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Asymmetric interdependent networks with multiple-dependence relation

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In this paper, we study the robustness of interdependent networks with multiple-dependence (MD) relation which is defined that a node is interdependent on several nodes on another layer, and this node will fail if any of these dependent nodes are failed. We propose a two-layered asymmetric interdependent network (AIN) model to address this problem, where the asymmetric feature is that nodes in one layer may be dependent on more than one node in the other layer with MD relation, while nodes in the other layer are dependent on exactly one node in this layer. We show that in this model the layer where nodes are allowed to have MD relation exhibits different types of phase transitions (discontinuous and hybrid), while the other layer only presents discontinuous phase transition. A heuristic theory based on message-passing approach is developed to understand the structural feature of interdependent networks and an intuitive picture for the emergence of a tricritical point is provided. Moreover, we study the correlation between the intralayer degree and interlayer degree of the nodes and find that this correlation has prominent impact to the continuous phase transition but has feeble effect on the discontinuous phase transition. Furthermore, we extend the two-layered AIN model to general multilayered AIN, and the percolation behaviors and properties of relevant phase transitions are elaborated.

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I. INTRODUCTION

Interdependent networks have become a powerful paradigm to characterize complex systems in which components may be dependent on each other [1–5]. Interdependent networks are described under the framework of multilayer networks [6], and an essential feature of them is that a node in one layer may be dependent on nodes in other layers, in the sense that the functioning of this node depends on the functioning of the depended nodes in other layers. It is shown that this feature of interdependent networks allows spreading the damage of nodes from one layer to others and an abrupt collapse may happen where the size of connected cluster changes to zero discontinuously upon a removal of a critical fraction of nodes [1,2]. This discontinuous percolation transition is essentially different from the continuous transition that is typical on single-layer networks. The robustness of the networks may be affected by delicate structure of interdependent networks including degree distribution [7,8], degree correlation [9–11], link overlap [12], clustering [13,14], intralayer dependency [15], intersimilarity [16,17], spatially embedded networks [18–21], and k -core percolation [22], etc.

A vast array of theoretical results characterize the critical percolation condition of interdependent networks, mainly under the assumption that a node in one layer is mutually dependent on exactly one node in the other layer [1]. Many

realistic systems, however, show that a component in a system may be dependent on multiple other components [23–25]. For instance, communication and power systems [26] are shown to have multiple-support relation, where a node (communication or power station) in one layer may be functioning if any of its supporting nodes (power or communication station) in other layers are functioning. The impact of this relation on interdependent networks has been studied in Refs. [26,27]. However, the opposite case where the functioning of a node relies on the simultaneous functioning of all its dependent nodes in other layers, here referred as multiple-dependence (MD) relation, is scarcely addressed.

In fact, the MD relation widely exists in the real world, including biological, technological, and social systems. For example, in gene regulatory networks, genes and proteins are multiply influenced by each other, reflected by the fact that the production of a protein requires the correct expression of relevant genes, and the expression of a gene requires the right condition involving the emergence of related protein as enzyme [28]. Other examples including memory or storage shared by multiple computing facilities, which may be corrupted when any of these facilities are compromised, or the development of a growing company depends on steady capital support from one or multiple resources.

In this paper, we investigate the impact of MD relation to the robustness of interdependent networks. The MD relation could make a dependent node to be rather vulnerable, and more seriously if the MD relation occurs to nodes in different layers, a feedback loop among the layers can be established which may induce a full collapse of the whole network even

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upon the failure of a single node, regardless of the connectivity within the layers. In order to prevent the emergence of this devasting condition, hence permitting quantitative investigation on the effect of MD relation, we propose a two-layered asymmetric interdependent network (AIN) model, where a node in one layer could be dependent on more than one node in the other layer with MD relation, while a node in the other layer is just dependent on exactly one node. By this model, the feedback loop is avoided and the extent of the MD relation could be controlled with different extent of asymmetry. We show that in the process where a network transits from a classic symmetric interdependent network to one with strong asymmetry under MD relation, different types of critical percolation behaviors emerge, including the hybrid and discontinuous phase transitions. A heuristic analysis built on the framework of message-passing approach [29–32] is developed to understand the transition behaviors. The percolation behaviors under general cases with intralayer and interlayer degree correlation and AIN with more than two layers are also elaborated. In addition, through the AIN model we provide an intuitive picture showing how a discontinuous phase transition changes to a continuous phase transition and the emergence of a tricritical point where the two types of transitions collide.

This paper is organized as follows. In Sec. II, we introduced the two-layered AIN model with MD relation. In Sec. III, we present the theoretical framework to analyze the proposed networks. In Sec. IV, we present the results of MD relation for a simple two-dependence case. In Sec. V, an intuitive picture of the emergence of hybrid phase transition is provided. In Sec. VI, results of a general case with power-law interdegree distribution are presented. In Sec. VII, the effect of intralayer and interlayer correlation is studied. In Sec. VIII, the general case of AIN with more than two layers is discussed. Finally, conclusions and discussions are presented in Sec. IX.

II. MODEL

We consider an asymmetric interdependent network (AIN) composed of two layers, namely A and B. The degree distributions of the two layers are denoted as $p_A(k_A)$ and $p_B(k_B)$, respectively. Both layers have the same number of nodes N . The asymmetric feature of the AIN is that one layer allows multiple dependence (MD) but the other not. Specifically, for a node in layer A, it can be dependent on an arbitrary number of nodes in layer B. That is, if a node i_A in the layer A is dependent on m nodes in layer B (m dependence), the functioning of i_A requires all the m -interdependent nodes to be functional. We use $q(m)$ to represent the fraction of m -dependence nodes in layer A and hence $\sum_{m=0} q(m) = 1$. Certainly, when $m = 0$, i_A is independent of the other layer. On the other hand, for a node in layer B, say i_B , it is dependent on exactly one node in layer A, so that every node in layer B is 1-dependence. A schematic example is provided in Fig. 1.

The constraint of 1-dependence on the nodes in the layer B requires the total number of interlinks to be the same as the number of nodes N in both layers. This conservation condition indicates the average number of interlinks the nodes have, i.e., the average interdegree, equals one for both layers, implying

FIG. 1. Schematic plot of an asymmetric interdependent network with multiple-dependence relation. Node a_1 is dependent on b_1 and b_2 . The failure of either b_1 or b_2 will cause the failure of a_1 . Since any pair of nodes interconnected by an interlink are mutually interdependent to each other, the failure of a_1 will further cause the failure of both b_1 and b_2 . a_2 is an independent node as it has no interlink. a_3 and a_4 are mutually interdependent with b_3 and b_4 on a one-to-one basis, respectively. In this network, nodes in layer A allow multiple dependence, while nodes in layer B are dependent on exactly one node in layer A. The intralayer structure is not shown for conciseness.

$\sum_{m=0} mq(m) = 1$. Therefore, if some nodes in layer A have more than one interlink, there must be some others that have none. Clearly, this model will return to a classic interdependent network under one-to-one correspondence when all the nodes in layer A are dependent on exactly one node in layer B, giving $q(m) = \delta_{m,1}$.

Additionally, aside from the interlayer condition given above, the functioning of a node is also required to belong to the giant component of the layer it stays in (intralayer condition). Combining the interlayer and intralayer conditions with a given initial structure, a cascading failure of the nodes in the networks may happen and the process continues until no further node fails. The remainder of the network after the cascading failure composes an extensive component, referred as mutually connected giant component (MCGC). The MCGC is a percolating cluster if it possesses a finite fraction of nodes in the network and nonpercolating otherwise.

III. THEORETICAL FRAMEWORK

Our methodology is built on message-passing technique [33–35], also known as belief propagation when applying on network models [36,37]. Here, we start from a single-layer network. In this paper, we consider bond percolation problem, where the survival probability of a link is indicated with $r \in [0, 1]$. In Appendix A, we show that for a treelike structure, a bond percolation problem could be directly mapped to a site percolation one and therefore the scheme of bond percolation adopted here will not lose its generality in the

obtained conclusions. We assume S' equals the possibility that a randomly chosen link connects to the giant component (GC) in the network. Suppose the network is locally treelike which could be a reasonable approximation of sparse networks, the possibility of a node with degree k being belonging to the giant component equals $1 - (1 - rS')^k$, as this event happens when at least one of its neighbors reached by an occupied link belongs to the GC. Denoting $p(k)$ as the degree distribution of the network and S the possibility of a randomly chosen node belongs to the GC, respectively, one gets

$$1 - S = \sum_k p(k)(1 - rS')^k, \quad (1a)$$

stating the possibility that a random chosen node does not belong to the GC if none of its link can reach it. With the generating function defined as $G_0(x) = \sum_k p(k)x^k$, Eq. (1a) could be reexpressed as

$$S = 1 - G_0(1 - rS'). \quad (1b)$$

The event pertinent to S' happens when the node reached by a randomly picked link has at least one of its excessive links (other than this chosen link) being occupied and connecting to the GC, which implies

$$S' = 1 - G_1(1 - rS'), \quad (1c)$$

where the generating function $G_1(x) = G'_0(x)/G'_0(1) = \sum_k k p(k)x^{k-1}/z$ with z being the average degree of the network. Taking the result of S' computed from Eq. (1c) into Eq. (1b), one obtains the value S , i.e., the fraction of the nodes in the GC, a key quantity indicating the extent of percolation.

Next, let us turn to two-layered interdependent networks with MD relation. We analogously denote the degree distributions of the layers A and B as $p_A(k_A)$ and $p_B(k_B)$, respective, and the generating functions as $G_0^A(x) = \sum_{k_A} p_A(k_A)x^{k_A}$, $G_1^A(x) = G_0^{A'}(x)/G_0^{A'}(1) = \sum_{k_A} k_A p_A(k_A)x^{k_A-1}/z_A$ and $G_0^B(x) = \sum_{k_B} p_B(k_B)x^{k_B}$, $G_1^B(x) = G_0^{B'}(x)/G_0^{B'}(1) = \sum_{k_B} k_B p_B(k_B)x^{k_B-1}/z_B$ with z_A and z_B being the average degrees of layer A and layer B, respectively. Further, we assume S_A and S_B (S'_A and S'_B) equal the possibilities that a randomly chosen node (edge) in the

layer A and B, respectively, belongs to (reaches) the MCGC, and r_A and r_B are the occupation probability of the links in layer A and layer B, respectively. With the definition of MCGC, a node belonging to the MCGC needs to meet both the intralayer and interlayer conditions. As for the intralayer condition, a node to be connected with the MCGC requires at least one of its occupied links goes to the GC of the proper layer. Hence, for a node in the layer A (B), such event happens with probability $[1 - G_0^A(1 - r_A S'_A)]$ ($[1 - G_0^B(1 - r_B S'_B)]$).

On the other hand, for the interlayer condition, as a node in layer A may be dependent on multiple nodes in layer B through MD relation, it requires all the dependent nodes in layer B to be functional. Hence, for a node in layer A, suppose it is dependent on m nodes in layer B, the possibility that all the m -dependent nodes belong to the GC of layer B equals $[1 - G_0^B(1 - r_B S'_B)]^m$, where we have adopted the assumption that these m nodes have no correlation when they are dependent on the node in layer A. Taking the intralayer and interlayer conditions of layer A together, we have

$$S_A = [1 - G_0^A(1 - r_A S'_A)] \left\{ \sum_m q(m) [1 - G_0^B(1 - r_B S'_B)]^m \right\} \\ = [1 - G_0^A(1 - r_A S'_A)] H_0[1 - G_0^B(1 - r_B S'_B)], \quad (2a)$$

$$S'_A = [1 - G_1^A(1 - r_A S'_A)] \left\{ \sum_m q(m) [1 - G_0^B(1 - r_B S'_B)]^m \right\} \\ = [1 - G_1^A(1 - r_A S'_A)] H_0[1 - G_0^B(1 - r_B S'_B)], \quad (2b)$$

where the generating function $H_0(x)$ is defined as $H_0(x) = \sum_{m=0} q(m)x^m$. Whereas, for a node in layer B, say i_B , though by definition it is dependent on exactly one node in layer A, say i_A , the i_A may, however, have a number of m interlinks. Since the functioning of i_A requires all the m -dependent nodes to be functional, the functioning of i_B will not only require the functioning of i_A , but also the functioning of the other $m - 1$ nodes interlinked by i_A . Further noticing that the possibility that i_B connects to i_A with m interlinks equals $mq(m)$ under noncorrelation assumption with the conservation condition $\sum_m mq(m) = 1$, the intralayer and interlayer conditions commonly give

$$S_B = [1 - G_0^B(1 - r_B S'_B)] [1 - G_0^A(1 - r_A S'_A)] \left\{ \sum_{m=1} mq(m) [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\} \\ = [1 - G_0^B(1 - r_B S'_B)] [1 - G_0^A(1 - r_A S'_A)] H_1[1 - G_0^B(1 - r_B S'_B)], \quad (3a)$$

$$S'_B = [1 - G_1^B(1 - r_B S'_B)] [1 - G_0^A(1 - r_A S'_A)] \left\{ \sum_{m=1} mq(m) [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\} \\ = [1 - G_1^B(1 - r_B S'_B)] [1 - G_0^A(1 - r_A S'_A)] H_1[1 - G_0^B(1 - r_B S'_B)], \quad (3b)$$

where $H_1(x) = H'_0(x) = \sum_{m=1} mq(m)x^{m-1}$ with $H_0(x) = \sum_{m=1} q(m)x^m$ and $\sum mq(m) = 1$.

IV. MULTIPLE DEPENDENCE WITH $m \leq 2$

In this section, we start from a simple case that a node in layer A is dependent on no more than two nodes in

layer B. In order to focus on the MD relation, we adopt random networks to represent the intralayer structure of the two layers with the same average degree z and the condition $r_A = r_B = r$ that gives $G_0^A(1 - rx) = G_1^A(1 - rx) = G_0^B(1 - rx) = G_1^B(1 - rx) = e^{-rx}$ [38]. We note that since in random networks $G_0^A(1 - rS'_A) = e^{-rS'_A}$, varying r at a given z is equivalent to directly changing the value of z . Hence, without

FIG. 2. Behaviors of S_A and S_B as functions of z for the case of (a) $q = 0$, (b) $q = 0.3$, and (c) $q = 0.5$. Comparisons of the behaviors of S_A and S_B among the three different q values are shown in (d) and (e), respectively. Symbols in (d) and (e) are obtained from numerical simulations with 100 different realizations and $N = 10^4$ where error bars stand for the standard deviation.

loss of generality, in the following when addressing the random networks we use $e^{-zS'_A}$ in place of $e^{-r z S'_A}$ for conciseness. This practice also applies to generating functions of other layers.

The case that nodes in layer A are dependent on no more than two nodes in layer B gives $q(m) = 0$ when $m \geq 3$. Due to the conservation in the number of interlinks equaling N , it is easy to see $q(0) = q(2)$ and, therefore, $q(1) = 1 - 2q(2)$. For brevity, here we denote $q = q(2)$ with $q \in [0, 0.5]$. When $q = 0$, the network returns to the classic interdependent network under one-to-one correspondence. When $q = 0.5$, in layer A half of the nodes are of 2-dependence and the remaining half are independent. Therefore, according to Eqs. (2) and (3), one immediately gets

$$S_A = (1 - e^{-zS'_A})[q + q(1 - e^{-zS'_B})^2 + (1 - 2q)(1 - e^{-zS'_B})], \quad (4a)$$

$$S_B = (1 - e^{-zS'_B})[(1 - 2q)(1 - e^{-zS'_A}) + 2q(1 - e^{-zS'_A})(1 - e^{-zS'_B})]. \quad (4b)$$

Figures 2(a)–2(c) show the typical behaviors of S_A and S_B as functions of z when $q = 0, 0.3$, and 0.5 , respectively. When $q = 0$, the behaviors of S_A and S_B are fully overlapped with a discontinuous phase transition at around $z = 2.4554$, which recovers the results of the classic interdependent networks [1,39]. When $q = 0.3$, S_A and S_B diverge and both have discontinuous phase transitions at the same place of $z = 2.676$. However, when $q = 0.5$, the percolation diagrams of S_A and S_B are essentially different. In this case, though, S_B still has

a discontinuous phase transition, the behavior of S_A has a hybrid phase transition, where a continuous phase transition first appears at $z = 2$ followed by a discontinuous one at $z = 2.878$, the same place of the discontinuous phase transition of layer B. Figures 2(d) and 2(e) present these results from the view of comparing S_A and S_B with different q , respectively. For S_A , one observes that the discontinuous phase transition happens at a larger z for a larger q . However, when q is large enough, here $q = 0.5$, a continuous phase transition appears at a z even smaller than the transition point of $q = 0$. However, the phase transition of S_B is always discontinuous with a larger q corresponding to a larger critical z , showing that a larger fraction of 2-dependence makes the layer B to be more vulnerable. The hybrid phase transition has been observed in various occasions of interdependent networks [40,41], however, a theoretical understanding pertinent to this phenomenon is still missing. In the next section, we will provide a heuristic scheme to understand the underlying mechanism of the phase diagram of different q values and provide a simple picture based on the framework of AIN to understand the emergence of hybrid phase transition.

V. HYBRID PHASE TRANSITION

We argue that the independent and dependent nodes have opposite effects of enhancing or weakening robustness of the network. This intuition is supported by the study of partial interdependence, where having a larger fraction of independent nodes may induce continuous phase transition with smaller critical point, whereas having a large fraction of dependent

FIG. 3. Behaviors of S_A , $T_{I,A}(z)$, and $T_{II}(z)$ as functions of z for the case of (a) $q = 0$, (b) $q = 0.3$, and (c) $q = 0.5$.

nodes may induce a discontinuous phase transition with larger critical point [2]. Hence, to characterize the competing effects of the two classes of nodes, here we separate independent nodes ($m = 0$) and dependent nodes ($m = 1, 2$) in layer A by cutting the connections between the two groups. Clearly, in the resulting networks, the average degrees of independent nodes and dependent nodes equal zq and $z(1 - q)$, respectively, where $r_A = r_B = r$ have been absorbed into z . Besides, since nodes in layer B only interlinked with the dependent nodes in layer A, this separation actually generates two networks: one contains the dependent nodes in layer A and the whole layer B, denoted as network I, and the other contains only the independent nodes in layer A, denoted as network II. The network I is a fully interdependent network with all the nodes being dependent on others, while network II is just a single-layered network.

For network I, to compensate the loss of intralayer connections from the separation from network II, we randomly add a fraction of links between the dependent nodes in layer A to

recover the average degree of them to z , which is equivalent to increase the wiring possibility between nodes in network I from z/N to $z/N(1 - q)$. We note that this connection-adding operation is only an approximation to the original network due to the fact that though the average degree is recovered in network I, it is expected to be more vulnerable compared to the original one since in this case all the nodes are dependent ones. On the other hand, as the contribution of independent nodes is based on intralayer connections, the intralayer average degree of network II is kept as qz to correctly reflect its impact on the robustness of network II.

We define $T'_{I,A}$, $T'_{I,B}$, and T'_{II} as the possibilities that a randomly chosen edge in layer A of network I, in layer B of network I, and in network II, respectively, connects to the giant component of the proper layer or network. Analogously, $T_{I,A}$, $T_{I,B}$, and T_{II} denote the possibility of a randomly chosen node that belongs to the giant component of the corresponding layer or network. With similar analysis of Eqs. (2) and (3), we have

$$T_{I,A} = [1 - G_0^{I,A}(1 - T'_{I,A})] \left\{ \left(\frac{q}{1 - q} \right) [1 - G_0^{I,B}(1 - T'_{I,B})]^2 + \left(\frac{1 - 2q}{1 - q} \right) [1 - G_0^{I,B}(1 - T'_{I,B})] \right\}, \quad (5a)$$

$$T_{I,B} = [1 - G_0^{I,B}(1 - T'_{I,B})] \{ (1 - 2q)[1 - G_0^{I,A}(1 - T'_{I,A})] + 2q[1 - G_0^{I,A}(1 - T'_{I,A})][1 - G_0^{I,B}(1 - T'_{I,B})] \}, \quad (5b)$$

$$T'_{I,A} = [1 - G_1^{I,A}(1 - T'_{I,A})] \left\{ \left(\frac{q}{1 - q} \right) [1 - G_0^{I,B}(1 - T'_{I,B})]^2 + \left(\frac{1 - 2q}{1 - q} \right) [1 - G_0^{I,B}(1 - T'_{I,B})] \right\}, \quad (5c)$$

$$T'_{I,B} = [1 - G_1^{I,B}(1 - T'_{I,B})] \{ (1 - 2q)[1 - G_1^{I,A}(1 - T'_{I,A})] + 2q[1 - G_0^{I,A}(1 - T'_{I,A})][1 - G_0^{I,B}(1 - T'_{I,B})] \}, \quad (5d)$$

where $G_0^{I,A}$ and $G_1^{I,A}$ ($G_0^{I,B}$ and $G_1^{I,B}$) are generating functions of layer A (B) in network I with similar definition of G_0^A and G_1^A . As layer A and layer B in the network I are still random networks with the same average degree z , we have $G_0^{I,A}(1 - x) = G_1^{I,A}(1 - x) = G_0^{I,B}(1 - x) = G_1^{I,B}(1 - x) = e^{-zx}$ and, hence, $T_{I,A} = T'_{I,A}$ and $T_{I,B} = T'_{I,B}$, giving $h_A(T_{I,A}) = h_B(T_{I,B}) = 0$ with the definition of $h_A(T_{I,A})$ and $h_B(T_{I,B})$ as follows:

$$h_A(T_{I,A}) = T_{I,A} - (1 - e^{-zT_{I,A}}) \left[\left(\frac{q}{1 - q} \right) (1 - e^{-zT_{I,B}})^2 + \left(\frac{1 - 2q}{1 - q} \right) (1 - e^{-zT_{I,B}}) \right], \quad (6a)$$

$$h_B(T_{I,B}) = T_{I,B} - (1 - e^{-zT_{I,B}}) [(1 - 2q)(1 - e^{-zT_{I,A}}) + 2q(1 - e^{-zT_{I,A}})(1 - e^{-zT_{I,B}})]. \quad (6b)$$

Furthermore, for network II, we define the possibility of a randomly chosen edges connecting to (a randomly chosen node belonging to) the GC of the network II as T'_{II} (T_{II}). Following the procedure of Eq. (1), one obtains $h_{II}(T_{II}) = 0$ with

$$h_{II}(T_{II}) = T_{II} - (1 - e^{-qzT_{II}}), \quad (7)$$

where the condition $T'_{II} = T_{II}$ has been adopted as the network II is a random network with average degree qz .

The behaviors of $T_{I,A}(z)$ and $T_{II}(z)$ of three typical q values are presented in Fig. 3. In addition, the behavior of S_A is also provided as comparison. Clearly, we can see that $T_{I,A}(z)$ [$T_{II}(z)$] always has a discontinuous (continuous) phase

transition and a larger q corresponds to a larger (smaller) transition point for $T_{I,A}(z)$ [$T_{II}(z)$], indicated as z^I (z^{II}). Specifically, when $q = 0$, $z^I = 2.4554 < z^{II} \rightarrow \infty$. In this case, the size of network II is actually zero, therefore, $T_{I,A} = S_A$ reflected by the collapse of the two curves. When $q = 0.3$, z^I is increased to 2.82, and z^{II} is about 3.33 much closer to z^I but still larger than it. Particularly, $z^I < z^{II}$ means the discontinuous phase transition of $T_{I,A}(z)$ happens before the continuous phase transition of $T_{II}(z)$. Therefore, at $z = z^I$, $T_{I,A}(z)$ will make a nonzero jump and $T_{II}(z)$ remains vanishing, which corresponds to a jump of S_A in nearby place. With z further growing to $z = z^{II}$, the S_A has been large enough and the transition of $T_{II}(z)$ from zero to nonzero is embedded in the nonzero value of S_A . When $q = 0.5$, the situation is reversed with $z^I = 3.09 > z^{II} = 2$. In this case, when $z = z^{II}$, network II starts to form a giant component and all the dependent nodes in network I remain failed, introducing a continuous phase transition of S_A . When z further grows to $z = z^I$, $T_{I,A}(z)$

will have a significant jump and this nonzero change is manifested by a discontinuous transition in S_A . As a result, the two transitions together produce the hybrid phase transition of S_A .

As discussed above, S_A is expected to be more robust than the $T_{I,A}$ because of the independent nodes. This intuition is indeed validated by the results that S_A jumps at a smaller z^I than $T_{I,A}$. Thus, in the process of q growing from 0 to 0.5, z^I increases from 2.4554 to 2.88 and z^{II} decreases from infinity to 2. Thus, in this process there should be a point of z at which $z^I = z^{II}$. This critical point, denoted as z^* , separates the phase transitions from discontinuous to continuous, known as the tricritical point. The critical point z^* could be directly estimated by the condition $z^I = z^{II}$, and z^I and z^{II} could be analytically calculated through the functions of h_A , h_B , h_{II} defined in Eqs. (6) and (7). Specifically, the discontinuous transition point z^I is obtained at the condition $h_A(T_{I,A}) = h_B(T_{I,B}) = h'_A(T_{I,A}) = h'_B(T_{I,B}) = 0$, giving

$$T_{I,A} = (1 - e^{-zT_{I,A}}) \left[\frac{q}{1-q} (1 - e^{-zT_{I,B}})^2 + \frac{1-2q}{1-q} (1 - e^{-zT_{I,B}}) \right] = 0, \quad (8a)$$

$$T_{I,B} = (1 - e^{-zT_{I,B}}) [(1-2q)(1 - e^{-zT_{I,A}}) + 2q(1 - e^{-zT_{I,A}})(1 - e^{-zT_{I,B}})] = 0, \quad (8b)$$

$$ze^{-zT_{I,B}}(1 - e^{-zT_{I,A}})[4q(1 - e^{-zT_{I,B}}) + 1 - 2q] + ze^{-zT_{I,A}}[2q(1 - e^{-zT_{I,B}})^2 + (1-2q)(1 - e^{-zT_{I,B}})] \frac{dT_{I,A}}{dT_{I,B}} = 1, \quad (8c)$$

$$\frac{dT_{I,A}}{dT_{I,B}} = \frac{zT_{I,A}e^{-zT_{I,A}}}{1 - e^{-zT_{I,A}}} \cdot \frac{dT_{I,A}}{dT_{I,B}} + \frac{zT_{I,B}e^{-zT_{I,B}}}{(1-q)(1 - e^{-zT_{I,B}})}. \quad (8d)$$

Then, for any given q , the solution of z^I could be computed numerically through Eq. (8). The continuous phase transition point z^{II} is obtained at the condition $h_{II}(T_{II}) = h'_{II}(T_{II}) = 0$. Simple calculations give its solution [39]

$$z^{II} = 1/q. \quad (9)$$

The behaviors of z^I and z^{II} as functions of q are presented as black solid curve and red dashed curve in Fig. 4, respectively. One observes that the intersection of z^I and z^{II} at

FIG. 4. Transition points z^I (solid black curve) and z^{II} (red dashed curve) as functions of q obtained from Eq. (9). The intersection implies the location of the tricritical point which separates the plane into two regimes with the left for discontinuous and the right for hybrid phase transition.

$q = q^* = 0.288$ and $z = z^* = 3.48$ separates the whole plane into two regimes: (i) when $q < q^*$, $z^I < z^{II}$ indicating discontinuous phase transition; (ii) when $q > q^*$, $z^I > z^{II}$ indicating hybrid phase transition.

Hence, the picture is now clear: Consider a two-layered interdependent network which can be regarded as a special case of partially interdependent network. We divide the first layer into two parts, say part I and part II, and there is no interdependency in part I and all the interlinks run between part II and the second layer. The independent part, i.e., part I, is a random network with average degree qz which percolation transition happens at $qz = 1$, giving $z^{II} = 1/q$. As to the interdependent part, i.e., part I and the second layer, originally when the network is fully interdependent, i.e., $q = 0$, we have discontinuous phase transition. With the growing of q , the independent part plays an increasingly important role. After a tipping point, impact of part I is evident enough which brings a hybrid phase transition: continuous phase transition of independent part accompanied with a discontinuous phase transition resulting from interdependent part.

VI. MULTIPLE DEPENDENCE WITH POWER-LAW DISTRIBUTION

Now, let us turn to a general case that the MD relation in layer A follows power-law distribution, so that $q(m) \sim m^{-\gamma}$ ($m \geq 1$). Following the conditions $\sum_{m=0}^{\infty} q(m) = 1$ and $\sum_{m=1}^{\infty} mq(m) = 1$, we have $q(m) = m^{-\gamma} / \sum_{m=1}^{\infty} m^{-\gamma+1}$ when $m \geq 1$ and $q(0) = 1 - \sum_{m=1}^{\infty} q(m)$. In addition, due to the finite average degree of $q(m)$, γ should satisfy $\gamma \geq 2$ [38].

FIG. 5. Behaviors of S_A and S_B as functions of z for the case of (a) $\gamma = 2.0$ and (b) $\gamma = 2.5$. Symbols are obtained from numerical simulations with 100 different realizations where $N = 10^4$ and error bars stand for standard deviation.

Similarly, following Eq. (4) and considering the layers A and B as random networks with average degree z , the self-sustained expressions of S_A and S_B can be written as

$$f_A(S_A) = S_A - (1 - e^{-zS_A}) \left[\sum_{m=0} q(m)(1 - e^{-zS_B})^m \right] = 0, \quad (10a)$$

$$f_B(S_B) = S_B - (1 - e^{-zS_B}) \left[\sum_{m=1} mq(m)(1 - e^{-zS_A}) \times (1 - e^{-zS_B})^{m-1} \right] = 0. \quad (10b)$$

Figure 5 shows the behaviors of S_A and S_B for $\gamma = 2.0$ and 2.5 . One can see that when $\gamma = 2.0$ S_A has a hybrid phase transition, and when $\gamma = 2.5$ S_A has a discontinuous

FIG. 6. Grayscale map of S_A on γ - z plane. The solid white curve describes the behavior of z^I obtained from Eq. (11) and the dashed black curve describes the behavior of z^{II} with the relation $z^{II}q(0) = 1$.

phase transition. The approach proposed in Sec. V helps us to understand the emergence of different types of phase transitions. However, the approximation in this approach could bring larger deviation in the scenario of MD with power-law distribution since in this case the network I may suffer even a larger extent of MD. Nevertheless, the spirit of this approach is still valid and we could analyze the critical behavior of the network if we could correctly predict the positions of the discontinuous and continuous phase transitions.

We notice that the discontinuous jump of S_A and S_B always happens at the same values of z . Indeed, a nonzero change in S_B will cause a nonzero change in S_A since one S_A does not allow two distinct solutions of S_B with a given set of $q(m)$ and z . This can be seen in Eq. (10a) that when S_A is given, only one possible solution of S_B is permitted due to the monotonicity in the term of $1 - e^{-zS_B}$. Hence, the location of z where S_A has a discontinuous jump z^I could be determined with the help of S_B . The critical point of S_B could be evaluated with the conditions of $f_B = df_B/dS_B = 0$ [39], further with the condition of $f_A = 0$ from which we could obtain the expression of dS_A/dS_B , the jumping point of S_B could then be calculated numerically through the following set of equations:

$$f_A(S_A) = S_A - (1 - e^{-zS_A}) \left[\sum_{m=0} q(m)(1 - e^{-zS_B})^m \right] = 0, \quad (11a)$$

$$f_B(S_B) = S_B - (1 - e^{-zS_B}) \left[\sum_{m=1} mq(m)(1 - e^{-zS_A})(1 - e^{-zS_B})^{m-1} \right] = 0, \quad (11b)$$

$$\frac{df_B}{dS_B} = 1 - \sum_{m=1} mq(m) \left[(1 - e^{-zS_B})^{m-1} (1 - e^{-zS_A}) m z e^{-zS_B} + (1 - e^{-zS_B})^m z e^{-zS_A} \frac{dS_A}{dS_B} \right] = 0, \quad (11c)$$

$$\frac{dS_A}{dS_B} = \sum_{m=0} q(m) \left[(1 - e^{-zS_A})(m-1)(1 - e^{-zS_B})^{m-2} z e^{-zS_B} + (1 - e^{-zS_B})^{m-1} z e^{-zS_A} \frac{dS_A}{dS_B} \right]. \quad (11d)$$

On the other hand, the continuous phase transition should be attributed to the independent nodes in layer A as discussed in Sec. V. Indeed, suppose S_A starts to grow from zero to nonzero through a continuous phase transition where

the discontinuous transition is yet happened, i.e., under the condition $z^I > z^{II}$, at this moment $S_B = 0$, indicating all the dependent nodes in layer B are failed that in turn causes all the dependent nodes in layer A to be failed as well, resulting in

the above judgment. Hence, the position of continuous phase transition of layer A, z^{II} , happens at $zq(0) = 1$ with the same calculations in Eq. (9).

Assembling these analyses, we could determine the tricritical point of by letting $z^{\text{I}} = z^{\text{II}}$. Figure 6 shows the behaviors of z^{I} and z^{II} on the z - γ plane with their intersection being the tricritical point. The grayscale indicates simulation results of S_A which conform to the borders z^{I} and z^{II} predicted analytically. We argue that in the simulation due to the finite size effect and fluctuations among different realizations, a discontinuous phase transition may happen earlier than z^{I} . As the resulting S_A pertinent to discontinuous jumps are

$$S'_A = \sum_{k_A, m} \frac{k_A}{z_A} p(k_A, m) [1 - (1 - r_A S'_A)^{k_A - 1}] [1 - G_0^B(1 - r_B S'_B)]^m, \quad (12a)$$

$$S_A = \sum_{k_A, m} p(k_A, m) [1 - (1 - r_A S'_A)^{k_A}] [1 - G_0^B(1 - r_B S'_B)]^m, \quad (12b)$$

$$S'_B = \left[1 - \sum_{k_B} \frac{k_B}{z_B} p_B(k_B) (1 - r_B S'_B)^{k_B - 1} \right] \left\{ \sum_{k_A, m} m p(k_A, m) [1 - (1 - r_A S'_A)^{k_A}] \left[1 - \sum_{k_B} p_B(k_B) (1 - r_B S'_B)^{k_B} \right]^{m-1} \right\} \\ = [1 - G_1^B(1 - r_B S'_B)] \left\{ \sum_{k_A, m} m p(k_A, m) [1 - (1 - r_A S'_A)^{k_A}] [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\}, \quad (12c)$$

$$S_B = [1 - G_0^B(1 - r_B S'_B)] \left\{ \sum_{k_A, m} m p(k_A, m) [1 - (1 - r_A S'_A)^{k_A}] [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\}, \quad (12d)$$

where the right hand side of Eq. (12a) reflects the fact that following a randomly picked link in layer A, the possibility of reaching a node with joint degree distribution $p(k_A, m)$ equals $k_A p(k_A, m)/z_A$, and such a node belongs to the MCGC requires any of its excess $k_A - 1$ degree connects to the GC of layer A and all the m nodes in layer B reached by the m interlinks belong to the GC of layer B. The meaning of the other equations can be understood similarly.

In the following, we propose a method to generate the function $p(k_A, m)$ where the extent of correlation could span from maximally negative to maximally positive, meanwhile let its marginal distributions satisfying $\sum_m p(k_A, m) = p_A(k_A)$ and $\sum_{k_A} p(k_A, m) = q(m)$. For doing so, we first generate the maximally positive $p(k_A, m)$ with given $p_A(k_A)$ and $q(m)$. To do this, we assign degree k_A to a set of N nodes according to the distribution $p_A(k_A)$ and assign m to another set of N nodes according to the distribution $q(m)$. We list the two sets of nodes both in ascending order with respect to degrees k_A and m , and index them from 1 to N , respectively. Hence, the two lists have minimum k_A and m at one end, and maximum k_A and m at the other end, respectively. Then, we combine each pair of nodes in the two lists with the same index into a new node, meanwhile keeping their degrees k_A and m . The resulting N combined nodes satisfy the distribution with the maximally positive correlation, which is denoted as $p^+(k_A, m)$. The details of the algorithm of generating $p^+(k_A, m)$ are provided in Appendix B. Similarly, the maximally negative correlation distribution $p^-(k_A, m)$ could be obtained by listing

distinctive to zero, the solid curve of z^{I} is shown to be a bit buried in the light gray area.

VII. INTRALAYER AND INTERLAYER CORRELATION

In this section we consider the general case that for nodes in layer A, their intralayer degree k_A and the interlayer degree m are correlated, described by a distribution $p(k_A, m)$. Since nodes in layer B are interconnected with exactly one node in layer A, such correlation does not apply to layer B and the degree distribution of layer B can still be described with $p_B(k_B)$. Applying the similar spirit in Eqs. (2) and (3), we have

the two sets of nodes, one in ascending order and the other in descending order and then combining all pairs of nodes with the same index. Further, we denote the distribution function under no correlation condition as $p^0(k_A, m) = p_A(k_A)q(m)$. With the three benchmark distributions, we define a general correlation function with a controlling parameter α as follows:

$$p(k_A, m) = \begin{cases} \alpha p^+(k_A, m) + (1 - \alpha) p^0(k_A, m), & \text{when } \alpha \geq 0 \\ |\alpha| p^-(k_A, m) + (1 - |\alpha|) p^0(k_A, m), & \text{when } \alpha < 0. \end{cases} \quad (13)$$

Suppose the Pearson correlation coefficients of $p^+(k_A, m)$ and $p^-(k_A, m)$ are $\mathcal{P}^+$ and $\mathcal{P}^-$, respectively, it is easy to find the Pearson correlation coefficient of general $p(k_A, m)$ equals $\alpha \mathcal{P}^+$ and $|\alpha| \mathcal{P}^-$ depending on the sign of α [see the insets of Figs. 7(b) and 7(d)].

The procedure of evaluating the threshold of discontinuous and continuous phase transitions with a given $p(k_A, m)$ is in line with that in Sec. VI. Details are provided in Appendices C and D, respectively.

To examine the effect of the intralayer and interlayer degree correlation, aside from the synthetic network with the layers being composed of random networks, denoted as ER, as used above, we also adopt an empirical network introduced in Ref. [42] for evaluation. This network, denoted as BRAIN, is a functional brain network generated from functional magnetic resonance imaging (fMRI) data to describe brain connectivity

FIG. 7. Behaviors of order parameters S_A and S_B for synthetic networks (a) and (b) and empirical networks (c) and (d) for the case of $\alpha = -1$ (black), $\alpha = 0$ (red), and $\alpha = 1$ (blue), respectively. In the synthetic networks (ER), each layer is a random network with average degree $z = 5$. In the empirical brain networks (BRAIN), each layer satisfies degree distribution $p(k) \sim k^{-\beta}$ with $\beta = 2.85$, $k_{\min} = 2$, and $k_{\max} = 133$. The interlayer degree distribution follows $q(m) \sim m^{-\gamma}$ with $\gamma = 2.0$. The insets in (b) and (d) indicate the Pearson correlation coefficient as function of α for the two networks. Symbols are obtained from numerical simulation with 100 realizations where $N = 6 \times 10^4$ and error bar standing for standard deviation.

when the brain is under resting state, giving the degree distribution of both layers as $p(k) \sim k^{-\beta}$ with $\beta = 2.85$, $k_{\min} = 2$, and $k_{\max} = 133$, and the network size $N = 60\,000$. We note

that although in Ref. [42] two other kinds of degree-degree correlations (one is the correlation between interlayer degree and intralayer degree with multiple-support relation on the nodes, and the other is the degree-degree correlation of the interlinks connecting nodes in different layers) has been studied, here we focus our scope on intralayer degree and interlayer degree correlation with MD relation.

When the intralayer degree k_A and the interlayer degree m are negative correlated, a node in layer A having a larger intralayer degree k_A is more likely to be an independent node. Thus, in this case, the independent nodes in layer A are easier to form a GC due to the large average degree among them, leading to the threshold of r of the continuous phase transition, denoted as r^{II} , to be smaller than that of the discontinuous phase transition r^{I} . On the other hand, in the case of positive correlation, a node in layer A with larger intralayer degree tends to be dependent on more nodes in layer B, easily causing its disfunctioning, and the independent nodes are difficult to form a GC due to their relative small average degree, which may result $r^{\text{II}} > r^{\text{I}}$. This analysis is validated by Fig. 7, where in both networks when $\alpha = -1$ and 0, S_A has hybrid phase transition and when $\alpha = 1$, only discontinuous phase transition shows up.

As to the discontinuous phase transition, Fig. 8 shows that r^{I} (red solid curves) is not sensitive to the degree-degree correlation $p(k_A, m)$. Since this correlation portrays the degree characteristics of nodes in layer A, this phenomenon implies that the onset of the discontinuous phase transition is mainly dependent on the robustness of layer B. This result is compatible to our arguments in Sec. VI where the location

FIG. 8. Grayscale maps of the simulation results S_A on the α - r plane for the same synthetic networks (a)–(c) and empirical networks (d)–(f) as in Fig. 7. The left, medium, and right columns correspond to the cases of $r_A = r_B = r$, $r_A = r$, and $r_B = 1$, and $r_A = 1$ and $r_B = r$, respectively. The solid red curves indicate the discontinuous threshold calculated from Eq. (C1), and the dashed red curves in (a) and (b) and (d) and (e) indicate the continuous threshold calculated from Eq. (D3). The dotted red curves in (c) and (f) are also calculated from Eq. (D3) and their intersections with $r = 1$ indicate the continuous threshold for the case of $r_A = 1$ presented by the dashed red curves. Other parameters are the same as those in Fig. 7.

of discontinuous phase transition is largely attributed to layer B. Indeed, as shown in Fig. 8, when $r_B = r$, i.e., a fraction of links in layer B are damaged, r^l keeps to be relative large no matter $r_A = r$ (left columns) or $r_A = 1$ (right columns). While when $r_B = 1$ (medium columns), i.e., all the links in layer B are preserved in the beginning, r^l is remarkably smaller compared to the other two cases, indicating a strong robustness of layer B.

VIII. AIN WITH MORE THAN TWO LAYERS

For the two-layered interdependent networks studied above, only one layer is allowed to have MD relation. This restriction is to prevent the appearance of a feedback loop. A feedback loop may happen when node a in layer A is dependent on node b in layer B and the node b is dependent on node c in layer A where c is distinct from a . It can be seen that a feedback loop could cause a full collapse even regardless of the internal connectivity of the network. However, for a two-layered interdependent network, when nodes in both layers allow MD relation, a feedback loop is almost inevitable. That is why the AIN model is introduced where only one layer allows MD relation. With the same reason, for networks with more than two layers, the restriction that only one layer allows MD relation is still required to prevent the appearance of feedback loop. Hence, in this section we consider an AIN composed of $n + 1$ layer with one layer allows MD relation, denoted as master layer, and the rest n ones follow 1-dependence relation, denoted as normal layer. Specifically, when a normal layer is connected with the master layer, the nodes between the two layers are interdependent in the same way as defined in previous sections. While when two normal layers are connected, due to the 1-dependence relation for both layers and the no-feedback loop condition, the two normal layers need to be fully dependent under one-to-one correspondence.

For a network with n interdependent layers, also known as a network of networks (NON) from a graphic representation [43], when the n layers have a looplike structure and each pair of connected layers are fully dependent under one-to-one correspondence, a feedback loop may appear (for example, for a triangle structure, node a in layer A mutually interdependent with node b in layer B, the node b is also mutually interdependent with node c in layer C, and further the node c is mutually interdependent with node d in layer A which, however, is distinct from the node a). This looplike structure will cause the fully dependent networks to have only one trivial stable solution with zero size MCGC, which means that such an NON could suffer a full collapse upon the failure of even a single node [44]. In other words, for an n layer fully dependent network under one-to-one correspondence, only the treelike structure may afford a nontrivial solution in the size of MCGC. Moreover, it has been shown that the robustness of a treelike structure is independent of the detailed structure of the tree [45]. For example, when all the n layers are random networks with identical average degree z , the relative size of the MCGC, S , satisfies the condition $S = (1 - e^{-zS})^n$ for any treelike structure [43].

Here, based on the same reason, to avoid the feedback loop we consider the multiple layers of the AIN possessing

a treelike structure. We will first address the case of a starlike structure with the master layer being the hub, and then turn to the general treelike structure with the master layer being the root. In this section, we neglect the intralayer and interlayer correlation to focus on the effect of multiple layers. The condition of noncorrelation could be lifted by introducing a joint degree distribution similar to that in Sec. VII.

A. Starlike structure

For a starlike network, the master layer is the hub, denoted as layer 0, and the rest n normal layers are the leaves, denoted as layer i with $1 \leq i \leq n$ [see the schematic diagram in Fig. 9(a)]. The degree distribution of the $n + 1$ layers is denoted as $p_i(k)$ with $0 \leq i \leq n$. In addition, the nodes in the master layer could be dependent on nodes in all the normal layers with MD relation. The distribution of interlayer degree of the nodes in the master layer that connect to the nodes in layer i with MD relation is denoted as $q_i(m)$ with $1 \leq i \leq n$. Defining $S_i(S'_i)$ with $0 \leq i \leq n$ as the possibility of a randomly chosen node (a randomly picked link) in the layer i that belongs (goes) to the MCGC of the network, we have

$$S'_0 = [1 - G_1^0(1 - r_0 S'_0)] \prod_{i=1}^n H_0^i [1 - G_0^i(1 - r_i S'_i)], \quad (14a)$$

$$S_0 = [1 - G_0^0(1 - r_0 S'_0)] \prod_{i=1}^n H_0^i [1 - G_0^i(1 - r_i S'_i)], \quad (14b)$$

$$S'_i = [1 - G_1^i(1 - r_i S'_i)] [1 - G_0^0(1 - r_0 S'_0)] \times H_1^i [1 - G_0^i(1 - r_i S'_i)] \prod_{j=1, j \neq i}^n H_0^j [1 - G_0^j(1 - r_j S'_j)], \quad (14c)$$

$$S_i = [1 - G_0^i(1 - r_i S'_i)] [1 - G_0^0(1 - r_0 S'_0)] \times H_1^i [1 - G_0^i(1 - r_i S'_i)] \prod_{j=1, j \neq i}^n H_0^j [1 - G_0^j(1 - r_j S'_j)], \quad (14d)$$

where the generating functions $G_0^i(x)$, $G_1^i(x)$, $H_0^i(x)$, $H_1^i(x)$, and r_i are defined similarly as above. The key point in Eq. (14) is that the survival of a node in the master layer demands the functioning of all the dependent nodes in the rest layers. When all the layers are random network with the same average degree z , and all the $q_i(m)$ follows Poisson distribution which can be realized when every node in layer i is randomly interdependent on a node in the master layer regardless of its multiple dependency, and $r_0 = \dots = r_n = r$, then we have

$$S_0 = (1 - e^{-zS_0}) e^{-ne^{-zS_*}}, \quad (15a)$$

$$S_* = (1 - e^{-zS_*}) (1 - e^{-zS_0}) e^{-ne^{-zS_*}}, \quad (15b)$$

where $S_* = S_1 = \dots = S_n$, the relation $H_0^i(1 - x) = H_1^i(1 - x) = e^{-x}$ has been adopted, and rz has been absorbed as z for

FIG. 9. (a) Schematic diagram of a starlike structure. (b) Behaviors of S_0 for the case of $n = 1, \dots, 5$ (from left to right). (c) The threshold of discontinuous phase transition z^I as a function of n , where the inset shows a log-log plot of this relation indicating a scaling about 0.21, i.e., $z^I \sim n^{0.21}$. (d) Schematic diagram of a general treelike structure (notations are defined in the main text). (e) Behaviors of S_0 for the cases of $n = 1, h = 2, 5$ and $n = 2, h = 2, 5, 100$. (f) Log-log plots of $z^I - n$ relation for the cases of $h = 1, \dots, 5$ (from bottom to top). The inset shows the value of the scaling of different h . In (b) and (e), symbols are obtained from simulation results, where $N = 10^4$ and error bars stand for standard deviation from the results of 100 different realizations.

simplicity. Using a similar procedure in Eq. (11) to determine the location of the first order phase transition, we have

$$f(S_0) = S_0 - (1 - e^{-zS_0})e^{-ne^{-zS_0}} = 0, \quad (16a)$$

$$f(S_*) = S_* - (1 - e^{-zS_*})(1 - e^{-zS_0})e^{-ne^{-zS_*}} = 0, \quad (16b)$$

$$\frac{df(S_*)}{dS_*} = 1 - \left\{ ze^{-zS_*}e^{-ne^{-zS_*}}(1 - e^{-zS_0})[1 + n(1 - e^{-zS_*})] + (1 - e^{-zS_*})ze^{-zS_0}e^{-ne^{-zS_*}} \frac{dS_0}{dS_*} \right\}, \quad (16c)$$

$$\frac{dS_0}{dS_*} = e^{-ne^{-zS_*}} \left[ze^{-zS_0} \frac{dS_0}{dS_*} + (1 - e^{-zS_0})ne^{-zS_*} \right]. \quad (16d)$$

The location of the discontinuous phase transition z^I of the network is then obtained by solving Eq. (16).

As to the continuous phase transition induced by the independent nodes in the master layer, since these kinds of nodes are not interlinked with the nodes in any of the normal layer, the fraction of these kinds of nodes simply equals $\prod_{i=1}^n q_i(0) = (e^{-1}/0!)^n = e^{-n}$ where the no-correlation condition is used. Applying the same arguments for Eq. (9), we get the location of the continuous phase transition of layer A, which is

$$z^{II} = 1 / \prod_{i=1}^n q_i(0) = e^n. \quad (17)$$

Figure 9(b) presents the behaviors of S_0 for different n . We observe that a hybrid phase transition appears when $n = 1$ (though not prominent) which means in this case $z^{II} < z^I$. However, since z^{II} increases exponentially with n which quickly exceeds z^I , discontinuous phase transition happens in all the following $n > 1$. Figure 9(c) shows the relation

between z^I and n , which implies that though z^I always increases with n , the growing speed decays. To better exhibit the growth tendency of z^I , a log-log plot of their relation is shown in the inset where a scaling ~ 0.21 is suggested.

B. General treelike structure

For a general treelike structure, we take the master layer as the root which is connected by a number of branches. Each branch is composed of a group of normal layers that are dependent under one-to-one correspondence. In each branch, there is one normal layer directly connected to the master layer, and nodes in the master layer may depend on the nodes in these layers with MD relation. In this case, we denote the master layer as layer 0, and the layers in branch i as layer i_u ($1 \leq u \leq h_i$) with h_i being the number of layers in the branch i and the layer i_1 is the one connected to the master layer. Other notations are defined with the same rule as above. A schematic diagram of the treelike structure is provided in Fig. 9(d).

Before giving the self-consistent equations describing the size of MCGC, we would first present the possibilities of some related events. Suppose branch i is isolated from the rest of the network, then the possibility of a randomly chosen node in layer i_v ($1 \leq v \leq h_i$) belongs to the MCGC of this isolated branch i equals $\prod_{u=1}^{h_i} [1 - G_0^{i_u}(1 - r_{i_u} S_{i_u}^*)]$, denoted as g_0^i for short, which is identical for all the v values. Indeed, as pointed above for a fully dependent treelike structure, the functioning of a node in one layer requires the functioning of all the nodes through the one-to-one correspondence in other layers regardless of the detailed tree structure [43]. Similarly, the possibility of a randomly picked link in layer i_v ($1 \leq v \leq h_i$) connecting to the node belonging to the MCGC

of branch i equals $[1 - G_1^{i_v}(1 - r_{i_v}S'_{i_v})] \prod_{u=1, u \neq v}^{h_i} [1 - G_0^{i_u}(1 - r_{i_u}S'_{i_u})]$, denoted as $g_1^{i_v}$. Based on these facts, the equations pertinent to the MCGC of general treelike structure can be obtained as follows:

$$S'_0 = [1 - G_1^0(1 - r_0S'_0)] \prod_{i=1}^n [H_0^i(g_0^i)], \quad (18a)$$

$$S_0 = [1 - G_0^0(1 - r_0S'_0)] \prod_{i=1}^n [H_0^i(g_0^i)], \quad (18b)$$

$$S'_{i_v} = g_1^{i_v} [1 - G_0^0(1 - r_0S'_0)] H_1^i(g_0^i) \prod_{j=1, j \neq i}^n [H_0^j(g_0^j)], \quad (18c)$$

$$S_{i_v} = g_0^i [1 - G_0^0(1 - r_0S'_0)] H_1^i(g_0^i) \prod_{j=1, j \neq i}^n [H_0^j(g_0^j)]. \quad (18d)$$

In the case when all the layers are random networks with the same average degree z , $q_i(m)$ follows Poisson distribution $r_0 = \dots = r_n = r$, and each branch has the same number of layers, i.e., $h_1 = \dots = h_n = h$, we have $g_0^i = g_1^{i_v} = (1 - e^{-zS_v})^h$ for $\forall i, v$. Compared with the results in Sec. VIII A, we can see the starlike structure is the specific case of $h = 1$. With the same spirit in Sec. VIII A, the discontinuous phase transition point z^I could be deduced accordingly, and its behavior for different h is presented in Figs. 9(e) and 9(f).

Figure 9(e) shows the behaviors of S_0 for some typical cases. When $n = 1$, hybrid phase transition happens for all different h . This is because a larger h means more normal layers in a branch and therefore produces a larger z^I . Hence, with a given n a hybrid phase transition can be more evident for a larger h . On the other hand, since z^{II} grows quickly with n [noting that z^{II} depends only on n as indicated by Eq. (17) even in a treelike structure], when n increases, h requires to be very large for the emergence of a hybrid phase transition, as the example of $n = 2$ and $h = 100$ shown in Fig. 9(e). The tendency of the growth of z^I with the increasing of n for different h is shown in Fig. 9(f) as a log-log plot. The inset of Fig. 9(e) manifests the scaling of z^I - n relation under different h . The results imply that with the growing of h the behavior of z^I as a function of n tends to converge.

IX. DISCUSSION AND CONCLUSION

In this paper, we propose a two-layer asymmetric interdependent network (AIN) model to study the effect of multiple-dependence (MD) relation. This relation addresses the scenario that supposes a node in one layer is dependent on multiple nodes in the other layer, then the functioning of this node requires the functioning of all its dependent nodes in the other layer. In the AIN, only one layer allows the MD relation, while in the other layer, every node is dependent on exactly one node. We remark that if nodes in both layers could have MD relation, a feedback loop may emerge and this will cause the whole network into a full collapse upon the failure of a single node regardless of the internal connectivity. The proposed AIN model could avoid this extreme situation with a controllable manner allowing a systematic investigation on the effect of the MD relation.

In the process of controlling the extent of MD, we keep the number of interlinks unchanged, that is equal the number of nodes in each layer, and vary the distribution of interlayer degree distribution from homogeneous (one-to-one correspondence) to heterogeneous (power-law distribution). Interestingly, the layer that does not allow MD becomes increasing fragile during this process, while the layer that allows the MD relation has a complicated behavior. Specifically, this layer becomes more fragile in the beginning stage with a discontinuous phase transition and then after a tipping point (tricritical point) it becomes more robust with the appearance of a hybrid phase transition. This interesting behavior actually comes from the competitive impacts from the independent and dependent nodes. Owing to the conservation in the number of interlinks, when more interlinks are gathered at a certain fraction of nodes making this part to be more fragile, the remainder part will, however, be more robust. On the other hand, for the other layer that does not apply MD, with the increasing extent of MD by aggregating the interlinks to a smaller group of nodes, this layer will become more fragile throughout the process.

Besides, for the intriguing behavior that the phase transition goes from discontinuous to hybrid, we provide an instructive analysis to understand the underlying mechanism. Specifically, we cut off the connections between the independent nodes and dependent nodes and then study their effect separately. A heuristic picture by adding up the impacts of the two types of nodes could largely reproduce the observed phase diagram and thus explain the underlying reason for the appearance of the tricritical point. Further, according to this scheme, we analytically derive the critical points of both types of phase transition and the location of tricritical point, validated by the numerical simulations on the general case of power-law interlayer degree distribution.

We further study the effect of the correlation between the intralayer degree of the nodes in the layer allowing MD relation and their interlayer degree. It is found that this correlation has a direct impact to the continuous phase transition but has limited impact to the discontinuous phase transition. The reason is that the discontinuous phase transition is largely determined by the layer where nodes are not allowed MD relation.

Moreover, we extend the two-layer AIN model to general multilayer AIN. We show that because of the no-feedback loop condition, only one layer is allowed to have MD relation and it only allows no-loop structure among the layers. In other words, only a treelike structure with the unique layer allowing MD relation is feasible for a general multilayered AIN. The percolation behaviors and phase diagrams of a starlike AIN and general treelike AIN are exhibited.

We remark that our proposed model provides a feasible scheme to study the effect of MD relation in interdependent networks. This model could be a good starting point toward a more realistic representation of real systems. We do not expect to describe a real system through this simple model, but aim to emphasize the importance of the MD relation. The MD relation considered here could be adopted as a complementary feature in interdependent networks. As real systems are shown to have strong robustness against disturbance, our model can be further developed to incorporate those important features, such as redundant interdependency [46], intersection of

FIG. 10. A schematic illustration of (a) $p_A(k_A)$ and (b) $q(m)$ presented in ascending order of k_A and m from left to right, respectively. The corresponding cumulative distributions $P_A(k_A)$ and $Q(m)$ are indicated as vertical bars in (c). The shaded area in the dashed circle is an example of assigning $p^+(k_A, m)$ which equals $Q(m) - P_A(k_A - 1)$.

multiplex networks [12,41], and quality of edges in correlated interdependent networks [42] for the description of more realistic scenarios in specific situations of interest.

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APPENDIX A: RELATION BETWEEN BOND PERCOLATION AND SITE PERCOLATION

In this Appendix, we will show the mapping relation between the bond percolation and the site percolation on a treelike single-layer network, which has also been addressed from a quenched network perspective [41]. This mapping relation can be readily extended to interdependent networks.

For the bond percolation on a single-layer network, by borrowing the same notations in the Sec. II, we have

$$S' = 1 - G_1(1 - rS'), \quad (\text{A1a})$$

$$S = 1 - G_0(1 - rS'), \quad (\text{A1b})$$

as shown in Eq. (1), where r is the occupation possibility of a link in the network. Similarly, for the site percolation, suppose r is the occupation possibility of a node in the network, then we have

$$T' = r[1 - G_1(1 - T')], \quad (\text{A2a})$$

$$T = r[1 - G_0(1 - T')], \quad (\text{A2b})$$

where T' and T are defined similarly as S' and S . Equation (A2) comes from the fact that a node (or a node reached by a link) belongs to the GC in the condition (i) at least one of its neighbors (not from the incoming link) belongs to the GC, and (ii) the probability that the node is occupied. The comparison of (A1) and (A2) indicates that if we call $T' = rS'$ and $T = rS$, Eq. (A2) can be recovered to Eq. (A1) by eliminating the factor r on both sides of the equations. This mapping relation also implies that with a given network, the critical points of the phase transition are the same for both types of percolation, but merely differ by a multiplicative factor in the order parameters for the two cases.

APPENDIX B: ALGORITHM FOR GENERATING $p^+(k_A, m)$

By referring to the lists of $p_A(k_A)$ and $q(m)$ in ascending order as shown in Figs. 10(a) and 10(b), the idea of the algorithm is that for a pair of k_A and m in the corresponding places of the two lists, the length of their overlapped parts is assigned as the value of $p^+(k_A, m)$ where the whole length of the two lists is set as unit. For doing so, we first define two cumulative distribution functions $P_A(k_A) = \sum_{k'_A=k_{\min}}^{k_A} p_A(k'_A)$ and $Q(m) = \sum_{m'=0}^m q(m')$. The values of $P_A(k_A)$ and $Q(m)$ are shown as vertical bars in Fig. 10(c), which can be regarded as an increasing sequence from left to right. With the setting of this sequence, now we assign the value of $p^+(k_A, m)$ one by one along it from left to right. Generally, suppose in the last step $P_A(k_A - 1)$ has been reached, and the intralayer degree $k_A - 1$ has all been allocated. Then, the value $P_A(k_A - 1)$ is denoted as X and intralayer degree is updated to k_A . In the next step, $Q(m)$ is reached [as shown in the dashed circle in Fig. 10(c)] and then $p^+(k_A, m)$ is assigned as the increment $p^+(k_A, m) = Q(m) - X$ (shaded area). After this, update X as $Q(m)$ and interlayer degree from m to $m + 1$. Repeat this process until $X = 1$. As an example, this procedure gives

$$p^+(k_A, 0) = \begin{cases} p_A(k_A), & \text{when } P_A(k_A) < q(0) = Q(0) \\ q(0) - P_A(k_A - 1), & \text{when } P_A(k_A - 1) < q(0) < P_A(k_A) \end{cases} \quad (\text{B1})$$

Algorithm 1. The generation of distribution $p^+(k_A, m)$.

Input: $P_A(k_A), Q(m)$
Output: $p^+(k_A, m)$
Set $k_A = k_{\min}, m = 0, X = 0$
while $X < 1$ **do**
 if $P_A(k_A) < Q(m)$ **then**
 $p^+(k_A, m) = P_A(k_A) - X$
 $X = P_A(k_A)$
 $k_A = k_A + 1$
 else
 $p^+(k_A, m) = Q(m) - X$
 $X = Q(m)$
 $m = m + 1$
 end if
end while

where $p_A(k_{\min} - 1)$ is set as zero. The detail of the algorithm is summarized below.

The distribution $p^-(k_A, m)$ can be generated similarly.

APPENDIX C: ANALYSIS OF DISCONTINUOUS PHASE TRANSITION WITH GENERAL CORRELATION FUNCTION $p(k_A, m)$

To evaluate the threshold of discontinuous phase transition for the case of correlated intralayer and interlayer degree distribution $p(k_A, m)$, we utilize a similar procedure in Sec. VII. For general degree distribution $p_A(k_A)$ [$p_B(k_B)$], S'_A and S_A (S'_B and S_B) are generally different. However, as the discontinuous phase transitions obviously happen at the same place for S'_A and S_A (S'_B and S_B) due to the continuity of the functions in Eq. (12), we evaluate the threshold of discontinuous phase transition by analyzing S'_A and S'_B . First, we define the functions $f(S'_A)$ and $f(S'_B)$ shown in Eqs. (C1a) and (C1b):

$$f(S'_A) = S'_A - \sum_{k_A, m} \frac{k_A}{z_A} p(k_A, m) [1 - (1 - r_A S'_A)^{k_A - 1}] [1 - G_0^B(1 - r_B S'_B)]^m = 0, \quad (\text{C1a})$$

$$f(S'_B) = S'_B - [1 - G_1^B(1 - r_B S'_B)] \left\{ \sum_{k_A, m} m p(k_A, m) [1 - (1 - r_A S'_A)^{k_A}] [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\} = 0. \quad (\text{C1b})$$

Then, from Eq. (C1b) we obtain the expression of $df(S'_B)/dS'_B$ shown in Eq. (C1c):

$$\begin{aligned} \frac{df(S'_B)}{dS'_B} = & 1 - \left([r_B G_1^{B'}(1 - r_B S'_B)] \left\{ \sum_{k_A, m} m p(k_A, m) [(1 - r_A S'_A)^{k_A}] [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\} \right. \\ & + [1 - G_1^B(1 - r_B S'_B)] \left\{ \sum_{k_A, m} m p(k_A, m) \left[r_A k_A (1 - r_A S'_A)^{k_A - 1} \frac{dS'_A}{dS'_B} \right] [1 - G_0^B(1 - r_B S'_B)]^{m-1} \right\} \\ & \left. + [1 - G_1^B(1 - r_B S'_B)] \left\{ \sum_{k_A, m} m p(k_A, m) [(1 - r_A S'_A)^{k_A}] (m-1) [1 - G_0^B(1 - r_B S'_B)]^{m-2} r_B G_0^{B'}(1 - r_B S'_B) \right\} \right) = 0, \end{aligned} \quad (\text{C1c})$$

where $G_0^{B'}(x) = dG_0^B(x)/dx = \sum_k k p_B(k) x^{k_B - 1}$ and $G_1^{B'}(x) = G_0^{B''}(x)/z_B = \sum_k k(k-1) p_B(k) x^{k_B - 2}/z_B$. The condition $df(S'_B)/dS'_B = 0$ determines the position of discontinuous phase transition of S'_B . The calculation of $df(S'_B)/dS'_B = 0$ involves the term dS'_A/dS'_B which can be deduced from Eq. (C1a) given below:

$$\begin{aligned} \frac{dS'_A}{dS'_B} = & \sum_{k_A, m} \frac{k_A}{z_A} p(k_A, m) \left\{ [1 - (1 - r_A S'_A)^{k_A - 1}] m [1 - G_0^B(1 - r_B S'_B)]^{m-1} [r_B G_0^{B'}(1 - r_B S'_B)] \right. \\ & \left. + (k_A - 1) r_A (1 - r_A S'_A)^{k_A - 2} \frac{dS'_A}{dS'_B} [1 - G_0^B(1 - r_B S'_B)] \right\}. \end{aligned} \quad (\text{C1d})$$

APPENDIX D: ANALYSIS OF CONTINUOUS PHASE TRANSITION WITH GENERAL CORRELATION FUNCTION $p(k_A, m)$

In this Appendix we evaluate the location of continuous phase transition caused by independent nodes in layer A under general correlation function $p(k_A, m)$. The degree distribution of these independent nodes in layer A is $p(k_A|m=0)$, i.e., those nodes with no interlinks. With the relation of conditional

probability

$$p(k_A|m=0) = \frac{p(k_A, 0)}{q(0)}, \quad (\text{D1})$$

$p(k_A|m=0)$ can be obtained with $p(k_A, 0)$ calculated with the help of the algorithm provided in Appendix B. On the other hand, in layer A, the possibility that a link issued by an independent node reaches another independent node equals the

ratio between the degrees these independent nodes have and the degrees of all the nodes in the network, which therefore equals to $\sum_k k p(k, 0) / \sum_{k,m} k p(k, m)$, denoted as w_0 . Thus, with the theory of bond percolation on single-layer network, the possibility S'_{II} (S_{II}), denoting the event that a randomly chosen link reaches (a randomly chosen node belongs) to the GC formed by these independent nodes, can be written as

$$S'_{II} = 1 - G_1^{II}(1 - r_A w_0 S'_{II}), \quad (D2a)$$

$$S_{II} = 1 - G_0^{II}(1 - r_A w_0 S'_{II}), \quad (D2b)$$

where $G_0^{II}(x) = \sum_{k_A} p(k_A|0)x^{k_A}$ and $G_1^{II}(x) = \sum_{k_A} k_A p(k_A|0)x^{k_A-1} / \sum_{k_A} p(k_A|0)$. Defining $f(S'_{II}) =$

$S'_{II} - 1 + G_1^{II}(1 - r_A w_0 S'_{II})$, the continuous phase transition emerges at the condition $\frac{df(S'_{II})}{dS'_{II}}|_{S'_{II}=0} = 0$. Further employing Eq. (C1), this condition determines the critical r_A , denoted as r^{II} , for the phase transition as follows:

$$r^{II} = \frac{1}{w_0} \frac{\sum_{k_A} k_A p(k_A|0)}{\sum_{k_A} k_A (k_A - 1) p(k_A|0)}. \quad (D3)$$

Since the distribution $p(k_A|0)$ varies with α , with given $p_A(k) \sim k^{-\beta}$ and $q(m) \sim m^{-\gamma}$, the behavior of $r^{II}(\alpha)$ could then be obtained.

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