Earthquake location derived from P, PmP and S waves : feasibility and problems

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**EARTHQUAKE LOCATION DERIVED FROM
P, PMP AND S WAVES: FEASIBILITY AND
PROBLEMS**

CHENG YAU WING

**SCHOOL OF PHYSICAL AND
MATHEMATICAL SCIENCES**

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CHENG YAU WING

**SCHOOL OF PHYSICAL AND
MATHEMATICAL SCIENCES**

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirements for the degree of
Master of Science

2020

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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CHENG YAU WING

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13 August 2020

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Date

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CHENG YAU WING

Abstract

In this research, we explore variety of methods to locate earthquakes in Kyushu Japan and Southern California.

Locating earthquakes to identify a fault line, is one of the possible way to find a cause of an earthquake occurrence. Although an earthquake occurrence is inevitable and unpredictable in general, at least the scientist will know that sometime later, the similar earthquake will occur again somewhere within a fault line, and therefore need to plan and be prepared in advance to prevent similar casualties again.

To find each earthquake location, at least 4 seismic stations locations and their corresponding phase arrival times should be known. In practice, there are usually more than 4 seismic stations for each earthquake and each phase arrival time is manually picked with some error, and hence requires to find the best fit approximation of each earthquake location. In this research thesis, we use Geiger's Method, and Grid Search and Fast Marching Method to locate earthquake.

Overall, both Geiger's Method, and Grid Search and Fast Marching Method, are feasible for locating epicenter of earthquakes. However, for both methods, we need to be careful when locating earthquake depth. Because the earthquake depth estimate might be very uncertain and the error estimate must be checked.

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Most importantly, I would like to thank my parents, Wong Sau Yung (Mother) and Cheng Chi Wan (Father), beyond my deepest sincere gratitude, for being supportive towards me, emotionally and financially, regardless of my situation, difficulties and achievements. Hence, I will need to take even more responsibility to take care of my parents as they are growing older.

After my graduation, the next few chapters of my life will require me to step out of my comfort zone significantly, to face an uncertain future, to constantly upgrade and improve myself by independent learning, and to readily adapt to the fast-changing and complex world, be it as a working adult, a researcher, or a future me who may gain enough relevant work and research experience to pursue even further studies or to become a lecturer. By then, I am on my own.

Cheng Yau Wing, August 2020
鄭又榮 (郑又荣)

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Symbols and Acronyms

Mathematical Symbols

$\text{atan2}(\cdot, \cdot)$	the four-quadrant inverse tangent
$\mathbb{R}$	the real number set
$\mathbb{R}^{N_1 \times N_2}$	the N_1 by N_2 matrix set. If $N_1 = 1$ or $N_2 = 1$, then vector set.
$\ \cdot\ $	the 2-norm of a vector
$\mathbf{q}$	the general vector with proper dimension
$\mathbf{M}$	the general matrix with proper dimension
$\det(\cdot)$	the determinant of a matrix
$\frac{\partial}{\partial \cdot}$	the derivative with respect to a vector or the partial derivative with respect to a variable
$\mathbf{I}$	the identity matrix with proper dimension
$\mathbf{0}$	all-zeros vector or matrix with proper dimension
$\mathbf{1}$	all-ones vector or matrix with proper dimension
$=$	equals to

Special Symbols for Algorithms

$=$	assigns a variable with a value
$\equiv$	comparison operator to check if a variable is equals to a value

Acronyms

P Wave	Primary Wave
PmP Wave	Moho Reflected Primary Wave
S Wave	Secondary Wave
Hinet Japan	High Sensitivity Seismograph Network Japan
SCEDC	Southern California Earthquake Data Center
Lat	Latitude in degree (for Table)
Lon	Longitude in degree (for Table)
Dep	Depth in km (for Table)
GS	Grid Search
FMM	Fast Marching Method
PDE	Partial Differential Equation

Chapter 1

Introduction

The purpose of this chapter is to provide a general audience a brief introduction of Kyushu Japan and Southern California as the two main study regions, as examples of motivation of the research. This chapter will also discuss and describe about the seismic waves and data, main objective, such as variety of methods to locate earthquakes, and limitations of the master research thesis.

1.1 Motivation of research and its study regions

Locating earthquakes to identify a fault line, is one of the possible way to find a cause of an earthquake occurrence. Although an earthquake occurrence is inevitable and unpredictable in general, at least the scientist will know that sometime later, the similar earthquake will occur again somewhere within a fault line, and therefore need to plan and be prepared in advance to prevent similar casualties again. [5] This is the main motivation of the research.

The two main study regions are Kyushu Japan and Southern California, which are practical examples of such motivation. In Kyushu Japan, according to Nikkei Asia [1], the government committee in Japan concluded with common consensus that Futagawa earthquake fault, with magnitude of 7.3, is the main cause of the earthquake in Kyushu Japan. It also mentioned an interesting discovery that when a fault vibrates, leading to an earthquake, then it will cause other nearby faults vibrate and induce new earthquakes, this is known as earthquake swarms. [1]

FIGURE 1.1: Futagawa earthquake fault on map of Kyushu Japan by Nikkei Asia [1]

FIGURE 1.2: Ridgecrest earthquake on map of Southern California by Cable News Network [2]

FIGURE 1.3: Two earthquakes at Ridgecrest, 4th and 5th of July 2019, on map of Southern California by Cable News Network [3]

On top of that, according to New York Times [6], as of April 2016, such earthquake caused the death of 9 people and the injuries of 1000 people in Kyushu Japan. Also, the transport and building infrastructures were destroyed, and the electrical power was tripped. [6]

While in Southern California, according to Cable News Network [3], two earthquakes at Ridgecrest, each took place on 4th of July 2019 as a foreshock and 5th of July 2019 as a mainshock, with magnitude of 6.4 and 7.1 respectively. These earthquakes caused fires, broken gas lines, gas leakages, power outages, water main breaks and damage of some buildings. [3]

1.2 Seismic Data

FIGURE 1.4: An example of my manually picked P and PmP arrival times in vertical component of a seismograph for Kyushu Japan are 10.8921 seconds and 11.8686 seconds respectively, where the hypocenter given by Hinet Japan is Latitude 33°, Longitude 130.34°, Depth 12 km and the given seismic station is Latitude 32.63° and Longitude 130.75°.

Definition 1.1 (Seismic Station and Seismograph). *In the Seismic Station, it records the behaviour of the Earthquake waves into a Seismograph, which consists of graphs of three amplitudes, Horizontal (East/West) and Horizontal (North/South) and Vertical components, against the time in seconds.*

Definition 1.2 (Primary Wave (P Wave)). *Primary Wave (P Wave) wave has an estimated velocity of around $v = 6 \text{ km/s}$ and arrives first as it travels directly from the Earthquake focus point (hypocenter) to the Seismic Station. It is a longitudinal wave, that is a wave that travels in a direction parallel to the direction of vibration, which creates regions of compression and rarefaction. [7]*

Definition 1.3 (Moho Reflected Primary Wave (PmP Wave)). *Moho Reflected Primary Wave (PmP Wave) has a velocity same as the Primary Wave, and typically arrives the time in between Primary and Secondary Wave, as it travels from the focus point of the Earthquake to the Seismic station with a reflection at the Moho Discontinuity.*

FIGURE 1.5: An example of my manually picked S arrival time in horizontal (east) component of a seismograph for Southern California is 38.205 seconds, where the hypocenter given by SCEDC is Latitude 35.4733° , Longitude -118.4250° , Depth 4.80 km, the given seismic station is Latitude 36.5504° , Longitude -117.8030° and Elevation 2.795 km, and the given P arrival time is 21.751 seconds.

Definition 1.4 (Secondary Wave (S Wave)). *Secondary Wave (S Wave) has an estimated velocity of 3.6 km/s and arrives later, as it is much slower than both P and PmP Wave. It is a traverse wave, that is a wave that travels in a direction perpendicular to the direction of vibration, such wave consists of crest and trough, in terms of an amplitude. [7]*

The seismic waveforms of Kyushu Japan and Southern California are downloaded from the Hinet Japan and SCEDC respectively. For Kyushu Japan, I have manually picked 5396 P arrival times for 222 Earthquake Events and 55 PmP arrival times for 10 Earthquake Events. For Southern California, I have manually picked 97 S arrival times for 15 Earthquake Events.

1.3 Main Objective of the Research Thesis

The main objective of the thesis is to find each earthquake location, at least 4 seismic stations locations and their corresponding phase arrival times should be known. In practice, there are usually more than 4 seismic stations for each earthquake and each phase arrival time is manually picked with some error, and hence requires to find the best fit approximation of each earthquake location.

In the first part of the thesis, each earthquake in Kyushu Japan will be located by applying Geiger's Method to set up system of linear equations, solving these equations by damped least square method to find the change in spatial coordinate and origin time, making use of such solution to update hypocenter and its origin time, and repeat them until it converges to near to the final hypocenter and the origin time. The velocity models of Kyushu Japan are 1 and 2 layer P (also PmP) wave velocity models for simplification.

In the second part of the thesis, each earthquake in Southern California will be located by Grid Search Algorithm and Fast Marching Method. Grid Search Algorithm mainly involves finding all possible hypocenters and origin time, then select a particular hypocenter and origin time that has the least square error as the final hypocenter and origin time. Fast Marching Method involves calculating travel time between each seismic station and a guessing spatial hypocenter, by solving Eikonal Equation. The velocity model of Southern California is a 4 layer S wave velocity model.

Finally, I will compare each hypocenter located by Geiger's Method and Grid Search with each hypocenter given by Hinet Japan and SCEDC respectively to determine the feasibility of earthquake location methods with P, PmP and/or S arrival times.

However, since the data is a primary source of raw data from the actual earthquake site locations, the data is noisy and the velocity structure of the Earth Crust is always inaccurate. As the result, each manually picked phase arrival time is bound to have error. Hence, regardless of any method of locating earthquake, there will be some inaccuracy in determining an earthquake location.

Chapter 2

Literature Review

2.1 Earth Coordinates for a Small Region

FIGURE 2.1: Spherical Earth with a fixed radius of $R_E = 6371.010$ km [4]

The region on the Earth is sufficiently small, such that the region is almost flat, hence we can assume that Earth has a spherical shape with a fixed radius of:

$$R_E = 6371.010 \text{ km} \quad (2.1)$$

and so we can use the depth, h in kilometers (km), to describe a point inside or outside the Earth, instead of using z-coordinate in kilometers (km):

$$\begin{aligned} z &= R_E \sin \phi \\ z &= \sqrt{R_E^2 - x^2 - y^2} \end{aligned} \tag{2.2}$$

where ϕ is latitude of a point in radian, x is x-coordinate of a point and y is y-coordinate of a point.

A point, for example a hypocenter with origin time, can be expressed in Latitude and Longitude Form [4] as:

- Latitude in Degree and Radian respectively:

$$\begin{aligned} -90^\circ &\leq \phi^\circ \leq 90^\circ \\ -\frac{\pi}{2} &\leq \phi = \frac{\pi}{180} \phi^\circ \leq \frac{\pi}{2} \end{aligned} \tag{2.3}$$

- Longitude in Degree and Radian respectively:

$$\begin{aligned} -180^\circ &< \lambda^\circ \leq 180^\circ \\ -\pi &< \lambda = \frac{\pi}{180} \lambda^\circ \leq \pi \end{aligned} \tag{2.4}$$

- Depth (km): h
- Origin Time (s): t

Or, in Cartesian form as:

- x-coordinate (km):

$$x = R_E \cos \phi \cos \lambda \tag{2.5}$$

- y-coordinate (km):

$$y = R_E \cos \phi \sin \lambda \tag{2.6}$$

- Depth (km): h
- Origin Time (s): t

Definition 2.1 (Four-quadrant inverse tangent).

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & , \text{ if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & , \text{ if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & , \text{ if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & , \text{ if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & , \text{ if } x = 0 \text{ and } y < 0 \\ \text{undefined} & , \text{ if } x = 0 \text{ and } y = 0 \end{cases} \quad (2.7)$$

where the range of four-quadrant inverse tangent and typical inverse tangent respectively:

$$-\pi < \text{atan2}(y, x) \leq \pi \quad (2.8)$$

$$-\frac{\pi}{2} < \arctan\left(\frac{y}{x}\right) < \frac{\pi}{2} \quad (2.9)$$

To convert from a point, for example a hypocenter with origin time, back to Latitude and Longitude Form again [4]:

- Latitude in Radian and Degree respectively:

$$\begin{aligned} -\frac{\pi}{2} \leq \phi = \arcsin\left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E}\right) &\leq \frac{\pi}{2} \\ -90^\circ \leq \phi^\circ = \frac{180}{\pi}\phi &\leq 90^\circ \end{aligned} \quad (2.10)$$

- Longitude in Radian and Degree respectively:

$$\begin{aligned} -\pi < \lambda = \text{atan2}(y, x) &\leq \pi \\ -180^\circ < \lambda^\circ = \frac{180}{\pi}\lambda &\leq 180^\circ \end{aligned} \quad (2.11)$$

- Depth (km): h
- Origin Time (s): t

2.2 Damped Least Square Optimisation

The lemmas below state the assumed prior knowledge of undergraduate linear algebra and calculus required before proceeding to solve the inverse problem of locating earthquakes by Geiger's Method.

Lemma 2.1 (Previous Results from Undergraduate Linear Algebra).

$$\|\mathbf{q}_1\|^2 = \mathbf{q}_1^T \mathbf{q}_1 = \mathbf{q}_1^T \mathbf{I} \mathbf{q}_1 \quad (2.12)$$

$$(\mathbf{M}_1 \pm \mathbf{M}_2)^T = \mathbf{M}_1^T \pm \mathbf{M}_2^T \quad (2.13)$$

$$(\mathbf{M}_1 \mathbf{M}_2)^T = \mathbf{M}_2^T \mathbf{M}_1^T \quad (2.14)$$

$$\mathbf{M}^{TT} = \mathbf{M} \quad (2.15)$$

Lemma 2.2 (Previous Results from Undergraduate Calculus [8]).

$$\frac{\partial}{\partial \mathbf{q}_1} (\mathbf{q}_1^T \mathbf{M} \mathbf{q}_1) = (\mathbf{M} + \mathbf{M}^T) \mathbf{q}_1 \quad (2.16)$$

$$\frac{\partial}{\partial \mathbf{q}_1} (\mathbf{q}_2^T \mathbf{M} \mathbf{q}_1) = \mathbf{M}^T \mathbf{q}_2 \quad (2.17)$$

In Geiger's Method, over determined linear equations, $\mathbf{M} \mathbf{q}_1 = \mathbf{q}_2$, are often encountered, because usually each earthquake event has more than 4 seismic stations, that is a system of more than 4 linear equations with 4 unknown variables, to locate an earthquake in Kyushu Japan. Hence, we consider using the Damped Least Square Method to solve such linear equations.

Theorem 2.3 (Damped Least Square Method). *The solution for minimising*

$$\|\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2\|^2 + \alpha \|\mathbf{q}_1\|^2 \quad (2.18)$$

subject to

$$\mathbf{M}\mathbf{q}_1 = \mathbf{q}_2 \quad (2.19)$$

is

$$\mathbf{q}_1 = (\mathbf{M}^T \mathbf{M} + \alpha \mathbf{I})^{-1} \mathbf{M}^T \mathbf{q}_2 \quad (2.20)$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial \mathbf{q}_1} \|\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2\|^2 &= \frac{\partial}{\partial \mathbf{q}_1} [(\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2)^T (\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2)] \\ &= \frac{\partial}{\partial \mathbf{q}_1} [((\mathbf{M}\mathbf{q}_1)^T - \mathbf{q}_2^T) (\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2)] \\ &= \frac{\partial}{\partial \mathbf{q}_1} [(\mathbf{M}\mathbf{q}_1)^T \mathbf{M}\mathbf{q}_1 - (\mathbf{M}\mathbf{q}_1)^T \mathbf{q}_2 - \mathbf{q}_2^T \mathbf{M}\mathbf{q}_1 + \mathbf{q}_2^T \mathbf{q}_2] \\ &= \frac{\partial}{\partial \mathbf{q}_1} [\mathbf{q}_1^T (\mathbf{M}^T \mathbf{M}) \mathbf{q}_1 - \mathbf{q}_2^T \mathbf{M}\mathbf{q}_1 - \mathbf{q}_2^T \mathbf{M}\mathbf{q}_1 + \|\mathbf{q}_2\|^2] \\ &= \frac{\partial}{\partial \mathbf{q}_1} [\mathbf{q}_1^T (\mathbf{M}^T \mathbf{M}) \mathbf{q}_1 - 2\mathbf{q}_2^T \mathbf{M}\mathbf{q}_1 + \|\mathbf{q}_2\|^2] \\ &= [\mathbf{M}^T \mathbf{M} + (\mathbf{M}^T \mathbf{M})^T] \mathbf{q}_1 - 2\mathbf{M}^T \mathbf{q}_2 \\ &= (\mathbf{M}^T \mathbf{M} + \mathbf{M}^T \mathbf{M}^{TT}) \mathbf{q}_1 - 2\mathbf{M}^T \mathbf{q}_2 \\ &= (\mathbf{M}^T \mathbf{M} + \mathbf{M}^T \mathbf{M}) \mathbf{q}_1 - 2\mathbf{M}^T \mathbf{q}_2 \\ &= 2\mathbf{M}^T \mathbf{M}\mathbf{q}_1 - 2\mathbf{M}^T \mathbf{q}_2 \\ &= 2(\mathbf{M}^T \mathbf{M}\mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2) \end{aligned} \quad (2.21)$$

$$\begin{aligned} \frac{\partial}{\partial \mathbf{q}_1} \|\mathbf{q}_1\|^2 &= \frac{\partial}{\partial \mathbf{q}_1} (\mathbf{q}_1^T \mathbf{q}_1) \\ &= \frac{\partial}{\partial \mathbf{q}_1} (\mathbf{q}_1^T \mathbf{I} \mathbf{q}_1) \\ &= (\mathbf{I} + \mathbf{I}^T) \mathbf{q}_1 \\ &= (\mathbf{I} + \mathbf{I}) \mathbf{q}_1 \\ &= 2\mathbf{I} \mathbf{q}_1 \\ &= 2\mathbf{q}_1 \end{aligned} \quad (2.22)$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{q}_1} (\|\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2\|^2 + \alpha \|\mathbf{q}_1\|^2) &= \frac{\partial}{\partial \mathbf{q}_1} \|\mathbf{M}\mathbf{q}_1 - \mathbf{q}_2\|^2 + \alpha \frac{\partial}{\partial \mathbf{q}_1} \|\mathbf{q}_1\|^2 \\
&= 2 (\mathbf{M}^T \mathbf{M} \mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2) + \alpha (2\mathbf{I} \mathbf{q}_1) \\
&= 2 (\mathbf{M}^T \mathbf{M} \mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2) + 2\alpha \mathbf{I} \mathbf{q}_1 \\
&= 2 (\mathbf{M}^T \mathbf{M} \mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2 + \alpha \mathbf{I} \mathbf{q}_1) \\
&= 2 (\mathbf{M}^T \mathbf{M} \mathbf{q}_1 + \alpha \mathbf{I} \mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2) \\
&= 2 [(\mathbf{M}^T \mathbf{M} + \alpha \mathbf{I}) \mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2] \\
&= \mathbf{0}
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
(\mathbf{M}^T \mathbf{M} + \alpha \mathbf{I}) \mathbf{q}_1 - \mathbf{M}^T \mathbf{q}_2 &= \mathbf{0} \\
(\mathbf{M}^T \mathbf{M} + \alpha \mathbf{I}) \mathbf{q}_1 &= \mathbf{M}^T \mathbf{q}_2
\end{aligned} \tag{2.24}$$

$$\therefore \mathbf{q}_1 = (\mathbf{M}^T \mathbf{M} + \alpha \mathbf{I})^{-1} \mathbf{M}^T \mathbf{q}_2 \tag{2.25}$$

□

Due to data error caused by manual phase arrival time picks, such linear equations may be inconsistent. Moreover, the matrix $\mathbf{M}^T \mathbf{M}$ may be almost or completely singular, for example the matrix $\mathbf{M}$ can be a Jacobian Matrix. By adding a scaled identity matrix, $\alpha \mathbf{I}$ to the matrix $\mathbf{M}^T \mathbf{M}$, the possibility of the modified matrix $\mathbf{M}^T \mathbf{M} + \alpha \mathbf{I}$ being a singular matrix reduces significantly. Hence, the damped least square method is suitable for solving such linear equations for the earthquake location in Kyushu Japan by Geiger's Method.

Chapter 3

Geiger's Method and its application

3.1 Derive Epidistance and Arrival Time

Proposition 3.1 (Arrival Time: 1 layer P wave velocity model).

$$f_{111}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{111}) = t + \frac{1}{v_{111}} \|\mathbf{x}_{1s_1} - \mathbf{x}\| \quad (3.1)$$

$$\|\mathbf{x}_{1s_1} - \mathbf{x}\| = \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2 + (h_{1s_1} - h)^2} \quad (3.2)$$

Proof. The proof is intuitive. □

Proposition 3.2 (Arrival Time: 1 layer PmP wave velocity model).

$$f_{211}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L211}, v_{211}) = t + \frac{1}{v_{211}} \|\mathbf{c}_{211s_2} - \mathbf{x}\| \quad (3.3)$$

$$\mathbf{c}_{211s_2} = (x_{2s_2}, y_{2s_2}, 2h_{L211} - h_{2s_2}) \quad (3.4)$$

$$\|\mathbf{c}_{211s_2} - \mathbf{x}\| = \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2 + (2h_{L211} - h_{2s_2} - h)^2} \quad (3.5)$$

Proof. Ray Path Segment 1 ($\mathbf{b}_1$):

$$\|\mathbf{b}_{12}\| = h_{L211} - h, \quad \sin \theta_{211s_2} = \frac{\|\mathbf{b}_{11}\|}{\|\mathbf{b}_1\|} \quad (3.6)$$

Ray Path Segment 2 ($\mathbf{b}_2$):

$$\|\mathbf{b}_{22}\| = h_{L211} - h_{2s_2}, \quad \sin \theta_{211s_2} = \frac{\|\mathbf{b}_{21}\|}{\|\mathbf{b}_2\|} \quad (3.7)$$

Ray Parameter:

$$p_{211s_2} = \frac{\sin \theta_{211s_2}}{v_{211}} = \frac{\|\mathbf{b}_{11}\|}{v_{211} \|\mathbf{b}_1\|} = \frac{\|\mathbf{b}_{21}\|}{v_{211} \|\mathbf{b}_2\|} \quad (3.8)$$

$$\frac{1}{p_{211s_2} v_{211}} = \frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|} = \frac{\|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} \quad (3.9)$$

Segments of Epicentral Distance:

$$\begin{aligned} \|\mathbf{b}_{11}\| &= \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\|\mathbf{b}_{12}\|} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\sqrt{\|\mathbf{b}_1\|^2 - \|\mathbf{b}_{11}\|^2}} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\|\mathbf{b}_{11}\| \sqrt{\left(\frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|}\right)^2 - 1}} \\ &= \frac{h_{L211} - h}{\sqrt{\left(\frac{1}{p_{211s_2} v_{211}}\right)^2 - 1}} \end{aligned} \quad (3.10)$$

$$\begin{aligned} \|\mathbf{b}_{21}\| &= \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{22}\|} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\sqrt{\|\mathbf{b}_2\|^2 - \|\mathbf{b}_{21}\|^2}} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{21}\| \sqrt{\left(\frac{\|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|}\right)^2 - 1}} \\ &= \frac{h_{L211} - h_{2s_2}}{\sqrt{\left(\frac{1}{p_{211s_2} v_{211}}\right)^2 - 1}} \end{aligned} \quad (3.11)$$

Epicentral Distance:

$$\|\mathbf{r}_{2s_2} - \mathbf{r}\| = \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} = \frac{2h_{L211} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{211s_2} v_{211}}\right)^2 - 1}} \quad (3.12)$$

$$\frac{1}{p_{211s_2} v_{211}} = \sqrt{1 + \frac{(2h_{L211} - h_{2s_2} - h)^2}{\|\mathbf{r}_{2s_2} - \mathbf{r}\|^2}} \quad (3.13)$$

Segments of travel time:

$$\frac{1}{p_{211s_2} v_{211}} = \frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|} = \frac{\|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} \implies \frac{1}{p_{211s_2} v_{211}^2} = \frac{G_1}{\|\mathbf{b}_{11}\|} = \frac{G_2}{\|\mathbf{b}_{21}\|} \quad (3.14)$$

$$G_1 = \frac{\|\mathbf{b}_{11}\|}{p_{211s_2} v_{211}^2} = \frac{h_{L211} - h}{p_{211s_2} v_{211}^2 \sqrt{\left(\frac{1}{p_{211s_2} v_{211}}\right)^2 - 1}} \quad (3.15)$$

$$G_2 = \frac{\|\mathbf{b}_{21}\|}{p_{211s_2} v_{211}^2} = \frac{h_{L211} - h_{2s_2}}{p_{211s_2} v_{211}^2 \sqrt{\left(\frac{1}{p_{211s_2} v_{211}}\right)^2 - 1}} \quad (3.16)$$

PmP Travel Time $g_{211}(\mathbf{x}, \mathbf{x}_{2s_2}, h_{L211}, v_{211})$, from the hypocenter $\mathbf{x} = (\mathbf{r}, h) = (x, y, h)$ to the seismic station $\mathbf{x}_{2s_2} = (\mathbf{r}_{2s_2}, h_{2s_2}) = (x_{2s_2}, y_{2s_2}, h_{2s_2})$:

$$\begin{aligned} g_{211}(\mathbf{x}, \mathbf{x}_{2s_2}, h_{L211}, v_{211}) &= \frac{2h_{L211} - h_{2s_2} - h}{p_{211s_2} v_{211}^2 \sqrt{\left(\frac{1}{p_{211s_2} v_{211}}\right)^2 - 1}} \\ &= \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{211s_2} v_{211}^2} = \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{v_{211}} \frac{1}{p_{211s_2} v_{211}} \\ &= \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{v_{211}} \sqrt{1 + \frac{(2h_{L211} - h_{2s_2} - h)^2}{\|\mathbf{r}_{2s_2} - \mathbf{r}\|^2}} \\ &= \frac{1}{v_{211}} \sqrt{\|\mathbf{r}_{2s_2} - \mathbf{r}\|^2 + (2h_{L211} - h_{2s_2} - h)^2} \\ &= \frac{1}{v_{211}} \|\mathbf{c}_{211s_2} - \mathbf{x}\| \end{aligned} \quad (3.17)$$

where:

$$\mathbf{c}_{211s_2} = (x_{2s_2}, y_{2s_2}, 2h_{L211} - h_{2s_2}) \quad (3.18)$$

$$\|\mathbf{c}_{211s_2} - \mathbf{x}\| = \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2 + (2h_{L211} - h_{2s_2} - h)^2} \quad (3.19)$$

PmP arrival time $f_{211}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L211}, v_{211})$, where $\mathbf{w} = (x, y, h, t)$ is the guessing hypocenter and origin time:

$$\begin{aligned} f_{211}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L211}, v_{211}) &= t + g_{211}(\mathbf{x}, \mathbf{x}_{2s_2}, h_{L211}, v_{211}) \\ &= t + \frac{1}{v_{211}} \|\mathbf{c}_{211s_2} - \mathbf{x}\| \end{aligned} \quad (3.20)$$

□

Proposition 3.3 (Epidistance and Arrival Time: 2 layer P wave velocity model).

Case 1: The hypocenter is at layer 1

• *P arrival time $f_{121}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{121})$:*

$$f_{121}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{121}) = t + \frac{1}{v_{121}} \|\mathbf{x}_{1s_1} - \mathbf{x}\| \quad (3.21)$$

$$\|\mathbf{x}_{1s_1} - \mathbf{x}\| = \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2 + (h_{1s_1} - h)^2} \quad (3.22)$$

Case 2: The hypocenter is at layer 2

- Epicentral Distance $d_{122}(h, h_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1})$:

$$\begin{aligned} d_{122}(h, h_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) &= \frac{h_{L121} - h_{1s_1}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{121}}\right)^2 - 1}} + \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \\ &= \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2} \\ &= \|\mathbf{r}_{1s_1} - \mathbf{r}\| \end{aligned} \quad (3.23)$$

- P arrival time $f_{122}(\mathbf{w}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1})$:

$$\begin{aligned} f_{122}(\mathbf{w}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \\ = t + \frac{\|\mathbf{r}_{1s_1} - \mathbf{r}\|}{p_{122s_1} v_{121}^2} + \frac{1}{p_{122s_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \end{aligned} \quad (3.24)$$

Proof. **Case 1: The hypocenter is at layer 1**

- P arrival time $f_{121}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{121})$, where $\mathbf{w} = (x, y, h, t)$ is the guessing hypocenter and origin time:

$$f_{121}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{121}) = t + \frac{1}{v_{121}} \|\mathbf{x}_{1s_1} - \mathbf{x}\| \quad (3.25)$$

$$\|\mathbf{x}_{1s_1} - \mathbf{x}\| = \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2 + (h_{1s_1} - h)^2} \quad (3.26)$$

Case 2: The hypocenter is at layer 2

- Ray Path Segment 2 ($\mathbf{b}_2$):

$$\|\mathbf{b}_{22}\| = h - h_{L121}, \quad \sin \theta_{122s_1} = \frac{\|\mathbf{b}_{21}\|}{\|\mathbf{b}_2\|} \quad (3.27)$$

- Ray Path Segment 1 ($\mathbf{b}_1$):

$$\|\mathbf{b}_{12}\| = h_{L121} - h_{1s_1}, \quad \sin \theta_{121s_1} = \frac{\|\mathbf{b}_{11}\|}{\|\mathbf{b}_1\|} \quad (3.28)$$

- Snell's Law:

$$\frac{v_{122}}{v_{121}} = \frac{\sin \theta_{122s_1}}{\sin \theta_{121s_1}} \quad (3.29)$$

- Ray Parameter:

$$p_{122s_1} = \frac{\sin \theta_{122s_1}}{v_{122}} = \frac{\sin \theta_{121s_1}}{v_{121}} \quad (3.30)$$

$$p_{122s_1} = \frac{\|\mathbf{b}_{21}\|}{v_{122} \|\mathbf{b}_2\|} = \frac{\|\mathbf{b}_{11}\|}{v_{121} \|\mathbf{b}_1\|} \quad (3.31)$$

$$\frac{1}{p_{122s_1}} = \frac{v_{122} \|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} = \frac{v_{121} \|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|} \quad (3.32)$$

- Segments of Epicentral Distance:

$$\begin{aligned} \|\mathbf{b}_{21}\| &= \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{22}\|} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\sqrt{\|\mathbf{b}_2\|^2 - \|\mathbf{b}_{21}\|^2}} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{21}\| \sqrt{\left(\frac{\|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|}\right)^2 - 1}} \\ &= \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \end{aligned} \quad (3.33)$$

$$\begin{aligned} \|\mathbf{b}_{11}\| &= \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\|\mathbf{b}_{12}\|} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\sqrt{\|\mathbf{b}_1\|^2 - \|\mathbf{b}_{11}\|^2}} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\|\mathbf{b}_{11}\| \sqrt{\left(\frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|}\right)^2 - 1}} \\ &= \frac{h_{L121} - h_{1s_1}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{121}}\right)^2 - 1}} \end{aligned} \quad (3.34)$$

- Epicentral Distance:

$$\begin{aligned} d_{122}(h, h_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) &= \frac{h_{L121} - h_{1s_1}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{121}}\right)^2 - 1}} + \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \\ &= \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2} \\ &= \|\mathbf{r}_{1s_1} - \mathbf{r}\| \end{aligned} \quad (3.35)$$

- Segments of travel time:

$$\frac{1}{p_{122s_1}} = \frac{v_{122} \|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} = \frac{v_{121} \|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|} \implies \frac{1}{p_{122s_1}} = \frac{v_{122}^2 G_2}{\|\mathbf{b}_{21}\|} = \frac{v_{121}^2 G_1}{\|\mathbf{b}_{11}\|} \quad (3.36)$$

$$G_2 = \frac{\|\mathbf{b}_{21}\|}{p_{122s_1} v_{122}^2} = \frac{h - h_{L121}}{p_{122s_1} v_{122}^2 \sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \quad (3.37)$$

$$G_1 = \frac{\|\mathbf{b}_{11}\|}{p_{122s_1} v_{121}^2} = \frac{h_{L121} - h_{1s_1}}{p_{122s_1} v_{121}^2 \sqrt{\left(\frac{1}{p_{122s_1} v_{121}}\right)^2 - 1}} \quad (3.38)$$

- P travel time $g_{122}(\mathbf{x}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1})$, from the hypocenter $\mathbf{x} = (\mathbf{r}, h) = (x, y, h)$ to the seismic station $\mathbf{x}_{1s_1} = (\mathbf{r}_{1s_1}, h_{1s_1}) = (x_{1s_1}, y_{1s_1}, h_{1s_1})$:

$$\begin{aligned} & g_{122}(\mathbf{x}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \\ &= \frac{h_{L121} - h_{1s_1}}{p_{122s_1} v_{121}^2 \sqrt{\left(\frac{1}{p_{122s_1} v_{121}}\right)^2 - 1}} + \frac{h - h_{L121}}{p_{122s_1} v_{122}^2 \sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \\ &= \frac{1}{p_{122s_1} v_{121}^2} \left(\sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2} - \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \right) \\ &\quad + \frac{h - h_{L121}}{p_{122s_1} v_{122}^2 \sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \quad (3.39) \\ &= \frac{\sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2}}{p_{122s_1} v_{121}^2} + \frac{1}{p_{122s_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \\ &= \frac{\|\mathbf{r}_{1s_1} - \mathbf{r}\|}{p_{122s_1} v_{121}^2} + \frac{1}{p_{122s_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \end{aligned}$$

- P arrival time $f_{122}(\mathbf{w}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1})$, where $\mathbf{w} = (x, y, h, t)$ is the guessing hypocenter and origin time:

$$\begin{aligned} & f_{122}(\mathbf{w}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \\ &= t + g_{122}(\mathbf{x}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \\ &= t + \frac{\|\mathbf{r}_{1s_1} - \mathbf{r}\|}{p_{122s_1} v_{121}^2} + \frac{1}{p_{122s_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \quad (3.40) \end{aligned}$$

□

Proposition 3.4 (Epidistance and Arrival Time: 2 layer PmP wave velocity model). *Case 1: The hypocenter is at layer 1*

- Epicentral Distance $d_{221}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{221s_2})$:

$$\begin{aligned}
 d_{221}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{221s_2}) &= \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2}v_{221}}\right)^2 - 1}} + \frac{2(h_{L222} - h_{L221})}{\sqrt{\left(\frac{1}{p_{221s_2}v_{222}}\right)^2 - 1}} \\
 &= \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} \\
 &= \|\mathbf{r}_{2s_2} - \mathbf{r}\|
 \end{aligned} \tag{3.41}$$

- PmP arrival time $f_{221}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2})$:

$$\begin{aligned}
 f_{221}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2}) \\
 = t + \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{221s_2}v_{222}^2} + \frac{1}{p_{221s_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right) \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2}v_{221}}\right)^2 - 1}}
 \end{aligned} \tag{3.42}$$

Case 2: The hypocenter is at layer 2

- Epicentral Distance $d_{222}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2})$:

$$\begin{aligned}
 d_{222}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) &= \frac{h_{L221} - h_{2s_2}}{\sqrt{\left(\frac{1}{p_{222s_2}v_{221}}\right)^2 - 1}} + \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2}v_{222}}\right)^2 - 1}} \\
 &= \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} \\
 &= \|\mathbf{r}_{2s_2} - \mathbf{r}\|
 \end{aligned} \tag{3.43}$$

- PmP arrival time $f_{222}(\mathbf{w}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2})$:

$$\begin{aligned}
 f_{222}(\mathbf{w}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) \\
 = t + \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{222s_2}v_{221}^2} + \frac{1}{p_{222s_2}} \left(\frac{1}{v_{222}^2} - \frac{1}{v_{221}^2} \right) \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2}v_{222}}\right)^2 - 1}}
 \end{aligned} \tag{3.44}$$

Proof. **Case 1: The hypocenter is at layer 1**

- Ray Path Segment 1 ($\mathbf{b}_1$):

$$\|\mathbf{b}_{12}\| = h_{L221} - h, \quad \sin \theta_{221s_2} = \frac{\|\mathbf{b}_{11}\|}{\|\mathbf{b}_1\|} \quad (3.45)$$

- Ray Path Segment 2 ($\mathbf{b}_2$):

$$\|\mathbf{b}_{22}\| = h_{L222} - h_{L221}, \quad \sin \theta_{222s_2} = \frac{\|\mathbf{b}_{21}\|}{\|\mathbf{b}_2\|} \quad (3.46)$$

- By Reflection Law, Ray Path Segment 3 ($\mathbf{b}_3$):

$$\|\mathbf{b}_{32}\| = h_{L222} - h_{L221} = \|\mathbf{b}_{22}\|, \quad \sin \theta_{222s_2} = \frac{\|\mathbf{b}_{31}\|}{\|\mathbf{b}_3\|} = \frac{\|\mathbf{b}_{21}\|}{\|\mathbf{b}_2\|} \quad (3.47)$$

- Ray Path Segment 4 ($\mathbf{b}_4$):

$$\|\mathbf{b}_{42}\| = h_{L221} - h_{2s_2}, \quad \sin \theta_{221s_2} = \frac{\|\mathbf{b}_{41}\|}{\|\mathbf{b}_4\|} \quad (3.48)$$

- Snell's Law:

$$\frac{v_{221}}{v_{222}} = \frac{\sin \theta_{221s_2}}{\sin \theta_{222s_2}} \quad (3.49)$$

- Ray Parameter:

$$p_{221s_2} = \frac{\sin \theta_{221s_2}}{v_{221}} = \frac{\sin \theta_{222s_2}}{v_{222}} \quad (3.50)$$

$$p_{221s_2} = \frac{\|\mathbf{b}_{11}\|}{v_{221} \|\mathbf{b}_1\|} = \frac{\|\mathbf{b}_{21}\|}{v_{222} \|\mathbf{b}_2\|} = \frac{\|\mathbf{b}_{31}\|}{v_{222} \|\mathbf{b}_3\|} = \frac{\|\mathbf{b}_{41}\|}{v_{221} \|\mathbf{b}_4\|} \quad (3.51)$$

$$\frac{1}{p_{221s_2}} = \frac{v_{221} \|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|} = \frac{v_{222} \|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} = \frac{v_{222} \|\mathbf{b}_3\|}{\|\mathbf{b}_{31}\|} = \frac{v_{221} \|\mathbf{b}_4\|}{\|\mathbf{b}_{41}\|} \quad (3.52)$$

- Segments of Epicentral Distance:

$$\begin{aligned} \|\mathbf{b}_{11}\| &= \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\|\mathbf{b}_{12}\|} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\sqrt{\|\mathbf{b}_1\|^2 - \|\mathbf{b}_{11}\|^2}} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{12}\|}{\|\mathbf{b}_{11}\| \sqrt{\left(\frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|}\right)^2 - 1}} \\ &= \frac{h_{L221} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \end{aligned} \quad (3.53)$$

$$\begin{aligned}
\|\mathbf{b}_{31}\| &= \|\mathbf{b}_{21}\| = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{22}\|} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\sqrt{\|\mathbf{b}_2\|^2 - \|\mathbf{b}_{21}\|^2}} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{21}\| \sqrt{\left(\frac{\|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|}\right)^2 - 1}} \\
&= \frac{h_{L222} - h_{L221}}{\sqrt{\left(\frac{1}{p_{221s_2} v_{222}}\right)^2 - 1}}
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
\|\mathbf{b}_{41}\| &= \frac{\|\mathbf{b}_{41}\| \|\mathbf{b}_{42}\|}{\|\mathbf{b}_{42}\|} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{42}\|}{\|\mathbf{b}_{12}\|} = \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{42}\|}{\sqrt{\|\mathbf{b}_1\|^2 - \|\mathbf{b}_{11}\|^2}} \\
&= \frac{\|\mathbf{b}_{11}\| \|\mathbf{b}_{42}\|}{\|\mathbf{b}_{11}\| \sqrt{\left(\frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|}\right)^2 - 1}} = \frac{h_{L221} - h_{2s_2}}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}}
\end{aligned} \tag{3.55}$$

• Epicentral Distance:

$$\begin{aligned}
d_{221}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{221s_2}) &= \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} + \frac{2(h_{L222} - h_{L221})}{\sqrt{\left(\frac{1}{p_{221s_2} v_{222}}\right)^2 - 1}} \\
&= \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} \\
&= \|\mathbf{r}_{2s_2} - \mathbf{r}\|
\end{aligned} \tag{3.56}$$

• Segments of travel time:

$$\frac{1}{p_{221s_2}} = \frac{v_{221} \|\mathbf{b}_1\|}{\|\mathbf{b}_{11}\|} = \frac{v_{222} \|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} = \frac{v_{222} \|\mathbf{b}_3\|}{\|\mathbf{b}_{31}\|} = \frac{v_{221} \|\mathbf{b}_4\|}{\|\mathbf{b}_{41}\|} \tag{3.57}$$

$$\frac{1}{p_{221s_2}} = \frac{v_{221}^2 G_1}{\|\mathbf{b}_{11}\|} = \frac{v_{222}^2 G_2}{\|\mathbf{b}_{21}\|} = \frac{v_{222}^2 G_3}{\|\mathbf{b}_{31}\|} = \frac{v_{221}^2 G_4}{\|\mathbf{b}_{41}\|} \tag{3.58}$$

$$\|\mathbf{b}_{31}\| = \|\mathbf{b}_{21}\| \implies \frac{1}{p_{221s_2}} = \frac{v_{221}^2 G_1}{\|\mathbf{b}_{11}\|} = \frac{v_{222}^2 G_2}{\|\mathbf{b}_{21}\|} = \frac{v_{222}^2 G_3}{\|\mathbf{b}_{21}\|} = \frac{v_{221}^2 G_4}{\|\mathbf{b}_{41}\|} \implies G_2 = G_3 \tag{3.59}$$

$$G_1 = \frac{\|\mathbf{b}_{11}\|}{p_{221s_2} v_{221}^2} = \frac{h_{L221} - h}{p_{221s_2} v_{221}^2 \sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \tag{3.60}$$

$$G_3 = G_2 = \frac{\|\mathbf{b}_{21}\|}{p_{221s_2} v_{222}^2} = \frac{h_{L222} - h_{L221}}{p_{221s_2} v_{222}^2 \sqrt{\left(\frac{1}{p_{221s_2} v_{222}}\right)^2 - 1}} \tag{3.61}$$

$$G_4 = \frac{\|\mathbf{b}_{41}\|}{p_{221s_2} v_{221}^2} = \frac{h_{L221} - h_{2s_2}}{p_{221s_2} v_{221}^2 \sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \quad (3.62)$$

- PmP travel time $g_{221}(\mathbf{x}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2})$, from the hypocenter $\mathbf{x} = (\mathbf{r}, h) = (x, y, h)$ to the seismic station $\mathbf{x}_{2s_2} = (\mathbf{r}_{2s_2}, h_{2s_2}) = (x_{2s_2}, y_{2s_2}, h_{2s_2})$:

$$\begin{aligned} & g_{221}(\mathbf{x}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2}) \\ &= \frac{2h_{L221} - h_{2s_2} - h}{p_{221s_2} v_{221}^2 \sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} + \frac{2(h_{L222} - h_{L221})}{p_{221s_2} v_{222}^2 \sqrt{\left(\frac{1}{p_{221s_2} v_{222}}\right)^2 - 1}} \\ &= \frac{2h_{L221} - h_{2s_2} - h}{p_{221s_2} v_{221}^2 \sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \\ &+ \frac{1}{p_{221s_2} v_{222}^2} \left(\sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} - \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \right) \quad (3.63) \\ &= \frac{\sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2}}{p_{221s_2} v_{222}^2} + \frac{1}{p_{221s_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right) \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \\ &= \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{221s_2} v_{222}^2} + \frac{1}{p_{221s_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right) \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \end{aligned}$$

- PmP arrival time $f_{221}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2})$, where $\mathbf{w} = (\mathbf{x}, t) = (x, y, h, t)$ is the guessing hypocenter and the guessing origin time for the calculated arrival time:

$$\begin{aligned} & f_{221}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2}) \\ &= t + g_{221}(\mathbf{x}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2}) \\ &= t + \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{221s_2} v_{222}^2} + \frac{1}{p_{221s_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right) \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}}\right)^2 - 1}} \quad (3.64) \end{aligned}$$

Case 2: The hypocenter is at layer 2

- Ray Path Segment 1 ($\mathbf{b}_1$): Does not exist, because the Earthquake Hypocenter is located at Layer 2.

- Ray Path Segment 2 ($\mathbf{b}_2$):

$$\|\mathbf{b}_{22}\| = h_{L222} - h, \quad \sin \theta_{222s_2} = \frac{\|\mathbf{b}_{21}\|}{\|\mathbf{b}_2\|} \quad (3.65)$$

- By Reflection Law, Ray Path Segment 3 ($\mathbf{b}_3$):

$$\|\mathbf{b}_{32}\| = h_{L222} - h_{L221}, \quad \sin \theta_{222s_2} = \frac{\|\mathbf{b}_{31}\|}{\|\mathbf{b}_3\|} = \frac{\|\mathbf{b}_{21}\|}{\|\mathbf{b}_2\|} \quad (3.66)$$

- Ray Path Segment 4 ($\mathbf{b}_4$):

$$\|\mathbf{b}_{42}\| = h_{L221} - h_{2s_2}, \quad \sin \theta_{221s_2} = \frac{\|\mathbf{b}_{41}\|}{\|\mathbf{b}_4\|} \quad (3.67)$$

- Snell's Law:

$$\frac{v_{221}}{v_{222}} = \frac{\sin \theta_{221s_2}}{\sin \theta_{222s_2}} \quad (3.68)$$

- Ray Parameter:

$$p_{222s_2} = \frac{\sin \theta_{222s_2}}{v_{222}} = \frac{\sin \theta_{221s_2}}{v_{221}} \quad (3.69)$$

$$p_{222s_2} = \frac{\|\mathbf{b}_{21}\|}{v_{222} \|\mathbf{b}_2\|} = \frac{\|\mathbf{b}_{31}\|}{v_{222} \|\mathbf{b}_3\|} = \frac{\|\mathbf{b}_{41}\|}{v_{221} \|\mathbf{b}_4\|} \quad (3.70)$$

$$\frac{1}{p_{222s_2}} = \frac{v_{222} \|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} = \frac{v_{222} \|\mathbf{b}_3\|}{\|\mathbf{b}_{31}\|} = \frac{v_{221} \|\mathbf{b}_4\|}{\|\mathbf{b}_{41}\|} \quad (3.71)$$

- Segments of Epicentral Distance:

$$\begin{aligned} \|\mathbf{b}_{21}\| &= \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{22}\|} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\sqrt{\|\mathbf{b}_2\|^2 - \|\mathbf{b}_{21}\|^2}} = \frac{\|\mathbf{b}_{21}\| \|\mathbf{b}_{22}\|}{\|\mathbf{b}_{21}\| \sqrt{\left(\frac{\|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|}\right)^2 - 1}} \\ &= \frac{h_{L222} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \end{aligned} \quad (3.72)$$

$$\begin{aligned}
\|\mathbf{b}_{31}\| &= \frac{\|\mathbf{b}_{31}\| \|\mathbf{b}_{32}\|}{\|\mathbf{b}_{32}\|} = \frac{\|\mathbf{b}_{31}\| \|\mathbf{b}_{32}\|}{\sqrt{\|\mathbf{b}_3\|^2 - \|\mathbf{b}_{31}\|^2}} = \frac{\|\mathbf{b}_{31}\| \|\mathbf{b}_{32}\|}{\|\mathbf{b}_{31}\| \sqrt{\left(\frac{\|\mathbf{b}_3\|}{\|\mathbf{b}_{31}\|}\right)^2 - 1}} \\
&= \frac{h_{L222} - h_{L221}}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}}
\end{aligned} \tag{3.73}$$

$$\begin{aligned}
\|\mathbf{b}_{41}\| &= \frac{\|\mathbf{b}_{41}\| \|\mathbf{b}_{42}\|}{\|\mathbf{b}_{42}\|} = \frac{\|\mathbf{b}_{41}\| \|\mathbf{b}_{42}\|}{\sqrt{\|\mathbf{b}_4\|^2 - \|\mathbf{b}_{41}\|^2}} = \frac{\|\mathbf{b}_{41}\| \|\mathbf{b}_{42}\|}{\|\mathbf{b}_{41}\| \sqrt{\left(\frac{\|\mathbf{b}_4\|}{\|\mathbf{b}_{41}\|}\right)^2 - 1}} \\
&= \frac{h_{L221} - h_{2s_2}}{\sqrt{\left(\frac{1}{p_{222s_2} v_{221}}\right)^2 - 1}}
\end{aligned} \tag{3.74}$$

- Epicentral Distance:

$$\begin{aligned}
d_{222}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) &= \frac{h_{L221} - h_{2s_2}}{\sqrt{\left(\frac{1}{p_{222s_2} v_{221}}\right)^2 - 1}} + \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \\
&= \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} \\
&= \|\mathbf{r}_{2s_2} - \mathbf{r}\|
\end{aligned} \tag{3.75}$$

- Segments of travel time:

$$\frac{1}{p_{222s_2}} = \frac{v_{222} \|\mathbf{b}_2\|}{\|\mathbf{b}_{21}\|} = \frac{v_{222} \|\mathbf{b}_3\|}{\|\mathbf{b}_{31}\|} = \frac{v_{221} \|\mathbf{b}_4\|}{\|\mathbf{b}_{41}\|} \tag{3.76}$$

$$\frac{1}{p_{222s_2}} = \frac{v_{222}^2 G_2}{\|\mathbf{b}_{21}\|} = \frac{v_{222}^2 G_3}{\|\mathbf{b}_{31}\|} = \frac{v_{221}^2 G_4}{\|\mathbf{b}_{41}\|} \tag{3.77}$$

$$G_2 = \frac{\|\mathbf{b}_{21}\|}{p_{222s_2} v_{222}^2} = \frac{h_{L222} - h}{p_{222s_2} v_{222}^2 \sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \tag{3.78}$$

$$G_3 = \frac{\|\mathbf{b}_{31}\|}{p_{222s_2} v_{222}^2} = \frac{h_{L222} - h_{L221}}{p_{222s_2} v_{222}^2 \sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \tag{3.79}$$

$$G_4 = \frac{\|\mathbf{b}_{41}\|}{p_{222s_2} v_{221}^2} = \frac{h_{L221} - h_{2s_2}}{p_{222s_2} v_{221}^2 \sqrt{\left(\frac{1}{p_{222s_2} v_{221}}\right)^2 - 1}} \quad (3.80)$$

- PmP travel time $g_{222}(\mathbf{x}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2})$, from the hypocenter $\mathbf{x} = (\mathbf{r}, h) = (x, y, h)$ to the seismic station $\mathbf{x}_{2s_2} = (\mathbf{r}_{2s_2}, h_{2s_2}) = (x_{2s_2}, y_{2s_2}, h_{2s_2})$:

$$\begin{aligned} & g_{222}(\mathbf{x}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) \\ &= \frac{h_{L221} - h_{2s_2}}{p_{222s_2} v_{221}^2 \sqrt{\left(\frac{1}{p_{222s_2} v_{221}}\right)^2 - 1}} + \frac{2h_{L222} - h_{L221} - h}{p_{222s_2} v_{222}^2 \sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \\ &= \frac{1}{p_{222s_2} v_{221}^2} \left(\sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} - \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \right) \\ &\quad + \frac{2h_{L222} - h_{L221} - h}{p_{222s_2} v_{222}^2 \sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \quad (3.81) \\ &= \frac{\sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2}}{p_{222s_2} v_{221}^2} + \frac{1}{p_{222s_2}} \left(\frac{1}{v_{222}^2} - \frac{1}{v_{221}^2} \right) \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \\ &= \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{222s_2} v_{221}^2} + \frac{1}{p_{222s_2}} \left(\frac{1}{v_{222}^2} - \frac{1}{v_{221}^2} \right) \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \end{aligned}$$

- PmP arrival time $f_{222}(\mathbf{w}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2})$, where $\mathbf{w} = (\mathbf{x}, t) = (x, y, h, t)$ is the guessing hypocenter and the guessing origin time for the calculated arrival time.

$$\begin{aligned} & f_{222}(\mathbf{w}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) \\ &= t + g_{222}(\mathbf{x}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) \\ &= t + \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{222s_2} v_{221}^2} + \frac{1}{p_{222s_2}} \left(\frac{1}{v_{222}^2} - \frac{1}{v_{221}^2} \right) \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \quad (3.82) \end{aligned}$$

□

3.2 Algorithms of Geiger's Method

The Geiger's Method is adapted from Analysis of Source Location Algorithms by Maochen Ge [9], where he described Geiger's Method for Cartesian Coordinate only. However, in this thesis, the Geiger's Method is modified to accommodate Latitude-Longitude and Depth coordinate as well.

Algorithm 3.1 (General Description of Geiger's Method).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.83)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max \|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.84)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.85)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.86)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 5: Execute **Step 5.1** to **Step 5.4**.

- **Step 5.1:** Which Velocity Model? If P wave, $i = 1$, else if PmP wave, $i = 2$.
- **Step 5.2:** Number of Layers? If 1 layer, $j = 1$, else if 2 layers, $j = 2$.
- **Step 5.3:** If $j \equiv 1$, then the velocity model and discontinuity are:

$$\mathbf{v}_{ij} = (v_{ij1}) \quad (3.87)$$

$$\mathbf{h}_{Lij} = (h_{Lij1}) \quad (3.88)$$

Else, if $j \equiv 2$, then the velocity model and discontinuities are:

$$\mathbf{v}_{ij} = (v_{ij1}, v_{ij2}) \quad (3.89)$$

$$\mathbf{h}_{Lij} = (h_{Lij1}, h_{Lij2}) \quad (3.90)$$

• **Step 5.4:** Go to **Step 6**.

Step 6: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\mathbf{Y}_i = \begin{pmatrix} \mathbf{y}_{i1} \\ \vdots \\ \mathbf{y}_{is_i} \\ \vdots \\ \mathbf{y}_{in_i} \end{pmatrix} = \begin{pmatrix} \phi_{i1} & \lambda_{i1} & h_{i1} \\ \vdots & \vdots & \vdots \\ \phi_{is_i} & \lambda_{is_i} & h_{is_i} \\ \vdots & \vdots & \vdots \\ \phi_{in_i} & \lambda_{in_i} & h_{in_i} \end{pmatrix} = \begin{pmatrix} \frac{\pi \phi_{i1}^\circ}{180} & \frac{\pi \lambda_{i1}^\circ}{180} & h_{i1} \\ \vdots & \vdots & \vdots \\ \frac{\pi \phi_{is_i}^\circ}{180} & \frac{\pi \lambda_{is_i}^\circ}{180} & h_{is_i} \\ \vdots & \vdots & \vdots \\ \frac{\pi \phi_{in_i}^\circ}{180} & \frac{\pi \lambda_{in_i}^\circ}{180} & h_{in_i} \end{pmatrix} \in \mathbb{R}^{n_i \times 3}$$

$$\mathbf{Y}_i = \begin{pmatrix} \Phi_i & \mathbf{h}_i \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_i^\circ & \mathbf{h}_i \end{pmatrix} \in \mathbb{R}^{n_i \times 3}, \quad \Phi_i \in \mathbb{R}^{n_i \times 2}, \quad \mathbf{h}_i = \mathbf{0}_{n_i \times 1} \in \mathbb{R}^{n_i \times 1} \quad (3.91)$$

Step 7: Seismic Stations in Cartesian Coordinates:

$$\begin{aligned}
 \mathbf{X}_i &= \begin{pmatrix} \mathbf{x}_{i1} \\ \vdots \\ \mathbf{x}_{is_i} \\ \vdots \\ \mathbf{x}_{in_i} \end{pmatrix} = \begin{pmatrix} x_{i1} & y_{i1} & h_{i1} \\ \vdots & \vdots & \vdots \\ x_{is_i} & y_{is_i} & h_{is_i} \\ \vdots & \vdots & \vdots \\ x_{in_i} & y_{in_i} & h_{in_i} \end{pmatrix} \in \mathbb{R}^{n_i \times 3} \\
 \mathbf{X}_i &= \begin{pmatrix} R_E \cos \phi_{i1} \cos \lambda_{i1} & R_E \cos \phi_{i1} \sin \lambda_{i1} & h_{i1} \\ \vdots & \vdots & \vdots \\ R_E \cos \phi_{is_i} \cos \lambda_{is_i} & R_E \cos \phi_{is_i} \sin \lambda_{is_i} & h_{is_i} \\ \vdots & \vdots & \vdots \\ R_E \cos \phi_{in_i} \cos \lambda_{in_i} & R_E \cos \phi_{in_i} \sin \lambda_{in_i} & h_{in_i} \end{pmatrix} \in \mathbb{R}^{n_i \times 3} \quad (3.92) \\
 \mathbf{X}_i &= \begin{pmatrix} \mathbf{r}_{i1} & h_{i1} \\ \vdots & \vdots \\ \mathbf{r}_{is_i} & h_{is_i} \\ \vdots & \vdots \\ \mathbf{r}_{in_i} & h_{in_i} \end{pmatrix} \in \mathbb{R}^{n_i \times 3}, \quad \mathbf{R}_i \in \mathbb{R}^{n_i \times 2}, \quad \mathbf{h}_i \in \mathbb{R}^{n_i \times 1}
 \end{aligned}$$

Step 8: Observed arrival time for each seismic station with an unknown hypocenter and origin time:

$$\mathbf{f}_{0i} = \begin{pmatrix} f_{0i1} \\ \vdots \\ f_{0is_i} \\ \vdots \\ f_{0in_i} \end{pmatrix} \in \mathbb{R}^{n_i \times 1} \quad (3.93)$$

Step 9: Execute **Step 9.1** to **Step 9.3**.

- **Step 9.1:** Which k^{th} layer is the hypocenter at? $k = 1$ or $k = 2$
- **Step 9.2:** If $i \equiv 1$, $j \equiv 2$ and $k \equiv 2$, or $i \equiv 2$ and $j \equiv 2$, then find the

following ray parameter for each seismic station:

$$\mathbf{0}_{n_i \times 1} < \mathbf{p}_{ijk} = \begin{pmatrix} p_{ijk1} \\ \vdots \\ p_{ijk s_i} \\ \vdots \\ p_{ijk n_i} \end{pmatrix} = \frac{1}{v_{ijk}} \begin{pmatrix} \sin \theta_{ijk1} \\ \vdots \\ \sin \theta_{ijk s_i} \\ \vdots \\ \sin \theta_{ijk n_i} \end{pmatrix} < \frac{1}{v_{ijk}} \mathbf{1}_{n_i \times 1} \in \mathbb{R}^{n_i \times 1} \quad (3.94)$$

satisfying the epicentral distance of each seismic station, that is distance between the x-y components of epicenter, $\mathbf{r} = (x, y)$, and x-y components of each seismic station, $\mathbf{r}_{is_i} = (x_{is_i}, y_{is_i})$:

$$\mathbf{d}(\mathbf{r}, \mathbf{R}_i) = \begin{pmatrix} \|\mathbf{r}_{i1} - \mathbf{r}\| \\ \vdots \\ \|\mathbf{r}_{is_i} - \mathbf{r}\| \\ \vdots \\ \|\mathbf{r}_{in_i} - \mathbf{r}\| \end{pmatrix} \in \mathbb{R}^{n_i \times 1}, \quad \|\mathbf{r}_{is_i} - \mathbf{r}\| = \sqrt{(x_{is_i} - x)^2 + (y_{is_i} - y)^2} \quad (3.95)$$

$$\mathbf{d}_{ijk}(h, \mathbf{h}_i, \mathbf{h}_{Lij}, \mathbf{v}_{ij}, \mathbf{p}_{ijk}) = \begin{pmatrix} d_{ijk}(h, h_{i1}, \mathbf{h}_{Lij}, \mathbf{v}_{ij}, p_{ijk1}) \\ \vdots \\ d_{ijk}(h, h_{is_i}, \mathbf{h}_{Lij}, \mathbf{v}_{ij}, p_{ijk s_i}) \\ \vdots \\ d_{ijk}(h, h_{in_i}, \mathbf{h}_{Lij}, \mathbf{v}_{ij}, p_{ijk n_i}) \end{pmatrix} \in \mathbb{R}^{n_i \times 1} \quad (3.96)$$

$$\mathbf{d}_{ijk}(h, \mathbf{h}_i, \mathbf{h}_{Lij}, \mathbf{v}_{ij}, \mathbf{p}_{ijk}) \equiv \mathbf{d}(\mathbf{r}, \mathbf{R}_i) \in \mathbb{R}^{n_i \times 1} \quad (3.97)$$

by applying Bisection Method, and go to **Step 9.3**. Else, skip to **Step 9.3**.

• **Step 9.3:** Go to **Step 10**.

Step 10: Calculated arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$, without loss of generality, for

simplicity:

$$\mathbf{f}_{ijk}(\mathbf{w}, \mathbf{X}_i) = \begin{pmatrix} f_{ijk}(\mathbf{w}, \mathbf{x}_{i1}) \\ \vdots \\ f_{ijk}(\mathbf{w}, \mathbf{x}_{is_i}) \\ \vdots \\ f_{ijk}(\mathbf{w}, \mathbf{x}_{in_i}) \end{pmatrix} \in \mathbb{R}^{n_i \times 1} \quad (3.98)$$

Step 11: Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{ijk} = \mathbf{f}_{0i} - \mathbf{f}_{ijk}(\mathbf{w}, \mathbf{X}_i) = \begin{pmatrix} f_{0i1} - f_{ijk}(\mathbf{w}, \mathbf{x}_{i1}) \\ \vdots \\ f_{0is_i} - f_{ijk}(\mathbf{w}, \mathbf{x}_{is_i}) \\ \vdots \\ f_{0in_i} - f_{ijk}(\mathbf{w}, \mathbf{x}_{in_i}) \end{pmatrix} \in \mathbb{R}^{n_i \times 1} \quad (3.99)$$

Step 12: Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\begin{aligned} & \frac{\partial \mathbf{f}_{ijk}(\mathbf{w}, \mathbf{X}_i)}{\partial \mathbf{w}} \\ &= \begin{pmatrix} \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{i1})}{\partial x} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{i1})}{\partial y} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{i1})}{\partial h} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{i1})}{\partial t} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{is_i})}{\partial x} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{is_i})}{\partial y} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{is_i})}{\partial h} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{is_i})}{\partial t} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{in_i})}{\partial x} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{in_i})}{\partial y} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{in_i})}{\partial h} & \frac{\partial f_{ijk}(\mathbf{w}, \mathbf{x}_{in_i})}{\partial t} \end{pmatrix} \in \mathbb{R}^{n_i \times 4} \end{aligned} \quad (3.100)$$

Step 13: Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} &= \begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial h} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial h} & \frac{\partial y}{\partial t} \\ \frac{\partial h}{\partial \phi} & \frac{\partial h}{\partial \lambda} & \frac{\partial h}{\partial h} & \frac{\partial h}{\partial t} \\ \frac{\partial t}{\partial \phi} & \frac{\partial t}{\partial \lambda} & \frac{\partial t}{\partial h} & \frac{\partial t}{\partial t} \end{pmatrix} = \begin{pmatrix} -R_E \sin \phi \cos \lambda & -y & 0 & 0 \\ -R_E \sin \phi \sin \lambda & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{4 \times 4} \\ \frac{\partial \mathbf{w}}{\partial \mathbf{u}} &= \begin{pmatrix} -R_E \sin \phi \cos \lambda & -R_E \cos \phi \sin \lambda & 0 & 0 \\ -R_E \sin \phi \sin \lambda & R_E \cos \phi \cos \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{4 \times 4} \end{aligned} \quad (3.101)$$

Step 14: Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{ijk} = \frac{\partial \mathbf{f}_{ijk}(\mathbf{w}, \mathbf{X}_i)}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_i \times 4} \quad (3.102)$$

Step 15: Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{ijk} \Delta \mathbf{u} = \Delta \mathbf{f}_{ijk} \quad (3.103)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{ijk}^T \mathbf{A}_{ijk} + \alpha \mathbf{I})^{-1} \mathbf{A}_{ijk}^T \Delta \mathbf{f}_{ijk} \quad (3.104)$$

Step 16: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned}
 x &= R_E \cos(\phi + \Delta\phi) \cos(\lambda + \Delta\lambda) \\
 y &= R_E \cos(\phi + \Delta\phi) \sin(\lambda + \Delta\lambda) \\
 h &= h + \Delta h \\
 t &= t + \Delta t \\
 \mathbf{w} &= (x, y, h, t)
 \end{aligned} \tag{3.105}$$

Step 17: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned}
 \phi &= \arcsin\left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E}\right) \\
 \lambda &= \text{atan2}(y, x) \\
 \mathbf{u} &= (\phi, \lambda, h, t)
 \end{aligned} \tag{3.106}$$

Step 18: Convert change in latitude and longitude from radian to degree:

$$(\Delta\phi^\circ, \Delta\lambda^\circ) = \left(\frac{180}{\pi}\Delta\phi, \frac{180}{\pi}\Delta\lambda\right) \tag{3.107}$$

Step 19: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 19.1** to **Step 19.2**. Else, skip to **Step 20**.

• **Step 19.1:** Output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi}\phi, \frac{180}{\pi}\lambda, h, t\right) \tag{3.108}$$

• **Step 19.2:** Terminate.

Step 20: Repeat **Step 9** to **Step 19**.

Algorithm 3.2 (Geiger's Method: 1 layer P wave velocity model).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.109)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max \|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.110)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Discontinuity of 1 layer P wave velocity model: h_{L111}

Step 5: 1 layer P wave velocity model: v_{111}

Step 6: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.111)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.112)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 7: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\mathbf{Y}_1 = \begin{pmatrix} \Phi_1 & \mathbf{h}_1 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_1^\circ & \mathbf{h}_1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 3}, \quad \Phi_1^\circ \in \mathbb{R}^{n_1 \times 2}, \quad \mathbf{h}_1 = \mathbf{0}_{n_1 \times 1} \in \mathbb{R}^{n_1 \times 1} \quad (3.113)$$

Step 8: Seismic Stations in Cartesian Coordinates:

$$\mathbf{X}_1 = \begin{pmatrix} \mathbf{R}_1 & \mathbf{h}_1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 3}, \quad \mathbf{R}_1 \in \mathbb{R}^{n_1 \times 2} \quad (3.114)$$

Step 9: Observed P arrival time for each seismic station with an unknown hypocenter and origin time:

$$\mathbf{f}_{01} \in \mathbb{R}^{n_1 \times 1} \quad (3.115)$$

Step 10: If $0 \leq h \leq h_{L111}$, then execute **Step 10.1** to **Step 10.7**. Else, terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

• **Step 10.1:** Calculated P arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111}) = \begin{pmatrix} f_{111}(\mathbf{w}, \mathbf{x}_{11}, v_{111}) \\ \vdots \\ f_{111}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{111}) \\ \vdots \\ f_{111}(\mathbf{w}, \mathbf{x}_{1n_1}, v_{111}) \end{pmatrix} \in \mathbb{R}^{n_1 \times 1} \quad (3.116)$$

where:

$$f_{111}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{111}) = t + \frac{1}{v_{111}} \|\mathbf{x}_{1s_1} - \mathbf{x}\| \quad (3.117)$$

$$\|\mathbf{x}_{1s_1} - \mathbf{x}\| = \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2 + (h_{1s_1} - h)^2} \quad (3.118)$$

• **Step 10.2:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{111} = \mathbf{f}_{01} - \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111}) \in \mathbb{R}^{n_1 \times 1} \quad (3.119)$$

• **Step 10.3:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\begin{aligned} & \frac{\partial \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111})}{\partial \mathbf{w}} \\ &= \begin{pmatrix} \frac{x - x_{11}}{v_{111} \|\mathbf{x}_{11} - \mathbf{x}\|} & \frac{y - y_{11}}{v_{111} \|\mathbf{x}_{11} - \mathbf{x}\|} & \frac{h - h_{11}}{v_{111} \|\mathbf{x}_{11} - \mathbf{x}\|} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{1s_1}}{v_{111} \|\mathbf{x}_{1s_1} - \mathbf{x}\|} & \frac{y - y_{1s_1}}{v_{111} \|\mathbf{x}_{1s_1} - \mathbf{x}\|} & \frac{h - h_{1s_1}}{v_{111} \|\mathbf{x}_{1s_1} - \mathbf{x}\|} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{1n_1}}{v_{111} \|\mathbf{x}_{1n_1} - \mathbf{x}\|} & \frac{y - y_{1n_1}}{v_{111} \|\mathbf{x}_{1n_1} - \mathbf{x}\|} & \frac{h - h_{1n_1}}{v_{111} \|\mathbf{x}_{1n_1} - \mathbf{x}\|} & 1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 4} \end{aligned} \quad (3.120)$$

- **Step 10.4:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.121)$$

- **Step 10.5:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{111} = \frac{\partial \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_1 \times 4} \quad (3.122)$$

- **Step 10.6:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{111} \Delta \mathbf{u} = \Delta \mathbf{f}_{111} \quad (3.123)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{111}^T \mathbf{A}_{111} + \alpha \mathbf{I})^{-1} \mathbf{A}_{111}^T \Delta \mathbf{f}_{111} \quad (3.124)$$

- **Step 10.7:** Go to **Step 11**.

Step 11: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned} x &= R_E \cos(\phi + \Delta \phi) \cos(\lambda + \Delta \lambda) \\ y &= R_E \cos(\phi + \Delta \phi) \sin(\lambda + \Delta \lambda) \\ h &= h + \Delta h \\ t &= t + \Delta t \\ \mathbf{w} &= (x, y, h, t) \end{aligned} \quad (3.125)$$

Step 12: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned} \phi &= \arcsin \left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E} \right) \\ \lambda &= \text{atan2}(y, x) \\ \mathbf{u} &= (\phi, \lambda, h, t) \end{aligned} \quad (3.126)$$

Step 13: Convert change in latitude and longitude from radian to degree:

$$(\Delta\phi^\circ, \Delta\lambda^\circ) = \left(\frac{180}{\pi} \Delta\phi, \frac{180}{\pi} \Delta\lambda \right) \quad (3.127)$$

Step 14: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 14.1** to **Step 14.2**. Else, skip to **Step 15**.

• **Step 14.1:** If $0 \leq h \leq h_{L122}$ and $t \geq 0$, then output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi} \phi, \frac{180}{\pi} \lambda, h, t \right) \quad (3.128)$$

Else, declare that the value of the depth of the guessing hypocenter, h , or the value of the guessing origin time, t , or both, is/are invalid.

• **Step 14.2:** Terminate.

Step 15: Repeat **Step 10** to **Step 14**.

Algorithm 3.3 (Geiger's Method: 1 layer PmP wave velocity model).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.129)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.130)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Discontinuity of 1 layer PmP wave velocity model:

$$h_{L211} = h_{L111} \quad (3.131)$$

Step 5: 1 layer PmP wave velocity model:

$$v_{211} = v_{111} \quad (3.132)$$

Step 6: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.133)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.134)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 7: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\mathbf{Y}_2 = \begin{pmatrix} \Phi_2 & \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_2^\circ & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{n_2 \times 3}, \quad \Phi_2^\circ \in \mathbb{R}^{n_2 \times 2}, \quad \mathbf{h}_2 = \mathbf{0}_{n_2 \times 1} \in \mathbb{R}^{n_2 \times 1} \quad (3.135)$$

Step 8: Seismic Stations in Cartesian Coordinates:

$$\mathbf{X}_2 = \begin{pmatrix} \mathbf{R}_2 & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{n_2 \times 3}, \quad \mathbf{R}_2 \in \mathbb{R}^{n_2 \times 2} \quad (3.136)$$

Step 9: Observed PmP arrival time for each seismic station with an unknown hypocenter and origin time:

$$\mathbf{f}_{02} \in \mathbb{R}^{n_2 \times 1} \quad (3.137)$$

Step 10: If $0 \leq h < h_{L211}$, then execute **Step 10.1** to **Step 10.7**. Else, terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

- **Step 10.1:** Calculated PmP arrival time for each seismic station with the

guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211}) = \begin{pmatrix} f_{211}(\mathbf{w}, \mathbf{x}_{21}, h_{L211}, v_{211}) \\ \vdots \\ f_{211}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L211}, v_{211}) \\ \vdots \\ f_{211}(\mathbf{w}, \mathbf{x}_{2n_2}, h_{L211}, v_{211}) \end{pmatrix} \in \mathbb{R}^{n_2 \times 1} \quad (3.138)$$

where:

$$f_{211}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L211}, v_{211}) = t + \frac{1}{v_{211}} \|\mathbf{c}_{211s_2} - \mathbf{x}\| \quad (3.139)$$

$$\mathbf{c}_{211s_2} = (x_{2s_2}, y_{2s_2}, 2h_{L211} - h_{2s_2}) \quad (3.140)$$

$$\|\mathbf{c}_{211s_2} - \mathbf{x}\| = \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2 + (2h_{L211} - h_{2s_2} - h)^2} \quad (3.141)$$

• **Step 10.2:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{211} = \mathbf{f}_{02} - \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211}) \in \mathbb{R}^{n_2 \times 1} \quad (3.142)$$

• **Step 10.3:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\begin{aligned} & \frac{\partial \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211})}{\partial \mathbf{w}} \\ &= \begin{pmatrix} \frac{x - x_{21}}{v_{211} \|\mathbf{c}_{2111} - \mathbf{x}\|} & \frac{y - y_{21}}{v_{211} \|\mathbf{c}_{2111} - \mathbf{x}\|} & \frac{h + h_{21} - 2h_{L211}}{v_{211} \|\mathbf{c}_{2111} - \mathbf{x}\|} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{2s_2}}{v_{211} \|\mathbf{c}_{211s_2} - \mathbf{x}\|} & \frac{y - y_{2s_2}}{v_{211} \|\mathbf{c}_{211s_2} - \mathbf{x}\|} & \frac{h + h_{2s_2} - 2h_{L211}}{v_{211} \|\mathbf{c}_{211s_2} - \mathbf{x}\|} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{2n_2}}{v_{211} \|\mathbf{c}_{211n_2} - \mathbf{x}\|} & \frac{y - y_{2n_2}}{v_{211} \|\mathbf{c}_{211n_2} - \mathbf{x}\|} & \frac{h + h_{2n_2} - 2h_{L211}}{v_{211} \|\mathbf{c}_{211n_2} - \mathbf{x}\|} & 1 \end{pmatrix} \in \mathbb{R}^{n_2 \times 4} \end{aligned} \quad (3.143)$$

• **Step 10.4:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.144)$$

- **Step 10.5:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{211} = \frac{\partial \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_2 \times 4} \quad (3.145)$$

- **Step 10.6:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{211} \Delta \mathbf{u} = \Delta \mathbf{f}_{211} \quad (3.146)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{211}^T \mathbf{A}_{211} + \alpha \mathbf{I})^{-1} \mathbf{A}_{211}^T \Delta \mathbf{f}_{211}$$

- **Step 10.7:** Go to **Step 11**.

Step 11: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned} x &= R_E \cos(\phi + \Delta \phi) \cos(\lambda + \Delta \lambda) \\ y &= R_E \cos(\phi + \Delta \phi) \sin(\lambda + \Delta \lambda) \\ h &= h + \Delta h \\ t &= t + \Delta t \\ \mathbf{w} &= (x, y, h, t) \end{aligned} \quad (3.147)$$

Step 12: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned} \phi &= \arcsin \left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E} \right) \\ \lambda &= \text{atan2}(y, x) \\ \mathbf{u} &= (\phi, \lambda, h, t) \end{aligned} \quad (3.148)$$

Step 13: Convert change in latitude and longitude from radian to degree:

$$(\Delta \phi^\circ, \Delta \lambda^\circ) = \left(\frac{180}{\pi} \Delta \phi, \frac{180}{\pi} \Delta \lambda \right) \quad (3.149)$$

Step 14: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 14.1** to **Step 14.2**. Else, skip to **Step 15**.

• **Step 14.1:** If $0 \leq h \leq h_{L211}$ and $t \geq 0$, then output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi}\phi, \frac{180}{\pi}\lambda, h, t \right) \quad (3.150)$$

Else, declare that the value of the depth of the guessing hypocenter, h , or the value of the guessing origin time, t , or both, is/are invalid.

• **Step 14.2:** Terminate.

Step 15: Repeat **Step 10** to **Step 14**.

Algorithm 3.4 (Geiger's Method: 1 layer velocity model, P and PmP combined).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.151)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.152)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Discontinuity of 1 layer P and PmP wave velocity model:

$$h_{L211} = h_{L111} \quad (3.153)$$

Step 5: 1 layer P and PmP wave velocity model:

$$v_{211} = v_{111} \quad (3.154)$$

Step 6: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.155)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.156)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 7: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\begin{aligned} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} &= \begin{pmatrix} \Phi_1 & \mathbf{h}_1 \\ \Phi_2 & \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_1^\circ & \mathbf{h}_1 \\ \frac{\pi}{180} \Phi_2^\circ & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 3} \\ \begin{pmatrix} \Phi_1^\circ \\ \Phi_2^\circ \end{pmatrix} &\in \mathbb{R}^{(n_1+n_2) \times 2}, \quad \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n_1 \times 1} \\ \mathbf{0}_{n_2 \times 1} \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \end{aligned} \quad (3.157)$$

Step 8: Seismic Stations in Cartesian Coordinates:

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{h}_1 \\ \mathbf{R}_2 & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 3}, \quad \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 2} \quad (3.158)$$

Step 9: Observed P and PmP arrival time for each seismic station with an unknown hypocenter and origin time:

$$\begin{pmatrix} \mathbf{f}_{01} \\ \mathbf{f}_{02} \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.159)$$

Step 10: If $0 \leq h < h_{L111}$ and $0 \leq h < h_{L211}$, then execute **Step 10.1** to **Step 10.7**. Else, terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

• **Step 10.1:** Calculated P and PmP arrival time for each seismic station with

the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111}) \\ \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.160)$$

- **Step 10.2:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \Delta \mathbf{f}_{111} \\ \Delta \mathbf{f}_{211} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{01} \\ \mathbf{f}_{02} \end{pmatrix} - \begin{pmatrix} \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111}) \\ \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211}) \end{pmatrix} \in \mathbb{R}^{n_2 \times 1} \quad (3.161)$$

- **Step 10.3:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111}) \\ \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 4} \quad (3.162)$$

- **Step 10.4:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.163)$$

- **Step 10.5:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{211} = \frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{111}(\mathbf{w}, \mathbf{X}_1, v_{111}) \\ \mathbf{f}_{211}(\mathbf{w}, \mathbf{X}_2, h_{L211}, v_{211}) \end{pmatrix} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{(n_1+n_2) \times 4}$$

- **Step 10.6:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \begin{pmatrix} \mathbf{A}_{111} \\ \mathbf{A}_{211} \end{pmatrix} \Delta \mathbf{u} = \begin{pmatrix} \Delta \mathbf{f}_{111} \\ \Delta \mathbf{f}_{211} \end{pmatrix} \quad (3.164)$$

$$\Delta \mathbf{u} = \left(\begin{pmatrix} \mathbf{A}_{111} \\ \mathbf{A}_{211} \end{pmatrix}^T \begin{pmatrix} \mathbf{A}_{111} \\ \mathbf{A}_{211} \end{pmatrix} + \alpha \mathbf{I} \right)^{-1} \begin{pmatrix} \mathbf{A}_{111} \\ \mathbf{A}_{211} \end{pmatrix}^T \begin{pmatrix} \Delta \mathbf{f}_{111} \\ \Delta \mathbf{f}_{211} \end{pmatrix} \quad (3.165)$$

- **Step 10.7:** Go to **Step 11**.

Step 11: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned}
 x &= R_E \cos(\phi + \Delta\phi) \cos(\lambda + \Delta\lambda) \\
 y &= R_E \cos(\phi + \Delta\phi) \sin(\lambda + \Delta\lambda) \\
 h &= h + \Delta h \\
 t &= t + \Delta t \\
 \mathbf{w} &= (x, y, h, t)
 \end{aligned} \tag{3.166}$$

Step 12: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned}
 \phi &= \arcsin\left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E}\right) \\
 \lambda &= \text{atan2}(y, x) \\
 \mathbf{u} &= (\phi, \lambda, h, t)
 \end{aligned} \tag{3.167}$$

Step 13: Convert change in latitude and longitude from radian to degree:

$$(\Delta\phi^\circ, \Delta\lambda^\circ) = \left(\frac{180}{\pi}\Delta\phi, \frac{180}{\pi}\Delta\lambda\right) \tag{3.168}$$

Step 14: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 14.1** to **Step 14.2**. Else, skip to **Step 15**.

• **Step 14.1:** If $0 \leq h \leq h_{L111}$ and $0 \leq h \leq h_{L211}$ and $t \geq 0$, then output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi}\phi, \frac{180}{\pi}\lambda, h, t\right) \tag{3.169}$$

Else, declare that the value of the depth of the guessing hypocenter, h , or the value of the guessing origin time, t , or both, is/are invalid.

• **Step 14.2:** Terminate.

Step 15: Repeat **Step 10** to **Step 14**.

Algorithm 3.5 (Geiger's Method: 2 layer P wave velocity model).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.170)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max \|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.171)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Discontinuities of 2 layer P wave velocity model:

$$\mathbf{h}_{L12} = (h_{L121}, h_{L122}) \quad (3.172)$$

Step 5: 2 layer P wave velocity model:

$$\mathbf{v}_{12} = (v_{121}, v_{122}) \quad (3.173)$$

Step 6: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.174)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.175)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 7: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\mathbf{Y}_1 = \begin{pmatrix} \Phi_1 & \mathbf{h}_1 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_1^\circ & \mathbf{h}_1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 3}, \quad \Phi_1^\circ \in \mathbb{R}^{n_1 \times 2}, \quad \mathbf{h}_1 = \mathbf{0}_{n_1 \times 1} \in \mathbb{R}^{n_1 \times 1} \quad (3.176)$$

Step 8: Seismic Stations in Cartesian Coordinates:

$$\mathbf{X}_1 = \begin{pmatrix} \mathbf{R}_1 & \mathbf{h}_1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 3}, \quad \mathbf{R}_1 \in \mathbb{R}^{n_1 \times 2} \quad (3.177)$$

Step 9: Observed P arrival time for each seismic station with an unknown hypocenter and origin time:

$$\mathbf{f}_{01} \in \mathbb{R}^{n_1 \times 1} \quad (3.178)$$

Step 10: If $0 \leq h \leq h_{L121}$, then execute **Step 10.1** to **Step 10.7**. Else, if $h > h_{L121}$, then skip to **Step 11**. Otherwise, $h < 0$, then terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

• **Step 10.1:** Calculated P arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \in \mathbb{R}^{n_1 \times 1} \quad (3.179)$$

where:

$$f_{121}(\mathbf{w}, \mathbf{x}_{1s_1}, v_{121}) = t + \frac{1}{v_{121}} \|\mathbf{x}_{1s_1} - \mathbf{x}\| \quad (3.180)$$

$$\|\mathbf{x}_{1s_1} - \mathbf{x}\| = \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2 + (h_{1s_1} - h)^2} \quad (3.181)$$

• **Step 10.2:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{121} = \mathbf{f}_{01} - \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \in \mathbb{R}^{n_1 \times 1} \quad (3.182)$$

• **Step 10.3:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\begin{aligned} & \frac{\partial \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121})}{\partial \mathbf{w}} \\ &= \begin{pmatrix} \frac{x - x_{11}}{v_{121} \|\mathbf{x}_{11} - \mathbf{x}\|} & \frac{y - y_{11}}{v_{121} \|\mathbf{x}_{11} - \mathbf{x}\|} & \frac{h - h_{11}}{v_{121} \|\mathbf{x}_{11} - \mathbf{x}\|} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{1s_1}}{v_{121} \|\mathbf{x}_{1s_1} - \mathbf{x}\|} & \frac{y - y_{1s_1}}{v_{121} \|\mathbf{x}_{1s_1} - \mathbf{x}\|} & \frac{h - h_{1s_1}}{v_{121} \|\mathbf{x}_{1s_1} - \mathbf{x}\|} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{1n_1}}{v_{121} \|\mathbf{x}_{1n_1} - \mathbf{x}\|} & \frac{y - y_{1n_1}}{v_{121} \|\mathbf{x}_{1n_1} - \mathbf{x}\|} & \frac{h - h_{1n_1}}{v_{121} \|\mathbf{x}_{1n_1} - \mathbf{x}\|} & 1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 4} \end{aligned} \quad (3.183)$$

- **Step 10.4:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.184)$$

- **Step 10.5:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{121} = \frac{\partial \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_1 \times 4} \quad (3.185)$$

- **Step 10.6:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{121} \Delta \mathbf{u} = \Delta \mathbf{f}_{121} \quad (3.186)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{121}^T \mathbf{A}_{121} + \alpha \mathbf{I})^{-1} \mathbf{A}_{121}^T \Delta \mathbf{f}_{121} \quad (3.187)$$

- **Step 10.7:** Skip to **Step 12**.

Step 11: If $h_{L121} < h \leq h_{L122}$, then execute **Step 11.1** to **Step 11.8**. Else, if $h > h_{L122}$, then terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

- **Step 11.1:** Find ray parameter, $\mathbf{p}_{122} \in \mathbb{R}^{n_1 \times 1}$, by applying Bisection Method on Epicentral Distance, such that:

$$\mathbf{d}_{122}(h, \mathbf{h}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \equiv \mathbf{d}(\mathbf{r}, \mathbf{R}_1) \in \mathbb{R}^{n_1 \times 1} \quad (3.188)$$

$$\mathbf{0}_{n_1 \times 1} < \mathbf{p}_{122} < \frac{1}{v_{122}} \mathbf{1}_{n_1 \times 1} \quad (3.189)$$

where:

$$d_{122}(h, h_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) = \frac{h_{L121} - h_{1s_1}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{121}}\right)^2 - 1}} + \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}}\right)^2 - 1}} \quad (3.190)$$

$$\|\mathbf{r}_{1s_1} - \mathbf{r}\| = \sqrt{(x_{1s_1} - x)^2 + (y_{1s_1} - y)^2} \quad (3.191)$$

$$d_{122}(h, h_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \equiv \|\mathbf{r}_{1s_1} - \mathbf{r}\| \quad (3.192)$$

• **Step 11.2:** Observed P arrival time for each seismic station with an unknown hypocenter and origin time:

$$\mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) = \begin{pmatrix} f_{122}(\mathbf{w}, \mathbf{r}_{11}, h_{L121}, \mathbf{v}_{12}, p_{1221}) \\ \vdots \\ f_{122}(\mathbf{w}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \\ \vdots \\ f_{122}(\mathbf{w}, \mathbf{r}_{1n_1}, h_{L121}, \mathbf{v}_{12}, p_{122n_1}) \end{pmatrix} \in \mathbb{R}^{n_1 \times 1} \quad (3.193)$$

where:

$$\begin{aligned} & f_{122}(\mathbf{w}, \mathbf{r}_{1s_1}, h_{L121}, \mathbf{v}_{12}, p_{122s_1}) \\ &= t + \frac{\|\mathbf{r}_{1s_1} - \mathbf{r}\|}{p_{122s_1} v_{121}^2} + \frac{1}{p_{122s_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) \frac{h - h_{L121}}{\sqrt{\left(\frac{1}{p_{122s_1} v_{122}} \right)^2 - 1}} \end{aligned} \quad (3.194)$$

• **Step 11.3:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{122} = \mathbf{f}_{01} - \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \in \mathbb{R}^{n_1 \times 1} \quad (3.195)$$

- **Step 11.4:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\frac{\partial \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122})}{\partial \mathbf{w}} = \begin{pmatrix} \frac{x - x_{11}}{p_{1221} v_{121}^2 \|\mathbf{r}_{11} - \mathbf{r}\|} & \frac{y - y_{11}}{p_{1221} v_{121}^2 \|\mathbf{r}_{11} - \mathbf{r}\|} & \frac{1}{p_{1221}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{1s_1}}{p_{122s_1} v_{121}^2 \|\mathbf{r}_{1s_1} - \mathbf{r}\|} & \frac{y - y_{1s_1}}{p_{122s_1} v_{121}^2 \|\mathbf{r}_{1s_1} - \mathbf{r}\|} & \frac{1}{p_{122s_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{1n_1}}{p_{122n_1} v_{121}^2 \|\mathbf{r}_{1n_1} - \mathbf{r}\|} & \frac{y - y_{1n_1}}{p_{122n_1} v_{121}^2 \|\mathbf{r}_{1n_1} - \mathbf{r}\|} & \frac{1}{p_{122n_1}} \left(\frac{1}{v_{122}^2} - \frac{1}{v_{121}^2} \right) & 1 \end{pmatrix} \in \mathbb{R}^{n_1 \times 4} \quad (3.196)$$

- **Step 11.5:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.197)$$

- **Step 11.6:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{122} = \frac{\partial \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_1 \times 4} \quad (3.198)$$

- **Step 11.7:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{122} \Delta \mathbf{u} = \Delta \mathbf{f}_{122} \quad (3.199)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{122}^T \mathbf{A}_{122} + \alpha \mathbf{I})^{-1} \mathbf{A}_{122}^T \Delta \mathbf{f}_{122} \quad (3.200)$$

- **Step 11.8:** Go to **Step 12**.

Step 12: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned}
 x &= R_E \cos(\phi + \Delta\phi) \cos(\lambda + \Delta\lambda) \\
 y &= R_E \cos(\phi + \Delta\phi) \sin(\lambda + \Delta\lambda) \\
 h &= h + \Delta h \\
 t &= t + \Delta t \\
 \mathbf{w} &= (x, y, h, t)
 \end{aligned} \tag{3.201}$$

Step 13: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned}
 \phi &= \arcsin\left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E}\right) \\
 \lambda &= \text{atan2}(y, x) \\
 \mathbf{u} &= (\phi, \lambda, h, t)
 \end{aligned} \tag{3.202}$$

Step 14: Convert change in latitude and longitude from radian to degree:

$$(\Delta\phi^\circ, \Delta\lambda^\circ) = \left(\frac{180}{\pi}\Delta\phi, \frac{180}{\pi}\Delta\lambda\right) \tag{3.203}$$

Step 15: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 15.1** to **Step 15.2**. Else, skip to **Step 16**.

- **Step 15.1:** If $0 \leq h \leq h_{L122}$ and $t \geq 0$, then output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi}\phi, \frac{180}{\pi}\lambda, h, t\right) \tag{3.204}$$

Else, declare that the value of the depth of the guessing hypocenter, h , or the value of the guessing origin time, t , or both, is/are invalid.

- **Step 15.2:** Terminate.

Step 16: Repeat **Step 10** to **Step 15**.

Algorithm 3.6 (Geiger's Method: 2 layer PmP wave velocity model).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.205)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max \|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.206)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Discontinuities of 2 layer PmP wave velocity model:

$$\mathbf{h}_{L22} = (h_{L221}, h_{L222}) = \mathbf{h}_{L12} = (h_{L121}, h_{L122}) \quad (3.207)$$

Step 5: 2 layer PmP wave velocity model:

$$\mathbf{v}_{22} = (v_{221}, v_{222}) = \mathbf{v}_{12} = (v_{121}, v_{122}) \quad (3.208)$$

Step 6: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.209)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.210)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 7: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\mathbf{Y}_2 = \begin{pmatrix} \Phi_2 & \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_2^\circ & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{n_2 \times 3}, \quad \Phi_2^\circ \in \mathbb{R}^{n_2 \times 2}, \quad \mathbf{h}_2 = \mathbf{0}_{n_2 \times 1} \in \mathbb{R}^{n_2 \times 1} \quad (3.211)$$

Step 8: Seismic Stations in Cartesian Coordinates:

$$\mathbf{X}_2 = \begin{pmatrix} \mathbf{R}_2 & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{n_2 \times 3}, \quad \mathbf{R}_2 \in \mathbb{R}^{n_2 \times 2} \quad (3.212)$$

Step 9: Observed PmP arrival time for each seismic station with an unknown hypocenter and origin time:

$$\mathbf{f}_{02} \in \mathbb{R}^{n_2 \times 1} \quad (3.213)$$

Step 10: If $0 \leq h < h_{L221}$, then execute **Step 10.1** to **Step 10.8**. Else, Skip to **Step 11**.

• **Step 10.1:** Find ray parameter, $\mathbf{p}_{221} \in \mathbb{R}^{n_2 \times 1}$, by applying Bisection Method on Epicentral Distance, such that:

$$\mathbf{d}_{221}(h, \mathbf{h}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{221}) \equiv \mathbf{d}(\mathbf{r}, \mathbf{R}_2) \in \mathbb{R}^{n_2 \times 1} \quad (3.214)$$

$$\mathbf{0}_{n_2 \times 1} < \mathbf{p}_{221} < \frac{1}{v_{221}} \mathbf{1}_{n_2 \times 1} \quad (3.215)$$

where:

$$d_{221}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{221s_2}) = \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2}v_{221}}\right)^2 - 1}} + \frac{2(h_{L222} - h_{L221})}{\sqrt{\left(\frac{1}{p_{221s_2}v_{222}}\right)^2 - 1}} \quad (3.216)$$

$$\|\mathbf{r}_{2s_2} - \mathbf{r}\| = \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} \quad (3.217)$$

$$d_{221}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{221s_2}) \equiv \|\mathbf{r}_{2s_2} - \mathbf{r}\| \quad (3.218)$$

• **Step 10.2:** Calculated PmP arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221}) = \begin{pmatrix} f_{221}(\mathbf{w}, \mathbf{x}_{21}, h_{L221}, \mathbf{v}_{22}, p_{2211}) \\ \vdots \\ f_{221}(\mathbf{w}, \mathbf{x}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2}) \\ \vdots \\ f_{221}(\mathbf{w}, \mathbf{x}_{2n_2}, h_{L221}, \mathbf{v}_{22}, p_{221n_2}) \end{pmatrix} \in \mathbb{R}^{n_2 \times 1} \quad (3.219)$$

where:

$$f_{221}(\mathbf{w}, \mathbf{X}_{2s_2}, h_{L221}, \mathbf{v}_{22}, p_{221s_2}) = t + \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{221s_2} v_{22}^2} + \frac{1}{p_{221s_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right) \frac{2h_{L221} - h_{2s_2} - h}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}} \right)^2 - 1}} \quad (3.220)$$

- **Step 10.3:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{221} = \mathbf{f}_{02} - \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221}) \in \mathbb{R}^{n_2 \times 1} \quad (3.221)$$

- **Step 10.4:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\frac{\partial \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221})}{\partial \mathbf{w}} = \begin{pmatrix} \frac{x - x_{21}}{p_{2211} v_{22}^2 \|\mathbf{r}_{21} - \mathbf{r}\|} & \frac{y - y_{21}}{p_{2211} v_{22}^2 \|\mathbf{r}_{21} - \mathbf{r}\|} & \frac{\frac{1}{p_{2211}} \left(\frac{1}{v_{22}^2} - \frac{1}{v_{221}^2} \right)}{\sqrt{\left(\frac{1}{p_{2211} v_{221}} \right)^2 - 1}} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{2s_2}}{p_{221s_2} v_{22}^2 \|\mathbf{r}_{2s_2} - \mathbf{r}\|} & \frac{y - y_{2s_2}}{p_{221s_2} v_{22}^2 \|\mathbf{r}_{2s_2} - \mathbf{r}\|} & \frac{\frac{1}{p_{221s_2}} \left(\frac{1}{v_{22}^2} - \frac{1}{v_{221}^2} \right)}{\sqrt{\left(\frac{1}{p_{221s_2} v_{221}} \right)^2 - 1}} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{2n_2}}{p_{221n_2} v_{22}^2 \|\mathbf{r}_{2n_2} - \mathbf{r}\|} & \frac{y - y_{2n_2}}{p_{221n_2} v_{22}^2 \|\mathbf{r}_{2n_2} - \mathbf{r}\|} & \frac{\frac{1}{p_{221n_2}} \left(\frac{1}{v_{22}^2} - \frac{1}{v_{221}^2} \right)}{\sqrt{\left(\frac{1}{p_{221n_2} v_{221}} \right)^2 - 1}} & 1 \end{pmatrix} \in \mathbb{R}^{n_2 \times 4} \quad (3.222)$$

- **Step 10.5:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.223)$$

- **Step 10.6:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{221} = \frac{\partial \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_2 \times 4} \quad (3.224)$$

- **Step 10.7:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{221} \Delta \mathbf{u} = \Delta \mathbf{f}_{221} \quad (3.225)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{221}^T \mathbf{A}_{221} + \alpha \mathbf{I})^{-1} \mathbf{A}_{221}^T \Delta \mathbf{f}_{221} \quad (3.226)$$

- **Step 10.8:** Skip to **Step 12**.

Step 11: If $h_{L221} \leq h < h_{L222}$, then execute **Step 11.1** to **Step 11.8**. Else, terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

- **Step 11.1:** Find ray parameter, $\mathbf{p}_{222} \in \mathbb{R}^{n_2 \times 1}$, by applying Bisection Method on Epicentral Distance, such that:

$$\mathbf{d}_{222}(h, \mathbf{h}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \equiv \mathbf{d}(\mathbf{r}, \mathbf{R}_2) \in \mathbb{R}^{n_2 \times 1} \quad (3.227)$$

$$\mathbf{0}_{n_2 \times 1} < \mathbf{p}_{222} < \frac{1}{v_{222}} \mathbf{1}_{n_2 \times 1} \quad (3.228)$$

where:

$$d_{222}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) = \frac{h_{L221} - h_{2s_2}}{\sqrt{\left(\frac{1}{p_{222s_2} v_{221}}\right)^2 - 1}} + \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}}\right)^2 - 1}} \quad (3.229)$$

$$\|\mathbf{r}_{2s_2} - \mathbf{r}\| = \sqrt{(x_{2s_2} - x)^2 + (y_{2s_2} - y)^2} \quad (3.230)$$

$$d_{222}(h, h_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) \equiv \|\mathbf{r}_{2s_2} - \mathbf{r}\| \quad (3.231)$$

- **Step 11.2:** Calculated PmP arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \in \mathbb{R}^{n_2 \times 1} \quad (3.232)$$

where:

$$\begin{aligned} & f_{222}(\mathbf{w}, \mathbf{r}_{2s_2}, \mathbf{h}_{L22}, \mathbf{v}_{22}, p_{222s_2}) \\ &= t + \frac{\|\mathbf{r}_{2s_2} - \mathbf{r}\|}{p_{222s_2} v_{221}^2} + \frac{1}{p_{222s_2}} \left(\frac{1}{v_{222}^2} - \frac{1}{v_{221}^2} \right) \frac{2h_{L222} - h_{L221} - h}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}} \right)^2 - 1}} \end{aligned} \quad (3.233)$$

- **Step 11.3:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\Delta \mathbf{f}_{222} = \mathbf{f}_{02} - \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \in \mathbb{R}^{n_2 \times 1} \quad (3.234)$$

- **Step 11.4:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\begin{aligned} & \frac{\partial \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222})}{\partial \mathbf{w}} = \\ & \begin{pmatrix} \frac{x - x_{21}}{p_{2221} v_{221}^2 \|\mathbf{r}_{21} - \mathbf{r}\|} & \frac{y - y_{21}}{p_{2221} v_{221}^2 \|\mathbf{r}_{21} - \mathbf{r}\|} & \frac{\frac{1}{p_{2221}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right)}{\sqrt{\left(\frac{1}{p_{2221} v_{222}} \right)^2 - 1}} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{2s_2}}{p_{222s_2} v_{221}^2 \|\mathbf{r}_{2s_2} - \mathbf{r}\|} & \frac{y - y_{2s_2}}{p_{222s_2} v_{221}^2 \|\mathbf{r}_{2s_2} - \mathbf{r}\|} & \frac{\frac{1}{p_{222s_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right)}{\sqrt{\left(\frac{1}{p_{222s_2} v_{222}} \right)^2 - 1}} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x - x_{2n_2}}{p_{222n_2} v_{221}^2 \|\mathbf{r}_{2n_2} - \mathbf{r}\|} & \frac{y - y_{2n_2}}{p_{222n_2} v_{221}^2 \|\mathbf{r}_{2n_2} - \mathbf{r}\|} & \frac{\frac{1}{p_{222n_2}} \left(\frac{1}{v_{221}^2} - \frac{1}{v_{222}^2} \right)}{\sqrt{\left(\frac{1}{p_{222n_2} v_{222}} \right)^2 - 1}} & 1 \end{pmatrix} \\ & \in \mathbb{R}^{n_2 \times 4} \end{aligned} \quad (3.235)$$

- **Step 11.5:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.236)$$

- **Step 11.6:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\mathbf{A}_{222} = \frac{\partial \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_2 \times 4} \quad (3.237)$$

- **Step 11.7:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \mathbf{A}_{222} \Delta \mathbf{u} = \Delta \mathbf{f}_{222} \quad (3.238)$$

$$\Delta \mathbf{u} = (\mathbf{A}_{222}^T \mathbf{A}_{222} + \alpha \mathbf{I})^{-1} \mathbf{A}_{222}^T \Delta \mathbf{f}_{222} \quad (3.239)$$

- **Step 11.8:** Go to **Step 12**.

Step 12: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned} x &= R_E \cos(\phi + \Delta \phi) \cos(\lambda + \Delta \lambda) \\ y &= R_E \cos(\phi + \Delta \phi) \sin(\lambda + \Delta \lambda) \\ h &= h + \Delta h \\ t &= t + \Delta t \\ \mathbf{w} &= (x, y, h, t) \end{aligned} \quad (3.240)$$

Step 13: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned} \phi &= \arcsin \left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E} \right) \\ \lambda &= \text{atan2}(y, x) \\ \mathbf{u} &= (\phi, \lambda, h, t) \end{aligned} \quad (3.241)$$

Step 14: Convert change in latitude and longitude from radian to degree:

$$(\Delta\phi^\circ, \Delta\lambda^\circ) = \left(\frac{180}{\pi} \Delta\phi, \frac{180}{\pi} \Delta\lambda \right) \quad (3.242)$$

Step 15: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 15.1** to **Step 15.2**. Else, skip to **Step 16**.

• **Step 15.1:** If $0 \leq h \leq h_{L222}$ and $t \geq 0$, then output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi} \phi, \frac{180}{\pi} \lambda, h, t \right) \quad (3.243)$$

Else, declare that the value of the depth of the guessing hypocenter, h , or the value of the guessing origin time, t , or both, is/are invalid.

• **Step 15.2:** Terminate.

Step 16: Repeat **Step 10** to **Step 15**.

Algorithm 3.7 (Geiger's Method: 2 layer velocity model, P and PmP combined).

Step 1: Set the radius of the spherical Earth in kilometers (km) as:

$$R_E = 6371.010 \quad (3.244)$$

Step 2: Set the termination criteria with an appropriate value for:

$$\max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| \quad (3.245)$$

Step 3: Set Matrix Adjustment Constant, α , to an appropriate value.

Step 4: Discontinuities of 2 layer P and PmP wave velocity model:

$$\mathbf{h}_{L22} = (h_{L221}, h_{L222}) = \mathbf{h}_{L12} = (h_{L121}, h_{L122}) \quad (3.246)$$

Step 5: 2 layer P and PmP wave velocity model:

$$\mathbf{v}_{22} = (v_{221}, v_{222}) = \mathbf{v}_{12} = (v_{121}, v_{122}) \quad (3.247)$$

Step 6: Set initial guessing of the hypocenter and origin time:

$$\mathbf{u} = (\mathbf{y}, t) = (\phi, \lambda, h, t) = \left(\frac{\pi\phi^\circ}{180}, \frac{\pi\lambda^\circ}{180}, h, t \right) \quad (3.248)$$

$$\mathbf{w} = (\mathbf{x}, t) = (\mathbf{r}, h, t) = (x, y, h, t) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h, t) \quad (3.249)$$

where $\mathbf{r} = (x, y)$ are x-y components of an epicenter.

Step 7: Seismic Stations in Latitude (degree to radian), Longitude (degree to radian) and depth:

$$\begin{aligned} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} &= \begin{pmatrix} \Phi_1 & \mathbf{h}_1 \\ \Phi_2 & \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{180} \Phi_1^\circ & \mathbf{h}_1 \\ \frac{\pi}{180} \Phi_2^\circ & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 3} \\ \begin{pmatrix} \Phi_1^\circ \\ \Phi_2^\circ \end{pmatrix} &\in \mathbb{R}^{(n_1+n_2) \times 2}, \quad \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n_1 \times 1} \\ \mathbf{0}_{n_2 \times 1} \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \end{aligned} \quad (3.250)$$

Step 8: Seismic Stations in Cartesian Coordinates:

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{h}_1 \\ \mathbf{R}_2 & \mathbf{h}_2 \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 3}, \quad \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 2} \quad (3.251)$$

Step 9: Observed P and PmP arrival time for each seismic station with an unknown hypocenter and origin time:

$$\begin{pmatrix} \mathbf{f}_{01} \\ \mathbf{f}_{02} \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.252)$$

Step 10: If $0 \leq h < h_{L121}$ and $0 \leq h < h_{L221}$, then execute **Step 10.1** to **Step 10.8**. Else, Skip to **Step 11**.

• **Step 10.1:** Find ray parameter, $\mathbf{p}_{221} \in \mathbb{R}^{n_2 \times 1}$, by applying Bisection Method on Epicentral Distance, such that:

$$\mathbf{d}_{221}(h, \mathbf{h}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{221}) \equiv \mathbf{d}(\mathbf{r}, \mathbf{R}_2) \in \mathbb{R}^{n_2 \times 1} \quad (3.253)$$

$$\mathbf{0}_{n_2 \times 1} < \mathbf{p}_{221} < \frac{1}{v_{221}} \mathbf{1}_{n_2 \times 1} \quad (3.254)$$

• **Step 10.2:** Calculated P and PmP arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.255)$$

• **Step 10.3:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \Delta \mathbf{f}_{121} \\ \Delta \mathbf{f}_{221} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{01} \\ \mathbf{f}_{02} \end{pmatrix} - \begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.256)$$

• **Step 10.4:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 4} \quad (3.257)$$

• **Step 10.5:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.258)$$

• **Step 10.6:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{221} \end{pmatrix} = \frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{221}(\mathbf{w}, \mathbf{X}_2, h_{L221}, \mathbf{v}_{22}, \mathbf{p}_{221}) \end{pmatrix} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{(n_1+n_2) \times 4} \quad (3.259)$$

- **Step 10.7:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{221} \end{pmatrix} \Delta \mathbf{u} = \begin{pmatrix} \Delta \mathbf{f}_{121} \\ \Delta \mathbf{f}_{221} \end{pmatrix} \quad (3.260)$$

$$\Delta \mathbf{u} = \left(\begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{221} \end{pmatrix}^T \begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{221} \end{pmatrix} + \alpha \mathbf{I} \right)^{-1} \begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{221} \end{pmatrix}^T \begin{pmatrix} \Delta \mathbf{f}_{121} \\ \Delta \mathbf{f}_{221} \end{pmatrix} \quad (3.261)$$

- **Step 10.8:** Skip to **Step 13**.

Step 11: If $h \equiv h_{L121}$ and $h \equiv h_{L221}$, then execute **Step 11.1** to **Step 11.8**.
Else, Skip to **Step 12**.

- **Step 11.1:** Find ray parameter, $\mathbf{p}_{222} \in \mathbb{R}^{n_2 \times 1}$, by applying Bisection Method on Epicentral Distance, such that:

$$\mathbf{d}_{222}(h, \mathbf{h}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \equiv \mathbf{d}(\mathbf{r}, \mathbf{R}_2) \in \mathbb{R}^{n_2 \times 1} \quad (3.262)$$

$$\mathbf{0}_{n_2 \times 1} < \mathbf{p}_{222} < \frac{1}{v_{222}} \mathbf{1}_{n_2 \times 1} \quad (3.263)$$

- **Step 11.2:** Calculated P and PmP arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.264)$$

- **Step 11.3:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \Delta \mathbf{f}_{121} \\ \Delta \mathbf{f}_{222} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{01} \\ \mathbf{f}_{02} \end{pmatrix} - \begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.265)$$

- **Step 11.4:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 4} \quad (3.266)$$

- **Step 11.5:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.267)$$

- **Step 11.6:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{222} \end{pmatrix} = \frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{121}(\mathbf{w}, \mathbf{X}_1, v_{121}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{(n_1+n_2) \times 4} \quad (3.268)$$

- **Step 11.7:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{222} \end{pmatrix} \Delta \mathbf{u} = \begin{pmatrix} \Delta \mathbf{f}_{121} \\ \Delta \mathbf{f}_{222} \end{pmatrix} \quad (3.269)$$

$$\Delta \mathbf{u} = \left(\begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{222} \end{pmatrix}^T \begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{222} \end{pmatrix} + \alpha \mathbf{I} \right)^{-1} \begin{pmatrix} \mathbf{A}_{121} \\ \mathbf{A}_{222} \end{pmatrix}^T \begin{pmatrix} \Delta \mathbf{f}_{121} \\ \Delta \mathbf{f}_{222} \end{pmatrix} \quad (3.270)$$

- **Step 11.8:** Skip to **Step 13**.

Step 12: If $h_{L121} < h < h_{L122}$ and $h_{L221} < h < h_{L222}$, then execute **Step 12.1** to **Step 12.8**. Else, terminate and declare that the value of the depth of the guessing hypocenter, h , is invalid.

- **Step 12.1:** Find ray parameter,

$$\begin{pmatrix} \mathbf{p}_{122} \\ \mathbf{p}_{222} \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.271)$$

by applying Bisection Method on Epicentral Distance, such that:

$$\begin{pmatrix} \mathbf{d}_{122}(h, \mathbf{h}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \\ \mathbf{d}_{222}(h, \mathbf{h}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} = \begin{pmatrix} \mathbf{d}(\mathbf{r}, \mathbf{R}_1) \\ \mathbf{d}(\mathbf{r}, \mathbf{R}_2) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.272)$$

$$\begin{pmatrix} \mathbf{0}_{n_1 \times 1} \\ \mathbf{0}_{n_2 \times 1} \end{pmatrix} < \begin{pmatrix} \mathbf{p}_{122} \\ \mathbf{p}_{222} \end{pmatrix} < \begin{pmatrix} \frac{1}{v_{122}} \mathbf{1}_{n_1 \times 1} \\ \frac{1}{v_{222}} \mathbf{1}_{n_2 \times 1} \end{pmatrix} \quad (3.273)$$

• **Step 12.2:** Calculated P and PmP arrival time for each seismic station with the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.274)$$

• **Step 12.3:** Difference between the observed arrival time of an unknown hypocenter and origin time, and the calculated arrival time of the guessing hypocenter and origin time, $\mathbf{w} = (x, y, h, t)$:

$$\begin{pmatrix} \Delta \mathbf{f}_{122} \\ \Delta \mathbf{f}_{222} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{01} \\ \mathbf{f}_{02} \end{pmatrix} - \begin{pmatrix} \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 1} \quad (3.275)$$

• **Step 12.4:** Jacobian Matrix of arrival time with respect to $\mathbf{w} = (x, y, h, t)$:

$$\frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \in \mathbb{R}^{(n_1+n_2) \times 4} \quad (3.276)$$

• **Step 12.5:** Jacobian Matrix of transformation from $\mathbf{w} = (x, y, h, t)$ to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{4 \times 4} \quad (3.277)$$

• **Step 12.6:** Jacobian Matrix of arrival time with respect to $\mathbf{u} = (\phi, \lambda, h, t)$:

$$\begin{pmatrix} \mathbf{A}_{122} \\ \mathbf{A}_{222} \end{pmatrix} = \frac{\partial}{\partial \mathbf{w}} \begin{pmatrix} \mathbf{f}_{122}(\mathbf{w}, \mathbf{R}_1, h_{L121}, \mathbf{v}_{12}, \mathbf{p}_{122}) \\ \mathbf{f}_{222}(\mathbf{w}, \mathbf{R}_2, \mathbf{h}_{L22}, \mathbf{v}_{22}, \mathbf{p}_{222}) \end{pmatrix} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} \in \mathbb{R}^{n_2 \times 4} \quad (3.278)$$

- **Step 12.7:** Solve the linear equation:

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta \phi \\ \Delta \lambda \\ \Delta h \\ \Delta t \end{pmatrix} \in \mathbb{R}^{4 \times 1}, \quad \begin{pmatrix} \mathbf{A}_{122} \\ \mathbf{A}_{222} \end{pmatrix} \Delta \mathbf{u} = \begin{pmatrix} \Delta \mathbf{f}_{122} \\ \Delta \mathbf{f}_{222} \end{pmatrix} \quad (3.279)$$

$$\Delta \mathbf{u} = \left(\begin{pmatrix} \mathbf{A}_{122} \\ \mathbf{A}_{222} \end{pmatrix}^T \begin{pmatrix} \mathbf{A}_{122} \\ \mathbf{A}_{222} \end{pmatrix} + \alpha \mathbf{I} \right)^{-1} \begin{pmatrix} \mathbf{A}_{122} \\ \mathbf{A}_{222} \end{pmatrix}^T \begin{pmatrix} \Delta \mathbf{f}_{122} \\ \Delta \mathbf{f}_{222} \end{pmatrix} \quad (3.280)$$

- **Step 12.8:** Go to **Step 13**.

Step 13: Update hypocenter and origin time in cartesian coordinate form:

$$\begin{aligned} x &= R_E \cos(\phi + \Delta\phi) \cos(\lambda + \Delta\lambda) \\ y &= R_E \cos(\phi + \Delta\phi) \sin(\lambda + \Delta\lambda) \\ h &= h + \Delta h \\ t &= t + \Delta t \\ \mathbf{w} &= (x, y, h, t) \end{aligned} \quad (3.281)$$

Step 14: Updated hypocenter and origin time in latitude (radian), longitude (radian), depth and origin time:

$$\begin{aligned} \phi &= \arcsin \left(\frac{\sqrt{R_E^2 - x^2 - y^2}}{R_E} \right) \\ \lambda &= \text{atan2}(y, x) \\ \mathbf{u} &= (\phi, \lambda, h, t) \end{aligned} \quad (3.282)$$

Step 15: Convert change in latitude and longitude from radian to degree:

$$(\Delta\phi^\circ, \Delta\lambda^\circ) = \left(\frac{180}{\pi} \Delta\phi, \frac{180}{\pi} \Delta\lambda \right) \quad (3.283)$$

Step 16: If $\|(\Delta\phi^\circ, \Delta\lambda^\circ)\| < \max\|(\Delta\phi^\circ, \Delta\lambda^\circ)\|$, then execute **Step 16.1** to

Step 16.2. Else, skip to **Step 17**.

• **Step 16.1:** If $0 \leq h \leq h_{L122}$ and $0 \leq h \leq h_{L222}$ and $t \geq 0$, then output final hypocenter and origin time in latitude (degree), longitude (degree), depth and origin time:

$$\mathbf{u}^\circ = (\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi} \phi, \frac{180}{\pi} \lambda, h, t \right) \quad (3.284)$$

Else, declare that the value of the depth of the guessing hypocenter, h , or the value of the guessing origin time, t , or both, is/are invalid.

• **Step 16.2:** Terminate.

Step 17: Repeat **Step 10** to **Step 16**.

3.3 Application: Kyushu Japan

In general, Geiger's Method can locate latitude and longitude of hypocenter, also known as the epicenter, with a reasonable accuracy, provided that the initial guess of the epicenter is not too far away from the final epicenter. However, according to the IS 11.1 Seismic source location [10] from University of Bergen, Department of Earth Science, Geiger's Method has difficulty to properly locate hypocentral depth, because in general, the derivative of the arrival time function with respect to the hypocentral depth is usually very small. This means the hypocentral depth increase or decrease without affecting the arrival time much. Hence, to find the hypocentral depth with better accuracy, either a seismic station should not be too far away from the epicenter or the actual hypocentral depth itself should be sufficiently deep. Unfortunately, quite often a seismic station is quite far away from the epicenter and the hypocentral depth is often quite shallow. In such situation, the partial workaround is that both the initial hypocentral depth and origin time value should each have a carefully fixed typical value. [10]

Actual parameters for all Algorithms of Geiger's Method applied in Kyushu Japan:

- Radius of Earth (km):

$$R_E = 6371.010 \quad (3.285)$$

- For 1 layer P wave velocity model (km/s):

$$v_{211} = v_{111} = 6 \quad (3.286)$$

- For 2 layer P wave velocity model (km/s):

$$(v_{221}, v_{222}) = \mathbf{v}_{22} = \mathbf{v}_{12} = (v_{121}, v_{122}) = (5.8, 6.7) \quad (3.287)$$

- Conrad depth for 2 layer P (also PmP) wave velocity model only (km):

$$h_{L221} = h_{L121} = 10 \quad (3.288)$$

- Moho depth for 1 and 2 layer P (also PmP) wave velocity model (km):

$$h_{L211} = h_{L222} = h_{L111} = h_{L122} = 30 \quad (3.289)$$

- Initial Guess Hypocenter and Origin Time:

$$(\phi^\circ, \lambda^\circ, h, t) = (25^\circ, 125^\circ, 9 \text{ km}, 3 \text{ s}) \quad (3.290)$$

- Termination Criteria of Bisection Method for Ray Parameters:

$$\varepsilon = 10^{-10} = 0.0000000001 \quad (3.291)$$

- Matrix Adjustment Constant (Damping Constant) of Damped Least Square Method:

$$\alpha = 100 \quad (3.292)$$

- Termination Criteria of Geiger's Method:

$$\max \|(\Delta\phi^\circ, \Delta\lambda^\circ)\| = 0.01^\circ \quad (3.293)$$

FIGURE 3.1: GMT Earthquake Plot for all selected earthquake locations (222 Earthquake Events) provided by Hinet Japan. For more details, refer to Appendix A

FIGURE 3.2: GMT Earthquake Plot (222 Earthquake Events) for Algorithm of Geiger's Method, using P arrival time, with 1 layer velocity model. For more details, refer to Appendix A

FIGURE 3.3: GMT Earthquake Plot (222 Earthquake Events) for Algorithm of Geiger's Method, using P arrival time, with 2 layer velocity model. For more details, refer to Appendix A

FIGURE 3.4: GMT Earthquake Plot for some selected earthquake locations (10 Earthquake Events) provided by Hinet Japan

Hinet Japan		
Lat	Lon	Dep
32.70	130.79	10.0
32.79	130.62	12.0
32.88	130.81	12.7
32.92	130.85	11.0
32.65	130.61	11.0
32.79	131.03	6.0
33.00	130.34	12.0
32.51	130.58	10.0
32.96	130.94	11.0
32.39	130.64	10.0

TABLE 3.1: Selected 10 Earthquake Events: Hinet Japan provided

FIGURE 3.5: GMT Earthquake Plot (10 Earthquake Events) for Algorithm of Geiger's Method, using P arrival time, with 1 layer velocity model

Hinet Japan			Geiger: 1 Layer P		
Lat	Lon	Dep	Lat	Lon	Dep
32.70	130.79	10.0	32.6785	130.8372	9.0531
32.79	130.62	12.0	32.7753	130.6500	9.0561
32.88	130.81	12.7	32.8953	130.8614	9.0670
32.92	130.85	11.0	32.9508	130.9016	9.0741
32.65	130.61	11.0	32.6801	130.6797	9.0547
32.79	131.03	6.0	32.8133	131.0640	9.0663
33.00	130.34	12.0	33.0417	130.3480	9.0198
32.51	130.58	10.0	32.5542	130.6181	9.0756
32.96	130.94	11.0	33.0080	130.9858	9.0462
32.39	130.64	10.0	32.3666	130.6762	9.0606

TABLE 3.2: Hinet Japan provided vs Geiger's Method (1 layer P velocity model)

FIGURE 3.6: GMT Earthquake Plot (10 Earthquake Events) for Algorithm of Geiger's Method, using P arrival time, with 2 layer velocity model

Hinet Japan			Geiger: 2 Layer P		
Lat	Lon	Dep	Lat	Lon	Dep
32.70	130.79	10.0	32.6802	130.8401	9.0503
32.79	130.62	12.0	32.7775	130.6515	9.0532
32.88	130.81	12.7	32.8948	130.8627	9.0644
32.92	130.85	11.0	32.9484	130.9039	9.0700
32.65	130.61	11.0	32.6778	130.6796	9.0515
32.79	131.03	6.0	32.8128	131.0647	9.0622
33.00	130.34	12.0	33.0410	130.3559	9.0166
32.51	130.58	10.0	32.5505	130.6219	9.0727
32.96	130.94	11.0	33.0069	130.9881	9.0419
32.39	130.64	10.0	32.3658	130.6824	9.0562

TABLE 3.3: Hinet Japan provided vs Geiger's Method (2 layer P velocity model)

FIGURE 3.7: GMT Earthquake Plot (10 Earthquake Events) for Algorithm of Geiger's Method, using PmP arrival time, with 1 layer velocity model

Hinet Japan			Geiger: 1 Layer PmP		
Lat	Lon	Dep	Lat	Lon	Dep
32.70	130.79	10.0	32.8446	130.8076	9.0182
32.79	130.62	12.0	32.8862	130.5916	9.0506
32.88	130.81	12.7	33.0544	130.8381	9.2453
32.92	130.85	11.0	33.0418	130.8673	9.0277
32.65	130.61	11.0	32.3404	130.7218	9.1587
32.79	131.03	6.0	32.8529	131.1762	8.9970
33.00	130.34	12.0	32.5573	130.2666	9.1170
32.51	130.58	10.0	32.3699	130.6481	9.1120
32.96	130.94	11.0	33.0995	131.0285	8.9990
32.39	130.64	10.0	32.3279	130.7365	9.0677

TABLE 3.4: Hinet Japan provided vs Geiger's Method (1 layer PmP model)

FIGURE 3.8: GMT Earthquake Plot (10 Earthquake Events) for Algorithm of Geiger's Method, using PmP arrival time, with 2 layer velocity model

Hinet Japan			Geiger: 2 Layer PmP		
Lat	Lon	Dep	Lat	Lon	Dep
32.70	130.79	10.0	32.9565	130.8272	9.0150
32.79	130.62	12.0	32.9948	130.5866	9.0034
32.88	130.81	12.7	33.0882	130.8644	9.0371
32.92	130.85	11.0	33.0294	130.8506	8.9919
32.65	130.61	11.0	32.3622	130.7167	9.0140
32.79	131.03	6.0	32.9630	131.1353	9.0017
33.00	130.34	12.0	32.5652	130.2472	9.0089
32.51	130.58	10.0	32.3440	130.6703	8.9954
32.96	130.94	11.0	33.0393	130.8778	9.0035
32.39	130.64	10.0	32.2328	130.7337	9.0065

TABLE 3.5: Hinet Japan provided vs Geiger's Method (2 layer PmP model)

FIGURE 3.9: GMT Earthquake Plot (10 Earthquake Events) for Algorithm of Geiger's Method, using P and PmP arrival time, with 1 layer velocity model

Hinet Japan			Geiger: 1 Layer P & PmP		
Lat	Lon	Dep	Lat	Lon	Dep
32.70	130.79	10.0	32.6842	130.8328	9.0577
32.79	130.62	12.0	32.7739	130.6428	9.0622
32.88	130.81	12.7	32.9061	130.8639	9.1432
32.92	130.85	11.0	32.9495	130.8933	9.1007
32.65	130.61	11.0	32.6648	130.6763	9.1124
32.79	131.03	6.0	32.8099	131.0832	9.1090
33.00	130.34	12.0	32.9997	130.3407	9.0289
32.51	130.58	10.0	32.5525	130.6152	9.1146
32.96	130.94	11.0	33.0137	130.9890	9.0514
32.39	130.64	10.0	32.3660	130.6852	9.1032

TABLE 3.6: Hinet Japan provided vs Geiger's Method (1 layer P and PmP)

FIGURE 3.10: GMT Earthquake Plot (10 Earthquake Events) for Algorithm of Geiger's Method, using P and PmP arrival time, with 2 layer velocity model

Hinet Japan			Geiger: 2 Layer P & PmP		
Lat	Lon	Dep	Lat	Lon	Dep
32.70	130.79	10.0	32.7000	130.8414	8.9949
32.79	130.62	12.0	32.7868	130.6448	8.9889
32.88	130.81	12.7	32.9071	130.8651	9.0839
32.92	130.85	11.0	32.9453	130.8926	9.0324
32.65	130.61	11.0	32.6586	130.6773	9.0439
32.79	131.03	6.0	32.8150	131.0887	9.0129
33.00	130.34	12.0	32.9792	130.3434	8.9650
32.51	130.58	10.0	32.5445	130.6215	9.0564
32.96	130.94	11.0	33.0132	130.9803	8.9976
32.39	130.64	10.0	32.3537	130.6934	9.0568

TABLE 3.7: Hinet Japan provided vs Geiger's Method (2 layer P and PmP)

Regarding the P (also PmP) wave velocity model, whether 1 or 2 layer velocity model is used, for majority of algorithms of Geiger's Method, the located earthquakes in corresponding velocity model are about the same as each other, except the particular algorithm of Geiger's Method that involves using PmP arrival times only.

For algorithms involves P arrival times only and P and PmP combined arrival times, earthquake locations in corresponding algorithm are about the same as each other, and obviously both algorithms have better earthquake location accuracy than the particular algorithm of Geiger's Method that involves using PmP arrival times only. The main reason is algorithms involves P arrival times only and P and PmP combined arrival times have far more arrival time picks in each earthquake event than the particular algorithm of Geiger's Method that involves using PmP arrival times only, because picking PmP phase arrival time is far more difficult than picking P phase arrival time.

In general, majority of algorithms of Geiger's Method, when compared with the earthquake locations provided by Hinet Japan, the located earthquakes in all algorithms and provided by Hinet Japan are about the same as one another, except the particular algorithm of Geiger's Method that involves using PmP arrival times only, for the same reason mentioned above. Also, note that as previously mentioned, the initial guess of hypocentral depth is being selected a typical value carefully, to ensure that the final hypocentral depth in all algorithms of Geiger's Method are about the same as the one provided by Hinet Japan.

Chapter 4

Grid Search Algorithm and Fast Marching Method

4.1 Eikonal Equation

Theorem 4.1 (Eikonal Equation in Cartesian Form). *If $\mathbf{x} \neq \mathbf{x}_{\mathbf{S}n}$, then:*

$$\left(\frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial x}\right)^2 + \left(\frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial y}\right)^2 + \left(\frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial h}\right)^2 = \frac{1}{[v(\mathbf{x})]^2} \quad (4.1)$$

else, if $\mathbf{x} = \mathbf{x}_{\mathbf{S}n}$, then:

$$T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x}) = 0 \quad (4.2)$$

where $T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})$ is the shortest travel time from the n^{th} seismic station,

$$\mathbf{x}_{\mathbf{S}n} = (x_{S_n}, y_{S_n}, h_{S_n}) \quad (4.3)$$

to any guessing hypocenter,

$$\mathbf{x} = (x, y, z) \quad (4.4)$$

in cartesian coordinate.

The above theorem is very common in any area of applied mathematics, physics and engineering. [11] [12] However, the subsequent corollary is the coordinate transformation from the cartesian form to the latitude-longitude form of Eikonal Equation which is quite uncommon, hence this thesis will prove such corollary.

Corollary 4.2 (Eikonal Equation for Small Region in Latitude-Longitude). *If $\mathbf{y} \neq \mathbf{y}_{\mathbf{S}n}$, then:*

$$\left(\frac{1}{R_E \sin \phi} \frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \phi} \right)^2 + \left(\frac{1}{R_E \cos \phi} \frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \lambda} \right)^2 + \left(\frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial h} \right)^2 = \frac{1}{[v(\mathbf{y})]^2} \quad (4.5)$$

else, if $\mathbf{y} = \mathbf{y}_{\mathbf{S}n}$, then:

$$T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y}) = 0 \quad (4.6)$$

where $T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})$ is the shortest travel time from the n^{th} seismic station,

$$\mathbf{y}_{\mathbf{S}n} = (\phi_{S_n}, \lambda_{S_n}, h_{S_n}) \quad (4.7)$$

to any guessing hypocenter,

$$\mathbf{y} = (\phi, \lambda, h) \quad (4.8)$$

and latitude ϕ and longitude λ are in radian.

Proof. Derivative of travel time with respect to $\mathbf{x} = (x, y, h)$:

$$\frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial \mathbf{x}} = \left(\frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial x} \quad \frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial y} \quad \frac{\partial T(\mathbf{x}_{\mathbf{S}n}, \mathbf{x})}{\partial h} \right) \in \mathbb{R}^{1 \times 3} \quad (4.9)$$

Jacobian Matrix of transformation from $\mathbf{x} = (x, y, h)$ to $\mathbf{y} = (\phi, \lambda, h)$:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial h} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial h} \\ \frac{\partial h}{\partial \phi} & \frac{\partial h}{\partial \lambda} & \frac{\partial h}{\partial h} \end{pmatrix} = \begin{pmatrix} -R_E \sin \phi \cos \lambda & -R_E \cos \phi \sin \lambda & 0 \\ -R_E \sin \phi \sin \lambda & R_E \cos \phi \cos \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \quad (4.10)$$

where:

$$\mathbf{x} = (x, y, h) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h) \quad (4.11)$$

Derivative of travel time with respect to $\mathbf{y} = (\phi, \lambda, h)$:

$$\frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \mathbf{y}} = \left(\frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \phi} \quad \frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \lambda} \quad \frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial h} \right) \in \mathbb{R}^{1 \times 3} \quad (4.12)$$

Linear Equations to represent chain rule for derivative:

$$\frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial \mathbf{y}} \quad (4.13)$$

$$\begin{aligned} & \left(\frac{\partial \mathbf{x}}{\partial \mathbf{y}} \right)^T \left(\frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial \mathbf{x}} \right)^T = \left(\frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial \mathbf{y}} \right)^T \\ & \begin{pmatrix} -R_E \sin \phi \cos \lambda & -R_E \sin \phi \sin \lambda & 0 \\ -R_E \cos \phi \sin \lambda & R_E \cos \phi \cos \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial x} \\ \frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial y} \\ \frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial h} \end{pmatrix} = \begin{pmatrix} \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial \phi} \\ \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial \lambda} \\ \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial h} \end{pmatrix} \end{aligned} \quad (4.14)$$

Equivalently:

$$\begin{aligned} & \frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial h} = \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial h} \\ & \begin{pmatrix} -R_E \sin \phi \cos \lambda & -R_E \sin \phi \sin \lambda \\ -R_E \cos \phi \sin \lambda & R_E \cos \phi \cos \lambda \end{pmatrix} \begin{pmatrix} \frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial x} \\ \frac{\partial T(\mathbf{x}_{\mathbf{S}_n}, \mathbf{x})}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial \phi} \\ \frac{\partial T(\mathbf{y}_{\mathbf{S}_n}, \mathbf{y})}{\partial \lambda} \end{pmatrix} \end{aligned} \quad (4.15)$$

Find the determinant:

$$\begin{aligned} & \det \begin{pmatrix} -R_E \sin \phi \cos \lambda & -R_E \sin \phi \sin \lambda \\ -R_E \cos \phi \sin \lambda & R_E \cos \phi \cos \lambda \end{pmatrix} \\ & = - (R_E^2 \cos \phi \sin \phi) (\cos \lambda)^2 - (R_E^2 \cos \phi \sin \phi) (\sin \lambda)^2 \\ & = - (R_E^2 \cos \phi \sin \phi) [(\cos \lambda)^2 + (\sin \lambda)^2] \\ & = -R_E^2 \cos \phi \sin \phi \end{aligned} \quad (4.16)$$

Solve the linear equation:

$$\begin{aligned}
\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial h} &= \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial h} \\
\begin{pmatrix} \frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial x} \\ \frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{R_E \cos \phi \cos \lambda}{-R_E^2 \cos \phi \sin \phi} & \frac{R_E \sin \phi \sin \lambda}{-R_E^2 \cos \phi \sin \phi} \\ \frac{R_E \cos \phi \sin \lambda}{-R_E^2 \cos \phi \sin \phi} & \frac{-R_E \sin \phi \cos \lambda}{-R_E^2 \cos \phi \sin \phi} \end{pmatrix} \begin{pmatrix} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} \\ \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda} \end{pmatrix} \quad (4.17) \\
\begin{pmatrix} \frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial x} \\ \frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial y} \end{pmatrix} &= \begin{pmatrix} -\frac{\cos \lambda}{R_E \sin \phi} & -\frac{\sin \lambda}{R_E \cos \phi} \\ -\frac{\sin \lambda}{R_E \sin \phi} & \frac{\cos \lambda}{R_E \cos \phi} \end{pmatrix} \begin{pmatrix} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} \\ \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda} \end{pmatrix}
\end{aligned}$$

Partial Derivatives of travel time with respect to x , y and h respectively:

$$\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial x} = -\frac{\cos \lambda}{R_E \sin \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} - \frac{\sin \lambda}{R_E \cos \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda} \quad (4.18)$$

$$\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial y} = -\frac{\sin \lambda}{R_E \sin \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} + \frac{\cos \lambda}{R_E \cos \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda} \quad (4.19)$$

$$\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial h} = \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial h} \quad (4.20)$$

Squared partial derivatives:

$$\begin{aligned}
\left(\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial x} \right)^2 &= \frac{(\cos \lambda)^2}{(R_E \sin \phi)^2} \left(\frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} \right)^2 + \frac{(\sin \lambda)^2}{(R_E \cos \phi)^2} \left(\frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda} \right)^2 \\
&\quad + \frac{2 \cos \lambda \sin \lambda}{R_E^2 \cos \phi \sin \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda}
\end{aligned} \quad (4.21)$$

$$\begin{aligned}
\left(\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial y} \right)^2 &= \frac{(\sin \lambda)^2}{(R_E \sin \phi)^2} \left(\frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} \right)^2 + \frac{(\cos \lambda)^2}{(R_E \cos \phi)^2} \left(\frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda} \right)^2 \\
&\quad - \frac{2 \cos \lambda \sin \lambda}{R_E^2 \cos \phi \sin \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \phi} \frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial \lambda}
\end{aligned} \quad (4.22)$$

$$\left(\frac{\partial T(\mathbf{x}_{\text{Sn}}, \mathbf{x})}{\partial h} \right)^2 = \left(\frac{\partial T(\mathbf{y}_{\text{Sn}}, \mathbf{y})}{\partial h} \right)^2 \quad (4.23)$$

Remember the cartesian form of Eikonal Equation: *If $\mathbf{x} \neq \mathbf{x}_{\mathbf{Sn}}$, then:*

$$\left(\frac{\partial T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x})}{\partial x}\right)^2 + \left(\frac{\partial T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x})}{\partial y}\right)^2 + \left(\frac{\partial T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x})}{\partial h}\right)^2 = \frac{1}{[v(\mathbf{x})]^2} \quad (4.24)$$

else, if $\mathbf{x} = \mathbf{x}_{\mathbf{Sn}}$, then:

$$T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x}) = 0 \quad (4.25)$$

Sum of Squared Partial Derivatives (if $\mathbf{x} \neq \mathbf{x}_{\mathbf{Sn}}$ and $\mathbf{y} \neq \mathbf{y}_{\mathbf{Sn}}$):

$$\begin{aligned} & \frac{1}{[v(\mathbf{x})]^2} \\ &= \left(\frac{\partial T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x})}{\partial x}\right)^2 + \left(\frac{\partial T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x})}{\partial y}\right)^2 + \left(\frac{\partial T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x})}{\partial h}\right)^2 \\ &= \frac{(\cos \lambda)^2 + (\sin \lambda)^2}{(R_E \sin \phi)^2} \left(\frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \phi}\right)^2 + \frac{(\sin \lambda)^2 + (\cos \lambda)^2}{(R_E \cos \phi)^2} \left(\frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \lambda}\right)^2 \\ & \quad + \frac{2 \cos \lambda \sin \lambda}{R_E^2 \cos \phi \sin \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \lambda} - \frac{2 \cos \lambda \sin \lambda}{R_E^2 \cos \phi \sin \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \lambda} \\ & \quad + \left(\frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial h}\right)^2 \\ &= \left(\frac{1}{R_E \sin \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \phi}\right)^2 + \left(\frac{1}{R_E \cos \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \lambda}\right)^2 + \left(\frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial h}\right)^2 \end{aligned} \quad (4.26)$$

Moreover:

$$\mathbf{x} = (x, y, h) = (R_E \cos \phi \cos \lambda, R_E \cos \phi \sin \lambda, h) = \mathbf{x}(\phi, \lambda, h) = \mathbf{x}(\mathbf{y}) \quad (4.27)$$

$$\mathbf{x}_{\mathbf{Sn}} = \mathbf{x}(\mathbf{y}_{\mathbf{Sn}}) \quad (4.28)$$

$$\frac{1}{[v(\mathbf{x})]^2} = \frac{1}{[v(\mathbf{x}(\mathbf{y}))]^2} = \frac{1}{[v(\mathbf{y})]^2} \quad (4.29)$$

$$T(\mathbf{x}_{\mathbf{Sn}}, \mathbf{x}) = T(\mathbf{x}(\mathbf{y}_{\mathbf{Sn}}), \mathbf{x}(\mathbf{y})) = T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y}) \quad (4.30)$$

$\therefore$ The Eikonal Equation for Small Region in Latitude-Longitude: *If $\mathbf{y} \neq \mathbf{y}_{\mathbf{Sn}}$, then:*

$$\left(\frac{1}{R_E \sin \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \phi}\right)^2 + \left(\frac{1}{R_E \cos \phi} \frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial \lambda}\right)^2 + \left(\frac{\partial T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y})}{\partial h}\right)^2 = \frac{1}{[v(\mathbf{y})]^2} \quad (4.31)$$

else, if $\mathbf{y} = \mathbf{y}_{\mathbf{Sn}}$, then:

$$T(\mathbf{y}_{\mathbf{Sn}}, \mathbf{y}) = 0 \quad (4.32)$$

□

4.2 Fast Marching Method

Remark 4.1 (Previously proved: Eikonal Equation for Small Region in Latitude–Longitude). *If $\mathbf{y} \neq \mathbf{y}_{\mathbf{S}n}$, then:*

$$\left(\frac{1}{R_E \sin \phi} \frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \phi} \right)^2 + \left(\frac{1}{R_E \cos \phi} \frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial \lambda} \right)^2 + \left(\frac{\partial T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})}{\partial h} \right)^2 = \frac{1}{[v(\mathbf{y})]^2} \quad (4.33)$$

else, if $\mathbf{y} = \mathbf{y}_{\mathbf{S}n}$, then:

$$T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y}) = 0 \quad (4.34)$$

where $T(\mathbf{y}_{\mathbf{S}n}, \mathbf{y})$ is the shortest travel time from the n^{th} seismic station,

$$\mathbf{y}_{\mathbf{S}n} = (\phi_{S_n}, \lambda_{S_n}, h_{S_n}) \quad (4.35)$$

to any guessing hypocenter,

$$\mathbf{y} = (\phi, \lambda, h) \quad (4.36)$$

and latitude ϕ and longitude λ are in radian.

We can discretise the shortest travel time from the n^{th} seismic station to any guessing hypocenter as:

$$T(\mathbf{y}_{(\mathbf{i}_{\mathbf{S}n}, \mathbf{j}_{\mathbf{S}n}, \mathbf{k}_{\mathbf{S}n})}, \mathbf{y}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})}) = T(\phi_{i_{S_n}}, \lambda_{i_{S_n}}, h_{i_{S_n}}, \phi_i, \lambda_j, h_k) = T_{i,j,k} \quad (4.37)$$

$$\begin{aligned} \phi_0 &\leq \phi_i = \phi_0 + i\Delta\phi \leq \phi_0 + I\Delta\phi = \phi_I \\ \lambda_0 &\leq \lambda_j = \lambda_0 + j\Delta\lambda \leq \lambda_0 + J\Delta\lambda = \lambda_J \\ h_0 &\leq h_k = h_0 + k\Delta h \leq h_0 + K\Delta h = h_K \end{aligned} \quad (4.38)$$

where $T_{i,j,k}$ is the short form notation of the shortest travel time, with any guessing hypocenter $\mathbf{y}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})}$ as a destination, to be solved numerically by Fast Marching Method and stored in $T(\mathbf{y}_{(\mathbf{i}_{\mathbf{S}n}, \mathbf{j}_{\mathbf{S}n}, \mathbf{k}_{\mathbf{S}n})}, \mathbf{y}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})})$, for each n^{th} seismic station, $\mathbf{y}_{(\mathbf{i}_{\mathbf{S}n}, \mathbf{j}_{\mathbf{S}n}, \mathbf{k}_{\mathbf{S}n})}$, as a fixed starting point.

The velocity model is discretised as:

$$v(\mathbf{y}_{(\mathbf{i},\mathbf{j},\mathbf{k})}) = v(\phi_i, \lambda_j, h_k) = v_{i,j,k} \quad (4.39)$$

and also the eikonal equation can be discretised as: *If $\mathbf{y}_{(\mathbf{i},\mathbf{j},\mathbf{k})} \neq \mathbf{y}_{(\mathbf{i}_{\text{Sn}},\mathbf{j}_{\text{Sn}},\mathbf{k}_{\text{Sn}})}$, then:*

$$\begin{aligned} & \left[\max \left\{ D_{i,j,k}^{-\phi} T - D_{i,j,k}^{+\phi} T, 0 \right\} \right]^2 + \left[\max \left\{ D_{i,j,k}^{-\lambda} T - D_{i,j,k}^{+\lambda} T, 0 \right\} \right]^2 \\ & + \left[\max \left\{ D_{i,j,k}^{-h} T - D_{i,j,k}^{+h} T, 0 \right\} \right]^2 = \frac{1}{(v_{i,j,k})^2} \end{aligned} \quad (4.40)$$

$$\begin{aligned} D_{i,j,k}^{-\phi} T &= \frac{T_{i,j,k} - T_{i-1,j,k}}{(R_E \sin \phi_i) \Delta \phi}, \quad D_{i,j,k}^{+\phi} T = \frac{T_{i+1,j,k} - T_{i,j,k}}{(R_E \sin \phi_i) \Delta \phi} \\ D_{i,j,k}^{-\lambda} T &= \frac{T_{i,j,k} - T_{i,j-1,k}}{(R_E \cos \phi_i) \Delta \lambda}, \quad D_{i,j,k}^{+\lambda} T = \frac{T_{i,j+1,k} - T_{i,j,k}}{(R_E \cos \phi_i) \Delta \lambda} \\ D_{i,j,k}^{-h} T &= \frac{T_{i,j,k} - T_{i,j,k-1}}{\Delta h}, \quad D_{i,j,k}^{+h} T = \frac{T_{i,j,k+1} - T_{i,j,k}}{\Delta h} \end{aligned} \quad (4.41)$$

else, if $\mathbf{y}_{(\mathbf{i},\mathbf{j},\mathbf{k})} = \mathbf{y}_{(\mathbf{i}_{\text{Sn}},\mathbf{j}_{\text{Sn}},\mathbf{k}_{\text{Sn}})}$ then:

$$T_{i,j,k} = 0 \quad (4.42)$$

Note that the discretised eikonal equation mentioned above is different from the usual cartesian form of the discretised eikonal equation that is often mentioned in some other research papers. [11] [12]

Equivalently, to solve the discretised eikonal equation algebraically to find $T_{i,j,k}$, it can be re-written as: *If $\mathbf{y}_{(\mathbf{i},\mathbf{j},\mathbf{k})} \neq \mathbf{y}_{(\mathbf{i}_{\text{Sn}},\mathbf{j}_{\text{Sn}},\mathbf{k}_{\text{Sn}})}$, then:*

$$\begin{aligned} & \left[\max \left\{ \frac{2T_{i,j,k} - (T_{i-1,j,k} + T_{i+1,j,k})}{(R_E \sin \phi_i) \Delta \phi}, 0 \right\} \right]^2 \\ & + \left[\max \left\{ \frac{2T_{i,j,k} - (T_{i,j-1,k} + T_{i,j+1,k})}{(R_E \cos \phi_i) \Delta \lambda}, 0 \right\} \right]^2 \\ & + \left[\max \left\{ \frac{2T_{i,j,k} - (T_{i,j,k-1} + T_{i,j,k+1})}{\Delta h}, 0 \right\} \right]^2 = \frac{1}{(v_{i,j,k})^2} \end{aligned} \quad (4.43)$$

else, if $\mathbf{y}_{(\mathbf{i},\mathbf{j},\mathbf{k})} = \mathbf{y}_{(\mathbf{i}_{\text{Sn}},\mathbf{j}_{\text{Sn}},\mathbf{k}_{\text{Sn}})}$ then:

$$T_{i,j,k} = 0 \quad (4.44)$$

Now, suppose that $\mathbf{y}_{(i,j,k)} \neq \mathbf{y}_{(i_{Sn},j_{Sn},k_{Sn})}$, then there are 7 cases to consider when solving algebraically the discretised eikonal equation:

Algorithm 4.1 (Solving the discretised eikonal equation algebraically).

Case 1:

$$2T_{i,j,k} \geq \max \{ (T_{i-1,j,k} + T_{i+1,j,k}), (T_{i,j-1,k} + T_{i,j+1,k}), (T_{i,j,k-1} + T_{i,j,k+1}) \} \quad (4.45)$$

$$\begin{aligned} & \left[\frac{2T_{i,j,k} - (T_{i-1,j,k} + T_{i+1,j,k})}{(R_E \sin \phi_i) \Delta \phi} \right]^2 + \left[\frac{2T_{i,j,k} - (T_{i,j-1,k} + T_{i,j+1,k})}{(R_E \cos \phi_i) \Delta \lambda} \right]^2 \\ & + \left[\frac{2T_{i,j,k} - (T_{i,j,k-1} + T_{i,j,k+1})}{\Delta h} \right]^2 = \frac{1}{(v_{i,j,k})^2} \end{aligned} \quad (4.46)$$

$$\begin{aligned} a &= 4 \left[\frac{1}{(R_E \sin \phi_i)^2 (\Delta \phi)^2} + \frac{1}{(R_E \cos \phi_i)^2 (\Delta \lambda)^2} + \frac{1}{(\Delta h)^2} \right] \\ b &= -4 \left[\frac{T_{i-1,j,k} + T_{i+1,j,k}}{(R_E \sin \phi_i) \Delta \phi} + \frac{T_{i,j-1,k} + T_{i,j+1,k}}{(R_E \cos \phi_i) \Delta \lambda} + \frac{T_{i,j,k-1} + T_{i,j,k+1}}{\Delta h} \right] \\ c &= \frac{(T_{i-1,j,k} + T_{i+1,j,k})^2}{(R_E \sin \phi_i)^2 (\Delta \phi)^2} + \frac{(T_{i,j-1,k} + T_{i,j+1,k})^2}{(R_E \cos \phi_i)^2 (\Delta \lambda)^2} + \frac{(T_{i,j,k-1} + T_{i,j,k+1})^2}{(\Delta h)^2} - \frac{1}{(v_{i,j,k})^2} \end{aligned} \quad (4.47)$$

Case 2:

$$(T_{i,j,k-1} + T_{i,j,k+1}) \geq 2T_{i,j,k} \geq \max \{ (T_{i-1,j,k} + T_{i+1,j,k}), (T_{i,j-1,k} + T_{i,j+1,k}) \} \quad (4.48)$$

$$\left[\frac{2T_{i,j,k} - (T_{i-1,j,k} + T_{i+1,j,k})}{(R_E \sin \phi_i) \Delta \phi} \right]^2 + \left[\frac{2T_{i,j,k} - (T_{i,j-1,k} + T_{i,j+1,k})}{(R_E \cos \phi_i) \Delta \lambda} \right]^2 = \frac{1}{(v_{i,j,k})^2} \quad (4.49)$$

$$\begin{aligned} a &= 4 \left[\frac{1}{(R_E \sin \phi_i)^2 (\Delta \phi)^2} + \frac{1}{(R_E \cos \phi_i)^2 (\Delta \lambda)^2} \right] \\ b &= -4 \left[\frac{T_{i-1,j,k} + T_{i+1,j,k}}{(R_E \sin \phi_i) \Delta \phi} + \frac{T_{i,j-1,k} + T_{i,j+1,k}}{(R_E \cos \phi_i) \Delta \lambda} \right] \\ c &= \frac{(T_{i-1,j,k} + T_{i+1,j,k})^2}{(R_E \sin \phi_i)^2 (\Delta \phi)^2} + \frac{(T_{i,j-1,k} + T_{i,j+1,k})^2}{(R_E \cos \phi_i)^2 (\Delta \lambda)^2} - \frac{1}{(v_{i,j,k})^2} \end{aligned} \quad (4.50)$$

Case 3:

$$(T_{i,j-1,k} + T_{i,j+1,k}) \geq 2T_{i,j,k} \geq \max \{ (T_{i-1,j,k} + T_{i+1,j,k}), (T_{i,j,k-1} + T_{i,j,k+1}) \} \quad (4.51)$$

$$\left[\frac{2T_{i,j,k} - (T_{i-1,j,k} + T_{i+1,j,k})}{(R_E \sin \phi_i) \Delta \phi} \right]^2 + \left[\frac{2T_{i,j,k} - (T_{i,j,k-1} + T_{i,j,k+1})}{\Delta h} \right]^2 = \frac{1}{(v_{i,j,k})^2} \quad (4.52)$$

$$\begin{aligned}
a &= 4 \left[\frac{1}{(R_E \sin \phi_i)^2 (\Delta \phi)^2} + \frac{1}{(\Delta h)^2} \right] \\
b &= -4 \left[\frac{T_{i-1,j,k} + T_{i+1,j,k}}{(R_E \sin \phi_i) \Delta \phi} + \frac{T_{i,j,k-1} + T_{i,j,k+1}}{\Delta h} \right] \\
c &= \frac{(T_{i-1,j,k} + T_{i+1,j,k})^2}{(R_E \sin \phi_i)^2 (\Delta \phi)^2} + \frac{(T_{i,j,k-1} + T_{i,j,k+1})^2}{(\Delta h)^2} - \frac{1}{(v_{i,j,k})^2}
\end{aligned} \tag{4.53}$$

Case 4:

$$(T_{i-1,j,k} + T_{i+1,j,k}) \geq 2T_{i,j,k} \geq \max \{ (T_{i,j-1,k} + T_{i,j+1,k}), (T_{i,j,k-1} + T_{i,j,k+1}) \} \tag{4.54}$$

$$\left[\frac{2T_{i,j,k} - (T_{i,j-1,k} + T_{i,j+1,k})}{(R_E \cos \phi_i) \Delta \lambda} \right]^2 + \left[\frac{2T_{i,j,k} - (T_{i,j,k-1} + T_{i,j,k+1})}{\Delta h} \right]^2 = \frac{1}{(v_{i,j,k})^2} \tag{4.55}$$

$$\begin{aligned}
a &= 4 \left[\frac{1}{(R_E \cos \phi_i)^2 (\Delta \lambda)^2} + \frac{1}{(\Delta h)^2} \right] \\
b &= -4 \left[\frac{T_{i,j-1,k} + T_{i,j+1,k}}{(R_E \cos \phi_i) \Delta \lambda} + \frac{T_{i,j,k-1} + T_{i,j,k+1}}{\Delta h} \right] \\
c &= \frac{(T_{i,j-1,k} + T_{i,j+1,k})^2}{(R_E \cos \phi_i)^2 (\Delta \lambda)^2} + \frac{(T_{i,j,k-1} + T_{i,j,k+1})^2}{(\Delta h)^2} - \frac{1}{(v_{i,j,k})^2}
\end{aligned} \tag{4.56}$$

From **Case 1** to **Case 4**, solve the quadratic equation

$$a (T_{i,j,k})^2 + b (T_{i,j,k}) + c = 0 \tag{4.57}$$

with the quadratic formula:

$$T_{i,j,k} = \max \left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\} \tag{4.58}$$

Case 5:

$$\min \{ (T_{i,j-1,k} + T_{i,j+1,k}), (T_{i,j,k-1} + T_{i,j,k+1}) \} \geq 2T_{i,j,k} \geq (T_{i-1,j,k} + T_{i+1,j,k}) \tag{4.59}$$

$$\frac{2T_{i,j,k} - (T_{i-1,j,k} + T_{i+1,j,k})}{(R_E \sin \phi_i) \Delta \phi} = \frac{1}{v_{i,j,k}} \tag{4.60}$$

$$T_{i,j,k} = \frac{1}{2} \left[(T_{i-1,j,k} + T_{i+1,j,k}) + \frac{(R_E \sin \phi_i) \Delta \phi}{v_{i,j,k}} \right] \tag{4.61}$$

Case 6:

$$\min \{(T_{i-1,j,k} + T_{i+1,j,k}), (T_{i,j,k-1} + T_{i,j,k+1})\} \geq 2T_{i,j,k} \geq (T_{i,j-1,k} + T_{i,j+1,k}) \quad (4.62)$$

$$\frac{2T_{i,j,k} - (T_{i,j-1,k} + T_{i,j+1,k})}{(R_E \cos \phi_i) \Delta \lambda} = \frac{1}{v_{i,j,k}} \quad (4.63)$$

$$T_{i,j,k} = \frac{1}{2} \left[(T_{i,j-1,k} + T_{i,j+1,k}) + \frac{(R_E \cos \phi_i) \Delta \lambda}{v_{i,j,k}} \right] \quad (4.64)$$

Case 7:

$$\min \{(T_{i-1,j,k} + T_{i+1,j,k}), (T_{i,j,k-1} + T_{i,j,k+1})\} \geq 2T_{i,j,k} \geq (T_{i,j,k-1} + T_{i,j,k+1}) \quad (4.65)$$

$$\frac{2T_{i,j,k} - (T_{i,j,k-1} + T_{i,j,k+1})}{\Delta h} = \frac{1}{v_{i,j,k}} \quad (4.66)$$

$$T_{i,j,k} = \frac{1}{2} \left[(T_{i,j,k-1} + T_{i,j,k+1}) + \frac{\Delta h}{v_{i,j,k}} \right] \quad (4.67)$$

The algorithm shown below is the Fast Marching Method, which has a fast computational time, when solving numerically to find the shortest travel time from the n^{th} seismic station to any guessing hypocenter. [12] [13]

Algorithm 4.2 (Fast Marching Method).

Step 1: Let a starting point, (i_{Sn}, j_{Sn}, k_{Sn}) , representing n^{th} seismic station to have travel time value of 0, that is $T_{i_{Sn}, j_{Sn}, k_{Sn}} = 0$, and include the starting point in the accepted set A , equivalently:

$$A = \emptyset \cup \{(i_{Sn}, j_{Sn}, k_{Sn}) : T_{i_{Sn}, j_{Sn}, k_{Sn}} = 0\} \quad (4.68)$$

Also, for other points not in the accepted set, $(i, j, k) \notin A$, set them as:

$$T_{i,j,k} = +\infty \quad (4.69)$$

Step 2: Find neighbouring points around the starting point, and include them in the neighbour set B . Then, for each neighbouring points, $(i, j, k) \in B$, solve the algebraic equation of discrete eikonal equation (Algorithm 4.1) to calculate travel time, $T_{i,j,k}$.

Step 3: Select a neighbouring point with the smallest travel time value:

$$\left\{ (i, j, k) : \min_{(i,j,k) \in B} \{T_{i,j,k}\} \right\} \quad (4.70)$$

and include such neighbouring point in the accepted set:

$$A = A \cup \left\{ (i, j, k) : \min_{(i,j,k) \in B} \{T_{i,j,k}\} \right\} \quad (4.71)$$

and remove such neighbouring point from the neighbour set:

$$B = B - \left\{ (i, j, k) : \min_{(i,j,k) \in B} \{T_{i,j,k}\} \right\} \quad (4.72)$$

Step 4: Update the neighbour set B to include new neighbouring points, $(i, j, k) \in B'$, around the updated accepted set A , in other words:

$$B = B \cup B' \quad (4.73)$$

Step 5: For all new neighbouring points, $(i, j, k) \in B'$, around the updated accepted set A , solve the algebraic equation of discrete eikonal equation (Algorithm 4.1) to calculate travel time, $T_{i,j,k}$. Hence, each point (i, j, k) in updated neighbour set B , will have calculated travel time, $T_{i,j,k}$.

Step 6: Select a neighbouring point with the smallest travel time value:

$$\left\{ (i, j, k) : \min_{(i,j,k) \in B} \{T_{i,j,k}\} \right\} \quad (4.74)$$

and include such neighbouring point in the accepted set:

$$A = A \cup \left\{ (i, j, k) : \min_{(i,j,k) \in B} \{T_{i,j,k}\} \right\} \quad (4.75)$$

and remove such neighbouring point from the neighbour set:

$$B = B - \left\{ (i, j, k) : \min_{(i,j,k) \in B} \{T_{i,j,k}\} \right\} \quad (4.76)$$

Step 7: Repeat **Step 4** to **Step 6**, until all points (i, j, k) are in accepted set A , and none of the points (i, j, k) are in the neighbour set B .

4.3 Grid Search

After the shortest travel time between each n^{th} seismic station and a guessing hypocenter,

$$T(\mathbf{Y}_{(\mathbf{i}_{Sn}, \mathbf{j}_{Sn}, \mathbf{k}_{Sn})}, \mathbf{Y}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})}) = T(\phi_{i_{Sn}}, \lambda_{i_{Sn}}, h_{i_{Sn}}, \phi_i, \lambda_j, h_k) = T_{i,j,k} \quad (4.77)$$

is solved numerically by Fast Marching Method, where latitude and longitude are in radian, we will need to find the best fit hypocenter and origin time, such that the average of square of difference between observed arrival time, t_{Sn} , and calculated arrival time of each n^{th} seismic station should be minimal.

Definition 4.1 (Optimisation Formulation for Grid Search).

$$\min \left\{ \frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 \right\} \quad (4.78)$$

subject to:

$$\begin{aligned} \phi_0 &\leq \phi_i = \phi_0 + i\Delta\phi \leq \phi_0 + I\Delta\phi = \phi_I \\ \lambda_0 &\leq \lambda_j = \lambda_0 + j\Delta\lambda \leq \lambda_0 + J\Delta\lambda = \lambda_J \\ h_0 &\leq h_k = h_0 + k\Delta h \leq h_0 + K\Delta h = h_K \end{aligned} \quad (4.79)$$

where all latitude and longitude of the seismic stations and the Grid Search constraint mentioned above are in radian. Also, t_0 represents the guessing origin time for each guessing hypocenter, (ϕ_i, λ_j, h_k) .

Theorem 4.3 (Origin Time for Grid Search). *The minimum average of square of difference between observed arrival time, t_{Sn} , and calculated arrival time of each n^{th} seismic station, for each guessing hypocenter, (ϕ_i, λ_j, h_k) can be expressed as:*

$$t_0 = \frac{1}{N} \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \quad (4.80)$$

Proof. Find the derivative of sum of square of time residues and let it be equals to 0:

$$\begin{aligned}
& \frac{\partial}{\partial t_0} \left[\frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 \right] \\
&= \sum_{n=1}^N \frac{1}{2} \frac{\partial}{\partial t_0} [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 \\
&= \sum_{n=1}^N [-t_{Sn} + t_0 + T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \\
&= \sum_{n=1}^N t_0 + \sum_{n=1}^N [-t_{Sn} + T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \\
&= Nt_0 - \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \\
&= 0
\end{aligned} \tag{4.81}$$

Then, the corresponding guessing origin time for any guessing hypocenter:

$$t_0 = \frac{1}{N} \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \tag{4.82}$$

Perform the second derivative test at the derived t_0 equation:

$$\begin{aligned}
& \frac{\partial^2}{\partial t_0^2} \left[\frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 \right] \\
&= \frac{\partial}{\partial t_0} \left[\frac{\partial}{\partial t_0} \left[\frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 \right] \right] \\
&= \frac{\partial}{\partial t_0} \left[Nt_0 - \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \right] \\
&= \frac{\partial}{\partial t_0} (Nt_0) - \frac{\partial}{\partial t_0} \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \\
&= N \\
&\geq 1 \\
&> 0
\end{aligned} \tag{4.83}$$

$\therefore$ Since the second derivative is positive at the derived t_0 equation, the guessing origin time for any guessing hypocenter, that minimise sum of square of time

residues is:

$$t_0 = \frac{1}{N} \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)] \quad (4.84)$$

□

The algorithm shown below is the Grid Search, which is about finding all possible values of the guessing hypocenter, (ϕ_i, λ_j, h_k) and origin time t_0 within a constraint

$$\begin{aligned} \phi_0 &\leq \phi_i = \phi_0 + i\Delta\phi \leq \phi_0 + I\Delta\phi = \phi_I \\ \lambda_0 &\leq \lambda_j = \lambda_0 + j\Delta\lambda \leq \lambda_0 + J\Delta\lambda = \lambda_J \\ h_0 &\leq h_k = h_0 + k\Delta h \leq h_0 + K\Delta h = h_K \end{aligned} \quad (4.85)$$

until the global minimum of

$$\frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 \quad (4.86)$$

within the constraint is found, provided that the sampling interval size and the total number of samples are set appropriately. The latitude and longitude of the final hypocenter in radian with depth and origin time will be expressed as (ϕ, λ, h, t) and converted to degree as $(\phi^\circ, \lambda^\circ, h, t)$.

Algorithm 4.3 (Grid Search Algorithm).

Step 1: Execute the following, where latitude and longitude are in radian:

```

for  $i = 0, 1, \dots, I$ 
{
  for  $j = 0, 1, \dots, J$ 
  {
    for  $k = 0, 1, \dots, K$ 
    {
       $t_0 = \frac{1}{N} \sum_{n=1}^N [t_{Sn} - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]$ 
      if [ $i \equiv 0$  and  $j \equiv 0$  and  $k \equiv 0$ ]
      or  $\frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2 < E_{min}$ 
      {
         $E_{min} = \frac{1}{2} \sum_{n=1}^N [t_{Sn} - t_0 - T(\phi_{Sn}, \lambda_{Sn}, h_{Sn}, \phi_i, \lambda_j, h_k)]^2$ 
        Update accepted hypocenter and origin time as:
         $(\phi, \lambda, h, t) = (\phi_i, \lambda_j, h_k, t_0)$ 
        Latitude and longitude of accepted hypocenter are in radian.
      }
    }
  }
}

```

Step 2: The final hypocenter and origin time, where latitude and longitude are in radian:

$$(\phi, \lambda, h, t) \quad (4.87)$$

Step 3: Output the final hypocenter and origin time, where latitude and longitude are in degree:

$$(\phi^\circ, \lambda^\circ, h, t) = \left(\frac{180}{\pi} \phi, \frac{180}{\pi} \lambda, h, t \right) \quad (4.88)$$

4.4 Application: Southern California

We perform Grid Search and Fast Marching Method for Southern California with the following 4 layer S wave velocity model and constraint, where latitude and longitude of each seismic station and guessing hypocenter are to be converted from degree to radian when performing Grid Search and Fast Marching Method, also the algorithm will convert the latitude and longitude of final hypocenter from radian to degree.

4 layer S wave velocity model

- Depth 5.5 km or less: 3.1792 km/s
- Depth 5.5 to 16 km: 3.6416 km/s
- Depth 16 to 32 km: 3.8728 km/s
- Depth 32 km and beyond: 4.5086 km/s

Constraint for Grid Search

- Latitude range in Degree:

$$34.766^\circ = \phi_0^\circ \leq \phi_i^\circ \leq \phi_I^\circ = 36.766^\circ \quad (4.89)$$

- Longitude range in Degree:

$$-118.065^\circ = \lambda_0^\circ \leq \lambda_i^\circ \leq \lambda_J^\circ = -116.065^\circ \quad (4.90)$$

- Depth range in km:

$$-4 \text{ km} = h_0 \leq h_k \leq h_K = 40 \text{ km} \quad (4.91)$$

FIGURE 4.1: GMT Plot of all selected 15 Earthquake Events provided by SCEDC

FIGURE 4.2: GMT Plot of all selected 15 Earthquake Events for Grid Search and Fast Marching Method using S arrival times

SCEDC			GS and FMM: 4 Layer S		
Lat	Lon	Dep	Lat	Lon	Dep
35.7990	-118.0440	7.54	35.7993	-118.0410	6.40
35.7960	-118.0460	7.19	35.7953	-118.0370	5.40
35.9950	-117.8770	2.31	35.6033	-117.8850	4.40
35.9462	-117.3030	2.62	35.6633	-117.3130	-0.20
35.6218	-117.5880	7.93	35.9793	-117.5970	6.00
36.0710	-117.9020	2.98	35.5233	-117.9050	8.60
35.4733	-118.4250	4.80	36.1273	-118.4250	4.20
35.9713	-117.3200	2.46	35.6353	-117.3330	0.40
35.9712	-117.3220	2.33	35.6393	-117.3290	1.80
35.4135	-117.7860	7.42	36.1873	-117.8090	11.60
36.3977	-117.8640	7.59	35.2393	-117.8610	-1.80
35.6807	-117.6200	6.12	35.9353	-117.6290	3.20
35.8298	-116.8000	7.67	35.7673	-116.8650	-3.40
35.4860	-118.2870	5.54	36.1153	-118.2890	4.80
35.6758	-117.4850	8.53	35.9193	-117.5090	10.60

TABLE 4.1: Selected 15 Earthquake Events: SCEDC provided vs Grid Search and Fast Marching Method using S arrival times

Chapter 5

Conclusion

For Kyushu Japan, there are a total of 222 earthquake events, are relocated using 5396 P wave arrival times only, while 10 earthquake events can be relocated by using 383 P or 55 PmP or both arrival times (total of 438 arrival times), all by Geiger's Method. In Southern California, 15 earthquake events are relocated using 97 S wave arrival times by Grid Search and Fast Marching Method.

In Kyushu Japan, by Geiger's Method, to locate epicenter of earthquakes by using P arrival times and combined P and PmP arrival times, is more accurate than using PmP arrival times only, using epicenter of earthquakes provided by Hinet Japan as main reference for comparison. For depth of each earthquake, the initial guess of earthquake depth is being selected a typical value carefully, to ensure that the earthquake depth in all algorithms of Geiger's Method are about the same as the one provided by Hinet Japan. Therefore, by Geiger's Method, to locate epicenter of earthquakes by using P arrival times and combined P and PmP arrival times is more feasible than using PmP arrival times only, in order to locate epicenter of earthquakes. However, for all cases of Geiger's Method in this thesis, they are generally not feasible for locating earthquake depth.

In Southern California, the epicentre locations (latitude and longitude of hypocenter) in both SCEDC provided earthquake locations, and Grid Search and Fast Marching Method, are about the same as each other. However, earthquake depths in SCEDC provided earthquake locations, and Grid Search and Fast Marching Method, are different from each other. Actually, in Grid Search and Fast Marching Method, a very minority have earthquake depth less than 0, that is because the earthquake location problem in Grid Search and Fast Marching Method are ill-posed in general.

Overall, both Geiger's Method, and Grid Search and Fast Marching Method, are feasible for locating epicenter of earthquakes. However, for both methods, we need to be careful when locating earthquake depth. Because the earthquake depth estimate might be very uncertain and the error estimate must be checked [10].

Appendix A

Kyushu Japan with 222 Events

TABLE A.1: Kyushu Japan with 222 Events

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
Lat	Lon	Dep	Lat	Lon	Dep	Lat	Lon	Dep
31.43	131.80	18.0	31.3655	131.2052	9.0448	31.4031	131.1966	9.0456
31.63	130.22	2.4	31.3938	130.2428	9.0048	31.4187	130.2650	9.0057
32.09	129.91	9.5	31.9380	129.9078	9.0044	31.9583	129.9313	9.0049
32.50	130.55	10.0	32.5451	130.5920	9.0746	32.5432	130.5961	9.0719
32.93	130.55	13.0	32.9546	130.5800	9.0332	32.9514	130.5817	9.0288
32.50	130.56	10.0	32.5427	130.6024	9.0696	32.5415	130.6066	9.0668
32.50	130.56	9.0	32.5271	130.6022	9.0681	32.5263	130.6060	9.0650
31.33	130.53	4.0	31.0538	130.6191	8.9972	31.0907	130.6266	8.9969
32.49	130.58	7.6	32.5294	130.6296	9.0781	32.5269	130.6337	9.0735
31.25	130.04	4.0	30.8077	130.2454	8.9992	30.8429	130.2535	8.9994
32.89	130.72	8.1	32.9147	130.7309	9.0382	32.9139	130.7321	9.0352
32.54	129.14	4.0	32.0772	129.3930	8.9999	32.1170	129.4007	8.9999
31.93	131.18	12.0	31.8416	131.1229	9.0585	31.8526	131.1141	9.0535
31.93	131.17	12.0	31.8456	131.1137	9.0559	31.8579	131.1040	9.0509
31.93	131.18	12.0	31.8519	131.1212	9.0606	31.8591	131.1148	9.0503
31.96	130.13	10.0	31.9205	130.2316	9.0064	31.9381	130.2533	9.0077
32.50	130.55	11.0	32.5507	130.5977	9.0684	32.5485	130.6005	9.0667
32.97	130.86	13.0	33.0198	130.9230	9.0598	33.0179	130.9258	9.0526
32.02	130.33	13.0	32.1022	130.3888	9.0157	32.1012	130.4023	9.0134

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
32.67	130.94	11.7	32.5829	130.8763	9.1148	32.5838	130.8782	9.1032
31.99	130.00	11.0	31.9055	130.0735	9.0035	31.9252	130.0932	9.0040
31.99	130.00	10.0	31.8792	130.0586	9.0036	31.9000	130.0786	9.0042
32.13	129.45	2.0	31.8533	129.5461	9.0002	31.8672	129.5701	9.0003
32.13	129.46	3.0	31.8378	129.5368	9.0002	31.8500	129.5622	9.0003
32.28	129.73	10.0	32.0600	129.7991	9.0032	32.0854	129.8209	9.0040
32.28	129.73	11.0	32.0781	129.8182	9.0032	32.1010	129.8375	9.0037
32.29	129.74	11.0	32.0759	129.8118	9.0035	32.1018	129.8345	9.0042
32.27	129.75	12.0	32.1325	129.8300	9.0022	32.1519	129.8518	9.0026
32.97	130.15	6.2	33.0181	130.1421	9.0065	33.0198	130.1554	9.0040
32.10	129.97	9.0	32.0372	130.0648	9.0037	32.0529	130.0876	9.0041
32.44	129.98	13.0	32.4674	130.1748	9.0648	32.4726	130.1785	9.0600
31.58	130.29	8.0	31.3591	130.3222	9.0057	31.3912	130.3417	9.0063
32.17	129.51	8.0	31.8275	129.6013	9.0000	31.8521	129.6226	9.0001
32.77	131.31	14.0	32.7944	131.3110	9.0473	32.7940	131.3032	9.0410
32.49	130.59	12.0	32.5155	130.6509	9.0584	32.5117	130.6558	9.0516
32.10	129.89	12.0	31.8819	129.9757	9.0014	31.9086	129.9938	9.0016
32.09	129.89	11.0	31.9869	129.9685	9.0044	32.0058	129.9906	9.0049
32.08	129.88	12.0	32.4089	130.2157	9.0529	32.4019	130.2195	9.0456
32.08	129.88	12.0	32.0622	129.9562	9.0118	32.0777	129.9808	9.0130
31.97	130.46	9.0	31.9371	130.4996	9.0077	31.9458	130.5165	9.0072
32.47	129.84	12.0	32.2189	129.8418	9.0128	32.2662	129.8728	9.0176
32.10	129.89	12.0	31.8965	129.9766	9.0012	31.9229	129.9951	9.0014
32.39	130.66	10.0	32.3621	130.7027	9.0479	32.3623	130.7078	9.0438
32.72	130.71	14.0	32.6737	130.7738	9.0525	32.6771	130.7759	9.0506
32.08	129.88	12.0	32.0668	129.9537	9.0125	32.0824	129.9784	9.0135
32.08	129.88	3.8	32.0665	129.9490	9.0128	32.0820	129.9740	9.0138
32.87	130.62	12.0	32.8566	130.6501	9.0502	32.8590	130.6522	9.0484
32.47	129.84	12.0	32.3739	129.9558	9.0461	32.3787	129.9578	9.0409
32.09	129.88	10.0	31.8878	130.0002	9.0016	31.9144	130.0175	9.0020
32.44	129.82	12.0	32.1400	129.8371	9.0103	32.1808	129.8598	9.0120
31.60	129.55	11.0	31.2332	129.6754	9.0002	31.2506	129.7015	9.0003
32.87	130.62	12.0	32.8381	130.6549	9.0265	32.8360	130.6547	9.0234

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
31.90	131.02	11.0	31.8203	130.9926	9.0092	31.8345	130.9838	9.0059
31.97	130.47	5.0	32.0094	130.4778	9.0218	32.0082	130.4975	9.0211
32.00	130.09	9.0	31.9132	130.1300	9.0129	31.9372	130.1531	9.0147
32.82	130.93	12.0	32.8385	130.9577	9.0640	32.8374	130.9577	9.0565
32.70	130.79	10.0	32.6785	130.8372	9.0531	32.6802	130.8401	9.0503
32.11	129.80	11.0	31.8189	129.9146	9.0012	31.8505	129.9321	9.0015
32.51	129.82	7.3	32.4787	129.9504	9.0418	32.4790	129.9511	9.0331
31.64	130.02	7.0	31.4467	130.0684	9.0028	31.4617	130.0911	9.0037
31.65	130.02	8.0	31.4635	130.0640	9.0027	31.4777	130.0869	9.0034
31.97	130.40	2.0	32.0250	130.4464	9.0192	32.0212	130.4710	9.0204
32.39	130.66	11.0	32.3347	130.7059	9.0391	32.3371	130.7106	9.0355
32.93	130.39	11.0	32.9637	130.4114	9.0412	32.9625	130.4165	9.0340
32.06	130.00	10.0	32.0789	130.0937	9.0155	32.0980	130.1166	9.0172
31.96	130.15	6.0	31.9645	130.2071	9.0213	31.9786	130.2324	9.0225
31.96	130.15	6.0	31.9392	130.1775	9.0227	31.9574	130.2067	9.0282
32.14	130.38	7.0	32.1720	130.4775	9.0420	32.1741	130.4951	9.0419
31.97	130.43	7.0	31.9982	130.4756	9.0137	31.9986	130.4934	9.0121
31.64	130.02	7.0	31.4570	130.0732	9.0027	31.4718	130.0957	9.0035
32.86	130.77	13.0	32.8512	130.7949	9.0488	32.8507	130.7953	9.0487
32.59	130.81	5.1	32.6215	130.8871	9.0388	32.6202	130.8871	9.0356
32.58	130.79	10.0	32.6449	130.8869	9.0512	32.6439	130.8875	9.0501
32.62	130.83	5.0	32.5739	130.8674	9.0695	32.5778	130.8699	9.0671
32.96	129.86	12.0	33.0694	129.9428	9.0208	33.0655	129.9646	9.0187
32.09	130.51	9.4	32.0917	130.5505	9.0351	32.0947	130.5603	9.0286
32.92	131.01	5.0	32.9651	131.0342	9.0551	32.9638	131.0343	9.0502
32.57	129.89	12.0	32.6088	130.0251	9.0448	32.6114	130.0310	9.0381
31.74	130.00	12.0	31.6269	130.0927	9.0014	31.6382	130.1132	9.0018
32.79	130.63	13.0	32.7921	130.6636	9.0633	32.7940	130.6655	9.0603
32.60	129.97	13.0	32.6312	130.0656	9.0327	32.6372	130.0757	9.0272
32.60	129.98	7.3	32.6168	130.0706	9.0307	32.6189	130.0745	9.0224
31.74	130.00	11.0	31.6048	130.0694	9.0018	31.6164	130.0907	9.0023
31.23	129.83	5.0	30.7156	130.0054	9.0000	30.7513	130.0213	9.0000
31.23	129.83	6.0	30.6890	129.9822	9.0000	30.7255	129.9989	9.0000

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
32.87	130.62	12.0	32.8594	130.6523	9.0426	32.8619	130.6559	9.0368
32.08	130.54	7.0	32.0922	130.5678	9.0238	32.0953	130.5822	9.0207
32.67	130.04	15.0	32.6692	130.1401	9.0323	32.6813	130.1537	9.0309
32.56	129.96	10.6	32.6385	130.0949	9.0655	32.6431	130.1029	9.0628
32.29	129.77	13.0	32.0670	129.8052	9.0035	32.0926	129.8270	9.0043
31.23	129.83	7.0	30.7257	130.0220	8.9998	30.7611	130.0374	8.9999
32.15	131.91	16.9	33.0628	131.2115	9.0230	33.0216	131.2339	9.0346
31.22	129.82	2.0	30.6563	130.0524	8.9999	30.6951	130.0657	8.9999
32.38	130.68	10.0	32.3580	130.7168	9.0493	32.3588	130.7220	9.0486
31.97	130.96	9.0	31.9156	130.9320	9.0355	31.9220	130.9304	9.0264
32.26	130.52	8.0	32.2272	130.5655	9.0258	32.2307	130.5767	9.0212
32.79	130.62	12.0	32.7753	130.6500	9.0561	32.7775	130.6515	9.0532
32.39	130.65	11.0	32.3218	130.7056	9.0277	32.3238	130.7099	9.0244
31.63	130.01	5.6	31.4439	130.0569	9.0029	31.4588	130.0800	9.0037
31.97	130.84	2.0	31.9393	130.8078	9.0362	31.9458	130.8105	9.0306
32.90	130.09	10.0	32.9587	130.1469	9.0410	32.9604	130.1607	9.0374
32.79	130.63	12.0	32.7830	130.6579	9.0535	32.7853	130.6599	9.0507
32.17	129.44	5.0	31.7909	129.5314	9.0002	31.8165	129.5540	9.0002
32.17	129.44	3.0	31.7656	129.5043	9.0002	31.7916	129.5278	9.0002
32.65	129.77	8.0	32.4776	129.7584	9.0264	32.5060	129.7860	9.0287
32.88	130.81	12.7	32.8953	130.8614	9.0670	32.8948	130.8627	9.0644
31.95	130.13	7.0	31.9303	130.1644	9.0224	31.9496	130.1926	9.0274
32.56	129.99	15.0	32.6289	130.1242	9.0701	32.6281	130.1248	9.0633
32.61	130.71	6.0	32.6090	130.7592	9.0798	32.6100	130.7638	9.0773
32.22	131.55	17.0	32.1883	131.3903	9.1511	32.2090	131.3728	9.1765
31.24	130.63	8.0	31.0015	130.7490	8.9986	31.0403	130.7501	8.9985
32.22	131.55	18.0	32.2272	131.3493	9.3431	32.2263	131.3494	9.3153
32.22	130.30	9.0	32.2424	130.4065	9.0602	32.2466	130.4044	9.0511
32.22	130.30	9.0	32.2422	130.4155	9.0349	32.2476	130.4290	9.0347
32.54	130.95	12.0	32.5332	130.9672	9.0272	32.5320	130.9672	9.0224
31.51	129.65	13.0	31.0655	129.8365	9.0001	31.0946	129.8560	9.0002
32.06	129.84	10.0	32.0213	129.9294	9.0027	32.0350	129.9513	9.0032
32.11	130.76	10.0	32.0981	130.7600	9.0260	32.0978	130.7651	9.0225

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
32.57	129.96	7.0	32.6212	130.0962	9.0406	32.6290	130.1073	9.0390
32.97	129.92	11.2	33.1044	129.9609	9.0253	33.0987	129.9795	9.0198
32.98	129.90	12.0	33.0927	129.9727	9.0229	33.0877	129.9909	9.0168
32.98	129.90	12.0	33.0944	129.9396	9.0265	33.0901	129.9633	9.0238
32.97	129.90	8.1	33.0855	129.9557	9.0240	33.0811	129.9749	9.0182
32.99	129.90	12.0	33.1045	129.9668	9.0220	33.0990	129.9882	9.0198
31.72	130.19	8.0	31.6680	130.2443	9.0103	31.6843	130.2691	9.0133
31.95	131.98	17.9	33.2033	131.2288	9.0056	33.1639	131.2303	9.0044
32.31	130.52	7.4	32.3159	130.5979	9.0361	32.3155	130.6046	9.0328
32.08	129.73	11.0	31.9097	129.8217	9.0011	31.9243	129.8431	9.0018
31.30	130.57	5.0	31.0543	130.6905	8.9998	31.0926	130.6953	8.9999
32.51	130.91	11.0	32.4877	130.9401	9.0576	32.4883	130.9408	9.0524
32.32	130.50	9.0	32.2781	130.5684	9.0381	32.2808	130.5783	9.0331
32.32	130.50	8.0	32.2743	130.5698	9.0336	32.2778	130.5793	9.0284
32.32	130.50	8.0	32.2238	130.5619	9.0506	32.2295	130.5693	9.0418
32.32	130.53	7.0	32.3018	130.5841	9.0502	32.3020	130.5944	9.0458
31.02	130.29	9.0	30.6201	130.4726	8.9965	30.6520	130.4826	8.9967
32.08	129.93	11.0	31.8873	130.0393	9.0023	31.9158	130.0566	9.0027
32.08	129.93	12.0	31.9766	130.0201	9.0081	31.9970	130.0415	9.0092
31.28	131.89	7.8	30.6385	131.7215	9.0000	30.6784	131.7048	9.0000
32.99	130.82	10.0	33.0419	130.8660	9.0457	33.0401	130.8692	9.0424
32.20	130.20	13.0	32.1901	130.2845	9.0458	32.2016	130.2978	9.0446
32.92	132.00	16.0	32.9228	131.9897	9.0284	32.9324	131.9827	9.0252
32.91	130.70	15.0	32.8973	130.7110	9.0528	32.8991	130.7133	9.0495
32.96	131.08	6.0	33.0309	131.1030	9.0599	33.0266	131.0999	9.0552
32.79	130.71	10.0	32.7108	130.7444	8.9820	32.7135	130.7446	8.9776
32.64	130.05	13.0	32.6804	130.1613	9.0552	32.6862	130.1707	9.0547
32.64	130.05	13.0	32.6729	130.1586	9.0731	32.6742	130.1587	9.0662
32.09	129.93	11.0	31.9638	130.0081	9.0079	31.9889	130.0308	9.0097
31.97	130.42	9.0	32.0087	130.4660	9.0110	32.0091	130.4866	9.0103
32.49	131.97	17.2	32.2745	132.0197	9.0218	32.3093	132.0040	9.0243
31.42	129.78	11.0	31.0691	129.8890	9.0002	31.0955	129.9090	9.0002
32.55	130.60	16.8	32.6274	130.6636	9.0760	32.6248	130.6658	9.0689

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
32.23	130.40	5.0	32.2358	130.4928	9.0418	32.2365	130.5100	9.0399
32.86	130.74	8.0	32.8224	130.7665	9.0463	32.8275	130.7686	9.0421
32.86	130.74	9.0	32.8262	130.7681	9.0398	32.8311	130.7700	9.0358
32.86	130.74	8.0	32.8258	130.7693	9.0451	32.8308	130.7712	9.0410
32.77	131.32	13.0	32.7837	131.2665	9.0858	32.7836	131.2588	9.0791
32.88	130.91	11.0	32.9221	130.9626	9.0547	32.9200	130.9645	9.0507
32.65	131.65	16.0	32.6273	131.5369	9.0507	32.6372	131.5194	9.0580
32.65	131.65	16.0	32.6167	131.5618	9.0400	32.6240	131.5498	9.0432
32.59	130.65	12.0	32.5968	130.7024	9.0666	32.5965	130.7054	9.0629
31.61	129.63	6.0	31.1809	129.8031	9.0002	31.2083	129.8238	9.0003
32.96	130.99	10.0	33.0328	131.0270	9.0514	33.0300	131.0273	9.0465
32.91	130.85	9.0	32.9374	130.9135	9.0466	32.9362	130.9156	9.0423
32.92	130.86	9.0	32.9292	130.9127	9.0605	32.9293	130.9166	9.0562
31.97	130.38	5.8	31.9924	130.4822	9.0434	31.9957	130.4937	9.0400
31.97	130.43	9.0	32.0151	130.4521	9.0157	32.0146	130.4681	9.0123
32.91	130.85	9.0	32.9167	130.9108	9.0664	32.9148	130.9105	9.0589
32.56	130.05	14.0	32.6452	130.1941	9.0357	32.6492	130.2005	9.0311
31.91	131.00	9.0	31.8538	130.9561	9.0210	31.8645	130.9530	9.0185
32.93	130.85	11.0	32.9405	130.8957	9.0516	32.9401	130.8991	9.0473
32.92	130.85	11.0	32.9508	130.9016	9.0741	32.9484	130.9039	9.0700
32.73	130.74	11.0	32.6826	130.7876	9.0472	32.6862	130.7911	9.0439
32.76	130.58	12.5	32.7059	130.7132	9.0521	32.7098	130.7110	9.0496
32.89	131.70	14.0	32.9286	131.6572	9.1008	32.9296	131.6505	9.0983
32.93	130.97	9.0	32.9514	131.0090	9.0439	32.9503	131.0098	9.0389
32.09	130.93	11.0	32.0610	130.8784	9.0083	32.0629	130.8789	9.0038
31.37	129.72	5.0	30.8286	129.8917	9.0001	30.8657	129.9091	9.0001
31.71	131.87	17.1	32.7980	130.7630	9.0069	32.7700	130.7853	9.0077
31.64	131.89	13.8	32.8935	130.8540	9.0028	32.8559	130.8688	9.0032
32.41	131.49	13.0	32.3855	131.3865	9.0272	32.3963	131.3741	9.0247
32.08	130.54	7.0	32.0931	130.5884	9.0170	32.0964	130.6013	9.0140
32.77	130.51	12.0	32.7774	130.5361	9.0521	32.7805	130.5402	9.0479
32.06	130.94	9.0	32.0523	130.8994	9.0253	32.0552	130.8988	9.0205
32.06	130.93	9.0	32.0326	130.8873	9.0388	32.0403	130.8874	9.0386

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
32.78	130.72	12.0	32.8999	130.8247	9.0845	32.8956	130.8239	9.0822
32.60	129.84	11.0	32.5821	129.9788	9.0674	32.5857	129.9837	9.0624
32.37	131.97	19.0	32.1935	131.8756	9.0589	32.2241	131.8624	9.0598
31.88	131.08	9.0	31.8269	131.0471	9.0196	31.8380	131.0396	9.0156
31.98	130.00	11.0	31.8895	130.0748	9.0037	31.9096	130.0943	9.0043
31.81	131.10	8.8	31.8362	131.0384	9.0763	31.8379	131.0344	9.0693
32.06	130.92	8.0	32.0381	130.8757	9.0334	32.0421	130.8757	9.0289
32.39	130.67	9.0	32.3609	130.7102	9.0479	32.3613	130.7150	9.0439
32.01	129.83	10.0	31.9561	129.9565	9.0093	31.9760	129.9814	9.0130
32.18	130.52	8.0	32.1797	130.5424	9.0472	32.1825	130.5543	9.0435
32.09	131.97	14.1	33.0664	131.0454	9.0215	33.0404	131.0671	9.0247
31.88	131.08	10.0	31.8154	131.0447	9.0219	31.8244	131.0391	9.0161
32.18	130.52	8.0	32.1862	130.5592	9.0305	32.1878	130.5680	9.0268
32.03	129.41	2.0	31.6689	129.5280	9.0001	31.6911	129.5516	9.0001
32.06	130.00	11.0	32.0672	130.0623	9.0130	32.0894	130.0863	9.0155
31.97	130.42	8.0	32.0139	130.4698	9.0109	32.0135	130.4874	9.0087
32.65	130.61	11.0	32.6801	130.6797	9.0547	32.6778	130.6796	9.0515
33.00	130.76	10.0	33.0436	130.7821	9.0654	33.0415	130.7831	9.0628
32.47	129.83	13.0	32.3723	129.9533	9.0338	32.3764	129.9547	9.0263
32.07	130.94	8.0	32.0590	130.9040	9.0232	32.0609	130.9030	9.0179
32.76	130.94	12.0	32.7969	131.0640	8.9998	32.7966	131.0619	8.9944
32.79	131.03	6.0	32.8133	131.0640	9.0663	32.8128	131.0647	9.0622
33.00	130.34	12.0	33.0417	130.3480	9.0198	33.0410	130.3559	9.0166
32.08	129.87	12.0	32.0670	129.9481	9.0149	32.0834	129.9738	9.0162
32.96	130.95	11.0	33.0118	130.9925	9.0627	33.0099	130.9940	9.0587
32.51	130.54	11.0	32.5546	130.5929	9.0637	32.5512	130.5956	9.0597
32.08	129.87	12.0	32.0537	129.9182	9.0135	32.0711	129.9454	9.0159
32.08	129.87	12.0	32.0877	129.9647	9.0141	32.1017	129.9887	9.0149
32.08	129.87	12.0	32.0788	129.9529	9.0139	32.0941	129.9779	9.0148
32.08	129.85	9.0	31.9452	129.9405	9.0048	31.9664	129.9596	9.0061
31.22	130.69	5.0	30.9772	130.7951	8.9980	31.0137	130.7952	8.9978
32.80	130.52	12.0	32.8188	130.5511	9.0535	32.8200	130.5543	9.0493
32.51	130.58	10.0	32.5542	130.6181	9.0756	32.5505	130.6219	9.0727

Hinet Japan			Geiger: 1 Layer P			Geiger: 2 Layer P		
32.96	130.94	11.0	33.0080	130.9858	9.0462	33.0069	130.9881	9.0419
32.39	130.64	10.0	32.3675	130.6803	9.0528	32.3675	130.6866	9.0511
32.39	130.64	10.0	32.3666	130.6762	9.0606	32.3658	130.6824	9.0562
32.26	130.57	7.7	32.1968	130.6016	9.0323	32.2016	130.6104	9.0277
32.04	129.85	12.0	31.9515	129.9035	9.0044	31.9688	129.9269	9.0052

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