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Tidal Love numbers of braneworld black holes and wormholes

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We study the tidal deformations of various known black hole and wormhole solutions in a simple context of warped compactification—Randall-Sundrum theory—in which the four-dimensional spacetime geometry is that of a brane embedded in five-dimensional anti–de Sitter spacetime. The linearized gravitational perturbation theory generically reduces to either an inhomogeneous second-order ordinary differential equation (ODE) or a homogeneous third-order ODE for which indicial roots associated with an expansion about asymptotic infinity can be related to tidal Love numbers. We describe various tidal-deformed metrics, classify their indicial roots, and find, in particular, that the quadrupolar tidal Love number is generically nonvanishing. Thus, it could be a signature of a braneworld by virtue of its potential appearance in gravitational waveforms emitted in binary merger events.

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I. INTRODUCTION

Recent spectacular advances in gravitational-wave (GW) detection have brought new optimism and excitement in using GW astronomy to the exploration of a range of topics in black hole and neutron-star physics. In particular, the GW signal emitted by a neutron-star binary carries information about the nuclear equation of state through tidal Love numbers, which are fundamentally a set of quantities that measure the gravitational tidal deformation of the object. As was first explained in a seminal work by Flanagan and Hinderer [1], the (electric, quadrupolar) tidal Love number (TLN) makes its appearance as a phase in the GW waveform at the fifth post-Newtonian order during the inspiral stage of a binary merger. Various detection events could then measure or set bounds on the TLN and thus the parameters (e.g., equation of state parameters and theory couplings) that it depends on [see, for example, [2,3] for the most recent neutron binary (GW170817) data analysis performed by LIGO].

For black holes in general relativity, the TLN is conspicuously zero [4,5]. In linearized perturbation theory, one can derive the differential equations which metric perturbations have to obey, and we can read off the TLNs from their long distance behavior. For the Schwarzschild black hole, the metric perturbations can be analytically solved and, upon imposing regularity at the horizon, one could demonstrate the vanishing of its TLN. For neutron stars, we match the exterior spacetime to a suitable interior, and the TLN thus depends on the equation of state parameters associated with the stellar interior.

It is a natural and well-founded question to explore how deviations from ordinary general relativity (GR) may lead to the nonvanishing of TLNs, especially in view of the large amount of work dedicated to various theoretical models for its UV completion. Let us briefly point out recent related results regarding this issue. In [6], black hole solutions in an effective field theory framework encompassing various higher-order curvature extensions of the Einstein-Hilbert action were studied, and it was found that they have nonvanishing TLNs. A similar work [7] found the same phenomenon for black holes in a theory with R^3 terms. In [8], black hole solutions in Brans-Dicke theory, Chern-Simons gravity, and Einstein-Maxwell theory were studied for their TLNs, and nonvanishing quadrupolar and octopolar magnetic TLNs were found in the case of Chern-Simons gravity. Cardoso *et al.* [8] also computed the TLNs of compact exotic objects which are not black holes, such as wormholes and boson stars, which turn out to possess nonvanishing TLNs as well. Coupled with the anticipated increasing precision of GW detectors in the near future, this collection of recent results demonstrates that TLNs could be interesting and important indicators of new physics.

In this paper, we study the TLNs for various black hole and wormhole solutions in a simple context of warped compactification—Randall-Sundrum theory [9]—in which our four-dimensional spacetime geometry is that of a brane embedded in a five-dimensional spacetime which has five-dimensional anti–de Sitter spacetime (AdS_5) as its vacuum. The conception of this model was originally motivated by the gauge hierarchy problem, and it has been actively studied at various levels, including its embedding in string theory, relevance for AdS/CFT, etc. In our opinion, this braneworld scenario is a simple and well-founded backdrop where one could start to probe how extra dimensions affect

TLNs. In this case, the extra dimension is noncompact and gravity is localized on the brane by the warped geometry of the embedding spacetime.

Most of this paper will be devoted to the electric, quadrupolar TLN commonly termed k_2 since, as shown in [1], it is the phenomenologically relevant TLN that appears at the fifth post-Newtonian order in the waveform. We will also present some essential and useful results for the general TLN associated with polar perturbations, but we leave axial-type or magnetic TLNs [10] for future work. The background braneworld solutions include black holes, wormholes, and naked singularities, which were presented in the literature some time ago. They are exact solutions to a four-dimensional formulation of the Randall-Sundrum model with an effective energy-momentum tensor (capturing the effects of the extra dimension) as first derived by Shiromizu [11]. The perturbation equations yield differential equations which appear difficult for analytic solutions, but the computation of TLNs in this case can still proceed with appropriate series expansions about infinity and the metric singularities, provided that the singular points of the differential equations are regular.

The families of solutions we consider are all parametrically connected to some well-defined limits of a particular black hole solution which have the same metric form as the Reissner-Nordström solution but a negative tidal charge. This "tidal black hole" will be used as an anchor solution for us to check the consistency of various series solutions. In the vanishing charge limit, it reduces to the Schwarzschild metric [12]. We should point out that in [13] there was an attempt to compute the TLN for the tidal black hole, but upon review (see details in Sec. IV B) we find that there is, unfortunately, a major error in that calculation. Nonetheless, we refer the reader to [13] for a study of TLNs of braneworld stars [14], which this paper does not cover.

Here is a summary of our main results: (i) TLNs of braneworld black holes and wormholes studied here are generically nonvanishing, (ii) they can be expressed in most cases as polynomials in the parameter that characterizes each family of solution, and (iii) indicial roots associated with near-horizon and asymptotic expansions display some level of universality across various families of solutions.

The paper is organized as follows. In Secs. II and III, we furnish a review of the notions of TLNs and the effective field equations of the Randall-Sundrum braneworld model, respectively, establishing some conventions along the way. In Sec. IV, we derive the perturbation equations and show that we end up with either a homogeneous third-order ordinary differential equation (ODE) or an inhomogeneous second-order ODE to solve. In Secs. V and VI, we study each braneworld solution in detail, staying focused mostly on the case of the quadrupolar TLN. In each case, we will briefly state relevant aspects of the causal structure of each solution following the relevant references, to which the reader can refer for the Carter-Penrose (CP) diagrams and other elaborations. Finally, we end with some concluding remarks in Sec. VII. A pair of appendixes gather some explicit details on metric perturbation and indicial roots.

II. ON TIDAL LOVE NUMBERS

We begin by reviewing the notion of TLNs by following [15]. They are quantities which measure the effect of gravitational tidal deformation due to some external companion or field on the object, and their appearance in gravitational waveforms was first studied seriously in [1]. In Newtonian gravity, the TLN is a constant of proportionality between the tidal field applied to the body and the resulting multipole moment of its mass distribution. One can characterize the tidal field by the tidal moment

$$\xi_{ab}(t) = -\partial_a \partial_b U_{\rm ext}, \tag{2.1}$$

where $U_{\rm ext}$ is the Newtonian potential of some external body. This is evaluated at the body's c.m. and we are working in the object's local asymptotic rest frame, with the x^a in Eq. (2.1) being asymptotically c.m. centered Cartesian coordinates. It is a symmetric and trace-free tensor which can be covariantly described by the Weyl tensor.[1] On the other hand, we define the quadrupole moment as

$$Q^{ab} = \int \rho(x)\left(x^a x^b - \frac{1}{3}\delta^{ab} r^2\right) d^3x, \tag{2.2}$$

where $\rho(x)$ is the mass density within the body. In the absence of the tidal field, we assume the body to be spherical and Q^{ab} vanishes. In the presence of a weak tidal field, from dimensional analysis

$$Q_{ab} = -\frac{2}{3} k_2 R^5 \xi_{ab}, \tag{2.3}$$

where R is the body's radius and the rest are conventions. k_2 is the dimensionless tidal Love number for a quadrupolar deformation and is the focus of this paper.

More generally, we could have tidal moments of higher multipole orders and higher powers of x^a. The set of tidal Love numbers k_l then measures the body's response. A useful way of characterizing this definition can be obtained from the expression of the Newtonian potential

$$U = \frac{M}{r} - \frac{1}{(l-1)l}[1 + 2k_l (R/r)^{2l+1}]\xi_L(t) x^L, \tag{2.4}$$

where $L \sim a_1 a_2 \ldots a_l$ is a multi-index that contains l individual indices. The tidal moment is now defined as $\xi_L(t) = -\frac{1}{(l-2)!}\partial_L U_{\rm ext}$, and the l-pole moment of the mass distribution is the tensor

[1]In Fermi normal coordinates centered at $r = 0$, $\xi_{ab} = R_{0a0b}$. See, for example, [16].

$$Q^L = \int \rho x^{\langle L \rangle} d^3x, \quad Q_L = -\frac{2(l-1)!}{(2l-1)!!} k_l R^{2l+1} \xi_L, \tag{2.5}$$

where $\langle L \rangle$ denotes the removal of its trace. It is useful to formulate a working definition of the tidal Love numbers from the following metric ansatz:

$$\begin{aligned} g_{tt} &= -1 + \frac{2M}{r} + \sum_{l\geq 2} \frac{2}{r^{l+1}} (M_l Y^{l0} + \dots) \\ &\quad - \frac{2}{l(l-1)} r^l (\xi_l Y^{l0} + \dots), \end{aligned} \tag{2.6}$$

$$\begin{aligned} g_{t\varphi} &= \frac{2J}{r} \sin^2\theta + \sum_{l\geq 2} \frac{2}{r^l} \left(\frac{S_l}{l} S_\varphi^{l0} + \dots \right) \\ &\quad + \frac{2r^{l+1}}{3l(l-1)} (B_l S_\varphi^{l0} + \dots), \end{aligned} \tag{2.7}$$

where $S_\varphi^{l0} = \sin(\theta)\partial_\theta Y^{l0}$. The "electric" and "magnetic" tidal Love numbers are defined as

$$k_l^{(E)} = -\frac{l(l-1)}{2M^{2l+1}} \frac{M_l}{\xi_l}, \quad k_l^{(B)} = -\frac{3l(l-1)}{2(l+1)M^{2l+1}} \frac{S_l}{B_l}, \tag{2.8}$$

where we have absorbed a factor of $\sqrt{\frac{4\pi}{2l+1}}$ in M_l, S_l, with the two types of TLNs describing polar and axial perturbations. Typically, we use gravitational perturbation theory to construct the metric describing the tidal deformations. One can parametrize static perturbations of spherically symmetric and static geometries as follows (see, for example, [17]). Let $g_{\mu\nu} = g_{\mu\nu}^{(bg)} + h_{\mu\nu}$. Then in the chart of $\{t, r, \theta, \phi\}$ we can write

$$\begin{aligned} h_{tt} &= g_{tt}^{(bg)} \sum_l H_0^{(l)} P_l, \qquad h_{rr} = g_{rr}^{(bg)} \sum_l H_2^{(l)} P_l, \\ h_{\theta\theta} &= r^2 \sum_l K^{(l)} P_l, \qquad h_{\phi\phi} = r^2 \sin^2\theta \sum_l K^{(l)} P_l, \\ h_{t\varphi} &= \sum_l h_0^{(l)} P_l' \sin^2\theta, \end{aligned} \tag{2.9}$$

where $P_l' = \frac{dP_l(\cos\theta)}{d(\cos\theta)}$ and $P_l = P_l(\cos\theta)$ are Legendre polynomials used as basis functions for the angular dependence and where, due to the spherical symmetry of the ansatz, we suppress the degenerate azimuthal numbers. The above ansatz for static linearized perturbations was first studied by Thorne and co-workers [18,19] and has since been frequently invoked in studies of TLN as well as quasi-Schwarzschild solutions [17]. For solutions relevant for TLN, the metric is not asymptotically flat and is valid within a restricted domain that lies in the compact object's exterior and which captures the tidal effects of an external matter source.

In the remainder of our paper, we will focus mainly on $k_2^{(E)}$: the dimensionless quadrupolar TLN which carries crucial phenomenological value due to its appearance as a tidal phase correction in the gravitational waveform associated with the inspiral stage. Taking $l = 2$ in Eq. (2.8), we have

$$k_2^{(E)} = -\frac{1}{M^5} \frac{M_2}{\xi_2}, \qquad k_2^{(B)} = -\frac{1}{M^5} \frac{S_2}{B_2}. \tag{2.10}$$

We have suppressed other angular harmonics in our ansatz for the perturbation. Let us quickly note that, upon restoring them, we can write the metric component g_{tt} in the form

$$g_{tt} = -1 + \frac{2M}{r} + \frac{3Q_{ij}}{r^3} (n^i n^j) - \xi_{ij} x^i x^j + \dots, \tag{2.11}$$

where $n^i = x^i/r$ is the unit vector and the quarupole moment Q_{ij} is traceless. The r^2 term represents the tidal force and the $1/r^3$ term characterizes the compact object's response to the tidal force. To linear order in ξ_{ij}, the induced moment takes the form (see, for example, [20])

$$Q_{ij} = -\lambda \xi_{ij} \tag{2.12}$$

for some constant λ. The dimensionless TLN $k_2^{(E)}$ is related by $k_2^{(E)} = \frac{3\lambda}{2M^5}$. Using the ansatz for $l = 2$ and expanding in spherical harmonics, we can write

$$Q_{ij} n^i n^j = \sum_{m=-2}^{2} Q_m Y_{2m}, \quad \xi_{ij} n^i n^j = \sum_{m=-2}^{2} \tilde{\xi}_m Y_{2m}. \tag{2.13}$$

A comparison of Eqs. (2.11) and (2.6) yields

$$\lambda = -\frac{2M_2}{3\xi_2}, \qquad k_2^{(E)} = \frac{3\lambda}{2M^5} = -\frac{1}{M^5} \frac{M_2}{\xi_2}. \tag{2.14}$$

To extract $k_2^{(E)}$, we need the asymptotic series expansion of h_{tt}, from which we read off any nonvanishing pair of r^2 and $1/r^3$ terms. Since we are seeking the induced response, the relevant term of the form $1/r^3$ should vanish together with the tidal force term. In the following, for each braneworld solution, we will focus on computing

$$\lambda = \frac{1}{3} \frac{C_3}{C_2}, \tag{2.15}$$

where C_3, C_2 are the coefficients of the $1/r^3$, r^2 terms in an asymptotic series expansion of h_{tt}. We will refer to λ as the TLN from now on.

III. ON RANDALL-SUNDRUM THEORY WITH WARPED COMPACTIFICATION

In this section, we furnish some essential points concerning the background gravitational theory—Randall-Sundrum theory with a brane of positive tension on which gravity is localized via curvature rather than compactification [9]. We work with a five-dimensional bulk with AdS_5 as the vacuum and an energy-momentum tensor that is confined on the $3+1$D brane, in which the effective cosmological constant can be set to vanish with a choice of brane tension. Following [11], we find it useful to formulate the gravitational dynamics purely in terms of an effective four-dimensional theory which we review below. We will also derive how various metric components (including the perturbations) which are functions of the $3+1$D brane–world volume coordinates appear in the line element obtained by expanding the metric about the brane.

We begin with a parent 5D metric ansatz that reads

$$ds^2 = dy^2 + g_{\mu\nu}(x,y)dx^\mu dx^\nu, \tag{3.1}$$

where y is a Gaussian normal coordinate that is orthogonal to the brane. In the neighborhood of $y=0$ where the confining brane lies, we can express the metric perturbation as

$$\begin{aligned} g_{\mu\nu}(x,y) &= g^{bg}_{\mu\nu}(x,y) + h_{\mu\nu}(x,y) \\ &= g^{bg}_{\mu\nu}(x,y) + h_{\mu\nu}(x,0) + h'_{\mu\nu}(x,0)y \\ &\quad + \frac{1}{2}h''_{\mu\nu}(x,0)y^2 + \ldots, \end{aligned} \tag{3.2}$$

where we denote ∂_y with a prime for notational simplicity in this section. From Eq. (3.1), the extrinsic curvature describing the embedding of constant y surfaces reads

$$K_{\mu\nu} = (\delta^\alpha_\mu - n^\alpha n_\mu)(\delta^\beta_\nu - n^\beta n_\nu)\nabla_\alpha n_\beta = \frac{1}{2}g'_{\mu\nu}. \tag{3.3}$$

We can fine-tune the brane cosmological constant to vanish, with

$$\Lambda_5 = -\frac{\kappa_5^2\lambda^2}{6}, \qquad \kappa_4^2 = \frac{1}{6}\kappa_5^4\lambda, \tag{3.4}$$

where λ is the brane tension and $\kappa^2_{4,5} = 8\pi G_{4,5}$. In this paper, we will adopt this condition for simplicity. The ordinary 4D limit involves taking $\kappa_5 \to 0$, $\lambda \sim \kappa_5^{-4} \to \infty$ such that κ_4 is finite. Henceforth, we denote κ_4 simply as κ. The 5D field equations read

$$G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2(T_{AB} + T^{br}_{AB}\delta(y)). \tag{3.5}$$

For the energy content, we set

$$T_{AB} = 0, \qquad T^{br}_{AB} = -\lambda g_{AB}\delta^A_\mu\delta^B_\nu. \tag{3.6}$$

Together with the fine-tuned vanishing of Λ_4, it can be shown using Gauss-Codazzi equations (see, for example, [21]) that the field equations on the brane read

$$\begin{aligned} G_{\mu\nu} &= -E_{\mu\nu}, \\ E_{\mu\nu} &= C^\alpha{}_{\beta\rho\sigma}n_\alpha n^\rho q^\beta_\mu q^\sigma_\nu = K_{\mu\alpha}K^\alpha{}_\nu - \partial_y K_{\mu\nu} - \frac{\Lambda_5}{6}g_{\mu\nu}. \end{aligned} \tag{3.7}$$

Apart from the continuity of the metric across $y=0$, the Israel junction conditions impose the discontinuity of the extrinsic curvature as

$$K^+_{\mu\nu} - K^-_{\mu\nu} = -\kappa_5^2\left(T^{br}_{\mu\nu} - \frac{1}{3}T^{br}g_{\mu\nu}\right) = -\frac{1}{3}\kappa_5^2\lambda g_{\mu\nu}, \tag{3.8}$$

which implies that on the brane

$$K_{\mu\nu} = -\frac{1}{6}\kappa_5^2\lambda g_{\mu\nu} = \frac{1}{4}Kg_{\mu\nu}. \tag{3.9}$$

By virtue of the metric ansatz, we also have Eq. (3.3), which leads to the following useful relation on the brane:

$$g'_{\mu\nu}(x,0) = \frac{1}{2}Kg_{\mu\nu}(x,0). \tag{3.10}$$

With the metric perturbation switched on, the scalar $K = -\frac{2}{3}\kappa_5^2\lambda$ remains invariant and specifies the constraint on the brane

$$g^{\mu\nu}(x,0)g''_{\mu\nu}(x,0) = K^2. \tag{3.11}$$

The perturbations have to satisfy [from Eq. (3.10)]

$$h'_{\mu\nu}(x,0) = \frac{1}{2}Kh_{\mu\nu}(x,0). \tag{3.12}$$

Under the perturbation, we find that

$$\delta E_{\mu\nu} = -\frac{1}{2}h''_{\mu\nu} - \frac{1}{6}\Lambda_5 h_{\mu\nu} + \frac{1}{2}h'_{\mu\alpha}K^\alpha{}_\nu + \frac{1}{2}K_\mu{}^\beta h'_{\beta\nu}. \tag{3.13}$$

When restricted on the brane, we find that

$$\delta E_{\mu\nu}(x,0) = -\frac{1}{2}h''_{\mu\nu}(x,0) + \frac{3}{16}K^2 h_{\mu\nu}(x,0), \tag{3.14}$$

with a slightly different background relation

$$E_{\mu\nu}(x,0) = -\frac{1}{2}g''_{\mu\nu}(x,0) + \frac{1}{8}K^2 g_{\mu\nu}(x,0). \tag{3.15}$$

We note that Eqs. (3.14) and (3.15) imply that up to second order in y we have the following expansion of the metric components:

$$g^{bg}_{\mu\nu} = g^{bg}_{\mu\nu}(x,0) + \frac{1}{2} K g^{bg}_{\mu\nu}(x,0) y + \left[\frac{1}{8} K^2 g^{bg}_{\mu\nu}(x,0) - E_{\mu\nu}(x,0)\right] y^2 + \ldots, \qquad (3.16)$$

$$h_{\mu\nu} = h_{\mu\nu}(x,0) + \frac{1}{2} K h_{\mu\nu}(x,0) y + \left[\frac{3}{16} K^2 h_{\mu\nu}(x,0) - \delta E_{\mu\nu}(x,0)\right] y^2 + \ldots. \qquad (3.17)$$

In this paper, we will work with various classical solutions in an effective 3 + 1D description as backgrounds. We will not seek their 4 + 1D completion beyond what they imply for the bulk extension via Eqs. (3.16) and (3.17). Although it has proven to be difficult to find the full solution in the bulk, the effective 4D field equations were shown in [11] to be fully consistent.[2]

IV. ON THE PERTURBATION EQUATIONS

From Eq. (3.7), we see that $E_{\mu\nu}$ should be interpreted as an energy-momentum tensor satisfying the Bianchi identity. We proceed by adopting the following ansatz for it:

$$-\frac{1}{\kappa_4^2} E_{\mu\nu} = \rho\left(U_\mu U_\nu + \frac{1}{3} h_{\mu\nu}\right) + q_{(\mu} U_{\nu)} + \Pi\left(\frac{1}{3} h_{\mu\nu} - r_\mu r_\nu\right), \qquad (4.1)$$

where r_μ and U_μ are unit radial and timelike vectors, respectively, $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, and the quantities ρ and Π are the effective density and stress induced on the brane. We also set $\kappa_4 = 1$ from now on. This ansatz has turned out to be very useful in seeking classical solutions (see, for example, [21] for an extensive review). For the solutions we consider, we take q_μ to vanish and $\rho = \rho(r)$, $\Pi = \Pi(r)$, which generally correspond to spherically symmetric and static geometries. We adopt the metric ansatz

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (4.2)$$

The Bianchi identity can be further simplified to be a single ODE in the radial coordinate

$$\rho' + \frac{f'(r)}{f(r)}(-\Pi + 2\rho) - 2\Pi' - \frac{6}{r}\Pi = 0. \qquad (4.3)$$

The background field equations read

$$G^t_t == -\rho, \quad G^r_r = -\frac{1}{3}(2\Pi - \rho), \quad G^\theta_\theta = G^\phi_\phi = \frac{1}{3}(\Pi + \rho). \qquad (4.4)$$

In terms of the functions f, g,

$$G^t_t = \frac{g(r) - g(r)^2 - rg'(r)}{r^2 g(r)^2}, \qquad G^r_r = \frac{f(r)(1 - g(r)) + rf'(r)}{r^2 f(r) g(r)}, \qquad (4.5)$$

$$G^\theta_\theta = G^\phi_\phi = \frac{-rg(r)(f'(r))^2 - 2f^2(r)g'(r) + f(r)(-rf'(r)g'(r) + 2g(r)(f'(r) + rf''(r)))}{4rf^2(r)g^2(r)}. \qquad (4.6)$$

We express the perturbations in the basis of Legendre polynomials $P_l(\cos\theta)$:

$$f(r) \to f(r)\left(1 + \sum_l H_0^{(l)}(r) P_l\right), \qquad g(r) \to g(r)\left(1 + \sum_l H_2^{(l)}(r) P_l\right),$$
$$g_{\theta\theta} = g_{\phi\phi}/\sin^2\theta = r^2 \to r^2 \sum_l K^l(r) P_l,$$
$$\rho(r) \to \rho(r) + \sum_l \delta\rho^l(r) P_l, \qquad \Pi(r) \to \Pi(r) + \sum_l \delta\Pi^l(r) P_l. \qquad (4.7)$$

In addition, the Bianchi identity translates into an ODE for the perturbations of ρ and Π as follows. From Eq. (4.3),

$$\delta\rho' + 2\frac{f'(r)}{f(r)}\delta\rho + (2\rho - \Pi)H_0' = 2\delta\Pi' + \delta\Pi\left(\frac{f'(r)}{f(r)} + \frac{6}{r}\right). \qquad (4.8)$$

[2] In [11], it was shown that one can integrate into the bulk by complementing the effective 3 + 1D description by two more differential equations involving the Lie derivative of $E_{\mu\nu}$ and the Weyl tensor, as explained in the appendix of [11]. We also refer the interested reader to [22], which contains discussions of how Campbell-Magaard–type embedding theorems in differential geometry imply the existence of bulk solutions from solutions in the Shiromizu-Maeda-Sasaki formulation.

Since ρ and Π parametrize the 5D graviton perturbations, we take $\delta\rho, \delta\Pi$ to be of the same fluctuation order as the 4D metric perturbations $\{H_0(r), H_2(r), K(r)\}$.

A. Decoupling of angular modes and a third-order ODE

In the following, we show that the angular polar modes in the basis of Legendre polynomials $P_l(\cos\theta)$ decouple in the linearized equations, and we derive the ODE that determines $H_0(r)$, $H_2(r)$, and $K(r)$. From now on, we will omit the superscript (l) from these functions and their dependence on r for notational simplicity.

From $\delta G^r_\theta = 0$, we find the relation

$$K' = -H_0' + \frac{1}{r}(H_0 + H_2) + \frac{f'}{2f}(H_2 - H_0), \tag{4.9}$$

whereas from $\delta G^\theta_\theta - \delta G^\phi_\phi$ we obtain

$$\begin{aligned}\delta G^\theta_\theta - \delta G^\phi_\phi = \frac{1+l}{r^2\sin^2\theta}&(H_0 + H_2)\\ &\times[(4 + l + (2 + l)\cos(2\theta))P_l\\ &- 2(5 + 2l)\cos(\theta)P_{1+l} + 2(2 + l)P_{2+l}],\end{aligned} \tag{4.10}$$

which implies for $l \geq 2$ that[3]

$$H_2 = -H_0, \qquad K' = -H_0' - \frac{f'}{f}H_0. \tag{4.11}$$

We find that K is eliminated in $\delta G^r_r - \delta G^t_t$, which reads

$$\begin{aligned}\delta G^r_r - \delta G^t_t = &-\frac{1+l}{r^2\sin^2\theta}H_0[(3 + 2l)\cos(\theta)P_{1+l} - (2 + l)P_{2+l}]\\ &+ \frac{P_l}{2r^2f^2g^2}[-3r^2gH_0f'^2 + rf(-rH_0f'g' + g(6H_0f' + rf'H_0' + 2rH_0f''))\\ &+ f^2\left[(1 + l)(2 + l + l\cos(2\theta))\frac{g^2H_0}{\sin^2\theta} + rg'(2H_0 - rH_0') + 2rg(2H_0' + rH_0'')\right]\Bigg].\end{aligned} \tag{4.12}$$

The appearance of P_{1+l} and P_{2+l} may suggest possible couplings among different polar modes, but at this point we invoke Bonnet's recursion formula

$$(l + 2)P_{2+l} - (2l + 3)\cos(\theta)P_{l+1} + (l + 1)P_l = 0$$

to simplify Eq. (4.12) as

$$\begin{aligned}\delta G^r_r - \delta G^t_t = &-\frac{l(l+1)}{r^2}P_lH_0\\ &+ \frac{P_l}{2r^2f^2g^2}[-3r^2gH_0f'^2 + rf(-rH_0f'g'\\ &+ g(6H_0f' + rf'H_0' + 2rH_0f''))\\ &+ f^2[rg'(2H_0 - rH_0') + 2rg(2H_0' + rH_0'')]].\end{aligned} \tag{4.13}$$

Thus, we see there is decoupling of the polar modes since the angular dependence lies purely in $P_l(\cos(\theta))$. This should be equated to $\frac{2}{3}(2\delta\rho - \delta\Pi)P_l$. We are left with the tracelessness condition, which, however, involves $K(r)$.

We now use the field equations and Bianchi identity to write down a single ODE for H_0. First, from Eqs. (4.6) and (4.11), the field equations imply in particular that

$$\delta G^\theta_\theta = \delta G^\phi_\phi = \frac{1}{2rfg^2}H_0(gf)'P_l = \frac{1}{3}(\delta\Pi + \delta\rho)P_l, \tag{4.14}$$

which allows one to express $\delta\Pi$ in terms of $\delta\rho$. This equation is identical for all $l \geq 2$. Substituting Eq. (4.14) into

$$\delta\rho' + 2\frac{f'}{f}\delta\rho + (2\rho - \Pi)H_0' = 2\delta\Pi' + \delta\Pi\left(\frac{f'}{f} + \frac{6}{r}\right), \tag{4.15}$$

we find the following third-order ODE:

$$C_3H_0''' + C_2H_0'' + C_1H_0' + C_0H_0 = 0, \tag{4.16}$$

where

$$\begin{aligned}C_3 &= \frac{r^2f}{2g}, \qquad C_2 = \frac{r}{4g^2}[3rgf' + f(8g - 3rg')],\\ C_1 &= \frac{1}{4fg^3}(-3r^2g^2f'^2 + rfg(-3rf'g' + g(16f' + 3rf''))\\ &\quad - f^2[-4g^2 + 2l(l + 1)g^3 - 2r^2g'^2 + r^2gg'']),\\ C_0 &= \frac{1}{4f^2g^3}(3r^2g^2f'^3 + rgff'(3rf'g' - 2g(4f' + 3rf''))\\ &\quad + 4f^3(-2rg'^2 + g(2g' + rg''))\\ &\quad - f^2[2l(1 + l)g^3f' - 2r^2f'g'^2\\ &\quad + rg(3rg'f'' + f'(8g' + rg''))\\ &\quad - 2g^2(6f' + r(6f'' + rf'''))]).\end{aligned} \tag{4.17}$$

[3]For $l = 0$, 1, the rhs vanishes identically, but K decouples from all equations and can be solved by the same quadrature once we solve two coupled ODEs in H_0 and H_2.

B. The decoupled case $\Pi = 2\rho, fg = 1$

There is a distinguished case of

$$\Pi = 2\rho = -\frac{q}{r^4}, \qquad f = 1/g = 1 - \frac{2m}{r} + \frac{q}{r^2},$$

which corresponds to the important example of the tidal black hole. We will study this solution in detail in Sec. V. In this case, we find that $\delta\rho$ decouples from the differential equations. From Eq. (4.14), we obtain

$$\delta\rho = -\delta\Pi, \tag{4.18}$$

which, upon solving Eq. (4.15), leads to $\delta\rho \sim \frac{1}{r^2 f}$.

At this point, we should point out that in [13], while studying this black hole solution, Chakravarti *et al.* unfortunately took the (correct) relation $\Pi = 2\rho$ to imply the (incorrect) relation $\delta\Pi = 2\delta\rho$. Although it is true that there are no *a priori* constraints between ρ and Π, the linearized perturbations of these functions have to satisfy the field equations and Bianchi identity for consistency, and Eq. (4.14) simply leads to Eq. (4.18) for this solution (without imposing any further constraints by hand, for example, by assuming that $\delta\rho = C\delta\Pi$ for some constant C).

In this case, Eq. (4.13) simplifies to read

$$H_0'' + \frac{P(r)}{r} H_0' + \frac{Q(r)}{r^2} H_0 = -l(l+1)\alpha \frac{r^2}{(r(r-2m)+q)^2}, \tag{4.19}$$

where α parametrizes the arbitrary constant for $\delta\rho$[4] and

$$P(r) = -\frac{2r(m-r)}{r(r-2m)+q},$$
$$Q(r) = -\frac{qr((l^2+l-2)r-4m) + r^2(-2l(l+1)mr + l(l+1)r^2 + 4m^2) + 2q^2}{(r(r-2m)+q)^2}. \tag{4.20}$$

In Sec. V, we will discuss the near-horizon and asymptotic series solutions to Eq. (4.19), focusing mostly on the more phenomenologically relevant $l = 2$ case.

V. ON THE TIDAL BLACK HOLE

In this section, we study the tidal deformation of the following black hole solution first presented in [23] with metric

$$f(r) = g(r)^{-1} = 1 - \frac{2m}{r} + \frac{q}{r^2}, \qquad q < 0. \tag{5.1}$$

This is of the usual Reissner-Nordström form, but we note that the corresponding energy-momentum tensor is of course not of Maxwell's theory. The parameter q is known as the tidal charge parameter and is of the negative sign, which can be interpreted as a strengthening of the gravitational field. We will henceforth refer to this solution as the tidal black hole. There is a regular horizon at

$$r = r_h = m\left[1 + \sqrt{1 - \frac{q}{m^2}}\right]. \tag{5.2}$$

The other zero of f lies at a negative $r_- = m\left[1 - \sqrt{1 - \frac{q}{m^2}}\right]$. The background fields read

$$G_t^t = G_r^r = -G_\phi^\phi = -G_\theta^\theta = -\frac{q}{r^4}, \qquad \rho = -\frac{q}{r^4} = \frac{\Pi}{2}. \tag{5.3}$$

Setting $l = 2$ in Eq. (4.19), we obtain the equation

$$H_0'' + \frac{P(r)}{r} H_0' + \frac{Q(r)}{r^2} H_0 = \frac{-6\alpha r^2}{(q + r^2 - 2mr)^2} \equiv \mathcal{S}(r), \tag{5.4}$$

where

$$P(r) = -\frac{-2r(m-r)}{q + r(r-2m)},$$
$$Q(r) = \frac{(-2(q^2 + 2qr(-m+r) + r^2(2m^2 - 6mr + 3r^2)))}{(q + r(-2m+r))^2}. \tag{5.5}$$

By the general theory of ODEs, Eq. (5.4) has two independent homogeneous solutions and a particular solution. Their analytic forms are not known to us, yet in the $q = 0$ limit exact solutions are known. In the following, we study their near-horizon series expansion in parameter $x = (r - r_h)/m$ and their asymptotic series expansion in parameter $u = m/r$. They can be used to compute the TLN for the tidal black hole that turns out to be a simple function of mass and tidal charge under certain assumptions for their regularity at the horizon.

[4]The factor of $l(l+1)$ is introduced for a slightly simpler notations for related computations in Sec. V.

A. Expansion about $r = r_h$

We first carry out an expansion about $r = r_h$ and impose regularity at the horizon. Let $x = (r - r_h)/m$, and recast the ODE in variable x, writing Eq. (5.4) as

$$H_0'' + \frac{\tilde{P}(x)}{x} H_0' + \frac{\tilde{Q}(x)}{x^2} H_0 = m^2 \mathcal{S}(x), \tag{5.6}$$

where $\tilde{P}(x) = xP(x)/(x + r_h)$ and $\tilde{Q}(x) = x^2 Q(x)/(x + r_h)^2$. We find $x = 0$ to be a regular point with

$$\tilde{P}(0) = 1, \qquad \tilde{Q}(0) = -1.$$

The indicial equation now reads

$$\mathcal{R}(\mathcal{R}-1) + \mathcal{R}\tilde{P}(0) + \tilde{Q}(0) = \mathcal{R}(\mathcal{R}-1) + \mathcal{R} - 1 = 0, \tag{5.7}$$

which has roots being ± 1 independent of q. The two homogeneous solutions are thus of the form

$$\tilde{S}_r(x, q) = x(1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots), \tag{5.8}$$

$$\tilde{S}_d(x, q) = \mathcal{N}\tilde{S}_r(x, q)\log(x) + \frac{1}{x}(1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots). \tag{5.9}$$

Since $\tilde{S}_d(x, q)$ diverges as $x \to 0$, this implies that g_{rr} is divergent at the horizon—a feature which nullifies the series as part of the general solution regular at the horizon. We also find a particular solution of the following form:

$$\tilde{S}_p(x, q) = C_0 + (C_1 + \mathcal{K}_1 \log(x))x + (C_2 + \mathcal{K}_2 \log(x))x^2 + \ldots \tag{5.10}$$

In Appendix A 1, we collect several explicit expressions for some of the series coefficients in Eqs. (5.8), (5.9), and (5.10) as useful references. The coefficients a_i in $\tilde{S}_r(x, q)$ are uniquely determined as functions of q and m, while in $\tilde{S}_d(x, q)$ the coefficients b_i, $i \geq 2$ are defined up to an arbitrary multiple of $\tilde{S}_r(x, q)$ added to $\tilde{S}_d(x, q)$. The particular solution is defined such that it vanishes in the $\alpha \to 0$ limit.

The series expressions $\tilde{S}_r(x, q)$, $\tilde{S}_d(x, q)$, and $\tilde{S}_p(x, q)$ are the near-horizon series expansions of the two independent homogeneous solutions and the particular solution of Eq. (5.4) up to their linear combinations, with leading-order terms $\sim x^1$, x^{-1}, and x^0, respectively. In the $q = 0$ limit, these series solutions read

$$\begin{aligned}
\tilde{S}_r(x, q=0) &= x\left(1 + \frac{1}{2}x\right) = \left(\frac{r}{m} - 2\right)\frac{r}{2m}, \\
\tilde{S}_d(x, q=0) &= -3\tilde{S}_r(x, q=0)\log(x) \\
&\quad + \frac{1}{x}\left(1 - \frac{5}{2}x + \frac{13}{4}x^3 + \ldots\right), \\
\tilde{S}_p(x, q=0) &= \alpha\left(1 + (\log(x))x + \left(-\frac{13}{12} + \frac{1}{2}\log(x)\right)x^2 \right. \\
&\quad \left. - \frac{5}{48}x^3 + \ldots\right).
\end{aligned} \tag{5.11}$$

We can compare them with the Schwarzschild case in ordinary GR for which there is an exact solution for H_0 first found by Hinderer to be

$$\begin{aligned}
H_0 = c_1 \frac{5}{8}\left(\frac{r}{m}\right)^2\left(1 - \frac{2m}{r}\right)&\left[-\frac{m(m-r)(2m^2 + 6mr - 3r^2)}{r^2(2m-r)^2}\right. \\
&\left. - \frac{3}{2}\log\left(1 - \frac{2m}{r}\right)\right] + c_2\left(\frac{r}{m}\right)^2\left(1 - \frac{2m}{r}\right),
\end{aligned} \tag{5.12}$$

where c_1 and c_2 are arbitrary constants. Expanding about the horizon $r_h = 2m$, we find that Eq. (5.12) is equivalently

$$\begin{aligned}
H_0(x) = c_1 \frac{5}{8}\left(\tilde{S}_d(x, q=0) - \left(\frac{13}{4} - 3\log(2)\right)\tilde{S}_r(x, q=0)\right) \\
+ 2c_2 \tilde{S}_r(x, q=0).
\end{aligned} \tag{5.13}$$

For convenience, we define

$$\begin{aligned}
\tilde{y}_d(x, q) &= \frac{5}{8}\tilde{S}_d(x, q) - \left(\frac{13}{4} - 3\log(2)\right)\tilde{S}_r(x, q), \\
\tilde{y}_r(x, q) &= 2\tilde{S}_r(x, q), \qquad \tilde{y}_p(x, q) = \tilde{S}_p(x, q),
\end{aligned} \tag{5.14}$$

after which Eq. (5.13) can be expressed as

$$H_0(x) = c_1 \tilde{y}_d(x, q=0) + c_2 \tilde{y}_r(x, q=0). \tag{5.15}$$

At this point, we note that the TLN is computed from the solution that is regular at the horizon. Discarding the solution $\tilde{y}_d(x, q=0)$ which is divergent at the horizon, we are left with $\tilde{y}_r(x, q=0)$, which can be used to show that the Schwarzschild black hole in ordinary GR has a vanishing TLN. We note that the particular solution vanishes in the ordinary GR limit, since α parametrizes fluctuations of the matter density that effectively descend purely from the 5D brane embedding (and not from 4D matter). As a side note, for the higher values of l, we find that we have the same indicial equation as in the $l = 2$ case, with the roots being ± 1. There is also a particular solution of the same form as in the $l = 2$ case. This once again implies that there are two regular branches of solutions, with one of them relevant for capturing the tidal response. Although the expansion about the horizon allows us to match the general form of regular solution to the Schwarzschild case in the limit of ordinary GR, we need the expansion about infinity to study the TLN since the latter is most conveniently read off from such a series.

For each of the two independent functions of r in Eq. (5.12) or, equivalently, $\tilde{y}_{\{d,r\}}(x, q = 0)$, we can also perform a series expansion about infinity in the parameter $u = m/r$. We find

$$\tilde{y}_d(x, q = 0) = u^3 + 3u^4 + \frac{50}{7}u^5 + \ldots, \quad (5.16)$$

$$\tilde{y}_r(x, q = 0) = \frac{1}{u^2} - \frac{2}{u}. \quad (5.17)$$

In the next section, we will solve for the asymptotic series expansion and compare the solutions against Eqs. (5.16) and (5.17) in the $q = 0$ limit.

B. Expansion about $r = \infty$

We now perform a series expansion about $r = \infty$ to pick up any nonvanishing TLNs. After switching to dimensionless parameter $u = m/r$, we rewrite the ODE as

$$H_0'' + \frac{2 - P(u)}{u}H_0' + \frac{Q(u)}{u^2}H_0 = \frac{m^2\mathcal{S}(u)}{u^4} \quad (5.18)$$

and find the expansion behavior

$$\begin{aligned} 2 - P(u) &= -2u + (2q - 4)u^2 + \ldots, \\ Q(u) &= -6 - 12u + \ldots, \\ \frac{m^2\mathcal{S}(u)}{\alpha u^4} &= \frac{1}{u^2} + \frac{4}{u} + (12 - 2q) + \ldots, \end{aligned} \quad (5.19)$$

which reveal $r = \infty$ or $u = 0$ to be a regular point. The indicial equation reads

$$\mathcal{R}(\mathcal{R} - 1) + \mathcal{R}(2 - P(0)) + Q(0) = \mathcal{R}(\mathcal{R} - 1) - 6 = 0, \quad (5.20)$$

which has roots 3 or -2. This gives the leading-order indices for the two independent homogeneous solutions, which we find to be

$$y_d(u, q) = u^3\left(1 + 3u + \frac{1}{7m^2}(50m^2 - 7q)u^2 + \frac{1}{42}\left(660 - 217\frac{q}{m^2}\right)u^3 + \ldots\right), \quad (5.21)$$

$$y_r(u,q) = \frac{1}{u^2} - \frac{2}{u} + \frac{2q}{3m^2} + \frac{2qu}{3m^2} - \frac{2q^2u^2}{3m^4} - \frac{16}{15m^4}(q^2\log(u))u^3 + \frac{2q^2}{45m^4}\left(31 + 15\frac{q}{m^2} - 72\log(u)\right)u^4 + \ldots. \quad (5.22)$$

We also find the particular solution to be of the following form:

$$\frac{y_p(u, q)}{\alpha} = 1 + 2u + \left(4 - \frac{q}{m^2}\right)u^2 + \left(\frac{q^2}{m^4} - 8\right)u^4 + \mathcal{O}(u^5). \quad (5.23)$$

Let us first consider their $q = 0$ limit:

$$y_d(u, q = 0) = u^3\left(1 + 3u + \frac{50}{7}u^2 + \frac{110}{7}u^3 + \ldots\right), \quad (5.24)$$

$$y_r(u, q = 0) = \frac{1}{u^2}(1 - 2u), \quad (5.25)$$

$$y_p(u, q = 0) = \alpha(1 + 2u + 4u^2 - 8u^4 + \ldots). \quad (5.26)$$

We find that Eqs. (5.24) and (5.25) are precisely Eqs. (5.16) and (5.17), respectively. From the preceding section, we see that the near-horizon expansion of $y_d(u, q)$ can be expressed in the form

$$y_d(u, q) = \kappa_{(1)}(q)(x + \ldots) + \kappa_{(0)}(q)(1 + \ldots) + \kappa_{(-1)}(q)\frac{1}{x}(1 + \ldots), \quad (5.27)$$

where we have kept only the leading-order terms in $\tilde{y}_r(x, q)$, $\tilde{y}_p(x, q)$, and $\tilde{y}_d(x, q)$, and the κ's are constants with possible q dependence. The $q = 0$ limit implies that $\kappa_{(-1)}(0) = 1$ and, since $\kappa_{(-1)}(q)$ is nonvanishing for a general q, $y_d(u, q)$ diverges at the horizon.

For $y_r(u, q)$, we cannot, however, invoke the above argument to prove the absence of $\tilde{S}_d(x, q)$ or $\tilde{y}_d(x, q)$ in its near-horizon expansion, and thus prove its regularity. In the absence of analytic solutions, we proceed by assuming that $y_r(u, q)$ is regular at the horizon, noting the following:

(a) A direct way to prove whether $y_r(u, q)$ is regular at the horizon is to study whether one could perform some resummation of the infinite series in (5.22). For example, if we regard Eq. (5.22) as the asymptotic series of a function of q and r that is also perturbative in q/m^2 (which is our physical regime of interest), then one could write it as a series in q. Generically, we expect each coefficient to be an infinite series in u, and for Eq. (5.22) this turns out to be the case for all $q^k, k > 1$ terms but not for the linear term. At order $\mathcal{O}(q^2)$, $y_r(u, q)$ can be written as a finite sum of terms which reads

$$y_r(u, q) = \frac{1}{u^2} - \frac{2}{u} + \frac{2q}{3m^2}(1 + u) + \mathcal{O}(q^2). \quad (5.28)$$

This is regular and in fact vanishes at the horizon $r_h = 2m - \frac{q}{2m} + \mathcal{O}(q^2)$. Thus, at order $\mathcal{O}(q^2)$, we have

$$y_r(u,q) = (1 + C_r q)\tilde{y}_r(x,q), \tag{5.29}$$

with the rhs understood to be evaluated at linear order, the $q = 0$ limit fixing the unity in the prefactor, and C_r being an undetermined constant.[5] Although this is compatible with the conjecture that $y_r(u,q)$ is regular at the horizon, unfortunately, it does not rigorously complete a proof for its regularity since it does not extend naturally to higher orders. Nevertheless, it indicates that functions of r that are divergent at the horizon could enter into the q series of $y_r(u,q)$ starting only at or above order $\mathcal{O}(q^2)$, if at all.

(b) Suppose that, contrary to our assumption, $y_r(u,q)$ diverges at the horizon. Then the regular solution we seek is a suitable linear combination of $y_r(u,q)$ and $y_d(u,q)$. For the latter, the strict $q = 0$ limit implies that its near-horizon expansion is of the form

$$y_d(u,q) = (1 + \mathcal{O}(q))\tilde{y}_d(x,q) + \mathcal{O}(q)\tilde{y}_r(x,q) + \mathcal{O}(q)\tilde{y}_p(x,q). \tag{5.30}$$

Also, the preceding point in (a) implies that we can write

$$y_r(u,q) = (1 + C_r q + \mathcal{O}(q^2))\tilde{y}_r(x,q) + \mathcal{O}(q^2)\tilde{y}_d(x,q) + \mathcal{O}(q^2)\tilde{y}_p(x,q). \tag{5.31}$$

It is then straightforward to see that at order $\mathcal{O}(q^2)$, the linear combination of $y_d(u,q)$ and $y_r(u,q)$ that reduces to $y_r(u,q=0)$ in the $q = 0$ limit is of the form

$$(1 + N_r q)(y_r(u,q) + \mathcal{O}(q^2)y_d(u,q)), \tag{5.32}$$

where N_r is an arbitrary constant since the term $\tilde{y}_d(x,q)$ must be canceled out. Thus, at linear order, Eq. (5.32) is of the same form as the rhs of Eq. (5.29).

(c) As we reviewed in Sec. IV, in the effective 4D gravitational theory, the tidal black hole is a solution to the case where the energy density ρ and stress Π take the form $\Pi = 2\rho = -\frac{q}{r^4}$. If $\alpha = 0$ (excluding the particular solution), then $\delta\rho = \delta\Pi = 0$ and the perturbations are vacuum perturbations. Now if $y_r(u,q)$ diverges at the horizon, then formally the near-horizon limit and $q \to 0$ limit do not commute. Physically, this translates to a picture where adiabatically turning on a small tidal charge induces a large (vacuum) perturbation near the horizon which we find somewhat difficult to understand, at least by naive intuition.[6] The regularity of $y_r(u,q)$ at the horizon may appear to be more physically reasonable than its converse.

Henceforth, we will assume that $y_r(u,q)$ is regular at the horizon, with our choice motivated by points (a) and (c) above and urged by feasibility of computation. For future work, it will be important to study our assumption more rigorously by, for example, studying the q-series expansion of the ODE (5.4) and terms at higher order in q. If all of these can be appropriately resummed, we could then know, beyond the linear order expressed in Eq. (5.28), whether it is genuinely regular at the horizon.

Finally, we note that the particular solution does not contain the tidal moment (r^2 term), although there is a nonvanishing quadrupolar moment due to the 5D embedding. Since we are considering the induced response of the body to some tidal force, when reading off the TLN we consider purely $y_r(u,q)$, which contains the tidal moment term and possibly some induced quadrupolar moment.

Thus, taking $\alpha = 0$, we have the asymptotic series expansion

$$H_0^{(\mathrm{reg})} = \mathcal{C}\left(\left(\frac{r}{m}\right)^2 + \frac{16}{15}q^2 \log\left(\frac{r}{m}\right)\frac{m^3}{r^3} - 2\frac{r}{m} + \frac{2q}{3m^2} + \frac{2q}{3mr} - \frac{2q^2}{3m^2r^2} + \cdots + \sum_{k=4}(\tilde{C}_k + \tilde{D}_k \log(u))u^k\right), \tag{5.33}$$

where $\tilde{C}_k$ and $\tilde{D}_k$ are constant coefficients which can be computed straightforwardly whenever required. This expression does not contain the $1/r^3$ term on its own, but taking into account $f(r) = 1 - \frac{2m}{r} + \frac{q}{r^2}$, with $h_{tt} = f(r)H_0^{(\mathrm{reg})}$, we find from Eqs. (5.33) and (2.15) that

$$\lambda = \frac{2mq^2}{3}. \tag{5.34}$$

This is our first example of a braneworld black hole solution that has a nonvanishing TLN which clearly increases with mass and the tidal charge q.

For higher values of l, from the expansion about infinity, we find the same solution for $2 - P(u)$ but $Q(u)$ is l dependent and reads

$$Q(u) = -l(1+l) - 2l(1+l)u + (l^2(q-4) + l(q-4) + 2(q-2))u^2 + \ldots. \tag{5.35}$$

The indicial equation reads

[5] In principle, we can compute C_r if we expand $\tilde{y}_r(x,q)$ in q to linear order, but we find this difficult, with $x = (r - r_h)/m$ being an infinite series in q.

[6] On a slightly related context, we note that the tidal black hole has been found to be stable against different types of perturbations (see, for example, [24]).

$$\mathcal{R}(\mathcal{R}-1)-l(l+1)=0, \quad (5.36)$$

which has roots $\{-l, 1+l\}$, with the particular solutions identical for all $l \geq 2$. As in the $l=2$ case, the solution associated with the indicial root of $-l$ contains the correct leading-order tidal moment r^l. The solution associated with the other root of $1+l$ diverges at the horizon.

VI. OTHER EXAMPLES OF BRANEWORLD BLACK HOLES AND WORMHOLES

We now proceed to consider other black hole and wormhole solutions with a similar computational approach focusing on the $l=2$ case. For this work, we consider various black hole and wormhole solutions which are static and spherically symmetric.[7] As we noted earlier, in the generic case, there is no decoupling of $\delta\rho$ from the perturbation equations and one has to solve a homogeneous third-order ODE for H_0. Nonetheless, the computational approach for studying the TLN is identical. We begin by discussing some essential generic points before studying specific black hole and wormhole geometries.

A. Some preliminaries

As in the case of the tidal black hole solution, we do not know the analytic form for the metric perturbation $H_0(r)$ for each family of black hole and wormhole geometries but will use both its near-horizon series expansion and that about the asymptotic infinity to deduce the series solution relevant for computing the TLN.

For this work, the chosen geometries all enjoy points in their parameter spaces where they coincide with certain limits of the tidal black hole (see Table I). For the tidal black hole, we have identified the regular/divergent nature of each asymptotic series via the $q=0$ (Schwarzschild) limit, which will again serve as the anchor limit for our analysis of two families of geometries: the Casadio-Fabbri-Mazzacurati (CFM) black holes and γ wormholes (also found by the same authors). For a class of wormholes found by Bronnikov and Kim, a point in its modulus space coincides with an analytic continuation $q=m^2$ of the tidal black hole. Finally, for a family of massless geometries found by Bronnikov, Melnikov, and Dehnen, there is a choice of parameter for which it coincides with the massless limit of the tidal black hole.

Like the tidal black hole, we also consider series solutions relevant for the "near-horizon" regime. The solutions that we study in this paper include wormholes and naked singularities apart from black hole geometries, and in those cases we consider the expansion about the throat and singular loci. By the "near-horizon regime," we generally refer to the small neighborhood of spherical surfaces along which the metric appears to be singular. Although the series solutions constructed in the near-horizon regime will not be useful for directly reading off the TLN, obtaining their general form by computing the indicial roots serves as a necessary condition for their regularity at the horizon or throat, as we have seen in the case of the tidal black hole.

[7] See, for example, [25] for other braneworld black hole solutions.

TABLE I. In this table, we summarize the relevant parameter domains for which we could compute the TLN based on our method, and the associated tidal black hole limits for each family of black hole/wormhole geometry. See Table VII in Appendix B for a more complete characterization in terms of indicial roots for each geometry.

Black hole/wormhole geometry	Relevant parameter domain	Tidal black hole limits
CFM black holes	$\beta \neq \frac{5}{4}$	$\beta \to 1$ $(q=0)$
γ black holes	$\gamma > 0$	$\gamma \to 1$ $(q=0)$
Bronnikov-Kim wormholes	$r_0 > 1,\ r_0 \neq 2$	$r_0 \to 2$ $(q=m^2)$
Massless geometries	$C \in (-\infty, \infty)$	$C \to 1$ $(m=0)$

Expanding about a metric singularity in each case (for us, this is either a black hole horizon or a wormhole throat), we seek the near-horizon series solution for each family of geometries after ensuring that the metric singularity is a regular point for the Frobenius method. Let the expansion parameter be $x=\frac{r-r_h}{L}$, where r_h can be either the black hole throat or the wormhole throat and L is some length parameter of each family of solutions. We will also be treating a couple of cases of naked singularities, in which case r_h is simply the singular surface. In all cases, $r=r_h$ describes a codimension-2 metric singularity which may or may not cloak physical curvature singularities. Writing Eq. (4.16) in the form

$$H_0'''+\frac{\mathcal{P}_1(x)}{x}H_0''+\frac{\mathcal{P}_2(x)}{x^2}H_0'+\frac{\mathcal{P}_1(x)}{x^3}H_0=0, \quad (6.1)$$

we compute the following limits from the metric

$$\lim_{x\to 0}\mathcal{P}_1\equiv\mathcal{P}_{10}, \quad \lim_{x\to 0}\mathcal{P}_2\equiv\mathcal{P}_{20}, \quad \lim_{x\to 0}\mathcal{P}_3\equiv\mathcal{P}_{30}, \quad (6.2)$$

from which we compute the indicial roots or the roots of the cubic equation

$$\mathcal{R}(\mathcal{R}-1)(\mathcal{R}-2)+\mathcal{R}(\mathcal{R}-1)\mathcal{P}_{30}+\mathcal{R}\mathcal{P}_{20}+\mathcal{P}_{10}=0. \quad (6.3)$$

We find that typically there are points in each parameter space which correspond to singularities in the near-horizon series solutions. These "exotic" points are marked by discontinuities in the first derivative of the horizon radius as a function of the parameter and where indicial roots separate into different branches (see Table VII of Appendix B). We will not rigorously study these cases here but will briefly comment on them in Appendix B. Since we do not have an analytic solution to determine regularity at the horizon directly, we rely on smooth limits to the tidal black hole for this purpose. Each regular series solution (about the asymptotic infinity) that we construct for computing the TLN is smoothly connected to various limits of $y_{\{r,d,q\}}(u,q)$ (see Table I). There are domains of parameters where the indicial roots (for near-horizon expansion) are all non-negative, indicating that all independent solutions are regular at the horizon. For our work, we will nonetheless restrict ourselves to regular solutions which have a clear GR limit, and hence the final series solution we choose to compute the TLN is always that associated with (limits of) $y_r(u,q)$.

There are also domains of parameter spaces where one of the indicial roots is negative. To explicitly confirm that, in the relevant limit ($q \to 0$ or $m \to 0$), they are not associated with the asymptotic series solution we use to compute the TLN, we work out the series solution belonging to the negative root in the cases that it arises. All such cases have the set $\{-1, 0, 1\}$ for their indicial roots in the near-horizon expansion, apart from those not smoothly connected to any limits of the tidal black hole.

For expanding about $r = \infty$, we define $u = L/r$, where L is some length parameter in the undeformed solution, and recast the ODE in Eq. (4.16) in the following form:

$$H_0''' + \frac{1}{u}\mathcal{P}_1 H_0'' + \frac{1}{u^2}\mathcal{P}_2 H_0' + \frac{1}{u^3}\mathcal{P}_3 H_0 = 0, \quad (6.4)$$

where

$$\mathcal{P}_1 = 6 - \frac{LC_2}{C_3 u}, \quad \mathcal{P}_2 = 6 - \frac{2LC_2}{C_3 u} + \frac{L^2 C_1}{C_3 u^2}, \quad \mathcal{P}_3 = -\frac{L^3 C_0}{C_3 u^3}. \quad (6.5)$$

The indicial equation reads

$$\mathcal{R}(\mathcal{R}-1)(\mathcal{R}-2) + \mathcal{R}(\mathcal{R}-1)\mathcal{P}_{10} + \mathcal{R}\mathcal{P}_{20} + \mathcal{P}_{30} = 0, \quad (6.6)$$

in which the roots $\mathcal{R}_1 > \mathcal{R}_2 > \mathcal{R}_3$ determine the form of the general solution. In contrast to the near-horizon expansion for all the solutions considered in this work, we find the universal values

$$\mathcal{P}_{10} = 2, \qquad \mathcal{P}_{20} = -6, \qquad \mathcal{P}_{30} = 0,$$
$$R_1 = 3, \qquad R_2 = 0, \qquad R_3 = -2. \quad (6.7)$$

This leads to the general solution being the linear combination of

$$S_1(u) = u^3(1 + a_1 u + a_2 u^2 + a_3 u^3 + \ldots),$$
$$S_2(u) = V_1 S_1(u)\log(u) + (1 + b_1 u + b_2 u^2 + \ldots),$$
$$S_3(u) = \frac{1}{u^2}(1 + d_1 u + d_2 u^2 + \cdots + d_5 u^5 + \ldots) + W_2 \log(u)(c_0 + c_1 u + c_2 u^2 + \ldots) + W_3(\log(u))^2 S_1(u). \quad (6.8)$$

As mentioned earlier, each family of solutions has a certain limit within its moduli space which reduces uniquely to the Schwarzschild solution or certain limits of the regular tidal-deformed metric of the tidal black hole. This yields consistency checks for regularity of each solution at the horizon or wormhole throat whenever there are negative indicial roots in the near-horizon expansion. In all cases, we take $S_3(u)$ to be the series relevant for picking the TLN, as it is the one associated with $y_r(u,q)$. We also need the full metric component h_{tt} in each case, which is collected separately in Appendix B.

For higher values of l, in all other braneworld solutions that we consider in this work, we find the following universal values for the roots of the indicial equation:

$$\mathcal{P}_{10} = 2, \qquad \mathcal{P}_{20} = -l(1+l), \qquad \mathcal{P}_{30} = 0,$$
$$\mathcal{R}_1 = l+1, \qquad \mathcal{R}_2 = 0, \qquad \mathcal{R}_3 = -l. \quad (6.9)$$

As in the specific case of $l = 2$, this leads to the general solution being the linear combination of

$$S_1(u) = u^{l+1}(1 + a_1 u + a_2 u^2 + a_3 u^3 + \ldots),$$
$$S_2(u) = V_1 S_1(u)\log(u) + (1 + b_1 u + b_2 u^2 + \ldots),$$
$$S_3(u) = \frac{1}{u^l}(1 + d_1 u + d_2 u^2 + \cdots + d_5 u^5 + \ldots) + W_2 \log(u)(1 + c_1 u + c_2 u^2 + \ldots) + W_3(\log(u))^2 S_1(u). \quad (6.10)$$

In the following, we will compute the TLN for various black hole and wormhole geometries, first by working out the asymptotic series solutions, identifying their tidal black hole counterparts before presenting details concerning their near-horizon expansions and regular/divergent behavior at the horizon. As mentioned, the indicial roots for the asymptotic expansion about infinity take on the universal values $\{3, 0, -2\}$, and we denote their corresponding series solutions by $S_1(u)$, $S_2(u)$, and $S_3(u)$, respectively, with $S_3(u)$ being the one relevant for determining the TLN. The indicial roots for the near-horizon expansion exhibit a much richer branch structure that bifurcates at points where the horizon radius—as a function of some parameter—is discontinuous in its first/second derivatives (kinks). For each family of solutions, there are regions in the parameter space where all indicial roots are positive and hence all independent series solutions are regular at the horizon. For regions which share the same identical roots $\{1, 0, -1\}$ as the tidal black hole, we denote their corresponding series solutions by $\tilde{S}_r(x)$, $\tilde{S}_p(x)$, and $\tilde{S}_d(x)$, respectively (see Table II for a summary of nomenclature). Finally, for regions that are not smoothly connected to some limit of the tidal black hole, we briefly comment on them in a separate appendix which contains a summary of all the indicial roots.

B. CFM black holes

This family of black hole solutions was found by Casadio *et al.* in [26]. The metric components read

$$f(r) = 1 - \frac{2m}{r}, \qquad g(r) = \frac{1 - \frac{3m}{2r}}{(1 - \frac{2m}{r})(1 - \frac{m}{2r}(4\beta - 1))}.$$

The horizon is located at $r = r_h = 2m$, and the limit $\beta \to 1$ corresponds to the Schwarzschild ansatz.

We first present the asymptotic series solutions. For $S_1(u)$, we find that

TABLE II. For the sake of clarity, we tabulate the nomenclature for various series solutions computed for each class of black hole and wormhole geometry in Secs. VI B–VI E.

Series solution (infinity)	Indicial roots	Series solution (near horizon)	Indicial roots
$S_1(u)$	3	$\tilde{S}_d(x)$	−1
$S_2(u)$	0	$\tilde{S}_p(x)$	0
$S_3(u)$	−2	$\tilde{S}_r(x)$	1

$$S_1(u) = u^3 + \frac{1}{4}(13\beta - 1)u^4 + \frac{1}{70}(533\beta^2 - 112\beta + 79)u^5 + \ldots, \tag{6.11}$$

which in the limit $\beta \to 1$ becomes

$$S_1(u) \to u^3\left(1 + 3u + \frac{50}{7}u^2 + \ldots\right) = y_d(u, q=0). \tag{6.12}$$

This is precisely the series that we identify to be divergent at the horizon. The $S_2(u)$ and $S_3(u)$ series read

$$\begin{aligned} S_2(u) &= -(5 - 8\beta + 3\beta^2)y_1(u)\log(u) + 1 + 2u - \frac{1}{2}(3\beta - 11)u^2 + \ldots, \\ S_3(u) &= \frac{1}{4}(\beta - 1)^2(33\beta^2 - 64\beta + 15)y_1(u)\log^2(u) \\ &\quad + \log(u)\left(\frac{1}{2}(1 - \beta)(11\beta - 3)\left(1 + 2u - \frac{1}{2}(3\beta - 11)u^2 + \ldots\right)\right) \\ &\quad + \frac{1}{u^2}\left(1 - \frac{1}{2}(11\beta - 7)u + \frac{1}{12}(55\beta^3 - 180\beta^2 + 158\beta - 33)u^3 + \ldots\right. \end{aligned} \tag{6.13}$$

In the $\beta \to 1$ limit, the $\log(u)$ series in $S_2(u)$ vanishes and we find that

$$S_2(u) \to y_p(u, q = 0)/\alpha, \tag{6.14}$$

which is precisely the solution in the Schwarzschild limit.[8] In the case of the tidal black hole, the $\delta\rho$ equation decouples and we have an arbitrary constant characterizing the arbitrary strength of the quadrupole moment. But this is not induced by some tidal moment (there is no accompanying $1/u^2$). The coefficient for u^3 in $S_2(u)$ is not what we seek in the traditional TLN definition. We should focus on the solution branch with $1/u^2$ representing the presence of the tidal moment due to an external body. Now, in the same limit, both the $\log(u)$ and $(\log(u))^2$ terms in $S_3(u)$ vanish and we have

$$S_3(u) \to \frac{1}{u^2} - \frac{2}{u} = y_r(u, q = 0), \tag{6.15}$$

which is the $q = 0$ limit of the regular tidal-deformed Schwarzschild solution which has vanishing TLN in the absence of any r^3 term.[9] From h_{tt} (see Appendix A 3), we read off the TLN to be (see Figs. 1 and 2)

$$\lambda = m^5 \frac{1}{72}(\beta - 1)(220\beta^3 - 1963\beta^2 + 3358\beta - 1155). \tag{6.16}$$

Phenomenological interest lies in the small neighborhood of the critical value $\beta = 1$, where for solutions with $\beta > 1$

[8]We should note that in solving for the series $S_2(u)$, there is an arbitrary constant b_3 which parametrizes $S_1(u)$, and setting it to vanish implies that in the $q \to 0$ limit we have Eq. (6.14).

[9]Again, we should note that Eq. (6.15) is obtained by setting $d_2 = d_5 = 0$ and that these parameters are otherwise arbitrary.

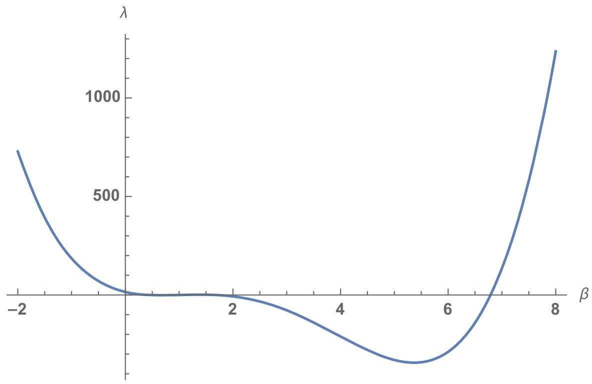

FIG. 1. Plot of the TLN λ (in units of m^5) vs the parameter β. The Schwarzschild limit corresponds to $\beta = 1$ and we have symmetric traversable wormhole geometries for $\beta \geq \frac{5}{4}$, although there is no transitional behavior of λ near this critical value. The other zeros do not correspond to any distinct causal structure of the solution. In Fig. 2, we zoom into the neighborhood of $\beta = 1$.

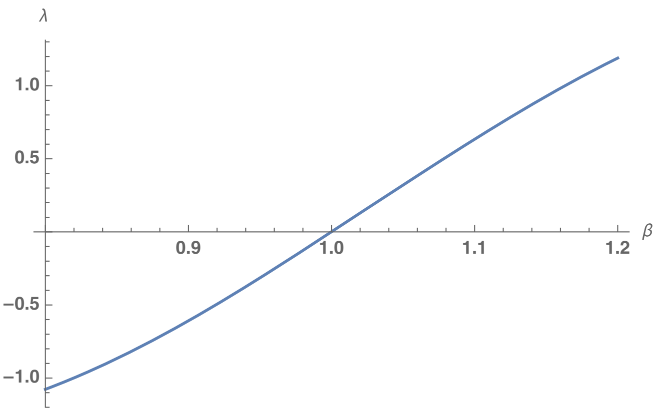

FIG. 2. Plot of the TLN λ (in units of m^5) vs the parameter β near $\beta \approx 1$. Near this transition value, solutions with $\beta > 1$ are nonsingular in nature, while those with $\beta < 1$ contain curvature singularities.

the TLN is positive, whereas it is negative for $\beta < 1$. The $\beta < 1$ solutions are black holes that are Schwarzschild-like in nature, with an event horizon at $r_h = 2m$, whereas the $\beta > 1$ solutions are nonsingular wormhole geometries. For $1 < \beta < \frac{5}{4}$, their Carter-Penrose diagrams resemble the form of the Kerr black hole, and for $\beta \geq \frac{5}{4}$ we have symmetric traversable wormholes. In this family of solutions, near the critical point, the sign of the TLN for solutions indicates the presence ($-$) or absence ($+$) of the black hole curvature singularity.

For the expansion about the horizon $r = r_h$, we note that as a function of βr_h has a discontinuity in the second derivative at $\beta = \frac{5}{4}$, beyond which it increases with β. In Table III, we summarize the physical interpretations of the geometry and the horizon for different β intervals. The indicial roots separate into two branches following this

TABLE III. In the table, we use the abbreviations WH (wormholes) and Sch BH (Schwarzschild black holes). Kerr WH refers to completely regular wormhole geometries in which the CP diagram resembles the form of Kerr wormholes, and RN WH refers to spacetimes with the casual structure of extremal Reissner-Nordström wormholes with a horizon cloaking a timelike singularity.

r_h	Indicial roots
Sch BH, $\beta < 1$, $r_h = 2m$	$\{-1, 0, 1\}$
Kerr WH, $\beta \in (1, \frac{5}{4})$, $r_h = 2m$	$\{-1, 0, 1\}$
RN WH, $\beta = \frac{5}{4}$, $r_h = 2m$	$\{-1, -\frac{1}{4}(1 \pm \sqrt{9 + 4l + 4l^2})\}$
WH, $\beta > \frac{5}{4}$, $r_h = \frac{m}{2}(4\beta - 1)$	$\{0, \frac{1}{2}, 1\}$

distinguished value, which is the set $\{-1, 0, 1\}$ for $\beta < \frac{5}{4}$ and $\{0, \frac{1}{2}, 1\}$ for $\beta > \frac{5}{4}$. The Schwarzschild limit lies at $\beta = 1$.

For $\beta > \frac{5}{4}$, the indicial roots are all positive, so all the solutions of the ODE are regular at the horizon. For $\beta < \frac{5}{4}$, the solutions are smoothly connected to the Schwarzschild $q = 0$ limit, and their indicial roots are identical. We explicitly check to see that, for the negative root, the $q = 0$ limit yields

$$\lim_{q=0} \tilde{S}_d(x) = \tilde{S}_d(x, q = 0) + 3\tilde{S}_p(x, q = 0), \tag{6.17}$$

and it is thus independent of the near-horizon solution $S_r(x; q = 0)$ corresponding to $S_3(u)$. We also find that

$$\lim_{q=0} \tilde{S}_r(x) = \tilde{S}_r(x, q = 0), \qquad \lim_{q=0} \tilde{S}_p(x) = \tilde{S}_p(x, q = 0). \tag{6.18}$$

Our method does not rigorously apply for the isolated case of $\beta = \frac{5}{4}$, at which the near-horizon series expansion develops a singularity paralleled by the kink in r_h as a function of β. In this case, there is nonetheless a positive indicial root and the asymptotic series $S_3(u)$ has a smooth limit at this point, so it is nevertheless possible that the solution of which asymptotic expansion is $S_3(u)$ is regular at the horizon.

C. γ wormholes

This family of solutions was studied by Casadio *et al.* in [26] and contains both pathological naked singularities and regular wormhole geometries. The metric components read

$$f(r) = \frac{1}{\gamma^2}\left(\gamma - 1 + \sqrt{1 - \frac{2\gamma m}{r}}\right)^2, \qquad g(r) = \left(1 - \frac{2\gamma m}{r}\right)^{-1}, \tag{6.19}$$

with $\gamma = 1$ being the Schwarzschild limit. For $\gamma < 1$, the metric is singular at

$$r_s = \begin{cases} \frac{2m}{2-\gamma} \equiv r_h, \\ 2m\gamma \equiv r_0, \end{cases} \tag{6.20}$$

where $r_h = \frac{2m}{2-\gamma}$ is a null and singular surface along which the Ricci scalar R diverges as $R \sim 1/(\sqrt{r-r_0} - \sqrt{r_h - r_0})$. For $\gamma > 1$, the only metric singularity lies at $r_0 = 2m\gamma$. There is a turning/minimum point (for all timelike geodesics) at $r = 2m\gamma$ where all curvature invariants are regular. The causal interpretation is that of a wormhole solution, as explained in [26].

We first present the asymptotic series solutions. Corresponding to the highest root of the indicial equation,

$$S_1(u) = u^3 + \frac{1}{4}(13\gamma - 1)u^4 + \frac{1}{70}(533\gamma^2 - 42\gamma + 9)u^5 + \mathcal{O}(u^6), \tag{6.21}$$

we find that in the limit $q \to 1$ this reduces precisely to the divergent branch of the tidal-deformed Schwarzschild solution, i.e.,

$$\lim_{\gamma\to1} S_1(u) = y_d(u, q=0).$$

Thus, we discard this solution branch for ensuring regularity at the horizon. We also find the following solution:

$$S_2(u) = \frac{1}{5}(-11\gamma^2 + 34\gamma - 23)y_1(u)\log(u) + 1 + 2u + \frac{9-\gamma}{2}u^2 + \dots. \tag{6.22}$$

We note that in solving for the series $S_2(u)$ there is an arbitrary constant b_3 which parametrizes $S_1(u)$, and setting it to vanish implies that in the $q \to 1$ limit this solution reduces exactly to the particular solution of the tidal-deformed Schwarzschild solution, i.e.,

$$\lim_{\gamma\to1} S_2(u) = y_p(u, q=0).$$

Finally, corresponding to the only negative root of the indicial equation, we have

$$S_3(u) = \frac{1}{20}(\gamma-1)^2(121\gamma^2 - 286\gamma + 69)y_1(u)\log^2(u) \quad \frac{1}{2}(-11\gamma^2 + 14\gamma - 3)\log(u)\left(1 + 2u + \frac{9-\gamma}{2}u^2 + \dots\right) \quad \frac{1}{u^2}\left(1 + \frac{1}{2}(7 - 11\gamma)u + \frac{1}{12}(55\gamma^3 - 196\gamma^2 + 189\gamma - 48)u^3 + \dots\right), \tag{6.23}$$

where we find that setting the arbitrary constants $d_2 = d_5 = 0$ allows us to identify $S_3(u)$ with the regular solution of the tidal-deformed Schwarzschild solution, i.e.,

$$\lim_{\gamma\to1} S_3(u) = y_r(u, q=0).$$

Thus, just as in the case of the CFM black holes, we find that the three series solutions can be parametrically connected to the corresponding series solutions of the $q = 0$ limit of the tidal-deformed tidal black hole in a unique way, and that the one corresponding to the negative root of the indicial equation $S_3(u)$ is the one relevant for computing the TLN.

Taking into account the background metric, from the series expansion of $h_{tt} = f(r)S_3(u)$ we can read off the TLN as (see Figs. 3 and 4)

$$\lambda = m^5 \frac{1}{144}(\gamma - 1)(233\gamma^3 - 1284\gamma^2 + 2232\gamma - 711). \tag{6.24}$$

In the neighborhood of the critical value $\gamma = 1$, we find that $\lambda > 0$ for $\gamma > 1$, which corresponds to nonsingular

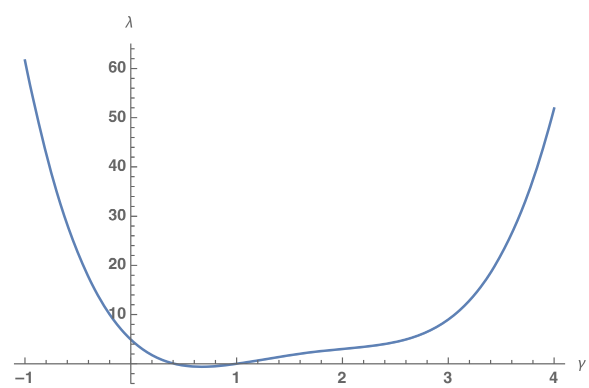

FIG. 3. Plot of λ (in units of m^5) vs γ for the family of γ-wormhole solutions. The solutions contain naked singularities for $\gamma < 1$, whereas we have regular wormhole geometries for $\gamma > 1$.

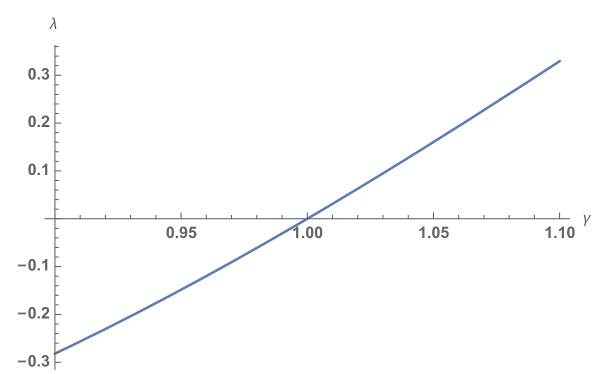

FIG. 4. Plot of λ (in units of m^5) vs γ near the Schwarzschild limit $\gamma = 1$, showing clearly that in the vicinity of the transition point the sign of γ corresponds to the presence (-)/absence (+) of curvature singularities.

TABLE IV. In the above, NS refers to naked singularities, for which r_h is the null surface where the Ricci scalar diverges.

r_h	Indicial roots
NS, $\gamma < 1$, $r_h = \frac{2m}{2-\gamma}$	$\{0, 1, 2\}$
Sch BH, $\gamma = 1$, $r_h = 2m$	$\{-1, 0, 1\}$
WH, $\gamma > 1$, $r_h = 2m\gamma$	$\{0, \frac{1}{2}, 1\}$

wormhole geometries, whereas $\lambda < 0$ for $\gamma < 1$, which corresponds to naked singularities. This is similar to the scenario in our study of tidal-deformed CFM black hole solutions, where a positive λ labels the nonsingular wormhole branch of the solution class, whereas negative λ pertains to Schwarzschild-type black holes near the critical Schwarzschild point. In both cases, λ is a degree-4 polynomial in the parameter with one of the polynomial roots associated with the Schwarzschild limit.

For the near-horizon series expansions, we find that the indicial roots separate into three branches, as shown in Table IV. Apart from the case of $\gamma = 1$, which coincides with the $q = 0$ Schwarzschild limit, all indicial roots are positive, and hence the solutions are all regular at the horizon.

D. Bronnikov-Kim wormholes

In [27], Bronnikov *et al.* presented a powerful solution-generating technique to construct exact braneworld black hole and wormhole solutions for the Randall-Sundrum theory considered in our work, covering the previous two families of solutions. In this and the subsequent sections, we study the two concrete examples mentioned in their seminal work.

We first study example 3 of [27], where the line element reads

$$f(r) = \left(1 - \frac{2\tilde{m}}{r}\right)^2, \qquad g(r) = \left(1 - \frac{r_0}{r}\right)^{-1}\left(1 - \frac{r_1}{r}\right)^{-1},$$
$$r_1 = \frac{\tilde{m} r_0}{r_0 - \tilde{m}},$$

with the parameter r_0 determining the causal structure as follows:
(a) $r_0 < \tilde{m}$: naked singularity at $r = 2\tilde{m}$,
(b) $\tilde{m} < r_0 < 2\tilde{m}$: wormhole with throat at $r_1 > 2\tilde{m}$,
(c) $r_0 = 2\tilde{m}$: extremal Reissner-Nordström form,
(d) $r_0 > 2\tilde{m}$: wormhole with throat at r_0.
Unlike in the previous two cases, for this solution, there is no natural limiting procedure to send it to the Schwarzschild limit, but to the $q = m^2$, $m = 2\tilde{m}$ limit of the tidal black hole or, equivalently, the extremal Reissner-Nordström solution (in ordinary GR) at some finite value of the radial coordinate $r_0 = 2\tilde{m}$. Since the tidal charge q was originally negative, this is an analytic continuation.

We find the following series solutions (taking the expansion variable to be $u = \tilde{m}/r$ and defining $R = r_0/\tilde{m}$),

$$\begin{aligned}
S_1(u) &= u^3 + \frac{13R^2 - 4R + 4}{8(R-1)}u^4 + \frac{533R^4 - 448R^3 + 592R^2 - 288R + 144}{280(R-1)^2}u^5 + \ldots \\
S_2(u) &= -\frac{(R-2)^2(15R^2 - 92R + 92)}{10(R-1)^2}y_1(u)\log(u) + 1 + 4u + \frac{3R^2 - 36R + 36}{2 - 2R}u^2 + \ldots \\
S_3(u) &= \frac{(R-2)^4(165R^4 - 1192R^3 + 2296R^2 - 2208R + 1104)}{160(R-1)^4}y_1(u)\log^2(u) \\
&\quad - (((-2+R)^2(12 - 12R + 11R^2))/(8(-1+R)^2))\log(u)\left(1 + 4u + \frac{3R^2 - 36R + 36}{2 - 2R}u^2 + \ldots\right) \\
&\quad + \frac{1}{u^2}\left(1 + \frac{11R^2 - 28R + 28}{4 - 4R}u + \frac{8}{3}u^2 + \ldots\right),
\end{aligned} \tag{6.25}$$

where for $S_2(u)$ and $S_3(u)$ we have picked the uniquely appropriate values $b_3 = 0$, $d_2 = \frac{8}{3}$, and $d_5 = 0$ such that, in the $R = 2$ limit, each reduces to a corresponding series solution in the $q = m^2 = 4\tilde{m}^2$ limit of the tidal black hole. Specifically, we find in this extremal limit

$$\begin{aligned}
\lim_{R\to 2} S_1(u) &= y_d(u, q = m^2), \\
\lim_{R\to 2} S_2(u) &= y_p(u, q = m^2)/\alpha, \\
\lim_{R\to 2} S_3(u) &= y_r(u, q = m^2),
\end{aligned} \tag{6.26}$$

with the last series solution being the regular solution from which one determines the TLN. Taking into account the background metric, from the series expansion of $h_{tt} = f(r)S_3(u)$ we read the TLN to be (see Figs. 5 and 6)

$$\lambda = \tilde{m}^5 \frac{55R^8 - 2073R^7 + 19389R^6 - 83064R^5 + 200948R^4 - 298640R^3 + 278448R^2 - 153344R + 38336}{144(R-1)^4}. \qquad (6.27)$$

We note that the vertical asymptote at $R = 1$ corresponds to the limit in which the wormhole geometry degenerates into a naked singularity. Another notable point is at $R = 2$, which corresponds to the extremal Reissner-Nordström solution. This turns out to be a local minimum point, with

$$\lim_{R\to 2} \lambda = \frac{64}{5}\tilde{m}^5 = \frac{2}{3}m^5,$$

which is in agreement with the $q = m^2$ limit of Eq. (5.22). Unlike the extremal black hole solution, which has a timelike curvature singularity behind a horizon, the wormhole geometries are globally regular.

For the near-horizon series expansions, we find that the indicial roots are separated into several branches (see Table V). They are all positive for $r_0 > 1$, $r_0 \neq 2$, where we can trust their regularity at the horizon and thus the computation of the TLN. At the distinguished point $R = 2$ that coincides with the $q = m^2$ analytic continuation of the tidal black hole, the indicial roots take on a different set of values, however, and the series solutions are not smoothly connected to those of the tidal black hole. There is a vertical asymptote at $R = 1$, and for $R < 1$ the indicial roots are $\{-2, 0, 2\}$. Thus, these series solutions cannot be smoothly deformed to any limit of the tidal black hole.

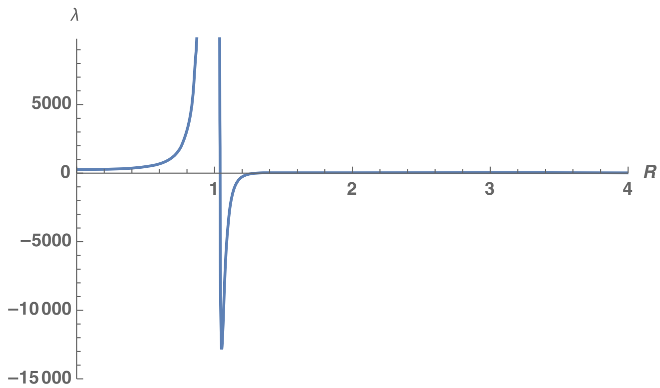

FIG. 5. Plot of λ (in units of $\tilde{m}^5$) vs R. We note that $R = 1$ marks the transition to a naked singularity with a vertical asymptote, with the left plot pertaining to naked singularities and the right plot associated with regular wormhole geometries. At $R = 2$, we have the extremal Reissner-Nordström metric.

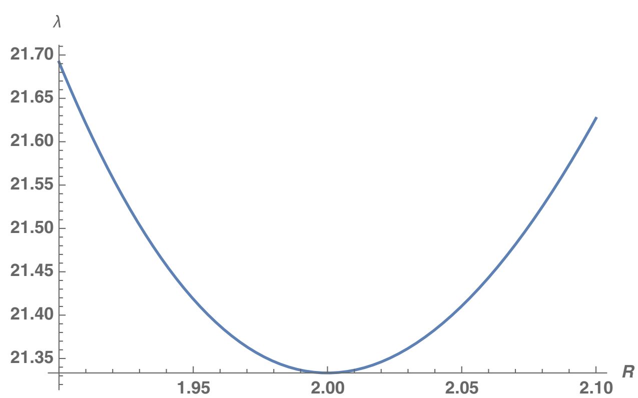

FIG. 6. Plot of λ (in units of $\tilde{m}^5$) vs R near the local minimum point $R = 2$ limit which pertains to the extremal Reissner-Nordström metric. This is a solution which is also part of the tidal black hole family, albeit an extremal solution with positive tidal charge $q = m^2$. The value of λ at $R = 2$ was checked and agrees with Eq. (5.34) in such a limit.

E. Massless geometries

Finally, we study a family of solutions which is example 2 of the Bronnikov-Melnikov-Dehnen solution-generating algorithm in [27]. There is a limiting procedure that takes them to the massless tidal black hole.. This family of spacetimes is interesting, as it has a subset of solutions which admit interpretations of wormholes, just like the CFM family of solutions. It is parametrized by a dimensionless constant C and a length parameter h which in the $C = 1$ limit can be interpreted as an imaginary charge of the massless Reissner-Nordström black hole:

$$f(r) = 1 - \frac{h^2}{r^2},$$
$$g^{-1}(r) = f(r)\left(1 + h\frac{C-1}{\sqrt{2r^2 - h^2}}\right), \qquad h > 0. \qquad (6.28)$$

TABLE V. In the table, RN refers to the extremal Reissner-Nordström metric, which is the $q = m^2$ analytic continuation of the tidal black hole.

r_h	Indicial roots
NS, $r_0 < \tilde{m}$, $r_s = 2\tilde{m}$	$\{-2, 0, 2\}$
WH, $r_0 \in [\tilde{m}, 2\tilde{m})$, $r_h = r_1$	$\{0, \frac{1}{2}, 1\}$
WH, $r_0 > 2\tilde{m}$, $r_h = r_0$	$\{0, \frac{1}{2}, 1\}$
RN, $r_0 = 2\tilde{m}$, $r_h = r_0$	$\{-2, -\frac{1}{2}(1 \pm \sqrt{9 + 4l + 4l^2})\}$

Metric singularity arises at the following radii:

$$r_h = \begin{cases} h, C \geq 0\,(\text{horizon}), \\ \sqrt{\frac{1}{2}(h^2 + h^2(1-C)^2)}, C < 0\,(\text{wormhole throat}). \end{cases} \tag{6.29}$$

In particular, we note that $C = 1$, $f = g^{-1}$ is the Reissner-Nordström metric with zero mass and imaginary charge or the massless limit ($m = 0$) of the tidal black hole. We find the asymptotic series solutions

$$\begin{aligned} S_1(u) &= u^3\left(1 - \frac{13(C-1)}{8\sqrt{2}}u + \frac{1}{560}(533C^2 - 1066C + 1093)u^2 + \ldots\right), \\ S_2(u) &= \frac{\sqrt{2}}{10}(1-C)u^3 y_1(u)\log(u) + 1 + u^2 + \frac{1}{960}(-91C^2 + 182C + 869)u^4 + \ldots, \\ S_3(u) &= \frac{11(C-1)^3}{160\sqrt{2}} y_1(u)\log^2(u) \\ &\quad - \frac{11(C-1)^2}{16}\log(u)\left(1 + u^2 - \frac{1375C^4 - 5500C^3 + 46912C^2 - 82824C + 68557}{13200\sqrt{2}(C-1)}u^3 + \ldots\right) \\ &\quad \frac{1}{u^2}\left(1 + \frac{11(C-1)}{4\sqrt{2}}u - \frac{2}{3}u + \ldots\right). \end{aligned} \tag{6.30}$$

We find that the various series solutions reduce to those of the massless tidal black hole upon setting $C = 1$. We find that with $q = -h^2$

$$\begin{aligned} \lim_{C\to 1} S_1(u) &= y_d(u, m = 0), \\ \lim_{C\to 1} S_2(u) &= y_p(u, m = 0)/\alpha, \\ \lim_{C\to 1} S_3(u) &= y_r(u, m = 0), \end{aligned} \tag{6.31}$$

where for $S_3(u)$ we have set $d_2 = -2h^2/3 = 2q/3$ and $d_5 = 0$. Taking into account the background metric, we find (see Fig. 7)

$$\lambda = h^5\frac{(C-1)(55C^2 - 110C - 1)}{576\sqrt{2}}. \tag{6.32}$$

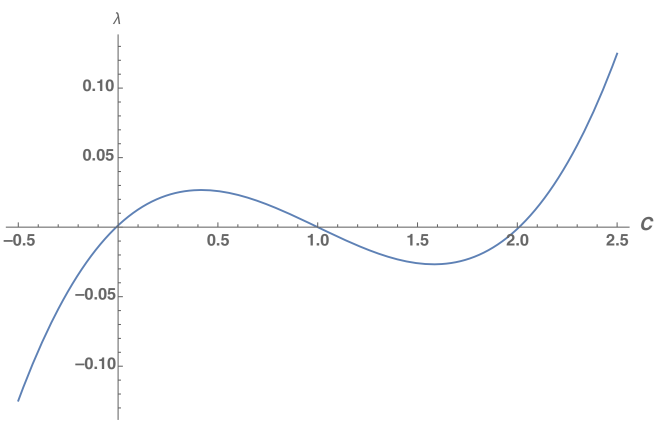

FIG. 7. Plot of λ (in units of h^5) vs C. This is a cubic curve with the zero at $C = 1$ corresponding to the transition case of the massless tidal black hole with $q = -h^2$. For $C > 1$, the causal structure is identical to that of Schwarzschild. For $C < 0$, we have globally regular wormholes, but $C = 0$ is not a polynomial root.

The vanishing value λ corresponds to the massless Reissner-Nordström black hole with imaginary charge/massless tidal black hole. For negative C, we have a wormhole solution. For $C \in [0, 1]$, we obtain a Kerr-like regular black hole. For $C > 1$, we have a Schwarzschild-like causal structure with the singularity at some finite value of $r = h/\sqrt{2}$.

For the series expansion about the horizon, we find that $C = 0$ splits the indicial roots into several branches, as summarized in Table VI. They are always positive for $C \leq 0$, and for $C > 0$ they take on the same set of values of $\{-1, 0, 1\}$ as in the massless Schwarzschild limit $C = 1$, where we find

$$\lim_{C\to 1}\tilde{S}_d(x) = \tilde{S}_d(x, m = 0) + 2\tilde{S}_p(x, m = 0)/\alpha. \tag{6.33}$$

Thus, it is independent of the near-horizon solution $\tilde{S}_r(x, m = 0)$ corresponding to $S_3(u)$. We also find that

TABLE VI. At $C = 1$, we recover the massless limit of the tidal black hole. $C = 0$ marks the point where the indicial roots separate into branches. All sets of roots are non-negative or equal to that of the tidal black hole.

r_h	Indicial roots
WH, $C < 0$, $r_{\text{th}} = \frac{1}{\sqrt{2}}\sqrt{h^2 + h^2(1-C)^2}$	$\{2, 1, 0\}$
Kerr WH, $C \in (0, 1)$, $r_h = h$	$\{-1, 0, 1\}$
Sch BH, $C \geq 1$, $r_h = h$	$\{-1, 0, 1\}$
RN WH $C = 0$, $r_h = h$	$\{0, 1, 5\}$

$$\lim_{C\to 1}\tilde{S}_r(x) = \tilde{S}_r(x, m = 0),$$
$$\lim_{C\to 1}\tilde{S}_p(x) = \frac{2}{3}\tilde{S}_p(x, m = 0). \tag{6.34}$$

VII. CONCLUSIONS

For each of the five classes of static and spherically symmetric braneworld solutions examined in this work, we computed the (quadrupolar) tidal Love number, with each being some rational function of the parameter that characterizes the family of solutions. They are generically nonvanishing and we derived them essentially by performing an asymptotic series expansion about the radial infinity. The gravitational perturbation equations were shown to reduce to a single third-order homogeneous ODE, and we found that the indicial equation associated with the regular singular point at infinity has a universal set of roots $\{3, 0, -2\}$ across all the braneworld solutions, with the tidal-deformed geometry described by the series solution with the negative root.

Among the braneworld solutions, there is a distinguished case in which the effective density fluctuation can be solved analytically, and the perturbation equation reduces to an inhomogeneous second-order ODE of which indicial roots associated with the asymptotic expansion are $\{3, -2\}$. This corresponds to the tidal black hole, which is one of the more popularly studied black hole solutions in Randall-Sundrum theory, and its TLN reads simply as $\lambda = \frac{2mq^2}{3}$, where q is the tidal charge. We completed the TLN calculation assuming that the asymptotic series solution associated with the negative indicial root $y_r(u, q))$ is regular at the horizon. This assumption is valid if the near-horizon limit commutes with the limit of vanishing tidal charge, which appears to be physically reasonable. We also found that, at linear order in the tidal charge, $y_r(u, q)$ has a finite number of terms and vanishes at the horizon. It would be important future work to study this assumption more carefully, possibly by examining the higher-order terms in the tidal charge expansion or numerically integrating the ODE that governs the perturbation.

For other braneworld solutions that we studied here, each can be parametrically connected to either the Schwarzschild or other limits of the tidal black hole, which thus serves as an anchor point. Once we established the case for tidal black holes, we performed various limits [specifically, (i) $q = 0$, (ii) $m = 0$, and (iii) $q = m^2$] to relate the series solutions in each family of solutions to those for tidal black holes and identify the one relevant for picking up the TLN. As a necessary consistency check, we also constructed series solutions by expanding about the metric singularity in each family of solutions, which can be interpreted as either horizon or throat surfaces. Unlike those in the asymptotic expansion, the indicial roots for the near-horizon expansion separate into branches at certain points in the parameter space and do not simply take on a universal set of values. In certain parameter domains, they are all positive, and thus all asymptotic series solutions are regular at the horizon. Our method also applies when they are equal to the set $\{-1, 0, 1\}$ associated with the tidal black hole, and their near-horizon series solutions can be smoothly connected to those of the tidal black hole.

It would be interesting to study the other cases in the parameter space where they cannot be smoothly deformed to limits of the tidal black hole. This would require other approaches of analysis beyond the limiting procedures that we used in this work. It is notable that in almost all cases the asymptotic series solutions themselves exhibit no singularity,[10] and it is possible that the expressions for the TLN extend to these regions of the parameter space. Another supporting point is that, in all cases, we find at least one positive root. But we will need other methods to study whether this corresponds to the near-horizon description of the series solution $S_3(u)$.

Although we focused on the computation of the quadrupolar $l = 2$ TLN for various braneworld solutions, this work also contains several results with regards to the general l case as described by Eqs. (4.16) and (4.19). For the near-horizon expansion, we found that the indicial roots are almost exclusively independent of l, with the exceptions being those with the causal structure of the extremal Reissner-Nordströ m metric. For the asymptotic expansion, the set of indicial roots $\{3, 0, -2\}$ generalizes to $\{l + 1, 0, -l\}$, whereas for the tidal black hole $\{3, -2\}$ generalizes to $\{l + 1, -l\}$. The methodology to compute the higher TLNs is similar—the relevant series solution is the one associated with the negative indicial root, and we can adopt the identical limiting procedure to check the consistency of this approach via the parametric connection of the various braneworld solutions to the tidal black hole. From the phenomenological point of view, the higher TLNs may not be as important since they should remain beyond the detectors' precision reach. But our study of the indicial roots in the near-horizon expansion suggests that certain aspects of the indicial equation may contain physical data related to global causal structures beyond horizon geometry. Thus, it would nevertheless be an interesting conjecture to explore, as it may yield further insights into the nature of gravitational degrees of freedom in braneworld black hole solutions.

A natural direction for future work would be to study the implication of the logarithmic terms and other terms which lie between r^2 and $\frac{1}{r^3}$ in the tidal-deformed metric, as collected in Appendix B. A number of them have natural interpretations. For example, in h_{tt}, terms of the order $\frac{1}{r}$, $\frac{1}{r^2}$ can be absorbed in renormalizing the

[10]The only exception is the naked singularity limit of the family of Bronnikov-Kim wormholes.

Arnowitt-Deser-Misner mass and tidal charge (the TLN expressions remain valid as functions of the bare parameters), whereas the constant term can be absorbed in an overall rescaling of the metric component without changing the TLN expressions. It was noted in [19] that the logarithmic terms may in principle be present generically but were not present for the Kerr solution in ordinary perturbative Einstein gravity. Also, it was argued that the r term can be gauged away. We have computed only the traditionally defined TLNs, but a more careful study might reveal the signature of these other terms in gravitational waves emitted in binary systems of these objects comparable to that of the TLN. It would be important to study these effects together with the TLN in characterizing gravitational waveforms, for example.

With regard to phenomenology, it was suggested in [8] that LIGO's precision could potentially bound the quadrupolar TLN k_2 to be of the order of 10 or less, with the Einstein Telescope possibly improving it by a factor of 100, or even greater with LISA's capabilities. The discovery of a nonvanishing k_2 in the GW waveform of the merger event presumed to be that of a black hole binary would provide exciting grounds for further study of whether or not the deviation is a signature of a braneworld, among other possibilities. It would be interesting to generalize our work to include realistic braneworld solutions arising from gravitational collapse and in the broader context of warped compactifications in string theory beyond the specific Randall-Sundrum model that we considered.

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APPENDIX A: ON SERIES SOLUTIONS FOR THE METRIC PERTURBATION

In this appendix, we collect some explicit terms in the near-horizon and asymptotic series expansions for the metric perturbation of the tidal black hole and geometries.

1. Near-horizon expansion of the tidal black hole

For notational convenience, we express q in units of mass parameter m below (to restore it, simply take $q \to q/m^2$). We define $x = (r - r_h)/m$, with r_h being the horizon of the tidal black hole. The two homogeneous solutions are thus of the form

$$\tilde{y}_r(x, q) = x(1 + a_1 x + a_2 x^2 + \ldots), \qquad \tilde{y}_d(x, q) = \mathcal{N}\tilde{y}_r(x, q)\log(x) + \frac{1}{x}(1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots), \tag{A1}$$

where

$$\begin{aligned}
a_1 &= \frac{\sqrt{1-q}q + 7q - 12\sqrt{1-q} - 12}{6(-q^2 + 3\sqrt{1-q}q + 5q - 4\sqrt{1-q} - 4)},\\
a_2 &= -\frac{q((3\sqrt{1-q} + 11)q^2 - 4(5\sqrt{1-q} + 7)q + 16(\sqrt{1-q} + 1))}{12(q-1)((\sqrt{1-q} + 7)q^3 - 8(3\sqrt{1-q} + 7)q^2 + 16(5\sqrt{1-q} + 7)q - 64(\sqrt{1-q} + 1))},\\
b_1 &= \frac{-3\sqrt{1-q}q - 13q + 20\sqrt{1-q} + 20}{2(-q^2 + 3\sqrt{1-q}q + 5q - 4\sqrt{1-q} - 4)}, \qquad b_2 = 0,\\
\mathcal{N} &= -\frac{6(-q^3 + 6(\sqrt{1-q} + 3)q^2 - 16(2\sqrt{1-q} + 3)q + 32(\sqrt{1-q} + 1))}{(q-1)((\sqrt{1-q} + 7)q^3 - 8(3\sqrt{1-q} + 7)q^2 + 16(5\sqrt{1-q} + 7)q - 64(\sqrt{1-q} + 1))}.
\end{aligned}$$

And the particular solution reads

$$\tilde{y}_p(x, q) = C_0 + (C_1 + \mathcal{K}_1 \log(x))x + (C_2 + \mathcal{K}_2 \log(x))x^2 + \ldots, \tag{A2}$$

where

$$
\begin{aligned}
C_0 &= -\frac{3\alpha(-q+2\sqrt{1-q}+2)}{2(q-1)}, \qquad C_1 = 0, \\
K_1 &= \frac{3\alpha(-q^2+4\sqrt{1-q}q+8q-8\sqrt{1-q}-8)}{(q-1)(q^2-3\sqrt{1-q}q-5q+4\sqrt{1-q}+4)}, \\
K_2 &= -\frac{\alpha(q^4-12(\sqrt{1-q}+5)q^3+16(11\sqrt{1-q}+24)q^2-64(8\sqrt{1-q}+11)q+384(\sqrt{1-q}+1))}{2(q-1)^2((\sqrt{1-q}+7)q^3-8(3\sqrt{1-q}+7)q^2+16(5\sqrt{1-q}+7)q-64(\sqrt{1-q}+1))}, \\
C_2 &= \alpha(-6q^8+(68\sqrt{1-q}+341)q^7+(238-798\sqrt{1-q})q^6-224(50\sqrt{1-q}+229)q^5 \\
&\quad +192(787\sqrt{1-q}+1903)q^4-256(2400\sqrt{1-q}+4211)q^3+512(2209\sqrt{1-q}+3083)q^2 \\
&\quad -4096(238\sqrt{1-q}+277)q+319488(\sqrt{1-q}+1)) \\
&\quad \times[6(q-1)^2(q^7-14(\sqrt{1-q}+7)q^6+224(2\sqrt{1-q}+7)q^5-1344(3\sqrt{1-q}+7)q^4 \\
&\quad +3840(4\sqrt{1-q}+7)q^3-5632(5\sqrt{1-q}+7)q^2+4096(6\sqrt{1-q}+7)q-8192(\sqrt{1-q}+1))]^{-1}.
\end{aligned}
$$

2. Near-horizon series expansions for CFM black holes ($\beta < 5/4$) and massless geometries ($C > 0$)

Corresponding to the indicial roots $\{1, 0, -1\}$, we have the ansatz

$$
\begin{aligned}
\tilde{S}_r(x) &= x(1+a_1x+a_2x^2+a_3x^3+\ldots), \\
\tilde{S}_p(x) &= V_1\tilde{S}_r(x)\log(x)+(1+b_1x+b_2x^2+\ldots), \\
\tilde{S}_d(x) &= \frac{1}{x}(1+d_1x+d_2x^2+\ldots)+W_2\log(x)(c_0+c_1x+c_2x^2+\ldots)+W_3\tilde{S}_r(x)(\log(x))^2.
\end{aligned}
$$

For the massless geometries, we have

$$
\begin{aligned}
a_1 &= \frac{2-C}{6C}, \qquad a_2 = \frac{-1+6C-4C^2}{4C^2}, \qquad a_3 = \frac{34-161C-200C^2+296C^3}{120C^3}, \\
V_1 &= -\frac{2(C-1)}{C}, \qquad b_1 = 0, \qquad b_2 = -\frac{2(19-33C+11C^2)}{9C^2}, \qquad b_3 = \frac{67-139C+6C^2+54C^3}{24C^3} \\
c_0 &= W_2 = W_3 = 0, \qquad d_1 = \frac{3}{2}, \qquad d_3 = \frac{23C-25}{12C}.
\end{aligned}
\tag{A3}
$$

For the CFM black holes, we have

$$
\begin{aligned}
a_1 &= \frac{4\beta-7}{6(4\beta-5)}, \qquad a_2 = \frac{-37+69\beta-32\beta^2}{3(4\beta-5)^2}, \qquad a_3 = \frac{-2511+6307\beta-5204\beta^2+1408\beta^3}{30(4\beta-5)^3}, \\
V_1 &= \frac{3-4\beta}{4\beta-5}, \qquad b_1 = 0, \qquad b_2 = \frac{-327+880\beta-592\beta^2}{36(-5+4\beta)^2}, \qquad b_3 = \frac{6592\beta^3-27552\beta^2+35192\beta-14217}{144(4\beta-5)^3}, \\
W_2 &= W_3 = 0, \qquad d_1 = \frac{1}{2}, \qquad d_2 = 0, \qquad d_3 = \frac{4(\beta-1)}{4\beta-5}.
\end{aligned}
\tag{A4}
$$

3. On the metric component h_{tt} for various tidal-deformed braneworld solutions

In this section, we collect various expressions for the metric perturbation $h_{tt} = f(r)H_0$ for $l = 2$ perturbations. The quadrupolar TLN is read off from the $1/u^2$ and u^3 coefficients in each case.

(I) CFM black holes:

$$h_{tt} = \frac{1}{u^2} + \frac{(3-11\beta)}{2u} + \left(-\frac{1}{2}(11\beta^2 - 14\beta + 3)\log(u) + (11\beta - 7)\right) + \frac{1}{12}u(55\beta^3 - 180\beta^2 + 158\beta - 33)$$
$$+ \frac{1}{48}u^2(-(\beta-1))(220\beta^3 - (36(11\beta^2 - 14\beta + 3))\log(u) - 1523\beta^2 + 2358\beta - 891)$$
$$+ \frac{1}{120}u^3(-(\beta-1))(-1100\beta^3 + 9815\beta^2 - (30(33\beta^3 - 97\beta^2 + 79\beta - 15))\log^2(u)$$
$$+ (1100\beta^4 - 11047\beta^3 + 25706\beta^2 - 23235\beta + 7191)\log(u) - 16790\beta + 5775) + O(u^4). \tag{A5}$$

(II) γ wormholes:

$$h_{tt} = \frac{2}{u^2} + \frac{5-11\gamma}{u} + (10\gamma + (-11\gamma^2 + 14\gamma - 3)\log(u) - 7)$$
$$+ \frac{1}{6}u(55\gamma^3 - 169\gamma^2 + 141\gamma - 6(11\gamma^2 - 14\gamma + 3)\log(u) - 15)$$
$$- \frac{1}{24}u^2((\gamma-1)(220\gamma^3 - 1453\gamma^2 + 2106\gamma - 24(11\gamma^2 - 36\gamma + 9)\log(u) - 639))$$
$$- \frac{1}{600}((\gamma-1)(-25(233\gamma^3 - 1284\gamma^2 + 2232\gamma - 711) + 2(12327 - 56665\gamma + 93611\gamma^2$$
$$+ 5500\gamma^4 - 45683\gamma^3)\log(u) - 60(121\gamma^3 - 407\gamma^2 + 355\gamma - 69)\log^2(u)))u^3 + O(u^4). \tag{A6}$$

(III) Bronnikov-Kim wormholes:

$$h_{tt} = \frac{1}{u^2} + \frac{11R^2 - 12R + 12}{(4-4R)u}$$
$$+ \frac{1}{24(R-1)^2}8(33R^3 - 97R^2 + 128R - 64) - 3(R-2)^2(11R^2 - 12R + 12)\log(u)$$
$$+ \frac{(55R^6 - 720R^5 + 2304R^4 - 3552R^3 + 2736R^2 - 1152R + 384)u}{96(R-1)^3}$$
$$+ \frac{1}{192(R-1)^4}(-24000 + 96000R - 176592R^2 + 193776R^3 - 134772R^4 + 58584R^5 - 14739R^6$$
$$+ 1743R^7 - 55R^8 + 36(-2+R)^4(-12 + 24R - 23R^2 + 11R^3)Log[u])u^2$$
$$+ \frac{1}{4800(R-1)^5}[100(55R^9 - 2128R^8 + 21462R^7 - 102453R^6 + 284012R^5 - 499588R^4$$
$$+ 577088R^3 - 431792R^2 + 191680R - 38336) + (-1375R^{10} + 60735R^9 - 718471R^8$$
$$+ 4174672R^7 - 14597128R^6 + 33515840R^5 - 52485312R^4 + 56357888R^3 - 40272512R^2$$
$$+ 17455360R - 3491072)\log(u)$$
$$+ 30(R-2)^4(165R^5 - 1357R^4 + 3488R^3 - 4504R^2 + 3312R - 1104)\log^2(u)]u^3 + O(u^4). \tag{A7}$$

(IV) Massless geometries:

$$h_{tt} = \frac{1}{u^2} + \frac{11(C-1)}{(4\sqrt{2})u} + \frac{1}{48}(-(33(C-1)^2)\log(u) - 80) + \frac{(-55C^3 + 165C^2 - 637C + 527)u}{192\sqrt{2}}$$
$$- \frac{1}{768}((C-1)^2(55C^2 - 110C + 1319))u^2$$
$$+ \frac{1}{19200\sqrt{2}}(C-1)u^3(100(55C^2 - 110C - 1) + (1375C^4 - 5500C^3 + 46912C^2$$
$$- 82824C + 68557)\log(u) + (1320(C-1)^2)\log^2(u)) + O(u^4). \tag{A8}$$

APPENDIX B: MORE ABOUT INDICIAL ROOTS

In the following, we compute and collect the indicial roots of all the braneworld geometries that we examined in this work by further studying Eq. (4.16). The near-horizon expansion in the decoupled case (tidal black hole) were studied separately in Sec. V, where we found an indicial equation of the form of Eq. (6.6),

$$\mathcal{R}(\mathcal{R}-1)(\mathcal{R}-2)+\mathcal{R}(\mathcal{R}-1)\mathcal{P}_{10}+\mathcal{R}\mathcal{P}_{20}+\mathcal{P}_{30}=0. \quad (B1)$$

In Table VII, we present the values for all the spacetime geometries considered in our paper. The roots of the indicial equation are, apart from a couple of cases, independent of l, and for all solutions there is at least one positive non-negative root, which is a necessary condition for the existence of series solutions regular at the metric singularity associated with either a horizon or a throat. For cases where all the roots are positive or if the set of roots is $\{-1,0,1\}$, we could identify the asymptotic series solution relevant for the TLN by taking appropriate limits. For other parameter domains, we cannot compute the TLN based rigorously on our method. Nevertheless, since there are no singularities in the asymptotic series solutions and there is at least one positive root in all cases, it is certainly possible that the various expressions for the TLN extend to some or all of these regions. Other methods of analysis beyond those used in this work have to be explored for this purpose.

For each family of solution characterized by some real parameter α (as summarized in Table I, $\alpha \in \{r_0, h, \gamma, \beta\}$), the coefficient functions $\{\mathcal{P}_1(x,\alpha), \mathcal{P}_2(x,\alpha), \mathcal{P}_3(x,\alpha)\}$ are discontinuous as bivariate functions of x and α at the regular singular point $x=0$ at values of α which mark transitions between different spacetime interpretations. This leads to different sets of roots characterizing different types of spacetime geometries. Interestingly, we find that, for the various solutions we study, the indicial roots tend to display some level of universality—similar sets of values are associated with the solutions belonging to different families but sharing identical causal structures. From Table VII, one can easily recognize the following pairings between the indicial roots and spacetime interpretations:

(a) $\{-1,0,1\}$~Schwarzschild black holes and globally regular Kerr-like wormholes.

(b) $\{0,\frac{1}{2},1\}$~ globally regular and traversable wormholes.

The only solution with l-dependent roots is the extremal Reissner-Nordström solution, which is a member of the Bronnikov-Kim family of solutions and a member of the CFM family of black hole solutions, where the CP diagram is identical to that of the extremal Reissner-Nordströ m metric but geometrically different. A subset of solutions defined by $C \leq 0$ in the "massless geometries" family appears to fall out of this classification. The case of $C=0$ has the extremal Reissner-Nordström-like causal structure,

TABLE VII. In the table, we use the abbreviations WH (wormholes), Sch BH (Schwarzschild black holes), NS (naked singularities), RN (extremal Reissner-Nordström metric). Kerr WH refers to completely regular wormhole geometries, of which the CP diagram resembles the form of the Kerr geometries. RN WH refers to spacetimes with the casual structure of the extremal Reissner-Nordström metric with a horizon cloaking a timelike singularity. The radius parameters r_s and r_{th} refer to the singular surface of a naked singularity and the radius of a wormhole throat, respectively. This table excludes the roots (± 1) for the tidal black hole, which are associated with a second-order ODE, as was presented in Sec. V.

	r_h	$\{\mathcal{P}_{10},\mathcal{P}_{20},\mathcal{P}_{30}\}$	Indicial roots
Bronnikov-Kim wormholes ($L=\tilde{m}$)	(NS) $r_0<\tilde{m},\ r_s=2\tilde{m}$	$\{3,-3,0\}$	$\{-2,0,2\}$
	(WH) $r_0\in[\tilde{m},2\tilde{m}),\ r_{\text{th}}=r_1$	$\{\frac{3}{2},0,0\}$	$\{0,\frac{1}{2},1\}$
	(WH) $r_0>2\tilde{m},\ r_{\text{th}}=r_0$	$\{\frac{3}{2},0,0\}$	$\{0,\frac{1}{2},1\}$
	(RN) $r_0=2\tilde{m},\ r_h=r_0$	$\{6,4-l-l^2,-2(2+l+l^2)\}$	$\{-2,-\frac{1}{2}(1\pm\sqrt{9+4l+4l^2})\}$
Massless geometries ($L=h$)	(WH) $C<0,\ r_{\text{th}}=\frac{1}{\sqrt{2}}\sqrt{h^2+h^2(1-C)^2}$	$\{0,0,0\}$	$\{2,1,0\}$
	(Kerr WH) $C\in(0,1),\ r_h=h$	$\{3,0,0\}$	$\{-1,0,1\}$
	(Sch BH) $C\geq 1,\ r_h=h$	$\{3,0,0\}$	$\{-1,0,1\}$
	(RN WH) $C=0,\ r_h=h$	$\{-3,0,0\}$	$\{0,1,5\}$
γ wormholes ($L=m$)	(NS) $\gamma<1,\ r_s=\frac{2m}{2-\gamma}$	$\{0,0,0\}$	$\{2,1,0\}$
	(Sch BH) $\gamma=1,\ r_h=2m$	$\{3,0,0\}$	$\{-1,0,1\}$
	(WH) $\gamma>1,\ r_{\text{th}}=2m\gamma$	$\{\frac{3}{2},0,0\}$	$\{0,\frac{1}{2},1\}$
CFM black holes ($L=m$)	(Sch BH) $\beta<1,\ r_h=2m$	$\{3,0,0\}$	$\{-1,0,1\}$
	(Kerr WH) $\beta\in(1,\frac{5}{4}),\ r_h=2m$	$\{3,0,0\}$	$\{-1,0,1\}$
	(RN WH) $\beta=\frac{5}{4},\ r_h=2m$	$\{\frac{9}{2},\frac{1}{4}(10-l-l^2),\frac{1}{4}(-2-l-l^2)\}$	$\{-1,-\frac{1}{4}(1\pm\sqrt{9+4l+4l^2})\}$
	(WH) $\beta>\frac{5}{4},\ r_{\text{th}}=\frac{m}{2}(4\beta-1)$	$\{\frac{3}{2},0,0\}$	$\{0,\frac{1}{2},1\}$

and matrices with $C < 0$ are globally regular and traversable wormholes. It would be interesting to study the relation between indicial roots and spacetime causal structures with more examples and in a deeper systematic classification. These roots reflect the tidal response of the object in the near-horizon regime and appear to capture aspects of causal structures of the various objects beyond their local horizon geometries.

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