigma_1(t, x)) \leq g_1(t, x, y, z\sigma_2(t, x)) \leq g_2(t, x, y, z\sigma_2(t, x)).$$

Combining the above with the inequality $z\mu_1(t, x) \leq z\mu_2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, one finds $f_1(t, x, y, z) \leq f_2(t, x, y, z)$, and by Theorem 4.4.2 we conclude that $\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})]$ for all convex non-decreasing functions $\phi(x)$ satisfying (4.7). $\square$

When the drift coefficients $\mu(t, x) = \mu_1(t, x) = \mu_2(t, x)$ are equal and $g_i(t, x, y, z)$ is independent of z , $i = 1, 2$, the following proposition can be proved for the convex g -ordering similarly to Corollary 4.4.4, by applying Theorem 4.4.1 which deals with convex ordering, instead of Theorem 4.4.2.

Corollary 4.4.5. (Convex order with equal drifts). Assume that $X_0^{(1)} = X_0^{(2)}$ and

$$\mu_1(t, x) = \mu_2(t, x), \quad \text{and} \quad 0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

(C'₁) $g_i(t, x, y, z) = g_i(t, x, y)$ is independent of $z \in \mathbb{R}$ for $i = 1, 2$, $t \in [0, T]$, $x, y \in \mathbb{R}$,

(C'₂) $g_1(t, x, y) \leq g_2(t, x, y)$, $t \in [0, T]$, $x, y \in \mathbb{R}$,

(C'₃) $(x, y) \mapsto f_i(t, x, y, z)$ and $(y, z) \mapsto f_i(t, x, y, z)$ are convex on $\mathbb{R}^2$ for $i = 1, 2$.

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7).

We note that the convexity of $u_1(t, x)$ and $u_2(t, x)$ is not needed in the proof of Theorem 4.4.1 when $\sigma_1(t, x) = \sigma_2(t, x)$, and in this case we can remove Condition (B'_2) in Theorem 4.4.2 as in the next corollary.

Corollary 4.4.6. (Monotonic order with equal volatilities). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma(t, x) := \sigma_1(t, x) = \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the following conditions:

$$(D_1) \quad \mu_1(t, x) \leq \mu_2(t, x), \quad x \in \mathbb{R}, \quad t \in [0, T],$$

$$(D_2) \quad g_1(t, x, y, z) \leq g_2(t, x, y, z) \text{ for all } (x, y, z) \in \mathbb{R}^2 \times \mathbb{R}_+, \quad t \in [0, T],$$

$$(D_3) \quad g_i(t, x, y, z) \text{ is non-decreasing in } x \text{ for } i = 1, 2 \text{ and } (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+.$$

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{mon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all non-decreasing functions $\phi(x)$ satisfying (4.7).

Proof. Similarly to the proof of Corollary 4.4.3, under the condition $\sigma_1(t, x) = \sigma_2(t, x)$ the convexity of $u_i(t, x)$ and the non-decreasing property of $g_i(t, x, y, z)$ with respect to z , $i = 1$ or $i = 2$, are no longer required. In addition, the condition

$$f_1(t, x, y, z) \leq f_2(t, x, y, z), \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T],$$

clearly holds from (D_1) – (D_2) , and we can conclude as in the proof of Corollary 4.4.4. $\square$

4.4.2 g -Stochastic Orderings with Partially Convex Drifts

Theorems 4.4.1 and 4.4.2 require the convexity assumptions (B_2) and (B'_2) on

$$(x, y, z) \mapsto f_i(t, x, y, z) := z\mu_i(t, x) + g_i(t, x, y, z\sigma_i(t, x))$$

in (x, y) and (y, z) to hold for both $i = 1, 2$. In this section, we develop different convex g -ordering results under weaker convexity conditions, based on a measurable function $\zeta(t, x)$ such that

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \left(\frac{\mu_i(t, X_t^{(i)}) - \zeta(t, X_t^{(i)})}{\sigma_i(t, X_t^{(i)})} \right)^2 dt \right) \right] < \infty, \quad i = 1, 2. \quad (4.35)$$

As in Section 4.4.1, the proofs of Theorems 4.4.7-4.4.8 are first stated for $\mathcal{C}_b^3$ BSDE coefficients as in Theorem 3.1.3, and then extended under (A.4.1)-(A.4.4) using Proposition 3.2.3.

Theorem 4.4.7. (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (4.36)$$

together with the conditions

$$(E_1) \quad f_1(t, x, y, z) \leq z\zeta(t, x) \leq f_2(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(E_2) \quad (x, y) \mapsto f_i(t, x, y, z) \text{ and } (y, z) \mapsto f_i(t, x, y, z) \text{ are convex on } \mathbb{R}^2 \text{ for } i = 1 \text{ or } i = 2.$$

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})], \quad (4.37)$$

for all convex functions $\phi(x)$ satisfying (4.7).

Proof. (i) We start by assuming that the function ϕ and the coefficients $\mu_i(t, \cdot)$, $\sigma_i(t, \cdot)$ and $g_i(t, \cdot, \cdot, \cdot)$ are $\mathcal{C}_b^3$ functions for all $t \in [0, T]$ as in Theorem 3.1.3, and that (E₂) holds with $i = 1$. Let

$$\theta_2(t, x) := \frac{\mu_2(t, x) - \zeta(t, x)}{\sigma_2(t, x)}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

By the Girsanov theorem, the process

$$\widetilde{W}_t := W_t + \int_0^t \theta_2(s, X_s^{(2)}) ds, \quad t \in [0, T],$$

is a standard Brownian motion under the probability measure $\mathbb{Q}_2$ defined by

$$\frac{d\mathbb{Q}_2}{d\mathbb{P}} := \exp \left(- \int_0^T \theta_2(s, X_s^{(2)}) dW_s - \frac{1}{2} \int_0^T (\theta_2(s, X_s^{(2)}))^2 ds \right),$$

and the forward SDEs (4.26a)-(4.26b) can be rewritten as

$$\begin{cases} dX_t^{(1)} = (\mu_1(t, X_t^{(1)}) - \theta_2(t, X_t^{(2)})\sigma_1(t, X_t^{(1)}))dt + \sigma_1(t, X_t^{(1)})d\widetilde{B}_t, & X_0^{(1)} = x_0^{(1)}, \\ dX_t^{(2)} = \zeta(t, X_t^{(2)})dt + \sigma_2(t, X_t^{(2)})d\widetilde{B}_t, & X_0^{(2)} = x_0^{(2)}, \end{cases}$$

with the associated BSDEs

$$\begin{cases} dY_t^{(1)} = -(g_1(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)}) + Z_t^{(1)}\theta_2(t, X_t^{(2)}))dt + Z_t^{(1)}d\widetilde{W}_t, & Y_T^{(1)} = \phi(X_T^{(1)}), \\ dY_t^{(2)} = -(g_2(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}) + Z_t^{(2)}\theta_2(t, X_t^{(2)}))dt + Z_t^{(2)}d\widetilde{W}_t, & Y_T^{(2)} = \phi(X_T^{(2)}). \end{cases}$$

By Theorem 3.1.3 we have $Y_t^{(1)} = u_1(t, X_t^{(1)})$ and $Y_t^{(2)} = u_2(t, X_t^{(2)})$, where the functions $u_1(t, x)$ and $u_2(t, x)$ are in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ and solve the PDEs

$$\frac{\partial u_i}{\partial t}(t, x) + \frac{1}{2}\sigma_i^2(t, x)\frac{\partial^2 u_i}{\partial x^2}(t, x) + f_i\left(t, x, u_i(t, x), \frac{\partial u_i}{\partial x}(t, x)\right) = 0, \quad (4.38)$$

with $u_i(T, x) = \phi(x)$, $i = 1, 2$. Applying the Itô formula to $u_1(t, X_t^{(2)})$ and using (4.38), we have

$$\begin{aligned} u_1(t, X_t^{(2)}) &= u_1(0, X_0^{(2)}) + \int_0^t \frac{\partial u_1}{\partial s}(s, X_s^{(2)})ds + \int_0^t \zeta(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)})ds \\ &\quad + \frac{1}{2} \int_0^t \sigma_2^2(s, X_s^{(2)})\frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)})ds + \int_0^t \sigma_2(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)})d\widetilde{W}_s \\ &= u_1(0, X_0^{(2)}) + \int_0^t \zeta(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)})ds \\ &\quad - \int_0^t f_1\left(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})\right)ds \\ &\quad + \frac{1}{2} \int_0^t (\sigma_2^2(s, X_s^{(2)}) - \sigma_1^2(s, X_s^{(2)}))\frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)})ds \\ &\quad + \int_0^t \sigma_2(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)})d\widetilde{W}_s. \end{aligned}$$

Taking expectation at time $t = T$ under $\mathbb{Q}_2$, we find

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_2} [\phi(X_T^{(2)})] &= u_1(0, X_0^{(2)}) + \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \zeta(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)})ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f_1\left(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})\right)ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T (\sigma_2^2(s, X_s^{(2)}) - \sigma_1^2(s, X_s^{(2)}))\frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)})ds \right]. \end{aligned}$$

Next, applying similarly the Itô formula to $u_2(t, X_t^{(2)})$ and then taking expectation at $t = T$ under $\mathbb{Q}_2$ we obtain, from (4.38),

$$\mathbb{E}_{\mathbb{Q}_2} [\phi(X_T^{(2)})] = u_2(0, X_0^{(2)}) - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f_2\left(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})\right)ds \right]$$

$$+ \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) ds \right].$$

From Assumption (E₁) and Condition (4.36) we get

$$\begin{aligned} u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) &= \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) ds \right] + \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T (\sigma_2^2(s, X_s^{(2)}) - \sigma_1^2(s, X_s^{(2)})) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) ds \right] \\ &\geq \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \left(f_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) - \zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \right) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \left(f_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})) - \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) \right) ds \right] \\ &\geq 0, \end{aligned} \tag{4.39}$$

where we have used (E₁) and the fact that $\frac{\partial^2 u_1}{\partial x^2}(t, x) \geq 0$, as follows from Theorem 3.2.6.

(ii) The case $i = 2$ in Assumption (E₂) is dealt with similarly by applying the Itô formula to $u_2(t, X_t^{(1)})$ and then to $u_1(t, X_t^{(1)})$, and by taking expectation at $t = T$ under the probability measure $\mathbb{Q}_1$ defined by

$$\frac{d\mathbb{Q}_1}{d\mathbb{P}} := \exp \left(- \int_0^T \theta_1(s, X_s^{(1)}) dW_s - \frac{1}{2} \int_0^T (\theta_1(s, X_s^{(1)}))^2 ds \right),$$

where

$$\theta_1(t, x) := \frac{\mu_1(t, x) - \zeta(t, x)}{\sigma_1(t, x)}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

In this case, from (4.36) and (E₁) we get

$$\begin{aligned} u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) &= \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T f_2(s, X_s^{(1)}, u_2(s, X_s^{(1)}), \frac{\partial u_2}{\partial x}(s, X_s^{(1)})) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T f_1(s, X_s^{(1)}, u_1(s, X_s^{(1)}), \frac{\partial u_1}{\partial x}(s, X_s^{(1)})) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \zeta(s, X_s^{(1)}) \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) ds \right] + \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \zeta(s, X_s^{(1)}) \frac{\partial u_1}{\partial x}(s, X_s^{(1)}) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T (\sigma_2^2(s, X_s^{(1)}) - \sigma_1^2(s, X_s^{(1)})) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) ds \right] \end{aligned}$$

$$\geq 0,$$

since $\frac{\partial^2 u_2}{\partial x^2}(t, x) \geq 0$ by Theorem 3.2.6. By the relations $Y_0^{(1)} = u_1(0, X_0^{(1)})$, $Y_0^{(2)} = u_2(0, X_0^{(2)})$ and $X_0^{(1)} = X_0^{(2)}$ we conclude to $Y_0^{(2)} - Y_0^{(1)} \geq 0$, which shows (4.37). The extension of (4.37) to coefficients satisfying (A.4.1)-(A.4.4) follows as in the proof of Theorem 4.4.1. $\square$

The next proposition deals with the increasing convex order, for which only the conditions (E'_1) -(E'_2) and $X_0^{(1)} \leq X_0^{(2)}$ are required in addition to Condition (4.40) and (E'_3) below.

Theorem 4.4.8. (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (4.40)$$

together with the conditions

$$(E'_1) \quad f_1(t, x, y, z) \leq z\zeta(t, x) \leq f_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(E'_2) \quad (x, y) \mapsto f_i(t, x, y, z) \text{ and } (y, z) \mapsto f_i(t, x, y, z) \text{ are respectively convex on } \mathbb{R}^2 \text{ and } \mathbb{R} \times \mathbb{R}_+ \text{ for } i = 1 \text{ or } i = 2,$$

$$(E'_3) \quad x \mapsto g_i(t, x, y, z) \text{ is non-decreasing on } \mathbb{R} \text{ for } i = 1, 2, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{iconv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all non-decreasing convex functions $\phi(x)$ satisfying (4.7).

Proof. As in the proof of Theorem 4.4.7 we start with $\mathcal{C}_b^3$ coefficients, and then extend the conclusion to coefficients satisfying (A.4.1)-(A.4.4) using continuous dependence Theorem 3.2.3. If $\phi(x)$ and $g_i(t, x, y, z)$ are non-decreasing in x by (E'_3) , $i = 1, 2$, then by Proposition 3.2.2 the solutions $u_1(t, x)$ and $u_2(t, x)$ of the PDE (4.38) are nondecreasing in x and satisfy

$$\frac{\partial u_1}{\partial x}(t, X_t^{(2)}) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t, X_t^{(2)}) \geq 0,$$

a.s., $t \in [0, T]$, hence conditions (E_1) -(E_2) only need to hold for $z \geq 0$, showing the sufficiency of (E'_i) , $i = 1, 2, 3$. In addition, we have $u_1(0, X_0^{(1)}) = Y_0^{(1)} \leq u_1(0, X_0^{(2)})$

by the assumption $X_0^{(1)} \leq X_0^{(2)}$, hence by repeating arguments in the proof of Theorem 4.4.7 for $i = 1$ we find by (4.39) that

$$\begin{aligned}
Y_0^{(2)} - Y_0^{(1)} &\geq u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \\
&= \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) ds \right] \\
&\quad - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T f_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})) ds \right] \\
&\quad - \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) ds \right] + \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) ds \right] \\
&\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}_2} \left[\int_0^T (\sigma_2^2(s, X_s^{(2)}) - \sigma_1^2(s, X_s^{(2)})) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) ds \right] \\
&\geq 0,
\end{aligned}$$

under Assumption (E'_2) for $i = 1$. The case $i = 2$ is treated similarly according to the proof of Theorem 4.4.7. $\square$

4.5 Comparison Results for g -Risk Measures

A g -risk measure is a mapping $\rho : L^2(\Omega, \mathcal{F}_T) \rightarrow \mathbb{R}$ satisfying the following conditions.

Definition 4.5.1. *Let g satisfy Conditions (A.4.2)-(A.4.3) and $X \in L^2(\Omega, \mathcal{F}_T)$.*

1) *The static g -risk measure is defined in terms of g -evaluation as*

$$\rho^g(X) := \mathcal{E}_g[-X].$$

2) *The dynamic g -risk measure is defined in terms of conditional g -evaluation as*

$$\rho_t^g(X) := \mathcal{E}_g[-X \mid \mathcal{F}_t], \quad t \in [0, T].$$

We refer to [32] for the relations between coherent and convex risk measures, and the g -expectations. We note that, taking $\phi(x) := -x$ in (4.9), $X^{(1)} \leq_{g_1, g_2}^{con} X^{(2)}$ implies $\rho^{g_1}(X^{(1)}) \leq \rho^{g_2}(X^{(2)})$. In addition, we have $\rho^{g_1}(\phi(X_T^{(1)})) \leq \rho^{g_2}(\phi(X_T^{(2)}))$ for all convex function $\phi(x)$ if and only if

$$X_T^{(2)} \leq_{g_1^{(-1)}, g_2^{(-1)}}^{conv} X_T^{(1)},$$

where

$$g_1^{(-1)}(t, x, y, z) := -g_1(t, x, -y, -z) \quad \text{and} \quad g_2^{(-1)}(t, x, y, z) = -g_2(t, x, -y, -z),$$

as from Remark 4.3.6 (or Lemma 4.2.4), we have $\mathcal{E}_{g^{(-1)}}[\phi(X_T)] = -\mathcal{E}_{g^{(1)}}[-\phi(X_T)]$. A stochastic ordering via sublinear expectations has also been defined in [92] by combining (4.9) with the inequality

$$-\rho^{g_1}(\phi(X^{(1)})) = -\mathcal{E}_{g_1}[-\phi(X^{(1)})] \leq -\mathcal{E}_{g_2}[-\phi(X^{(2)})] = -\rho^{g_2}(\phi(X^{(2)})),$$

where $-\mathcal{E}_{g_i}[-\phi(X_T^{(i)})]$ and $\mathcal{E}_{g_i}[\phi(X_T^{(i)})]$ respectively represent bid and ask prices, $i = 1, 2$.

Theorems 4.4.1 and 4.4.7 admit the following versions for the comparison of risks. First, we have the next consequence of Theorem 4.4.1 and Lemma 4.2.4, where we let

$$f_i^{(-1)}(t, x, y, z) := z\mu_i(t, x) + g_i^{(-1)}(t, x, y, z\sigma_1(t, x)), \quad t \in \mathbb{R}_+, \quad x, y, z \in \mathbb{R}, \quad i = 1, 2.$$

Corollary 4.5.2. *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(F_1) \quad f_1^{(-1)}(t, x, y, z) \leq f_2^{(-1)}(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(F_2) \quad (x, y) \mapsto f_i^{(-1)}(t, x, y, z) \quad \text{and} \quad (y, z) \mapsto f_i^{(-1)}(t, x, y, z) \quad \text{are convex on } \mathbb{R}^2 \quad \text{for} \\ i = 1, 2.$$

Then we have

$$-\mathcal{E}_{g_1}[-\phi(X_T^{(1)})] \leq -\mathcal{E}_{g_2}[-\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7).

Similarly, we have the next consequence of Theorem 4.4.7 and Lemma 4.2.4.

Corollary 4.5.3. *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(G_1) \quad f_1^{(-1)}(t, x, y, z) \leq z\zeta(t, x) \leq f_2^{(-1)}(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(G_2) \quad (x, y) \mapsto f_i^{(-1)}(t, x, y, z) \text{ and } (y, z) \mapsto f_i^{(-1)}(t, x, y, z) \text{ are convex on } \mathbb{R}^2 \text{ for } i = 1 \text{ or } i = 2.$$

Then we have

$$-\mathcal{E}_{g_1}[-\phi(X_T^{(1)})] \leq -\mathcal{E}_{g_2}[-\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7).

We note that Corollaries 4.5.2 and 4.5.3 can be applied to Example 4.6.1 below for the comparison of g_i -risk measures with $g_i(t, x, y, z)$ linear in y and z , in which case $g_i^{(-1)}(t, x, y, z) = g_i(t, x, y, z)$ and the bid and ask prices $-\mathcal{E}_{g_i}[-\phi(X_T^{(i)})] = \mathcal{E}_{g_i}[\phi(X_T^{(i)})]$ are equal, $i = 1, 2$.

Furthermore, we can also derive results for the increasing convex and monotonic orderings of g -risk measures under Conditions (F_1) - (F_2) and (G_1) - (G_2) . For example, if (F_1) - (F_2) or (G_1) - (G_2) only holds for $z \geq 0$ and g_i is non-decreasing in x , $i = 1, 2$, we then get versions of Corollaries 4.5.2-4.5.3 for the increasing convex g -risk comparisons as in Theorems 4.4.2 and 4.4.8. Similarly, under additional the assumption $\sigma_1(t, x) = \sigma_2(t, x)$ and by removing (F_2) and (G_2) , we can obtain versions of Corollaries 4.5.2-4.5.3 for the monotonic g -risk ordering as in Corollary 4.4.3.

4.6 Comparisons of Option Prices under Different Constraints

In the following examples we aim at comparing option prices of the form $Y_0^{(i)} := \mathcal{E}_{g_i}[\phi(X_T^{(i)})]$ for two risky assets with positive prices $(X_t^{(i)})_{t \in [0, T]}$, $i = 1, 2$, given by

$$dX_t^{(i)} = X_t^{(i)} a_i(t, X_t^{(i)}) dt + X_t^{(i)} b_i(t, X_t^{(i)}) dW_t, \quad i = 1, 2,$$

where the coefficients $\mu_i(t, x) = x a_i(t, x)$ and $\sigma_i(t, x) = x b_i(t, x)$ satisfy (A.4.1), $i = 1, 2$, for example $a_i(t, x)$ and $b_i(t, x)$ can be bounded functions. Example 4.6.1 compares option prices in classical expectation for standard self-financing portfolios with $X_t^{(1)} = X_t^{(2)} := X_t$, $a_1(t, x) = a_2(t, x) := r$, $\sigma(t) = b_1(t, X_t)$, with a misspecified volatility coefficient $b_2(t, x)$ such that $\sigma(t) \leq b_2(t, X_t)$, a.s., and is consistent with Theorem 6.2 in [26].

Example 4.6.1. *Taking*

$$g_i(t, x, y, z) := -ry - z \frac{a_i(t, x) - r}{b_i(t, x)}, \quad i = 1, 2, \quad (4.41)$$

under the conditions

$$X_0^{(1)} = X_0^{(2)} \quad \text{and} \quad 0 < b_1(t, x) \leq b_2(t, x), \quad t \in [0, T], \quad x > 0, \quad (4.42)$$

we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e. the values of the self-financing portfolios hedging the claim payoffs $\phi(X_T^{(1)})$ and $\phi(X_T^{(2)})$ satisfy

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7).

Proof. Consider the risk-free asset $E_t := E_0 e^{rt}$ and the portfolio valued

$$V_t^{(i)} := p_t^{(i)} X_t^{(i)} + q_t^{(i)} E_t, \quad t \in \mathbb{R}_+,$$

where $p_t^{(i)}$ is the quantity of risky assets and $q_t^{(i)}$ is the quantity of risk-free assets. When the strategy $(p_t^{(i)}, q_t^{(i)})_{t \in \mathbb{R}_+}$ is self-financing, we have

$$\begin{aligned} dV_t^{(i)} &= q_t^{(i)} dE_t + p_t^{(i)} dX_t^{(i)} \\ &= (rV_t^{(i)} + \theta_i(t, X_t^{(i)}) p_t^{(i)} X_t^{(i)} b_i(t, X_t^{(i)})) dt + p_t^{(i)} X_t^{(i)} b_i(t, X_t^{(i)}) dW_t, \end{aligned} \quad (4.43)$$

where

$$\theta_i(t, x) := \frac{a_i(t, x) - r}{b_i(t, x)}, \quad i = 1, 2.$$

Hence, letting

$$Z_t^{(i)} := p_t^{(i)} X_t^{(i)} b_i(t, X_t^{(i)}),$$

and discounting as

$$\tilde{V}_t^{(i)} := e^{-rt} V_t^{(i)}, \quad \tilde{X}_t^{(i)} := e^{-rt} X_t^{(i)}, \quad \text{and} \quad \tilde{Z}_t^{(i)} := e^{-rt} Z_t^{(i)},$$

with $V_T^{(i)} = \phi(X_T^{(i)})$, we find the linear BSDE

$$\tilde{V}_t^{(i)} = \tilde{V}_T^{(i)} + \int_t^T \tilde{g}_i(s, \tilde{X}_s^{(i)}, \tilde{V}_s^{(i)}, \tilde{Z}_s^{(i)}) ds - \int_t^T \tilde{Z}_s^{(i)} dW_s,$$

with $\tilde{g}_i(t, x, y, z) = -z\theta_i(t, xe^{rt})$, $i = 1, 2$. Since

$$xz(a_i(t, xe^{rt}) - r) + \tilde{g}_i(t, x, y, zxb_i(t, xe^{rt})) = 0, \quad x, y, z \in (0, \infty) \times \mathbb{R}^2, \quad t \in [0, T],$$

$i = 1, 2$, we check that Conditions (B_1) and (E_1) are satisfied (with $\zeta(t, x) = 0$) together with (B_2) and (E_2) , hence, under (4.42), Theorems 4.4.1 and 4.4.7 show that

$$\tilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1} [e^{-rT} \phi(e^{rT} \tilde{X}_T^{(1)})] \leq \tilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2} [e^{-rT} \phi(e^{rT} \tilde{X}_T^{(2)})],$$

that is $\tilde{X}_T^{(1)} \leq_{\tilde{g}_1, \tilde{g}_2}^{\text{conv}} \tilde{X}_T^{(2)}$. Therefore, the price of the first claim is upper bounded by the price of the second claim, i.e.,

$$\mathcal{E}_{g_1} [\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2} [\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7), or $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, with $g_i(t, x, y, z)$ as in (4.41), $i = 1, 2$. $\square$

The next example considers the kernel $g(t, x, y, z) := \alpha(t)|z|$ with $\alpha(t) > 0$, $t \in [0, T]$, and shows that the second risky asset would be preferred over the first one by risk-seeking investors whose preferences are modeled by g since in this case $\mathcal{E}_g[\cdot]$ is represented as in (4.11a), see also Theorem 1.37 in [90].

Example 4.6.2. *Consider the kernel*

$$g_i(t, x, y, z) = g(t, x, y, z) := \alpha(t)|z|, \quad z \in \mathbb{R}, \quad i = 1, 2,$$

where $\alpha(t)$ is a positive bounded function on $[0, T]$. Then, under the conditions

$$X_0^{(1)} \leq X_0^{(2)}, \quad a_1(t, x) \leq a_2(t, x), \quad \text{and} \quad b_1(t, x) \leq b_2(t, x), \quad t \in [0, T], \quad x > 0, \quad (4.44)$$

together with conditions $x \mapsto xa_i(t, x)$ and $x \mapsto xb_i(t, x)$ are convex on $\mathbb{R}_+$, for $i = 1, 2$ and $t \in [0, T]$, we have

$$\mathcal{E}_g [\phi(X_T^{(1)})] \leq \mathcal{E}_g [\phi(X_T^{(2)})], \quad (4.45)$$

for all non-decreasing convex functions $\phi(x)$ satisfying (4.7).

Proof. Under (4.44), we check that when $\alpha(t) > 0$, $t \in [0, T]$, we have

$$xza_1(t, x) + x|z|\alpha(t)b_1(t, x) \leq xza_2(t, x) + x|z|\alpha(t)b_2(t, x), \quad x, z \in \mathbb{R}_+,$$

and we conclude by Theorem 4.4.2. $\square$

In the case $\alpha(t) < 0$, $t \in [0, T]$, the function $z \mapsto g(t, x, y, z) = \alpha(t)|z|$ is concave on $\mathbb{R}$ and a similar result can be obtained for risk-averse investors by taking $b_1(t, x) \geq b_2(t, x)$ in (4.44), with assuming $x \mapsto xa_i(t, x)$ and $x \mapsto xb_i(t, x)$ are concave on $\mathbb{R}_+$, for $i = 1, 2$ and ϕ non-decreasing concave in (4.45), where $\mathcal{E}_g[\cdot]$ is represented as in (4.11b).

In Example 4.6.3 we consider the comparison of option prices for a standard self-financing portfolio and a second self-financing hedging portfolio in which borrowing occurs at the rate $R \geq r$, as in Example 1.1 in [27].

Example 4.6.3. *Taking*

$$g_1(t, x, y, z) := -ry - z \frac{a_1(t, x) - r}{b_1(t, x)}$$

and

$$g_2(t, x, y, z) := -ry - z \frac{a_2(t, x) - r}{b_2(t, x)} + (R - r) \left(y - \frac{z}{b_2(t, x)} \right)^-,$$

where $w^- = -\min(w, 0)$, and under the conditions

$$X_0^{(1)} = X_0^{(2)} \quad \text{and} \quad 0 < b_1(t, x) \leq b_2(t, x), \quad t \in [0, T], \quad x > 0, \quad (4.46)$$

we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e.

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7), i.e. the fair price of the unconstrained portfolio is less than that of the one with constraints.

Proof. The first portfolio value is the discounted wealth process of Example 4.6.1, which satisfies the BSDE

$$\tilde{V}_t^{(1)} = e^{-rT} \phi(e^{rT} \tilde{X}_T^{(1)}) + \int_t^T \tilde{g}_1(s, \tilde{X}_s^{(1)}, \tilde{V}_s^{(1)}, \tilde{Z}_s^{(1)}) ds - \int_t^T \tilde{Z}_s^{(1)} dW_s,$$

where $\theta_1(t, x) := (a_1(t, x) - r)/b_1(t, x)$, with the generator $\tilde{g}_1(t, x, y, z) := -z\theta_1(t, x)e^{rt}$. In the second portfolio the investor is only allowed to borrow at the rate $R > r$, which yields the discounted wealth process

$$\tilde{V}_t^{(2)} = e^{-rT} \phi(e^{rT} \tilde{X}_T^{(2)}) + \int_t^T \tilde{g}_2(s, \tilde{X}_s^{(2)}, \tilde{V}_s^{(2)}, \tilde{Z}_s^{(2)}) ds - \int_t^T \tilde{Z}_s^{(2)} dW_s,$$

which is a BSDE with the generator

$$\tilde{g}_2(t, x, y, z) := -z\theta_2(t, xe^{rt}) + (R - r) \left(y - \frac{z}{b_2(t, xe^{rt})} \right)^-.$$

We check that

$$\begin{aligned} xz(a_1(t, xe^{rt}) - r) + \tilde{g}_1(t, x, y, zxb_1(t, xe^{rt})) &= 0 \\ &\leq (R - r)(y - xz)^- \\ &= zx(a_2(t, xe^{rt}) - r) + \tilde{g}_2(t, x, y, zxb_2(t, xe^{rt})), \end{aligned} \quad (4.47)$$

$x, y, z \in (0, \infty) \times \mathbb{R}^2$, $t \in [0, T]$, where both functions $(x, y) \mapsto (R - r)(y - xz)^-$ and $(y, z) \mapsto (R - r)(y - xz)^-$ are convex, hence (B_2) and (E_2) are satisfied. In addition, (B_1) and (E_1) hold (with $\zeta(t, x) = 0$) hence by (4.46), Theorems 4.4.1 and 4.4.7 both show that

$$\tilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1}[e^{-rT}\phi(e^{rT}\tilde{X}_T^{(1)})] \leq \tilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2}[e^{-rT}\phi(e^{rT}\tilde{X}_T^{(2)})],$$

or

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

with $g_i(t, x, y, z) = -ry + \tilde{g}_i(t, xe^{-rt}, y, z)$, $i = 1, 2$, for all convex functions $\phi(x)$ satisfying (4.7). $\square$

In particular, when $X_t^{(1)} = X_t^{(2)} := X_t$, $a_1(t, x) = a_2(t, x) := a(t, x)$ and $b_1(t, x) = b_2(t, x) := b(t, x)$, Example 4.6.3 shows that

$$\mathcal{E}_{g_1}[\phi(X_T)] \leq \mathcal{E}_{g_2}[\phi(X_T)],$$

for all convex functions $\phi(x)$ satisfying (4.7). Here, using the same underlying risky asset $(X_t)_{t \in [0, T]}$, the first agent hedges the contingent claim $\phi(X_T)$ by a self-financing strategy without borrowing money, while the second agent hedges the same claim under constraints on the difference between the borrowing and lending rates. In this case, the initial investment of the second agent is higher as we have $V_0^{(2)} \geq V_0^{(1)}$.

In the next example we assume that both self-financing hedging portfolios require borrowing at the rate $R \geq r$.

Example 4.6.4. *Taking*

$$g_i(t, x, y, z) := -ry - z \frac{a_i(t, x) - r}{b_i(t, x)} + (R - r) \left(y - \frac{z}{b_i(t, x)} \right)^-, \quad i = 1, 2,$$

under the conditions

$$X_0^{(1)} = X_0^{(2)} \quad \text{and} \quad 0 < b_1(t, x) \leq b_2(t, x), \quad t \in [0, T], \quad x > 0, \quad (4.48)$$

we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e.

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7).

Proof. We consider two portfolios constructed as in Example 4.6.3, with discounted wealth processes given by the BSDEs

$$\tilde{V}_t^{(i)} = e^{-rT} \phi(e^{rT} \tilde{X}_T^{(i)}) + \int_t^T \tilde{g}_i(s, \tilde{X}_s^{(i)}, \tilde{V}_s^{(i)}, \tilde{Z}_s^{(i)}) ds - \int_t^T \tilde{Z}_s^{(i)} dW_s, \quad i = 1, 2,$$

with the generators

$$\tilde{g}_i(t, x, y, z) := -z\theta_i(t, xe^{rt}) + (R - r) \left(y - \frac{z}{b_i(t, xe^{rt})} \right)^-, \quad i = 1, 2.$$

We check that

$$\begin{aligned} xz(a_1(t, xe^{rt}) - r) + \tilde{g}_1(t, x, y, zxb_1(t, xe^{rt})) &= (R - r)(y - xz)^- \\ &= zx(a_2(t, e^{rt}x) - r) + \tilde{g}_2(t, x, y, zxb_2(t, xe^{rt})), \end{aligned}$$

$x, y, z \in (0, \infty) \times \mathbb{R}^2$, $t \in [0, T]$, hence (B_1) is satisfied. Here, Theorem 4.4.7 cannot be applied because (E_1) is not satisfied. However, (B_2) is satisfied since both functions $(x, y) \mapsto (R - r)(y - xz)^-$ and $(y, z) \mapsto (R - r)(y - xz)^-$ are convex, hence by (4.48), Theorem 4.4.1 shows that

$$\tilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1}[e^{-rT} \phi(e^{rT} \tilde{X}_T^{(1)})] \leq \tilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2}[e^{-rT} \phi(e^{rT} \tilde{X}_T^{(2)})],$$

or

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

with $g_i(t, x, y, z) = -ry + \tilde{g}_i(t, xe^{-rt}, y, z)$, $i = 1, 2$, for all convex functions $\phi(x)$ satisfying (4.7). $\square$

The next Example 4.6.5 is based on three risky asset prices $(X_t^{(1)})_{t \in [0, T]}$, $(X_t^{(2)})_{t \in [0, T]}$ and $(X_t^{(3)})_{t \in [0, T]}$, see §3.2 of [50]. The portfolio of the first investor is based on the risky asset $X_t^{(1)}$ and on the risk-free asset $E_t := E_0 e^{rt}$ as in Example 4.6.1. On the

other hand, the second investor is longing $X_t^{(2)}$ and E_t while shorting $X_t^{(3)}$ and the risk-free asset $\bar{E}_t := \bar{E}_0 e^{\bar{r}t}$, with $\bar{r} \geq r$. Following [50], we assume further that

$$\frac{a_2(t, x) - r_0}{b_2(t, x)} \leq \frac{a_3(t, x) - r_0}{b_3(t, x)}, \quad x > 0, \quad t \in [0, T], \quad (4.49)$$

for some $r_0 \in [r, \bar{r}]$.

Example 4.6.5. In addition to $(X_t^{(1)})_{t \in [0, T]}$ and $(X_t^{(2)})_{t \in [0, T]}$, consider a third asset with positive price $(X_t^{(3)})_{t \in [0, T]}$ given by

$$dX_t^{(3)} = X_t^{(3)} a_3(t, X_t^{(2)}) dt + X_t^{(3)} b_3(t, X_t^{(2)}) dW_t, \quad t \in [0, T].$$

Let

$$g_1(t, x, y, z) := -ry - z \frac{a_1(t, x) - r}{b_1(t, x)},$$

and

$$g_2(t, x, y, z) := -ry - z^+ \frac{a_2(t, x) - r}{b_2(t, x)} + z^- \frac{a_3(t, x) - r}{b_3(t, x)} + (\bar{r} - r) \left(y - \frac{z^+}{b_2(t, x)} + \frac{z^-}{b_3(t, x)} \right)^-,$$

and assume that

$$\theta_2(t, x) := \frac{a_2(t, x) - r}{b_2(t, x)} \leq \theta_3(t, x) := \frac{a_3(t, x) - r}{b_3(t, x)}, \quad t \in [0, T], \quad x > 0. \quad (4.50)$$

Then, under the conditions

$$X_0^{(1)} = X_0^{(2)} \quad \text{and} \quad 0 < b_1(t, x) \leq b_2(t, x), \quad t \in [0, T], \quad x > 0, \quad (4.51)$$

we have $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, i.e.

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

for all convex functions $\phi(x)$ satisfying (4.7).

Proof. Under Condition (4.50) the model is without arbitrage by Theorems 3.1 and 3.2 in [50]. In the optimal solution of Theorem 3.2 therein, the optimal hedging strategy at time t for the second investor is to long

$$\Delta_t^{(2)} := \left(\frac{\partial v}{\partial x}(t, X_t^{(2)}) \right)^+$$

units of $X_t^{(2)}$, and to short

$$\Delta_t^{(3)} := -\frac{b_2(t, X_t^{(2)})X_t^{(2)}}{b_3(t, X_t^{(2)})X_t^{(3)}} \left(\frac{\partial v}{\partial x}(t, X_t^{(2)}) \right)^-,$$

units of $X_t^{(3)}$, while longing $(\Delta_t^{(0)})^+$ units of E_t , and shorting $-(\Delta_t^{(0)})^-$ units of $\bar{E}_t$, where

$$\Delta_t^{(0)} := v(t, X_t^{(2)}) - X_t^{(2)}\Delta_t^{(2)} - X_t^{(3)}\Delta_t^{(3)}, \quad t \in [0, T].$$

In other words, the discounted portfolio asset price processes $\tilde{V}_t^{(i)} := e^{-rt}V_t^{(i)}$ and $\tilde{X}_t^{(i)} := e^{-rt}X_t^{(i)}$ satisfy the BSDEs

$$\tilde{V}_t^{(i)} = e^{-rT}\phi(e^{rT}\tilde{X}_T^{(i)}) + \int_t^T \tilde{g}_i(s, \tilde{X}_s^{(i)}, \tilde{V}_s^{(i)}, \tilde{Z}_s^{(i)})ds - \int_t^T \tilde{Z}_s^{(i)}dW_s, \quad i = 1, 2,$$

with

$$\tilde{Z}_t^{(2)} := e^{-rt}X_t^{(2)}b_2(t, X_t^{(2)})\frac{\partial v}{\partial x}(t, X_t^{(2)}), \quad t \in [0, T],$$

and the generators $\tilde{g}_1(t, x, y, z) := -z\theta_1(t, xe^{rt})$ and

$$\begin{aligned} \tilde{g}_2(t, x, y, z) &:= -z\theta_2(t, xe^{rt}) + z^-(\theta_3(t, xe^{rt}) - \theta_2(t, xe^{rt})) \\ &\quad + (\bar{r} - r) \left(y - \frac{z^+}{b_2(t, xe^{rt})} + \frac{z^-}{b_3(t, xe^{rt})} \right)^-. \end{aligned}$$

Hence the second portfolio price $V_t^{(2)} = v(t, X_t^{(2)})$ satisfies the PDE

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) + rx\frac{\partial v}{\partial x}(t, x) + \frac{1}{2}x^2b_2(t, x)\frac{\partial^2 v}{\partial x^2}(t, x) + x(\theta_3(t, x) - \theta_2(t, x))b_2(t, x)\left(\frac{\partial v}{\partial x}(t, x)\right)^- \\ - rv(t, x) + (\bar{r} - r) \left(v(t, x) - x\left(\frac{\partial v}{\partial x}(t, x)\right)^+ + x\frac{b_2(t, x)}{b_3(t, x)}\left(\frac{\partial v}{\partial x}(t, x)\right)^- \right) = 0. \end{aligned}$$

Then, by Conditions (4.50)-(4.51) we have

$$\begin{aligned} &xz(a_1(t, xe^{rt}) - r) + \tilde{g}_1(t, x, y, zb_1(t, xe^{rt})) = 0 \\ &\leq xz^-b_2(t, x)(\theta_3(t, xe^{rt}) - \theta_2(t, xe^{rt})) + (\bar{r} - r) \left(y - xz^+ + \frac{b_2(t, xe^{rt})}{b_3(t, xe^{rt})}xz^- \right)^- \\ &= xz(a_2(t, xe^{rt}) - r) + \tilde{g}_2(t, x, y, zb_2(t, xe^{rt})), \quad (x, y, z) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad t \in [0, T], \end{aligned}$$

hence (E₂) is satisfied. Condition (E₁) is satisfied with $\zeta(t, x) = 0$ hence, under Condition (4.51), Theorem 4.4.7 shows that

$$\tilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1}[e^{-rT}\phi(e^{rT}\tilde{X}_T^{(1)})] \leq \tilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2}[e^{-rT}\phi(e^{rT}\tilde{X}_T^{(2)})],$$

or

$$\mathcal{E}_{g_1}[\phi(X_T^{(1)})] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)})],$$

with $g_i(t, x, y, z) = -ry + \tilde{g}_i(t, xe^{-rt}, y, z)$, $i = 1, 2$, for all convex functions $\phi(x)$ satisfying (4.7), that is $X_T^{(1)} \leq_{g_1, g_2}^{\text{conv}} X_T^{(2)}$. Note that here, Theorem 4.4.1 cannot be applied since the function

$$(x, y) \mapsto xz(a_2(t, xe^{rt}) - r) + \tilde{g}_2(t, x, y, zxb_2(t, xe^{rt}))$$

may not be convex, hence (B₂) is not satisfied. $\square$

Remark 4.6.6. • We note that under the conditions as in Example 4.6.2, it also implies the comparison of terminal portfolios values of the above examples

$$\mathcal{E}_g[u(V_T^{(1)})] \leq \mathcal{E}_g[u(V_T^{(2)})],$$

for all non-decreasing convex utility and payoff functions u and ϕ , since $u \circ \phi(x)$ is also non-decreasing convex. Hence, in this case the second portfolio would be preferred over the first portfolio by risk-seeking investors whose preferences are described by g .

- It is worthy to emphasize that our comparison of portfolios values can hold for at any time $t \in [0, T)$, i.e.,

$$\mathcal{E}_g[u(V_t^{(1)})] \leq \mathcal{E}_g[u(V_t^{(2)})], \quad \text{for each } t \in [0, T),$$

for all non-decreasing convex utility u , since one can also obtain similar comparisons under $\mathcal{F}_t$ -conditional g -evaluations (g -expectations)

$$V_t^{(1)} := \mathcal{E}_{g_1}[\phi(X_T^{(1)}) \mid \mathcal{F}_t] \leq \mathcal{E}_{g_2}[\phi(X_T^{(2)}) \mid \mathcal{F}_t] := V_t^{(2)}, \text{ a.s.,}$$

for each $t \in [0, T)$ and for all non-decreasing convex payoff functions ϕ .

Chapter 5

Stochastic Orderings under G -Expectation Framework

Chapter 5 is developed by motivations from the results of the g -stochastic orderings in Chapter 4. Here, we study stochastic orderings by a nonlinear expectation denoted by $\mathcal{E}_G$, which is generated by a solution of the G -Backward Stochastic Differential Equation (G -BSDE). Firstly, in Section 5.2, we define the so-called G -stochastic orderings by using the nonlinear expectations $\mathcal{E}_G$, including monotonic, convex and increasing convex orderings. We then derive several sufficient conditions with two different approaches for the convex, increasing convex and monotonic G -stochastic orderings of G -diffusion stochastic processes driven by G -Brownian motion. In the first approach, given in Section 5.3.1, we simply apply comparison results of solutions of the associated HJB-type equations, also called G -PDEs. This first approach requires convexity of the functions

$$(x, y, z) \mapsto M_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)),$$

and

$$(x, y, z) \mapsto N_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x)),$$

for both $i = 1$ and $i = 2$. Next, in the second approach presented in Section 5.3.2, we use stochastic analysis arguments under the G -expectation framework to obtain comparison results for the G -stochastic orderings that only needs convexity of the functions $M_i(t, x, y, z)$ and $N_i(t, x, y, z)$, for $i = 1$ or $i = 2$. As consequences, in Section 5.4 we derive results for comparing G -risk measures which are induced by the $\mathcal{E}_G$ -expectations. Here, we also note that our approaches rely on comparison

results for the G -FBSDEs and on several extensions of convexity, monotonicity and continuous dependence property for the solutions of associated HJB equations, which have been shown in Section 3.4 of Chapter 3. In the last Section 5.5, we study the effects of ambiguous drift and volatility coefficients of G -FSDEs to associated contingent claim superhedging prices and portfolio values by applying G -stochastic ordering results.

5.1 Introduction

With the aim of dealing with financial problems under uncertain volatility, Peng [77] introduced a new family of sublinear expectations $\hat{\mathbb{E}}$ that can be represented as a supremum of linear expectations. Since then, a new form of stochastic calculus has been developed on sublinear expectation spaces which generalize usual probability spaces. In particular, Peng [77] constructed a sublinear expectation $\mathbb{E}_G$ called the G -expectation and the corresponding G -Brownian motion, G -Itô's formula, G -Itô stochastics integrals, and G -stochastic differential equations. In addition, sublinear expectations have been used to establish a new central limit theorem for G -normal distributions [78], and for the study of dynamic risk measures [76], [82]. Those results have been applied to contingent claim pricing in financial markets with uncertain volatility [41], [95], as well as to stochastic control [97], [44] and to robust mean-variance hedging [10]. Recently, Song [87, 88] has extended Stein's method to G -normal approximation under sublinear expectations.

The goal of this chapter is to study stochastic orderings under a nonlinear G -expectation framework, which leads to more general concepts of stochastic orderings in comparison with the standard linear expectation setting. Uncertainty orders of G -normal, G -maximal and G -distributed random variables have been studied in [92] on the sublinear expectation space. In addition, in Chapter 4, a stochastic ordering has also been studied under the g -expectations and g -evaluations. In this chapter, we extend these approaches to the settings of nonlinear expectations generated by G -BSDEs which are able to model volatility uncertainty, making them specially suited for financial modeling. Namely, we use the $\mathcal{E}_G$ -expectation and $\mathcal{E}_G$ -evaluation $\mathcal{E}_G[\xi]$ of a random variable ξ , defined by [43], as the initial value Y_0 for a triple $(Y_t, Z_t, K_t)_{t \in [0, T]}$ of adapted processes which solves a G -BSDE of the form

$$-dY_t = f(t, X_t, Y_t, Z_t)dt + g(t, X_t, Y_t, Z_t)d\langle B \rangle_t - Z_t dB_t - dK_t, \quad 0 \leq t \leq T, \quad (5.1)$$

with terminal condition $Y_T = \xi$, $(B_t)_{t \in \mathbb{R}_+}$ is a G -Brownian motion, $(X_t)_{t \in \mathbb{R}_+}$ is a

G -diffusion process driven by $(B_t)_{t \in \mathbb{R}_+}$ and K_t is a decreasing G -martingale.

Main results

In Theorems 5.3.1 and 5.3.7, we derive sufficient conditions on two G -BSDE generators $f_i(t, x, y, z)$, $g_i(t, x, y, z)$, $i = 1, 2$ for the convex ordering

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})], \quad (5.2)$$

for all convex functions $\varphi(x)$ with polynomial growth, where $X_T^{(1)}$ and $X_T^{(2)}$ are the terminal values of the solutions of the G -FSDEs

$$dX_t^{(1)} = b_1(t, X_t^{(1)})dt + h_1(t, X_t^{(1)})d\langle B \rangle_t + \sigma_1(t, X_t^{(1)})dB_t,$$

$$dX_t^{(2)} = b_2(t, X_t^{(2)})dt + h_2(t, X_t^{(2)})d\langle B \rangle_t + \sigma_2(t, X_t^{(2)})dB_t,$$

with $X_0^{(1)} = X_0^{(2)}$, under the bound

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}.$$

In an option pricing setting, the forward processes $(X_t^{(i)})_{t \geq 0}$, $i = 1, 2$, represents the prices of risky assets driven by G -Brownian motion under uncertain volatility, and the $(Y_t^{(i)})_{t \geq 0}$ in (5.1), $i = 1, 2$, represents the portfolio wealth processes or superhedging prices of the associated contingent claim. The processes $(Z_t^{(i)})_{t \geq 0}$ in (5.1), $i = 1, 2$, play a crucial role in hedging the claim with payoff $\varphi(X_T^{(i)})$ at maturity time T , as $\pi_t^{(i)} := Z_t^{(i)}/\sigma_i(t, X_t^{(i)})$ is the amount invested in the risky asset $X_t^{(i)}$ at time $t \in [0, T]$, see e.g. [95]. In addition, the quantity $\mathcal{E}_G[-\varphi(X_T^{(i)})]$, makes sense as a risk measure. Here, the choice of generator functions f_i and g_i determines the investor's portfolio strategy and the corresponding risk measures. Inequalities in *G-stochastic orderings* (5.2) can be interpreted as comparisons of portfolio values, option prices, and risk measures relating to underlying assets with uncertain volatility.

The G -stochastic orderings can be also used for the comparison of expected utilities when φ represents a utility function with $X_t^{(i)}$ represents for state process and $Y_t^{(i)}$ is used as value function of a stochastic control problem, see [44, 97, 84, 93]. and as such they are applicable to risk management. Precisely, using the convexity (resp. concavity) of φ , they enable one to study the behavior of risk seeking (resp. risk averse) investors, see e.g. [90].

First, in Theorem 5.3.1 we show that the convex ordering (5.2) can be derived as a consequence of the comparison Theorem 2.4 in Appendix C of [80], provided that

$$zb_1(t, x) + f_1(t, x, y, z\sigma_1(t, x)) \leq zb_2(t, x) + f_2(t, x, y, z\sigma_2(t, x)), \quad x, y, z \in \mathbb{R}, t \in [0, T],$$

and

$$zh_1(t, x) + g_1(t, x, y, z\sigma_1(t, x)) \leq zh_2(t, x) + g_2(t, x, y, z\sigma_2(t, x)), \quad x, y, z \in \mathbb{R}, t \in [0, T],$$

and both functions $(x, y, z) \mapsto zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x))$ and $(x, y, z) \mapsto zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x))$ are convex in (x, y) and in (y, z) on $\mathbb{R}^2$ for both $i = 1, 2$ and $t \in [0, T]$. Then in Theorem 5.3.7, we show, using G -stochastic calculus arguments, that this condition can be relaxed into a single convexity assumption in (x, y) and in (y, z) for the functions

$$(x, y, z) \mapsto N_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x)), \quad i = 1 \text{ or } i = 2,$$

and

$$(x, y, z) \mapsto M_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)), \quad i = 1 \text{ or } i = 2.$$

Increasing convex ordering is dealt with in Theorems 5.3.2, and in Theorem 5.3.4 under ordering conditions on the drifts $b_i(t, x)$, $h_i(t, x)$, $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$. In Theorem 5.3.8, increasing convex ordering is considered under weaker convexity conditions on the functions $M_i(t, x, y, z)$ and $N_i(t, x, y, z)$ which only need to be convex for $i = 1$ or $i = 2$. Monotonic ordering is considered in Corollary 5.3.3, and the particular cases of equal drifts and equal volatilities are treated in Corollaries 5.3.5 and 5.3.6 for the convex and monotonic orderings.

The proofs of Theorems 5.3.1-5.3.2 and 5.3.7-5.3.8 rely on an extension of convexity property of the solutions of associated HJB equations types which is proved in Theorem 3.4.6 in Chapter 3. The convexity properties of solutions of nonlinear PDEs have been studied by several authors, see e.g. Theorem 3.1 in [60], Theorem 2.1 in [33], and Theorem 1.1 in [11], see also Theorem 1 in [1] in the elliptic case. Those works typically require global convexity of the nonlinear drifts $M(t, x, y, z)$ and $N(t, x, y, z)$ in all state variables (x, y, z) , a condition which is too strong for even the Black-Scholes-Barenblatt equation to be satisfied. For this reason, in Theorem 3.4.6 we extended Theorem 1.1 of [11] in dimension one, by replacing the global convexity of the nonlinear drift $M_i(t, x, y, z)$ and $N_i(t, x, y, z)$ in (x, y, z) with its convexity in (x, y) and (y, z) , $i = 1, 2$.

Comparisons with g -Stochastic Orderings

In comparison with g -stochastic orderings presented in [62] or in Chapter 4, we obtain results of G -stochastic orderings that involve more techniques under new G -stochastic calculus. First, we need to establish the existence of classical solutions for HJB (G -PDEs) equations, instead of semilinear parabolic PDEs, which is done under additional boundedness conditions in Theorems 3.3.5-3.3.6. In order to deal with supremum terms in HJB equation to get the convexity for its solution as shown in Theorem 3.4.6, we have to consider mollified HJB equations, and then using approximation arguments with the help of a continuous dependence (stability) Theorem 3.4.3. This stability theorem relies on a dominated convergent theorem under the G -expectation, which does not always hold under general sublinear expectations, see e.g. [46]. Also, the proofs of Theorems 5.3.7-5.3.8 use G -Girsanov transformation and G -martingale representation theorem on extended G -expectation spaces.

5.2 G -Stochastic Orderings

G -Forward-Backward Stochastic Differential Equations

Let $(\Omega, L_G^1(\Omega, \mathcal{F}), \mathbb{E}_G)$ be a sublinear G -expectation space. Let $(B_t)_{t \geq 0}$ be G -Brownian motion with $-\mathbb{E}_G[-B_1^2] = \underline{\sigma}^2$ and $\mathbb{E}_G[B_1^2] = \bar{\sigma}^2$. Consider a G -FSDE of the form

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + h(s, X_s^{t,x})d\langle B \rangle_s + \sigma(s, X_s^{t,x})dB_s, \text{ with } X_t^{t,x} = x, 0 \leq t \leq s \leq T, \quad (5.4)$$

and an associated G -BSDE of the form

$$\begin{aligned} Y_s^{t,x} = & \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\langle B \rangle_r \\ & - \int_s^T Z_r^{t,x} dB_r - (K_T^{t,x} - K_s^{t,x}), \end{aligned} \quad (5.5)$$

where $\varphi(X_T^{t,x}) \in L_G^2(\Omega, \mathcal{F}_T)$ and $K_s^{t,x}$ is a decreasing G -martingale, $b, h, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $f, g : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are deterministic functions and satisfy the Conditions (A.5.1)-(A.5.3)

(A.5.1) The functions $b(t, x)$, $h(t, x)$ and $\sigma(t, x)$ are continuous in $t \in [0, T]$ and Lipschitz continuous in x , i.e., there is a constant $C > 0$ such that for all $x, y \in \mathbb{R}$,

$$(i) \quad |b(t, x) - b(t, y)| \leq C |x - y|,$$

- (ii) $|h(t, x) - h(t, y)| \leq C |x - y|$,
 (iii) $|\sigma(t, x) - \sigma(t, y)| \leq C |x - y|$.

In particular, we work on the linear growth conditions

$$b(t, x) \leq C(1 + |x|), \quad \sigma(t, x) \leq C(1 + |x|), \quad h(t, x) \leq C(1 + |x|),$$

on $b(t, x)$, $\sigma(t, x)$ and $h(t, x)$.

(A.5.2) The functions $f(t, x, y, z)$ and $g(t, x, y, z)$ are also continuous in $t \in [0, T]$ and there exist a positive integer m and constant $C > 0$ such that for $x, x', y, y', z, z' \in \mathbb{R}$,

(i)

$$|f(t, x, y, z) - f(t, x', y', z')| \leq C \left[(1 + |x|^m + |x'|^m) |x - x'| + |y - y'| + |z - z'| \right],$$

(ii)

$$|g(t, x, y, z) - g(t, x', y', z')| \leq C \left[(1 + |x|^m + |x'|^m) |x - x'| + |y - y'| + |z - z'| \right],$$

$$(iii) \quad |\varphi(x) - \varphi(y)| \leq C (1 + |x|^m + |y|^m) |x - y|.$$

(A.5.3) Furthermore, we assume that

$$f(t, x, 0, 0) = g(t, x, 0, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Under the Conditions (A.5.2)-(A.5.3), and by Theorem 4.1 in [42] the G -BSDEs (5.5) admit a unique solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s \in [t, T]}$. In addition, if we define $u(t, x) := Y_t^{t,x}$ for $(t, x) \in [0, T] \times \mathbb{R}$, then under (A.5.1)-(A.5.2), $u(t, x)$ is the unique viscosity solution of the following G -PDEs (HJB equations type), see e.g. [80],

$$\frac{\partial u}{\partial t}(t, x) + F \left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x) \right) = 0, \quad (5.6)$$

with $u(T, x) = \varphi(x)$, where

$$F(t, x, y, z, w) = 2G(H(t, x, y, z, w)) + zb(t, x) + f(t, x, y, z\sigma(t, x)),$$

and

$$H(t, x, y, z, w) = \frac{1}{2} \sigma^2(t, x) w + zh(t, x) + g(t, x, y, z\sigma(t, x)).$$

Recall that in one dimension, the function G has the following form

$$G(a) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \bar{\sigma}^2} (\sigma^2 a) = \frac{1}{2} (\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \quad a \in \mathbb{R}.$$

$\mathcal{E}_G$ -evaluation and $\mathcal{E}_G$ -expectation

We state the definition of the $\mathcal{E}_G$ -evaluation generated by a solution of the G -BSDE as follows:

Definition 5.2.1. *Given $\xi \in L_G^2(\Omega, \mathcal{F}_T)$ and the G -Backward SDE*

$$\begin{cases} dY_t^{0,x} = -f(t, X_t^{0,x}, Y_t^{0,x}, Z_t^{0,x})dt - g(t, X_t^{0,x}, Y_t^{0,x}, Z_t^{0,x})d\langle B \rangle_t + Z_t^{0,x}dB_t + dK_t^{0,x}, \\ Y_T^{0,x} = \xi, \end{cases}$$

for $0 \leq t \leq T$, we respectively call

$$\mathcal{E}_G[\xi] := Y_0^{0,x} \quad \text{and} \quad \mathcal{E}_G[\xi \mid \mathcal{F}_t] := Y_t^{0,x}$$

the $\mathcal{E}_G$ -evaluation and the Ω_t -conditional $\mathcal{E}_G$ -evaluation of ξ , $t \in [0, T]$.

Under (A.5.3), one can show in addition that the map $\xi \mapsto \mathcal{E}_G[\xi]$ preserves all properties of the G -expectation $\widehat{\mathbb{E}}_G$, except for the sublinearity and the constant preserving property, see [43]. Under the (stronger than (A.5.3)) condition

$$(A.5.3') \quad f(t, x, y, 0) = g(t, x, y, 0) = 0 \text{ for all } t \in [0, T], x, y \in \mathbb{R},$$

the $\mathcal{E}_G$ -evaluation satisfies the property $\mathcal{E}_G[c] = c$ for constant $c \in \mathbb{R}$. Moreover, if both $f(t, x, y, z)$ and $g(t, x, y, z)$ are sublinear in (y, z) , then the $\mathcal{E}_G$ -evaluation becomes a G -expectation, see e.g. [43]. We note that the results of Sections 5.3 and 5.4 remain valid for $\mathcal{E}_G$ -expectations if we assume (A.5.3') instead of (A.5.3).

G -Stochastic Orderings

In Chapter 4, the g -stochastic orderings have been studied using the g -expectations $\mathcal{E}_g$. In this chapter, we extend this approach to the settings of $\mathcal{E}_G$ -expectations and $\mathcal{E}_G$ -evaluations, which are also not sublinear in general.

Definition 5.2.2. *Under hypotheses (A.5.2)-(A.5.3). For any $X^{(1)}, X^{(2)} \in L_G^2(\Omega, \mathcal{F}_T)$, we say that*

1) $X^{(1)}$ is dominated by $X^{(2)}$ in the monotonic G -ordering, i.e. $X^{(1)} \leq_G^{\text{mon}} X^{(2)}$, if

$$\mathcal{E}_G[\varphi(X^{(1)})] \leq \mathcal{E}_G[\varphi(X^{(2)})], \quad (5.7)$$

for all non-decreasing $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

- 2) $X^{(1)}$ is dominated by $X^{(2)}$ in the convex G -ordering, i.e. $X^{(1)} \leq_G^{\text{conv}} X^{(2)}$, if (5.7) holds for all convex functions $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.
- 3) $X^{(1)}$ is dominated by $X^{(2)}$ in the increasing convex G -ordering, i.e. $X^{(1)} \leq_G^{\text{icon}} X^{(2)}$, if (5.7) holds for all non-decreasing convex functions $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Remark 5.2.3. In case the generators $f(t, x, y, z)$ and $g(t, x, y, z)$ are sublinear or convex in (y, z) , the $\mathcal{E}_G$ -expectation becomes a sublinear expectation. Hence, the G -stochastic orderings also have similar characterizations as in the case of the g -stochastic orderings, which have been investigated in Section 4.3.

5.3 Comparison Results for G -Stochastic Orderings

Recall that we define a type of G -expectation of $\varphi(X_T)$ generated by G -BSDEs as

$$\mathcal{E}_G[\varphi(X_T)] := Y_0^{0, X_0} = u(0, X_0),$$

and in this section we propose comparison results based on that expectation by considering two system of G -FBSDEs

$$dX_s^{(i)} = b_i(s, X_s^{(i)})ds + h_i(s, X_s^{(i)})d\langle B \rangle_s + \sigma_i(s, X_s^{(i)})dB_s, \text{ with } X_0^{(i)} = x_0^{(i)}, \quad (5.8)$$

$$\begin{aligned} Y_s^{(i)} &= \varphi(X_T^{(i)}) + \int_s^T f_i(r, X_r^{(i)}, Y_r^{(i)}, Z_r^{(i)}) dr + \int_s^T g_i(r, X_r^{(i)}, Y_r^{(i)}, Z_r^{(i)}) d\langle B \rangle_r \\ &\quad - \int_s^T Z_r^{(i)} dB_r - (K_T^{(i)} - K_s^{(i)}), \quad i = 1, 2. \end{aligned} \quad (5.9)$$

In the sequel, we will use the following functions

$$M_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)), \quad i = 1, 2, \quad (5.10)$$

and

$$N_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x)), \quad i = 1, 2. \quad (5.11)$$

In the following theorems, all coefficients b_i , h_i , σ_i , f_i , g_i , and φ are assumed throughout this chapter to satisfy Conditions (A.5.1)-(A.5.3).

5.3.1 G -Stochastic Orderings with Convex Drifts

Theorem 5.3.1. (Convex order). Assume that $X_0^{(1)} = X_0^{(2)}$ and

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(BB_1) \quad M_1(t, x, y, z) \leq M_2(t, x, y, z), \quad x, y, z \in \mathbb{R}, t \in [0, T],$$

$$(BB_2) \quad N_1(t, x, y, z) \leq N_2(t, x, y, z), \quad x, y, z \in \mathbb{R}, t \in [0, T],$$

$$(BB_3) \quad (x, y, z) \mapsto M_i(t, x, y, z) \text{ and } (x, y, z) \mapsto N_i(t, x, y, z) \text{ are both convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R}^2, \text{ for } i = 1, 2, t \in [0, T].$$

Then we have $X_T^{(1)} \leq_G^{conv} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all convex functions $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. We start by assuming that the function φ and the coefficients $b_i(t, \cdot)$, $h_i(t, \cdot)$, $\sigma_i(t, \cdot)$, $f_i(t, \cdot, \cdot, \cdot)$ and $g_i(t, \cdot, \cdot, \cdot)$ are bounded functions in $\mathcal{C}_b^3(\mathbb{R})$ and $\mathcal{C}_b^3(\mathbb{R}^3)$, respectively, for all $t \in [0, T]$. By Theorem 3.3.5, the functions $u_1(t, x) := Y_t^{(1), t, x}$ and $u_2(t, x) := Y_t^{(2), t, x}$ are solutions of the backward PDEs (5.12) below which are continuous in t and x ,

$$\frac{\partial u_i}{\partial \tau}(\tau, x) = F_i\left(\tau, x, u_i(\tau, x), \frac{\partial u_i}{\partial x}(\tau, x), \frac{\partial^2 u_i}{\partial x^2}(\tau, x)\right) \text{ with } u_i(0, x) = \varphi(x), \quad i = 1, 2. \quad (5.12)$$

by setting $\tau := T - t$, where

$$\begin{aligned} F_i(\tau, x, y, z, w) &= 2G(H_i(\tau, x, y, z, w)) + N_i(\tau, x, y, z), \\ H_i(\tau, x, y, z, w) &= \frac{w}{2}\sigma_i^2(\tau, x) + M_i(\tau, x, y, z). \end{aligned} \quad (5.13)$$

We also assume that there exists constants $c, C' > 0$ such that

$$0 < c \leq \sigma_1(t, x) \leq \sigma_2(t, x) \leq C', \quad t \in [0, T], \quad x \in \mathbb{R}. \quad (5.14)$$

In this case, by Theorem 3.3.5 the second derivative $\left| \frac{\partial^2 u_i}{\partial x^2}(t, x) \right|$ is bounded by $C'' > 0$. In addition, under (BB_3) , both solutions $u_1(t, x)$ and $u_2(t, x)$ of (5.12) are

convex functions of x by Theorem 3.4.6, hence we have $\frac{\partial^2 u_i}{\partial x^2}(\tau, x) \geq 0$, $\tau \in [0, T]$, $x \in \mathbb{R}$. Therefore, in (5.12) we can replace $H_i(\tau, x, y, z, w)$ with

$$\tilde{H}_i(\tau, x, y, z, w) := M_i(\tau, x, y, z) + \frac{(\min(w, C''))^+}{2} \sigma_i^2(t, x), \quad i = 1, 2, \quad (5.15)$$

where $w^+ = \max(w, 0)$, and rewrite the backward PDEs (5.12) as

$$\frac{\partial u_i}{\partial \tau}(\tau, x) = \tilde{F}_i \left(\tau, x, u_i(\tau, x), \frac{\partial u_i}{\partial x}(\tau, x), \frac{\partial^2 u_i}{\partial x^2}(\tau, x) \right) \text{ with } u_i(0, x) = \varphi(x), \quad i = 1, 2.$$

where

$$\tilde{F}_i(t, x, y, z, w) = 2G(\tilde{H}_i(t, x, y, z, w)) + N_i(t, x, y, z).$$

Next, for all $\tau \in [0, T]$ and $x_1, x_2, y, z \in \mathbb{R}$, we check that

$$\begin{aligned} |M_i(\tau, x_2, y, z) - M_i(\tau, x_1, y, z)| & \leq |z| |h_i(\tau, x_2) - h_i(\tau, x_1)| + |g_i(\tau, x_2, y, z\sigma_i(\tau, x_2)) - g_i(\tau, x_1, y, z\sigma_i(\tau, x_1))| \\ & \leq C|z||x_2 - x_1| + C((1 + x_1^m + x_2^m)|x_2 - x_1| + |z| |\sigma_i(\tau, x_2) - \sigma_i(\tau, x_1)|) \\ & \leq C|z||x_2 - x_1| + C(1 + x_1^m + x_2^m + |z|)|x_2 - x_1|, \quad m > 1, i = 1, 2, \end{aligned}$$

and similarly

$$|N_i(\tau, x_2, y, z) - N_i(\tau, x_1, y, z)| \leq C|z||x_2 - x_1| + C(1 + x_1^m + x_2^m + |z|)|x_2 - x_1|,$$

for $i = 1, 2$. Thus, given $x_1, x_2, y, z, w \in \mathbb{R}$ and by (A.5.1)-(A.5.2), we have

$$\begin{aligned} |\tilde{F}_i(\tau, x_2, y, z, w) - \tilde{F}_i(\tau, x_1, y, z, w)| & \leq \bar{\sigma}^2 |\tilde{H}_i(t, x_2, y, z, w) - \tilde{H}_i(t, x_1, y, z, w)| \\ & \quad + |N_i(t, x_2, y, z, w) - N_i(t, x_1, y, z, w)| \\ & \leq \frac{C''\bar{\sigma}^2}{2} |\sigma_i^2(\tau, x_2) - \sigma_i^2(\tau, x_1)| + C|z||x_2 - x_1| + C(1 + x_1^m + x_2^m + |z|)|x_2 - x_1| \\ & \leq \frac{C''\bar{\sigma}^2}{2} |\sigma_i(\tau, x_2) - \sigma_i(\tau, x_1)| (\sigma_i(\tau, x_1) + \sigma_i(\tau, x_2)) + \\ & \quad + C|z||x_2 - x_1| + C(1 + x_1^m + x_2^m + |z|)|x_2 - x_1| \\ & \leq \bar{\sigma}^2 C C' C'' |x_2 - x_1| + C(1 + |x_1|^m + |x_2|^m + 2|z|)|x_2 - x_1| \\ & \leq (\bar{\sigma}^2 C C' C'' + C)|x_2 - x_1|(2 + |x_1|^m + |x_2|^m + 2|z|), \quad m > 1, \quad i = 1, 2, \end{aligned}$$

which shows that Condition (G) of Theorem 2.4 in Appendix C of [80] is satisfied.

By the conditions $0 < \sigma_1(t, x) \leq \sigma_2(t, x)$, and (BB₁)-(BB₂), we check that

$$\tilde{F}_2(\tau, x, y, z, w) - \tilde{F}_1(\tau, x, y, z, w)$$

$$\begin{aligned}
&\geq \underline{\sigma}^2(M_2(\tau, x, y, z) - M_1(\tau, x, y, z)) + \frac{(\min(w, C'''))^+}{2} \underline{\sigma}^2(\sigma_2^2(\tau, x) - \sigma_1^2(\tau, x)) \\
&\quad + (N_2(\tau, x, y, z) - N_1(\tau, x, y, z)) \\
&\geq 0, \quad x, y, z, w \in \mathbb{R}, \quad \tau \in [0, T].
\end{aligned}$$

Besides, we have $\tilde{F}_2(\tau, x, y, z, w_1) \leq \tilde{F}_2(\tau, x, y, z, w_2)$ when $w_1 \leq w_2$, and

$$\begin{aligned}
|\tilde{F}_2(\tau, x, y_1, z, w_1) - \tilde{F}_2(\tau, x, y_2, z, w_2)| &\leq \frac{C'''\bar{\sigma}^2}{2} \sigma_2^2(\tau, x) |w_1 - w_2| \\
&\quad + \bar{\sigma}^2 |M_2(\tau, x, y_1, z) - M_2(\tau, x, y_2, z)| \\
&\quad + C |N_2(\tau, x, y_1, z) - N_2(\tau, x, y_2, z)| \\
&\leq C' (|y_1 - y_2| + |w_1 - w_2|),
\end{aligned}$$

$(\tau, x) \in [0, T] \times \mathbb{R}$, $(y_1, z, w_1), (y_2, z, w_2) \in \mathbb{R}^3$, hence $\tilde{F}_2(t, x, y, z, w)$ is Lipschitz in y and w . Therefore, by the comparison Theorem 2.4 in Appendix C of [80] it follows that $u_1(t, x) \leq u_2(t, x)$ for all $t \in [0, T]$, from which we conclude to

$$Y_0^{(1)} = u_1(0, X_0^{(1)}) \leq Y_0^{(2)} = u_2(0, X_0^{(1)}) = u_2(0, X_0^{(2)}),$$

hence

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})], \quad (5.16)$$

for all bounded convex functions $\varphi(x) \in \mathcal{C}_b^3(\mathbb{R})$.

In order to extend (5.16) to coefficients satisfying (A.5.1)-(A.5.3) without assuming the bound (5.14), we apply the above argument to sequences $(b_{n,i})_{n \geq 1}, (h_{n,i})_{n \geq 1}, (\sigma_{n,i})_{n \geq 1}, (f_{n,i})_{n \geq 1}, (g_{n,i})_{n \geq 1}$ and $(\varphi_n)_{n \geq 1}$ of $\mathcal{C}_b^3$ bounded functions converging pointwise respectively to $b_i, h_i, \sigma_i, f_i, g_i$, and $\varphi(x)$, $i = 1, 2$, satisfying (A.5.1)-(A.5.3) for a same constant $C > 0$ for all $n \geq 1$, and such that

$$0 < c_n \leq \sigma_{n,1}(t, x) \leq \sigma_{n,2}(t, x) \leq C_n, \quad t \in [0, T], \quad x \in \mathbb{R}, \quad n \geq 1, \quad (5.17)$$

for some constants $c_n, C_n > 0$. We note that the approximations $(\varphi_n)_{n \geq 1}, (M_{n,i})_{n \geq 1}$ and $(N_{n,i})_{n \geq 1}$ defined by (5.10) and (5.11), can be chosen to be convex, see e.g. [4] and Problem 1.4.14 in [99]. The continuous dependence Theorem 3.4.3 then yields the convergence of the corresponding sequences $(Y_{n,0}^{(i)})_{n \geq 1}$ of G -BSDE solutions, concluding the proof. $\square$

By similar arguments, we derive the following statement for the increasing convex order. The proof of Theorem 5.3.2 is first stated for $\mathcal{C}_b^3$ bounded coefficients $\varphi, b_i(t, x), h_i(t, x), \sigma_i(t, x)$ and $g_i(t, x, y, z)$ under (5.17), and then extended to coefficients satisfying (A.5.1)-(A.5.3) by applying the continuous dependence Theorem 3.4.3 as in the proof of Theorem 5.3.1.

Theorem 5.3.2. (Increasing convex order). Assume that $X_0^{(1)} \leq X_0^{(2)}$ and

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(BB'_1) \quad M_1(t, x, y, z) \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(BB'_2) \quad N_1(t, x, y, z) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(BB'_3) \quad (x, y, z) \mapsto M_i(t, x, y, z) \text{ and } (x, y, z) \mapsto N_i(t, x, y, z) \text{ are both convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R} \times \mathbb{R}_+, \text{ for } i = 1, 2, \quad t \in [0, T],$$

$$(BB'_4) \quad x \mapsto f_i(t, x, y, z) \text{ and } x \mapsto g_i(t, x, y, z) \text{ are non-decreasing on } \mathbb{R}, \text{ for } i = 1, 2, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_G^{\text{icon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing convex functions $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. Under (BB'_4) , when $\varphi(x)$, $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$, $i = 1, 2$, are non-decreasing in x , Theorem 3.4.2 tells us that the PDE solutions $u_1(t, x)$ and $u_2(t, x)$ satisfy

$$\frac{\partial u_1}{\partial x}(t, x) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t, x) \geq 0, \quad t \in [0, T],$$

hence Conditions (BB_1) – (BB_3) only need to hold for $z \geq 0$, and the conclusion follows by repeating the arguments in the proof of Theorem 5.3.1. $\square$

We note that in case $\sigma_1(t, x) = \sigma_2(t, x)$, the convexity of $u_i(t, x)$, $i = 1, 2$, is no longer required in the proofs of Theorems 5.3.1–5.3.2, and one can then remove Condition (BB'_3) to obtain a result for the monotonic order.

Corollary 5.3.3. (Monotonic order with equal volatilities). Assume that $X_0^{(1)} \leq X_0^{(2)}$ and

$$0 < \sigma(t, x) := \sigma_1(t, x) = \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(BB''_1) \quad M_1(t, x, y, z) \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(BB''_2) \quad N_1(t, x, y, z) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

(BB_3'') $x \mapsto f_i(t, x, y, z)$ and $x \mapsto g_i(t, x, y, z)$ are non-decreasing on $\mathbb{R}$, for $i = 1, 2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0, T]$.

Then we have $X_T^{(1)} \leq_G^{\text{mon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. When $\sigma_1(t, x) = \sigma_2(t, x)$, we can repeat the proof of Theorem 5.3.1 by using H_i in (5.13), without defining $\tilde{H}_i$ in (5.15) and without assuming (BB_3), and then follow the proof argument of Theorem 5.3.2 without requiring the convexity of $u_i(t, x)$, $i = 1, 2$. $\square$

Ordered drifts

Theorem 5.3.2 also admits the following version in the case of ordered drifts.

Theorem 5.3.4. (Increasing convex order). Assume that $X_0^{(1)} \leq X_0^{(2)}$ and

$$b_1(t, x) \leq b_2(t, x), \quad h_1(t, x) \leq h_2(t, x), \quad \text{and} \quad 0 < \sigma_1(t, x) \leq \sigma_2(t, x),$$

for $t \in [0, T]$, $x \in \mathbb{R}$, together with the following conditions:

$$(CC_1) \quad f_1(t, x, y, z) \leq f_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(CC_2) \quad g_1(t, x, y, z) \leq g_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(CC_3) \quad z \mapsto f_i(t, x, y, z) \text{ and } z \mapsto g_i(t, x, y, z) \text{ are both non-decreasing on } \mathbb{R}_+, \text{ for } i = 1 \text{ or } i = 2, \quad x, y \in \mathbb{R}, \quad t \in [0, T],$$

$$(CC_4) \quad x \mapsto f_i(t, x, y, z) \text{ and } x \mapsto g_i(t, x, y, z) \text{ are both non-decreasing on } \mathbb{R}, \text{ for } i = 1, 2, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T],$$

$$(CC_5) \quad (x, y, z) \mapsto M_i(t, x, y, z) \text{ and } (x, y, z) \mapsto N_i(t, x, y, z) \text{ are both convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R} \times \mathbb{R}_+, \text{ for } i = 1, 2, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_G^{\text{icon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing convex functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. Under (CC_4) , since $\varphi(x)$, $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$, $i = 1, 2$, are non-decreasing in x , by Theorem 3.4.2 the solutions $u_1(t, x)$ and $u_2(t, x)$ of (5.12) are non-decreasing in x and, as in the proof of Theorem 5.3.2, one can take $z \geq 0$ since $\frac{\partial u_i}{\partial x}(t, x) \geq 0$. Assuming that e.g. $f_1(t, x, y, z)$ and $g_1(t, x, y, z)$ is non-decreasing in z under (CC_3) , then by $z\sigma_1(t, x) \leq z\sigma_2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, and (CC_1) – (CC_2) , we have

$$\begin{aligned} f_1(t, x, y, z\sigma_1(t, x)) &\leq f_1(t, x, y, z\sigma_2(t, x)) \leq f_2(t, x, y, z\sigma_2(t, x)), \\ g_1(t, x, y, z\sigma_1(t, x)) &\leq g_1(t, x, y, z\sigma_2(t, x)) \leq g_2(t, x, y, z\sigma_2(t, x)). \end{aligned}$$

Combining the above with the inequality $zb_1(t, x) \leq zb_2(t, x)$ and $zh_1(t, x) \leq zh_2(t, x)$, for $(t, x) \in [0, T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, one finds

$$M_1(t, x, y, z) \leq M_2(t, x, y, z) \quad \text{and} \quad N_1(t, x, y, z) \leq N_2(t, x, y, z),$$

and by Theorem 5.3.2 we conclude that $\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})]$, for all convex non-decreasing locally Lipschitz functions $\varphi(x)$. $\square$

When the drift coefficients, $b(t, x) := b_1(t, x) = b_2(t, x)$ and $h(t, x) := h_1(t, x) = h_2(t, x)$, are equal and $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$ are independent of z , $i = 1, 2$, the following theorem can be proved for the convex G -ordering similarly to Theorem 5.3.4, by applying Theorem 5.3.1 which deals with convex ordering, instead of Theorem 5.3.2.

Corollary 5.3.5. (Convex order with equal drifts). *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$b_1(t, x) = b_2(t, x), \quad h_1(t, x) = h_2(t, x), \quad \text{and} \quad 0 < \sigma_1(t, x) \leq \sigma_2(t, x),$$

for $t \in [0, T]$, $x \in \mathbb{R}$, together with the conditions

(CC'_1) $f_i(t, x, y, z) = f_i(t, x, y)$ and $g_i(t, x, y, z) = g_i(t, x, y)$ are independent of $z \in \mathbb{R}$,
for $i = 1, 2$, $t \in [0, T]$, $x, y \in \mathbb{R}$,

(CC'_2) $f_1(t, x, y) \leq f_2(t, x, y)$ and $g_1(t, x, y) \leq g_2(t, x, y)$, $t \in [0, T]$, $x, y \in \mathbb{R}$,

(CC'_3) $(x, y, z) \mapsto M_i(t, x, y, z)$ and $(x, y, z) \mapsto N_i(t, x, y, z)$ are both convex in $(x, y) \in \mathbb{R}^2$ and in $(y, z) \in \mathbb{R}^2$, for $i = 1, 2$, $t \in [0, T]$.

Then we have $X_T^{(1)} \leq_G^{\text{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all convex functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

We note that the convexity of $u_1(t, x)$ and $u_2(t, x)$ is not needed in the proof of Theorem 5.3.1 when $\sigma_1(t, x) = \sigma_2(t, x)$, and in this case we can remove Condition (BB'_2) in Theorem 5.3.2 as in the next corollary.

Corollary 5.3.6. (Monotonic order with equal volatilities). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma(t, x) := \sigma_1(t, x) = \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the following conditions:

$$(DD_1) \quad b_1(t, x) \leq b_2(t, x) \text{ and } h_1(t, x) \leq h_2(t, x), \quad x \in \mathbb{R}, \quad t \in [0, T],$$

$$(DD_2) \quad f_1(t, x, y, z) \leq f_2(t, x, y, z) \text{ and } g_1(t, x, y, z) \leq g_2(t, x, y, z), \text{ for all } (x, y, z) \in \mathbb{R}^2 \times \mathbb{R}_+, \quad t \in [0, T],$$

$$(DD_3) \quad x \mapsto f_i(t, x, y, z) \text{ and } x \mapsto g_i(t, x, y, z) \text{ are non-decreasing on } \mathbb{R}, \text{ for } i = 1, 2 \text{ and } (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+.$$

Then we have $X_T^{(1)} \leq_G^{\text{mon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. Similarly to the proof of Corollary 5.3.3, under the condition $\sigma_1(t, x) = \sigma_2(t, x)$ the convexity of $u_i(t, x)$ and the non-decreasing property of $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$ with respect to z , $i = 1$ or $i = 2$, are no longer required. In addition, the condition

$$M_1(t, x, y, z) \leq M_2(t, x, y, z), \quad N_1(t, x, y, z) \leq N_2(t, x, y, z),$$

for all $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0, T]$, clearly holds from (DD_1) – (DD_2) , and we can conclude as in the proof of Theorem 5.3.4. $\square$

5.3.2 G -Stochastic Orderings with Partially Convex Drifts

Theorems 5.3.1 and 5.3.2 require the convexity assumptions (BB_2) and (BB'_2) on

$$(t, x, y, z) \mapsto M_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x))$$

and

$$(t, x, y, z) \mapsto N_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x))$$

in (x, y) and (y, z) to hold for both $i = 1, 2$. In this section, we develop different convex G -ordering results under weaker convexity conditions, based on a measurable functions $\zeta(t, x)$ and $\eta(t, x)$ such that

$$\theta_i(t, X_t^{(i)}) := \frac{b_i(t, X_t^{(i)}) - \zeta(t, X_t^{(i)})}{\sigma_i(t, X_t^{(i)})} \quad \text{and} \quad \lambda_i(t, X_t^{(i)}) := \frac{h_i(t, X_t^{(i)}) - \eta(t, X_t^{(i)})}{\sigma_i(t, X_t^{(i)})}, \quad (5.18)$$

for $i = 1, 2$, are bounded processes. As in Section 5.3.1, the proofs of Theorems 5.3.7-5.3.8 are first stated for $\mathcal{C}_b^3$ bounded coefficients as in Theorem 3.3.5, and then extended under (A.5.1)-(A.5.3) using the continuous dependence Theorem 3.4.3.

Theorem 5.3.7. (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (5.19)$$

together with the conditions

$$(EE_1) \quad M_1(t, x, y, z) \leq z\eta(t, x) \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(EE_2) \quad N_1(t, x, y, z) \leq z\zeta(t, x) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(EE_3) \quad (x, y, z) \mapsto M_i(t, x, y, z) \text{ and } (x, y, z) \mapsto N_i(t, x, y, z) \text{ are convex in } (x, y) \text{ and } (y, z) \in \mathbb{R}^2 \text{ for } i = 1 \text{ or } i = 2, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_G^{\text{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})], \quad (5.20)$$

for all convex functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. (i) We start by assuming that the function φ and the coefficients $b_i(t, \cdot)$, $h_i(t, \cdot)$, $\sigma_i(t, \cdot)$, $f_i(t, \cdot, \cdot, \cdot)$ and $g_i(t, \cdot, \cdot, \cdot)$ are $\mathcal{C}_b^3$ bounded functions for all $t \in [0, T]$ as in Theorem 3.3.5, and that (EE₃) holds with $i = 1$. Let

$$\theta_2(t, x) := \frac{b_2(t, x) - \zeta(t, x)}{\sigma_2(t, x)} \quad \text{and} \quad \lambda_2(t, x) := \frac{h_2(t, x) - \eta(t, x)}{\sigma_2(t, x)}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

are bounded functions. By the G -Girsanov transformation, see e.g. Theorems 3.2 and 5.2 in [43], the process

$$\tilde{B}_t^G := B_t + \int_0^t \theta_2(s, X_s^{(2)}) ds + \int_0^t \lambda_2(s, X_s^{(2)}) d\langle B \rangle_s, \quad t \in [0, T],$$

is a G -Brownian motion under the following G -expectation $\tilde{\mathbb{E}}_2$ defined by

$$\begin{aligned} \tilde{\mathbb{E}}_2[X] := & \tilde{\mathbb{E}}_G \left[X \exp \left(- \int_0^T \lambda_2(s, X_s^{(2)}) dB_s - \frac{1}{2} \int_0^T (\lambda_2(s, X_s^{(2)}))^2 d\langle B \rangle_s \right. \right. \\ & - \int_0^T \theta_2(s, X_s^{(2)}) \lambda_2(s, X_s^{(2)}) ds - \int_0^T \theta_2(s, X_s^{(2)}) d\tilde{B}_s \\ & \left. \left. - \frac{1}{2} \int_0^T (\theta_2(s, X_s^{(2)}))^2 d\langle \tilde{B} \rangle_s \right) \right], \end{aligned}$$

where $(B, \tilde{B})$ is an auxiliary extended $\tilde{G}$ -Brownian motion and

$$\tilde{G}(A) := \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \nu \leq \bar{\sigma}^2} \text{tr} \left[A \begin{pmatrix} \nu & 1 \\ 1 & \nu^{-1} \end{pmatrix} \right],$$

for A in the set $\mathbb{S}_2$ of 2×2 symmetric matrices. The forward SDEs (5.8) can be rewritten as

$$\begin{cases} dX_t^{(1)} = (b_1(t, X_t^{(1)}) - \theta_2(t, X_t^{(2)})\sigma_1(t, X_t^{(1)}))dt \\ \quad + (h_1(t, X_t^{(1)}) - \lambda_2(t, X_t^{(2)})\sigma_1(t, X_t^{(1)}))d\langle B \rangle_t + \sigma_1(t, X_t^{(1)})d\tilde{B}_t^G, \\ dX_t^{(2)} = \zeta(t, X_t^{(2)})dt + \eta(t, X_t^{(2)})d\langle B \rangle_t + \sigma_2(t, X_t^{(2)})d\tilde{B}_t^G, \end{cases}$$

with the associated BSDEs

$$\begin{cases} dY_t^{(1)} = -(f_1(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)}) + Z_t^{(1)}\theta_2(t, X_t^{(2)}))dt \\ \quad - (g_1(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)}) + Z_t^{(1)}\lambda_2(t, X_t^{(2)}))d\langle B \rangle_t + Z_t^{(1)}d\tilde{B}_t^G + dK_t^{(1)}, \\ dY_t^{(2)} = -(f_2(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}) + Z_t^{(2)}\theta_2(t, X_t^{(2)}))dt \\ \quad - (g_2(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}) + Z_t^{(2)}\lambda_2(t, X_t^{(2)}))d\langle B \rangle_t + Z_t^{(2)}d\tilde{B}_t^G + dK_t^{(2)}, \end{cases}$$

where at terminal time, $Y_T^{(1)} = \varphi(X_T^{(1)})$, and $Y_T^{(2)} = \varphi(X_T^{(2)})$. By Theorem 3.3.5 we have $Y_t^{(1)} = u_1(t, X_t^{(1)})$ and $Y_t^{(2)} = u_2(t, X_t^{(2)})$, where the functions $u_1(t, x)$ and $u_2(t, x)$ are in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ and solve the PDEs

$$\begin{aligned} \frac{\partial u_i}{\partial t}(t, x) + 2G \left(\frac{1}{2} \sigma_i^2(t, x) \frac{\partial^2 u_i}{\partial x^2}(t, x) + M_i \left(t, x, u_i(t, x), \frac{\partial u_i}{\partial x}(t, x) \right) \right) \\ + N_i \left(t, x, u_i(t, x), \frac{\partial u_i}{\partial x}(t, x) \right) = 0, \end{aligned} \quad (5.21)$$

with $u_i(T, x) = \varphi(x)$, $i = 1, 2$ and $G(a) = \frac{1}{2} \sup_{\sigma^2 \leq \sigma^2 \leq \bar{\sigma}^2} (\sigma^2 a)$. Applying the G -Itô formula to $u_1(t, X_t^{(2)})$ and using (5.21), we have

$$\begin{aligned}
u_1(t, X_t^{(2)}) &= u_1(0, X_0^{(2)}) + \int_0^t \frac{\partial u_1}{\partial s}(s, X_s^{(2)}) ds + \int_0^t \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) ds \\
&\quad + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s + \\
&\quad + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G \\
&= u_1(0, X_0^{(2)}) + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G \\
&\quad + \int_0^t \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) ds - \int_0^t N_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})) ds \\
&\quad - 2 \int_0^t G\left(\frac{1}{2} \sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) + M_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)}))\right) ds \\
&\quad + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\
&\geq u_1(0, X_0^{(2)}) + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G \\
&\quad - 2 \int_0^t G\left(\frac{1}{2} \sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) + M_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)}))\right) ds \\
&\quad + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s,
\end{aligned}$$

where we used the first part of (EE₂). Since $u_2(t, x)$ is a solution of the equation (5.21), we know that $Y_s^{(2)} = u_2(s, X_s^{(2)})$ and $Z_s^{(2)} = \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)})$, and hence, by plugging those terms in the second G -BSDE, we have

$$\begin{aligned}
u_2(t, X_t^{(2)}) &= u_2(0, X_0^{(2)}) - \int_0^t f_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) ds \\
&\quad - \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \theta_2(s, X_s^{(2)}) ds + \\
&\quad - \int_0^t g_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) d\langle B \rangle_s \\
&\quad - \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \lambda_2(s, X_s^{(2)}) d\langle B \rangle_s + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G + K_t^{(2)} \\
&= u_2(0, X_0^{(2)}) + \int_0^t \left(\zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) - N_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) \right) ds
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) - M_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) \right) d\langle B \rangle_s \\
& + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G + K_t^{(2)}.
\end{aligned}$$

Next, given that $u_1(T, X_T^{(2)}) = u_2(T, X_T^{(2)}) = \varphi(X_T^{(2)})$ a.s. at the terminal time, we have

$$\begin{aligned}
& u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq \tag{5.22} \\
& \int_0^T \left(N_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) - \zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \right) ds \\
& + \int_0^T \left(M_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) - \eta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\
& - 2 \int_0^T G\left(\frac{1}{2}\sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) + M_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)}))\right) ds \\
& + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2}\sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\
& + \int_0^T \sigma_1(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - \int_0^T \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - K_T^{(2)} \\
& \geq -2 \int_0^T G\left(\frac{1}{2}\sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) + M_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)}))\right) ds \\
& + \int_0^t \left(M_1(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})) + \frac{1}{2}\sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\
& + \int_0^T \sigma_1(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - \int_0^T \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - K_T^{(2)}, \tag{5.23}
\end{aligned}$$

where we applied the second parts of (EE₁)- (EE₂) and then first part of (EE₁) and $\sigma_2(t, x)^2 \geq \sigma_1^2(t, x)$ with $\frac{\partial^2 u_1}{\partial x^2}(t, x) \geq 0$. Finally, we take the expectation under $\tilde{\mathbb{E}}_2$ and noting that $K_T^{(2)} \leq 0$, a.s. and that $\int_0^t \eta_s d\langle B \rangle_s - 2 \int_0^t G(\eta_s) ds$ is a G -martingale for any $\eta_s \in M_G^1(0, T)$, see Proposition 1.4 of [80] page 72, we obtain

$$u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq 0.$$

Since $X_0^{(2)} = X_0^{(1)}$ and $Y_0^{(i)} = u_i(0, X_0^{(i)})$, $i = 1, 2$, we get $Y_0^{(2)} \geq Y_0^{(1)}$. Therefore, by definition, we have

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all convex function $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

(ii) The case $i = 2$ in Assumption (EE₃) can be proved similarly. The extension of (5.20) to coefficients satisfying (A.5.1)-(A.5.3) follows as in the proof of Theorem 5.3.1. $\square$

Next, we deal with the increasing convex order, for which only the conditions (EE'_1) - (EE'_3) and $X_0^{(1)} \leq X_0^{(2)}$ are required in addition to Condition (5.24) and (EE'_4) below.

Theorem 5.3.8. (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (5.24)$$

together with the conditions

$$(EE'_1) \quad M_1(t, x, y, z) \leq z\eta(t, x) \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(EE'_2) \quad N_1(t, x, y, z) \leq z\zeta(t, x) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(EE'_3) \quad (x, y) \mapsto M_i(t, x, y, z) \text{ and } (x, y) \mapsto N_i(t, x, y, z) \text{ are convex on } \mathbb{R}^2; \quad (y, z) \mapsto M_i(t, x, y, z) \text{ and } (y, z) \mapsto N_i(t, x, y, z) \text{ are convex on } \mathbb{R} \times \mathbb{R}_+ \text{ for } i = 1 \text{ or } i = 2,$$

$$(EE'_4) \quad x \mapsto f_i(t, x, y, z) \text{ and } x \mapsto g_i(t, x, y, z) \text{ are non-decreasing on } \mathbb{R} \text{ for } i = 1, 2, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_G^{\text{icon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing convex functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. As in the proof of Theorem 5.3.7 we start with $\mathcal{C}_b^3$ bounded coefficients, and then extend the conclusion to coefficients satisfying (A.5.1)-(A.5.3) using the continuous dependence Theorem 3.4.3. If $\varphi(x)$, $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$ are non-decreasing in x by (E'_4) , $i = 1, 2$, then by Theorem 3.4.2 the solutions $u_1(t, x)$ and $u_2(t, x)$ of the PDE (5.21) are nondecreasing in x and satisfy

$$\frac{\partial u_1}{\partial x}(t, X_t^{(2)}) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t, X_t^{(2)}) \geq 0,$$

a.s., $t \in [0, T]$, hence conditions (EE_1) - (EE_3) only need to hold for $z \geq 0$, showing the sufficiency of (EE'_i) , $i = 1, 2, 3$. In addition, we have $Y_0^{(1)} = u_1(0, X_0^{(1)}) \leq u_1(0, X_0^{(2)})$ by the assumption $X_0^{(1)} \leq X_0^{(2)}$, hence by repeating arguments in the proof of Theorem 5.3.7 for $i = 1$ we find by (5.23) that

$$Y_0^{(2)} - Y_0^{(1)} \geq u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq 0,$$

under Assumption (EE'_3) for $i = 1$. The case $i = 2$ is treated similarly according to the proof of Theorem 5.3.7. $\square$

Corollary 5.3.9. (Increasing order). Assume that $X_0^{(1)} \leq X_0^{(2)}$ and

$$0 < \sigma_1(t, x) = \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(EE_1'') \quad M_1(t, x, y, z) \leq z\eta(t, x) \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(EE_2'') \quad N_1(t, x, y, z) \leq z\zeta(t, x) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R}, \quad z \in \mathbb{R}_+,$$

$$(EE_3'') \quad x \mapsto f_i(t, x, y, z) \text{ and } x \mapsto g_i(t, x, y, z) \text{ are non-decreasing on } \mathbb{R} \text{ for } i = 1, 2, \\ y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad t \in [0, T].$$

Then we have $X_T^{(1)} \leq_G^{\text{mon}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_G[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Proof. If $\sigma_1(t, x) = \sigma_2(t, x)$, we can check that the convexity of $u_i(t, x)$, $i = 1, 2$ is no longer required, so one can remove Condition (EE_3') . Then this result is a directly consequence of Theorem 5.3.8. $\square$

5.4 Comparison Results for G -Risk Measures

In this section, we introduce a risk measure induced by the nonlinear expectation $\mathcal{E}_G$ as we did for the g -expectation in Section 4.5 of Chapter 4. Moreover, we study coherent and convex risk measures relied on properties of the functions $f(t, x, y, z)$ and $g(t, x, y, z)$.

Definition 5.4.1. Let Conditions (A.5.2)-(A.5.3) hold true and $\xi \in L_G^2(\Omega, \mathcal{F}_T)$.

1) The static G -risk measure is defined by

$$\rho^G(\xi) := \mathcal{E}_G[-\xi].$$

2) The dynamic G -risk measure is defined as

$$\rho_t^G(\xi) := \mathcal{E}_G[-\xi \mid \mathcal{F}_t], \quad t \in [0, T].$$

Below, we show coherent and convex properties for the risk measures ρ^G and $(\rho_t^G)_{t \in [0, T]}$, which rely on particular properties of generator functions $f(t, y, x, z)$ and $g(t, x, y, z)$ of the G -BSDEs (5.5).

Proposition 5.4.2. *Let Conditions (A.5.2)-(A.5.3) hold true.*

- (i) *If f and g are independent of y , positively homogeneous and sub-additive in $z \in \mathbb{R}$, then ρ^G and $(\rho_t^G)_{t \in [0, T]}$ are static coherent and dynamic coherent risk measures, respectively.*
- (ii) *If f and g are independent of y , positively homogeneous and convex in $z \in \mathbb{R}$, then ρ^G and $(\rho_t^G)_{t \in [0, T]}$ are static convex and dynamic convex risk measures, respectively.*

Proof. (i) It is sufficient to prove that $\mathcal{E}_G$ satisfies (b) Monotonicity, (d) Translatability, and (e) Sublinearity in the definition 2.4.1 of Chapter 2. First, monotonicity of $\mathcal{E}_G$ is clear from the comparison theorem of G -BSDEs. Since $G(a) = \frac{1}{2}(a^+ \bar{\sigma}^2 - a^- \underline{\sigma}^2)$, $a \in \mathbb{R}$, it is easy to check that

$$G(a) + G(-a) = \frac{1}{2}(a^+ + a^-)(\bar{\sigma}^2 - \underline{\sigma}^2) > 0, \quad a \neq 0.$$

Hence, by Proposition 18 of [39], $\mathcal{E}_G$ holds translatability, i.e., $\mathcal{E}_G[\xi + c] = \mathcal{E}_G[\xi] + c$, for $c \in \mathbb{R}$, $\xi \in L_G^2(\Omega, \mathcal{F}_T)$ if and only if f and g are independent of y , i.e.,

$$f(t, x, y, z) = f(t, x, z), \quad g(t, x, y, z) = g(t, x, z), \quad x, y, z \in \mathbb{R}, t \in [0, T].$$

In addition, if $f(t, x, \lambda z) = \lambda f(t, x, z)$ and $g(t, x, \lambda z) = \lambda g(t, x, z)$, for $\lambda \geq 0$, then we obviously have

$$f(t, x, \lambda z) - \lambda f(t, x, z) = 2G(g(t, x, \lambda z) - \lambda g(t, x, z)) = 0, \quad \lambda \in \mathbb{R}_+, x, z \in \mathbb{R}, t \in [0, T].$$

Thus, by the assertion (4) of Proposition 17 in [39], we get $\mathcal{E}_G[\lambda \xi] = \lambda \mathcal{E}_G[\xi]$, for all $\lambda \geq 0$, $\xi \in L_G^2(\Omega, \mathcal{F}_T)$. Finally, we can check sub-additivity by the assertion (2) of Proposition 17 in [39] and because of monotonicity of $G(a)$ that

$$\mathcal{E}_G[\xi_1 + \xi_2] \leq \mathcal{E}_G[\xi_1] + \mathcal{E}_G[\xi_2], \quad \xi_1, \xi_2 \in L_G^2(\Omega, \mathcal{F}_T),$$

if $f(t, x, z + z') \leq f(t, x, z) + f(t, x, z')$ and $g(t, x, z + z') \leq g(t, x, z) + g(t, x, z')$ for all $x, z, z' \in \mathbb{R}$, $t \in [0, T]$. We conclude the static coherent risk measure ρ^G . Similarly, one can prove the dynamic coherent risk measure $(\rho_t^G)_{t \in [0, T]}$.

Part (ii) is proved in the same fashion as in the part (i) with additionally using the assertion (3) of Proposition 17 in [39]. $\square$

Next, we denote by $(\bar{Y}_t^{(a)}, \bar{Z}_t^{(a)}, K_t^{(a)})_{t \in [0, T]}$ the solution of the following G -BSDEs

$$\begin{aligned} \bar{Y}_t^{(a)} = & \xi + \int_t^T f^{(a)}\left(s, X_s, \bar{Y}_s^{(a)}, \bar{Z}_s^{(a)}\right) ds + \int_t^T g^{(a)}\left(s, X_s, \bar{Y}_s^{(a)}, \bar{Z}_s^{(a)}\right) d\langle B \rangle_s \\ & - \int_t^T \bar{Z}_s^{(a)} dB_s - K_T^{(a)}, \end{aligned} \quad (5.25)$$

where $\xi \in L_G^2(\Omega, \mathcal{F}_T)$ and the generators

$$f^{(a)}(t, x, y, z) := af\left(t, x, \frac{y}{a}, \frac{z}{a}\right), \quad g^{(a)}(t, x, y, z) := ag\left(t, x, \frac{y}{a}, \frac{z}{a}\right), \quad (5.26)$$

for all $a \in \mathbb{R} \setminus \{0\}$, $x, y, z \in \mathbb{R}$, $t \in [0, T]$. We also define a conditional expectation

$$\bar{\mathcal{E}}_G^{(a)}[\xi | \mathcal{F}_t] := \bar{Y}_t^{(a)}, \quad t \in [0, T]. \quad (5.27)$$

The Lemma 5.4.3 below plays an important role in deriving comparison results for the G -risk measures presented in Corollaries 5.4.5-5.4.6. It shows in particular by (5.28) that the inequality

$$\rho^G(X_T^{(1)}) \leq \rho^G(X_T^{(2)}) \quad \text{holds if and only if} \quad \bar{\mathcal{E}}_G^{(-1)}[X_T^{(2)}] \leq \bar{\mathcal{E}}_G^{(-1)}[X_T^{(1)}].$$

Lemma 5.4.3. *Consider the respective solutions $(X_t)_{t \in [0, T]}$ and $(Y_t, Z_t)_{t \in [0, T]}$ of (5.4) and (5.5) with the terminal condition $Y_T = \varphi(X_T) \in L_G^2(\Omega, \mathcal{F}_T)$. Then, for any $a \neq 0$ we have the relation*

$$\bar{\mathcal{E}}_G^{(a)}[a\varphi(X_T)] = a\mathcal{E}_G[\varphi(X_T)].$$

where $\bar{\mathcal{E}}_G^{(a)}$ is defined by (5.27).

Proof. Let $(\bar{Y}_t^{(a)}, \bar{Z}_t^{(a)}, \bar{K}_t^{(a)})_{t \in [0, T]}$ be the solution of the G -BSDE (5.25) with terminal $\xi := a\varphi(X_T)$, i.e.,

$$\begin{aligned} \frac{\bar{Y}_t^{(a)}}{a} = & \varphi(X_T) + \int_t^T f\left(s, X_s, \frac{\bar{Y}_s^{(a)}}{a}, \frac{\bar{Z}_s^{(a)}}{a}\right) ds + \int_t^T g\left(s, X_s, \frac{\bar{Y}_s^{(a)}}{a}, \frac{\bar{Z}_s^{(a)}}{a}\right) d\langle B \rangle_s \\ & - \frac{1}{a} \int_t^T \bar{Z}_s^{(a)} dB_s - (1/a)\bar{K}_T^{(a)}. \end{aligned}$$

By uniqueness of the solution $(Y_t, Z_t, K_t)_{t \in [0, T]}$ of the G -BSDEs

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds + \int_t^T g(s, X_s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - K_T,$$

we get that $\bar{Y}_t^{(a)}/a = Y_t$, $\bar{Z}_t^{(a)}/a = Z_t$ and $\bar{K}_t^{(a)}/a = K_t$, $t \in [0, T]$. Thus, at $t = 0$, we have that $\bar{\mathcal{E}}_G^{(a)}[a\varphi(X_T)] = a\mathcal{E}_G[\varphi(X_T)]$. $\square$

Remark 5.4.4. From Lemma 5.4.3, we see that $\bar{\mathcal{E}}_G^{(a)}[a^2\varphi(X_T)] = a\mathcal{E}_G[a\varphi(X_T)]$. Hence, by taking $a = -1$, one obtains that

$$\bar{\mathcal{E}}_G^{(-1)}[\varphi(X_T)] = -\mathcal{E}_G[-\varphi(X_T)]. \quad (5.28)$$

In a call option pricing problem, $-\mathcal{E}_G[-\varphi(X_T)]$ and $\mathcal{E}_G[\varphi(X_T)]$ represent for bid (subhedging) and ask (superhedging) prices, respectively. By taking $\varphi(x) := -x$ in (5.7), then $X^{(1)} \leq_G^{conv} X^{(2)}$ implies that $\rho^{g_1}(X^{(1)}) \leq \rho^{g_2}(X^{(2)})$. In addition, we have $\rho^{g_1}(\varphi(X_T^{(1)})) \leq \rho^{g_2}(\varphi(X_T^{(2)}))$ if and only if

$$-\bar{\mathcal{E}}_G^{(-1)}[\varphi(X_T^{(1)})] \leq -\bar{\mathcal{E}}_G^{(-1)}[\varphi(X_T^{(2)})].$$

In the sequel, we will use the following functions

$$f_i^{(-1)}(t, x, y, z) := -f_i(t, x, -y, -z) \quad \text{and} \quad g_i^{(-1)}(t, x, y, z) = -g_i(t, x, -y, -z),$$

and

$$\begin{aligned} M_i^{(-1)}(t, x, y, z) &:= zh_i(t, x) + g_i^{(-1)}(t, x, y, z\sigma_i(t, x)), \\ N_i^{(-1)}(t, x, y, z) &:= zb_i(t, x) + f_i^{(-1)}(t, x, y, z\sigma_i(t, x)), \end{aligned}$$

for $i = 1, 2$ and $x, y, z \in \mathbb{R}$, $t \in [0, T]$.

Theorems 5.3.1 and 5.3.7 admit the following versions for the comparison of G -risk measures or concave G -stochastic ordering. Firstly, we have a consequence of Theorem 5.3.1 and Lemma 5.4.3.

Corollary 5.4.5. Assume that $X_0^{(1)} = X_0^{(2)}$ and

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(FF_1) \quad M_1^{(-1)}(t, x, y, z) \leq M_2^{(-1)}(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(FF_2) \quad N_1^{(-1)}(t, x, y, z) \leq N_2^{(-1)}(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(FF_3) \quad (x, y, z) \mapsto M_i^{(-1)}(t, x, y, z) \text{ and } (x, y, z) \mapsto N_i^{(-1)}(t, x, y, z) \text{ are both convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R}^2, \text{ for } i = 1, 2 \text{ and } t \in [0, T].$$

Then we have

$$-\mathcal{E}_G[-\varphi(X_T^{(1)})] \leq -\mathcal{E}_G[-\varphi(X_T^{(2)})],$$

for all convex functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Similarly, we have a consequence of Theorem 5.3.7 and Lemma 5.4.3.

Corollary 5.4.6. *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1(t, x) \leq \sigma_2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

together with the conditions

$$(GG_1) \quad M_1^{(-1)}(t, x, y, z) \leq z\eta(t, x) \leq M_2^{(-1)}(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(GG_2) \quad N_1^{(-1)}(t, x, y, z) \leq z\zeta(t, x) \leq N_2^{(-1)}(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

$$(GG_3) \quad (x, y, z) \mapsto M_i^{(-1)}(t, x, y, z) \text{ and } (x, y, z) \mapsto N_i^{(-1)}(t, x, y, z) \text{ are both convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R}^2, \text{ for } i = 1 \text{ or } i = 2 \text{ and } t \in [0, T].$$

Then we have

$$-\mathcal{E}_G[-\varphi(X_T^{(1)})] \leq -\mathcal{E}_G[-\varphi(X_T^{(2)})],$$

for all convex functions $\varphi(x) \in \mathcal{C}_{l.Lip}(\mathbb{R})$.

Furthermore, we can also derive results for the increasing convex and monotonic orderings of the G -risk measures under Conditions (FF_1) - (FF_3) and (GG_1) - (GG_3) . For example, if (FF_1) - (FF_3) or (GG_1) - (GG_3) only holds for $z \geq 0$ along with f_i and g_i are non-decreasing in x , $i = 1, 2$, we then get versions of Corollaries 5.4.5-5.4.6 for the increasing convex G -risk comparisons as in Theorems 5.3.2 and 5.3.8. Similarly, under an additional assumption $\sigma_1(t, x) = \sigma_2(t, x)$ and by removing (FF_3) and (GG_3) , we can obtain versions of Corollaries 5.4.5-5.4.6 for the monotonic G -risk ordering as in Corollary 5.3.3.

5.5 Comparisons of Option Superhedging Prices under Ambiguous Coefficients

The G -stochastic ordering results of Section 5.2 can be used to derive comparisons of option superhedging prices and portfolio values with underlying risky assets driven by G -FSDEs, as in the same manner of Section 4.6 in Chapter 4. In this section we instead study the effects of ambiguous drift and volatility coefficients of G -FSDEs on associated contingent claim superhedging prices and portfolio values. Specially, we consider a risk-free asset priced $E_t := E_0 e^{rt}$ and a risky asset with non-negative prices modeled by the G -FSDE:

$$dX_s = X_s \alpha(s, X_s) ds + X_s \beta(s, X_s) d\langle B \rangle_s + X_s \gamma(s, X_s) dB_s, \quad X_0 = x_0, \quad (5.29)$$

for $s \in [0, T]$, where the coefficients $b(t, x) := x\alpha(t, x)$, $h(t, x) := x\beta(t, x)$ and $\sigma(t, x) := x\gamma(t, x)$ satisfy (A.5.1), for example $\alpha(t, x)$, $\beta(t, x)$ and $\gamma(t, x)$ can be bounded functions. Denote by $(p_t, q_t)_{t \in [0, T]}$ the quantities (strategy processes) invested in the risk-free asset and the risky asset at time $t \in [0, T]$, we consider the wealth process defined by $Y_t := p_t E_t + q_t X_t - C_t$, where $(C_t)_{t \geq 0}$ is a cumulative consumption process with $C_0 = 0$, such that

$$\begin{aligned} dY_t &= rp_t E_t dt + q_t dX_t - dC_t \\ &= r(Y_t - q_t X_t)dt + q_t \left(X_t \alpha(t, X_t)dt + X_t \beta(t, X_t)d\langle B \rangle_t + X_t \gamma(t, X_t)dB_t \right) - dC_t \\ &= rY_t dt + \frac{\alpha(t, x) - r}{\gamma(t, X_t)} q_t X_t \gamma(t, X_t)dt + \frac{\beta(t, X_t)}{\gamma(t, X_t)} q_t X_t \gamma(t, X_t)d\langle B \rangle_t \\ &\quad + q_t X_t \gamma(t, X_t)dB_t - dC_t, \end{aligned} \quad (5.30)$$

Setting $\theta(t, x) := \frac{\alpha(t, x) - r}{\gamma(t, x)}$, $\lambda(t, x) := \frac{\beta(t, x)}{\gamma(t, x)}$, $Z_t := q_t X_t \gamma(t, X_t)$, and $K_t := -C_t$, for $0 \leq t \leq T$, the wealth process can be rewritten as the G-BSDE

$$\begin{aligned} Y_t &= Y_T - \int_t^T (rY_s + \theta(s, X_s)Z_s)ds - \int_t^T \lambda(s, X_s)Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s \\ &\quad - dK_t, \quad t \in [0, T]. \end{aligned} \quad (5.31)$$

Discounting processes $\tilde{Y}_t = e^{-rt}Y_t$, $\tilde{Z}_t = e^{-rt}Z_t$, $\tilde{K}_t = e^{-rt}K_t$, and by applying the G -Itô, we get the G -BSDE

$$\tilde{Y}_t = \tilde{Y}_T - \int_t^T \theta(s, X_s)\tilde{Z}_s ds - \int_t^T \lambda(s, X_s)\tilde{Z}_s d\langle B \rangle_s - \int_t^T \tilde{Z}_s dB_s - d\tilde{K}_t, \quad (5.32)$$

for $t \in [0, T]$, where generator functions of the G -BSDE $\tilde{f}(t, x, y, z) := -\theta(t, x)z$ and $\tilde{g}(t, x, y, z) := -\lambda(t, x)z$, for $t \in [0, T]$, $x \in \mathbb{R}_+$, and $y, z \in \mathbb{R}$.

Next, we apply our G -stochastic ordering results to a model misspecification problem where we estimate the impact of uncertain or ambiguous bounded drift coefficients $\alpha(t, x)$, $\beta(t, x)$ and volatility coefficient $\gamma(t, x)$ on the superhedging prices of a contingent call claim with convex payoff $\tilde{Y}_T = \varphi(X_T)$.

Scenario 1: Suppose that the volatility $\gamma(t, x)$ is ambiguous or uncertain, i.e., $\underline{\gamma}(t) \leq \gamma(t, x) \leq \bar{\gamma}(t)$, and assume for simplicity that the drift coefficients $\alpha(t, x)$ and $\beta(t, x)$ are precisely known to the hedgers. In this case, the ambiguity of $\gamma(t, x)$ yields a bound on the superhedging price of the contingent claim by Theorems 5.3.1

and 5.3.7. Letting $\theta_{\underline{\gamma}}(t, x) := \frac{\alpha(t, x) - r}{\underline{\gamma}(t)}$, and

$$\theta_{\bar{\gamma}}(t, x) := \frac{\alpha(t, x) - r}{\bar{\gamma}(t)}, \quad \lambda_{\underline{\gamma}}(t, x) := \frac{\beta(t, x)}{\underline{\gamma}(t)}, \quad \lambda_{\bar{\gamma}}(t, x) := \frac{\beta(t, x)}{\bar{\gamma}(t)},$$

by the relations $N(t, x, y, z) := xz\alpha(t, x) - xz\theta(t, x)\gamma(t, x) = rxz$ and

$$\begin{aligned} N_{\underline{\gamma}}(t, x, y, z) &:= xz\alpha(t, x) - xz\underline{\gamma}(t)\theta_{\underline{\gamma}}(t, x) = rxz, \\ N_{\bar{\gamma}}(t, x, y, z) &:= xz\alpha(t, x) - xz\bar{\gamma}(t)\theta_{\bar{\gamma}}(t, x) = rxz, \end{aligned}$$

we check that the function $N(t, x, y, z) = N_{\underline{\gamma}}(t, x, y, z) = N_{\bar{\gamma}}(t, x, y, z)$ is convex in $x \in \mathbb{R}$ and in $z \in \mathbb{R}$ separately. Similarly, we have $M(t, x, y, z) := xz\beta(t, x) - xz\gamma(t, x)\lambda(t, x) = 0$, and

$$\begin{aligned} M_{\underline{\gamma}}(t, x, y, z) &:= xz\beta(t, x) - xz\underline{\gamma}(t)\lambda_{\underline{\gamma}}(t, x) = 0, \\ M_{\bar{\gamma}}(t, x, y, z) &:= xz\beta(t, x) - xz\bar{\gamma}(t)\lambda_{\bar{\gamma}}(t, x) = 0, \end{aligned}$$

$t \in [0, T]$, $x, y, z \in \mathbb{R}$, hence $M(t, x, y, z) = M_{\underline{\gamma}}(t, x, y, z) = M_{\bar{\gamma}}(t, x, y, z)$, $t \in [0, T]$, $x, y, z \in \mathbb{R}$, and $x\underline{\gamma}(t) \leq x\gamma(t, x) \leq x\bar{\gamma}(t)$, $(t, x) \in [0, T] \times \mathbb{R}_+$. Hence, all conditions in Theorem 5.3.1 hold true and Conditions (EE₁)-(EE₂) of Theorem 5.3.7 are satisfied with $\eta(t, x) = 0$ and $\zeta(t, x) = rx$. We deduce the portfolio value inequalities

$$\mathcal{E}_G[\varphi(X_T^{\underline{\gamma}})] \leq \mathcal{E}_G[\varphi(X_T)] \leq \mathcal{E}_G[\varphi(X_T^{\bar{\gamma}})],$$

for all convex functions $\varphi(x)$, where $(X_t^{\underline{\gamma}})_{t \in [0, T]}$ and $(X_t^{\bar{\gamma}})_{t \in [0, T]}$ represent the mis-specified risky price processes defined using $\underline{\gamma}(t)$ and $\bar{\gamma}(t)$ in (5.29), respectively.

Scenario 2: Suppose that the ambiguous or uncertain drift $\alpha(t, x)$ satisfies $\underline{\alpha}(t) \leq \alpha(t, x) \leq \bar{\alpha}(t)$, while the coefficients $\beta(t, x)$, and $\gamma(t, x)$ are precisely known to the hedgers. Here, in contrast to the first scenario, the superhedging price of the contingent claim are not affected by the ambiguity of the drift $\alpha(t, x)$. Let $(X_t^{\alpha})_{t \in [0, T]}$ and $(X_t^{\bar{\alpha}})_{t \in [0, T]}$ denote the risky asset price processes given by

$$\begin{aligned} dX_s^{\alpha} &= \underline{\alpha}(t)X_s^{\alpha}ds + X_s^{\alpha}\beta(s, X_s^{\alpha})d\langle B \rangle_s + X_s^{\alpha}\gamma(s, X_s^{\alpha})dB_s, \\ dX_s^{\bar{\alpha}} &= \bar{\alpha}(t)X_s^{\bar{\alpha}}ds + X_s^{\bar{\alpha}}\beta(s, X_s^{\bar{\alpha}})d\langle B \rangle_s + X_s^{\bar{\alpha}}\gamma(s, X_s^{\bar{\alpha}})dB_s, \end{aligned}$$

where $X_0^{\alpha} = X_0^{\bar{\alpha}} = x_0$, and let

$$\tilde{Y}_t^{\alpha} = \varphi(X_t^{\alpha}) - \int_t^T \theta^{\alpha}(s, X_s^{\alpha})\tilde{Z}_s^{\alpha}ds - \int_t^T \lambda(s, X_s^{\alpha})\tilde{Z}_s^{\alpha}d\langle B \rangle_s - \int_t^T \tilde{Z}_s^{\alpha}dB_s - d\tilde{K}_s^{\alpha},$$

$$\tilde{Y}_t^{\bar{\alpha}} = \varphi(X_T^{\bar{\alpha}}) - \int_t^T \theta^{\bar{\alpha}}(s, X_s^{\bar{\alpha}}) \tilde{Z}_s^{\bar{\alpha}} ds - \int_t^T \lambda(s, X_s^{\bar{\alpha}}) \tilde{Z}_s^{\bar{\alpha}} d\langle B \rangle_s - \int_s^T \tilde{Z}_s^{\bar{\alpha}} dB_s - d\tilde{K}_s^{\bar{\alpha}}, \quad (5.33)$$

denote the corresponding discounted wealth processes, where $\theta^\alpha(t, x) := (\underline{\alpha}(t) - r)/\gamma(t, x)$ and $\theta^{\bar{\alpha}}(t, x) := (\bar{\alpha}(t) - r)/\gamma(t, x)$. The true superhedging price of the contingent claim at $t = 0$, is defined by $\tilde{Y}_0 := \mathcal{E}_G[\varphi(X_T)]$, where $(\tilde{Y}_t)_{t \in [0, T]}$ solves the G -BSDE (5.32). We check that the functions $N(t, x, y, z) := xz\alpha(t, x) - xz\theta(t, x)\gamma(t, x)$, $N_{\underline{\alpha}}(t, x, y, z) := xz\underline{\alpha}(t)(t, x) - xz\theta_{\underline{\alpha}}(t, x)\gamma(t, x)$, and

$$N_{\bar{\alpha}}(t, x, y, z) := xz\bar{\alpha}(t) - xz\theta_{\bar{\alpha}}(t, x)\gamma(t, x),$$

are separately convex in $x \in \mathbb{R}$ and in $z \in \mathbb{R}$, as they satisfy

$$N(t, x, y, z) = N_{\underline{\alpha}}(t, x, y, z) = N_{\bar{\alpha}}(t, x, y, z) = rxz,$$

$t \in [0, T]$, $x, y, z \in \mathbb{R}$. We also have

$$M(t, x, y, z) = M_{\underline{\alpha}}(t, x, y, z) = M_{\bar{\alpha}}(t, x, y, z) := xz\beta(t, x) - xz\gamma(t, x)\lambda(t, x) = 0,$$

$t \in [0, T]$, $x, y, z \in \mathbb{R}$. Therefore (BB₁)-(BB₂) hold as equalities and, since $X_0^{(1)} = X_0^{(2)}$ and the volatility $\gamma(t, x)$ is precisely known, the G -PDE (5.12) is the same for $\underline{\alpha}(t)$ and $\bar{\alpha}(t)$, hence we have $\mathcal{E}_G[\varphi(X_T)] = \mathcal{E}_G[\varphi(X_T^{\underline{\alpha}})] = \mathcal{E}_G[\varphi(X_T^{\bar{\alpha}})]$ for all convex functions φ .

Similarly, if the second drift $\beta(t, x)$ is ambiguous or uncertain, i.e., $\underline{\beta}(t) \leq \beta(t, x) \leq \bar{\beta}(t)$, while $\alpha(t, x)$ and $\gamma(t, x)$ are precisely determined by the hedgers. As in Scenario 2, we find that the ambiguity of $\beta(t, x)$ does not affect the superhedging price of the contingent claim. In fact, letting $(X_t^{\underline{\beta}})_{t \in [0, T]}$ and $(X_t^{\bar{\beta}})_{t \in [0, T]}$ represent the risky price processes corresponding to $\beta(t, x) = \underline{\beta}(t)$ and $\beta(t, x) = \bar{\beta}(t)$, respectively, then $\mathcal{E}_G[\varphi(X_T)] = \mathcal{E}_G[\varphi(X_T^{\underline{\beta}})] = \mathcal{E}_G[\varphi(X_T^{\bar{\beta}})]$, for all convex functions φ .

Chapter 6

Stochastic Orderings in Linear, Sublinear and Nonlinear expectations

In this chapter, we derive comparison results in the sense of convex and increasing convex orders between the linear expectation $\mathbb{E}$, sublinear expectation $\widehat{\mathbb{E}}$, g -expectation $\mathcal{E}_g$, G -expectation $\mathbb{E}_G$ and nonlinear expectation $\mathcal{E}_G$, including both kinds of lower means $-\mathcal{E}[-\varphi(\cdot)]$ and upper means $\mathcal{E}[\varphi(\cdot)]$, where $\mathcal{E}$ represents for one of the expectations $\mathbb{E}$, $\mathbb{E}_G$, $\widehat{\mathbb{E}}$, $\mathcal{E}_g$ and $\mathcal{E}_G$ above. In Section 6.1, we first start by recalling a concrete representation for the G -expectations $\mathbb{E}_G$, which shows a connection to linear expectations. Next, in Section 6.2, we show the monotonicity, convexity and concavity of solutions of G -heat PDEs, which are important properties to constitute comparison results in later sections. Then in Section 6.3, we derive sufficient conditions for several comparisons between diffusion stochastic processes $(X_t)_{t \in [0, T]}$ at terminal time and G -normal distributed random variables $\widehat{X}$, i.e., between $\mathcal{E}_g[\varphi(X_T)]$ and $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$, a consequence between $\mathbb{E}[\varphi(X_T)]$ and $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$, as well as between $-\mathcal{E}_g[-\varphi(X_T)]$ and $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})]$, and between $\mathbb{E}[\varphi(X_T)]$ and $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})]$. Such inequalities can offer us lower and upper bounds information and the concentration of measure inequalities for the g -expectations of terminal values of the diffusion SDEs. Moving to Section 6.4, we show analogous results for comparing between terminal values of G -diffusion stochastic processes $(X_t^G)_{t \in [0, T]}$ and G -normal distributed random variables, i.e., between $\mathbb{E}_G[\varphi(X_T^G)]$ and $\mathbb{E}_G[\varphi(\widehat{X})]$, and between $-\mathbb{E}_G[-\varphi(X_T^G)]$ and $-\mathbb{E}_G[-\varphi(\widehat{X})]$. In the last Section 6.5, we devote to present some comparisons between the nonlinear expectations $\mathcal{E}_g$ and $\mathcal{E}_G$ that enable us to have some orderings of $(X_t)_{t \in [0, T]}$ and $(X_t^G)_{t \in [0, T]}$, specifically between $\mathcal{E}_g[\varphi(X_T)]$ and $\mathcal{E}_G[\varphi(X_T^G)]$. This chapter also reuses comparison results for

forward, for forward-backward, and for G -forward-backward stochastic differential equations, together with the monotonicity, convexity and continuous dependence property of solutions of the associated semilinear parabolic PDEs and HJB equations (G -PDEs).

6.1 A Representation of G -Expectation

As introduced in Section 2.5 of Chapter 2, the G -expectation and corresponding G -Brownian motion are constructed on the sublinear expectation space, not on the probability space. In the next sequel, we introduce the ramification with probability theory. As before, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(W_t)_{t \geq 0}$ be a classical Brownian motion on this space. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the augmented filtration such that $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\} \cup \mathcal{N}$, for any $t \geq 0$, where $\mathcal{N}$ is the collection of all $\mathbb{P}$ -null sets. For fixed $t \geq 0$, we define $\mathcal{F}_s^t := \sigma\{W_{t+u} - W_t | 0 \leq u \leq s\} \cup \mathcal{N}$. We also let $L^2(\Omega, \mathcal{F}_t) = L^2(\Omega, \mathcal{F}_t, \mathbb{P})$, with $t \in [0, T]$, denote the space of all real-valued, $\mathcal{F}_t$ -measurable and square integrable random variables applying with the L^2 -norm. Let $L^2(\Omega \times [0, T])$ denote the space of real-valued, $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted processes $(V_t)_{t \in [0, T]}$ with $\mathbb{E} \int_0^T V_t^2 dt < \infty$.

Let $\Theta := [\underline{\sigma}, \bar{\sigma}]$ such that $G(x) = \frac{1}{2} \sup_{\theta \in \Theta} (x\theta^2)$ and $\mathcal{A}_{t,T}^\Theta$ be the collection of all Θ -valued $\mathcal{F}_s^t$ -adapted processes on $[t, T]$. For any $\theta \in \mathcal{A}_{t,T}^\Theta$, we define the process

$$B_T^{t,\theta} := \int_t^T \theta_s dW_s,$$

Let P^θ be the law of $B_t^{0,\theta} := \int_0^t \theta_s dW_s$ and defining $\mathcal{P}_1 := \{P^\theta | \theta \in \mathcal{A}_{t,T}^\Theta\}$ and the closure $\mathcal{P} = \overline{\mathcal{P}_1}$ under the topology of weak convergence. Then for any $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $n \in \mathbb{N}$, $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, the G -expectations can be represented as follows, see e.g. [21, 86]:

$$\mathbb{E}_G[\varphi(B_{t_1}, \dots, B_{t_n} - B_{n-1})] = \sup_{P^\theta \in \mathcal{P}} \mathbb{E}_{P^\theta}[\varphi(B_{t_1}, \dots, B_{t_n} - B_{n-1})]. \quad (6.1)$$

We can extend this representation for all $\xi \in L_G^1(\Omega, \mathcal{F}_T)$ by $\mathbb{E}_G[\xi] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\xi]$, where $\mathcal{P}$ is called a set that represents $\mathbb{E}_G$. Similarly, the conditional G -expectations can be represented in terms of ess sup, see e.g. [86]. In particular, we have

$$\mathbb{E}_G[\xi | \mathcal{F}_t] = \text{ess sup}_{Q' \in \mathcal{P}(t, Q)} \mathbb{E}_{Q'}[\xi | \mathcal{F}_t], \quad \mathcal{P}(t, Q) := \{Q' \in \mathcal{P} | Q = Q' \text{ on } \mathcal{F}_t\}.$$

For each fixed $T \geq 0$, let $(B_t)_{t \geq [0, T]}$ be a G -Brownian motion under $\mathbb{E}_G$. We consider a G -Itô integral of the form

$$I(\eta) := \int_0^t \eta_s dB_s, \quad \eta_t \in L_G^1(\Omega, \mathcal{F}).$$

It is well-known that $\mathbb{E}_G[I(\eta)] = -\mathbb{E}_G[-I(\eta)] = 0$. This also holds for each $P \in \mathcal{P}$, i.e., $\mathbb{E}_P[I(\eta)] = 0$ since $-\mathbb{E}_G[-I(\eta)] \leq \mathbb{E}_P[I(\eta)] \leq \mathbb{E}_G[I(\eta)]$.

6.2 Monotonicity, Convexity and Concavity of G -heat PDEs

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sublinear expectation space. Recall that $\widehat{X}$ follows G -normal distribution, denoted by $\widehat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2])$, if $\widehat{u}(t, x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}\widehat{X})]$ is a solution of the following G -heat PDEs:

$$\frac{\partial \widehat{u}}{\partial t}(t, x) - G\left(\frac{\partial^2 \widehat{u}}{\partial x^2}(t, x)\right) = 0, \quad \widehat{u}(0, x) = \varphi(x),$$

where $G(a) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \overline{\sigma}^2} (a\sigma^2) = \frac{1}{2}(a^+\overline{\sigma}^2 - a^-\underline{\sigma}^2)$, $a \in \mathbb{R}$, $0 < \underline{\sigma}^2 < \overline{\sigma}^2$.

If $\varphi(x)$ is a convex function then it is known that the solution $\widehat{u}(t, x)$ is convex in $x \in \mathbb{R}$, see e.g. [80]. Here we state some useful lemmas, including both monotonicity and convexity as well as concavity of $\widehat{u}(t, x)$ which are needed for the proofs of propositions in later sections.

Lemma 6.2.1. *Suppose that $\widehat{u}(t, x)$ is the solution of the following G -heat PDEs:*

$$\begin{cases} \frac{\partial \widehat{u}}{\partial t}(t, x) - \frac{1}{2} \left(\overline{\sigma}^2 \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(t, x) \right)^+ - \underline{\sigma}^2 \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(t, x) \right)^- \right) = 0, \\ \widehat{u}(0, x) = \varphi(x). \end{cases} \quad (6.2)$$

If $\varphi(x)$ is non-decreasing (resp. convex), then $\widehat{u}(t, x)$ is also non-decreasing (resp. convex) in $x \in \mathbb{R}$ for every fixed $t > 0$.

Proof. (i) By [80], the solution $\widehat{u}(t, x)$ of the G -heat PDE (6.2) has the form $\widehat{u}(t, x) = \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}\widehat{X})]$, where $\widehat{X}$ is G -normal distributed, i.e., $\widehat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2])$. Letting $x_1 \leq x_2$, and since $\varphi(x)$ is a non-decreasing function, we have

$$\widehat{u}(t, x_1) = \widehat{\mathbb{E}}[\varphi(x_1 + \sqrt{t}\widehat{X})] \leq \widehat{\mathbb{E}}[\varphi(x_2 + \sqrt{t}\widehat{X})] = \widehat{u}(t, x_2),$$

showing that $\hat{u}(t, x)$ is non-decreasing in $x \in \mathbb{R}$.

(ii) For proving the convexity, we take $\lambda \in [0, 1]$ and by the convexity of φ , we have

$$\begin{aligned}\hat{u}(t, \lambda x + (1 - \lambda)y) &= \hat{\mathbb{E}}[\varphi(\lambda x + (1 - \lambda)y + \sqrt{t}\hat{X})] \\ &= \hat{\mathbb{E}}[\varphi(\lambda(x + \sqrt{t}\hat{X}) + (1 - \lambda)(y + \sqrt{t}\hat{X}))] \\ &\leq \hat{\mathbb{E}}[\lambda\varphi(x + \sqrt{t}\hat{X}) + (1 - \lambda)\varphi(y + \sqrt{t}\hat{X})] \\ &\leq \lambda\hat{\mathbb{E}}[\varphi(x + \sqrt{t}\hat{X})] + (1 - \lambda)\hat{\mathbb{E}}[\varphi(y + \sqrt{t}\hat{X})] \\ &= \lambda\hat{u}(t, x) + (1 - \lambda)\hat{u}(t, y),\end{aligned}$$

for all $x, y \in \mathbb{R}$, hence $\hat{u}(t, x)$ is convex in $x \in \mathbb{R}$. $\square$

Lemma 6.2.2. *Suppose that $\hat{v}(t, x)$ is the solution of the following G-heat PDEs:*

$$\begin{cases} \frac{\partial \hat{v}}{\partial t}(t, x) - \frac{1}{2} \left(\bar{\sigma}^2 \left(\frac{\partial^2 \hat{v}}{\partial x^2}(t, x) \right)^+ - \underline{\sigma}^2 \left(\frac{\partial^2 \hat{v}}{\partial x^2}(t, x) \right)^- \right) = 0, \\ \hat{v}(0, x) = -\varphi(x). \end{cases} \quad (6.3)$$

If $\varphi(x)$ is non-decreasing (resp. convex), then $\hat{v}(t, x)$ is non-increasing (resp. concave) in $x \in \mathbb{R}$ for every fixed $t > 0$.

Proof. (i) By [80], the solution $\hat{v}(t, x)$ has the form $\hat{v}(t, x) = \hat{\mathbb{E}}[-\varphi(x + \sqrt{t}\hat{X})]$, where $\hat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$. Letting $x_1 \leq x_2$, and since $-\varphi(x)$ is non-increasing, we have

$$\hat{v}(t, x_1) = \hat{\mathbb{E}}[-\varphi(x_1 + \sqrt{t}\hat{X})] \geq \hat{\mathbb{E}}[-\varphi(x_2 + \sqrt{t}\hat{X})] = \hat{v}(t, x_2),$$

showing that $\hat{v}(t, x)$ is non-increasing in $x \in \mathbb{R}$.

(ii) Suppose that φ is convex. We take $\lambda \in [0, 1]$, and by the concavity of $-\varphi$ and $\hat{X}$ is G-normal distributed, we have

$$\hat{\mathbb{E}}[-\varphi(\lambda x + (1 - \lambda)y + \sqrt{t}\hat{X})] = \mathbb{E}[-\varphi(\lambda x + (1 - \lambda)y + \sqrt{t}N(0, \underline{\sigma}^2))].$$

Thus,

$$\begin{aligned}\hat{v}(t, \lambda x + (1 - \lambda)y) &= \hat{\mathbb{E}}[-\varphi(\lambda x + (1 - \lambda)y + \sqrt{t}\hat{X})] \\ &= \mathbb{E}[-\varphi(\lambda(x + \sqrt{t}N(0, \underline{\sigma}^2)) + (1 - \lambda)(y + \sqrt{t}N(0, \underline{\sigma}^2)))] \\ &\geq \mathbb{E}[-\lambda\varphi(x + \sqrt{t}N(0, \underline{\sigma}^2)) - (1 - \lambda)\varphi(y + \sqrt{t}N(0, \underline{\sigma}^2))]\end{aligned}$$

$$\begin{aligned}
&= \lambda \mathbb{E} \left[-\varphi(x + \sqrt{t}N(0, \underline{\sigma}^2)) \right] + (1 - \lambda) \mathbb{E} \left[-\varphi(y + \sqrt{t}N(0, \underline{\sigma}^2)) \right] \\
&= \lambda \widehat{\mathbb{E}} \left[-\varphi(x + \sqrt{t}\widehat{X}) \right] + (1 - \lambda) \widehat{\mathbb{E}} \left[-\varphi(y + \sqrt{t}\widehat{X}) \right] \\
&= \lambda \widehat{v}(t, x) + (1 - \lambda) \widehat{v}(t, y),
\end{aligned}$$

for all $x, y \in \mathbb{R}$, hence $\widehat{v}(t, x)$ is concave in $x \in \mathbb{R}$. $\square$

In the sequel, we define the following function

$$\widetilde{G}(a) := \frac{1}{2} \inf_{\underline{\sigma}^2 \leq \sigma^2 \leq \bar{\sigma}^2} (a\sigma^2), \quad a \in \mathbb{R},$$

and a nonlinear expectation $\widetilde{\mathbb{E}}[X] := -\widehat{\mathbb{E}}[-X]$, $X \in L_G^1(\Omega, \mathcal{F})$. It is clear that $\widetilde{G}(a) = \frac{1}{2} \widetilde{\mathbb{E}}[a\widehat{X}]$, where $\widehat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$. Thus, we can rewrite $\widetilde{G}(a)$ as

$$\widetilde{G}(a) := \frac{1}{2} (a^+ \underline{\sigma}^2 - a^- \bar{\sigma}^2), \quad a \in \mathbb{R}.$$

Following [97], $\widetilde{\mathbb{E}}$ is called a superlinear expectation, i.e., it satisfies the monotonicity, constant-preserving, positive homogeneity property of Definition 2.5.1, as well as the super-additivity property:

$$\widetilde{\mathbb{E}}[X + Y] \geq \widetilde{\mathbb{E}}[X] + \widetilde{\mathbb{E}}[Y], \quad X, Y \in L_G^1(\Omega, \mathcal{F}).$$

We also define $\widetilde{u}(t, x) := \widetilde{\mathbb{E}}[\varphi(x + \sqrt{t}\widehat{X})]$, then $\widetilde{u}(t, x)$ is the solution of the following $\widetilde{G}$ -heat PDEs, see e.g. [97]:

$$\frac{\partial \widetilde{u}}{\partial t}(t, x) - \widetilde{G}\left(\frac{\partial^2 \widetilde{u}}{\partial x^2}(t, x)\right) = 0, \quad \widetilde{u}(0, x) = \varphi(x). \quad (6.4)$$

We obtain the following lemmas as consequences.

Lemma 6.2.3. *Suppose that $\widetilde{u}(t, x)$ is the solution of the following $\widetilde{G}$ -heat PDEs:*

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(t, x) - \frac{1}{2} \left(\underline{\sigma}^2 \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}(t, x) \right)^+ - \bar{\sigma}^2 \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}(t, x) \right)^- \right) = 0, \\ \widetilde{u}(0, x) = \varphi(x). \end{cases} \quad (6.5)$$

If $\varphi(x)$ is non-decreasing (resp. convex), then $\widetilde{u}(t, x)$ is also non-decreasing (resp. convex) in $x \in \mathbb{R}$ for every fixed $t > 0$.

Proof. We have $\widetilde{u}(t, x) := \widetilde{\mathbb{E}}[\varphi(x + \sqrt{t}\widehat{X})] = -\widehat{\mathbb{E}}[-\varphi(x + \sqrt{t}\widehat{X})] = -\widehat{v}(t, x)$, where $\widehat{v}(t, x)$ is the solution of the G -heat PDE (6.3), which allow us to conclude the announced result from Lemma 6.2.2. $\square$

Lemma 6.2.4. *Suppose that $\tilde{v}(t, x)$ is the solution of the following $\tilde{G}$ -heat PDEs:*

$$\begin{cases} \frac{\partial \tilde{v}}{\partial t}(t, x) - \frac{1}{2} \left(\frac{\sigma^2}{\partial x^2} \left(\frac{\partial^2 \tilde{v}}{\partial x^2}(t, x) \right)^+ - \bar{\sigma}^2 \left(\frac{\partial^2 \tilde{v}}{\partial x^2}(t, x) \right)^- \right) = 0, \\ \tilde{v}(0, x) = -\varphi(x). \end{cases} \quad (6.6)$$

If $\varphi(x)$ is non-decreasing (resp. convex), then $\tilde{u}(t, x)$ is non-increasing (resp. concave) in $x \in \mathbb{R}$ for every fixed $t > 0$.

Proof. We have $\tilde{v}(t, x) := \tilde{\mathbb{E}}[-\varphi(x + \sqrt{t}\hat{X})] = -\hat{\mathbb{E}}[\varphi(x + \sqrt{t}\hat{X})] = -\hat{u}(t, x)$, where $\hat{u}(t, x)$ is the solution of the G -heat PDE (6.2), which allow us to conclude the announced property from Lemma 6.2.1. $\square$

6.3 Comparisons of g -Expectations with G -Normal Distribution

Let $(X_t)_{t \in [0,1]}$ be the solution of the following diffusion SDEs

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad 0 \leq t \leq T := 1, \quad (6.7)$$

where $\mu(t, x)$ and $\sigma(t, x)$ satisfy the Condition (A₁) in Chapter 4 and here we fix the terminal time $T := 1$. Let $(Y_t, Z_t)_{t \in [0,1]}$ be solution of the associated BSDEs

$$-dY_t = g(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \quad 0 \leq t \leq 1, \quad Y_1 = \varphi(X_1) \in L^2(\Omega, \mathcal{F}_T), \quad (6.8)$$

where $g(t, x, y, z)$ satisfy the Conditions (A.4.2)-(A.4.3) in Chapter 4, and $\varphi \in C_{l.Lip}(\mathbb{R})$. For a Markovian case, we know that $Y_t = u(t, X_t)$ and $Z_t = \sigma(t, X_t) \frac{\partial u}{\partial x}(t, X_t)$, where $u(t, x)$ solves the following PDE

$$\frac{\partial u}{\partial t}(t, x) + \mu(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 u}{\partial x^2}(t, x) + g\left(t, x, u(t, x), \sigma(t, x) \frac{\partial u}{\partial x}(t, x)\right) = 0, \quad (6.9)$$

with $u(1, x) = \varphi(x)$. Recall that $\mathcal{E}_g[\varphi(X_1)]$, the g -expectation of $\varphi(X_1)$ is defined by

$$\mathcal{E}_g[\varphi(X_1)] := Y_0 = u(0, X_0).$$

From the point of view of the nonlinear expectation theory, see Section 2.5, $\mathcal{E}_g[\varphi(X_1)]$ can also be regarded as the distribution function of X_1 under the g -expectation

framework with the test functions φ . Now we let $\hat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$ be a G -normal random variable on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then distribution function of $\hat{X}$ is defined by $\hat{\mathbb{E}}[\varphi(\hat{X})]$ which can be characterized by the G -heat PDE

$$\begin{cases} \frac{\partial \hat{u}}{\partial t}(t, x) - G\left(\frac{\partial^2 \hat{u}}{\partial x^2}(t, x)\right) = 0, \\ \hat{u}(0, x) = \varphi(x), \end{cases} \quad (6.10)$$

where $G(a) = \frac{1}{2} \hat{\mathbb{E}}[a\hat{X}] = \frac{1}{2} (\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, $a \in \mathbb{R}$, and its viscosity solution $\hat{u}(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}\hat{X})]$. In particular, we have $\hat{\mathbb{E}}[\varphi(\hat{X})] := \hat{u}(1, 0)$. Noting that since we take the function G is non-degenerate, i.e., in case of $\bar{\sigma}^2 > \underline{\sigma}^2 > 0$, so the viscosity solution becomes classical solution $\hat{u}(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$, see e.g. [80].

In this section, we derive sufficient conditions for comparing $\hat{\mathbb{E}}[\varphi(\hat{X})] \leq \mathcal{E}_g[\varphi(X_1)]$ in the sense that the distribution of X_1 under the g -expectation is *stronger* than the G -normal distribution of $\hat{X}$ under the sublinear expectation $\hat{\mathbb{E}}$, for all convex test functions $\varphi \in C_{l.Lip}(\mathbb{R})$. It is clear that this orderings will offer us lower bounds for the g -expectation $\mathcal{E}_g[\varphi(X_1)]$ when we take a certain convex function φ . Here we propose comparison results between the random variable X_1 and the G -normal random variable $\hat{X}$ based on a measurable function $\zeta(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}$ such that

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^1 \frac{(\mu(t, X_t) - \zeta(t, X_t))^2}{\sigma^2(t, X_t)} dt \right) \right] < \infty. \quad (6.11)$$

In the sequel, we will also use the following conditions:

$$(K_1) \quad z\mu(t, x) + g(t, x, y, z\sigma(t, x)) \geq z\zeta(t, x), \text{ for } x, y, z \in \mathbb{R}, \text{ and } t \in [0, 1].$$

$$(K'_1) \quad z\mu(t, x) + g(t, x, y, z\sigma(t, x)) \leq z\zeta(t, x), \text{ for } (x, y, z) \in \mathbb{R}^3, \text{ and } t \in [0, 1].$$

$$(K_2) \quad z\mu(t, x) + g^{(-1)}(t, x, y, z\sigma(t, x)) \leq z\zeta(t, x), \text{ for } (x, y, z) \in \mathbb{R}^3, \text{ and } t \in [0, 1], \\ \text{where } g^{(-1)}(t, x, y, z) = -g(t, x, -y, -z), \text{ } (x, y, z) \in \mathbb{R}^3.$$

$$(K'_2) \quad z\mu(t, x) + g^{(-1)}(t, x, y, z\sigma(t, x)) \geq z\zeta(t, x) \text{ for } (x, y, z) \in \mathbb{R}^3, \text{ and } t \in [0, 1].$$

$$(K_3) \quad x \mapsto g(t, x, y, z) \text{ is non-decreasing on } \mathbb{R}, \text{ for } (y, z) \in \mathbb{R}^2 \text{ and } t \in [0, 1].$$

Comparisons of upper means

Proposition 6.3.1 (Convex order). *Assume that $X_0 = 0$ and*

$$0 < \bar{\sigma}^2 \leq \sigma^2(t, x), \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Condition (K_1) . Then we have

$$\widehat{\mathbb{E}}[\varphi(\widehat{X})] \leq \mathcal{E}_g[\varphi(X_1)], \quad (6.12)$$

for all convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. (i) We start by assuming that the coefficients $\varphi(x)$, $\mu(t, \cdot)$, $\sigma(t, \cdot)$, and $g(t, \cdot, \cdot, \cdot)$ are $\mathcal{C}_b^3$ functions for all $t \in [0, T]$ as in Theorem 3.1.3. Setting

$$\theta(t, x) := \frac{\mu(t, x) - \zeta(t, x)}{\sigma(t, x)}, \quad x \in \mathbb{R}, \quad t \in [0, 1].$$

By the Girsanov theorem, the process

$$\widetilde{W}_t := W_t + \int_0^t \theta(s, X_s) ds, \quad t \in [0, 1],$$

is a standard Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp \left(- \int_0^1 \theta(s, X_s) dW_s - \frac{1}{2} \int_0^1 (\theta(s, X_s))^2 ds \right),$$

and the forward SDE can be rewritten as

$$dX_t = \zeta(t, X_t) dt + \sigma(t, X_t) d\widetilde{W}_t, \quad 0 \leq t \leq 1, \quad X_0 = 0,$$

with the associated BSDE

$$-dY_t = [g(t, X_t, Y_t, Z_t) + Z_t \theta(t, X_t)] dt - Z_t d\widetilde{W}_t, \quad 0 \leq t \leq 1, \quad Y_1 = \varphi(X_1),$$

We have $Y_t = u(t, X_t)$, where the functions $u(t, x)$ solves the PDE (6.9). By applying the Itô formula to $u(t, X_t)$ and using (6.9), we find

$$\begin{aligned} u(t, X_t) &= u(0, X_0) + \int_0^t \zeta(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds + \int_0^t \mu(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \\ &\quad - \int_0^t g(s, X_s, u(s, X_s), \sigma(s, X_s) \frac{\partial u}{\partial x}(s, X_s)) ds + \int_0^t \sigma(s, X_s) \frac{\partial u}{\partial x}(s, X_s) d\widetilde{W}_s. \end{aligned}$$

Taking expectation under $\mathbb{Q}$ at time $t = 1$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[\varphi(X_1)] &= u(0, X_0) + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \end{aligned}$$

$$- \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, u(s, X_s), \sigma(s, X_s) \frac{\partial u}{\partial x}(s, X_s)) ds \right], \quad (6.13)$$

where $u(0, X_0) = Y_0 = \mathcal{E}_g[\varphi(X_1)]$. On the other hand, given φ a convex function in $C_{l.Lip}(\mathbb{R})$, we consider the G -heat equation

$$\begin{cases} \frac{\partial \hat{u}}{\partial t}(t, x) - \frac{1}{2} \left(\bar{\sigma}^2 \left(\frac{\partial^2 \hat{u}}{\partial x^2}(t, x) \right)^+ - \underline{\sigma}^2 \left(\frac{\partial^2 \hat{u}}{\partial x^2}(t, x) \right)^- \right) = 0, \\ \hat{u}(0, x) = \varphi(x), \end{cases} \quad (6.14)$$

with its solution $\hat{u}(t, x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}\hat{X})]$, $0 \leq t \leq 1$. Since $\hat{u}(t, x)$ is convex in x by Lemma 6.2.1, the above G -heat equation (6.14) becomes

$$\begin{cases} \frac{\partial \hat{u}}{\partial \tau}(\tau, x) + \frac{\bar{\sigma}^2}{2} \left(\frac{\partial^2 \hat{u}}{\partial x^2}(\tau, x) \right)^+ = 0, \\ \hat{u}(1, x) = \varphi(x), \end{cases} \quad (6.15)$$

after replacing t by $\tau := 1 - t$. Applying the Itô formula to $\hat{u}(\tau, X_\tau)$, and using (6.15), we have

$$\begin{aligned} \hat{u}(\tau, X_\tau) &= \hat{u}(0, X_0) + \frac{1}{2} \int_0^\tau (\sigma^2(s, X_s) - \bar{\sigma}^2) \left(\frac{\partial^2 \hat{u}}{\partial x^2}(s, X_s) \right)^+ ds \\ &\quad + \int_0^\tau \sigma(s, X_s) \frac{\partial \hat{u}}{\partial x}(s, X_s) d\widetilde{W}_s, \end{aligned}$$

where $\hat{u}(0, X_0) = \hat{u}(0, 0) = \widehat{\mathbb{E}}[\varphi(\hat{X})]$ at time $\tau = 0$, i.e. at $t = 1$, which yields

$$\mathbb{E}_{\mathbb{Q}}[\varphi(X_1)] = \widehat{\mathbb{E}}[\varphi(\hat{X})] + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \bar{\sigma}^2) \left(\frac{\partial^2 \hat{u}}{\partial x^2}(s, X_s) \right)^+ ds \right], \quad (6.16)$$

after taking expectation under $\mathbb{Q}$ at $\tau = 1$. Hence, from (6.13) and (6.16) we obtain

$$\begin{aligned} \mathcal{E}_g[\varphi(X_1)] - \widehat{\mathbb{E}}[\varphi(\hat{X})] &= u(0, X_0) - \hat{u}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, u(s, X_s), \frac{\partial u}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \bar{\sigma}^2) \left(\frac{\partial^2 \hat{u}}{\partial x^2}(s, X_s) \right)^+ ds \right] \\
& \geq 0,
\end{aligned}$$

where we used the assumptions (K_1) and $\sigma^2(t, x) \geq \bar{\sigma}^2$.

(ii) The extension of (6.12) to coefficients satisfying (A.4.1)-(A.4.3) follows as in the proof of Theorem 4.4.7. $\square$

Remark 6.3.2. (i) Assuming conditions of Proposition 6.3.1 hold, then we can derive some bounds for uncertain mean, variance and other moments for the g -expectation of X_1 . In fact, we know from G -normal distribution that $\hat{\mathbb{E}}[\hat{X}] = \hat{\mathbb{E}}[-\hat{X}] = 0$ and $\hat{\mathbb{E}}[\hat{X}^2] = \bar{\sigma}^2$, $\hat{\mathbb{E}}[\hat{X}^4] = 3\bar{\sigma}^4$, see e.g. [80]. Hence by taking $\varphi = -x, x, x^2$, and x^4 , respectively, we have

$$-\mathcal{E}_g[-X_1] \leq 0, \quad \mathcal{E}_g[X_1] \geq 0, \quad \mathcal{E}_g[X_1^2] \geq \bar{\sigma}^2, \quad \mathcal{E}_g[X_1^4] \geq 3\bar{\sigma}^4.$$

(ii) In addition to conditions of Proposition 6.3.1, we assume that the generator $g(t, x, y, z)$ is sublinear in $(y, z) \in \mathbb{R}^2$. Then we have

$$\sup_{P \in \mathcal{P}} P(X_1 > \varepsilon) \geq \mathbb{P}(\bar{Z} > \varepsilon), \quad \varepsilon > 0, \quad (6.17)$$

or equivalently

$$\inf_{P \in \mathcal{P}} P(X_1 \leq \varepsilon) \leq \mathbb{P}(\bar{Z} \leq \varepsilon), \quad \varepsilon > 0, \quad (6.18)$$

where $\mathcal{P}$ is the set that represents $\mathcal{E}_g$ and $\bar{Z} \sim N(0, \bar{\sigma}^2)$.

In fact, we obtain (6.17) by taking $\varphi(x) = \mathbb{1}_{\{x > \varepsilon\}}$, for $x \in \mathbb{R}, \varepsilon > 0$ in the inequality (6.12) with noting that $\hat{\mathbb{E}}[\varphi(\hat{X})] = \mathbb{E}[\varphi(N(0, \bar{\sigma}^2))]$ in the case convex function φ , and together with $\mathcal{E}_g[\varphi(\hat{X})] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\varphi(X_1)]$ when $(y, z) \mapsto g(t, x, y, z)$ sublinear on $\mathbb{R}^2$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}$.

From the proposition above, we also get an additional version for the increasing convex order.

Proposition 6.3.3 (Increasing convex order). Assume that $X_0 \geq 0$ and

$$0 < \bar{\sigma}^2 \leq \sigma^2(t, x), \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Conditions (K_3) and (K_1'') :

(K_1'') $z\mu(t, x) + g(t, x, y, z\sigma(t, x)) \geq z\zeta(t, x)$, for $(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}_+$, and $t \in [0, 1]$.

Then we have

$$\widehat{\mathbb{E}}[\varphi(\widehat{X})] \leq \mathcal{E}_g[\varphi(X_1)],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. If φ is a non-decreasing convex, then $\widehat{u}(\tau, x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{1 - \tau}\widehat{X})]$, $\tau = 1 - t$, is a non-decreasing convex in x by Lemma 6.2.1. Then at time $\tau = 0$ or $t = 1$ we have that $\widehat{u}(0, X_0) = \widehat{\mathbb{E}}[\varphi(X_0 + \widehat{X})] \geq \widehat{\mathbb{E}}[\varphi(\widehat{X})]$, provided that $X_0 \geq 0$. In addition, by the Condition (K_3) and Proposition 3.2.2, the solution $u(t, x)$ of the PDE (6.9) satisfies $\frac{\partial u}{\partial x}(t, x) \geq 0$, so the Condition (K_1) only needs to hold for $z \geq 0$, for all $x, y \in \mathbb{R}, t \in [0, 1]$, i.e., it becomes the Condition (K_1''). By repeating arguments in the proof of Proposition 6.3.1, we obtain

$$\begin{aligned} \mathcal{E}_g[\varphi(X_1)] - \widehat{\mathbb{E}}[\varphi(\widehat{X})] &\geq u(0, X_0) - \widehat{u}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, u(s, X_s), \frac{\partial u}{\partial x}(s, X_s)\sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \bar{\sigma}^2) \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s) \right)^+ ds \right] \\ &\geq 0, \end{aligned}$$

Hence, we get the announced result. $\square$

Remark 6.3.4. In Proposition 6.3.3, if we require that the Condition (K_1'') holds for all $(x, y, z) \in \mathbb{R}^3$, i.e., it becomes (K_1), then we can remove the Condition (K_3). Specifically, assume that $X_0 \geq 0$ and $\sigma^2(t, x) \geq \bar{\sigma}^2$, for all $x \in \mathbb{R}, t \in [0, 1]$, together with the Condition (K_1), we still have

$$\widehat{\mathbb{E}}[\varphi(\widehat{X})] \leq \mathcal{E}_g[\varphi(X_1)],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

When the generator function $g(t, x, y, z) = 0$ then $\mathcal{E}_0 \equiv \mathbb{E}$ and hence we have the following corollary from Proposition 6.3.3.

Corollary 6.3.5 (Increasing convex order). *Assume that $X_0 \geq 0$, $\mu(t, x) \geq \zeta(t, x)$, and $\sigma^2(t, x) \geq \bar{\sigma}^2 > 0$, for all $x \in \mathbb{R}, t \in [0, 1]$. Then we have*

$$\widehat{\mathbb{E}}[\varphi(\widehat{X})] \leq \mathbb{E}[\varphi(X_1)],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. For the PDE (6.9) with $g \equiv 0$, the solution $u(t, x)$ satisfies $\frac{\partial u}{\partial x}(t, x) \geq 0$, provided that $x \mapsto \varphi(x)$ is non-decreasing on $\mathbb{R}$. Hence, we can take $z \geq 0$ and check with $g = 0$ and $\mu(t, x) \geq \zeta(t, x)$ that

$$(K_1'') \quad z\mu(t, x) + g(t, x, y, z\sigma(t, x)) = z\mu(t, x) \geq z\zeta(t, x), \quad z \in \mathbb{R}_+, x \in \mathbb{R} \text{ and } t \in [0, 1].$$

Therefore, Proposition 6.3.3 admits $\widehat{\mathbb{E}}[\varphi(\widehat{X})] \leq \mathbb{E}[\varphi(X_1)]$ as a consequence. $\square$

Comparisons of lower means

Now we turn to compare lower mean of G -normal random variables, i.e., comparisons between $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})]$ and $-\mathcal{E}_g[-\varphi(X_1)]$. Recall that $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})] = \widehat{\mathbb{E}}[\varphi(\widehat{X})]$ and $-\mathcal{E}_g[-\varphi(X_1)] = \mathcal{E}_{g^{(-1)}}[\varphi(X_1)]$, where $g^{(-1)}(t, x, y, z) = -g(t, x, -y, -z)$, for $(x, y, z) \in \mathbb{R}^3$ and $t \in \mathbb{R}_+$.

Proposition 6.3.6 (Convex order). *Assume that $X_0 = 0$ and*

$$0 < \sigma^2(t, x) \leq \underline{\sigma}^2, \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Condition (K_1) or (K_2) . Then we have

$$-\mathcal{E}_g[-\varphi(X_1)] \leq -\widehat{\mathbb{E}}[-\varphi(\widehat{X})], \quad (6.19)$$

for all convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. (i) Under the Condition (K_1) , we still use the Brownian motion $(\widetilde{W})_{t \in [0, 1]}$ under the measure $\mathbb{Q}$ defined as in the proof of Proposition 6.3.1. Denoting by $(\overline{Y}_t, \overline{Z}_t)_{t \in [0, 1]}$ the solution of following BSDE

$$-d\overline{Y}_t = (g(t, X_t, \overline{Y}_t, \overline{Z}_t) + \overline{Z}_t \theta(t, X_t)) dt - \overline{Z}_t d\widetilde{W}_t, \quad 0 \leq t \leq 1, \quad \overline{Y}_1 = -\varphi(X_1),$$

We know that $\overline{Y}_t = v(t, X_t)$, where the function $v(t, x)$ solves the PDE

$$\frac{\partial v}{\partial t}(t, x) + \mu(t, x) \frac{\partial v}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 v}{\partial x^2}(t, x) + g\left(t, x, v(t, x), \sigma(t, x) \frac{\partial v}{\partial x}(t, x)\right) = 0, \quad (6.20)$$

with $v(1, x) = -\varphi(x)$. In the same fashion to the proof of Proposition 6.3.1, applying the Itô formula to $v(t, X_t)$ and using (6.20) we have

$$\begin{aligned} v(t, X_t) &= v(0, X_0) - \int_0^t g\left(s, X_s, v(s, X_s), \sigma(s, X_s) \frac{\partial v}{\partial x}(s, X_s)\right) ds \\ &\quad - \int_0^t \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds + \int_0^t \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \\ &\quad + \int_0^t \sigma(s, X_s) \frac{\partial u}{\partial x}(s, X_s) d\widetilde{W}_s. \end{aligned}$$

Taking expectation under $\mathbb{Q}$ at time $t = 1$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[-\varphi(X_1)] &= v(0, X_0) - \mathbb{E}_{\mathbb{Q}}\left[\int_0^1 g\left(s, X_s, v(s, X_s), \sigma(s, X_s) \frac{\partial v}{\partial x}(s, X_s)\right) ds\right] \\ &\quad - \mathbb{E}_{\mathbb{Q}}\left[\int_0^1 \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds\right] + \mathbb{E}_{\mathbb{Q}}\left[\int_0^1 \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds\right], \end{aligned} \quad (6.21)$$

where $v(0, X_0) = \bar{Y}_0 = \mathcal{E}_g[-\varphi(X_1)]$. On the other hand, given φ a convex function in $C_{l.Lip}(\mathbb{R})$, we consider the G -heat equation

$$\begin{cases} \frac{\partial \widehat{v}}{\partial t}(t, x) - \frac{1}{2} \left(\bar{\sigma}^2 \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(t, x) \right)^+ - \underline{\sigma}^2 \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(t, x) \right)^- \right) = 0, \\ \widehat{v}(0, x) = -\varphi(x), \end{cases} \quad (6.22)$$

with its solution $\widehat{v}(t, x) := \widehat{\mathbb{E}}[-\varphi(x + \sqrt{t}\widehat{X})]$, $0 \leq t \leq 1$. Since $\widehat{v}(t, x)$ is concave in x by Lemma 6.2.2, the G -heat equation (6.22) becomes

$$\begin{cases} \frac{\partial \widehat{v}}{\partial \tau}(\tau, x) - \frac{\bar{\sigma}^2}{2} \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(\tau, x) \right)^- = 0, \\ \widehat{v}(1, x) = -\varphi(x), \end{cases} \quad (6.23)$$

after replacing t by $\tau := 1 - t$. Applying the Itô formula to $\widehat{v}(\tau, X_\tau)$, and using (6.23), we have

$$\begin{aligned} \widehat{v}(\tau, X_\tau) &= \widehat{v}(0, X_0) + \frac{1}{2} \int_0^\tau (\underline{\sigma}^2 - \sigma^2(s, X_s)) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s) \right)^- ds \\ &\quad + \int_0^\tau \sigma(s, X_s) \frac{\partial \widehat{v}}{\partial x}(s, X_s) d\widetilde{W}_s, \end{aligned}$$

where $\widehat{v}(0, X_0) = \widehat{v}(0, 0) = \widehat{\mathbb{E}}[-\varphi(\widehat{X})]$ at time $\tau = 0$, i.e. at $t = 1$, which yields

$$\mathbb{E}_{\mathbb{Q}}[-\varphi(X_1)] = \widehat{\mathbb{E}}[-\varphi(\widehat{X})] + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\underline{\sigma}^2 - \sigma^2(s, X_s)) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s) \right)^- ds \right], \quad (6.24)$$

after taking expectation under $\mathbb{Q}$ at $\tau = 1$. Hence, from (6.21) and (6.24) we obtain

$$\begin{aligned} \mathcal{E}_g[-\varphi(X_1)] - \widehat{\mathbb{E}}[-\varphi(\widehat{X})] &= v(0, X_0) - \widehat{v}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, v(s, X_s), \frac{\partial v}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\underline{\sigma}^2 - \sigma^2(s, X_s)) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s) \right)^- ds \right] \\ &\geq 0, \end{aligned}$$

where we used the assumptions (K_1) and $\sigma^2(t, x) \leq \underline{\sigma}^2$.

(ii) Under the Condition (K_2) , the proof can be done in a similar way, where we apply the Itô formula to $\bar{u}(\tau, X_\tau)$ and then to $\tilde{u}(\tau, X_\tau)$, here $\tilde{u}(t, x)$ is the solution of the $\tilde{G}$ -heat PDE (6.5) and $\bar{u}(t, x)$ solves the PDEs:

$$\frac{\partial \bar{u}}{\partial t}(t, x) + \mu(t, x) \frac{\partial \bar{u}}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 \bar{u}}{\partial x^2}(t, x) + g^{(-1)}\left(t, x, \bar{u}(t, x), \sigma(t, x) \frac{\partial \bar{u}}{\partial x}(t, x)\right) = 0, \quad (6.25)$$

with $\bar{u}(1, x) = \varphi(x)$. We will find that

$$\begin{aligned} \mathcal{E}_{g^{(-1)}}[\varphi(X_1)] - \widehat{\mathbb{E}}[\varphi(\widehat{X})] &= \bar{u}(0, X_0) - \tilde{u}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g^{(-1)}(s, X_s, \bar{u}(s, X_s), \frac{\partial \bar{u}}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial \bar{u}}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial \bar{u}}{\partial x}(s, X_s) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \underline{\sigma}^2) \left(\frac{\partial^2 \tilde{u}}{\partial x^2}(s, X_s) \right)^+ ds \right] \end{aligned}$$

$$\leq 0,$$

where we utilized the assumptions (K_2) and $\sigma^2(t, x) \leq \underline{\sigma}^2$. $\square$

Remark 6.3.7. (i) As before, we can derive some bounds for upper mean, variance and other moments for the g -expectation of X_1 by assuming conditions of Proposition 6.3.6 hold. In fact, we know from G -normal distribution that $\widehat{\mathbb{E}}[\widehat{X}] = \widehat{\mathbb{E}}[-\widehat{X}] = 0$ and $-\widehat{\mathbb{E}}[-\widehat{X}^2] = \underline{\sigma}^2$, $-\widehat{\mathbb{E}}[-\widehat{X}^4] = 3\underline{\sigma}^4$, see e.g. [80]. Hence by taking $\varphi = -x$, x , x^2 , and x^4 , respectively, we find

$$\mathcal{E}_g[X_1] \geq 0, \quad -\mathcal{E}_g[-X_1] \leq 0, \quad -\mathcal{E}_g[-X_1^2] \leq \underline{\sigma}^2, \quad -\mathcal{E}_g[-X_1^4] \leq 3\underline{\sigma}^4.$$

(ii) In addition to conditions of Proposition 6.3.6, we assume that the generator $g(t, x, y, z)$ is sublinear in $(y, z) \in \mathbb{R}^2$. Then we have

$$\inf_{P \in \mathcal{P}} P(X_1 > \varepsilon) \leq \mathbb{P}(\underline{Z} > \varepsilon) \leq \frac{2\sigma}{\varepsilon} \exp\left(-\frac{\varepsilon}{2\underline{\sigma}^2}\right), \quad \varepsilon > 0, \quad (6.26)$$

or equivalently

$$\sup_{P \in \mathcal{P}} P(X_1 \leq \varepsilon) \geq \mathbb{P}(\underline{Z} \leq \varepsilon) \geq 1 - \frac{2\sigma}{\varepsilon} \exp\left(-\frac{\varepsilon}{2\underline{\sigma}^2}\right), \quad \varepsilon > 0, \quad (6.27)$$

where $\mathcal{P}$ is the set that represents $\mathcal{E}_g$ and $\underline{Z} \sim N(0, \underline{\sigma}^2)$.

Indeed, we get the bound (6.26) by taking $\varphi(x) = \mathbb{1}_{\{x > \varepsilon\}}$, for $x \in \mathbb{R}$, $\varepsilon > 0$ in the inequality (6.19) with noting that $-\widehat{\mathbb{E}}[-\varphi(\widehat{X})] = \mathbb{E}\left[\varphi(N(0, \underline{\sigma}^2))\right]$ in the case convex function φ , and together with $-\mathcal{E}_g[-\varphi(\widehat{X})] = \inf_{P \in \mathcal{P}} \mathbb{E}_P[\varphi(X_1)]$ when $(y, z) \mapsto g(t, x, y, z)$ is sublinear on $\mathbb{R}^2$, for $t \in \mathbb{R}_+$, $x \in \mathbb{R}$.

Proposition 6.3.6 admits a version of the increasing convex order for comparing lower means in which we change assumption of $X_0 = 0$ to be $X_0 \leq 0$ and still keep the Conditions (K_1) or (K_2) .

Proposition 6.3.8 (Increasing convex order). Assume that $X_0 \leq 0$ and

$$0 < \sigma^2(t, x) \leq \underline{\sigma}^2, \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Condition (K_1) or (K_2) . Then we have

$$-\mathcal{E}_g[-\varphi(X_1)] \leq -\widehat{\mathbb{E}}[\varphi(-\widehat{X})],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. (i) We first prove under the Condition (K_1) . If φ is a non-decreasing convex, then $\widehat{v}(\tau, x) := \widehat{\mathbb{E}} \left[-\varphi(x + \sqrt{1-\tau}\widehat{X}) \right]$ $\tau : 1-t$, is a non-increasing concave in x by Lemma 6.2.2. Then at time $\tau = 0$ or $t = 1$, we have $\widehat{v}(0, X_0) = \widehat{\mathbb{E}} \left[-\varphi(X_0 + \widehat{X}) \right] \geq \widehat{\mathbb{E}} \left[-\varphi(\widehat{X}) \right]$, provided that $X_0 \leq 0$. By repeating arguments in the proof of Proposition 6.3.6 and by (K_1) , together with $\sigma^2(t, x) \leq \underline{\sigma}^2$, we find that

$$\begin{aligned} \mathcal{E}_g \left[-\varphi(X_1) \right] - \widehat{\mathbb{E}} \left[-\varphi(\widehat{X}) \right] &\geq v(0, X_0) - \widehat{v}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, v(s, X_s), \frac{\partial v}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\underline{\sigma}^2 - \sigma^2(s, X_s)) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s) \right)^- ds \right] \\ &\geq 0. \end{aligned}$$

(ii) We now prove the proposition under the Condition (K_2) . Given φ a non-decreasing convex, $\widetilde{u}(\tau, x) := \widetilde{\mathbb{E}} \left[\varphi(x + \sqrt{1-\tau}\widehat{X}) \right]$, with $\tau := 1-t$, is a non-decreasing convex in x by Lemma 6.2.3. Then at time $\tau = 0$ or $t = 1$, we have $\widetilde{u}(0, X_0) = \widetilde{\mathbb{E}} \left[\varphi(X_0 + \widehat{X}) \right] \leq \widetilde{\mathbb{E}} \left[\varphi(\widehat{X}) \right]$, provided that $X_0 \leq 0$. By repeating arguments in the proof of Proposition 6.3.6 with the use of PDEs (6.25), (6.5) and Lemma 6.2.3, we find that

$$\begin{aligned} \mathcal{E}_{g^{(-1)}} \left[\varphi(X_1) \right] - \widetilde{\mathbb{E}} \left[\varphi(\widehat{X}) \right] &\leq \bar{u}(0, X_0) - \widetilde{u}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g^{(-1)}(s, X_s, v(s, X_s), \frac{\partial v}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \underline{\sigma}^2) \left(\frac{\partial^2 \widetilde{u}}{\partial x^2}(s, X_s) \right)^+ ds \right] \\ &\leq 0, \end{aligned}$$

which yields by the Condition (K_2) and $\sigma^2(t, x) \leq \underline{\sigma}^2$. Hence, we conclude the announced result. $\square$

Corollary 6.3.9 (Increasing convex order). *Assume that $X_0 \leq 0$, $\mu(t, x) \leq \zeta(t, x)$, and $0 < \sigma^2(t, x) \leq \underline{\sigma}^2$, for $x \in \mathbb{R}$, and $t \in [0, 1]$. Then we have*

$$\mathbb{E} [\varphi(X_1)] \leq -\widehat{\mathbb{E}} [-\varphi(\widehat{X})],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. For the PDE (6.20) with $g \equiv 0$, one can prove by using similar arguments in the proof of Proposition 3.2.2 that the solution $v(t, x)$ satisfies $\frac{\partial v}{\partial x}(t, x) \leq 0$, provided that $x \mapsto \varphi(x)$ is non-decreasing on $\mathbb{R}$. Hence, the Condition (K_1) only needs to hold for $z \leq 0$ and we check with $g = 0$ and $\mu(t, x) \leq \zeta(t, x)$ that

$$(K_1) \quad z\mu(t, x) + g(t, x, y, z\sigma(t, x)) = z\mu(t, x) \geq z\zeta(t, x), \quad z \in \mathbb{R}_-, x \in \mathbb{R}, \text{ and } t \in [0, 1].$$

Therefore, Proposition 6.3.8 admits $\mathbb{E} [\varphi(X_1)] = -\mathbb{E} [-\varphi(X_1)] \leq -\widehat{\mathbb{E}} [-\varphi(\widehat{X})]$ as a consequence. $\square$

As can be seen in previous propositions, we treat volatility in case of $\sigma^2(t, x) \geq \bar{\sigma}^2$ or $\sigma^2(t, x) \leq \underline{\sigma}^2$. For the completeness, we also derive stochastic orderings for the case the volatility is bounded on the interval $[\underline{\sigma}, \bar{\sigma}]$, i.e., $\underline{\sigma}^2 \leq \sigma^2(t, x) \leq \bar{\sigma}^2$.

Comparisons of both lower means and upper means

Proposition 6.3.10 (Convex order). *Assume that $X_0 = 0$ and*

$$0 < \underline{\sigma}^2 \leq \sigma^2(t, x) \leq \bar{\sigma}^2, \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Condition (K'_1). Then we have

$$-\widehat{\mathbb{E}} [\varphi(-\widehat{X})] \leq -\mathcal{E}_g [-\varphi(X_1)] \quad \text{and} \quad \mathcal{E}_g [\varphi(X_1)] \leq \widehat{\mathbb{E}} [\varphi(\widehat{X})],$$

for all convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. (i) Repeating arguments in the proof of Proposition 6.3.1, we have

$$\begin{aligned} \mathcal{E}_g [\varphi(X_1)] - \widehat{\mathbb{E}} [\varphi(\widehat{X})] &= u(0, X_0) - \widehat{u}(0, X_0) \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, u(s, X_s), \frac{\partial u}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \end{aligned}$$

$$\begin{aligned}
& -\mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\
& + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \bar{\sigma}^2) \left(\frac{\partial^2 \hat{u}}{\partial x^2}(s, X_s) \right)^+ ds \right] \\
& \leq 0,
\end{aligned}$$

where we used the assumptions (K'_1) and $\sigma^2(t, x) \leq \bar{\sigma}^2$.

(ii) Similarly, by repeating arguments in the proof of Proposition 6.3.3, we find

$$\begin{aligned}
\mathcal{E}_g[-\varphi(X_1)] - \widehat{\mathbb{E}}[-\varphi(\widehat{X})] &= v(0, X_0) - \widehat{v}(0, X_0) \\
&= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, v(s, X_s), \frac{\partial v}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\
&+ \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\
&- \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\
&+ \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\underline{\sigma}^2 - \sigma^2(s, X_s)) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s) \right)^- ds \right] \\
&\leq 0,
\end{aligned}$$

by the assumptions (K'_1) and $\sigma^2(t, x) \geq \underline{\sigma}^2$. □

Remark 6.3.11. Assuming the conditions of the Proposition 6.3.10 hold, and taking $\varphi = -x$, x , x^2 , and x^4 , respectively, we have

$$-\mathcal{E}_g[-X_1] = \mathcal{E}_g[X_1] = 0, \quad \mathcal{E}_g[X_1^2] \leq \bar{\sigma}^2, \quad \text{and} \quad \mathcal{E}_g[X_1^4] \leq 3\bar{\sigma}^4.$$

For the increasing convex order of upper means, one can replace the assumption $X_0 = 0$ in the Proposition 6.3.10 to be $X_0 \leq 0$ as in the following proposition.

Proposition 6.3.12 (Increasing convex order). Assume that $X_0 \leq 0$ and

$$0 < \sigma^2(t, x) \leq \bar{\sigma}^2, \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Condition (K'_1) . Then we have

$$\mathcal{E}_g[\varphi(X_1)] \leq \widehat{\mathbb{E}}[\varphi(\widehat{X})],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. By following arguments in the proof of Proposition 6.3.10 with noting that given non-decreasing φ and $X_0 \leq 0$, we have

$$\begin{aligned}
\mathcal{E}_g[\varphi(X_1)] - \widehat{\mathbb{E}}[\varphi(\widehat{X})] &\leq u(0, X_0) - \widehat{u}(0, X_0) \\
&= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g \left(s, X_s, u(s, X_s), \frac{\partial u}{\partial x}(s, X_s) \sigma(s, X_s) \right) ds \right] \\
&\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\
&\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial u}{\partial x}(s, X_s) ds \right] \\
&\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\sigma^2(s, X_s) - \bar{\sigma}^2) \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s) \right)^+ ds \right] \\
&\leq 0,
\end{aligned}$$

by the assumptions (K'_1) and $\sigma^2(t, x) \leq \bar{\sigma}^2$. □

Proposition 6.3.13 (Increasing convex order). *Assume that $X_0 \geq 0$ and*

$$0 < \bar{\sigma}^2 \leq \sigma^2(t, x), \quad x \in \mathbb{R}, \quad t \in [0, 1],$$

together with the Condition (K'_1) . Then we have

$$-\mathcal{E}_g[-\varphi(X_1)] \geq -\widehat{\mathbb{E}}[\varphi(-\widehat{X})],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$.

Proof. By following arguments in the proof of Proposition 6.3.10 with noting that given non-decreasing φ and $X_0 \geq 0$, we have

$$\begin{aligned}
\mathcal{E}_g[-\varphi(X_1)] - \widehat{\mathbb{E}}[-\varphi(\widehat{X})] &\leq v(0, X_0) - \widehat{v}(0, X_0) \\
&= \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 g(s, X_s, v(s, X_s), \frac{\partial v}{\partial x}(s, X_s) \sigma(s, X_s)) ds \right] \\
&\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \mu(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\
&\quad - \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 \zeta(s, X_s) \frac{\partial v}{\partial x}(s, X_s) ds \right] \\
&\quad + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left[\int_0^1 (\underline{\sigma}^2 - \sigma^2(s, X_s)) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s) \right)^- ds \right] \\
&\leq 0,
\end{aligned}$$

which yields by the assumptions (K'_1) and $\sigma^2(t, x) \geq \underline{\sigma}^2$. □

6.4 Comparisons of G -Expectations with G -Normal Distribution

In this section, we study comparison results under G -expectation for the G -FSDEs of the form

$$dX_t^G = b(t, X_t^G)dt + h(t, X_t^G)d\langle B \rangle_t + \sigma(t, X_t^G)dB_s, \quad 0 \leq t \leq T := 1, \quad (6.28)$$

where $b(t, x)$, $h(t, x)$ and $\sigma(t, x)$ satisfy the usual Lipschitz condition (A.5.1), $(B_t)_{t \geq 0}$ and $(\langle B \rangle_t)_{t \geq 0}$ are G -Brownian motion and quadratic variance process of $(B_t)_{t \geq 0}$, respectively, defined on the G -expectation space $(\Omega, L_G^1(\Omega, \mathcal{F}), \mathbb{E}_G)$. In this case, we are to compare between $\mathbb{E}_G[\varphi(X_1^G)]$ and $\mathbb{E}_G[\varphi(\hat{X})]$, where $\hat{X}$ is G -normal distributed under $\mathbb{E}_G$, still denoted by $\hat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$. Recall that the G -expectation $\mathbb{E}_G$ is constructed based on the sublinear expectation $\hat{\mathbb{E}}$, see Section 2.5 in Chapter 2. In addition, $\mathbb{E}_G$ has a representation $\mathbb{E}_G[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X]$ for all $X \in L_G^1(\Omega, \mathcal{F})$, where $\mathcal{P}$ is a family of weakly compact probability measure on $(\Omega, \mathcal{F})$, see Section 6.1 above. First, we have a result for the increasing convex order stated in the following proposition.

Proposition 6.4.1 (Increasing convex order). *Let $(X_t^G)_{t \in [0,1]}$ be solution of the G -Forward SDEs (6.28) and let $\hat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$ be a G -normal random variable on $(\Omega, L_G^1(\Omega, \mathcal{F}), \mathbb{E}_G)$. Assuming that the following conditions hold*

- (i) $X_0^G \geq 0$.
- (ii) $b(t, x) \geq 0$, $h(t, x) \geq 0$, for $x \in \mathbb{R}$, and $t \in [0, 1]$.
- (iii) $\underline{\sigma}^2 \sigma^2(t, x) \geq \bar{\sigma}^2 > 0$, for $x \in \mathbb{R}$, and $t \in [0, 1]$.

Then we have

$$\mathbb{E}_G[\varphi(X_1^G)] \geq \mathbb{E}_P[\varphi(X_1^G)] \geq \mathbb{E}_G[\varphi(\hat{X})],$$

and

$$\mathbb{E}_P[\varphi(X_1^G)] \geq -\mathbb{E}_G[-\varphi(X_1^G)] \geq -\mathbb{E}_G[-\varphi(\hat{X})],$$

for all non-decreasing convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$, and $P \in \mathcal{P}$.

Proof. Since $\varphi(x)$ is nondecreasing convex, $\hat{X}$ is characterized via the G -heat equation

$$\begin{cases} \frac{\partial \hat{u}}{\partial \tau}(\tau, x) + \frac{\bar{\sigma}^2}{2} \left(\frac{\partial^2 \hat{u}}{\partial x^2}(\tau, x) \right)^+ = 0, \\ \hat{u}(1, x) = \varphi(x). \end{cases} \quad (6.29)$$

where $\tau := 1 - t$. Applying the G -Itô formula to $\widehat{u}(\tau, X_\tau^G)$, which is q.s. nondecreasing convex in $X_\tau^G, \tau \in [0, 1]$ and using (6.29), we have

$$\begin{aligned} \widehat{u}(\tau, X_\tau^G) - \widehat{u}(0, X_0^G) &= \int_0^\tau \left(b(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) - \frac{\overline{\sigma}^2}{2} \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s^G) \right)^+ \right) ds \\ &\quad + \int_0^\tau \left(h(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) + \frac{1}{2} \sigma^2(s, X_s^G) \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s^G) \right)^+ \right) d\langle B \rangle_s \\ &\quad + \int_0^\tau \sigma(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) dB_s. \end{aligned}$$

By Conditions (ii)-(iii) and $\frac{\partial \widehat{u}}{\partial x}(t, x) \geq 0$, for all $x \in \mathbb{R}, t \in [0, 1]$, we derive that

$$\begin{aligned} \widehat{u}(\tau, X_\tau^G) - \widehat{u}(0, X_0^G) &\geq \frac{-\overline{\sigma}^2}{2} \int_0^\tau \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s^G) \right)^+ ds + \frac{\overline{\sigma}^2}{2\underline{\sigma}^2} \int_0^\tau \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s^G) \right)^+ d\langle B \rangle_s \\ &\quad + \int_0^\tau \sigma(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) dB_s. \end{aligned}$$

By the bounded property (2.31), i.e., $\int_t^T \eta_s d\langle B \rangle_s \geq \underline{\sigma}^2 \int_t^T \eta_s ds$ if $\eta_s \in M_G^1(0, T)$ with $\eta_s \geq 0$, q.s. (quasi-surely), we obtain

$$\widehat{u}(\tau, X_\tau^G) - \widehat{u}(0, X_0^G) \geq \int_0^\tau \sigma(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) dB_s.$$

Taking expectation under $\mathbb{E}_P, P \in \mathcal{P}$ at $\tau = 1$ and noting that $\widehat{u}(1, X_1^G) = \varphi(X_1^G)$ and at the time $\tau = 0$, i.e., $t = 1$, $\widehat{u}(0, X_0^G) = \mathbb{E}_G[\varphi(X_0^G + \widehat{X})] \geq \mathbb{E}_G[\varphi(\widehat{X})]$ by Condition (i) and because φ nondecreasing. Then, we get

$$\mathbb{E}_P[\varphi(X_1^G)] - \mathbb{E}_G[\varphi(\widehat{X})] \geq \mathbb{E}_P[\widehat{u}(1, X_1^G)] - \widehat{u}(0, X_0^G) \geq 0.$$

Thus, we obtain

$$\mathbb{E}_G[\varphi(X_1^G)] - \mathbb{E}_G[\varphi(\widehat{X})] \geq \mathbb{E}_P[\varphi(X_1^G)] - \mathbb{E}_G[\varphi(\widehat{X})] \geq 0. \quad (6.30)$$

Similarly, we consider the G -heat equation

$$\begin{cases} \frac{\partial \widehat{v}(\tau, x)}{\partial \tau} - \frac{1}{2} \underline{\sigma}^2 \left(\frac{\partial^2 \widehat{v}(\tau, x)}{\partial x^2} \right)^- = 0, \\ \widehat{v}(1, x) = -\varphi(x). \end{cases} \quad (6.31)$$

Applying the G -Itô formula to $\widehat{v}(\tau, X_\tau^G)$, which is q.s. non-increasing concave in $X_\tau^G, \tau \in [0, 1]$ and by using (6.31) with Conditions (ii)-(iii), we have

$$\widehat{v}(\tau, X_\tau^G) - \widehat{v}(0, X_0^G) = \int_0^\tau \left(b(s, X_s^G) \frac{\partial \widehat{v}}{\partial x}(s, X_s^G) + \frac{\sigma^2}{2} \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s^G) \right)^- \right) ds$$

$$\begin{aligned}
& + \int_0^\tau \left(h(s, X_s^G) \frac{\partial \widehat{v}}{\partial x}(s, X_s^G) - \frac{1}{2} \sigma^2(X_s^G) \left(\frac{\partial^2 \widehat{v}}{\partial x^2}(s, X_s^G) \right)^- \right) d\langle B \rangle_s \\
& + \int_0^\tau \sigma(s, X_s^G) \frac{\partial \widehat{v}}{\partial x}(s, X_s^G) dB_s \\
& \leq -\frac{\bar{\sigma}^2}{2} \int_0^\tau \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s^G) \right)^- ds - \frac{1}{2} \frac{\bar{\sigma}^2}{\underline{\sigma}^2} \int_0^\tau \left(\frac{\partial^2 \widehat{u}}{\partial x^2}(s, X_s^G) \right)^- d\langle B \rangle_s \\
& \quad + \int_0^\tau \sigma(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) dB_s \\
& \leq \int_0^\tau \sigma(s, X_s^G) \frac{\partial \widehat{u}}{\partial x}(s, X_s^G) dB_s.
\end{aligned}$$

Taking expectation under $\mathbb{E}_P, P \in \mathcal{P}$, at the time $\tau = 1$, and noting that $\widehat{v}(1, X_1^G) = -\varphi(X_1^G)$ and at the time $\tau = 0$, i.e., $t = 1$, we have $\widehat{v}(0, X_0^G) = \mathbb{E}_G[-\varphi(X_0^G + \widehat{X})] \leq \mathbb{E}_G[-\varphi(\widehat{X})]$ by Condition (i) and because $-\varphi$ non-increasing. Then, we find

$$\mathbb{E}_P[-\varphi(X_1^G)] - \mathbb{E}_G[-\varphi(\widehat{X})] \leq \mathbb{E}_P(-\varphi(X_1^G)) - \widehat{v}(0, X_0^G) \leq 0.$$

It implies that $\mathbb{E}_P[-\varphi(X_1^G)] \leq \mathbb{E}_G[-\varphi(\widehat{X})]$. Then, by taking the supremum over $\mathcal{P}$, we obtain $\sup_{P \in \mathcal{P}} \mathbb{E}_P[-\varphi(X_1^G)] \leq \mathbb{E}_G[-\varphi(\widehat{X})]$. Therefore, we reach

$$-\mathbb{E}_P[-\varphi(X_1^G)] \geq -\sup_{P \in \mathcal{P}} \mathbb{E}_P[-\varphi(X_1^G)] \geq -\mathbb{E}_G[-\varphi(\widehat{X})], \quad (6.32)$$

where we note that $-\sup_{P \in \mathcal{P}} \mathbb{E}_P[-\varphi(X_1^G)] = -\mathbb{E}_G[-\varphi(X_1^G)]$. $\square$

The next result is a convex ordering version of Proposition 6.4.1 when the initial value and the drift coefficients of $(X_t^G)_{t \in [0,1]}$ are assumed to be equal to zero.

Proposition 6.4.2 (Convex order). *Let X_t^G be solution of the G -Forward SDEs (6.28) and let $\widehat{X} \sim \mathcal{N}(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$ be a G -normal random variable on the G -expectation space $(\Omega, L_G^1(\Omega, \mathcal{F}), \mathbb{E}_G)$. Assuming that the following conditions hold*

$$(i) \ X_0^G = 0.$$

$$(ii) \ b(t, x) = 0, \text{ and } h(t, x) = 0, \text{ for } x \in \mathbb{R}, \text{ and } t \in [0, 1].$$

$$(iii) \ \underline{\sigma}^2 \sigma^2(t, x) \geq \bar{\sigma}^2 > 0, \text{ for } x \in \mathbb{R}, t \in [0, 1].$$

Then we have

$$\mathbb{E}_G[\varphi(X_1^G)] \geq \mathbb{E}_P[\varphi(X_1^G)] \geq \mathbb{E}_G[\varphi(\widehat{X})], \quad (6.33)$$

and

$$\mathbb{E}_P[\varphi(X_1^G)] \geq -\mathbb{E}_G[-\varphi(X_1^G)] \geq -\mathbb{E}_G[-\varphi(\hat{X})], \quad (6.34)$$

for all convex functions $\varphi \in C_{l.Lip}(\mathbb{R})$ and $P \in \mathcal{P}$.

Proof. If $X_0^G = 0$ and $b(t, x) = h(t, x) = 0$ then the non-decreasing property of $\varphi(x)$ and $\hat{u}(t, x)$ are no longer required. Then, by repeating arguments of the proof of Proposition 6.4.1, we find that

$$\mathbb{E}_G[\varphi(X_1^G)] - \mathbb{E}_G[\varphi(\hat{X})] \geq \mathbb{E}_P[\varphi(X_1^G)] - \mathbb{E}_G[\varphi(\hat{X})] = \mathbb{E}_P[\hat{u}(1, X_1^G)] - \hat{u}(0, X_0^G) \geq 0,$$

and

$$\mathbb{E}_P[-\varphi(X_1^G)] - \mathbb{E}_G[-\varphi(\hat{X})] = \mathbb{E}_P(-\varphi(X_1^G)) - \hat{v}(0, X_0^G) \leq 0.$$

Hence, we conclude the convex order result. $\square$

Remark 6.4.3. (i) Under the Conditions in the Proposition 6.4.2, we can derive lower bounds for $\mathbb{E}_G[\varphi(X_1^G)]$ as in the Remark 6.3.2:

$$-\mathbb{E}_G[-X_1^G] \leq 0, \quad \mathbb{E}_G[X_1^G] \geq 0, \quad \mathbb{E}_G[(X_1^G)^2] \geq \bar{\sigma}^2, \quad \mathbb{E}_G[(X_1^G)^4] \geq 3\bar{\sigma}^4.$$

(ii) We note that $\hat{\mathbb{E}}[\varphi(\hat{X})] = \mathbb{E}[\varphi(\bar{Z})]$, and $-\hat{\mathbb{E}}[-\varphi(\hat{X})] = \mathbb{E}[\varphi(\underline{Z})]$, in the case convex function φ , where $\bar{Z} \sim N(0, \bar{\sigma}^2)$ and $\underline{Z} \sim N(0, \underline{\sigma}^2)$, see e.g. [80]. Also, we have

$$\mathbb{E}_G[\varphi(X_1^G)] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\varphi(X_1^G)], \quad \text{and} \quad -\mathbb{E}_G[-\varphi(X_1^G)] = \inf_{P \in \mathcal{P}} \mathbb{E}_P[\varphi(X_1^G)],$$

where $\mathcal{P}$ is the set that represents $\mathbb{E}_G$. Hence, under the Conditions in the Proposition 6.4.2 (or Proposition 6.4.1), we get

$$\sup_{P \in \mathcal{P}} P(X_1^G > \varepsilon) \geq \mathbb{P}(\bar{Z} > \varepsilon), \quad \text{and} \quad \inf_{P \in \mathcal{P}} P(X_1^G > x) \geq \mathbb{P}(\underline{Z} > \varepsilon), \quad \varepsilon > 0,$$

by applying (6.33) and (6.34) with the function $\varphi(x) = \mathbb{1}_{\{x > \varepsilon\}}$, for $x \in \mathbb{R}$, $\varepsilon > 0$.

6.5 Comparisons of Nonlinear Expectations $\mathcal{E}_g$ and $\mathcal{E}_G$

In this section, we derive sufficient conditions for the comparison of $\mathcal{E}_g[\varphi(X_T^{(1)})]$ and $\mathcal{E}_G[\varphi(X_T^{(2)})]$, where $\varphi(X_T^{(1)})$ and $\varphi(X_T^{(2)})$ are terminal values of BSDEs and G -BSDEs, respectively. First, we consider FBSDEs of the form:

$$X_t^{(1)} = X_0^{(1)} + \int_0^t \mu_1(s, X_s^{(1)})ds + \int_0^t \sigma_1(s, X_s^{(1)})dW_s, \quad (6.35a)$$

$$Y_t^{(1)} = \varphi(X_T^{(1)}) + \int_t^T g_1(s, X_s^{(1)}, Y_s^{(1)}, Z_s^{(1)})ds - \int_t^T Z_s^{(1)}dW_s, \quad (6.35b)$$

where the coefficients μ_1 , σ_1 , and g_1 satisfy (A.4.1)-(A.4.3) and $(W_t)_{t \in [0, T]}$ is a usual Brownian motion. We also consider the following G -FBSDEs

$$X_t^{(2)} = X_0^{(2)} + \int_0^t b_2(s, X_s^{(2)})ds + \int_0^t h_2(s, X_s^{(2)})d\langle B \rangle_s + \int_0^t \sigma_2(s, X_s^{(2)})dB_s, \quad (6.36a)$$

$$\begin{aligned} Y_t^{(2)} = & \varphi(X_T) + \int_t^T f_2(s, X_s^{(2)}, Y_s^{(2)}, Z_s^{(2)})ds + \int_t^T g_2(s, X_s^{(2)}, Y_s^{(2)}, Z_s^{(2)})d\langle B \rangle_s \\ & - \int_t^T Z_s^{(2)}dB_s - \int_t^T dK_s^{(2)}, \end{aligned} \quad (6.36b)$$

where the coefficients $b_2, h_2, \sigma_2, f_2, g_2$ and φ satisfy (A.5.1)-(A.5.3) and $(B_t)_{t \in [0, T]}$ is a G -Brownian motion satisfying $0 < \underline{\sigma}^2 = -\mathbb{E}_G[-B_1^2] < \mathbb{E}_G[B_1^2] = \bar{\sigma}^2$. We note that $K_t^{(2)}$ is a decreasing G -martingale under $\mathbb{E}_G$.

We recall that nonlinear expectations $\mathcal{E}_{g_1}[\varphi(X_T^{(1)})] := Y_0^{(1)}$ and $\mathcal{E}_G[\varphi(X_T^{(2)})] := Y_0^{(2)}$. In the sequel, we will also utilize the following functions:

$$f_1(t, x, y, z) := z\mu_1(t, x) + g_1(t, x, y, z\sigma_1(t, x)), \quad (6.37)$$

$$M_2(t, x, y, z) := zh_2(t, x) + g_2(t, x, y, z\sigma_2(t, x)), \quad (6.38)$$

$$N_2(t, x, y, z) := zb_2(t, x) + f_2(t, x, y, z\sigma_2(t, x)). \quad (6.39)$$

Throughout this section, we derive comparisons relying on the following conditions based on measurable functions $\zeta(t, x)$ and $\eta(t, x)$ such that

$$\theta_1(t, X_t^{(1)}) := \frac{\mu_1(t, X_t^{(1)}) - \zeta(t, X_t^{(1)})}{\sigma_1(t, X_t^{(1)})}, \quad (6.40)$$

and

$$\theta_2(t, X_t^{(2)}) := \frac{b_2(t, X_t^{(2)}) - \zeta(t, X_t^{(2)})}{\sigma_2(t, X_t^{(2)})} \quad \text{and} \quad \lambda_2(t, X_t^{(2)}) := \frac{h_2(t, X_t^{(2)}) - \eta(t, X_t^{(2)})}{\sigma_2(t, X_t^{(2)})}, \quad (6.41)$$

are bounded processes.

Proposition 6.5.1. (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$0 < \sigma_1^2(t, x)/\underline{\sigma}^2 \leq \sigma_2^2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (6.42)$$

together with the conditions

$$(L_1) \quad z\eta(t, x) \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R} \text{ and } z \in \mathbb{R}_+.$$

$$(L_2) \quad f_1(t, x, y, z) \leq z\zeta(t, x) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y \in \mathbb{R} \text{ and } z \in \mathbb{R}_+.$$

$$(L_3) \quad \text{Convexity of } (L_{3a}) \text{ or } (L_{3b}) \text{ as in the following:}$$

$$(L_{3a}) \quad (x, y, z) \mapsto f_1(t, x, y, z) \text{ is convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R} \times \mathbb{R}_+, \quad t \in [0, T].$$

$$(L_{3b}) \quad (x, y, z) \mapsto M_2(t, x, y, z) \text{ and } (x, y, z) \mapsto N_2(t, x, y, z) \text{ are convex in } (x, y) \in \mathbb{R}^2 \text{ and in } (y, z) \in \mathbb{R} \times \mathbb{R}_+, \quad t \in [0, T].$$

$$(L_4) \quad x \mapsto g_i(t, x, y, z), \quad i = 1, 2 \text{ and } x \mapsto f_2(t, x, y, z) \text{ are non-decreasing on } \mathbb{R} \text{ for all } (y, z) \in \mathbb{R} \times \mathbb{R}_+ \text{ and } t \in [0, T].$$

$$(L_5) \quad \eta(t, x) \geq 0, \text{ for } x \in \mathbb{R} \text{ and } t \in [0, T].$$

Then we have

$$\mathcal{E}_{g_1}[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})], \quad (6.43)$$

for all non-decreasing convex functions $\varphi(x) \in C_{l.Lip}(\mathbb{R})$.

Proof. (i) We start by assuming that the coefficients $\mu_1(t, \cdot)$, $\sigma_1(t, \cdot)$, and $g_1(t, \cdot, \cdot, \cdot)$ are $\mathcal{C}_b^3$ functions for all $t \in [0, T]$ as in Theorem 3.1.3; and the function φ with coefficients $b_2(t, \cdot)$, $h_2(t, \cdot)$, $\sigma_2(t, \cdot)$, $f_2(t, \cdot, \cdot, \cdot)$ and $g_2(t, \cdot, \cdot, \cdot)$ are $\mathcal{C}_b^3$ bounded functions for all $t \in [0, T]$ as in Theorem 3.3.5, and that (L_{3a}) holds true. Letting

$$\theta_2(t, x) := \frac{b_2(t, x) - \zeta(t, x)}{\sigma_2(t, x)} \quad \text{and} \quad \lambda_2(t, x) := \frac{h_2(t, x) - \eta(t, x)}{\sigma_2(t, x)}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

are bounded functions. By the G -Girsanov transformation, see e.g. Theorems 3.2 and 5.2 in [43], the process

$$\tilde{B}_t^G := B_t + \int_0^t \theta_2(s, X_s^{(2)}) ds + \int_0^t \lambda_2(s, X_s^{(2)}) d\langle B \rangle_s, \quad t \in [0, T],$$

is a G -Brownian motion under the following G -expectation $\tilde{\mathbb{E}}_2$ defined by

$$\begin{aligned} \tilde{\mathbb{E}}_2[X] := & \tilde{\mathbb{E}}_G \left[X \exp \left(- \int_0^T \lambda_2(s, X_s^{(2)}) dB_s - \frac{1}{2} \int_0^T (\lambda_2(s, X_s^{(2)}))^2 d\langle B \rangle_s \right. \right. \\ & - \int_0^T \theta_2(s, X_s^{(2)}) \lambda_2(s, X_s^{(2)}) ds - \int_0^T \theta_2(s, X_s^{(2)}) d\tilde{B}_s \\ & \left. \left. - \frac{1}{2} \int_0^T (\theta_2(s, X_s^{(2)}))^2 d\langle \tilde{B} \rangle_s \right) \right], \end{aligned}$$

where $(B_t, \tilde{B}_t)_{t \in [0, T]}$ is an auxiliary extended $\tilde{G}$ -Brownian motion and

$$\tilde{G}(A) := \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \nu \leq \bar{\sigma}^2} \text{tr} \left[A \begin{pmatrix} \nu & 1 \\ 1 & \nu^{-1} \end{pmatrix} \right], \quad A \in \mathbb{S}_2.$$

The G -FSDEs (6.36a) can be rewritten as

$$dX_t^{(2)} = \zeta(t, X_t^{(2)}) dt + \eta(t, X_t^{(2)}) d\langle B \rangle_t + \sigma_2(t, X_t^{(2)}) d\tilde{B}_t^G, \quad (6.44)$$

with the associated G -BSDEs

$$\begin{aligned} dY_t^{(2)} = & -(f_2(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}) + Z_t^{(2)} \theta_2(t, X_t^{(2)})) dt \\ & - (g_2(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}) + Z_t^{(2)} \lambda_2(t, X_t^{(2)})) d\langle B \rangle_t + Z_t^{(2)} d\tilde{B}_t^G + dK_t^{(2)}, \end{aligned} \quad (6.45)$$

where at terminal time, $Y_T^{(2)} = \varphi(X_T^{(2)})$. By Theorem 3.1.3 and Theorem 3.3.5 we have $Y_t^{(1)} = u_1(t, X_t^{(1)})$ and $Y_t^{(2)} = u_2(t, X_t^{(2)})$, where the functions $u_1(t, x)$ and $u_2(t, x)$ are in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ and solve the PDEs

$$\frac{\partial u_1}{\partial t}(t, x) + \frac{1}{2} \sigma_1^2(t, x) \frac{\partial^2 u_1}{\partial x^2}(t, x) + f_1\left(t, x, u_1(t, x), \frac{\partial u_1}{\partial x}(t, x)\right) = 0, \quad (6.46)$$

and

$$\frac{\partial u_2}{\partial t}(t, x) + 2G\left(\frac{1}{2} \sigma_2^2(t, x) \frac{\partial^2 u_2}{\partial x^2}(t, x) + M_2\left(t, x, u_2(t, x), \frac{\partial u_2}{\partial x}(t, x)\right)\right)$$

$$+ N_2\left(t, x, u_2(t, x), \frac{\partial u_2}{\partial x}(t, x)\right) = 0, \quad (6.47)$$

with $u_i(T, x) = \varphi(x)$, $i = 1, 2$ and $G(a) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \bar{\sigma}^2} (\sigma^2 a)$. Applying the G -Itô formula to $u_1(t, X_t^{(2)})$ and using (6.46), we have

$$\begin{aligned} u_1(t, X_t^{(2)}) &= u_1(0, X_0^{(2)}) + \int_0^t \frac{\partial u_1}{\partial s}(s, X_s^{(2)}) ds + \int_0^t \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) ds \\ &\quad + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s + \\ &\quad + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G \\ &= u_1(0, X_0^{(2)}) + \int_0^t \zeta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) ds \\ &\quad - \int_0^t f_1\left(s, X_s^{(2)}, u_1(s, X_s^{(2)}), \frac{\partial u_1}{\partial x}(s, X_s^{(2)})\right) ds - \int_0^t \frac{1}{2} \sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) ds \\ &\quad + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\ &\quad + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G. \\ &\geq u_1(0, X_0^{(2)}) - \int_0^t \frac{1}{2} \sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) ds \\ &\quad + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\ &\quad + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G, \end{aligned}$$

where we used the first part of (L₂). Since $u_2(t, x)$ is a solution of the equation (6.47), we know that $Y_s^{(2)} = u_2(s, X_s^{(2)})$ and $Z_s^{(2)} = \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)})$, and hence, by plugging those terms in the second G -BSDE (6.45), we have

$$\begin{aligned} u_2(t, X_t^{(2)}) &= u_2(0, X_0^{(2)}) - \int_0^t f_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) ds \\ &\quad - \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \theta_2(s, X_s^{(2)}) ds + \\ &\quad - \int_0^t g_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) d\langle B \rangle_s \end{aligned}$$

$$\begin{aligned}
& - \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \lambda_2(s, X_s^{(2)}) d\langle B \rangle_s + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G + K_t^{(2)} \\
& = u_2(0, X_0^{(2)}) + \int_0^t \left(\zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) - N_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) \right) ds \\
& + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) - M_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) \right) d\langle B \rangle_s \\
& + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G + K_t^{(2)}.
\end{aligned}$$

Noting that at the terminal time $u_1(T, X_T^{(2)}) = u_2(T, X_T^{(2)}) = \varphi(X_T^{(2)})$ a.s. and by (L₄) we have $\frac{\partial u_1}{\partial x}(t, x) \geq 0$ and $\frac{\partial u_2}{\partial x}(t, x) \geq 0$ for all $t \in [0, T]$ and $x \in \mathbb{R}$, see Proposition 3.2.2 and 3.4.2. Hence, from (L₁), (L₂), (L₅), and $K_T^{(2)} \leq 0, q.s.$, we get that

$$\begin{aligned}
& u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq \\
& \int_0^T \left(N_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) - \zeta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \right) ds \\
& + \int_0^T \left(M_2(s, X_s^{(2)}, u_2(s, X_s^{(2)}), \frac{\partial u_2}{\partial x}(s, X_s^{(2)})) - \eta(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\
& - \int_0^T \frac{1}{2} \sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) ds \\
& + \int_0^t \left(\eta(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B \rangle_s \\
& + \int_0^T \sigma_1(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - \int_0^T \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - K_T^{(2)} \\
& \geq - \int_0^T \frac{1}{2} \sigma_1^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) ds + \int_0^t \frac{1}{2} \sigma_2^2(s, X_s^{(2)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) d\langle B \rangle_s \\
& + \int_0^T \sigma_1(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G - \int_0^T \sigma_2(s, X_s^{(2)}) \frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\tilde{B}_s^G \quad (6.48)
\end{aligned}$$

By the Condition (L_{3a}), we know that $\frac{\partial u_1}{\partial x^2}(t, x) \geq 0$, see Theorem 3.2.6. Now we take expectation under $\mathbb{E}_2$ and apply the property $\int_0^t \eta_s d\langle B \rangle_s \geq \underline{\sigma}^2 \int_0^t \eta_s ds$ for any non-negative $\eta_s \in M_G^1(0, T)$, see (2.31) in Chapter 2, together with $\sigma_2^2(t, x) \geq \sigma_1^2(t, x)/\underline{\sigma}^2 > 0$, we obtain

$$u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq 0.$$

Since $X_0^{(2)} \geq X_0^{(1)}$, we have $Y_0^{(2)} = u_2(0, X_0^{(2)}) \geq u_2(0, X_0^{(1)}) \geq u_1(0, X_0^{(1)}) = Y_0^{(1)}$. Thus, we get $Y_0^{(2)} \geq Y_0^{(1)}$. Therefore, by definition we have

$$\mathcal{E}_{g_1}[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})],$$

for all non-decreasing convex function $\varphi \in C_{l.Lip}(\mathbb{R})$.

(ii) The case with the use of Assumption (L_{3b}) can be proved similarly. In this case, we define

$$\theta_1(t, x) := \frac{\mu_1(t, x) - \zeta(t, x)}{\sigma_1(t, x)}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

By the Girsanov theorem, the process

$$\widetilde{W}_t := W_t + \int_0^t \theta_1(s, X_s^{(1)}) ds, \quad t \in [0, T],$$

is a standard Brownian motion under the probability measure $\mathbb{Q}_1$ defined by

$$\frac{d\mathbb{Q}_1}{d\mathbb{P}} := \exp \left(- \int_0^T \theta_1(s, X_s^{(1)}) dW_s - \frac{1}{2} \int_0^T (\theta_1(s, X_s^{(1)}))^2 ds \right),$$

and the FSDE (6.35a) can be rewritten as

$$dX_t^{(1)} = \zeta(t, X_t^{(1)}) dt + \sigma_1(t, X_t^{(1)}) d\widetilde{W}_t, \quad (6.49)$$

with the associated BSDEs

$$\begin{aligned} dY_t^{(1)} &= -(g_1(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)}) + Z_t^{(1)} \theta_1(t, X_t^{(1)})) dt + Z_t^{(1)} d\widetilde{W}_t, \\ Y_T^{(1)} &= \phi(X_T^{(1)}). \end{aligned} \quad (6.50)$$

Similarly, we apply the Itô formula to $u_2(t, X_t^{(1)})$ and using (6.47), we have

$$\begin{aligned} u_2(t, X_t^{(1)}) &= u_2(0, X_0^{(1)}) + \int_0^t \frac{\partial u_2}{\partial s}(s, X_s^{(1)}) ds + \int_0^t \zeta(s, X_s^{(1)}) \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) ds \\ &\quad + \frac{1}{2} \int_0^t \sigma_1^2(s, X_s^{(1)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(1)}) ds + \int_0^t \sigma_1(s, X_s^{(1)}) \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) d\widetilde{W}_s \\ &= u_2(0, X_0^{(1)}) + \int_0^t \zeta(s, X_s^{(1)}) \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) ds \\ &\quad - \int_0^t N_2(s, X_s^{(1)}, u_2(s, X_s^{(1)}), \frac{\partial u_2}{\partial x}(s, X_s^{(1)})) ds \\ &\quad - 2 \int_0^t G\left(\frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) + M_2(s, X_s^{(1)}, u_2(s, X_s^{(1)}), \frac{\partial u_2}{\partial x}(s, X_s^{(1)}))\right) ds \\ &\quad + \frac{1}{2} \int_0^t \sigma_1^2(s, X_s^{(1)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(1)}) ds + \int_0^t \sigma_2(s, X_s^{(2)}) \frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\widetilde{W}_s. \end{aligned}$$

Taking expectation under $\mathbb{Q}_1$ at time $t = T$, we find

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_1} [\varphi(X_T^{(1)})] &= u_2(0, X_0^{(1)}) + \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \zeta(s, X_s^{(1)}) \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) ds \right] \\ &\quad - \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T N_2(s, X_s^{(1)}, u_2(s, X_s^{(1)}), \frac{\partial u_2}{\partial x}(s, X_s^{(1)})) ds \right] \end{aligned}$$

$$\begin{aligned}
& - \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T 2G \left(\frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) \right. \right. \\
& \quad \left. \left. + M_2(s, X_s^{(1)}, u_2(s, X_s^{(1)}), \frac{\partial u_2}{\partial x}(s, X_s^{(1)})) \right) ds \right] \\
& + \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^t \frac{1}{2} \sigma_1^2(s, X_s^{(1)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(1)}) ds \right] \tag{6.51}
\end{aligned}$$

Next, applying similarly the Itô formula to $u_1(t, X_t^{(1)})$ and then taking expectation under $\mathbb{Q}_1$ at $t = T$ we obtain from (6.46),

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_1} [\varphi(X_T^{(1)})] &= u_1(0, X_0^{(1)}) - \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T f_1 \left(s, X_s^{(1)}, u_1(s, X_s^{(1)}), \frac{\partial u_1}{\partial x}(s, X_s^{(1)}) \right) ds \right] \\
&+ \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \zeta(s, X_s^{(1)}) \frac{\partial u_1}{\partial x}(s, X_s^{(1)}) ds \right].
\end{aligned}$$

By Conditions (L₁), (L₄) and (L₅), we have $M_2(t, x, y, z) \geq 0$ and noting that $G(a)$ is a monotonic function, we get

$$\begin{aligned}
u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) &\geq \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T N_2 \left(s, X_s^{(1)}, u_2(s, X_s^{(1)}), \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) \right) ds \right] \\
&- \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \zeta(s, X_s^{(1)}) \frac{\partial u_2}{\partial x}(s, X_s^{(1)}) ds \right] \\
&+ \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \left(2G \left(\frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) \right) - \frac{1}{2} \sigma_1^2(s, X_s^{(1)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(1)}) \right) ds \right] \\
&+ \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \left(\zeta(s, X_s^{(1)}) \frac{\partial u_1}{\partial x}(s, X_s^{(1)}) - f_1 \left(s, X_s^{(1)}, u_1(s, X_s^{(1)}), \frac{\partial u_1}{\partial x}(s, X_s^{(1)}) \right) \right) ds \right] \\
&\geq \mathbb{E}_{\mathbb{Q}_1} \left[\int_0^T \left(2G \left(\frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) \right) - \frac{1}{2} \sigma_1^2(s, X_s^{(1)}) \frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(1)}) \right) ds \right]
\end{aligned}$$

where we have used (L₂). Since $\frac{\partial^2 u_2}{\partial x^2}(t, x) \geq 0$, as follows from (L_{3b}) by Theorem 3.2.6, we have

$$\begin{aligned}
2G \left(\frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) \right) &= \bar{\sigma}^2 \frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) \\
&\geq \underline{\sigma}^2 \frac{1}{2} \sigma_2^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}) \\
&\geq \frac{1}{2} \sigma_1^2(s, X_s^{(1)}) \frac{\partial^2 u_2}{\partial x^2}(s, X_s^{(1)}),
\end{aligned}$$

by the Condition (6.42). In addition, by the non-decreasing of $u_1(t, x)$, together with $X_0^{(1)} \leq X_0^{(2)}$, we have

$$u_2(0, X_0^{(2)}) - u_1(0, X_0^{(1)}) \geq u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq 0,$$

which allows us to conclude.

(iii) The extension of (6.43) to coefficients satisfying (A.4.1)-(A.4.3) along with (A.5.1)-(A.5.3) follows as in the proof of Theorem 4.4.1 and Theorem 5.3.1. $\square$

As can be seen in the proof of Proposition 6.5.1, if we set the function $\eta(t, x) = 0$ in the Condition (L₅), then one can remove Condition (L₅) as the non-decreasing property of the solutions $u_1(t, x)$ and $u_2(t, x)$ are no longer required. In this case, we have a version of convex order as follows:

Proposition 6.5.2. (convex order). *Assume that the condition (6.41) holds with $\eta(t, x) = 0$, $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1^2(t, x) \leq \underline{\sigma}^2 \sigma_2^2(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \quad (6.52)$$

together with the conditions

$$(L'_1) \quad 0 \leq M_2(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R}.$$

$$(L'_2) \quad f_1(t, x, y, z) \leq z\zeta(t, x) \leq N_2(t, x, y, z), \quad t \in [0, T], \quad x, y, z \in \mathbb{R}.$$

$$(L'_3) \quad \text{Convexity of } (L'_{3a}) \text{ or } (L'_{3b}) \text{ as in the following:}$$

$$(L'_{3a}) \quad (x, y, z) \mapsto f_1(t, x, y, z) \text{ is convex in } (x, y) \text{ and in } (y, z) \in \mathbb{R}^2, \quad t \in [0, T].$$

$$(L'_{3b}) \quad (x, y, z) \mapsto M_2(t, x, y, z) \text{ and } (x, y, z) \mapsto N_2(t, x, y, z) \text{ are convex in } (x, y) \text{ and in } (y, z) \in \mathbb{R}^2, \quad t \in [0, T].$$

Then we have

$$\mathcal{E}_{g_1}[\varphi(X_T^{(1)})] \leq \mathcal{E}_G[\varphi(X_T^{(2)})], \quad (6.53)$$

for all convex functions $\varphi(x) \in C_{l.Lip}(\mathbb{R})$.

Proof. By repeating arguments in the proof of Proposition 6.5.1 with the use of the assumptions $X_0^{(1)} = X_0^{(2)}$ and $\eta(t, x) \equiv 0$, one can finally get that

$$u_2(0, X_0^{(2)}) - u_1(0, X_0^{(1)}) = u_2(0, X_0^{(2)}) - u_1(0, X_0^{(2)}) \geq 0,$$

which yields the conclusion of (6.53). $\square$

Remark 6.5.3. *If we respectively replace the function $f_1(t, x, y, z)$, $M_2(t, x, y, z)$ and $N_2(t, x, y, z)$ by the following functions*

$$f_1^{(-1)}(t, x, y, z) := z\mu_1(t, x) - g_1(t, x, -y, -z\sigma_1(t, x)), \quad (6.54)$$

and

$$M_2^{(-1)}(t, x, y, z) := zh_2(t, x) - g_2(t, x, -y, -z\sigma_2(t, x)), \quad (6.55)$$

$$N_2^{(-1)}(t, x, y, z) := zb_2(t, x) - f_2(t, x, -y, -z\sigma_2(t, x)), \quad (6.56)$$

in the Proposition 6.5.1 (resp. Proposition 6.5.2), then we have

$$-\mathcal{E}_{g_1}[-\varphi(X_T^{(1)})] \leq -\mathcal{E}_G[-\varphi(X_T^{(2)})],$$

for all non-decreasing convex (resp. convex) functions $\varphi(x) \in \mathcal{C}_{\text{l.Lip}}(\mathbb{R})$. We obtain this result by applying Lemma 4.2.4 in Section 4.5 and Lemma 5.4.3 in Section 5.4, together with Proposition 6.5.1 (resp. Proposition 6.5.2).

Conclusion and Future Work

In this dissertation, first of all, we have extended several properties such as monotonicity, convexity and continuous dependence of solutions of the usual FBSDEs, G -FBSDEs and corresponding solutions of nonlinear parabolic equations and Hamilton-Jacobi-Bellman equations type. We then study the stochastic orderings from the point of view of the nonlinear expectations such as $\mathcal{E}_g$ generated by BSDEs and $\mathcal{E}_G$ generated by G -BSDEs, which generalize the concepts of classical stochastic orders in the linear expectations framework. In a summary of theoretical contributions, we have established several sufficient conditions for the g - and G -stochastic orderings in the sense of convex, increasing convex and monotonic orders. Analogous comparison results for the g - and G -risk measures have been proposed as consequences, in terms of concave g - and G -stochastic orderings. In addition, we have also derived comparison results between the expectations $\mathbb{E}, \mathcal{E}_g, \widehat{\mathbb{E}}, \mathbb{E}_G$ and $\mathcal{E}_G$ in the convex and increasing convex orders, in terms of the forms: $\mathbb{E}_g[\varphi(X_T)]$ versus $\widehat{\mathbb{E}}[\varphi(\widehat{X})]$, $\mathbb{E}_G[\varphi(X_T^G)]$ versus $\mathbb{E}_G[\varphi(\widehat{X})]$, and $\mathcal{E}_g[\varphi(X_T)]$ versus $\mathcal{E}_G[\varphi(X_T^G)]$, where $\widehat{X}$ denotes the G -normal distributed random variable, X_T and X_T^G are terminal values of diffusion SDEs driven by the standard Brownian motion and by the G -Brownian motion, respectively. For applications, we have provided comparisons of option prices under the different portfolio hedging constraints by the g -stochastic orderings as well as comparisons of option superhedging prices under ambiguous coefficients by the G -stochastic orderings, together with applications to bounds in expected values and to tail bounds. As can be seen, the g -stochastic orderings can be interpreted as comparisons of portfolio values, option prices, and risk measures. Such interpretations are possibly applied for the G -stochastic orderings if the underlying assets are captured by G -FSDEs with uncertain volatility. Also, the inequalities in the g -

and G -stochastic orderings can offer us lower and upper bound information and the concentration of measure inequalities. As such, they are applicable to risk management and decision theory. Besides, our results can serve as alternative comparison theorems for solutions of the FBSDEs and G -FBSDEs.

Future Work

So far, we have only studied the g - and G -stochastic orderings in one dimensional setting and hence there are still many possible works to extend in the next time. Firstly, a natural extension is to propose analogous comparison results in multi-dimensional setting. In this direction, however, there are some challenging issues that we need to carry out. The most difficult is to extend the convexity of multivariate solutions of PDEs and HJB equations without the requirement of global convexity of the generator functions as we did in the case of univariate solutions. Another main issue is the monotonicity of solutions of associated PDEs and the test functions due to there is no natural total order in multi-dimensional setting. Thus, we have to consider appropriate partial orders such as directionally convex order, supermodular order, lower and upper orthant order, etc. Also, one has to deal with related time inconsistency problems. The second potential research is to work on BSDEs and G -BSDEs with jumps, with quadratic growth and reflected BSDEs as well as G -BSDEs. We then can correspondingly define nonlinear expectations. This approach will surely provide more real applications, for instance in insurance, stochastic control theory and option pricing with the optimal stopping time. Thirdly, we study robust hedging problems, portfolio selection and the choice of consumption plans via the utility maximization with the help of the g - and G -stochastic orderings, see e.g. [2, 89, 59, 54, 24, 17] which are closely related to this formulation. Fourthly, the stochastic dominance under the linear expectations framework has recently provided as an alternative tool for option pricing, see e.g. many proposed interesting results in the book [83]. Hence, we want to apply the g - and G -stochastic orderings in this field under the nonlinear expectations framework instead. Last but not least, our comparison results rely on the use of nonlinear Feynman-Kac formula, e.g. $Y_s^{t,x} = u(s, X_s^{t,x})$, $s \in [t, T]$ and thus, it requires the smoothness solutions of the PDEs in order to apply the Itô formula. This approach may have some shortcomings. In order to overcome this issue, one can think of using predictable representation approach to obtain convex orderings as shown in the article [3].

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