Isogeometric analysis-based approaches for modeling solid materials and structures with defects

Li, Weidong

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**ISOGOMETRIC ANALYSIS-BASED APPROACHES
FOR MODELING SOLID MATERIALS AND STRUCTURES
WITH DEFECTS**

LI WEIDONG

SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

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SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2021

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research, is free of plagiarised materials, and has not been submitted for a higher degree to any other University or Institution.

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Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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Date

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Assoc Prof Zhou Kun

Authorship Attribution Statement

This thesis contains material from 4 papers published in the following peer-reviewed journals in which I am listed as an author.

Chapter 3 is published as **W. Li**, N. Nguyen-Thanh, and K. Zhou. Geometrically nonlinear analysis of thin-shell structures based on an isogeometric-meshfree coupling approach. *Computer Methods in Applied Mechanics and Engineering*, 2018. **336**: 111-134.

The contributions of the co-authors are as follows:

- Associate Professor Zhou proposed the research idea, analyzed the numerical results and revised the manuscript.
- I developed the numerical approach, investigated a series of benchmark problems, analyzed the numerical results and drafted the manuscript.
- Dr. Nguyen-Thanh provided technique support of the research and revised the manuscript.

Chapter 4 is published as **W. Li**, N. Nguyen-Thanh, and K. Zhou. An isogeometric-meshfree collocation approach for two-dimensional elastic fracture problems with contact loading. *Engineering Fracture Mechanics*, 2020. **223**: 106779.

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Chapter 5 is published as **W. Li**, N. Nguyen-Thanh, J. Huang, and K. Zhou. Adaptive analysis of crack propagation in thin-shell structures via an isogeometric-meshfree moving least-squares approach. *Computer Methods in Applied Mechanics and Engineering*, 2020. **358**: 112613.

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- I developed the numerical approach, investigated a series of benchmark problems, analyzed the numerical results and drafted the manuscript.
- Dr. Nguyen-Thanh provided technique support of the research and revised the manuscript.
- Dr. Huang revised the manuscript.

Chapter 6 is published as **W. Li**, N. Nguyen-Thanh, and K. Zhou. Phase-field modeling of brittle fracture in a 3D polycrystalline material via an adaptive isogeometric-meshfree approach. *International Journal for Numerical Methods in Engineering*, 2020. **121**: 5042-5065.

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- Associate Professor Zhou proposed the research idea, analyzed the numerical results and revised the manuscript.
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- Dr. Nguyen-Thanh provided technique support of the research and revised the manuscript.

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Li Weidong

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Date

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Li Weidong

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Abstract

Solid materials and structures usually undergo a complex process of fracture, such as crack initiation, propagation and coalescence before experiencing final failure. The failure of materials and structures can be induced by either micro- or macro-defects, including cracks, voids, dislocations and inclusions. The fundamental understanding of fracture mechanics remains a big challenge, particularly for thin-shell structures. Researchers have mainly investigated the deformation and fracture mechanisms of solids and structures via analytical and numerical approaches. Analytical approaches show their limitations when being implemented to investigate geometrically complicated structures with various boundary conditions and to treat cracks at arbitrary positions. Numerical approaches demonstrate their efficiency and robustness on tackling these complex engineering problems over analytical ones. As a recently proposed numerical method, isogeometric analysis (IGA) exhibits significant advantages in terms of geometry exactness and higher-order approximation. Therefore, this Ph.D. research focuses on the development of IGA-based approaches and their applications to the modeling of solid materials and structures with defects.

Firstly, a novel IGA-meshfree coupling approach is developed to perform both the linear and geometrically nonlinear analyses of shell structures. The analyses are based upon the assumption of the Kirchhoff-Love thin-shell theory. Both parametric and physical domains are utilized for the thin-shell structures, where the former one is used to couple the IGA and meshfree methods and to obtain the latter one via mapping. The entire domain is divided into three subdomains: the subdomain described by the IGA method to ensure geometry exactness, the subdomain described by the meshfree method to achieve local refinement and the coupling subdomain described by both methods. In the coupling subdomain, the coupling formulation is obtained based on the

consistency conditions to realize the smoothness between the IGA and meshfree subdomains. The coupling approach can achieve a higher convergence rate than the IGA and meshfree methods because of the realization of local refinement. The accuracy and robustness of the coupling approach are validated by solving a series of shell benchmark problems.

Secondly, a strong form-based IGA-meshfree moving least-squares collocation (IMMLS-C) approach is developed for two-dimensional linear elastic fracture problems with contact loading. The IMMLS-C approach uses reproducing conditions to establish the equivalence between MLS shape functions and isogeometric basis functions, thus exhibiting advantages including the exact geometry representation, convenient crack modeling and flexible adaptive refinement. The IMMLS-C approach focuses on solving fracture problems based on strong formulations without requiring the numerical integration of Galerkin weak forms. Traction-free boundary conditions are enforced over a set of collocation points located on both sides of a crack surface. The displacement discontinuity along the crack surface is modeled by the visibility criterion, and the singularity of near-tip stress fields is captured by adaptive mesh refinement without adding tip-enrichment functions, thereby reducing the degrees of freedom. Moreover, contact constraints are enforced by introducing a penalty algorithm to the strong formulations. The numerical results demonstrate that the adaptive refinement can achieve a high convergence rate at a low computational cost.

Thirdly, crack propagation in thin-shell structures is investigated via an IGA-meshfree moving least-squares (MLS) approach. The approach provides an effective strategy of adaptive mesh refinement for IGA in a straightforward meshfree manner. The adaptivity of the mesh refinement is achieved by utilizing a gradient-based error estimator to identify the meshes that need to be refined by adding linear reproducing

points. The Kirchhoff-Love theory is further applied in the IGA-meshfree MLS formulation to simplify the modeling of cracked thin-shell structures by neglecting the rotational degrees of freedom. In this way, the singularity of the stress fields near the crack tip and the discontinuity of the displacement fields around the crack surface can be efficiently captured by the adaptive mesh refinement to produce accurate results. A series of two-dimensional static and quasi-static crack propagation problems about thin-shell structures are investigated. The predicted propagation paths are in good agreement with the reference results.

Finally, an adaptive IGA-meshfree phase-field approach is developed for the modeling of brittle fracture in three-dimensional polycrystalline materials. In this approach, the flexible local mesh refinement scheme that is inherited from a meshfree method is combined with an error estimator that includes both the phase field and its gradient to achieve adaptive refinement. In this way, propagating cracks can be dynamically tracked, and the mesh near cracks is refined in a meshfree manner without requiring *a priori* knowledge of crack paths. The intergranular and transgranular crack propagation patterns in polycrystalline materials can be simulated by this approach. A series of numerical examples that deal with isotropic and anisotropic fracture are investigated to demonstrate the robustness and effectiveness of the developed approach.

The development of efficient and accurate IGA-based approaches provides a promising way to investigate the deformation and fracture mechanisms of solid materials and structures with defects. The understanding of these mechanisms is of significant importance to engineering structure design and failure prevention.

List of Publications

- [1] **W. Li**, N. Nguyen-Thanh, and K. Zhou. Phase-field modeling of brittle fracture in a 3D polycrystalline material via an adaptive isogeometric-meshfree approach. *International Journal for Numerical Methods in Engineering*, 2020. **121**: 5042-5065
- [2] **W. Li**, N. Nguyen-Thanh, J. Huang, and K. Zhou. Adaptive analysis of crack propagation in thin-shell structures via an isogeometric-meshfree moving least-squares approach. *Computer Methods in Applied Mechanics and Engineering*, 2020. **358**: 112613.
- [3] **W. Li**, N. Nguyen-Thanh, and K. Zhou. An isogeometric-meshfree collocation approach for two-dimensional elastic fracture problems with contact loading. *Engineering Fracture Mechanics*, 2020. **223**: 106779.
- [4] **W. Li**, N. Nguyen-Thanh, and K. Zhou. Geometrically nonlinear analysis of thin-shell structures based on an isogeometric-meshfree coupling approach. *Computer Methods in Applied Mechanics and Engineering*, 2018. **336**: 111-134.
- [5] J. Huang, N. Nguyen-Thanh, **W. Li**, and K. Zhou. An adaptive isogeometric-meshfree coupling approach for the limit analysis of cracked structures. *Theoretical and Applied Fracture Mechanics*, 2020. **106**: 102426.
- [6] N. Nguyen-Thanh, **W. Li**, J. Huang, N. Srikanth, and K. Zhou. An adaptive isogeometric analysis meshfree collocation method for elasticity and frictional contact problems. *International Journal for Numerical Methods in Engineering*, 2019. **120**(2): 209-230.
- [7] N. Nguyen-Thanh, **W. Li**, and K. Zhou. Static and free-vibration analyses of cracks in thin-shell structures based on an isogeometric-meshfree coupling approach. *Computational Mechanics*, 2018. **62**(6): 1287-1309.

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Nomenclature

Abbreviations

1D	One-dimensional
2D	Two-dimensional
3D	Three-dimensional
AMR	Adaptive mesh refinement
CAD	Computer-aided design
CAE	Computer-aided engineering
CPM	Cracking particle method
CTOD	Crack tip opening displacement
DEM	Diffusive element method
DOFs	Degrees of freedom
EFGM	Element-free Galerkin method
EPFM	Elastic-plastic fracture mechanics
FEM	Finite element method
G-RKCM	Gradient reproducing kernel collocation method
HB-splines	Hierarchical B-splines
IGA	Isogeometric analysis
IMMLS-C	IGA-meshfree moving least-squares collocation
KL	Kirchhoff-Love
LEFM	Linear elastic fracture mechanics
LME	Local maximum entropy
LR	Locally refined
MLPG	Meshfree local Petrov-Galerkin
MLS	Moving least-squares
MR	Mindlin-Reissner

NEM	Natural element method
NURBS	Non-uniform rational B-splines
PDEs	Partial differential equations
PFM	Phase-field model
PHT-splines	Polynomial splines over hierarchical T-meshes
PUM	Partition of unity method
RBCM	Radial basis collocation method
RBFs	Radial basis functions
RK	Reproducing kernel
RKCM	Reproducing kernel collocation method
RKPM	Reproducing kernel particle method
RPHT-splines	Rational PHT-splines
SIFs	Stress intensity factors
SPH	Smoothed particle hydrodynamics
THB-splines	Truncated hierarchical B-splines
XFEM	Extended finite element method
XIGA	Extended IGA

Symbols

δ	Variation operator
Ω	Problem domain
Γ	Domain boundary
Γ_t	Natural boundary
Γ_u	Essential boundary
$\mathbf{b}$	Body force
$\bar{\mathbf{t}}$	Prescribed force
$\bar{\mathbf{u}}$	Prescribed displacement
$\boldsymbol{\varepsilon}$	Strain
$\boldsymbol{\sigma}$	Stress
$\mathbf{u}$	Displacement
E	Young's modulus
ν	Poisson's ratio
μ	Shear modulus
$\mathbf{S}$	Second Piola–Kirchhoff stress
$\mathbf{E}$	Green–Lagrange strain
$\mathbf{I}$	Second-order identify tensor
G_c	Energy release rate

Chapter 1 Introduction

Numerical methods have become a promising alternative to both analytical and experimental methods for investigating the mechanisms of deformation and failure in solid materials and structures with defects. Developments of efficient and accurate numerical methods are of significant importance to understanding the mechanisms. Therefore, this Ph.D. dissertation aims to develop isogeometric analysis-based approaches for modeling solid materials and structures with defects. In this chapter, the background of the research is briefly introduced, followed by the research motivations and objectives. The organization of the whole dissertation is finally outlined.

1.1. Background

Investigations of the deformation and failure mechanisms of solid materials and structures with defects are significantly important in engineering. As illustrated in Fig. 1.1, in industrial engineering, the existence of cracks in structures such as storage tanks, wind turbine blades and aircraft fuselages can lead to fatigue crack propagation until the final structural failure occurs. The stress-strain curves of brittle and ductile materials are visualized in Fig. 1.2. To understand the mechanisms, experimental, analytical and numerical methods have been developed. Numerical methods exhibit advantages such as high efficiency and low cost as compared with the other two. Moreover, due to the development of high-performance computing technologies, numerical methods have prevailed in many engineering problems [1, 2]. The finite element method (FEM), which is a type of numerical method, has been widely developed and integrated into commercial software. Recently, the development of

robust, efficient and accurate numerical methods has attracted significant attention from researchers for modeling solids and structures [3, 4].

Fig. 1.1. Fractures of engineering structures: (a) a water tank [5], (b) wind turbine blades, (c) water pipes and (d) aircraft fuselages.

Fig. 1.2. Stress-strain curves of brittle and ductile materials.

Commonly used engineering structures include beams, plates and shells. A beam is defined as a structural element with a significantly large length as compared with other dimensions such as the width. A plate or shell is a structural element with a

relatively small thickness as compared with other dimensions. The major difference between a plate and a shell is that the plate is initially flat, whereas the shell has a curved surface. Due to the curvature in shell structures, shells are capable of providing efficient loading carrying mechanisms as compared with plates and beams and have been used in numerous engineering applications, such as turbine disks and aircraft.

Investigations into the deformation mechanisms of shell structures under various loading conditions are vital to the structural integrity analysis and failure. Two types of shell theories have been developed: the Kirchhoff-Love (KL) theory [6] and the Mindlin-Reissner (MR) theory [7]. The former theory, which is also called the classical theory, neglects the shear deformation and is typical for shell structures with a very small thickness. The latter one considers the shear deformation in the thickness direction and is especially suitable for thick shells. Based on the two theories, shell structures have been numerically analyzed within the framework of the FEM. The KL theory is more computationally efficient than the MR one, whereas the KL one gives the fourth-order derivatives in the governing partial differential equations (PDEs), thus requiring the C^1 continuity of displacements in the Galerkin weak formulations. As the C^1 continuity is not convenient to be implemented within the framework of the FEM [8], the developments of numerical methods for achieving the higher-order continuities are expected to be carried out.

Engineering structures usually contain defects such as cracks, voids and inclusions that are normally caused by manufacturing processes. In addition, crack initiation occurs when the stress-based or fracture mechanics-based criteria are met. Cracks can not only propagate, causing progressive damage throughout structures, but also interact with preexistent cracks, influencing the near-tip stress fields and crack propagation paths. The fracture behaviors, including crack initiation, propagation,

coalescence and branching, have a remarkable effect on structural integrity and failure [9, 10]. Thus, understanding the fracture mechanism of structures and solids is of significant importance.

Fracture mechanics mainly deals with the investigations of fracture behaviors in solids and structures with or without preexistent cracks. In brittle fracture, only elastic deformation is investigated within the context of linear elastic fracture mechanics (LEFM), in which the energy-based failure theory proposed by Griffith [11] is considered as the fundamental. Ductile fracture that considers both elastic and plastic deformation is investigated by elastic-plastic fracture mechanics (EPFM). Three modes of fracture associated with material failure are classified: the Mode-I fracture representing crack opening, the Mode-II fracture representing crack sliding and the Mode-III fracture representing crack tearing. These fracture models are capable of describing both the near-tip displacement and stress that are used to investigate the fracture processes.

Numerical simulations of fracture processes are based on two types of crack models, including the discrete model and the diffusive model. The former model treats a crack as a geometrical discontinuity, and the latter one considers a cracked solid as a continuum by introducing a crack field that is smeared throughout the problem domain. The discrete crack model requires explicit or implicit representations of crack topologies. The requirement poses an obstacle to the FEM-based numerical implementations as edges of finite elements need to align with crack topologies, resulting in low numerical efficiency in treating crack propagation, especially in three-dimensional (3D) problems. To avoid remeshing, the extended finite element method (XFEM) was proposed by Belytschko and Black [12] and then applied to model internal boundaries, such as cracks and inclusions. However, the XFEM suffers from

two critical limitations, including the approximated geometry and low-order continuity that need to be tackled in modeling fracture in thin shells.

Compared with the discrete crack model, the diffusive one can conveniently simulate complicated fracture behaviors, such as crack initiation, propagation and coalescence. The phase-field model (PFM), which is treated as a diffusive crack model, exhibits advantages particularly in the 3D fracture modeling, such as without tracking crack topologies or defining enrichment functions, as compared with the discrete crack model. Numerous PFMs have been developed to investigate brittle and ductile fracture in solids and structures. However, PFMs suffer from high computational cost because they require a fine mesh for the accurately regularized representations of crack topologies. Research works in the developments of computationally efficient numerical frameworks have been carried out, such as the adaptive mesh refinement (AMR) strategy [13] and strong form-based formulations [14].

Apart from the above-mentioned FEM-based numerical framework, isogeometric analysis (IGA) proposed by Hughes *et al.* [3] connects computer-aided design (CAD) and computer-aided engineering (CAE) by adopting non-uniform rational B-splines (NURBS) basic functions for both defining geometries and approximating displacement fields. Compared with the FEM, IGA exhibits advantages in the exact representation of geometries and flexible construction of higher-order and smooth basis functions. In structural modeling, Kiendl *et al.* [15] first developed the isogeometric formulations of thin shells, in which the curve mid-surface was exactly represented and the C^1 continuity was conveniently obtained. In fracture modeling, Ghorashi *et al.* [16] proposed the extended IGA (XIGA) by incorporating the enrichment functions into IGA within the framework of the partition of unity. The extended formulations can obtain smooth stress fields by using higher-order

approximations. The developments of IGA-based approaches provide a promising alternative for modeling solid materials and structures.

1.2. Motivations

IGA has prevailed in both engineering design and analysis due to its advantages such as the smooth approximation and exact geometry representation over the FEM. However, the traditional NURBS-based IGA suffers from the capability of local mesh refinement that is of particular importance for accurate and efficient numerical implementations. Although several basis functions have been developed to tackle these limitations, such as the locally refined (LR) B-splines [17], T-splines [18] and polynomial splines over hierarchical T-meshes (PHT-splines) [19], the basis functions are complex to be implemented. Compare with IGA, meshfree methods are flexible for implementing local mesh refinement by directly adding nodes.

It is desirable to couple the IGA and meshfree methods because of such advantages as the exact geometry, the local refinement flexibility and the higher-order approximation. The idea of coupling approaches is to preserve the advantages of numerical methods and circumvent their disadvantages [20-22]. In the coupling approaches, a problem domain is divided into three subdomains, including the outer subdomain to obtain the boundary exactness, the interior subdomain to provide a flexible way to process problems such as the cracks and inclusions, and the coupling subdomain to realize smoothness. Belytschko *et al.* [22] proposed the coupling approach of the element-free Galerkin method (EFGM) and the FEM for elastodynamic problems. The coupling approach utilized three subdomains: the meshfree subdomain to consider crack growth, the FEM subdomain to reduce

computational cost and the coupling subdomain to achieve smoothness between the other two subdomains.

Developments of efficient coupling approaches can facilitate the investigations of deformation and fracture mechanisms of solids and structures. Firstly, the smooth and higher-order approximations give more accurate field variables. In the KL-shell modeling, the C^1 continuity is conveniently implemented by the arbitrary higher-order shape functions. The insensitivity to mesh distortion, which is inherited from the meshfree methods, is particularly suitable for modeling large deformations in solid materials and structures. Secondly, the flexibility of local mesh refinement is particularly important to the problems involving singularity or discontinuity. Shell structures subjected to a concentrated point load will experience stress concentration, where a fine mesh is required. In fracture modeling, the near-tip regions need to be locally refined for capturing the singular crack-tip stress. Moreover, the local mesh refinement is of particular importance to investigating complex fracture behaviors within the framework of the PFM. Thus, developments of the IGA-based approaches are desirable for the understanding of deformation and fracture mechanisms of solids and structures.

1.3. Objectives

The Ph.D. study aims to develop robust, efficient and accurate IGA-based approaches for modeling solid materials and structures with defects. These approaches are capable of investigating the deformation and fracture behaviors of solids and structures, thus providing a comprehensive understanding of the deformation and failure mechanisms. The present research has the following main objectives:

- To develop an IGA-meshfree coupling approach based on the reproducing conditions for investigating both the linear and nonlinear large deformation behaviors of shells;
- To develop an adaptive IGA-meshfree moving least-squares (MLS) approach by introducing a gradient-based error estimator for investigating the fracture behaviors of both two-dimensional (2D) plane structures and thin-shell structures;
- To investigate the brittle fracture behaviors of 3D polycrystalline materials by developing an adaptive IGA-meshfree phase-field approach;
- To provide guidance for applications of solid materials and structures with cracks under various loading conditions.

1.4. Dissertation outline

This Ph.D. dissertation contains seven chapters. After the introduction, the literature review on the numerical methods, shell theory, discrete fracture modeling and phase-field fracture modeling are conducted in Chapter 2. Chapter 3 investigates both the linear and nonlinear large deformation behaviors of shells. In Chapter 4, cracking behaviors of 2D plane structures subjected to contact loading are investigated via the strong form-based formulations. Afterwards, the analysis of crack propagation in thin shells is conducted within an adaptive IGA-meshfree-based numerical framework in Chapter 5. Chapter 6 investigates the anisotropic brittle fracture behaviors of 3D polycrystals based on the phase-field modeling. Chapter 7 concludes the main contributions of this Ph.D. work and recommends future research.

Chapter 2 Literature Review

This chapter first reviews the numerical methods, including the FEM, meshfree methods and IGA. The fundamentals of fracture mechanics are then introduced, and two types of fracture modeling for solids and structures are presented. Finally, a summary is given at the end of this chapter.

2.1. Finite element method

FEM, a widely used numerical method to solve a set of PDEs, has been applied in numerous engineering applications. The idea of FEM was first adopted by Courant [23] for the structural analysis, in which triangular elements were used. Clough [24] proposed the name “finite element” and applied it to plane stress problems. A history of the beginning of FEM was briefly introduced by Gupta and Meek [25]. In FEM, the solution domain is divided into many interconnected subdomains (elements) with the vertices (nodes) of the elements. In each element, the field variables such as the displacements are interpolated by both the nodal variables and shape functions. The basis of FEM is expressed as follows:

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega = \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma. \quad (2.1)$$

The discretization of Eq. (2.1) gives the following form:

$$\delta \mathbf{u}_e^T \left(\int_{\Omega_e} \mathbf{B}_e^T \mathbf{D} \mathbf{B}_e \mathbf{u}_e d\Omega_e - \int_{\Omega_e} \mathbf{N}_e^T \mathbf{b}_e d\Omega_e - \int_{\Gamma_e} \mathbf{N}_e^T \bar{\mathbf{t}}_e d\Gamma_e \right) = 0, \quad (2.2)$$

where Ω_e is the domain of an element, $\mathbf{B}_e$ is the matrix representing strain-displacement relations, $\mathbf{D}$ is the matrix representing stress-strain relations, and $\mathbf{N}_e$ is the matrix describing the nodal shape functions. Due to the arbitrary virtual displacement $\delta \mathbf{u}_e$, the second part in Eq. (2.2) equals zero, and the stiffness matrix $\mathbf{K}_e$ and force vector $\mathbf{F}_e$ can be obtained as

$$\mathbf{K}_e = \int_{\Omega_e} \mathbf{B}_e^T \mathbf{D} \mathbf{B}_e d\Omega_e, \quad (2.3)$$

$$\mathbf{F}_e = \int_{\Omega_e} \mathbf{N}_e^T \mathbf{b}_e d\Omega_e + \int_{\Gamma_e} \mathbf{N}_e^T \bar{\mathbf{t}}_e d\Gamma_e. \quad (2.4)$$

The integration over each element and its boundary can be obtained by the Gauss integration. The general solution procedures of engineering problems within the framework of FEM have been detailed in [26].

Although FEM has been applied to numerous engineering problems, it suffers from limitations in terms of solving the large deformation problems and moving boundary problems. Moreover, due to the nature of FEM, it is challenging to obtain the accurate geometry models of complex structures such as the thin-shell structures and to construct the higher-order approximation of field variables. Thus, the alternatives to FEM, such as meshfree methods and IGA, have been proposed to tackle these limitations.

2.2. Meshfree methods

Compared to the mesh-based methods that discretize a problem domain into many small elements, meshfree methods discretize a problem domain into many scattered nodes and thus can avoid the mesh alignment and remeshing issues in moving boundary problems. Moreover, the meshfree methods exhibit the following advantages over the FEM: (i) the initial mesh generation is avoided, and the pre-processing time is saved in modeling complex geometries; (ii) the smooth and arbitrary higher-order shape functions are obtained; (iii) AMR is conveniently implemented by adding nodes at arbitrary regions, such as the stress concentration and crack-tip regions, without considering the element connectivity.

Meshfree methods originate from the smoothed particle hydrodynamics (SPH) [27], in which a problem domain is discretized into many moving particles that possess both the masses and volumes. Applications of the SPH have been introduced, such as interfacial flows [28], explosion [29] and nonlinear dynamics [30]. However, the SPH has limitations in terms of the tensile stability and treatments of both the material interface and solid boundary [31].

In addition to the SPH, many meshfree methods have been developed, and these methods are classified into two types: the Galerkin weak form and the strong form. The weak form-based meshfree methods use Gaussian quadrature throughout the problem domain, and the strong form-based methods directly solve the governing equations at collocation points. The two types of meshfree methods are reviewed as follows.

(a) Weak form-based Galerkin methods

Nayroles *et al.* [32] proposed the diffusive element method (DEM) based on the Galerkin method without requiring mesh generation. To improve the accuracy of DEM, Belytschko *et al.* [33] proposed the EFGM that used the MLS approximation. As the EFGM lacks the interpolation property, it utilized the Lagrange multiplier method to impose Dirichlet boundary conditions. In addition, the reproducing kernel (RK) approximation proposed by Liu *et al.* [34] was utilized to construct formulations of the reproducing kernel particle method (RKPM), in which displacement fields are approximated by the RK formulations. Developments in the RKPM have been carried out to improve the numerical accuracy [35] and convergence [36]. The EFGM and RKPM have been used in many fields, such as the linear and nonlinear analysis [37, 38], fracture mechanics [39, 40] and plate and shell problems [41, 42].

The EFGM and RKPM, which belong to the global weak form-based methods, require the background mesh for integration over the problem domain, and thus these methods are not truly meshfree. To avoid the background mesh, Atluri *et al.* [43] developed the meshfree local Petrov-Galerkin (MLPG) method, in which the support domains were treated as the subdomains, and the integration was carried out over each subdomain and its boundary. Different choices of test functions have been introduced in the MLPG. For example, when the MLS approximation is used in the test functions, the local weak form-based EFGM is obtained and equivalent to the MLPG. Other choices to construct test functions have been investigated, such as the local boundary integral equation method [44] and the least-squares meshfree method [45].

In addition to the above-mentioned methods, the *hp*-Clouds method proposed by Duarte *et al.* [46] introduced *h*- and *p*-refinements in the approximation space. In this method, the *p*-refinement gives higher-order polynomial shape functions without introducing the discontinuity as compared with the EFGM or RKPM, and the *h*-refinement locally adds nodes at arbitrary regions. The natural element method (NEM) proposed by Sukumar [47] used a Voronoi diagram to determine the neighbor nodes for the natural neighbor interpolation. Compared to other meshfree methods, the NEM possesses the interpolation property and can straightforwardly impose the Dirichlet boundary conditions.

The weak form-based meshfree methods have the following major limitations: (i) many meshfree nodes are required for field approximations, leading to the high computational cost; (ii) Dirichlet boundary conditions are not convenient to enforce due to the lack of the interpolation property; (iii) Gaussian integration results in low computational efficiency due to the requirement of many integration points. Attempts have been carried out to tackle the limitations. For example, Breiokopf *et al.* [48]

proposed a fast algorithm to calculate the shape functions without requiring matrix inversion. To straightforwardly enforce the Dirichlet boundary conditions, Krongauz and Belytschko [49] developed a coupled meshfree-finite element method that divided the problem domain into two subdomains: the boundary and inner subdomains, which were respectively described by the FEM and meshfree method. Recent developments of the meshfree methods have been introduced in [50-53].

(b) Strong form-based collocation methods

Compared to weak formulations, the strong formulations use many scattered nodes, namely the collocation points, at which the governing equations are satisfied. Moreover, the strong ones eliminate domain integration, thus saving computational time. The earliest works on the collocation methods are introduced in [54-56]. Later, Franke [57, 58] established the theoretical foundations of the radial basis collocation method (RBCM) and demonstrated the convergence of solutions. In the RBCM, the displacement and its derivative were approximated by differentiable radial basis functions (RBFs) [59]. Developments of the RBCM have been investigated for obtaining better-conditioned discrete systems. For example, Wong *et al.* [60] and Kansa and Hon [61] improved the numerical stability by proposing a domain decomposition strategy. Fasshauer [62] introduced the smoothing methods and multilevel algorithms in the RBFs. Chen *et al.* [63] developed the RK-enhanced local RBCM with a similar convergence rate to that of the traditional RBCM.

In addition to the RBFs, the MLS and RK shape functions have been developed for the implementation of the collocation methods. Oñate *et al.* [64] used the weighted MLS-based approximations together with the point collocation scheme to develop the finite point method. Aluru [65] developed the reproducing kernel collocation method (RKCM) based on RK approximations. Hu *et al.* [66] investigated the convergence

and stability of RKCM and demonstrated that a high convergence rate was obtained at the cost of computational time. To improve the computational efficiency and remain the optimal convergence rate, Chi *et al.* [4] developed the gradient reproducing kernel collocation method (G-RKCM) based on the gradient reproducing conditions.

2.3. Isogeometric analysis

As a mesh-based numerical method, the IGA first proposed by Hughes *et al.* [3] integrates CAD and CAE. Compared to the FEM that uses Lagrange polynomials, IGA utilizes basis functions to model geometries and approximate field variables, such as the displacement. Thus, IGA avoids the errors related to the approximations of geometries. Moreover, IGA performs better than the FEM in terms of the following aspects: (i) smooth and higher-order basis functions improve the accuracy of solutions; (ii) the coarse or fine mesh is capable of exactly representing geometries without being influenced by mesh refinement.

(a) Basis functions

In IGA, basis functions, such as NURBS, T-splines and LR NURBS, have been applied to the field approximation and geometry representation. As the NURBS basis functions are widely utilized in CAD models, the basis formulations of B-splines and NURBS are briefly introduced. Details of deriving the basis functions have been provided in [67, 68].

IGA treats both the parametric and physical spaces as a single patch that contains multiple elements. In the one-dimensional (1D) parametric space, a B-spline is constructed by a knot vector $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ with ξ_i being the knot, p being the polynomial order and n being the number of control points. The 1D basis functions $N_{i,p}(\xi)$ are defined as

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi_i \leq \xi < \xi_{i+1} \\ 0, & \xi < \xi_i \text{ or } \xi \geq \xi_{i+1} \end{cases} \quad (p=0), \quad (2.5)$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (p \geq 1). \quad (2.6)$$

It is noted that $N_{i,p}(\xi)$ satisfy the compact support property.

A B-spline curve $\mathbf{C}(\xi)$ is constructed by combining $N_{i,p}(\xi)$ and corresponding control points $\mathbf{P}_i$ as follows:

$$\mathbf{C}(\xi) = \sum_{i=1}^n N_{i,p}(\xi) \mathbf{P}_i. \quad (2.7)$$

Two knot vectors $\Xi_1 = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ and $\Xi_2 = \{\eta_1, \eta_2, \dots, \eta_{m+q+1}\}$ are considered. A

B-spline surface is obtained based on the tensor product as follows:

$$\mathbf{S}(\xi, \eta) = \sum_{i=1}^n \sum_{j=1}^m N_{i,p}(\xi) M_{j,q}(\eta) \mathbf{P}_{i,j}, \quad (2.8)$$

where $\mathbf{P}_{ij}$ are the control points. Similar to a B-spline surface, a solid is obtained as follows:

$$\mathbf{B}(\xi, \eta, \zeta) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l N_{i,p}(\xi) M_{j,q}(\eta) L_{k,r}(\zeta) \mathbf{P}_{i,j,k}. \quad (2.9)$$

As the B-spline basis functions show limitations in terms of modeling a conic section, the NURBS basis functions that are constructed based on the projections can tackle the issues. A 1D NURBS basis function can be expressed as

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi) w_i}{\sum_{\hat{i}=1}^n N_{\hat{i},p}(\xi) w_{\hat{i}}}, \quad (2.10)$$

where w_i is the weight. A NURBS curve $\mathbf{C}(\xi)$ and a NURBS surface $\mathbf{S}(\xi, \eta)$ are constructed as follows:

$$\mathbf{C}(\xi) = \sum_{i=1}^n R_{i,p}(\xi) \mathbf{P}_i, \quad (2.11)$$

$$\mathbf{S}(\xi, \eta) = \sum_{i=1}^n \sum_{j=1}^m R_{i,j}^{p,q}(\xi, \eta) \mathbf{P}_{i,j}, \quad (2.12)$$

with the 2D NURBS basis function being as

$$R_{i,j}^{p,q}(\xi, \eta) = R_i(\xi) = \frac{N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j}}{\sum_{\hat{i}=1}^n \sum_{\hat{j}=1}^m N_{\hat{i},p}(\xi) M_{\hat{j},q}(\eta) w_{\hat{i},\hat{j}}}. \quad (2.13)$$

Fig. 2.1 illustrates the representation of a NURBS surface within the framework of IGA.

Fig. 2.1. Schematic illustrations of a NURBS surface.

In IGA, the basis functions can be enriched without changing the geometry or parametrization of a physical model. Three types of refinement schemes have been introduced, including the knot insertion, order elevation and k -refinement [3]. In the

first scheme, elements are divided into small ones in the same way as the h -refinement of FEM. By adding knots in the knot spans, both the new elements and corresponding control points are obtained. The second scheme is similar to the p -refinement in FEM, which increases orders of polynomial approximations. Moreover, the order elevation requires adding knots to keep both the number of elements and the continuity between elements intact. The third scheme that is unique to IGA is implemented by combining both the order elevation and knot insertion, and it gives a smaller number of basis functions than the other two, thus saving computational cost.

(b) Developments of IGA

IGA has been utilized in many fields, such as the fracture analysis [69], contact analysis [70] and plate and shell problems [15]. However, IGA faces several challenging issues that hinder its wide applications. The issues are related to the AMR, quadrature rule, collocation method, complex geometry modeling and volumetric parameterization.

AMR can increase the accuracy of solutions at the low cost of computational time. However, it is not convenient to implement AMR in NURBS-based IGA due to the tensor product. The T-splines that allow T-junctions by adding extra vertices were first applied to IGA [71] and used for the investigation of local mesh refinement [18, 72]. To guarantee the line dependence and partition of unity, the analysis-suitable T-splines were developed [73, 74]. Hierarchical B-splines (HB-splines) [75] were incorporated into the framework of IGA for achieving AMR [76] and extended to the truncated hierarchical B-splines (THB-splines) [77] that guaranteed the partition of unity as well as the stability and accuracy of solutions. PHT-splines [78], which generalize the B-splines over hierarchical T-splines and preserve their advantages such as adaptivity, linear dependence and non-negativity, were extended to the rational PHT-splines

(RPHT-splines) [79] for modeling complex geometries with the consideration of local refinement [80]. The comparison of local mesh refinement strategies in IGA has been reviewed in [81, 82].

In addition, the idea of coupling the IGA and meshfree methods has been investigated because of such advantages as the exact geometry, the local refinement flexibility and the higher-order approximation. Two different types of coupling approaches of the IGA and meshfree methods are implemented in the parametric domain and physical domain, respectively. One approach coupling the IGA and the RKPM in the parametric domain was developed by Wang and Zhang [83]. The other approach for coupling two methods in the physical domain was developed by Valizadeh *et al.* [84] to couple the IGA and RKPM and by Rosolen and Arroyo [85] to couple the IGA and the local maximum entropy (LME) meshfree method. These coupling approaches have been utilized in the structural analysis [86, 87] and fracture analysis [69, 88]. A recent work by Zhang and Wang [89] further demonstrated the equivalence between the isogeometric and RK approximations, thus providing a convenient way to implement local refinement in IGA.

Numerical quadrature rules, which influence the accuracy, efficiency and stability of numerical solutions, have attracted considerable attention in the community of IGA. For example, Hughes *et al.* first [90] demonstrated that the higher-order continuity across elements influenced the numerical accuracy and proposed an efficient quadrature rule for the NURBS-based IGA. The quadrature rule was further extended to structured NURBS [91] and unstructured NURBS patches [92]. Developments of the quadrature rules in IGA have been reviewed in [93].

In addition to the Gaussian quadrature rules, the collocation methods have been investigated in IGA to achieve both the high computational accuracy and efficiency.

For example, Auricchio *et al.* [94] developed the NURBS-based collocation method that treated the Greville abscissae [95] as collocation points. Schillinger *et al.* [96] further developed an adaptive isogeometric collocation method and demonstrated its advantages including accuracy and efficiency as compared with both the isogeometric Galerkin method and standard FEM. Unlike the collocation points by using Greville abscissae or Demko abscissae, Anitescu *et al.* [97] developed a superconvergent IGA collocation method that considered the superconvergence points as collocation points. Applications of the IGA collocation methods have been introduced in [98-101].

Modeling complex geometries is not convenient in NURBS-based IGA due to the continuity requirements between multiple patches. The subdivision surfaces [102], which can naturally model complex geometries, were incorporated into the framework of IGA [103]. Moreover, the trimming technique, which is used for modeling complex geometries in CAD, was utilized to represent trimmed surfaces in IGA [104]. In addition, the CAD models define the boundary surfaces of solids in which the interior information required for IGA such as control points and weights are not available. The transition of a bivariate surface representation into a trivariate solid, namely the volumetric parameterization, has been investigated. The harmonic functions were adopted to construct the trivariate B-spline representations in [105]. The numerical frameworks to construct trivariate NURBS [106, 107] and T-splines [108] volumetric parametrizations have been further proposed.

2.4. Discrete fracture modeling

Two crack models have been developed, including the discrete model [109] and the diffusive model [110]. The former model treats a crack as a geometrical

discontinuity, and the latter one treats a cracked solid as a continuum. Fig. 2.2 shows the 2D crack representations based on the two models.

Fig. 2.2. Schematic representations of a solid with cracks: (a) the discrete crack model and (b) the diffusive crack model [111].

2.4.1. Fundamentals of fracture mechanics

Fracture studies can be dated back to several centuries ago when the fracture toughness of glass rods was experimentally investigated. Griffith studied the effect of flaw sizes on the fracture strength and formulated an energy-based theory, which is regarded as the beginning of LEFM [11]. In Griffith's theory, the potential energy of a system should be reduced to produce the new crack as follows:

$$-\frac{d\Pi}{dA} = \frac{dW_s}{dA} \quad (2.14)$$

with Π being the total potential energy, A being the area of crack surfaces and W_s being the energy related to crack surfaces. The energy release rate G that is capable of determining conditions of crack propagation is expressed as

$$G = -\frac{d\Pi}{dA}. \quad (2.15)$$

Griffith's theory can be utilized for the analysis of ideally brittle materials, such as ceramics. However, it is not feasible for ductile materials such as metals that undergo significant plastic deformation. By considering plastic deformation, the plastic energy is introduced in Eq. (2.15) as follows:

$$-\frac{d\Pi}{dA} = \frac{dW_s}{dA} + \frac{dU_p}{dA}. \quad (2.16)$$

The stress intensity factors (SIFs) proposed by Irwin [112] are used to define asymptotic stress as

$$\lim_{r \rightarrow 0} \sigma_{ij}^m = \frac{K_m}{\sqrt{2\pi r}} f_{ij}^m(\theta), \quad i, j \in \{1, 2, 3\}, \quad m \in \{I, II, III\}, \quad (2.17)$$

where σ_{ij} is the stress, K_m is the SIF, (r, θ) is the coordinate system locally centered at a crack front, and f_{ij} is a dimensionless function. Fig. 2.3 visualizes the Mode-I fracture, the Mode-II fracture and the Mode-III fracture. The first fracture mode is related to the in-plane tension loading normal to a crack plane. The second one is related to the in-plane shear loading. The third one is related to the out-of-plane loading. Three types of SIFs are defined as follows [113]:

$$\begin{aligned} K_I &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{22}(r, 0) \\ K_{II} &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{21}(r, 0) . \\ K_{III} &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{23}(r, 0) \end{aligned} \quad (2.18)$$

Irwin [112] discovered that SIFs were related to the energy release rate as follows:

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}, \quad (2.19)$$

where E' is given by E or $E/(1-\nu^2)$ under the plane-stress or plane-strain condition, respectively.

Fig. 2.3. Three different types of fracture modes [114].

LEFM is typically suitable for the linear elastic and small-scale plastic deformation analysis, and EPFM is capable of describing the significant nonlinear deformation such as yielding. In EPFM, the crack tip opening displacement (CTOD) and the J-integral are utilized to describe the crack-tip behaviors. CTOD was proposed by Wells [115] to measure the fracture toughness and has been calculated by analytical and experimental methods [116, 117]. The J-integral proposed by Rice [117] is expressed as a line-independent integral:

$$J = \int_{\Gamma} \left(w dy - T_i \frac{\partial u_i}{\partial x} ds \right), \quad (2.20)$$

where ds is the increment along Γ , u_i is the displacement, w is the strain energy density, and T_i is the traction.

Structures are generally subjected to combined loading including the tension, shear and out-of-plane torsion, resulting in the mixed-mode fracture. The crack propagation direction in the mixed-mode fracture can be determined by three criteria, including the maximum principal stress criterion [118], the maximum energy release rate criterion [119] and the minimum strain energy density criterion [120]. The first criterion states that a crack grows along the angle normal to the maximum principal

stress. In the second criterion, the crack propagation occurs along the angle where the maximum energy release rate is obtained. The equivalence between the first and the second criterion was introduced in [119]. In the third criterion, the crack propagation occurs along the angle where the minimum strain energy density is obtained.

2.4.2. Fracture modeling of 2D plane structures

Fracture in 2D plane structures was initially investigated using the analytical methods. For example, Inglis [121] first investigated the stress fields around corners or holes and discovered the stress concentration phenomenon. Westergaard [122] studied the problem of an infinitely extended plane structure with a center crack and developed the formulations of near-tip stress fields. William [123] investigated different types of stress distributions of a stationary crack. Analytical methods, however, show their limitations when being applied to the problems with complex geometries or boundary conditions.

Numerical methods exhibit the accuracy and feasibility in solving fracture problems as compared with analytical ones. The FEM has been used for the fracture simulations of engineering structures. For example, Ngo and Scordelis [109] used the FEM to study the crack patterns of a reinforced concrete structure. Anderson *et al.* [124] proposed a 2D finite element formulation to study the fracture behavior of rocket motors. To describe the stress singularities at crack tips, different types of finite elements have been developed. Byskov *et al.* [125] proposed the four-node triangular cracked elements containing a linear crack. Henshell *et al.* [126] found that the quarter-point method could achieve the singularity without special crack-tip elements. The quarter-point method requires remeshing to ensure the edges of elements align with crack surfaces and identifying positions of mid-side nodes during crack propagation. The requirement can be met by the automatic remeshing algorithm

proposed by Ingraffea and Saouma [127]. However, the algorithm leads to computationally inefficient solutions due to the ever-changing crack geometries.

In order to minimize remeshing, Belytschko and Black [12] proposed the XFEM by introducing enrichment functions into the finite element approximations and modeled arbitrary cracks within elements via the XFEM. Moes and Belytschko [128] further developed the formulations by incorporating the Heaviside function for the discontinuous displacement across crack surfaces and the near-tip asymptotic functions for the singular stress at crack tips. The formulations use the level set approach [129] to explicitly represent crack geometries that are independent of element edges.

In the XFEM, the finite element approximations are enriched by integrating discontinuous enrichment functions as follows [128]:

$$u(\mathbf{x}) = \sum_{i=1}^n N_i(\mathbf{x}) \left(u_i + \sum_{j=1}^{n_E(i)} A_{ji} F_j(r, \theta) \right), \quad (2.21)$$

where N_i is the finite element shape function, and A_{ji} is the parameter related to the i th node. The parameter $n_E(i)$ equals 4 when all nodes are within the support domain of crack tips and zero otherwise. $F_j(r, \theta)$ is the in-plane asymptotic crack-tip enrichment function. The displacement approximation is obtained by introducing the discontinuous functions and the enrichment functions as follows:

$$u(\mathbf{x}) = \sum N_i(\mathbf{x}) u_i + \sum b_j N_j(\mathbf{x}) H(\mathbf{x}) + \sum N_k(\mathbf{x}) \left(\sum_{l=1}^4 c_k^l F_l(\mathbf{x}) \right) \quad (2.22)$$

with

$$H(\mathbf{x}) = \begin{cases} 1 & (\mathbf{x} - \mathbf{x}') \cdot \mathbf{e}_n > 0 \\ -1 & (\mathbf{x} - \mathbf{x}') \cdot \mathbf{e}_n < 0 \end{cases}, \quad (2.23)$$

where $\mathbf{x}'$ is the point obtained by projecting a point $\mathbf{x}$ to the crack surface, and $\mathbf{e}_n$ is the norm vector.

XFEM has been applied to the analysis of cracks due to its ability to readily handle the discontinuities. For example, Dolbow *et al.* [130] introduced the contact-friction law into a fracture model to investigate frictional contact for crack propagation problems. Belytschko *et al.* [131] proposed a dynamic fracture model within the framework of the XFEM for rate-independent materials. Réthoré *et al.* [132] investigated both stationary and propagating cracks via a general fracture model that satisfied energy conservation conditions. Remmers *et al.* [133] proposed a model that treated cracks as a set of overlapping cohesive segments to simulate crack nucleation and coalescence. Budyn *et al.* [134] developed a model for investigating multiple propagating cracks with the consideration of the interactions among cracks and micro-defects. Recent applications of the XFEM to engineering structures have been introduced [135, 136].

Meshfree methods have been proposed to investigate the stationary and propagating cracks because of the following advantages: (i) remeshing processes are avoided; (ii) local mesh refinement around crack tips can be implemented by directly adding meshfree nodes without considering the nodal connectivity. Belytschko *et al.* [12, 137, 138] first used the EFGM to study 2D crack propagation and dynamic crack problems. Rabczuk *et al.* [139] developed the cracking particle method (CPM) for the 2D crack growth analysis, in which a set of discontinuous segments was used to model arbitrary crack geometries. To improve the computational efficiency, the meshfree point collocation method that required no Gauss integration was proposed for 2D elastic crack problems [140]. Moreover, the AMR strategy was incorporated into the framework of the EFGM for investigating crack propagation problems, in which a higher convergence rate was obtained than the global uniform refinement [141].

In addition to the XFEM and meshfree methods, IGA has been developed for the modeling of fracture in structures. For example, Verhoosel *et al.* [142] proposed the knot insertion strategy to model the discontinuity in plane structures. However, the strategy suffers from distorted meshes in modeling complex crack geometries and fracture behaviors. De Luycker *et al.* [143] incorporated enrichment functions into the NURBS-based isogeometric formulations by using the partition of unity. The enriched formulations were capable of conveniently describing crack topologies and giving a higher convergence rate and computational efficiency than the XFEM. Later, Ghorashi *et al.* [16] developed the partition-of-unity-based XIGA, in which both the enrichment functions and Heaviside functions were incorporated into the standard IGA approximations, and investigated the 2D quasi-static crack propagation problems via XIGA.

2.4.3. Fracture modeling of shell structures

(a) Shell theory

Shell-like structures, in which the thickness is remarkably smaller than other parameters such as the radius of curvature, have been used in numerous applications, such as turbine disks and aircraft, due to their good load-carrying capacity and lightweight property. Based on the value of t/R with t and R respectively being the thickness and radius, shells can be classified into two different types: thin shells ($t/R \leq 0.05$) and thick shells ($t/R > 0.05$). Shell theories and their numerical implementations have been reviewed by Bischoff *et al.* [144]. The 3D theory of elasticity that is a natural way to model shells was first proposed by Ericksen and Truesdell [145] and then monographed by Naghdi [146]. The 3D model, however, has issues such as the

low numerical efficiency and accuracy that are caused by a large number of elements and their high aspect ratios.

To avoid these limitations, the 3D theory has been simplified into two widely used 2D theories including the KL [6] and MR [7] theories, in which a solid shell is modeled by the combination of its mid-surface and director. A shell geometry represented by the mid-surfaces in the reference configuration and the deformed configuration is visualized in Fig. 2.4.

Fig. 2.4. Mid-surfaces of a shell in two configurations [114].

In the KL theory, both transverse shear strains and the change of curvature are neglected, thus avoiding the shear locking phenomenon. The KL theory is typically applied to very thin shells, in which the value of t/R is negligible as compared with unity. As the KL theory gives the fourth-order governing PDEs, the C^1 continuity of displacements needs to be satisfied in the Galerkin weak formulations [8]. This requirement poses an obstacle to the FEM-based numerical framework, in which the higher-order interpolations are not convenient to construct by the traditional Lagrange polynomials.

In order to satisfy the requirement of the C^1 continuity, various methods have been proposed to formulate C^1 conforming thin-shell finite elements. For example, rectangular shell elements constructed with bicubic Hermite polynomials and triangular shell elements with quintic polynomials were introduced to interpolate field variables, whereas the computational cost was rather high [147]. Other shell finite elements such as reduced-integration Lagrange elements [148], triangular elements using the subdivision surface for the non-local interpolation [149] and quadrilateral elements satisfying the KL hypothesis at discrete nodes [8] have been developed.

Similar to the KL theory, the MR theory assumes that the vector perpendicular to middle surfaces remains straight in the deformed configuration except for the consideration of shear deformations in the thickness. The MR theory requires only the C^0 continuity of displacements [150], which is convenient to implement in the FEM-based framework as compared to the KL theory. However, MR theory has two major limitations. First, the computational effort is increased due to the consideration of the rotational degrees of freedom (DOFs). Second, the MR theory suffers from the shear locking phenomenon in dealing with very thin shells. Numerous strategies have been proposed to overcome these locking issues. The intuitive way is to increase the order of polynomial shape functions [151, 152], whereas the method significantly increases the computational cost when very higher-order approximations are used. The reduced integration methods, which decrease the total number of Gauss points related to the shear parts, have been proposed [153, 154]. Alternatives such as the assumed strain method [155], B-bar projection method [156] and adjusted approximation spaces [157] have been introduced to solve the locking issues.

In addition to the FEM, both the meshfree methods and IGA, which can construct smooth and arbitrary higher-order approximations, have been introduced for the

numerical implementations of the KL and MR theories. For example, the EFGM was proposed to study the cracking behaviors of shell structures in [42]. Shell formulations were developed by using the NURBS basis functions [15]. Recent developments of the shell theories based on the IGA and meshfree methods have been introduced [158, 159].

(b) Fracture modeling of shells

FEM has been developed for modeling fractures in shells [160-164]. For example, a FEM-based numerical framework was developed to study crack propagation in pressurized aircraft fuselage structures [160]. In the framework, through-the-thickness cracks were analyzed with the consideration of the geometrically nonlinear stiffening effect, and their fracture parameters including the SIFs and crack growth direction were calculated. As the XFEM is more flexible and efficient than FEM in the fracture modeling, Areias *et al.* [161] proposed the XFEM-based numerical framework to investigate arbitrary crack growth in thick-shell structures. In the framework, the enhanced assumed strain formulation was utilized to tackle the locking issue. Later, Areias *et al.* [162] incorporated the KL theory into the XFEM-based numerical framework and conducted the elastoplastic analysis of crack propagation.

Meshfree methods that require no remeshing in the fracture modeling have been utilized for modeling shell structures with cracks. For example, Rabczuk *et al.* [165] investigated multiple crack propagation in thin shells based on a meshfree numerical framework, which treated cracks as a set of cracked particles and could flexibly simulate crack branching. Rabczuk *et al.* [166] further extended the numerical framework to investigate nonlinear dynamic crack propagation in thin shells. Qian *et al.* [167] analyzed the nonlinear response of cylindrical shell structures based on the viscoplastic theory. To implement the adaptive local refinement for crack problems,

Lee *et al.* [141] proposed an error indicator within the framework of the EFGM and investigated crack propagation behaviors. Applications of the meshfree methods and their developments for the fracture modeling can be referred to [40, 150, 168].

XIGA has been utilized for modeling fracture behaviors of shells [169-171]. For example, Ghorashi *et al.* [170] proposed a numerical framework based on the XIGA for the investigations of curved cracks. The framework used sub-triangles techniques for integration over the elements that are intersected by a crack. Nguyen-Thanh *et al.* [114] developed the XIGA-based numerical framework to study fracture parameters of shells. Yang *et al.* [171] developed an AMR scheme based on PHT-splines to investigate the fracture behaviors of shell structures. In addition, the IGA-meshfree coupling approach has been developed to retain the advantages, such as the refinement flexibility and higher-order approximation, and it has been utilized for investigating fracture behaviors of shells [69, 172]. Recently, this coupling approach was further improved by introducing a unified formulation throughout the problem domain without requiring the domain subdivision and applied to study crack growth in shell structures [173].

2.5. Phase-field modeling of fracture

The PFM, which is classified as a typical kind of diffusive model [110], exhibits advantages in modeling complex fracture behaviors as compared to the discrete crack model. The PFM has been developed in both the physics and mechanics communities. In the physics community, the PFM was developed within the context of the Landau-Ginzburg theory, and the model in the mechanics community was developed within the framework of the variational theory. Herein, the variational-based PFM and its applications are reviewed.

The PFM was first proposed by Francfort and Marigo [174] to simulate crack growth in brittle materials based on the variational principle. Subsequently, Bourdin *et al.* [175] improved the variational formulation by regularizing sharp crack surfaces with both a phase-field variable and a length-scale parameter for convenient numerical implementations. In the variational formulation, the total energy functional including both the bulk energy and the crack surface energy is minimized to obtain a set of PDEs that automatically governs the fracture behaviors such as crack initiation, propagation and branching without tracking crack topologies. The coupled PDEs can be solved by a monolithic scheme [176] or a staggered scheme [177], in which the displacement and phase-field variables are calculated simultaneously or sequentially, respectively. To avoid unrealistic fracture behaviors under compression loading, Amor *et al.* [178] improved the PFM by decomposing a strain tensor into volumetric and deviatoric components. Miehe *et al.* [179] decomposed a strain tensor into positive and negative components and further introduced a history field of the maximum positive strain energy to enforce irreversibility conditions. Borden *et al.* [111] proposed a higher-order PFM for obtaining the exact regularized representations of sharp crack topologies and increasing the convergence rate.

In addition, the PFM has been developed for ductile fracture [180-184]. For example, Borden [180] developed a PFM for large deformation ductile fracture problems. Ulmer *et al.* [181] developed a formulation to investigate ductile fracture in elastoplastic solids within a thermodynamically consistent framework. The formulation was capable of predicting the brittle-to-ductile failure mode transition under dynamic loading. Ambati *et al.* [182] investigated ductile fracture in elastoplastic solids based on a PFM that included the coupling effect between the degradation function and the plastic strain state.

Numerical implementations of the PFM have been investigated within the framework of FEM [177, 179]. However, the PFM suffers from high computational cost because a fine mesh is required around crack surfaces. A locally pre-refined mesh can improve computational efficiency, whereas crack propagation paths are unknown. AMR is a good alternative because it can automatically refine mesh without requiring the crack growth paths to be known *a priori*. Until now, numerous AMR schemes have been developed via the FEM [185-188]. For example, an adaptive refinement strategy combined with a goal-oriented error estimator was proposed for the phase-field modeling of a two-phase system [185]. To allow the mesh to grow without requiring *a priori* mesh refinement, a predictor-corrector mesh adaptivity scheme was developed, and a primal-dual active set strategy was used to enforce irreversibility conditions [186].

In addition to FEM, the meshfree methods and IGA have been developed for the phase-field fracture modeling. For example, Rosolen *et al.* investigated the brittle fracture via a PFM within the numerical framework of the LME meshfree method. Shao *et al.* [189] developed an EFGM-based algorithm to investigate brittle fracture in both 2D and 3D problems. Moreover, Borden *et al.* [190] developed a second-order PFM for dynamic fracture based on IGA, in which T-splines basis functions were introduced to discretize the domain and to approximate displacement fields. Later, Borden *et al.* [111] proposed a fourth-order model that utilized higher-order basis functions and generated higher regularity in the phase-field solutions. To obtain computationally efficient solutions, AMR strategies using T-splines [190], PHT-Splines [191] and THB-splines [13] have been investigated in 2D and 3D phase-field fracture simulations.

2.6. Summary

This chapter firstly reviews three numerical methods, including the FEM, meshfree methods and IGA. Secondly, the discrete fracture modeling of 2D plane and thin-shell structures is elaborated. Finally, the phase-field modeling of both brittle and ductile fracture is presented. The discrete model treats a crack as a discontinuity by remeshing or partition-of-unity-based schemes, and the PFM treats a cracked body as a continuum. Numerical implementations of both the discrete and phase-field fracture modeling are discussed in detail.

Chapter 3 An IGA-Meshfree Coupling Approach for Linear and Geometrically Nonlinear Analyses of Shells

An IGA-meshfree coupling approach is developed to investigate linear and geometrically nonlinear deformation behaviors of shells. The coupling approach divides a problem domain into three subdomains, in which the isogeometric subdomain ensures the geometry exactness, the meshfree one achieves the local refinement, and the coupling one ensures the smooth transition between the other two subdomains. The KL shell theory is employed without the consideration of rotational DOFs. The coupling approach can simulate the linear and nonlinear behaviors of shells with a higher convergence rate than the IGA and meshfree methods because of the realization of local refinement.

3.1. Problem description and solution approach

3.1.1. Equilibrium equations of thin shells

The KL theory is utilized for investigating the deformation behaviors of thin shells, and their kinematics are introduced in Appendix A. For shells, the total sum of the internal and external work equals zero:

$$\delta W = \delta W_{\text{int}} + \delta W_{\text{ext}} = 0 \quad (3.1)$$

with δW_{int} being as

$$\delta W_{\text{int}} = - \int_V \delta \mathbf{E} : \mathbf{S} dV. \quad (3.2)$$

The integration along the thickness gives the δW_{int} as follows [192]:

$$- \int_V \delta \mathbf{E} : \mathbf{S} dV = - \int_A (\mathbf{n} : \delta \boldsymbol{\varepsilon} + \mathbf{m} : \delta \boldsymbol{\kappa}) dA, \quad (3.3)$$

where the membrane stress $\mathbf{n} = t\mathbf{C}\boldsymbol{\varepsilon}$ and the bending stress $\mathbf{m} = t^3/12\mathbf{C}\boldsymbol{\kappa}$ are expressed

by the Voigt's notation, and dA denotes the area of a mid-surface given as

$$dA = \bar{j}_0 d\xi^1 d\xi^2, \quad \bar{j}_0 = \|\mathbf{A}_1 \times \mathbf{A}_2\| \quad (3.4)$$

with

$$\mathbf{C} = \frac{E}{1-\nu^2} \begin{bmatrix} (A^{11})^2 & \nu A^{11} A^{22} + (1-\nu)(A^{12})^2 & A^{11} A^{12} \\ \vdots & (A^{22})^2 & A^{22} A^{12} \\ \text{Sym.} & \dots & \frac{1}{2}[(1-\nu)A^{11} A^{22} + (1+\nu)(A^{12})^2] \end{bmatrix}, \quad (3.5)$$

where $A^{a\beta}$ is the contravariant metric coefficient.

By taking the first derivative of Eq. (3.1), the residual force is obtained as

$$\mathbf{R}_r = \left(\frac{\partial W_{\text{int}}}{\partial u_r} + \frac{\partial W_{\text{ext}}}{\partial u_r} \right) = F_r^{\text{int}} + F_r^{\text{ext}}. \quad (3.6)$$

The second derivative yields the internal stiffness K_{rs}^{int} :

$$K_{rs}^{\text{int}} = \int_A \left(\frac{\partial \mathbf{n}}{\partial u_s} : \frac{\partial \boldsymbol{\varepsilon}}{\partial u_r} + \mathbf{n} : \frac{\partial^2 \boldsymbol{\varepsilon}}{\partial u_r \partial u_s} + \frac{\partial \mathbf{m}}{\partial u_s} : \frac{\partial \boldsymbol{\kappa}}{\partial u_r} + \mathbf{m} : \frac{\partial^2 \boldsymbol{\kappa}}{\partial u_r \partial u_s} \right) dA. \quad (3.7)$$

For the linear deformation, the stiffness matrix can be reduced as

$$\bar{K}_{rs}^{\text{int}} = \int_A \left(\frac{\partial \mathbf{n}}{\partial u_s} : \frac{\partial \boldsymbol{\varepsilon}}{\partial u_r} + \frac{\partial \mathbf{m}}{\partial u_s} : \frac{\partial \boldsymbol{\kappa}}{\partial u_r} \right) dA. \quad (3.8)$$

3.1.2. IGA-meshfree coupling approach

In the parametric domain, the isogeometric and MLS shape functions are coupled based on the reproducing conditions. The parametric domain includes the IGA subdomain $\mathcal{Q}_\xi^{\text{IGA}}$, the meshfree subdomain $\mathcal{Q}_\xi^{\text{MF}}$ and the coupling subdomain $\mathcal{Q}_\xi^{\text{CP}}$, and their corresponding physical subdomains are represented by $\mathcal{Q}^{\text{IGA}}$, $\mathcal{Q}^{\text{MF}}$ and $\mathcal{Q}^{\text{CP}}$, respectively. The IGA subdomain is responsible for the outer part that contains the

boundary, and the meshfree subdomain is accountable for the interior part with the implementation of local mesh refinement.

The displacement $u_I(\mathbf{x})$ can be approximated based on the coupling approach as follows:

$$u_I(\mathbf{x}) = \sum_{I=1}^{N_S} \Phi_I(\xi) u_I, \quad (3.9)$$

where u_I denotes the coefficient related to control points, N_S denotes the number of basis functions, and $\Phi_I(\xi)$ is the basis function given as

$$\Phi_I(\xi) = \begin{cases} R_I(\xi) & \xi_I \in \Omega_\xi^{\text{IGA}} \\ R_I(\xi) + \Psi_I^{\text{CP}}(\xi) & \xi_I \in \Omega_\xi^{\text{CP}} \\ \Psi_I(\xi) & \xi_I \in \Omega_\xi^{\text{MF}} \end{cases}, \quad (3.10)$$

where $\Psi_I(\xi)$ is the MLS shape function introduced in Appendix B, and $R_I(\xi)$ is the NURBS basis function.

The coupling shape function $\Psi_j^{\text{CP}}(\xi)$ can be obtained based on the consistency conditions expressed in Eqs. (B.12) and (C.6). The NURBS basis function is treated as a constant in the coupling subdomain. The p th order consistency condition is defined as

$$\sum_{I=1}^{N_S} \Phi_I(\xi) \mathbf{p}(\xi_I^{[l]}) = \sum_{J=1}^{N_M} \Psi_J^{\text{CP}}(\xi) \mathbf{p}(\xi_J^{[l]}) + \sum_{I=1}^{N_B} R_I(\xi) \mathbf{p}(\xi_I^{[l]}) = \mathbf{p}(\xi), \quad (3.11)$$

where $\mathbf{p}(\xi_I^{[l]})$ is the reproducing point vector defined in Eq. (C.7).

The meshfree shape function based on the MLS approximation is expressed as

$$\Psi_j^{\text{CP}}(\xi) = \mathbf{p}^T(\xi_j^{[l]}) \mathbf{c}(\xi) \widehat{W}(\xi - \xi_j^{[l]}). \quad (3.12)$$

The following expression can be obtained by substituting Eq. (3.12) for the meshfree shape functions in Eq. (3.11):

$$\mathbf{c}(\xi) = \bar{\mathbf{C}}^{-1}(\xi) [\mathbf{p}(\xi) - \mathbf{q}(\xi)], \quad (3.13)$$

with

$$\bar{\mathbf{C}}(\xi) = \sum_{J=1}^{N_M} \mathbf{p}(\xi_J^{[I]}) \mathbf{p}^T(\xi_J^{[I]}) \hat{W}(\xi - \xi_J^{[I]}), \quad (3.14)$$

$$\mathbf{q}(\xi) = \sum_{I=1}^{N_B} R_I(\xi) \mathbf{p}(\xi_I^{[I]}). \quad (3.15)$$

The meshfree shape function $\Psi_J^{CP}(\xi)$ in the coupling subdomain is obtained as

$$\Psi_J^{CP}(\xi) = \mathbf{p}^T(\xi_J^{[I]}) \bar{\mathbf{C}}^{-1}(\xi) [\mathbf{p}(\xi) - \mathbf{q}(\xi)] \hat{W}(\xi - \xi_J^{[I]}). \quad (3.16)$$

A 2D problem is provided herein to demonstrate the modeling steps. Two knot vectors $\Xi_1 = \Xi_2 = \{0, 0, 0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1, 1, 1\}$ are defined to construct NURBS basis functions. Based on Eq. (C.4), the linear and quadratic reproducing points of two directions are calculated as

$$\xi^{[1]} = \eta^{[1]} = \left\{0, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}, 1\right\}, \quad (3.17)$$

$$\xi^{[2]} = \eta^{[2]} = \left\{0, 0, \sqrt{\frac{1}{32}}, \sqrt{\frac{3}{32}}, \sqrt{\frac{3}{16}}, \sqrt{\frac{5}{16}}, \sqrt{\frac{15}{32}}, \sqrt{\frac{21}{32}}, \sqrt{\frac{7}{8}}, 1\right\}. \quad (3.18)$$

The shape functions can be obtained according to Eq. (3.10), as shown in Fig. 3.1. It is noted that the smooth transition is achieved among the IGA subdomain, coupling subdomain and meshfree subdomain. Fig. 3.2 shows the discretization of a circle in both the parametric and physical domains.

This coupling approach retains the local refinement flexibility of the meshfree method and implements local refinement within the meshfree subdomain. Taking the local refinement strategy in 2D as an example: each background mesh is subdivided into 4 submeshes, and 5 nodes are inserted. Further mesh refinement can be treated in the same way until the stopping criterion is met.

Fig. 3.1. Quadratic shape functions of the 2D coupling approach.

Fig. 3.2. The 2D coupling approximation: (a) the parametric and (b) physical domains.

Fig. 3.3 shows the refinement processes of the first two steps. The initial mesh is subdivided into four meshes in the first step of refinement. The mesh subdivision is further implemented at the top-right mesh. It is noted that the original nodes and newly inserted nodes are denoted by the solid circle and hollow circle, respectively. The new support sizes are calculated for the inserted nodes to guarantee the smooth transition between the coarse and fine meshes [193].

Fig. 3.3. The mesh refinement scheme: (a) an initial mesh, (b) the mesh in the 1st refinement step and (c) the mesh in the 2nd refinement step.

3.1.3. Numerical implementation based on the coupling approach

The mid-surfaces in the reference and deformed configurations are defined in the same way as the displacement represented by the shape function $\Phi_I(\xi)$ in Eq. (3.10) as follows:

$$\boldsymbol{\varphi}(\xi^1, \xi^2) = \sum_{I=1}^{N_S} \Phi_I(\xi) p_I^* \quad \text{and} \quad \boldsymbol{\varphi}_0(\xi^1, \xi^2) = \sum_{I=1}^{N_S} \Phi_I(\xi) p_I \quad (3.19)$$

with the control points p_I and p_I^* being in the reference and deformed configurations, respectively.

Based on linearized kinematics, the membrane strain $\boldsymbol{\varepsilon}$ and the bending strain $\boldsymbol{\kappa}$ are obtained as follows [149]:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \sum_{I=1}^{N_S} \mathbf{B}_n^I u_I \quad \text{and} \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{bmatrix} = \sum_{I=1}^{N_S} \mathbf{B}_m^I u_I, \quad (3.20)$$

where $\mathbf{B}_n^I$ and $\mathbf{B}_m^I$ are obtained as

$$\mathbf{B}_n^I = \begin{bmatrix} (\mathbf{A}_1 \Phi_{I,1})^T \\ (\mathbf{A}_2 \Phi_{I,2})^T \\ (\mathbf{A}_1 \Phi_{I,2} + \mathbf{A}_2 \Phi_{I,1})^T \end{bmatrix} \quad \text{and} \quad \mathbf{B}_m^I = \begin{bmatrix} (\mathbf{b}_{m11}^I)^T \\ (\mathbf{b}_{m22}^I)^T \\ (2\mathbf{b}_{m12}^I)^T \end{bmatrix} \quad (3.21)$$

with

$$\begin{aligned} \mathbf{b}_{\alpha\beta}^I = & \mathbf{A}_{\alpha\beta} \cdot \mathbf{A}_3 \frac{1}{\bar{j}_0} \left[\Phi_{I,1} (\mathbf{A}_2 \times \mathbf{A}_3) + \Phi_{I,2} (\mathbf{A}_3 \times \mathbf{A}_1) \right] \\ & - \frac{1}{\bar{j}_0} \left[\Phi_{I,1} (\mathbf{A}_2 \times \mathbf{A}_{\alpha\beta}) + \Phi_{I,2} (\mathbf{A}_{\alpha\beta} \times \mathbf{A}_1) \right] - \Phi_{I,\alpha\beta} \mathbf{A}_3 \end{aligned} \quad (3.22)$$

The global stiffness matrices are obtained by substituting strain and stress into Eqs. (3.7) and (3.8). The linearized systems are given as

$$\mathbf{K}_{rs}^{\text{int}} \Delta \mathbf{u} = \mathbf{R}, \quad (3.23)$$

$$\bar{\mathbf{K}}_{rs}^{\text{int}} \mathbf{u} = \mathbf{F}^{\text{ext}}. \quad (3.24)$$

The nonlinear deformation $\Delta \mathbf{u}$ is obtained based on the Newton-Raphson method [194], and the linear deformation can be directly calculated by Eq. (3.24).

3.2. Linear analysis of shells

The coupling approach is first applied to solve thin-shell benchmark problems that include a pinched cylindrical shell with rigid diaphragms, a Scordelis–Lo roof and a hemispherical shell [195]. In the coupling approach, the quartic spline weight function is used with support sizes 2, 3 and 4 for the quadratic, cubic and quartic approximations, respectively. The 4×4 Gauss quadrature scheme is utilized in the coupling approximations. Local refinement is implemented by directly inserting meshfree nodes around the point loadings where stress concentrations occur.

The errors in the displacement and strain energy that are respectively denoted by err_{disp} and err_{ener} are calculated as

$$err_{\text{disp}} = \frac{\sqrt{\int_{\Omega} (\mathbf{u} - \mathbf{u}_{\text{ect}}) \cdot (\mathbf{u} - \mathbf{u}_{\text{ect}}) d\Omega}}{\sqrt{\int_{\Omega} \mathbf{u}_{\text{ect}} \cdot \mathbf{u}_{\text{ect}} d\Omega}}, \quad (3.25)$$

$$err_{\text{ener}} = \frac{\sqrt{\int_{\Omega} \mathbf{E}^T (\mathbf{u} - \mathbf{u}_{\text{ect}}) : \mathbf{C} : \mathbf{E} (\mathbf{u} - \mathbf{u}_{\text{ect}}) d\Omega}}{\sqrt{\int_{\Omega} \mathbf{E}^T (\mathbf{u}_{\text{ect}}) : \mathbf{C} : \mathbf{E} (\mathbf{u}_{\text{ect}}) d\Omega}}, \quad (3.26)$$

where $\mathbf{u}_{\text{ect}}$ are the estimated exact displacements. Since there are no exact results for these shell problems, the higher-order NURBS approximation ($p = q = 9$) is applied to calculate $\mathbf{u}_{\text{ect}}$ and estimate the error norms.

3.2.1. A pinched cylindrical shell

This example deals with a cylindrical shell, which is supported by fixed diaphragms and is subjected to two opposite point loadings $P = 1$ N (Fig. 3.4(a)). The geometry parameters include the radius $R = 300$ mm, the length $L = 600$ mm and the thickness $t = 1$ mm. The material properties are $E = 3 \times 10^6$ N/mm² and $\nu = 0.3$. One-eighth of the cylindrical shell is modeled because of the symmetry (Fig. 3.4(b)).

Fig. 3.4. The pinched cylindrical shell with rigid diaphragms: (a) the schematic configuration and (b) the locally refined mesh in the physical domain.

Fig. 3.5(a) shows the contour plots of displacements in the deformed configuration with the scaling factor of 1×10^7 . The results converge to the analytical

solution 1.8248×10^{-5} mm as global refinement is conducted [195]. The errors in the strain energy using the quadratic shape function are shown in Fig. 3.5(b). The coupling approach realizes the exact boundaries and meanwhile converges faster than the meshfree method. Moreover, the coupling approach with local refinement produces a higher convergence rate than global refinement.

Fig. 3.5. The pinched cylindrical shell: (a) the contour plots of displacements in the deformed configuration (scaling factor of 1×10^7) and (b) the errors in the strain energy.

3.2.2. A Scordelis–Lo roof shell

This example deals with a Scordelis–Lo roof, which is a section of the cylindrical shell supported by rigid diaphragms (Fig. 3.6(a)). The roof is subjected to the uniform gravity load $P = 90$ N. The geometry parameters include the length $L = 50$ mm, the radius $R = 25$ mm and the thickness $t = 0.25$ mm. The material properties are $E = 4.32 \times 10^8$ N/mm² and $\nu = 0.0$.

Fig. 3.7(a) shows the contour plots of displacements in the deformed configuration with a scaling factor of 10. Fig. 3.7(b) shows the errors in the displacement using the quadratic approximation. The coupling approach with local

refinement achieves a higher convergence rate than global refinement. Moreover, the coupling approach (with local refinement) performs better than the IGA method (with global refinement) when the meshes are refined. The force and moment resultants are shown in Fig. 3.8.

Fig. 3.6. The Scordelis-Lo roof: (a) the schematic configuration and (b) the mesh with local refinement.

Fig. 3.7. The Scordelis-Lo roof: (a) the contour plots of displacements in the deformed configuration (scaling factor of 10) and (b) the errors in the displacement.

Fig. 3.8. The contour plots of resultants.

3.2.3. A hemispherical shell

This example deals with a hemispherical shell subjected to two opposite point forces $F = 2$ N (Fig. 3.9(a)). The geometry parameters are the radius $R = 10$ mm and the thickness $t = 0.04$ mm. The material properties include $E = 6.825 \times 10^7$ N/mm² and $\nu = 0.3$. The locally refined meshes for one-quarter of the hemispherical shell are plotted in the physical domain (Fig. 3.9(b)).

Fig. 3.9. Schematics of the hemispherical shell: (a) one-quarter of the geometry and (b) the locally refined meshes.

Fig. 3.10(a) shows the contour plots of displacements in the deformed configuration. Fig. 3.10(b) shows the errors in the strain energy using the quadratic approximation. The coupling approach (with global refinement) achieves the same convergence rate as the IGA method (with global refinement) when meshes are refined. Furthermore, the coupling approach with local refinement converges faster than global refinement.

Fig. 3.10. The hemispherical shell: (a) the contour plots of displacements in the deformed configuration (scaling factor of 30) and (b) the errors in the strain energy.

3.3. Geometrically nonlinear analysis of shells

The benchmark problems including a cantilever plate, a hemispherical shell, a pinched cylindrical shell and an open-ended cylindrical shell are analyzed by the coupling approach. The coupling approximation regards the outer part of geometry as the IGA subdomain and the interior part of the geometry as the meshfree subdomain.

3.3.1. A cantilever plate

A cantilever plate subjected to the shear force $P = 4$ N is considered, as visualized in Fig. 3.11(a). The geometry parameters include the length $L = 10$ mm, the thickness $t = 0.1$ mm and the width $b = 1$ mm. The material properties include $E = 1.2 \times 10^6$ N/mm² and $\nu = 0.3$. The quadratic basis function is used to approximate the displacement variables. The shear force is equally applied in 40 load steps.

The contour plots of displacements at different load steps are visualized in Fig. 3.11(b). Fig. 3.12 shows the load-deflection curves of the cantilever plate. The

numerical results are consistent with the reference solutions [196], demonstrating the accuracy of the geometrically nonlinear analysis with the coupling approach.

Fig. 3.11. The cantilever plate: (a) the geometry and (b) contour plots of displacements at several load steps.

Fig. 3.12. The load-deflection curves of a cantilever plate.

3.3.2. A hemispherical shell

This example deals with a hemispherical shell subjected to two opposite forces $P = 400$ N (Fig. 3.14(a)). The geometry parameters include the radius $R = 10$ mm and the thickness $t = 0.04$ mm. The material properties include $E = 6.825 \times 10^7$ N/mm² and $\nu = 0.3$. The quadratic and cubic basis functions are adopted for the coupling approximation, and 40 equal load steps are applied herein.

Fig. 3.13 visualizes the load-deflection curves of the hemispherical shell, which are consistent with reference solutions [196]. Furthermore, the cubic basis function converges faster than the quadratic basis function, whereas it is more time-consuming. The results presented by Duong *et al.* [197] show the same influence of different-order basis functions as present results. Fig. 3.14 shows the contour plots of displacements in the deformed configurations.

Fig. 3.13. The load-deflection curves of a hemispherical shell.

Fig. 3.14. The hemispherical shell: (a) one-quarter of the geometry; the contour plots of displacements in the deformed configurations at the load steps (b) 1, (c) 10 and (d) 40.

3.3.3. A pinched cylindrical shell

A pinched cylindrical shell subjected to the point loading $P = 1.2 \times 10^4$ N is considered. The geometry parameters are the radius $R = 200$ mm, the length $L = 200$ mm and the thickness $t = 1$ mm. The material properties include $E = 3 \times 10^4$ N/mm² and $\nu = 0.3$. The quadratic basis function is employed to approximate the displacement field, and 40 load steps are equally applied.

Fig. 3.15 visualizes the deformed configurations at different load steps. Fig. 3.16 shows the load-deflection curves of the cylindrical shell, which are consistent with the reference solutions [196].

Fig. 3.15. The contour plots of displacements in the deformed configurations at the load steps (a) 1, (b) 5, (c) 30 and (d) 40.

Fig. 3.16. The load-deflection curves of a cylindrical shell.

3.3.4. An open-ended cylindrical shell

An open-ended cylindrical shell subjected to the radial force $P = 4 \times 10^4$ N is shown in Fig. 3.17(a). The geometry parameters are the radius $R = 4.953$ mm, length $L = 10.35$ mm and thickness $t = 0.094$ mm. The material properties include $E = 10.5 \times 10^6$ N/mm² and $\nu = 0.3125$. One-eighth of the shell is modeled because of the symmetry. The quadratic basis function and 20 equal load steps are applied.

The deformed cylindrical shells at different load steps are visualized in Fig. 3.17(b)-(d). Displacements at points A, B, and C are consistent with the reference solutions [196] (Fig. 3.18), verifying the accuracy of the coupling approach.

Fig. 3.17. The open-ended cylindrical shell: (a) the geometry; the contour plots of displacements in the deformed configurations at the load steps (b) 5, (c) 10 and (d) 20.

Fig. 3.18. The load-deflection curves of an open-ended cylindrical shell.

3.4. Summary

A coupling approach of the IGA and meshfree methods is developed to investigate the linear and nonlinear large deformation behaviors of thin-shell structures. The coupling approach utilizes three subdomains: the IGA subdomain to ensure exact geometry representation, the meshfree subdomain to obtain local refinement flexibility and the coupling subdomain to guarantee smoothness between the other two subdomains. The coupling approach can converge faster than the IGA and meshfree methods by implementing local refinement. The accuracy of this coupling approach is demonstrated by solving benchmark problems containing both linear and nonlinear deformation.

The coupling approach with global refinement can produce more accurate results, higher convergence rates and less computational cost than the meshfree method while achieving the same convergence rate as the IGA method. Furthermore, with the successful implementation of local refinement, the coupling approach can achieve higher convergence rates than the IGA and meshfree methods (with global refinement).

Chapter 4 A Strong Form-Based IGA-Meshfree MLS Approach for 2D Elastic Fracture Problems

In this chapter, a strong form-based IGA-meshfree MLS collocation (IMMLS-C) approach is developed for 2D linear elastic fracture problems with contact loading. In the IMMLS-C approach, the equivalence between MLS shape functions and isogeometric basis functions is established, which provides a local mesh refinement scheme for IGA in a meshfree manner. Strong formulations for fracture problems are derived within the framework of the IMMLS-C approach. Contact constraints are enforced by introducing a penalty algorithm to the strong formulations. The simulated fracture parameters, such as SIFs in 2D plane structures, agree with the reference solutions.

4.1. Problem description and solution approach

4.1.1. Equilibrium equations

Within the context of LEFM, the natural boundary conditions on crack surfaces are imposed to obtain the analytical solutions of fracture problems. Fig. 4.1 shows a cracked body with a red dashed line denoting a crack. The cracked body is governed by the following equations:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega, \quad (4.1)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t, \quad (4.2)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u, \quad (4.3)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{c+} = \mathbf{t}_{c+} \quad \text{on } \Gamma_{c+}, \quad (4.4)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{c-} = \mathbf{t}_{c-} \quad \text{on } \Gamma_{c-}. \quad (4.5)$$

The outward surface normal vector of the natural boundary is denoted by $\mathbf{n}$. Eqs. (4.4)

and (4.5) represent the traction-free boundary conditions along the crack surface Γ_c ($\Gamma_c = \Gamma_{c+} \cup \Gamma_{c-}$) with the unit normal vectors $\mathbf{n}_{c+}$ and $\mathbf{n}_{c-}$. The traction vectors $\mathbf{t}_{c+}$ and $\mathbf{t}_{c-}$ are considered to be zero.

Fig. 4.1. A cracked body in two dimensions: (a) the schematic configuration and (b) the discretization by collocation points.

The constitutive equation and strain-displacement relations are represented by

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\text{tr}(\boldsymbol{\varepsilon})\mathbf{I}, \quad (4.6)$$

$$\boldsymbol{\varepsilon} = \nabla_{\text{sym}} \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad (4.7)$$

where $\text{tr}(\boldsymbol{\varepsilon})$ is the trace of Cauchy's strain tensor. Under the plane-strain condition, the Lamé constants λ and μ are expressed by both E and ν as follows:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = G = \frac{E}{2(1+\nu)}. \quad (4.8)$$

The displacement discontinuity due to cracks can be modeled by several approaches, such as the visibility method [137], diffraction method [198], transparency method [198] and the method based on the extrinsic partition of unity enrichment [128, 199]. Herein, the visibility method is employed to model the displacement discontinuity. As shown in Fig. 4.2, when the influence domain of the point J is fully

cut by a crack surface, the nodes which are located on the same side of the crack surface as the point J are considered for field approximations. When the influence domain of a point is partially cut, the diffraction method can be employed for obtaining smooth stress fields around crack tips.

Fig. 4.2. Discontinuity modeling along the crack surface.

Two level sets, ϕ and φ , are used in the visibility criterion to determine visible nodes. Fig. 4.3 shows a crack surface and a local tip coordinate system x - y that is used to measure ϕ and φ . ϕ denotes the shortest distance normal to a crack surface, and φ denotes the shortest distance tangential to a crack surface. When the points J and I are located on the same side of a crack surface, namely the condition $\phi_J \phi_I > 0$ is satisfied, the point I is involved in the field approximations. On the other hand, the condition is not met at point K , indicating the field approximations without considering point K .

Fig. 4.3. A crack representation by the level set method.

4.1.2. IGA-meshfree MLS approach

By using the reproducing conditions [83], a reproducing point vector is obtained as $\mathbf{p}^T(\xi_I^{[.]}) = \left\{ 1, \xi_I^{[1]}, \eta_I^{[1]}, (\xi_I^{[2]})^2, \xi_I^{[1]} \eta_I^{[1]}, (\eta_I^{[2]})^2, \dots, (\xi_I^{[p]})^p, \dots, (\eta_I^{[q]})^q \right\}$, which is detailed in Appendix C. By considering the reproducing point vector within the framework of MLS approximation, the following matrices are obtained:

$$\bar{\mathbf{A}}(\xi) = \sum_{J=1}^{N_M} \hat{W}(\xi - \xi_J^{[1]}) \mathbf{p}(\xi_J^{[.]}) \mathbf{p}^T(\xi_J^{[.]}) \quad (4.9)$$

$$\mathbf{B}(\xi) = \begin{bmatrix} \hat{W}(\xi - \xi_1^{[1]}) \mathbf{p}(\xi_1^{[.]}) & \hat{W}(\xi - \xi_2^{[1]}) \mathbf{p}(\xi_2^{[.]}) & \dots & \hat{W}(\xi - \xi_{N_M}^{[1]}) \mathbf{p}(\xi_{N_M}^{[.]}) \end{bmatrix} \quad (4.10)$$

The MLS basis function is achieved by replacing the matrices $\bar{\mathbf{A}}(\xi)$ and $\mathbf{B}(\xi)$ in Eq. (B.10) as follows:

$$\begin{aligned} \Phi_J(\xi) &= \mathbf{p}^T(\xi) (\bar{\mathbf{A}}^{-1} \mathbf{B})_J \\ &= \mathbf{p}^T(\xi) \left(\sum_{J=1}^{N_M} \hat{W}(\xi - \xi_J^{[1]}) \mathbf{p}(\xi_J^{[.]}) \mathbf{p}^T(\xi_J^{[.]}) \right)^{-1} \hat{W}(\xi - \xi_J^{[1]}) \mathbf{p}(\xi_J^{[.]}) \end{aligned} \quad (4.11)$$

where the linear reproducing points $\xi_J^{[1]}$ are treated as meshfree nodes. It is noted that the MLS-based formulation is equivalent to the RK formulation in [89]. Thus, Eq. (4.11) can reproduce isogeometric basis functions. B-spline basis functions are reproduced via Eq. (4.11) when the support size equals $(p + 1)/2$. NURBS basis functions are reproduced by combining weights w_J with the meshfree reproducing formulations as follows:

$$\psi_J(\xi) = \frac{\Phi_J(\xi) w_J}{\sum_{K=1}^N \Phi_K(\xi) w_K} \quad (4.12)$$

It is noted that, in the meshfree methods, the larger support domain is used to include enough nodes for an accurate approximation, which leads to the high computational cost. The IMMLS-C approach reproduces the B-spline and NURBS

basis functions by using a small support domain. Thus, the computation cost of the approach is less than that of the meshfree methods. Moreover, the present approach utilizes Eq. (4.11), the unified meshfree formulation, throughout the whole domain without subdivisions compared with the previous coupling approach in Chapter 3. Therefore, the IMMLS-C approach simplifies the implementation of adaptivity in the analysis of crack propagation.

4.1.2. Numerical implementation

Fig. 4.1(b) shows the discretization of a cracked body with collocation points that are classified into three types: the interior, natural boundary and essential boundary points. The points on both Γ_t and Γ_c and the points on Γ_u belong to the sets Z_{tc} and Z_u , respectively. Other points in the interior domain belong to the set Z_{int} .

The displacement fields are approximated by the shape function Φ_i and nodal displacement $\mathbf{u}_i$ as follows:

$$\mathbf{u} = \sum_{i=1}^n \Phi_i \mathbf{u}_i . \quad (4.13)$$

Compared with the classical form of displacement approximation based on the idea of XFEM, the present approximation in Eq. (4.13) avoids adding additional enrichment functions, thus reducing the total DOFs and computational cost. The substitution of Eq. (4.13) in Eqs. (4.1)-(4.5) yields the following discretized equations:

$$\mathbf{L}(\Phi(\xi_l))\mathbf{u} = -\mathbf{b}(\xi_l), \quad \xi_l \in \Omega, \quad (4.14)$$

$$\mathbf{B}_{tc}(\Phi(\xi_j))\mathbf{u} = \mathbf{t}(\xi_j), \quad \xi_j \in \Gamma_t \cup \Gamma_c, \quad (4.15)$$

$$\mathbf{B}_u(\Phi(\xi_k))\mathbf{u} = \bar{\mathbf{u}}(\xi_k), \quad \xi_k \in \Gamma_u. \quad (4.16)$$

$\mathbf{L}$ is the differential operator of the equilibrium equation. $\mathbf{B}_{tc}$ and $\mathbf{B}_u$ are the boundary operators associated with the prescribed traction and displacement boundary

conditions, respectively. These operators in 2D elasticity are expressed as follows:

$$\mathbf{L} = \begin{bmatrix} (\lambda + 2\mu)\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2} & (\lambda + \mu)\frac{\partial^2}{\partial x\partial y} \\ (\lambda + \mu)\frac{\partial^2}{\partial x\partial y} & (\lambda + 2\mu)\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2} \end{bmatrix}, \quad (4.17)$$

$$\mathbf{B}_{tc} = \begin{bmatrix} (\lambda + 2\mu)n_x\frac{\partial}{\partial x} + \mu n_y\frac{\partial}{\partial y} & \lambda n_x\frac{\partial}{\partial y} + \mu n_y\frac{\partial}{\partial x} \\ \lambda n_y\frac{\partial}{\partial x} + \mu n_x\frac{\partial}{\partial y} & (\lambda + 2\mu)n_y\frac{\partial}{\partial y} + \mu n_x\frac{\partial}{\partial x} \end{bmatrix}, \quad (4.18)$$

$$\mathbf{B}_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.19)$$

By rearranging Eqs. (4.14)-(4.16), the discretized system is obtained as

$$\begin{bmatrix} \mathbf{K}^{int} \\ \mathbf{K}^{tc} \\ \mathbf{K}^u \end{bmatrix} \begin{pmatrix} \mathbf{u}^{int} \\ \mathbf{u}^{tc} \\ \mathbf{u}^u \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{int} \\ \mathbf{f}^{tc} \\ \mathbf{f}^u \end{pmatrix}, \quad (4.20)$$

where $\mathbf{K}^{int}$, $\mathbf{K}^{tc}$ and $\mathbf{K}^u$ denote the matrices for the points in the sets Z_{int} , Z_{tc} and Z_u , respectively; $\mathbf{f}^{int}$, $\mathbf{f}^{tc}$ and $\mathbf{f}^u$ are the corresponding force vectors.

4.2. Numerical results

The IMMLS-C approach is applied to the 2D crack problems within the framework of LEFM. The linear reproducing points are considered as the collocation points in the IMMLS-C approach. The meshfree nodes are treated as the collocation points in the strong form-based EFGM. The SIFs of cracks are calculated by the interaction integral [128], which is introduced in Appendix D. The last two examples consider contact conditions. The contact constraints are imposed by the penalty method, as introduced in Appendix F. Furthermore, the plane-strain condition is assumed for all the examples. It should also be noted that the crack path is represented by a red line.

4.2.1. An infinite plate with a center crack

This example deals with the isotropic infinite center-cracked plate subjected to a remote tension. Fig. 4.4(a) shows the geometrical configuration of the simplified square edge-cracked domain that is subjected to a uniform tension $\sigma = 10^4 \text{ N/cm}^2$. The geometrical parameters of the square domain include the crack length $a = 5 \text{ cm}$ and width of square domain $2a$. The material parameters include $E = 10^7 \text{ N/cm}^2$ and $\nu = 0.3$. The analytical solutions are [200]

$$\begin{aligned} u_x(r, \theta) &= \frac{2(1+\nu)}{\sqrt{2\pi}} \frac{K_I}{E} \sqrt{r} \cos \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \\ u_y(r, \theta) &= \frac{2(1+\nu)}{\sqrt{2\pi}} \frac{K_I}{E} \sqrt{r} \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \\ \sigma_{xx}(r, \theta) &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \sigma_{yy}(r, \theta) &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \sigma_{xy}(r, \theta) &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \end{aligned} \quad (4.21)$$

with $K_I = \sigma\sqrt{\pi a}$. The error norm in terms of the relative strain energy error is calculated by

$$err_E = \sqrt{\frac{\int_A (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\text{ext}})^T : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\text{ext}}) dA}{\int_A (\boldsymbol{\varepsilon}_{\text{ext}})^T : \mathbf{C} : (\boldsymbol{\varepsilon}_{\text{ext}}) dA}}, \quad (4.22)$$

where $\boldsymbol{\varepsilon}_{\text{ext}}$ are the exact results, and $\mathbf{C}$ is the fourth-order tensor. Herein, background meshes are used to calculate the error norm for the convergence study. Fig. 4.4(b) shows the initial meshes and collocation points.

The contour plot of von Mises stress is shown in Fig. 4.5, in which the near-tip stresses are captured via adaptive refinement. The adaptively refined meshes are shown in Fig. 4.6. After the first step in which the whole meshes are refined, the near-

tip meshes are locally refined, whereas the meshes away from the crack tip remain coarse. Fig. 4.7 illustrates the error norm under three schemes: the EFG collocation approach with uniform meshes, and the present approach with both uniform and adaptive meshes. For the strong form-based EFG approach, the collocation points are the same as the meshfree nodes. The present approach with adaptive refinement achieves the highest convergence rate. Fig. 4.8 visualizes the comparison of numerical efficiency between the proposed approach and EFG. The present approach requires less computational time to yield equally accurate results due to the implementation of adaptive refinement.

Fig. 4.4. An infinite plate with a center crack: (a) the schematic configuration and (b) initial meshes.

Fig. 4.5. Contour plots of the infinite plate with a center crack: (a) the von Mises stress and (b) a close-up of the crack-tip region.

Fig. 4.6. Adaptively refined meshes: (a) second-level meshes, (b) third-level meshes, (c) fourth-level meshes and (d) fifth-level meshes.

Fig. 4.7. Errors in the strain energy norm.

Fig. 4.8. The comparison of numerical efficiency between the proposed approach and EFG.

4.2.2. An edge-cracked plate subjected to shear loading

This example deals with an edge-cracked rectangular plate subjected to a shear loading $P = 1 \text{ N/cm}^2$. Fig. 4.9 shows the geometry and initial meshes. The geometrical parameters include the height $H = 16 \text{ cm}$, width $W = 7 \text{ cm}$ and crack length $a = 3.5 \text{ cm}$. The material parameters include $E = 3 \times 10^7 \text{ N/cm}^2$ and $\nu = 0.25$. The reference solutions of SIFs are $K_I = 34 \text{ N}/\sqrt{\text{cm}^3}$ and $K_{II} = 4.55 \text{ N}/\sqrt{\text{cm}^3}$ [128].

The adaptively refined meshes and the stress contour plot that is relevant to the refined meshes are illustrated in Fig. 4.10. It is observed that the most dense nodes are generated near crack tips based on a gradient-based error indicator. Table 4.1 lists the values of K_I and K_{II} calculated by the proposed approach and other numerical methods. In the meshfree point collocation method [140], the meshfree nodes are considered as the collocation points. The errors of both K_I and K_{II} are less than 1%, demonstrating the accuracy of the proposed approach.

Fig. 4.9. An edge-cracked plate: (a) the schematic configuration and (b) initial meshes.

Fig. 4.10. Adaptive refinement of the edge-cracked plate: (a) meshes, (b) nodes and (c) the contour plot of von Mises stress.

Table 4.1. Comparison of SIFs for an edge-cracked plate subjected to shear loading.

Methods	Mode-I SIFs		Mode-II SIFs	
	K_I	Error (%)	K_{II}	Error (%)
Present	33.68	0.93	4.55	0.009
Reference [201]	34.12	0.35	4.76	4.62
Reference [140]	33.78	0.65	4.48	1.54

4.2.3. An edge-cracked plate subjected to tensile loading

Fig. 4.11 shows the configuration of the geometry, loading and initial meshes for an edge-cracked plate that is subjected to the uniform tension $P = 1 \text{ N/cm}^2$. The geometry parameters include the width $W = 8 \text{ cm}$, crack length $a = 0.5W$ and height $H = 16 \text{ cm}$. The material parameters include $E = 3 \times 10^7 \text{ N/cm}^2$ and $\nu = 0.25$. Exact values of Mode-I SIFs are calculated by

$$K_I = P\sqrt{\pi a} \left[1.12 - 0.23\left(\frac{a}{W}\right) + 10.55\left(\frac{a}{W}\right)^2 - 21.72\left(\frac{a}{W}\right)^3 + 30.39\left(\frac{a}{W}\right)^4 \right]. \quad (4.23)$$

As shown in Fig. 4.12(a), the near-tip meshes are locally refined due to high gradients. The contour plot of the von Mises stress that is relevant to the final refined meshes is illustrated in Fig. 4.12(c). The high-density collocation points are generated near crack tips to capture the stress singularity. To validate the proposed approach, different ratios of a/W (0.2, 0.3, 0.4, 0.5 and 0.6) are used to calculate the Mode-I SIF, and the comparison of the Mode-I SIF values is shown in Table 4.2. The present approach achieves a more accurate K_I with a relative error less than 1%. Fig. 4.13 shows the convergence rate of K_I under three schemes: the weak form-based Galerkin approach with uniform meshes [202] and the present collocation approach with both uniform and adaptive meshes. The relative errors of K_I at different refinement steps are

12.01%, 11.71%, 3.41%, 1.15% and 0.31%, and the corresponding DOFs are 1224, 1728, 2234, 2862 and 3378, respectively. It should be noted that the weak form-based approaches are more accurate than their strong counterparts when both approaches apply uniform meshes, which is observed in Fig. 4.13. Furthermore, the present approach with AMR achieves a high convergence rate of K_I with respect to the DOFs.

Fig. 4.11. An edge-cracked plate subjected to tensile loading: (a) the schematic configuration and (b) initial meshes.

Fig. 4.12. Adaptive refinement of the edge-cracked plate: (a) meshes, (b) nodes and (c) the contour plot of σ_{VM} .

Table 4.2. Comparison of the Mode-I SIF values at different crack lengths.

a/W	Analytical values	Reference solutions [202]	Error (%)	Present results	Error (%)
0.2	3.07	3.05	0.62	3.06	0.47
0.3	4.56	4.58	0.48	4.55	0.24
0.4	6.67	6.66	0.18	6.65	0.32
0.5	10.02	9.93	0.90	9.99	0.31
0.6	15.64	15.48	1.02	15.60	0.26

Fig. 4.13. The convergence of Mode-I SIF values with respect to DOFs.

4.2.4. A rigid cylinder in contact with a central cracked block

This problem deals with the frictional contact between a central cracked block and a rigid cylinder. Fig. 4.14(a) shows the geometry configuration with the radius $R = 1$ mm, length $L = 2$ mm, width $W = 2$ mm, crack length $2a = 1/4W$ and distance $d = 1/4W$. The lower block is linearly elastic with the material parameters including $E = 1.333$ kN/mm² and $\nu = 0.333$. The penalty parameters, $\varepsilon_N = 10^4$ and $\varepsilon_T = 10^3$, and the coefficient of friction $\mu = 0.3$ are used to enforce the contact constraints. A vertical force $P = 0.005$ kN is distributed uniformly on the half-cylinder.

Fig. 4.15 illustrates the contour plot of the von Mises stress that is relevant to the final meshes shown in Fig. 4.14(b). Both the contact and crack-tip regions are densified with collocation points to capture the accurate field variables. The effects of different radiuses and distances on the values of SIFs are further investigated in Fig. 4.16. Due to the pressure loading, both the K_I and K_{II} are negative at the left crack tip, which indicates that an opening crack will not be observed. The absolute values of both K_I and K_{II} increase with an increase in the radius due to the larger contact areas. The results under the condition of the different distances ($1/8W$, $1/4W$ and $3/8W$) show that the absolute values of both K_I and K_{II} decrease with an increase in the distance. The numerical normalized contact pressures without any cracks are visualized in Fig. 4.17, which demonstrates good agreement with the analytical values. By considering a crack, the normal contact tractions under the different crack lengths and distances are shown in Fig. 4.18. Moreover, under the condition $2a = 1/24W$, the effect of the distance on the normal contact traction becomes more significant when the crack is near contact areas.

Fig. 4.14. A rigid cylinder in contact with a central cracked block: (a) the schematic configuration and (b) refined meshes.

Fig. 4.15. Contour plots of the central cracked block in contact with a rigid cylinder: (a) the von Mises stress and (b) a close-up of the crack-tip region.

Fig. 4.16. SIFs of the left crack tip ($2a = 1/4W$): (a) effects of the radiuses and (b) effects of the distances.

Fig. 4.17. Analytical and numerical normalized contact tractions without any cracks.

Fig. 4.18. Effects of (a) the different crack lengths with $d = 1/8W$ and (b) the different distances with $2a = 1/24W$ on the normal contact traction.

4.2.5. Frictional contact between an elastic cylinder and a central cracked block

Fig. 4.19(a) illustrates the geometry and boundary configurations of a central cracked block in contact with a half elastic cylinder. The upper geometry is loaded with a vertical displacement $u_y = 0.002$ mm and horizontal displacement $u_x = 0.00075$ mm. The geometrical parameters are the radius $R = 1$ mm, height $h = 0.3$ mm, width $W = 0.5$ mm, length $L = 0.5$ mm, crack length $2a = 1/4W$ and distance $d = 1/4W$. The material parameters of the two bodies include the coefficient of friction $\mu = 0.2$, $E = 1$ kN/mm² and $\nu = 0.3$. The penalty parameters, $\varepsilon_N = 1000$ and $\varepsilon_T = 100$, are used to enforce the contact constraints.

Fig. 4.19(b) visualizes the adaptively refined meshes after three levels of refinement. The most dense mesh at both the contact and crack-tip regions is generated according to a gradient-based error indicator. Fig. 4.20 visualizes the contour plot of the von Mises stress and its close-up at the contact and crack-tip regions. Contour plots of stress components are also investigated, as shown in Fig. 4.21. Furthermore, the distance effect on both K_I and K_{II} is studied via both the present approach and the strong form-based EFG, as listed in Table 4.3. Under the different distances ($1/8W$,

$1/4W$ and $3/8W$), the absolute values of both K_I and K_{II} decrease with an increase in d for the two approaches. It is also observed that the values of K_I are negative due to the compression loading. The similar negative SIFs for a cracked specimen under compression have been obtained in [203] that treats the value of negative K_I as zero. Other strategies that aim to keep the negative SIFs significant utilize the combined SIF for the analysis of a subsurface crack under cyclic contact loading [204, 205].

Fig. 4.19. Two deformable bodies with a central crack: (a) the schematic configuration and (b) refined meshes.

Fig. 4.20. The contour plot of two deformable bodies with a central crack: (a) the von Mises stress and (b) a close-up of the contact and crack-tip regions.

Fig. 4.21. Contour plots of stresses: (a) σ_{xx} , (b) σ_{xy} and (c) σ_{yy} .

Table 4.3. Values of SIFs for the crack at various distances.

d	Present ($\times 10^{-3}$)				EFG-Collocation ($\times 10^{-3}$)			
	Right crack tip		Left crack tip		Right crack tip		Left crack tip	
	K_I	K_{II}	K_I	K_{II}	K_I	K_{II}	K_I	K_{II}
$1/8W$	-1.42	0.68	-1.38	-0.59	-1.47	0.61	-1.40	-0.54
$1/4W$	-1.05	0.28	-1.01	-0.23	-1.06	0.29	-1.02	-0.20
$3/8W$	-0.80	0.17	-0.78	-0.08	-0.84	0.19	-0.81	-0.12

4.3. Summary

An adaptive IMMLS-C approach is developed for the 2D linear elastic fracture problems with contact loading. The concept of the proposed approach relies on the usage of reproducing conditions to establish the equivalence between the isogeometric basis functions and MLS shape functions. Due to this equivalence, the present approach not only possesses the advantages of IGA, including the geometric exactness

and higher-order approximation, but also inherits the flexibility of adaptive refinement from meshfree methods. The adaptive refinement is automatically implemented by using a gradient-based error estimator without requiring the exact stress or strain fields. This approach is further applied to fracture problems for obtaining strong formulations, thus avoiding the time-consuming numerical integration. Within the strong formulations, the displacement continuity along the traction-free crack surface is treated by the visibility method without adding additional enrichment functions.

The accuracy of the present approach is validated by solving a series of numerical examples. In the first example, the simulation results demonstrate that the present approach achieves both a higher convergence rate and computational efficiency compared with the strong form-based EFGM that treats meshfree nodes as collocation points. Furthermore, the singularity of stress fields near a crack tip and the discontinuity of displacement fields around a crack surface are captured by AMR. In the second and third examples, the values of SIFs calculated by the proposed approach are in good agreement with the reference results. The simulation results of Hertzian contact problems are verified by the reference results, which further demonstrate the robustness of the IMMLS-C approach.

Chapter 5 An IGA-Meshfree MLS Approach for Analyses of Crack Propagation in Shells

In this chapter, an IGA-meshfree MLS approach is developed to investigate the crack propagation in shells. This proposed approach achieves the equivalence between IGA basis functions and MLS basis functions and implements AMR by using a gradient-based error indicator. This IGA-meshfree MLS formulation is enriched for modeling cracked shells based on the partition of unity method (PUM). The singularity of the stress fields and the discontinuity of the displacement fields can be efficiently captured. The simulated behaviors of both the stationary and propagating cracks are verified with the reference solutions.

5.1. Problem description and solution approach

5.1.1. Numerical implementation

On the cracked mid-surface of a shell, the displacements are expressed based on PUM as follows:

$$\begin{aligned} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} &= \sum_{I=1}^{N_S} \Phi_I(\xi) \begin{Bmatrix} u_{I1} \\ u_{I2} \end{Bmatrix} + \sum_{J=1}^{N_H} \Phi_J(\xi) [H(\xi) - H(\xi_J)] \begin{Bmatrix} c_{J1} \\ c_{J2} \end{Bmatrix} \\ &\quad + \sum_{K=1}^{N_K} \Phi_K(\xi) \left\{ \sum_{\gamma=1}^4 [Q_K^\gamma(\xi) - Q_K^\gamma(\xi_K)] \begin{Bmatrix} d_{K1}^\gamma \\ d_{K2}^\gamma \end{Bmatrix} \right\} \end{aligned} \quad (5.1)$$

$$\begin{aligned} u_3 &= \sum_{I=1}^{N_S} \Phi_I(\xi) u_{I3} + \sum_{J=1}^{N_H} \Phi_J(\xi) [H(\xi) - H(\xi_J)] c_{J3} \\ &\quad + \sum_{K=1}^{N_K} \Phi_K(\xi) \left\{ \sum_{\gamma=1}^4 [F_K^\gamma(\xi) - F_K^\gamma(\xi_K)] d_{K3}^\gamma \right\} \end{aligned} \quad (5.2)$$

where Φ_I is calculated by Eq. (4.11); $\mathbf{u} = \{u_{I1}, u_{I2}, u_{I3}\}$ are the standard DOFs; $\mathbf{c} = \{c_{J1}, c_{J2}, c_{J3}\}$ and $\mathbf{d} = \{d_{K1}, d_{K2}, d_{K3}\}$ are the DOFs related to the crack surfaces and tips, respectively; H , Q and F are the Heaviside, in-plane and out-of-plane enrichment

functions, respectively. Fig. 5.1 shows the 1D and 2D step-enriched basis functions, respectively.

Fig. 5.1. The step-enriched quadratic basis functions: (a) 1D functions and (b) the 2D function of Φ_4 .

By utilizing the local polar coordinate system, the in-plane enrichment function can be expressed as

$$Q(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right\}. \quad (5.3)$$

The out-of-plane enrichment function can be expressed as

$$F(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), r^{\frac{3}{2}} \sin\left(\frac{\theta}{2}\right), r^{\frac{3}{2}} \cos\left(\frac{\theta}{2}\right), r^{\frac{3}{2}} \sin\left(\frac{3\theta}{2}\right), r^{\frac{3}{2}} \cos\left(\frac{3\theta}{2}\right) \right\}. \quad (5.4)$$

It is noted that the shifted Heaviside step functions and shifted tip-enrichment functions were introduced by Belytschko *et al.* [206] for compatibility enforcement. By shifting, the blending of enriched elements into the standard ones is simplified. Herein, the same discretization of the displacement fields as that in [114] is used.

The mid-surface in a reference configuration can be expressed as

$$\boldsymbol{\varphi}_0(\xi^1, \xi^2) = \sum_{I=1}^{N_s} \Phi_I(\xi) \mathbf{P}_I, \quad (5.5)$$

where $\mathbf{P}_I \in \mathbb{R}^3$ are meshfree nodes in the physical space. The mid-surface in a deformed configuration is defined in the same way.

The matrices related to the membrane and bending strains can be divided into standard and enriched parts. The standard membrane and bending strains are described by the membrane matrix $\mathbf{B}_n^{ul}$ and the bending matrix $\mathbf{B}_m^{ul}$ in Eq. (3.20), respectively. The membrane matrix related to the Heaviside enrichment function can be expressed as

$$\mathbf{B}_n^{cJ} = \begin{bmatrix} (\mathbf{A}_1 (H(\xi) - H(\xi_J)) \Phi_{J,1})^T \\ (\mathbf{A}_2 (H(\xi) - H(\xi_J)) \Phi_{J,2})^T \\ ((\mathbf{A}_1 \Phi_{J,2} + \mathbf{A}_2 \Phi_{J,1}) (H(\xi) - H(\xi_J)))^T \end{bmatrix}. \quad (5.6)$$

The membrane matrix associated with the tip enrichment function is

$$\mathbf{B}_n^{dK} = \begin{bmatrix} (\mathbf{A}_1 \{(\mathcal{Q}_K'(\xi) - \mathcal{Q}_K'(\xi_K)) \Phi_{K,1} + \Phi_K \mathcal{Q}_{K,1}'(\xi)\})^T \\ (\mathbf{A}_2 \{(\mathcal{Q}_K'(\xi) - \mathcal{Q}_K'(\xi_K)) \Phi_{K,2} + \Phi_K \mathcal{Q}_{K,2}'(\xi)\})^T \\ ((\mathbf{A}_2 \Phi_{K,1} + \mathbf{A}_1 \Phi_{K,2}) (\mathcal{Q}_K'(\xi) - \mathcal{Q}_K'(\xi_K)) \\ + \Phi_K (\mathbf{A}_2 \mathcal{Q}_{K,1}'(\xi) + \mathbf{A}_1 \mathcal{Q}_{K,2}'(\xi)))^T \end{bmatrix}. \quad (5.7)$$

The bending matrix associated with the Heaviside enrichment function is

$$\mathbf{B}_m^{cJ} = (H(\xi) - H(\xi_J)) \begin{bmatrix} (\mathbf{b}_{m11}^{cJ})^T \\ (\mathbf{b}_{m22}^{cJ})^T \\ (2\mathbf{b}_{m12}^{cJ})^T \end{bmatrix}, \quad (5.8)$$

where $\mathbf{b}_{m11}^{cJ}$, $\mathbf{b}_{m22}^{cJ}$ and $\mathbf{b}_{m12}^{cJ}$ are the same as the standard bending parts. The bending matrix associated with the tip enrichment function is

$$\mathbf{B}_m^{dK} = \begin{bmatrix} (\mathbf{b}_{m11}^{dK})^T \\ (\mathbf{b}_{m22}^{dK})^T \\ (2\mathbf{b}_{m12}^{dK})^T \end{bmatrix}, \quad (5.9)$$

where $\mathbf{b}_{m11}^{dK}$, $\mathbf{b}_{m22}^{dK}$ and $\mathbf{b}_{m12}^{dK}$ are obtained by replacing $\Phi_{I,1}$, $\Phi_{I,2}$, $\Phi_{I,12}$, $\Phi_{I,11}$ and $\Phi_{I,22}$ in Eq. (3.22) as follows:

$$\begin{aligned} \Phi_{I,1} &= \Phi_{K,1} (Q_K^\gamma(\xi) - Q_K^\gamma(\xi_K)) + \Phi_I Q_{K,1}^\gamma(\xi), \\ \Phi_{I,2} &= \Phi_{K,2} (Q_K^\gamma(\xi) - Q_K^\gamma(\xi_K)) + \Phi_I Q_{K,2}^\gamma(\xi), \\ \Phi_{I,11} &= \Phi_{K,11} (Q_K^\gamma(\xi) - Q_K^\gamma(\xi_K)) + 2\Phi_{I,1} Q_{K,1}^\gamma(\xi) + \Phi_I Q_{K,11}^\gamma(\xi), \\ \Phi_{I,22} &= \Phi_{K,22} (Q_K^\gamma(\xi) - Q_K^\gamma(\xi_K)) + 2\Phi_{I,2} Q_{K,2}^\gamma(\xi) + \Phi_I Q_{K,22}^\gamma(\xi), \\ \Phi_{I,12} &= \Phi_{K,12} (Q_K^\gamma(\xi) - Q_K^\gamma(\xi_K)) + \Phi_{I,1} Q_{K,2}^\gamma(\xi) + \Phi_{I,2} Q_{K,1}^\gamma(\xi) + \Phi_I Q_{K,12}^\gamma(\xi). \end{aligned} \quad (5.10)$$

By discretizing the virtual work in Eq. (3.1), the linearized system is obtained as

$$\mathbf{KU} = \mathbf{F}, \quad (5.11)$$

where the unknown displacement fields are defined as $\mathbf{U} = \{\mathbf{u}, \mathbf{c}, \mathbf{d}\}^T$; the stiffness matrix $\mathbf{K}$ is expressed as

$$\mathbf{K}_{IJ} = \begin{bmatrix} \mathbf{K}_{IJ}^{uu} & \mathbf{K}_{IJ}^{uc} & \mathbf{K}_{IJ}^{ud} \\ \mathbf{K}_{IJ}^{cu} & \mathbf{K}_{IJ}^{cc} & \mathbf{K}_{IJ}^{cd} \\ \mathbf{K}_{IJ}^{du} & \mathbf{K}_{IJ}^{dc} & \mathbf{K}_{IJ}^{dd} \end{bmatrix}, \quad (5.12)$$

where the superscript u is related to the standard parts; c and d are related to the enriched parts. For these three parts, the corresponding stiffness matrix is expressed as follows:

$$\mathbf{K}_{IJ}^{ij} = \int_A t (\mathbf{B}_n^{ij})^T : \mathbf{C} : \mathbf{B}_n^{ij} dA + \int_A \frac{t^3}{12} (\mathbf{B}_m^{ij})^T : \mathbf{C} : \mathbf{B}_m^{ij} dA, \quad i, j = u, c, d. \quad (5.13)$$

The force vector $\mathbf{F}$ is described as

$$\begin{aligned} \mathbf{F}_I &= \{\mathbf{F}_I^u \ \mathbf{F}_I^c \ \mathbf{F}_I^{d\gamma}\}^T \quad \text{with} \quad \mathbf{F}_I^u = \int_A \Phi_I \mathbf{b} dA + \int_\Gamma \Phi_I \bar{\mathbf{t}} d\Gamma, \\ \mathbf{F}_I^c &= \int_A \Phi_I H \mathbf{b} dA + \int_\Gamma \Phi_I H \bar{\mathbf{t}} d\Gamma, \quad \text{and} \quad \mathbf{F}_I^{d\gamma} = \int_A \Phi_I Q^\gamma \mathbf{b} dA + \int_\Gamma \Phi_I Q^\gamma \bar{\mathbf{t}} d\Gamma. \end{aligned} \quad (5.14)$$

5.1.2. Adaptive refinement procedure

The procedure of AMR includes two parts: error estimation and mesh refinement. The meshfree local refinement strategy is introduced in Appendix G. The gradient-based error estimation strategy is used to identify elements associated with large errors. As discussed in [207], the strain energy density and external potential energy density influence the accuracy of the solutions, while the gradient of strain energy density determines the global errors that are used for the AMR.

According to the extended displacement approximation of a cracked shell surface in Eqs. (5.1) and (5.2), the strain energy density can be divided into three parts: a standard part and two enriched parts. The strain energy density of an element k is calculated as

$$\begin{aligned} D_k^{\text{SE}} &= \frac{1}{2} (\mathbf{n}^T : \boldsymbol{\varepsilon} + \mathbf{m}^T : \boldsymbol{\kappa}) = \frac{1}{2} \left(t \boldsymbol{\varepsilon}^T : \mathbf{C} : \boldsymbol{\varepsilon} + \frac{t^3}{12} \boldsymbol{\kappa}^T : \mathbf{C} : \boldsymbol{\kappa} \right) \\ &= \frac{1}{2} \left(t \mathbf{u}^T (\mathbf{B}_n)^T : \mathbf{C} : (\mathbf{B}_n) \mathbf{u} + \frac{t^3}{12} \mathbf{u}^T (\mathbf{B}_m)^T : \mathbf{C} : (\mathbf{B}_m) \mathbf{u} \right). \end{aligned} \quad (5.15)$$

To circumvent the second-order derivatives of the shape functions, a convenient implementation for the gradient of strain energy density is applied, as introduced in [208]. Within an element k , the average gradient of strain energy density can be obtained as follows:

$$G_k^{\text{SED}} = \frac{\sum_{i=1}^{N_k-1} \sum_{j=i+1}^{N_k} \frac{|D_i^{\text{SE}} - D_j^{\text{SE}}|}{l_{ij}}}{N_k!}, \quad (5.16)$$

where l_{ij} is the distance between nodes i and j , and N_k is the number of nodes. The strain energy densities at nodes i and j are denoted by D_i^{SE} and D_j^{SE} , respectively. The gauss points within an element are used to calculate the strain energy density.

A mesh density D_k^M that represents the number of nodes within a unit area is calculated as

$$D_k^M = \frac{N_k}{A_k}, \quad (5.17)$$

where A_k is the area of an element k . The average mesh intensity of an element k is calculated as follows:

$$r_k^{MI} = \frac{G_k^{SED}}{D_k^M}. \quad (5.18)$$

Ideal meshes can be obtained when the average mesh intensity is the same throughout the domain. However, the ideal meshes for most of the problems are rather difficult to achieve. To obtain accurate solutions, the maximum and minimum mesh intensities R_{max} and R_{min} are used to determine the threshold R_{thre} for identifying elements associated with large errors.

The average mesh intensities of elements are calculated via Eq. (5.18). The mesh refinement on the element k is performed when its average mesh intensity exceeds the threshold R_{thre} , which is defined as follows:

$$\text{if } r_k^{MI} > R_{thre}, \quad \text{refine } (R_{min} \leq R_{thre} \leq R_{max}). \quad (5.19)$$

Herein, the threshold is defined as $R_{thre} = kR_{max}$ with k being a constant number and R_{max} being the maximum value of average mesh intensities.

5.2. Numerical results

Within the framework of LEFM, the IGA-meshfree MLS approach is applied to investigate the stationary and propagating cracks in shells. The maximum circumferential stress criterion is utilized for determining whether a crack would grow and the growth angle in the quasi-static crack propagation. In the propagation analysis,

the sufficiently large quasi-static loading is enforced, and only the propagation angle needs to be calculated. The quadratic basis function is employed with 8×8 gauss integration for meshes around crack tips, 6×6 gauss integration for meshes around crack surfaces and 4×4 for the rest domain. The Lagrange multiplier method is applied to enforce boundary conditions. The crack propagation path is denoted by a red line in the following examples.

Table 5.1 describes the computational procedure of adaptive crack propagation analysis.

Table 5.1. Flow chart of the adaptive crack propagation analysis.

<i>Step 1.</i> Load the geometry data of the initial NURBS surface to obtain the initial knot vectors, control points, weights and meshes.
<i>Step 2.</i> Check the crack propagation step k . If $k \leq N_{\text{step}}$, go to <i>Step 3</i> . Otherwise, go to <i>Step 6</i> .
<i>Step 3.</i> According to the current meshes, calculate basis functions by Eq. (4.11), assemble $\mathbf{K}$ and $\mathbf{F}$ based on Eqs. (5.13) and (5.14), respectively, and then solve the displacement $\mathbf{U}$ by Eq. (5.11).
<i>Step 4.</i> Calculate the gradient-based error estimator and mark elements accordingly. If the current refinement step j exceeds threshold N_{refine} , exit the mesh refinement. Otherwise, obtain the new meshes by refining the marked elements and return to <i>Step 3</i> .
<i>Step 5.</i> Calculate the fracture parameters including the SIFs and propagation angle and update the crack geometry. Then return to <i>Step 2</i> .
<i>Step 6.</i> Output the displacement fields for postprocessing.

5.2.1. An infinite plate with a center crack

This example investigates an isotropic infinite plate with a center crack subjected

to remote tension. As shown in Fig. 5.2(a), this problem domain is simplified into an edge-cracked square domain subjected to a uniform tension $\sigma = 10^4$ N/cm². The material parameters include $E = 10^7$ N/cm² and $\nu = 0.3$. The geometry parameters are set as follows: the crack length a equals 5 cm, and the width of the square ABCD is $2a$. Under the plane-strain condition, analytical solutions are expressed in Eq. (4.21). The error in the strain energy is calculated by Eq. (4.22). The initial isogeometric meshes and meshfree nodes in the physical space are shown in Fig. 5.2(b). The threshold R_{thre} for error estimation is set to be 1% of R_{max} , and the maximum refinement step is 6 to ensure that the smallest distance between the nodes is less than 1/64 of the initial nodal spacing.

Fig. 5.2. A center-cracked infinite plate: (a) the schematic diagram and (b) initial configurations.

Due to the singular stresses at crack tips, meshes near crack tips are required to be refined for obtaining accurate results. As shown in Fig. 5.3, meshes and corresponding nodes around the crack tip are automatically refined, whereas those away from tip-regions keep constant. Under the pure Mode-I fracture, an opening crack is visualized in Fig. 5.4, in which the contour plots of displacement and stress σ_{xx} are described,

respectively. The discontinuous displacements across crack surfaces and singular crack-tip stresses are captured.

Fig. 5.3. The refined (a) meshes and (b) nodes at the last step of AMR.

Fig. 5.4. Contour plots (magnification factor of 10): (a) the displacement u_y and (b) the stress σ_{xx} .

To further demonstrate the advantages of the present approach, the errors in terms of the strain energy norm under four conditions: present approach with and without AMR, as well as IGA and the meshfree method with uniform refinement, are calculated. As shown in Fig. 5.5, the present approach with AMR achieves a higher convergence rate. Moreover, the results of the present approach with uniform refinement are more accurate than those of the meshfree method, although they have

the same convergence rate. Fig. 5.6 shows the computational time versus the strain energy error for the different methods. The present approach with adaptivity costs the least amount of time while the EFG-based meshfree method costs the most, while both approaches yield equally accurate results.

Fig. 5.5. Errors in the strain energy norm.

Fig. 5.6. The comparison of the computational efficiency between the present approach and other approaches.

5.2.2. Crack propagation in a double cantilever beam

This example deals with a double cantilever beam under the plane stress condition. Fig. 5.7(a) shows the geometry and its parameters, including the length $L =$

60 cm, height $H = 20$ cm, distance off the mid-plane $\Delta H = 0.14$ cm, crack length $a = 20$ cm and loading $P = 1.0 \times 10^5$ N. The material parameters include $E = 3.0 \times 10^7$ N/cm² and $\nu = 0.3$. Fig. 5.7(b) visualizes the initial isogeometric meshes and meshfree nodes in the physical space. Crack propagation is simulated in 11 steps, and the increment at each step equals 1 cm. The threshold R_{thre} for error estimation is set to be 5% of R_{max} in the AMR.

Fig. 5.7. A double cantilever beam: (a) the geometry and (b) initial configurations.

Fig. 5.8 visualizes the crack propagation paths at different steps. As expected, the crack grows towards the upper side due to the distance off the mid-plane. In each propagation step, meshes are automatically refined near the crack tip. The final propagation path and its comparison with the reference solutions [16] are visualized in Fig. 5.9(a), which demonstrates that the predicted crack propagation path agrees well with the reference results. To further investigate the effect of ΔH on propagation paths, several different values of ΔH (0, 0.035, 0.07 and 0.3 cm) are considered. Crack propagation paths at different values of ΔH are shown in Fig. 5.9(b). When ΔH equals 0, this problem is a pure Mode-I fracture, and thus the crack should propagate in a straight line. Otherwise, a mixed-mode fracture should be considered, and the crack propagates upwards along a larger inclined angle as ΔH increases. Fig. 5.10 visualizes

contour plots of the displacement and stress fields ($\Delta H = 0.14$ cm) at the final propagation step, in which the singular crack-tip stress fields are captured via AMR. In addition, the negative stress fields at the top edge lead to higher gradients of the strain energy density, and this situation influences mesh refinement at the top edge, as shown in Fig. 5.8(d).

Fig. 5.8. Crack propagation paths and refined meshes: (a) the 1st step, (b) the 5th step, (c) the 7th step and (d) the 11th step.

Fig. 5.9. Crack paths of the double cantilever beam: (a) the comparison with the reference solutions [16] ($\Delta H = 0.14$ cm) and (b) the paths at different ΔH .

Fig. 5.10. Contour plots of the double cantilever beam at $\Delta H = 0.14$ cm (magnification factor of 2): (a) displacement in the y -direction and (b) stress σ_{xx} .

5.2.3. A cylindrical shell with a longitudinal crack

As shown in Fig. 5.11(a), a cylindrical shell with a longitudinal crack subjected to an internal pressure $P = 1$ N/cm² is considered. The fixed boundary conditions at both ends are enforced. The geometry parameters include the radius $R = 20$ cm and length $L = 100$ cm. The material parameters include $E = 1 \times 10^3$ N/cm² and $\nu = 0.3$. With the crack length a and the thickness t , the normalized crack length can be defined as

$$\rho = \frac{a}{\sqrt{Rt}}. \quad (5.20)$$

Fig. 5.11(b) shows the initial isogeometric meshes and meshfree nodes in the physical space. The threshold R_{thre} for error estimation is set to be 2% of R_{max} in the mesh refinement.

As shown in Fig. 5.12, the crack opening displacements are calculated for different ρ (0.5, 1, 2 and 3) and ratios of R to t (5, 10, 20 and 30). The simulated crack opening displacements are in good agreement with the reference solutions [209]. The meshes and nodes at the last refinement step under the conditions where $R/t = 5$ and $\rho = 3$ are visualized in Fig. 5.13, in which the near-tip meshes and nodes are

automatically refined due to the high gradient of strain energy density. Fig. 5.14 shows contour plots of the displacement under two different conditions. It is found that an opening crack is achieved under the pure Mode-I fracture, in which the crack should theoretically propagate along the axial direction [10].

Fig. 5.11. The cracked cylindrical shell: (a) the geometry and (b) initial configurations.

Fig. 5.12. Crack opening displacements at different conditions.

Fig. 5.13. Meshes and nodes of the cylindrical shell at $R/t = 5$ and $\rho = 3$.

Fig. 5.14. Contour plots of the displacement (magnification factor of 3): (a) at $R/t = 5$ and $\rho = 3$ and (b) at $R/t = 10$ and $\rho = 3$.

5.2.4. Crack propagation in an edge-cracked cylindrical shell

This example deals with an edge-cracked cylindrical shell (Fig. 5.15(a)). The geometry and loading parameters include the radius $R = 1.5$ m, length $L = 15$ m, crack length $a = 4.25$ m, distance $d = 1.17$ m, thickness $t = 0.1$ m and concentrated force $P = 2 \times 10^4$ kN. Material parameters include $E = 210$ Gpa and $\nu = 0.3$. The boundary condition of one fixed end is enforced for the crack analysis, in which 9 steps of the

crack propagation with a crack increment of 4% of the initial crack length are adopted.

Due to the mixed-mode fracture, the crack propagation path would not be a straight line. Fig. 5.16 shows the propagation path with both the adaptively refined meshes and nodes at the last propagation step. The crack initially propagates along the circumferential direction and gradually deflects to the axial direction. The predicted crack growth path is in good agreement with the reference result [210]. In addition, a higher density of nodes and meshes at the crack tips is produced for the accuracy and efficiency of solutions.

The contour plot of von Mises stress at the last step is visualized in Fig. 5.17, in which high stress gradients are captured at the crack tip. To further investigate the influence of loading positions on the propagation path, two different distances d (0.59 m and 1.77 m) are considered. As shown in Fig. 5.18, the crack initially propagates to the loading side for three different distances. After several steps, the crack with the largest d propagates very closely to the circumferential direction, whereas the crack with the smallest d tends to grow along the axial direction. The different propagation paths are caused by the different loading positions where the loading components along the x -direction and z -direction are different. For smaller distances, the loading in the z -direction is larger than that in the x -direction, and this situation plays a major role in the crack propagation, which leads to the propagation path along the axial direction.

Fig. 5.15. An edge-cracked cylindrical shell: (a) the geometry and (b) initial configurations.

Fig. 5.16. The refined meshes and nodes at the last loading step.

Fig. 5.17. The contour plot of von Mises stress.

Fig. 5.18. Crack paths projected to the X-Y plane at different distances d .

5.2.5. Crack propagation in a pressurized cylindrical shell with tear straps

Fig. 5.19 shows an internally pressurized cylinder with two circumferential tear straps, which has been analyzed in the experiment conducted by Keesecker *et al.* [211]. The tear straps influence the crack propagation and induce crack turning when crack tips are near the tear straps. Table 5.2 shows the geometric and material parameters. Due to symmetry, a half-cylinder is modeled with the symmetry boundary conditions applied to the edges in the axial directions. As for the two circumferential ends, the axial line load, which equals one half of the product of pressure and radius, is uniformly applied along the circumference.

Fig. 5.20 shows the curve of the pressure versus the half crack length based on the crack propagation criterion introduced in [212] and the comparison with the reference results [161, 162, 211]. Since the reference results were obtained from the geometrically and materially nonlinear analysis, and the present results are obtained within the framework of LEFM, only the linear part of the reference results is chosen for comparison. The simulated results agree well with the envelope of peak values of

the nonlinear analysis.

Fig. 5.21 shows the adaptively refined meshes at different propagation steps. Elements near the crack tips are refined, whereas elements far away from tips remain coarse. The crack initially propagates along the axial direction until it is near the tear straps, after which the crack travels in the circumferential direction. This change occurs at the point where the axial normal stress exceeds the hoop stress [211]. Fig. 5.22 and Fig. 5.23 visualize the contour plots of displacement and von Mises stress at different propagation steps, respectively.

Fig. 5.19. The cylindrical shell with two tear straps: (a) the geometry and (b) initial configurations.

Table 5.2. Geometric and material parameters of the cylindrical shell with tear straps.

Geometric parameters (mm)			Material parameters	
Crack length	Width of tear straps	Thickness of tear straps	Young's modulus	Poisson's ratio
101.6	50.8	3.96	71422 MPa	0.3

Fig. 5.20. Pressure versus the half crack length.

Fig. 5.21. AMR of the cylindrical shell at different propagation steps: (a) $a = 115.8$ mm and (b) $a = 197.1$ mm and $a_{\text{cir}} = 52.5$ mm.

Fig. 5.22. Contour plots of the displacement at different propagation steps: (a) $a = 115.8$ mm (magnification factor of 3) and (b) $a = 197.1$ mm and $a_{\text{cir}} = 52.5$ mm.

Fig. 5.23. Contour plots of the von Mises stress at different propagation steps: (a) $a = 115.8$ mm (magnification factor of 3) and (b) $a = 197.1$ mm and $a_{\text{cir}} = 52.5$ mm.

5.3. Summary

The adaptive analysis of crack propagation in thin-shell structures is implemented via the IGA-meshfree MLS approach that utilizes the MLS basis functions to achieve the equivalence between the meshfree basis functions and isogeometric basis functions. With this equivalence, the present approach not only inherits the advantages

of IGA but also achieves the flexibility of AMR for IGA. The AMR is implemented in a meshfree manner by adding linear reproducing points directly to the meshes that are identified by a gradient-based error estimator. With these advantages, the present approach is extended based on the idea of XFEM and applied to the adaptive crack propagation analysis of shells. Subsequently, the accuracy of this approach is validated by solving a series of numerical examples. The simulation results demonstrate that the present approach achieves both a higher convergence rate and computational efficiency compared with IGA and the meshfree method. Moreover, the singularity of stress fields near crack tips and the discontinuity of displacement fields around crack surfaces are captured by AMR. Furthermore, the predicted propagation paths in thin shells are in good agreement with the reference results.

Chapter 6 An Adaptive IGA-Meshfree Phase-Field Approach for Modeling Brittle Fracture in 3D Polycrystalline Materials

In this chapter, an adaptive IGA-meshfree approach is developed for the phase-field modeling of brittle fracture in a 3D polycrystalline material. This approach is capable of flexibly achieving AMR by introducing an error estimator that includes both the phase field and its gradient. Within the proposed numerical framework, the anisotropic brittle fracture is investigated based on the PFM. Propagating cracks are automatically tracked with a locally refined mesh around crack geometries. Moreover, the intergranular and transgranular crack propagation behaviors in polycrystals can be simulated.

6.1. Problem description and solution approach

6.1.1. Phase-field modeling of brittle fracture

The regularized representations of cracks in a polycrystalline material are introduced. The anisotropy within grains is modeled by the crack surface functional with an orientation-dependent parameter that is related to the preferential cleavage planes. Subsequently, governing equations of an anisotropic brittle fracture problem are derived based on the variational theory.

(a) Regularized representations of cracks

Consider a 1D infinite bar with an initial crack at $x = 0$ as shown in Fig. 6.1, where discrete and diffusive representations of the crack are compared. The discrete crack is modeled by the Dirac delta function, and the diffusive crack can be modeled by the exponential function as follows [177]:

$$d(x) = e^{-|x|/l}, \quad (6.1)$$

where d is a phase-field variable describing the fully broken ($d = 1$) and intact ($d = 0$) materials, and l is a length scale parameter representing a crack width. Eq. (6.1) can be determined by solving the differential equation

$$d - l^2 d'' = 0 \quad (6.2)$$

which is subjected to the boundary conditions: $d(0) = 1$ and $d(\pm\infty) = 0$. It is noted that Eq. (6.2) is the Euler-Lagrange equation associated with the variational problem $d = \text{Arg}\{\inf_{d \in W} (I(d))\}$ with $W = \{d | d(0) = 1, d(\pm\infty) = 0\}$ and $I(d) = \int [d^2 + l^2 d'^2] / 2 dV$.

The area of a crack surface is represented as

$$\Gamma_c(d) = \frac{1}{2l} \int [d^2 + l^2 d'^2] dV. \quad (6.3)$$

For a higher-dimensional crack problem, Eq. (6.3) can be expressed as

$$\Gamma_c(d) = \int \gamma(d, \nabla d) dV \quad (6.4)$$

with the crack surface density function being as follows:

$$\gamma(d, \nabla d) = \frac{1}{2l} (d^2 + l^2 |\nabla d|^2). \quad (6.5)$$

It is noted that the substitution of Eq. (6.1) for d in Eq. (6.4) gives the unit area of a crack surface.

In a polycrystalline material, each grain has corresponding preferential cleavage planes. The anisotropy is implemented by introducing an orientation-dependent parameter in Eq. (6.5) as follows:

$$\gamma(d, \nabla d, \boldsymbol{\omega}) = \frac{1}{2l} [d^2 + l^2 \boldsymbol{\omega} : (\nabla d \otimes \nabla d)] \quad (6.6)$$

with the symmetric second-order tensor $\boldsymbol{\omega}$ describing the material anisotropy as [213]

$$\boldsymbol{\omega} = \mathbf{I} + \alpha (\mathbf{I} - \mathbf{M} \otimes \mathbf{M}), \quad (6.7)$$

where $\mathbf{M}$ is a unit vector perpendicular to a fracture plane, α is a penalty parameter to

control the crack growth paths along a preferential cleavage plane, and $\otimes$ is a dyadic operator. It is noted that isotropic fracture surface energy can be obtained when α equals 0. Fig. 6.2 illustrates the vector $\mathbf{M}$ norm to the preferential crack propagation path. By defining $\mathbf{M} = [\cos(\theta) \sin(\theta)]^T$ with the preferential crack growth angle θ , Eq. (6.7) can be expressed as

$$\omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \right). \quad (6.8)$$

Fig. 6.1. A 1D crack: (a) the discrete representation and (b) the diffusive representation.

Fig. 6.2. The schematic representation of a vector $\mathbf{M}$ norm to the preferential growth path.

(b) Equilibrium equations

According to the variational formulation of brittle fracture [174], the free energy of a cracked body is composed of the bulk energy E_b and the crack surface energy E_s as follows:

$$E(\mathbf{u}, \Gamma) = E_b(\mathbf{u}, \Gamma) + E_s(\Gamma) = \int_{\Omega} \psi(\boldsymbol{\varepsilon}(\mathbf{u})) d\Omega + \int_{\Gamma_c} G_c d\Gamma, \quad (6.9)$$

where $\psi(\boldsymbol{\varepsilon}(\mathbf{u}))$ is the strain energy density. By substituting Eq. (6.4) for the regularized representation of a sharp crack topology Γ_c in Eq. (6.9), the free energy functional for anisotropic materials is modified as

$$E(\mathbf{u}, d) = \int_{\Omega} \psi(\boldsymbol{\varepsilon}(\mathbf{u}), d) d\Omega + \int_{\Omega} G_c \frac{1}{2l} [d^2 + l^2 \boldsymbol{\omega} : (\nabla d \otimes \nabla d)] d\Omega. \quad (6.10)$$

To prevent the crack growth under compression, the strain energy density is decomposed into the positive and negative components with the positive part degraded by a stress degradation function $g(d)$ as follows:

$$\psi(\boldsymbol{\varepsilon}(\mathbf{u}), d) = g(d) \psi^+(\boldsymbol{\varepsilon}) + \psi^-(\boldsymbol{\varepsilon}) \quad (6.11)$$

with

$$g(d) = (1 - d)^2 + k, \quad (6.12)$$

where k is a parameter to control numerical stabilities. Herein, we consider k as 10^{-8} in the numerical simulations. By decomposing the trace of a strain tensor into positive and negative parts: $\langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_{\pm} = (\text{tr}(\boldsymbol{\varepsilon}) \pm |\text{tr}(\boldsymbol{\varepsilon})|)/2$, the strain energy density is recast as follows [178]:

$$\begin{aligned} \psi(\boldsymbol{\varepsilon}(\mathbf{u}), d) &= g(d) \psi^+(\boldsymbol{\varepsilon}) + \psi^-(\boldsymbol{\varepsilon}) \\ &= g(d) \left(\frac{K}{2} \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_+^2 + \mu (\boldsymbol{\varepsilon}_{\text{dev}} : \boldsymbol{\varepsilon}_{\text{dev}}) \right) + \frac{K}{2} \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_-^2, \end{aligned} \quad (6.13)$$

where $K = \lambda + 2/3\mu$ denotes the bulk modulus with Lamé constants λ and μ , and $\boldsymbol{\varepsilon}_{\text{dev}}$ is the deviatoric part of a strain tensor. The Cauchy stress tensor is defined as

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d) = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = g(d) \left(K \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_+ \mathbf{I} + 2\mu \boldsymbol{\varepsilon}_{\text{dev}} \right) + K \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_- \mathbf{I}. \quad (6.14)$$

The elastic fourth-order tensor is expressed as

$$\begin{aligned} \mathbb{C}(\boldsymbol{\varepsilon}, d) &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} \\ &= g(d) \left(H^+(\text{tr}(\boldsymbol{\varepsilon})) K \mathbf{I} \otimes \mathbf{I} + 2\mu \left(\mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) \right) + H^-(\text{tr}(\boldsymbol{\varepsilon})) K \mathbf{I} \otimes \mathbf{I} \end{aligned} \quad (6.15)$$

where $\mathbb{I}$ is a fourth-order identity tensor for symmetric second-order tensors, and the sign functions $H^-(\text{tr}(\boldsymbol{\varepsilon}))$ and $H^+(\text{tr}(\boldsymbol{\varepsilon}))$ are defined as follows:

$$H^-(\text{tr}(\boldsymbol{\varepsilon})) = \begin{cases} 1 & \text{if } \text{tr}(\boldsymbol{\varepsilon}) < 0 \\ 0 & \text{if } \text{tr}(\boldsymbol{\varepsilon}) \geq 0 \end{cases}; \quad H^+(\text{tr}(\boldsymbol{\varepsilon})) = \begin{cases} 0 & \text{if } \text{tr}(\boldsymbol{\varepsilon}) < 0 \\ 1 & \text{if } \text{tr}(\boldsymbol{\varepsilon}) \geq 0 \end{cases}. \quad (6.16)$$

The external work is expressed as

$$W_{\text{ext}} = \int_{\Omega} \mathbf{b} \cdot \mathbf{u} d\Omega + \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \mathbf{u} d\Gamma. \quad (6.17)$$

The total potential energy of a cracked body is represented as $\Pi(\mathbf{u}, d) = E(\mathbf{u}, d) - W_{\text{ext}}$.

Thus, governing equations are obtained as follows:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \quad (6.18)$$

$$2(1-d)H - \frac{G_c}{l} \left\{ d - l^2 \left[\Delta d + \alpha (\Delta d - \mathbf{M} \otimes \mathbf{M} : \nabla \nabla d) \right] \right\} = 0 \quad (6.19)$$

with the boundary conditions being as

$$\begin{cases} \boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} & \text{on } \Gamma_t \\ \mathbf{u} = \bar{\mathbf{u}} & \text{on } \Gamma_u \\ \nabla d \cdot \mathbf{n} = 0 & \text{on } \Gamma_c \end{cases} \quad (6.20)$$

A local history field of the maximum positive strain energy H is used to enforce the irreversibility condition as follows [179]:

$$H(\mathbf{x}, \tau) = \max_{\tau \in [0, T]} \psi^+(\boldsymbol{\varepsilon}(\mathbf{x}, \tau)). \quad (6.21)$$

The pre-existing cracks can be modeled in the following manners: (i) discrete

representations implemented by meshes; (ii) Dirichlet boundary conditions of the phase-field variable with $d = 1$ on the crack surface; (iii) initial strain-history field [190]. Herein, we adopt the third manner to model an initial crack as

$$H(\mathbf{x}, 0) = B \begin{cases} \frac{G_c}{2l} \left(1 - \frac{2d'(\mathbf{x}, \Gamma_c)}{l} \right) & d'(\mathbf{x}, \Gamma_c) \leq \frac{l}{2} \\ 0 & d'(\mathbf{x}, \Gamma_c) > \frac{l}{2} \end{cases} \quad (6.22)$$

with $B = 1/(1-d)$, where $d'(\mathbf{x}, \Gamma_c)$ is the shortest distance from a point $\mathbf{x}$ to the crack surface. We consider $B = 1000$ for all the numerical simulations.

6.1.2. Numerical implementation

The decoupled governing Eqs. (6.18) and (6.19) for the phase-field and displacement problems, respectively, are discretized within a numerical framework based on the IGA-meshfree approach, and then solved using a staggered scheme.

(a) Discretization of the phase-field equation

The weak form of Eq. (6.19) for a phase-field problem is expressed as

$$\int_{\Omega} 2(1-d)H\delta d d\Omega - \int_{\Omega} \frac{G_c}{l} \left\{ d - l^2 \left[\Delta d + \alpha(\Delta d - \mathbf{M} \otimes \mathbf{M} : \nabla \nabla d) \right] \right\} \delta d d\Omega = 0, \quad (6.23)$$

where δd is a test function. By using the divergence theorem, Eq. (6.23) can be rearranged as

$$\begin{aligned} & \int_{\Omega} \left(2H + \frac{G_c}{l} \right) d \delta d d\Omega + \int_{\Omega} G_c l \nabla d \cdot \left[\mathbf{I} + \alpha(\mathbf{I} - \mathbf{M} \otimes \mathbf{M}) \right] \cdot \nabla \delta d d\Omega \\ & = \int_{\Omega} 2H \delta d d\Omega \end{aligned} \quad (6.24)$$

The phase-field variable d and test function δd are approximated by the IGA-meshfree basis function Φ_i and nodal phase-field variable as follows:

$$d(\xi) = \sum_{i=1}^n \Phi_i(\xi) d_i(\xi) = \mathbf{\Phi} \mathbf{d}, \quad \delta d(\xi) = \sum_{i=1}^n \Phi_i(\xi) \delta d_i(\xi) = \mathbf{\Phi} \delta \mathbf{d}. \quad (6.25)$$

By substituting Eq. (6.25) for both d and δd in Eq. (6.24), a linear phase-field equation is obtained as

$$\mathbf{K}_d \mathbf{d} = \mathbf{F}_d \quad (6.26)$$

with

$$\begin{cases} \mathbf{K}_d = \int_{\Omega} \left(2H + \frac{G_c}{l} \right) \boldsymbol{\Phi}^T \boldsymbol{\Phi} d\Omega + \int_{\Omega} G_c l \mathbf{B}_d^T [\mathbf{I} + \alpha (\mathbf{I} - \mathbf{M} \otimes \mathbf{M})] \mathbf{B}_d d\Omega \\ \mathbf{F}_d = \int_{\Omega} 2 \boldsymbol{\Phi}^T H d\Omega \end{cases}, \quad (6.27)$$

where the matrix $\mathbf{B}_d$ is expressed as $\mathbf{B}_d = [\mathbf{B}_d^1 \quad \mathbf{B}_d^2 \quad \cdots \quad \mathbf{B}_d^n]$ with

$$\mathbf{B}_d^i = \begin{cases} \begin{bmatrix} \frac{\partial \Phi_i}{\partial x} & \frac{\partial \Phi_i}{\partial y} \end{bmatrix}^T & \text{for two dimensions} \\ \begin{bmatrix} \frac{\partial \Phi_i}{\partial x} & \frac{\partial \Phi_i}{\partial y} & \frac{\partial \Phi_i}{\partial z} \end{bmatrix}^T & \text{for three dimensions} \end{cases}. \quad (6.28)$$

The Gaussian quadrature rule is used to calculate both $\mathbf{K}_d$ and $\mathbf{F}_d$ in Eq. (6.27). Herein, the quadrilateral mesh is used for the domain integration, and the Voronoi mesh is used to define the grain geometry in polycrystals. To consider the anisotropic fracture energy of polycrystals in integrating the stiffness matrix, the following steps are conducted. First, the Voronoi tessellation algorithms are utilized to generate grains with random sizes and shapes for polycrystalline materials [214, 215]. Second, the material anisotropy related to the preferential cleavage plane is defined by Eq. (6.8) for each grain. Third, the anisotropy corresponding to a grain i is applied to the quadrature points that are located inside the grain i . Thus, the anisotropic fracture energy associated with each quadrature point is considered in the domain integration.

(b) Discretization of the displacement equation

By using the Lagrange multiplier method to enforce the essential boundary condition in Eq. (6.20), the weak form of Eq. (6.18) for a displacement problem is

expressed as

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d) : \boldsymbol{\varepsilon}(\delta \mathbf{u}) d\Omega - \int_{\Omega} \mathbf{b} \cdot \delta \mathbf{u} d\Omega - \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \delta \mathbf{u} d\Gamma = \\ \int_{\Gamma_u} \boldsymbol{\lambda} \cdot \delta \mathbf{u} d\Gamma + \int_{\Gamma_u} \delta \boldsymbol{\lambda} \cdot (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma \end{aligned}, \quad (6.29)$$

where $\boldsymbol{\lambda}$ is a Lagrange multiplier.

The displacement field $\mathbf{u}$ and test function $\delta \mathbf{u}$ are approximated as follows:

$$\mathbf{u}(\boldsymbol{\xi}) = \sum_{i=1}^n \Phi_i(\boldsymbol{\xi}) u_i(\boldsymbol{\xi}) = \boldsymbol{\Phi} \mathbf{u}, \quad \delta \mathbf{u}(\boldsymbol{\xi}) = \sum_{i=1}^n \Phi_i(\boldsymbol{\xi}) \delta u_i(\boldsymbol{\xi}) = \boldsymbol{\Phi} \delta \mathbf{u}. \quad (6.30)$$

The Lagrange multiplier $\boldsymbol{\lambda}$ is approximated as

$$\boldsymbol{\lambda} = \sum_{i=1}^m Z_i(\boldsymbol{\xi}) \lambda_i(\boldsymbol{\xi}) \quad (6.31)$$

with $Z_i(\boldsymbol{\xi})$ being a set of boundary basis functions. Three choices of the interpolation space for the Lagrange multiplier have been investigated [216]: (i) the finite element interpolation on Γ_u , (ii) the meshfree approximation on Γ_u , and (iii) the point collocation method with Dirac delta function. Due to the convenient implementation, the point collocation method is used to construct the basis function $Z_i(\boldsymbol{\xi}) = \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_i)$ where $\boldsymbol{\xi}_i$ is a set of meshfree points along Γ_u , and δ is the Dirac delta function.

The substitution of the variables in Eq. (6.29) by the corresponding expressions in Eqs. (6.30) and (6.31) yields a linear displacement system

$$\begin{bmatrix} \mathbf{K}_u & \mathbf{G} \\ \mathbf{G} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ \mathbf{Q} \end{Bmatrix} \quad (6.32)$$

with

$$\begin{cases} \mathbf{K}_u = \int_{\Omega} \mathbf{B}_u^T \mathbf{C} \mathbf{B}_u d\Omega \\ \mathbf{F}_u = \int_{\Omega} \boldsymbol{\Phi}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \boldsymbol{\Phi}^T \bar{\mathbf{t}} d\Gamma \\ \mathbf{G}_{ij} = - \int_{\Gamma_u} \Phi_i(\boldsymbol{\xi}) Z_j(\boldsymbol{\xi}) d\Gamma = - \Phi_i(\boldsymbol{\xi}_j) \mathbf{R} \\ \mathbf{Q}_j = - \int_{\Gamma_u} Z_j(\boldsymbol{\xi}) \bar{\mathbf{u}}(\boldsymbol{\xi}) d\Gamma = - \mathbf{R} \bar{\mathbf{u}}(\boldsymbol{\xi}_j) \end{cases}, \quad (6.33)$$

where $\mathbf{R}$ is a diagonal matrix with $R_{ii} = 1$ and $R_{ij} = 0$ ($i \neq j$) if the essential boundary condition is enforced in the x_i direction, and the matrix $\mathbf{B}_u$ is described as

$\mathbf{B}_u = [\mathbf{B}_u^1 \quad \mathbf{B}_u^2 \quad \cdots \quad \mathbf{B}_u^n]$ with

$$\mathbf{B}_u^i = \begin{cases} \begin{bmatrix} \frac{\partial \Phi_i}{\partial x} & 0 & \frac{\partial \Phi_i}{\partial y} \\ 0 & \frac{\partial \Phi_i}{\partial y} & \frac{\partial \Phi_i}{\partial x} \end{bmatrix}^T & \text{for two dimensions} \\ \begin{bmatrix} \frac{\partial \Phi_i}{\partial x} & 0 & 0 & \frac{\partial \Phi_i}{\partial y} & 0 & \frac{\partial \Phi_i}{\partial z} \\ 0 & \frac{\partial \Phi_i}{\partial y} & 0 & \frac{\partial \Phi_i}{\partial x} & \frac{\partial \Phi_i}{\partial z} & 0 \\ 0 & 0 & \frac{\partial \Phi_i}{\partial z} & 0 & \frac{\partial \Phi_i}{\partial y} & \frac{\partial \Phi_i}{\partial x} \end{bmatrix}^T & \text{for three dimensions} \end{cases} \quad (6.34)$$

6.1.3. Adaptive mesh refinement

In this section, an error estimator consisting of both the phase field and its gradient is introduced to determine the crack surfaces and initiation regions, where mesh refinement is implemented by directly adding nodes in a meshfree manner. It is to be noted that the refinement criterion introduced in Chapter 5 is adopted.

The phase-field variable that is treated as an error estimator can be used directly to determine the elements around crack surfaces, whereas the estimator is not efficient for crack initiation [189]. Mesh refinement before crack initiation or growth can give accurate solutions. Consider the high gradient of a phase-field variable at the crack tip and initiation regions. Furthermore, the gradient of a phase-field variable needs to be sufficiently resolved for accurate crack presentations. Thus, the gradient of a phase-field variable is used as an error estimator for identifying elements at crack surfaces and initiation regions.

For an element k with the area A_k and the number of gauss integration points N_k ,

its averaged mesh intensity is defined as

$$I_k^M = G_k^{\text{PF}} \frac{A_k}{N_k}, \quad (6.35)$$

where G_k^{PF} is the averaged gradient of a phase-field variable:

$$G_k^{\text{PF}} = \frac{\sum_{a=1}^{N_k-1} \sum_{b=a+1}^{N_k} \frac{|d_a - d_b|}{l_{ab}}}{N_k!}, \quad (6.36)$$

where l_{ab} is the distance between two integration points. The phase-field variable is approximated by Eq. (6.25). By comparing the values of averaged mesh intensities with a threshold, the elements with large phase-field gradients can be determined.

6.1.4. Solution procedure

This section summarizes the solution procedure for the phase-field modeling of the polycrystalline materials. In the solution procedure, the staggered scheme without iteration is used to solve the phase field and displacement variables sequentially. It is noted that small loading increments are required to ensure accurate results when the iteration is not considered [217]. The procedure includes the following five main steps:

- (I) Definition of the geometry via the data in IGA including knot vectors, control points and weights;
- (II) Definition of pre-existing cracks via the initial history field;
- (III) Definition of the geometry of grains via Voronoi tessellation algorithms;
- (IV) AMR and update on IGA-meshfree basis functions;
- (V) Staggered scheme to solve both displacement and phase-field variables.

In step I, both the mesh and geometry are defined by the control points and weights in IGA. Linear reproducing points of knot vectors are defined as the meshfree nodes with the normalized support size of $(p + 1)/2$. Step II deals with the pre-existing

cracks that are defined by the initial history field in Eq. (6.22). The Voronoi mesh is generated as the geometry of grains in polycrystalline materials in step III. Grains with different sizes and locations are randomly generated.

Step IV deals with the AMR. First, the error indicator based on both the phase-field variable and its gradient is defined. Second, the elements that are identified by the error estimator are refined, and corresponding knot vectors are updated. Third, the linear reproducing points associated with the updated knot vectors are calculated as the updated meshfree nodes. Basis functions related to the meshfree nodes are calculated by Eq. (4.11).

Step V deals with the phase-field and displacement variables that are solved in a staggered manner within a displacement controlled incremental analysis. Consider the total loading steps N_{step} , maximum AMR level N_{ref} and the known variables ($\mathbf{u}_i$, d_i , H_i and M_i) at the i th loading step. The variables ($\mathbf{u}_{i+1}$, d_{i+1} , H_{i+1} and M_{i+1}) at the $(i+1)$ th loading step can be updated in the following processes. By initializing mesh as $M_{i+1,1} = M_i$, the history field and phase-field variable on mesh $M_{i+1,j}$ ($j = 1, 2, \dots, N_{\text{ref}}$) are, respectively, obtained as $H_{i+1,j}$ and $d_{i+1,j}$ with the frozen displacement $\mathbf{u}_i$ until the refinement stopping criterion is met ($j = N_{\text{ref}}$). It is noted that $\mathbf{u}_i$ is used to calculate the sign functions in Eq. (6.16). Then $d_{i+1,j}$ is updated as the phase field variable d_{i+1} at the $(i+1)$ th loading step. Subsequently, the displacement $\mathbf{u}_{i+1}$ is solved by using d_{i+1} . The staggered scheme with AMR is summarized in Algorithm I as follows:

Algorithm I: Staggered scheme with AMR at the $(i+1)$ th loading step.

Input: displacement $\mathbf{u}_i$, phase-field variable d_i , history field H_i and mesh M_i .

(i) Initialize mesh: $M_{i+1,1} = M_i$.

(ii) **for** $j = 1, 2, \dots, N_{\text{ref}}$

Calculate the history field $H_{i+1,j} = \max \left(\psi_j^+ \left(\boldsymbol{\varepsilon}(\mathbf{u}_i) \right), \bar{H}_i \right)$, where $\bar{H}_i$ is projected from mesh M_i to refined mesh $M_{i+1,j}$ via data transfer [13];

Calculate the phase-field variable $d_{i+1,j}$ via the equation

$$2(1-d_{i+1,j})H_{i+1,j} - \frac{G_c}{l} \left\{ d_{i+1,j} - l^2 \left[\Delta d_{i+1,j} + \alpha (\Delta d_{i+1,j} - \mathbf{M} \otimes \mathbf{M} : \nabla \nabla d_{i+1,j}) \right] \right\} = 0,$$

which is discretized as

$$\left\{ \int_{\Omega} \left(2H_{i+1,j} + \frac{G_c}{l} \right) \boldsymbol{\Phi}^T \boldsymbol{\Phi} d\Omega + \int_{\Omega} G_c l \mathbf{B}_{d_{i+1,j}}^T \left[\mathbf{I} + \alpha (\mathbf{I} - \mathbf{M} \otimes \mathbf{M}) \right] \mathbf{B}_{d_{i+1,j}} d\Omega \right\} \mathbf{d}_{i+1,j};$$

$$= \int_{\Omega} 2\boldsymbol{\Phi}^T H_{i+1,j} d\Omega$$

Implement AMR if $j < N_{\text{ref}}$. Refined mesh $M_{i+1,j+1}$ and its corresponding meshfree nodes and basis functions are obtained (see Section 5.3).

end

(iii) Update the variables at the $(i+1)$ th step: $H_{i+1} = H_{i+1,N_{\text{ref}}}$; $M_{i+1} = M_{i+1,N_{\text{ref}}}$; $d_{i+1} = d_{i+1,N_{\text{ref}}}$.

(iv) Calculate the displacement $\mathbf{u}_{i+1}$ via the equation

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_{i+1}, d_{i+1}) + \mathbf{b} = \mathbf{0}.$$

Output: displacement $\mathbf{u}_{i+1}$, phase-field variable d_{i+1} , history field H_{i+1} and mesh M_{i+1} at the $(i+1)$ th loading step.

6.2 Numerical results

The IGA-meshfree approach is applied to the adaptive phase-field modeling of fracture. First, the isotropic brittle fracture is simulated to demonstrate the robustness of the present approach. Then, the anisotropic brittle fracture in polycrystalline materials is investigated. The penalty parameter α is chosen as 10 to control the crack

paths along the preferential cleavage planes. Plane-strain condition is assumed for 2D examples. Furthermore, the Dirichlet boundary conditions are enforced at a set of points along Γ_u based on the Lagrange multiplier method.

6.2.1. An edge-cracked plate subjected to tensile or shear loading

A 2D edge-cracked plate subjected to tension or shear loading is considered (Fig. 6.3). The material and geometry parameters include Lamé constants $\lambda = 121.15 \times 10^3$ N/mm², $\mu = 80.77 \times 10^3$ N/mm² and energy release rate $G_c = 2.7$ N/mm. The prescribed displacement is enforced at the top edge along the y and x directions for the tension and shear tests, respectively.

In the tension test, a pure Mode-I fracture is simulated with the displacement control strategy [179] that uses the displacement increment $\Delta u = 10^{-5}$ mm until $u = 5 \times 10^{-3}$ mm and $\Delta u = 10^{-6}$ mm until final failure. The length scale parameter l is set as 0.0075 mm. Fig. 6.4 shows the adaptively refined mesh and corresponding contour plot of the phase-field variable. During crack propagation, elements are dynamically identified and refined until element sizes near the crack surface are less than $l/2$. Furthermore, the AMR results in 4434 elements with 13668 DOFs at the end of simulations, which is more computationally efficient than the uniform refinement with 20000 triangular elements [179] and 12735 quadrilateral elements [218]. The contour plot demonstrates a straight crack propagation path that is consistent with the theoretical results. Fig. 6.5 visualizes the load-displacement curves that are in good agreement with the reference solutions [179, 219].

Fig. 6.3. Geometry (units in mm) and boundary conditions of an edge-cracked plate subjected to (a) tensile loading and (b) shear loading.

Fig. 6.4. The tension test at the final displacement step with $l = 0.0075$ mm: (a) the adaptively refined mesh and (b) corresponding phase-field contour plot.

Fig. 6.5. The load-displacement curves of a plate subjected to tensile loading.

The shear test aims to simulate a pure Mode-II fracture with a displacement increment $\Delta u = 10^{-5}$ mm until final failure and the length scale parameter $l = 0.008$ mm. Fig. 6.6 visualizes the refined mesh and relevant contour plot of the phase-field variable at the final displacement step. The simulated results indicate that the crack grows towards the bottom right corner until the final fracture, which agrees well with reference solutions [179, 219]. Moreover, AMR results in 5292 elements with 16383 DOFs, thus saving the computational cost compared to both the uniform refinement with 30000 triangular elements [179] and adaptive refinement with 8664 quadrilateral elements [220] implemented via FEM. Fig. 6.7 visualizes the load-displacement curves of a shear test. It is noted that the traction suddenly drops around $u_x = 0.016$ mm since a crack is initiated at the bottom right corner, and this phenomenon has been discussed in [218].

Fig. 6.6. The shear test at the final displacement step with $l = 0.008$ mm: (a) the adaptively refined mesh and (b) corresponding phase-field contour plot.

Fig. 6.7. The load-displacement curves of a plate subjected to shear loading.

6.2.2. A bi-material plate subjected to tensile loading

An edge-cracked bi-material specimen subjected to tensile loading is investigated, as shown in Fig. 6.8. The top part (material 1) includes following material properties: $E = 377$ kN/mm², $\nu = 0.3$, $G_c = 0.01$ kN/mm and length scale $l = 0.3$ mm. The bottom part (material 2) that is softer than the top part includes following material properties: $E = 37.7$ kN/mm², $\nu = 0.3$, $G_c = 0.001$ kN/mm and length scale $l = 0.3$ mm. The displacement is enforced at the right edge along the x -direction with an increment $\Delta u =$

10^{-4} mm until $u = 4 \times 10^{-2}$ mm and then $\Delta u = 10^{-5}$ mm until the end of simulations.

Fig. 6.8. Geometry (units in mm) and boundary conditions of the cracked bi-material plate.

Fig. 6.9 shows the force-displacement curve with a kink point where crack branching occurs. Fig. 6.10 visualizes the refined mesh and contour plot of the phase-field variable at different crack propagation stages. It is evident that the crack initially grows along the vertical direction up to the interface of the two materials, after which the crack would branch and grow along the interface because material 1 is harder than material 2. The present fracture pattern is the same as the reference results [219, 221].

Fig. 6.9. The load-displacement curve of the bi-material plate subjected to tensile loading.

Fig. 6.10. The adaptively refined mesh and corresponding phase-field contour plot for an edge-cracked bi-material plate at different propagation stages.

6.2.3. A three-point bending test on a 3D skew notched beam

This example considers a 3D skew notched beam subjected to three-point

bending. Fig. 6.11 visualizes the geometry and boundary conditions of the notched beam. The Dirichlet boundary conditions are enforced at a set of points along Γ_u as follows: $u_x = u_y = u_z = 0$ along the edge ($y = 250$ and $z = 0$), $u_x = u_z = 0$ along the edge ($y = 10$ and $z = 0$), and $u_z = -u$ along the edge ($y = 130$ and $z = 60$). The material and PFM parameters are as follows: Lamé constants $\lambda = 3.21$ kN/mm², $\mu = 1.02$ kN/mm², $G_c = 5 \times 10^{-4}$ kN/mm and the length scale parameter $l = 1.8$ mm. Herein, the notch is treated as an initial crack modeled by the history field H .

Fig. 6.11. Geometry (units in mm) and boundary conditions of the 3D skew notched beam.

The load-displacement curve of a three-point bending test is visualized in Fig. 6.12. As this notched beam is skew-symmetric with respect to the middle planes, the crack would propagate under the combination of Mode-I and Mode-III loading conditions, causing a non-planar crack surface. Fig. 6.13 shows the adaptively refined meshfree points with red denoting the crack topology during crack propagation. It is found that the crack initially twists along the central vertical axis until the crack surface is parallel to the transversal plane, and then grows upwards along the transversal plane until the end of simulations. The simulated crack surface represented by points ($d > 0.9$) is visualized in Fig. 6.14 where the crack propagation patterns are in good agreement with reference solutions [189, 222, 223].

Fig. 6.12. The load-displacement curve of a 3D skew notched beam.

Fig. 6.13. Adaptively refined meshfree points for a skew notched beam at different propagation stages: (a) $u = 0.64$ mm, (b) $u = 0.72$ mm, (c) $u = 0.84$ mm and (d) $u = 0.86$ mm.

Fig. 6.14. The simulated crack surface represented by points for a 3D skew notched beam: (a) global view, (b) local x - y - z view and (c) local y - z view.

6.2.4. Fracture simulation of the Brazilian splitting test

A Brazilian disc under compression is investigated to obtain the tensile strength. The geometry and boundary conditions of the Brazilian splitting test are visualized in Fig. 6.15. The material parameters include $E = 5 \times 10^4$ N/mm², $\nu = 0.2$, $G_c = 0.07$ N/mm and length scale $l = 1$ mm.

Since the maximum tensile stress observed in the center of the disc controls the crack initiation [224], the principal stress criterion is used in the simulation. Fig. 6.16 visualizes the fracture behaviors including crack initiation and propagation, in which a crack initiates at the center and propagates along a straight line until final fracture. Fig. 6.17 shows the stress-strain curve where the tensile stress and strain are calculated as

follows:

$$\begin{aligned}\sigma &= \frac{2F}{\pi DL} \\ \varepsilon &= \frac{2u}{D}\end{aligned}\tag{6.37}$$

with F being the loading. The maximum tensile stress is 19.84 MPa at the strain of 0.0023 that corresponds to a displacement of 0.0058 mm. The simulated results are in good agreement with the reference results [224], including maximum tensile stress of 19 MPa and displacement of 0.006 mm.

Fig. 6.15. Geometry (units in mm) and boundary conditions of the Brazilian splitting test.

Fig. 6.16. Fracture behaviors of the Brazilian test: (a) crack initiation and (b) final fracture.

Fig. 6.17. The stress-strain curve of a Brazilian splitting test.

6.2.5. Crack propagation in a 2D polycrystalline material

Anisotropic crack propagation in a 2D polycrystalline structure with 54 grains is investigated. Fig. 6.18(a) and (b) depict the geometry, boundary conditions and angles corresponding to anisotropy for this polycrystalline structure. The material properties are set as follows: $\lambda = 121.15 \times 10^3 \text{ N/mm}^2$, $\mu = 80.77 \times 10^3 \text{ N/mm}^2$, $G_c = 2.7 \text{ N/mm}$, crack length $a = 1/15 \text{ mm}$ and length scale $l = 0.01 \text{ mm}$. The prescribed displacement is enforced at the top edge along the y -direction with an increment $\Delta u = 10^{-5} \text{ mm}$ until $u = 6.5 \times 10^{-3} \text{ mm}$ and $\Delta u = 10^{-6} \text{ mm}$ until final fracture.

The mesh at the end of simulations is shown in Fig. 6.18(c) where the elements near a crack surface are refined until element sizes are small enough for the regularized representation of a sharp crack topology. The load-displacement curve of a 2D polycrystalline structure is shown in Fig. 6.18(d), where the traction increases linearly before the sudden drop at the final failure. Fig. 6.19 visualizes the contour plots of a phase-field variable at different propagation stages. Both the transgranular and intergranular fracture patterns are captured, and the latter fracture occurs

depending on the angles of preferential cleavage planes.

Fig. 6.18. A 2D polycrystalline structure with 54 grains: (a) the geometry (units in mm) and boundary conditions; (b) angles of the direction of anisotropy; (c) the refined mesh; (d) the load-displacement curve.

Fig. 6.19. Phase-field contour plots for the 2D polycrystalline structure: (a) $u = 6.5 \times 10^{-3}$ mm, (b) $u = 7.65 \times 10^{-3}$ mm, (c) $u = 8.25 \times 10^{-3}$ mm and (d) $u = 8.6 \times 10^{-3}$ mm.

6.2.6. Crack propagation in a 3D polycrystalline material

This example investigates crack propagation in a 3D polycrystalline unit cube with 200 grains. Fig. 6.20(a) and (b) show the geometry, boundary conditions and anisotropic angles within each grain for a 3D polycrystalline structure. The prescribed displacement is enforced at the top surface along the z -direction. The material properties include $\lambda = 165.7 \times 10^3$ N/mm², $\mu = 63.9 \times 10^3$ N/mm², length scale $l = 0.015$ mm and $G_c = 2.04$ N/mm.

The contour plot of a phase-field variable at the end of simulations is shown in

Fig. 6.20 (c), in which cracks are dynamically tracked with refined meshfree points around the crack surfaces. As the mesh away from crack surfaces remains coarse, AMR is more computationally efficient than global uniform refinement. Fig. 6.20(d) shows the load-displacement curve where the traction along the top surface initially increases and then suddenly drops until the end of simulations. Crack surfaces described by the refined nodes ($d > 0.9$) are visualized in Fig. 6.21, where the complex fracture patterns, including crack initiation and coalescence, are captured for the 3D polycrystalline structure.

Fig. 6.20. A 3D polycrystalline structure with 200 grains: (a) the geometry (units in

mm) and boundary conditions; (b) angles of the direction of anisotropy; (c) the contour plot of the phase-field variable; (d) the load-displacement curve.

Fig. 6.21. Crack surfaces of the 3D polycrystalline structure at different propagation stages: (a) $u = 5 \times 10^{-4}$ mm, (b) $u = 1.65 \times 10^{-2}$ mm, (c) $u = 1.95 \times 10^{-2}$ mm and (d) $u = 2.1 \times 10^{-2}$ mm.

6.3. Summary

An adaptive IGA-meshfree approach is developed for the phase-field modeling of crack propagation in a 3D polycrystalline material. The IGA-meshfree approach implements the 3D formulations that obtain the equivalence between IGA and the EFGM, thus inheriting the advantage of flexible mesh refinement from the meshfree

method and exact geometry representations from IGA. The 3D formulations are combined with an AMR strategy that utilizes a phase-field-based error estimator. Subsequently, the present approach is applied for the phase-field modeling that introduces the anisotropy by the crack density functional. The locally refined mesh around crack surfaces helps to capture the regularized representations of sharp crack topologies. In this way, propagating cracks are dynamically tracked without requiring *a priori* knowledge of crack growth paths.

Brittle fracture in both 2D and 3D isotropic materials is simulated to validate the present approach. The expected crack propagation paths are accurately predicted with AMR. Moreover, the present approach can capture complex crack propagation patterns, such as twisting and branching, especially for the 3D simulations without describing crack topologies. Anisotropic brittle fracture in 2D and 3D polycrystalline materials is then investigated. The intergranular and transgranular fracture behaviors are well captured in the simulation results, which further demonstrate the robustness of the present approach.

Chapter 7 Conclusions and Recommendations

7.1. Conclusions

This Ph.D. study develops robust, efficient and accurate IGA-based approaches for modeling the deformation and fracture behaviors of solid materials and structures with defects. The IGA-based approaches integrate the advantages of the IGA and EFGM, such as the higher-order field approximations, exact geometry representations and flexible mesh refinement. The gradient-based error indicator is proposed for achieving AMR within the framework of the proposed approaches without requiring exact stress or strain. The IGA-based approaches can not only accurately perform the structural analysis but also efficiently predict the fracture behaviors of the solids and structures.

Firstly, a coupling approach of the IGA and EFGM is developed for the linear and geometrically nonlinear elastic analyses of shell structures. The coupling approach utilizes three subdomains: the IGA subdomain for exact geometry representation, the meshfree subdomain for local refinement flexibility and the coupling subdomain for smoothness. The IGA subdomain and meshfree subdomain are respectively approximated by the NURBS and MLS shape functions, while the coupling subdomain is approximated by both functions based on the reproducing conditions. This coupling approach is capable of conveniently achieving the C^1 continuity required in the KL-shell theory. Linear and nonlinear behaviors of shell structures are investigated via this coupling approach with local mesh refinement. The numerical results are verified by the reference solutions, demonstrating that the coupling approach can produce a higher convergence rate than the IGA and meshfree methods.

Secondly, a strong form-based IMMLS-C approach is developed to investigate 2D linear elastic fracture problems with contact loading. The IMMLS-C approach

inherits advantages from both IGA and meshfree methods similarly to the previous coupling approach. Moreover, the IMMLS-C approach uses a unified formulation throughout the problem domain and achieves equivalence between the isogeometric basis functions and MLS shape functions without requiring domain subdivisions. The IMMLS-C approach treats the linear reproducing points as collocation points and achieves AMR by using a gradient-based error estimator. The approach can circumvent numerical integration and conveniently treat essential boundary conditions as compared with the weak-form approaches. In modeling cracks, the displacement discontinuity along crack surfaces is introduced by the visibility criterion. The singularity around crack tips is captured by AMR without adding extra enrichment functions, leading to a decrease in the DOFs as compared with the partition-of-unity-based methods, such as the XFEM. In modeling two contacting bodies with cracks, the contact constraints are regularized by a penalty algorithm, and the contact surfaces are exactly represented. Fracture parameters such as the SIFs in 2D plane structures are investigated. The numerical results demonstrate the accuracy and efficiency of the proposed approach.

Thirdly, crack propagation in thin-shell structures within the context of LEFM is investigated via a weak form-based IGA-meshfree MLS approach. Like the previous strong formulations, the weak ones are capable of conveniently implementing AMR based on a gradient-based error estimator. In modeling cracked thin shells, the approach is extended by introducing both the enrichment and Heaviside functions based on the PUM. SIFs of cracks in thin shells are derived from the equivalent domain integral. The crack propagation angle is determined by the maximum circumferential stress criterion. Crack-tip behaviors of the stationary and propagating cracks in shell structures are investigated. Both the SIFs and predicted propagation

paths are verified with reference solutions. In addition, the mesh is adaptively refined and moves along with the crack tips. The adaptive refinement strategy makes the approach achieve a higher convergence rate and computational efficiency than IGA and the meshfree method with global uniform mesh refinement.

Finally, the adaptive phase-field modeling of brittle fracture in a 3D polycrystalline material is investigated via a numerical framework based on the IGA-meshfree MLS approach. The 3D numerical framework is capable of achieving AMR by introducing a phase-field-based error estimator. Within the numerical framework, the anisotropy of fracture resistance within each grain generated by Voronoi tessellation algorithms is introduced for the adaptive phase-field modeling. In this way, propagating cracks are automatically determined, and the regularized representations of sharp crack topologies are obtained by AMR without requiring *a priori* knowledge of crack propagation paths. Fracture behaviors of both the 2D and 3D materials with the isotropic and anisotropic surface energy are studied, in which crack initiation, branching, twisting and coalescence are captured. Furthermore, the intergranular and transgranular crack propagation patterns in polycrystalline materials are simulated.

This Ph.D. study provides a robust, efficient and accurate numerical framework based on the IGA-based approaches for investigating the deformation and fracture mechanisms of solid materials and structures with cracks. The understanding of the mechanisms is of significant importance to the structural analysis and design.

7.2. Recommendations

The developed IGA-based approaches provide a promising numerical framework for modeling solid materials and structures with cracks. Although the linear elastic behavior has been modeled within the IGA-based numerical framework, plastic

deformation processes that occur in most of engineering materials and structures are of significant importance to understanding the failure mechanisms. In addition, micro-defects such as cracks, inclusions and voids may occur in solid materials and structures during the manufacturing or operation processes. Micro-defects have significant effects on failure behaviors because of their interactions with the neighboring cracks. Thus, the numerical framework can be further developed in the following aspects.

Most of engineering materials, in particular metals, experience plastic deformation before their final failures [225, 226]. In addition, the crack-tip stress is significantly large, resulting in a plastic zone ahead of a crack tip. To consider plastic deformation and its effect on fracture patterns, an elastoplastic model that contains a yield function, a hardening law and an evolution equation should be constructed. The plastic model may result in low computational efficiency and lead to convergence issues due to material nonlinearity. These issues can be solved by the numerical framework based on the IGA-based approaches with the consideration of efficient iteration schemes.

Micro-defects such as inclusions can interact with their neighboring cracks and have a significant influence on the displacement and stress fields in the vicinity of the crack tips, thus influencing the crack propagation path. To clearly understand the effect of micro-defects on crack growth, various methods have been developed [227, 228], in which a very fine mesh around crack tips is needed to capture the accurate stress field. This requirement results in low computational efficiency when many micro-defects are considered for investigating their interactions with the cracks. This issue can be solved by integrating the multiscale method with the adaptive IGA-based approaches.

Appendix A. Kinematics of Thin Shells

In the initial configuration, a material point has the following positions based on the KL theory:

$$\mathbf{X}_0(\xi^1, \xi^2, \xi^3) = \boldsymbol{\varphi}_0(\xi^1, \xi^2) + \xi^3 \mathbf{A}_3(\xi^1, \xi^2), \quad (\text{A.1})$$

where ξ^1 and ξ^2 represent curvilinear coordinates in the mid-surfaces, ξ^3 represents the coordinate in the thickness, $\boldsymbol{\varphi}_0$ is the position in the mid-surfaces, and $\mathbf{A}_3$ is a vector normal to the mid-surfaces. Similarly, the position in the deformed configuration is obtained as

$$\mathbf{x}(\xi^1, \xi^2, \xi^3) = \boldsymbol{\varphi}(\xi^1, \xi^2) + \xi^3 \mathbf{a}_3(\xi^1, \xi^2). \quad (\text{A.2})$$

The displacement of a material point on the middle surface is calculated as

$$\mathbf{u}(\xi^1, \xi^2) = \boldsymbol{\varphi}(\xi^1, \xi^2) - \boldsymbol{\varphi}_0(\xi^1, \xi^2). \quad (\text{A.3})$$

By taking the derivatives of point $\mathbf{X}_0$ with respect to ξ^i ($i = 1, 2, 3$), the covariant basis vectors in the initial configuration are obtained as follows:

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{X}_0}{\partial \xi^\alpha} = \frac{\partial \boldsymbol{\varphi}_0}{\partial \xi^\alpha} + \xi^3 \frac{\partial \mathbf{A}_3}{\partial \xi^\alpha} = \mathbf{A}_\alpha + \xi^3 \frac{\partial \mathbf{A}_3}{\partial \xi^\alpha} \quad (\alpha = 1, 2), \quad (\text{A.4})$$

$$\mathbf{G}_3 = \frac{\partial \mathbf{X}_0}{\partial \xi^3} = \mathbf{A}_3. \quad (\text{A.5})$$

The covariant basis vectors in the deformed configuration are given as

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{x}}{\partial \xi^\alpha} = \frac{\partial \boldsymbol{\varphi}}{\partial \xi^\alpha} + \xi^3 \frac{\partial \mathbf{a}_3}{\partial \xi^\alpha} = \mathbf{a}_\alpha + \xi^3 \frac{\partial \mathbf{a}_3}{\partial \xi^\alpha}, \quad (\text{A.6})$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{x}}{\partial \xi^3} = \mathbf{a}_3. \quad (\text{A.7})$$

The unit normal vectors of the middle surface in the two configurations are obtained as follows:

$$\mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{\|\mathbf{A}_1 \times \mathbf{A}_2\|}, \quad \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}. \quad (\text{A.8})$$

The covariant metric coefficients satisfy the following equations:

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad (\text{A.9})$$

$$A_{ij} = \mathbf{A}_i \cdot \mathbf{A}_j, \quad a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j. \quad (\text{A.10})$$

The contravariant basis vectors and corresponding metric coefficients in two configurations are obtained as follows:

$$\mathbf{G}^i = G^{ij} \mathbf{G}_j, \quad G^{ij} = \frac{1}{G_{ij}}, \quad (\text{A.11})$$

$$\mathbf{A}^i = A^{ij} \mathbf{A}_j, \quad A^{ij} = \frac{1}{A_{ij}}, \quad (\text{A.12})$$

$$\mathbf{g}^i = g^{ij} \mathbf{g}_j, \quad g^{ij} = \frac{1}{g_{ij}}, \quad (\text{A.13})$$

$$\mathbf{a}^i = a^{ij} \mathbf{a}_j, \quad a^{ij} = \frac{1}{a_{ij}}. \quad (\text{A.14})$$

The Green–Lagrange strain based on the KL theory can be obtained as

$$\mathbf{E} = E_{\alpha\beta} \mathbf{G}^\alpha \otimes \mathbf{G}^\beta, \quad E_{\alpha\beta} = \varepsilon_{\alpha\beta} + \zeta^3 \kappa_{\alpha\beta} \quad (\alpha, \beta = 1, 2), \quad (\text{A.15})$$

where $\varepsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ denote the membrane strain and the change of curvature, respectively [229]:

$$\varepsilon_{\alpha\beta} = \frac{1}{2} (\mathbf{a}_\alpha \cdot \mathbf{a}_\beta - \mathbf{A}_\alpha \cdot \mathbf{A}_\beta), \quad (\text{A.16})$$

$$\kappa_{\alpha\beta} = \mathbf{A}_{\alpha,\beta} \cdot \mathbf{A}_3 - \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3. \quad (\text{A.17})$$

Appendix B. MLS Approximation

The displacement fields are obtained based on the MLS approximation [33]:

$$u^h(\xi) = \sum_{J=1}^{\bar{m}} p_J(\xi) \bar{a}_J(\xi) = \mathbf{p}^T(\xi) \bar{\mathbf{a}}(\xi) \quad (\text{B.1})$$

with $\bar{m}$ being the total number of basis functions, $\mathbf{p}(\xi)$ being the basis functions and $\bar{\mathbf{a}}(\xi)$ being the coefficients. The 2D basis function with the p th order is given as

$$\mathbf{p}(\xi) = \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \dots, \xi^p, \dots, \eta^p\}^T. \quad (\text{B.2})$$

The quadratic, cubic and quartic polynomial basis vectors are given as

$$\mathbf{p}(\xi) = \begin{cases} \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2\}^T & (p=2) \\ \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3\}^T & (p=3) \\ \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4\}^T & (p=4) \end{cases} \quad (\text{B.3})$$

The vector $\bar{\mathbf{a}}(\xi)$ in Eq. (B.1) is obtained by

$$J = \sum_{I=1}^{N_M} \hat{W}(\xi - \xi_I) [\mathbf{p}^T(\xi_I) \bar{\mathbf{a}}(\xi) - u_I]^2 \quad (\text{B.4})$$

with N_M being the number of points and u_I being the nodal parameter. The field nodes and support domains of a square plate are shown in Fig. B.1.

Fig. B.1. The discretization of a square plate: field nodes and support domains.

The derivative of J with respect to $\bar{\mathbf{a}}(\xi)$ gives the following equation:

$$\bar{\mathbf{A}}(\xi) \bar{\mathbf{a}}(\xi) = \mathbf{B}\mathbf{U}, \quad (\text{B.5})$$

where $\mathbf{U} = \{u_1, u_2, \dots, u_{N_M}\}^T$ is the vector consisting of nodal parameters, and the weighted moment matrix $\bar{\mathbf{A}}(\xi)$ and matrix $\mathbf{B}(\xi)$ are calculated as

$$\bar{\mathbf{A}}(\xi) = \sum_{I=1}^{N_M} \hat{W}(\xi - \xi_I) \mathbf{p}(\xi_I) \mathbf{p}^T(\xi_I), \quad (\text{B.6})$$

$$\mathbf{B}(\xi) = \begin{bmatrix} \hat{W}(\xi - \xi_1) \mathbf{p}(\xi_1) & \hat{W}(\xi - \xi_2) \mathbf{p}(\xi_2) & \dots & \hat{W}(\xi - \xi_{N_M}) \mathbf{p}(\xi_{N_M}) \end{bmatrix}. \quad (\text{B.7})$$

The vector $\bar{\mathbf{a}}(\xi)$ is expressed as

$$\bar{\mathbf{a}}(\xi) = \bar{\mathbf{A}}(\xi)^{-1} \mathbf{B}\mathbf{U}. \quad (\text{B.8})$$

The MLS approximant can be obtained by substituting $\bar{\mathbf{a}}(\xi)$ into Eq. (B.1) as

$$u^h(\xi) = \sum_{I=1}^{N_M} \Psi_I(\xi) u_I(\xi) = \Psi(\xi) \mathbf{U}, \quad (\text{B.9})$$

where $\Psi_I(\xi)$ is expressed as

$$\Psi_I(\xi) = \sum_{J=1}^{\bar{m}} p_J(\xi) (\bar{\mathbf{A}}^{-1}(\xi) \mathbf{B}(\xi))_{JI} = \mathbf{p}^T(\xi) (\bar{\mathbf{A}}^{-1} \mathbf{B})_I. \quad (\text{B.10})$$

The quartic spline weight function $\hat{W}(\xi - \xi_I)$ is given as

$$\hat{W}(\xi - \xi_I) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & (r \leq 1) \\ 0 & (r > 1) \end{cases}, \quad r = \frac{|\xi - \xi_I|}{d_{max}^I}, \quad d_{max}^I = \lambda d_c, \quad (\text{B.11})$$

where λ is the dimensionless size, and d_c is the spacing between nodes. The commonly used range of λ is 2-4 [230].

By using the MLS approximation, the 2D consistency condition, which means that any complete order of basis functions is exactly reproduced, is defined as

$$\sum_{I=1}^{N_M} \Psi_I(\xi) \mathbf{p}(\xi_I) = \mathbf{p}(\xi). \quad (\text{B.12})$$

Appendix C. Reproducing Conditions for IGA

The consistency conditions for univariate B-spline basis functions are defined as

$$\sum_{i=1}^n N_i^p(\xi) \mathbf{p}(\xi_i^{[l]}) = \mathbf{p}(\xi), \quad (\text{C.1})$$

where $\mathbf{p}(\xi) = \{1, \xi, \xi^2, \dots, \xi^p\}^T$ denotes the 1D polynomial basis. The 1D

reproducing point vector $\mathbf{p}(\xi_i^{[l]})$ is defined as

$$\mathbf{p}(\xi_i^{[l]})^T = \left\{ 1, \xi_i^{[1]}, (\xi_i^{[2]})^2, \dots, (\xi_i^{[p]})^p \right\}, \quad (\text{C.2})$$

and reproducing points $\xi_i^{[l]} (l=1, 2, \dots, p)$ for the monomial ξ^l are calculated as follows [231]:

$$\xi_i^{[l]} = \sqrt[l]{\frac{S_p^l[G_{i+1}^{i+p}]}{C_p^l}}; \quad C_p^l = \frac{p!}{l!(p-l)!}, \quad (\text{C.3})$$

where $G_{i+1}^{i+p} = \{\xi_{i+1}, \xi_{i+2}, \dots, \xi_{i+p}\}$ is the knot vector, the operator S_p^l means selecting l knots from G and multiplying them as one term and then adding all the terms together.

When p equals 2, the reproducing points $\xi_i^{[l]}$ can be calculated as

$$\xi_i^{[l]} = \begin{cases} \xi_i^{[1]} = \frac{\xi_{i+1} + \xi_{i+2}}{2} \\ \xi_i^{[2]} = \sqrt{\xi_{i+1}\xi_{i+2}} \end{cases}. \quad (\text{C.4})$$

For 2D basis functions, the following consistency conditions can be readily obtained based on the tensor product of B-spline basis functions:

$$\sum_{i=1}^n \sum_{j=1}^m N_{i,p}(\xi) M_{j,q}(\eta) (\xi_i^{[\alpha]})^\alpha (\eta_j^{[\beta]})^\beta = \xi^\alpha \eta^\beta, \quad (\alpha + \beta \leq p + q). \quad (\text{C.5})$$

Eq. (C.5) can be simplified as

$$\sum_{l=1}^{N_B} N_l^{pq}(\xi) \mathbf{p}(\xi_l^{[l]}) = \mathbf{p}(\xi) \quad (\text{C.6})$$

with $N_I^{pq}(\xi) = N_{i,p}(\xi)M_{j,q}(\eta)$, where $\mathbf{p}(\xi_I^{[l]})$ is the 2D reproducing point vector:

$$\mathbf{p}(\xi_I^{[l]})^T = \left\{ 1, \xi_I^{[1]}, \eta_I^{[1]}, \left(\xi_I^{[2]}\right)^2, \xi_I^{[1]}\eta_I^{[1]}, \left(\eta_I^{[2]}\right)^2, \dots, \left(\xi_I^{[p]}\right)^p, \dots, \left(\eta_I^{[q]}\right)^q \right\} \quad (\text{C.7})$$

with $\xi_I^{[l]}$ being the 2D reproducing points.

Appendix D. Calculations of SIFs

The interaction integral is applied to extract the mixed-mode SIFs [128]. By defining two states of a cracked body, the present state $(\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)})$ and the auxiliary state $(\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)})$, the J-integral is expressed as

$$J^{(1+2)} = \int_{\Gamma} \left[\frac{1}{2} (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) (\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}) \delta_{1j} - (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) \frac{\partial (u_i^{(1)} + u_i^{(2)})}{\partial x_1} \right] n_j d\Gamma, \quad (D.1)$$

where δ_{1j} is the Kronecker delta, and n_j is the unit vector norm to Γ . Eq. (D.1) can be rearranged as follows:

$$J^{(1+2)} = J^{(1)} + J^{(2)} + I^{(1,2)} = J^{(1)} + J^{(2)} + \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}), \quad (D.2)$$

$$I^{(1,2)} = \int_A \left[\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA, \quad (D.3)$$

where $I^{(1,2)}$ is the interaction integral; A is the integral domain enclosed by Γ ; q is a smooth weight function; $W^{(1,2)} = \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)}$ is the interaction strain energy density; E' is given by $E/(1-\nu^2)$ under the plane-strain condition.

By considering a pure Mode-I crack with $K_I^{(2)} = 1$, the Mode-I SIF is obtained as

$$K_I^{(1)} = \frac{E'}{2} I^{(1,2)}. \quad (D.4)$$

The Mode-II SIF can be obtained in the same manner.

The SIFs of the crack in thin plates and shells can be divided into two parts: the membrane SIFs associated with the plane stress state and the bending SIFs associated with the plate theory. The membrane SIFs, K_I and K_{II} , are calculated by using the 2D plane stress elasticity, and the bending SIFs, k_1 and k_2 , are calculated by using the KL-plate theory as follows [232]:

$$\begin{aligned} K_I &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{\theta\theta}(r, 0), & K_{II} &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{r\theta}(r, 0), \\ k_1 &= \lim_{r \rightarrow 0} \sqrt{2r} \sigma_{\theta\theta}\left(r, 0, \frac{t}{2}\right), & k_2 &= \lim_{r \rightarrow 0} \frac{3+\nu}{1+\nu} \sqrt{2r} \sigma_{r\theta}\left(r, 0, \frac{t}{2}\right), \end{aligned} \quad (D.5)$$

where (r, θ, z) is the local polar coordinate at a crack tip.

For a shell with through-thickness cracks, the J-integral can be redefined as [233]

$$fJ = - \int_A \left(W n_1 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j \right) q dA. \quad (D.6)$$

Based on the divergence theorem, Eq.(D.6) is simplified as [163]

$$J = - \frac{1}{h} \int_V \left(W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \frac{\partial q}{\partial x_j} dV - \frac{1}{h} \int_V \left[\frac{\partial W}{\partial x_1} - \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \right] q dV. \quad (D.7)$$

Within the context of KL theory, the J-integral is related to the SIFs as follows [232]:

$$J = \frac{1}{E} (K_I^2 + K_{II}^2) + \frac{\pi}{3E} \left(\frac{1+\nu}{3+\nu} \right) (k_1^2 + k_2^2). \quad (D.8)$$

The mixed-mode SIFs for 2D and 3D problems can be calculated by the interaction integral [128, 233], in which both the present state with fields $(\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)})$ and auxiliary state with fields $(\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)})$ are utilized. The summation of J-integral can be expressed as follows:

$$\begin{aligned} J^{(1+2)} &= - \frac{1}{h} \int_V \left[\frac{1}{2} (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) (\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}) \delta_{1j} - (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) \frac{\partial (u_i^{(1)} + u_i^{(2)})}{\partial x_1} \right] \frac{\partial q}{\partial x_j} dV \\ &\quad - \frac{1}{h} \int_V \left\{ \frac{\partial}{\partial x_1} \left[\frac{1}{2} (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) (\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}) \right] - \frac{\partial}{\partial x_j} \left[(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) \frac{\partial (u_i^{(1)} + u_i^{(2)})}{\partial x_1} \right] \right\} q dV. \end{aligned} \quad (D.9)$$

Eq. (D.9) can be expanded and rearranged as

$$J^{(1+2)} = J^{(1)} + J^{(2)} + I^{(1,2)} \quad (D.10)$$

with the interaction integral I being expressed as

$$I^{(1,2)} = \frac{1}{h} \int_V \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV. \quad (D.11)$$

The summation of J-integral using Eq. (D.8) is expressed as

$$J^{(1+2)} = J^{(1)} + J^{(2)} + \frac{2}{E} \left(K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right) + \frac{2\pi}{3E} \left(\frac{1+\nu}{3+\nu} \right) \left(k_1^{(1)} k_1^{(2)} + k_2^{(1)} k_2^{(2)} \right). \quad (D.12)$$

By comparing Eqs. (D.10) and (D.12), the relationship between SIFs and interaction integral is expressed as

$$I^{(1,2)} = \frac{2}{E} \left(K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right) + \frac{2\pi}{3E} \left(\frac{1+\nu}{3+\nu} \right) \left(k_1^{(1)} k_1^{(2)} + k_2^{(1)} k_2^{(2)} \right). \quad (D.13)$$

By following the same extraction of SIFs in a 2D case, the SIFs of thin shells can be obtained by Eqs. (D.4) and (D.13).

Appendix E. Calculations of Crack Propagation Angles

A crack propagation direction can be obtained by the maximum circumferential stress criterion [118], in which a crack will grow along the direction θ_c that is normal to the angle of a maximum circumferential stress $\sigma_{\theta\theta}$ as follows [160]:

$$K_I \sin(\theta_c) + K_{II} (3 \cos \theta_c - 1) = 0, \quad (\text{E.1})$$

where the angle θ_c is defined by $\partial \sigma_{\theta\theta} / \partial \theta = 0$. By solving the above equation, the angle θ_c is given as

$$\theta_c = 2 \arctan \left(\frac{1 - \sqrt{1 + 8(K_{II} / K_I)^2}}{4(K_{II} / K_I)} \right). \quad (\text{E.2})$$

Appendix F. Contact Kinematics and Constraints

For the frictional contact between two deformable bodies, a penalty method is used to enforce contact constraints [234]. The following notations are used for convenient representations: the superposed dot indicates a derivative with respect to time, the superscript s indicates a slave surface, the superscript m indicates a master surface, and the symbol $(\bar{\cdot})$ indicates the point projected from a slave to a master surface.

The normal gap between two bodies is defined as

$$g_N = (\mathbf{x}^s - \bar{\mathbf{x}}^m) \cdot \bar{\mathbf{n}}^m, \quad (\text{F.1})$$

where $\bar{\mathbf{n}}^m$ is the normal to the master surface at the projection point $\bar{\mathbf{x}}^m$ of a point $\mathbf{x}^s$ on the slave surface. The normal contact constraints that aim to implement the impenetrability between contact bodies are

$$g_N \geq 0, \quad t_N \leq 0, \quad \text{and} \quad g_N t_N = 0, \quad (\text{F.2})$$

where t_N is the normal contact traction.

By assuming that the master surface is parameterized by a connective coordinate ξ , the tangential slit increment is defined as

$$\dot{\mathbf{g}}_T = \dot{\xi} \boldsymbol{\tau}_1^m \quad (\text{F.3})$$

with the covariant vector being $\boldsymbol{\tau}_1^m = \mathbf{x}_{,\xi}^m$. The contact constraints in the tangential direction are

$$\|\mathbf{t}_T\| - \mu t_N \leq 0, \quad \dot{\mathbf{g}}_T = \dot{\gamma} \frac{\mathbf{t}_T}{\|\mathbf{t}_T\|}, \quad \gamma \geq 0, \quad \text{and} \quad \gamma (\|\mathbf{t}_T\| - \mu t_N) = 0, \quad (\text{F.4})$$

where μ is the coefficient of friction, and γ is the amount of slip.

The contact traction which is decomposed into normal and tangential components is defined as

$$\mathbf{t} = \mathbf{t}_N + \mathbf{t}_T = t_N \bar{\mathbf{n}}^m + t_T^1 \bar{\boldsymbol{\tau}}_1^m \quad (\text{F.5})$$

with the normal component being $t_N = \mathbf{t} \cdot \bar{\mathbf{n}}^m$.

By using the penalty algorithm, the normal contact constraints are regularized as

$$t_N = \begin{cases} \varepsilon_N g_N & \text{if } g_N \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (\text{F.6})$$

with the normal penalty parameter $\varepsilon_N > 0$. According to a backward Euler formulation, the regularized tangential constraints are

$$t_T^1 = t_{Tn}^1 + \varepsilon_T \left[\left(\bar{\xi} - \bar{\xi}_n \right) - \gamma \frac{t_T^1}{|t_T^1|} \right], \quad \|\mathbf{t}_T\| - \mu t_N \leq 0, \quad \gamma \geq 0, \quad \text{and} \quad \gamma \left(\|\mathbf{t}_T\| - \mu t_N \right) = 0, \quad (\text{F.7})$$

where the tangential penalty parameter satisfies the condition $\varepsilon_T > 0$. A return mapping algorithm is adopted for the regularized tangential constraints. More details on the linearization of contact variables are introduced in [70, 235].

Appendix G. Meshfree Local Refinement Procedure for IGA

A meshfree local refinement strategy that was proposed by Wang *et al.* [89] is employed. This strategy achieves the local refinement in a meshfree manner based on the equivalence between RK basis functions and isogeometric basis functions. To implement mesh refinement in a specified element, the initial knot interval corresponding to the element is subdivided by inserting additional knots, after which both the newly added meshfree nodes and the basis functions are calculated. The 1D and 2D local refinement procedures are briefly introduced.

(a) 1D local mesh refinement

Fig. G.1 shows the initial knot vector $\Xi_1 = \{0, 0, 0, 1/3, 2/3, 1, 1, 1\}$, its linear reproducing points $\xi^{[1]} = \{0, 1/6, 1/2, 5/6, 1\}$ and the basis functions. The meshfree shape functions are equivalent to the isogeometric B-spline basis functions. After the insertion of $1/2$ into the initial knot vector, the new meshfree nodes $\xi^{[1]} = \{0, 1/6, 5/12, 7/12, 5/6, 1\}$ are calculated by Eq. (C.3), which are visualized in Fig. G.2(a). It is found that the basis functions related to the inserted nodes are denoted by red dashed lines, and the basis functions related to the original nodes remain constant. The second level of refinement is further implemented, as shown in Fig. G.2(b).

Fig. G.1. Basis functions before refinement.

Fig. G.2. Basis functions: (a) first-level refinement and (b) second-level refinement.

(b) 2D local mesh refinement

By implementing mesh refinement on the initial meshes that are visualized in Fig. G.3, the refined meshes, inserted nodes and basis functions are obtained. As shown in Fig. G.3(a) and (b), the nodes within the middle element are refined, and the nodes outside this element remain constant. The contour lines at the middle nodes and four newly added nodes are shown in Fig. G.3(e) and (f), respectively. Fig. G.4 shows the second-level refinement within the bottom left segment of the middle element.

(c) 3D local mesh refinement

The local mesh refinement in a 3D case is achieved in the same way as 1D or 2D procedures. Fig. G.5 illustrates both the mesh and nodes corresponding to the first three steps of mesh refinement, in which eight red nodes are added for the interior elements at the third step.

Fig. G.4. Second-level refinement: (a) meshes and (b) corresponding basis functions.

Fig. G.5. The mesh and nodes in a 3D case ($p = q = r = 2$): (a) before refinement, (b) the 1st step of mesh refinement, (c) the 2nd step of mesh refinement and (d) the 3rd step of mesh refinement.

References

- [1] M.-C. Hsu, I. Akkerman, and Y. Bazilevs, High-performance computing of wind turbine aerodynamics using isogeometric analysis. *Computers & Fluids*, 2011. **49**(1): 93-100.
- [2] G.J. Kennedy and J.R. Martins, A parallel finite-element framework for large-scale gradient-based design optimization of high-performance structures. *Finite Elements in Analysis and Design*, 2014. **87**: 56-73.
- [3] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering*, 2005. **194**(39-41): 4135-4195.
- [4] S.-W. Chi, J.-S. Chen, H.-Y. Hu, and J.P. Yang, A gradient reproducing kernel collocation method for boundary value problems. *International Journal for Numerical Methods in Engineering*, 2013. **93**(13): 1381-1402.
- [5] R.W. Peters and P. Kuhn, *Bursting strength of unstiffened pressure cylinders with slits*. 1957: National Advisory Committee for Aeronautics.
- [6] A.E.H. Love, XVI. The small free vibrations and deformation of a thin elastic shell. *Philosophical Transactions of the Royal Society A*, 1888(179): 491-546.
- [7] R. Mindlin, Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *Journal of Applied Mechanics*, 1951. **18**: 31-38.
- [8] P. Areias, J.H. Song, and T. Belytschko, A finite-strain quadrilateral shell element based on discrete Kirchhoff–Love constraints. *International Journal for Numerical Methods in Engineering*, 2005. **64**(9): 1166-1206.
- [9] Y. Bazilevs, X. Deng, A. Korobenko, F. Lanza di Scalea, M.D. Todd, and S.G. Taylor, Isogeometric Fatigue Damage Prediction in Large-Scale Composite Structures Driven by Dynamic Sensor Data. *Journal of Applied Mechanics*, 2015. **82**(9): 091008.
- [10] J. Kiendl, M. Ambati, L. De Lorenzis, H. Gomez, and A. Reali, Phase-field description of brittle fracture in plates and shells. *Computer Methods in Applied Mechanics and Engineering*, 2016. **312**: 374-394.
- [11] G. AA and E.M. VI, The Phenomena of Rupture and Flow in Solids. *Philosophical Transactions of the Royal Society A*, 1921. **221**(582-593): 163-198.
- [12] T. Belytschko and T. Black, Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering*, 1999. **45**(5): 601-620.
- [13] P. Hennig, M. Ambati, L. De Lorenzis, and M. Kästner, Projection and transfer operators in adaptive isogeometric analysis with hierarchical B-splines.

- Computer Methods in Applied Mechanics and Engineering*, 2018. **334**: 313-336.
- [14] D. Schillinger, M.J. Borden, and H.K. Stolarski, Isogeometric collocation for phase-field fracture models. *Computer Methods in Applied Mechanics and Engineering*, 2015. **284**: 583-610.
- [15] J. Kiendl, K.U. Bletzinger, J. Linhard, and R. Wüchner, Isogeometric shell analysis with Kirchhoff–Love elements. *Computer Methods in Applied Mechanics and Engineering*, 2009. **198**(49-52): 3902-3914.
- [16] S.S. Ghorashi, N. Valizadeh, and S. Mohammadi, Extended isogeometric analysis for simulation of stationary and propagating cracks. *International Journal for Numerical Methods in Engineering*, 2012. **89**(9): 1069-1101.
- [17] K.A. Johannessen, T. Kvamsdal, and T. Dokken, Isogeometric analysis using LR B-splines. *Computer Methods in Applied Mechanics and Engineering*, 2014. **269**: 471-514.
- [18] M.R. Dörfel, B. Jüttler, and B. Simeon, Adaptive isogeometric analysis by local h-refinement with T-splines. *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(5-8): 264-275.
- [19] N. Nguyen-Thanh, H. Nguyen-Xuan, S.P.A. Bordas, and T. Rabczuk, Isogeometric analysis using polynomial splines over hierarchical T-meshes for two-dimensional elastic solids. *Computer Methods in Applied Mechanics and Engineering*, 2011. **200**(21-22): 1892-1908.
- [20] B. Rao and S. Rahman, A coupled meshless-finite element method for fracture analysis of cracks. *International Journal of Pressure Vessels and Piping*, 2001. **78**(9): 647-657.
- [21] T. Rabczuk, S.P. Xiao, and M. Sauer, Coupling of mesh-free methods with finite elements: basic concepts and test results. *International Journal for Numerical Methods in Biomedical Engineering*, 2006. **22**(10): 1031-1065.
- [22] T. Belytschko, D. Organ, and Y. Krongauz, A coupled finite element-element-free Galerkin method. *Computational Mechanics*, 1995. **17**(3): 186-195.
- [23] R. Courant, Variational methods for the solution of problems of equilibrium and vibrations. *Bulletin of the American Mathematical Society*, 1943. **49**(1): 1-23.
- [24] R.W. Clough, The finite element method in plane stress analysis, in *Second ASCE Conference on Electronic Computation*. 1960, American Society of Civil Engineers: Pittsburgh. 345-378.
- [25] K. Gupta and J. Meek, A brief history of the beginning of the finite element method. *International Journal for Numerical Methods in Engineering*, 1996. **39**(22): 3761-3774.

-
- [26] S.S. Rao, *The finite element method in engineering*. 2017: Butterworth-Heinemann.
- [27] L.B. Lucy, A numerical approach to the testing of the fission hypothesis. *The Astronomical Journal*, 1977. **82**: 1013-1024.
- [28] A. Colagrossi and M. Landrini, Numerical simulation of interfacial flows by smoothed particle hydrodynamics. *Journal of Computational Physics*, 2003. **191**(2): 448-475.
- [29] M. Liu, G. Liu, K. Lam, and Z. Zong, Smoothed particle hydrodynamics for numerical simulation of underwater explosion. *Computational Mechanics*, 2003. **30**(2): 106-118.
- [30] J. Chen and J. Beraun, A generalized smoothed particle hydrodynamics method for nonlinear dynamic problems. *Computer Methods in Applied Mechanics and Engineering*, 2000. **190**(1-2): 225-239.
- [31] M. Liu and G. Liu, Smoothed particle hydrodynamics (SPH): an overview and recent developments. *Archives of Computational Methods in Engineering*, 2010. **17**(1): 25-76.
- [32] B. Nayroles, G. Touzot, and P. Villon, Generalizing the finite element method: diffuse approximation and diffuse elements. *Computational Mechanics*, 1992. **10**(5): 307-318.
- [33] T. Belytschko, Y.Y. Lu, and L. Gu, Element-free Galerkin methods. *International Journal for Numerical Methods in Engineering*, 1994. **37**(2): 229-256.
- [34] W.K. Liu, S. Jun, and Y.F. Zhang, Reproducing kernel particle methods. *International Journal for Numerical Methods in Fluids*, 1995. **20**(8-9): 1081-1106.
- [35] W. Han and X. Meng, Error analysis of the reproducing kernel particle method. *Computer Methods in Applied Mechanics and Engineering*, 2001. **190**(46-47): 6157-6181.
- [36] W.-K. Liu, S. Li, and T. Belytschko, Moving least-square reproducing kernel methods (I) methodology and convergence. *Computer Methods in Applied Mechanics and Engineering*, 1997. **143**(1): 113-154.
- [37] Y. Lu, T. Belytschko, and L. Gu, A new implementation of the element free Galerkin method. *Computer Methods in Applied Mechanics and Engineering*, 1994. **113**(3-4): 397-414.
- [38] J.-S. Chen, C. Pan, and C.-T. Wu, Large deformation analysis of rubber based on a reproducing kernel particle method. *Computational Mechanics*, 1997. **19**(3): 211-227.

-
- [39] T. Belytschko, Y. Lu, L. Gu, and M. Tabbara, Element-free Galerkin methods for static and dynamic fracture. *International Journal of Solids and Structures*, 1995. **32**(17-18): 2547-2570.
- [40] B. Ren and S. Li, Modeling and simulation of large-scale ductile fracture in plates and shells. *International Journal of Solids and Structures*, 2012. **49**(18): 2373-2393.
- [41] P. Krysl and T. Belytschko, Analysis of thin plates by the element-free Galerkin method. *Computational Mechanics*, 1995. **17**(1-2): 26-35.
- [42] P. Krysl and T. Belytschko, Analysis of thin shells by the element-free Galerkin method. *International Journal of Solids and Structures*, 1996. **33**(20-22): 3057-3080.
- [43] S.N. Atluri and T. Zhu, A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational Mechanics*, 1998. **22**(2): 117-127.
- [44] T. Zhu, J.-D. Zhang, and S. Atluri, A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach. *Computational Mechanics*, 1998. **21**(3): 223-235.
- [45] S.H. Park and S.K. Youn, The least-squares meshfree method. *International Journal for Numerical Methods in Engineering*, 2001. **52**(9): 997-1012.
- [46] C.A. Duarte and J.T. Oden, An hp adaptive method using clouds. *Computer Methods in Applied Mechanics and Engineering*, 1996. **139**(1-4): 237-262.
- [47] N. Sukumar, B. Moran, and T. Belytschko, The natural element method in solid mechanics. *International Journal for Numerical Methods in Engineering*, 1998. **43**(5): 839-887.
- [48] P. Breitkopf, A. Rassineux, G. Touzot, and P. Villon, Explicit form and efficient computation of MLS shape functions and their derivatives. *International Journal for Numerical Methods in Engineering*, 2000. **48**(3): 451-466.
- [49] Y. Krongauz and T. Belytschko, Enforcement of essential boundary conditions in meshless approximations using finite elements. *Computer Methods in Applied Mechanics and Engineering*, 1996. **131**(1-2): 133-145.
- [50] J.-S. Chen, M. Hillman, and S.-W. Chi, Meshfree Methods: Progress Made after 20 Years. *Journal of Engineering Mechanics*, 2017. **143**(4): 04017001.
- [51] D. Wang and J. Wu, An efficient nesting sub-domain gradient smoothing integration algorithm with quadratic exactness for Galerkin meshfree methods. *Computer Methods in Applied Mechanics and Engineering*, 2016. **298**: 485-519.

-
- [52] J.-S. Chen, S.-W. Chi, and H.-Y. Hu, Recent developments in stabilized Galerkin and collocation meshfree methods. *Computer Assisted Methods in Engineering and Science*, 2017. **18**(1/2): 3-21.
- [53] J.J. Koester and J.-S. Chen, Conforming window functions for meshfree methods. *Computer Methods in Applied Mechanics and Engineering*, 2019. **347**: 588-621.
- [54] R. Franke, Scattered data interpolation: tests of some methods. *Mathematics of Computation*, 1982. **38**(157): 181-200.
- [55] E.J. Kansa, Multiquadrics—A scattered data approximation scheme with applications to computational fluid-dynamics—I surface approximations and partial derivative estimates. *Computers & Mathematics with Applications*, 1990. **19**(8-9): 127-145.
- [56] E.J. Kansa, Multiquadrics—A scattered data approximation scheme with applications to computational fluid-dynamics—II solutions to parabolic, hyperbolic and elliptic partial differential equations. *Computers & Mathematics with Applications*, 1990. **19**(8-9): 147-161.
- [57] C. Franke and R. Schaback, Solving partial differential equations by collocation using radial basis functions. *Applied Mathematics and Computation*, 1998. **93**(1): 73-82.
- [58] C. Franke and R. Schaback, Convergence order estimates of meshless collocation methods using radial basis functions. *Advances in Computational Mathematics*, 1998. **8**(4): 381-399.
- [59] R.L. Hardy, Multiquadric equations of topography and other irregular surfaces. *Journal of Geophysical Research*, 1971. **76**(8): 1905-1915.
- [60] A. Wong, Y. Hon, T. Li, S. Chung, and E. Kansa, Multizone decomposition for simulation of time-dependent problems using the multiquadric scheme. *Computers & Mathematics with Applications*, 1999. **37**(8): 23-43.
- [61] E. Kansa and Y. Hon, Circumventing the ill-conditioning problem with multiquadric radial basis functions: applications to elliptic partial differential equations. *Computers & Mathematics with Applications*, 2000. **39**(7-8): 123-137.
- [62] G.E. Fasshauer, Solving differential equations with radial basis functions: multilevel methods and smoothing. *Advances in Computational Mathematics*, 1999. **11**(2-3): 139-159.
- [63] J. Chen, W. Hu, and H. Hu, Reproducing kernel enhanced local radial basis collocation method. *International Journal for Numerical Methods in Engineering*, 2008. **75**(5): 600-627.
- [64] E. Oñate, S. Idelsohn, O. Zienkiewicz, R. Taylor, and C. Sacco, A stabilized finite point method for analysis of fluid mechanics problems. *Computer Methods in Applied Mechanics and Engineering*, 1996. **139**(1-4): 315-346.

- [65] N.R. Aluru, A point collocation method based on reproducing kernel approximations. *International Journal for Numerical Methods in Engineering*, 2000. **47**(6): 1083-1121.
- [66] H.-Y. Hu, J.-S. Chen, and W. Hu, Error analysis of collocation method based on reproducing kernel approximation. *Numerical Methods for Partial Differential Equations*, 2011. **27**(3): 554-580.
- [67] J.A. Cottrell, T.J. Hughes, and Y. Bazilevs, *Isogeometric analysis: toward integration of CAD and FEA*. 2009: John Wiley & Sons.
- [68] L. Piegl and W. Tiller, *The NURBS book*. 2012: Springer Science & Business Media.
- [69] N. Nguyen-Thanh, W. Li, and K. Zhou, Static and free-vibration analyses of cracks in thin-shell structures based on an isogeometric-meshfree coupling approach. *Computational Mechanics*, 2018. **62**: 1287-1309.
- [70] R. Kruse, N. Nguyen-Thanh, L. De Lorenzis, and T.J.R. Hughes, Isogeometric collocation for large deformation elasticity and frictional contact problems. *Computer Methods in Applied Mechanics and Engineering*, 2015. **296**: 73-112.
- [71] Y. Bazilevs, V.M. Calo, J.A. Cottrell, J.A. Evans, T.J.R. Hughes, S. Lipton, M.A. Scott, and T.W. Sederberg, Isogeometric analysis using T-splines. *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(5): 229-263.
- [72] L. Chen and R. de Borst, Locally Refined T-splines. *International Journal for Numerical Methods in Engineering*, 2018. **114**(6): 637-659.
- [73] M.A. Scott, X. Li, T.W. Sederberg, and T.J. Hughes, Local refinement of analysis-suitable T-splines. *Computer Methods in Applied Mechanics and Engineering*, 2012. **213**: 206-222.
- [74] J. Zhang and X. Li, Local refinement for analysis-suitable++ T-splines. *Computer Methods in Applied Mechanics and Engineering*, 2018. **342**: 32-45.
- [75] D.R. Forsey and R.H. Bartels. *Hierarchical B-spline refinement*. in *Proceedings of the 15th annual conference on Computer graphics and interactive techniques*. 1988.
- [76] E.M. Garau and R. Vázquez, Algorithms for the implementation of adaptive isogeometric methods using hierarchical B-splines. *Applied Numerical Mathematics*, 2018. **123**: 58-87.
- [77] M. Carraturo, C. Giannelli, A. Reali, and R. Vázquez, Suitably graded THB-spline refinement and coarsening: Towards an adaptive isogeometric analysis of additive manufacturing processes. *Computer Methods in Applied Mechanics and Engineering*, 2019. **348**: 660-679.
- [78] J. Deng, F. Chen, X. Li, C. Hu, W. Tong, Z. Yang, and Y. Feng, Polynomial splines over hierarchical T-meshes. *Graphical Models*, 2008. **70**(4): 76-86.

-
- [79] P. Wang, J. Xu, J. Deng, and F. Chen, Adaptive isogeometric analysis using rational PHT-splines. *Computer-Aided Design*, 2011. **43**(11): 1438-1448.
- [80] N. Nguyen-Thanh, J. Kiendl, H. Nguyen-Xuan, R. Wuchner, K.U. Bletzinger, Y. Bazilevs, and T. Rabczuk, Rotation free isogeometric thin shell analysis using PHT-splines. *Computer Methods in Applied Mechanics and Engineering*, 2011. **200**(47-48): 3410-3424.
- [81] P. Hennig, M. Kästner, P. Morgenstern, and D. Peterseim, Adaptive mesh refinement strategies in isogeometric analysis—A computational comparison. *Computer Methods in Applied Mechanics and Engineering*, 2017. **316**: 424-448.
- [82] K.A. Johannessen, F. Remonato, and T. Kvamsdal, On the similarities and differences between Classical Hierarchical, Truncated Hierarchical and LR B-splines. *Computer Methods in Applied Mechanics and Engineering*, 2015. **291**: 64-101.
- [83] D. Wang and H. Zhang, A consistently coupled isogeometric-meshfree method. *Computer Methods in Applied Mechanics and Engineering*, 2014. **268**: 843-870.
- [84] N. Valizadeh, Y. Bazilevs, J.S. Chen, and T. Rabczuk, A coupled IGA-Meshfree discretization of arbitrary order of accuracy and without global geometry parameterization. *Computer Methods in Applied Mechanics and Engineering*, 2015. **293**: 20-37.
- [85] A. Rosolen and M. Arroyo, Blending isogeometric analysis and local maximum entropy meshfree approximants. *Computer Methods in Applied Mechanics and Engineering*, 2013. **264**: 95-107.
- [86] W. Li, N. Nguyen-Thanh, and K. Zhou, Geometrically nonlinear analysis of thin-shell structures based on an isogeometric-meshfree coupling approach. *Computer Methods in Applied Mechanics and Engineering*, 2018. **336**: 111-134.
- [87] P. Tan, N. Nguyen-Thanh, T. Rabczuk, and K. Zhou, Static, dynamic and buckling analyses of 3D FGM plates and shells via an isogeometric-meshfree coupling approach. *Composite Structures*, 2018. **198**: 35-50.
- [88] N. Nguyen-Thanh, J. Huang, and K. Zhou, An isogeometric-meshfree coupling approach for analysis of cracks. *International Journal for Numerical Methods in Engineering*, 2018. **113**(10): 1630-1651.
- [89] H. Zhang and D. Wang, Reproducing kernel formulation of B-spline and NURBS basis functions: A meshfree local refinement strategy for isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, 2017. **320**: 474-508.

-
- [90] T.J.R. Hughes, A. Reali, and G. Sangalli, Efficient quadrature for NURBS-based isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(5): 301-313.
- [91] C. Adam, T.J. Hughes, S. Bouabdallah, M. Zarroug, and H. Maitournam, Selective and reduced numerical integrations for NURBS-based isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, 2015. **284**: 732-761.
- [92] R.R. Hiemstra, F. Calabro, D. Schillinger, and T.J. Hughes, Optimal and reduced quadrature rules for tensor product and hierarchically refined splines in isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, 2017. **316**: 966-1004.
- [93] F. Calabrò, G. Loli, G. Sangalli, and M. Tani, *Quadrature Rules in the Isogeometric Galerkin Method: State of the Art and an Introduction to Weighted Quadrature*, in *Advanced Methods for Geometric Modeling and Numerical Simulation*. 2019, Springer. 43-55.
- [94] F. Auricchio, L.B. Da Veiga, T.J.R. Hughes, A. Reali, and G. Sangalli, Isogeometric collocation methods. *Mathematical Models and Methods in Applied Sciences*, 2010. **20**(11): 2075-2107.
- [95] C. De Boor, C. De Boor, E.-U. Mathématicien, C. De Boor, and C. De Boor, *A practical guide to splines*. Vol. 27. 1978: Springer-Verlag New York.
- [96] D. Schillinger, J.A. Evans, A. Reali, M.A. Scott, and T.J.R. Hughes, Isogeometric collocation: Cost comparison with Galerkin methods and extension to adaptive hierarchical NURBS discretizations. *Computer Methods in Applied Mechanics and Engineering*, 2013. **267**: 170-232.
- [97] C. Anitescu, Y. Jia, Y.J. Zhang, and T. Rabczuk, An isogeometric collocation method using superconvergent points. *Computer Methods in Applied Mechanics and Engineering*, 2015. **284**: 1073-1097.
- [98] N. Nguyen-Thanh, W. Li, J. Huang, N. Srikanth, and K. Zhou, An adaptive isogeometric analysis meshfree collocation method for elasticity and frictional contact problems. *International Journal for Numerical Methods in Engineering*, 2019. **120**(2): 209-230.
- [99] L. De Lorenzis, J.A. Evans, T.J.R. Hughes, and A. Reali, Isogeometric collocation: Neumann boundary conditions and contact. *Computer Methods in Applied Mechanics and Engineering*, 2015. **284**: 21-54.
- [100] W. Li, N. Nguyen-Thanh, and K. Zhou, An isogeometric-meshfree collocation approach for two-dimensional elastic fracture problems with contact loading. *Engineering Fracture Mechanics*, 2020. **223**: 106779.
- [101] F. Maurin, F. Greco, L. Coox, D. Vandepitte, and W. Desmet, Isogeometric collocation for Kirchhoff–Love plates and shells. *Computer Methods in Applied Mechanics and Engineering*, 2018. **329**: 396-420.

-
- [102] J. Stam. *Exact evaluation of Catmull-Clark subdivision surfaces at arbitrary parameter values*. in *Proceedings of the 25th annual conference on Computer graphics and interactive techniques*. 1998.
- [103] X. Wei, Y. Zhang, T.J. Hughes, and M.A. Scott, Truncated hierarchical Catmull–Clark subdivision with local refinement. *Computer Methods in Applied Mechanics and Engineering*, 2015. **291**: 1-20.
- [104] H.-J. Kim, Y.-D. Seo, and S.-K. Youn, Isogeometric analysis with trimming technique for problems of arbitrary complex topology. *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(45-48): 2796-2812.
- [105] T. Martin, E. Cohen, and R.M. Kirby, Volumetric parameterization and trivariate B-spline fitting using harmonic functions. *Computer Aided Geometric Design*, 2009. **26**(6): 648-664.
- [106] G. Xu, B. Mourrain, A. Galligo, and T. Rabczuk, High-quality construction of analysis-suitable trivariate NURBS solids by reparameterization methods. *Computational Mechanics*, 2014. **54**(5): 1303-1313.
- [107] H. Al Akhras, T. Elguedj, A. Gravouil, and M. Rochette, Isogeometric analysis-suitable trivariate NURBS models from standard B-Rep models. *Computer Methods in Applied Mechanics and Engineering*, 2016. **307**: 256-274.
- [108] J. Escobar, J. Cascón, E. Rodríguez, and R. Montenegro, A new approach to solid modeling with trivariate T-splines based on mesh optimization. *Computer Methods in Applied Mechanics and Engineering*, 2011. **200**(45-46): 3210-3222.
- [109] D. Ngo and A.C. Scordelis. *Finite Element Analysis of Reinforced Concrete Beams*. in *Journal Proceedings*. 1967.
- [110] Y.R. Rashid, Analysis of reinforced concrete pressure vessels. *Nuclear Engineering and Design*, 1968. **7**(4): 334-344.
- [111] M.J. Borden, T.J.R. Hughes, C.M. Landis, and C.V. Verhoosel, A higher-order phase-field model for brittle fracture: Formulation and analysis within the isogeometric analysis framework. *Computer Methods in Applied Mechanics and Engineering*, 2014. **273**: 100-118.
- [112] G.R. Irwin, Onset of fast crack propagation in high strength steel and aluminum alloys. 1956, Naval Research Lab Washington DC.
- [113] T.L. Anderson, *Fracture mechanics: fundamentals and applications*. 2017: CRC Press.
- [114] N. Nguyen-Thanh, N. Valizadeh, M.N. Nguyen, H. Nguyen-Xuan, X. Zhuang, P. Areias, G. Zi, Y. Bazilevs, L. De Lorenzis, and T. Rabczuk, An extended isogeometric thin shell analysis based on Kirchhoff–Love theory. *Computer Methods in Applied Mechanics and Engineering*, 2015. **284**: 265-291.

-
- [115] A.A. Wells. *Unstable crack propagation in metals: cleavage and fast fracture*. in *Proceedings of the crack propagation symposium*. 1961.
- [116] G. Irwin. *Plastic zone near a crack and fracture toughness*. in *Seventh Sagamore Ordnance Materials Conference*. 1960.
- [117] J.R. Rice, A path independent integral and the approximate analysis of strain concentration by notches and cracks. *Journal of Applied Mechanics*, 1968. **35**(2): 379-386.
- [118] F. Erdogan and G. Sih, On the crack extension in plates under plane loading and transverse shear. *Journal of Basic Engineering*, 1963. **85**(4): 519-525.
- [119] M. Hussain, S. Pu, and J. Underwood. *Strain energy release rate for a crack under combined mode I and mode II*. in *Fracture Analysis: Proceedings of the 1973 National Symposium on Fracture Mechanics, Part II*. 1974. ASTM International.
- [120] G.C. Sih, Strain-energy-density factor applied to mixed mode crack problems. *International Journal of Fracture*, 1974. **10**(3): 305-321.
- [121] C.E. Inglis, Stresses in a plate due to the presence of cracks and sharp corners. *Trans Inst Naval Archit*, 1913. **55**: 219-241.
- [122] H.M. Westergaard, Bearing pressures and cracks. *Journal of Applied Mechanics*, 1939. **6**(2): A49-A53.
- [123] M.L. Williams, On the Stress Distribution at the Base of a Stationary Crack. *Journal of Applied Mechanics*, 1956. **24**: 109-114.
- [124] G.P. Anderson, V.L. Ruggles, and G.S. Stibor, Use of finite element computer programs in fracture mechanics. *International Journal of Fracture Mechanics*, 1971. **7**(1): 63-76.
- [125] E. Byskov, The calculation of stress intensity factors using the finite element method with cracked elements. *International Journal of Fracture Mechanics*, 1970. **6**(2): 159-167.
- [126] R. Henshell and K. Shaw, Crack tip finite elements are unnecessary. *International Journal for Numerical Methods in Engineering*, 1975. **9**(3): 495-507.
- [127] A. Ingraffea and V. Saouma, *Numerical modeling of discrete crack propagation in reinforced and plain concrete*, in *Fracture mechanics of concrete: structural application and numerical calculation*. 1985, Springer. 171-225.
- [128] N. Moes, J. Dolbow, and T. Belytschko, A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering*, 1999. **46**(1): 131-150.

-
- [129] S. Osher and J.A. Sethian, Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, 1988. **79**(1): 12-49.
- [130] J. Dolbow, N. Moes, and T. Belytschko, An extended finite element method for modeling crack growth with frictional contact. *Computer Methods in Applied Mechanics and Engineering*, 2001. **190**(51-52): 6825-6846.
- [131] T. Belytschko, H. Chen, J. Xu, and G. Zi, Dynamic crack propagation based on loss of hyperbolicity and a new discontinuous enrichment. *International Journal for Numerical Methods in Engineering*, 2003. **58**(12): 1873-1905.
- [132] J. Réthoré, A. Gravouil, and A. Combescure, An energy-conserving scheme for dynamic crack growth using the eXtended finite element method. *International Journal for Numerical Methods in Engineering*, 2005. **63**(5): 631-659.
- [133] J. Remmers, R. de Borst, and A. Needleman, A cohesive segments method for the simulation of crack growth. *Computational Mechanics*, 2003. **31**(1-2): 69-77.
- [134] E. Budyn, G. Zi, N. Moës, and T. Belytschko, A method for multiple crack growth in brittle materials without remeshing. *International Journal for Numerical Methods in Engineering*, 2004. **61**(10): 1741-1770.
- [135] J. Lasry, Y. Renard, and M. Salaun, Stress intensity factors computation for bending plates with extended finite element method. *International Journal for Numerical Methods in Engineering*, 2012. **91**(9): 909-928.
- [136] A. Afshar, A. Daneshyar, and S. Mohammadi, XFEM analysis of fiber bridging in mixed-mode crack propagation in composites. *Composite Structures*, 2015. **125**: 314-327.
- [137] T. Belytschko, L. Gu, and Y.Y. Lu, Fracture and crack growth by element free Galerkin methods. *Modelling and Simulation in Materials Science and Engineering*, 1994. **2**(3a): 519-534.
- [138] T. Belytschko and M. Tabbara, Dynamic fracture using element-free Galerkin methods. *International Journal for Numerical Methods in Engineering*, 1996. **39**(6): 923-938.
- [139] T. Rabczuk and T. Belytschko, Cracking particles: a simplified meshfree method for arbitrary evolving cracks. *International Journal for Numerical Methods in Engineering*, 2004. **61**(13): 2316-2343.
- [140] S.H. Lee and Y.C. Yoon, Meshfree point collocation method for elasticity and crack problems. *International Journal for Numerical Methods in Engineering*, 2004. **61**(1): 22-48.
- [141] G.H. Lee, H.J. Chung, and C.K. Choi, Adaptive crack propagation analysis with the element-free Galerkin method. *International Journal for Numerical Methods in Engineering*, 2003. **56**(3): 331-350.

-
- [142] C.V. Verhoosel, M.A. Scott, R. De Borst, and T.J.R. Hughes, An isogeometric approach to cohesive zone modeling. *International Journal for Numerical Methods in Engineering*, 2011. **87**(1-5): 336-360.
- [143] E. De Luycker, D.J. Benson, T. Belytschko, Y. Bazilevs, and M.C. Hsu, X-FEM in isogeometric analysis for linear fracture mechanics. *International Journal for Numerical Methods in Engineering*, 2011. **87**(6): 541-565.
- [144] M. Bischoff, E. Ramm, and J. Irslinger, Models and finite elements for thin-walled structures. *Encyclopedia of Computational Mechanics Second Edition*, 2018: 1-86.
- [145] J. Ericksen and C. Truesdell, Exact theory of stress and strain in rods and shells. *Archive for Rational Mechanics and Analysis*, 1957. **1**(1): 295-323.
- [146] P.M. Naghdi, *The theory of shells and plates*, in *Linear Theories of Elasticity and Thermoelasticity*. 1973, Springer. 425-640.
- [147] Y. Basar and Y. Ding, Finite-element analysis of hyperelastic thin shells with large strains. *Computational Mechanics*, 1996. **18**(3): 200-214.
- [148] T.J.R. Hughes and W.K. Liu, Nonlinear finite element analysis of shells: Part I. Three-dimensional shells. *Computer Methods in Applied Mechanics and Engineering*, 1981. **26**(3): 331-362.
- [149] F. Cirak, M. Ortiz, and P. Schroder, Subdivision surfaces: a new paradigm for thin-shell finite-element analysis. *International Journal for Numerical Methods in Engineering*, 2000. **47**(12): 2039-2072.
- [150] T. Chau-Dinh, G. Zi, P.-S. Lee, T. Rabczuk, and J.-H. Song, Phantom-node method for shell models with arbitrary cracks. *Computers & Structures*, 2012. **92-93**: 242-256.
- [151] E. Rank, R. Krause, and K. Preusch, On the accuracy of p-version elements for the Reissner–Mindlin plate problem. *International Journal for Numerical Methods in Engineering*, 1998. **43**(1): 51-67.
- [152] I. Babuška and M. Suri, On locking and robustness in the finite element method. *SIAM Journal on Numerical Analysis*, 1992. **29**(5): 1261-1293.
- [153] O. Zienkiewicz, R. Taylor, and J. Too, Reduced integration technique in general analysis of plates and shells. *International Journal for Numerical Methods in Engineering*, 1971. **3**(2): 275-290.
- [154] T.J. Hughes, M. Cohen, and M. Haroun, Reduced and selective integration techniques in the finite element analysis of plates. *Nuclear Engineering and Design*, 1978. **46**(1): 203-222.
- [155] J.C. Simo and M. Rifai, A class of mixed assumed strain methods and the method of incompatible modes. *International Journal for Numerical Methods in Engineering*, 1990. **29**(8): 1595-1638.

-
- [156] R. Bouclier, T. Elguedj, and A. Combescure, Locking free isogeometric formulations of curved thick beams. *Computer Methods in Applied Mechanics and Engineering*, 2012. **245**: 144-162.
- [157] G. Kikis, W. Dornisch, and S. Klinkel, Adjusted approximation spaces for the treatment of transverse shear locking in isogeometric Reissner–Mindlin shell analysis. *Computer Methods in Applied Mechanics and Engineering*, 2019. **354**: 850-870.
- [158] M. Ambati, J. Kiendl, and L. De Lorenzis, Isogeometric Kirchhoff–Love shell formulation for elasto-plasticity. *Computer Methods in Applied Mechanics and Engineering*, 2018. **340**: 320-339.
- [159] D. Wang, J. Wang, and J. Wu, Arbitrary order recursive formulation of meshfree gradients with application to superconvergent collocation analysis of Kirchhoff plates. *Computational Mechanics*, 2020. **65**(3): 877-903.
- [160] D.O. Potyondy, P.A. Wawrzynek, and A.R. Ingraffea, Discrete crack growth analysis methodology for through cracks in pressurized fuselage structures. *International Journal for Numerical Methods in Engineering*, 1995. **38**(10): 1611-1633.
- [161] P.M.A. Areias and T. Belytschko, Non-linear analysis of shells with arbitrary evolving cracks using XFEM. *International Journal for Numerical Methods in Engineering*, 2005. **62**(3): 384-415.
- [162] P.M.A. Areias, J.H. Song, and T. Belytschko, Analysis of fracture in thin shells by overlapping paired elements. *Computer Methods in Applied Mechanics and Engineering*, 2006. **195**(41-43): 5343-5360.
- [163] Q. Zeng, Z. Liu, D. Xu, H. Wang, and Z. Zhuang, Modeling arbitrary crack propagation in coupled shell/solid structures with X-FEM. *International Journal for Numerical Methods in Engineering*, 2016. **106**(12): 1018-1040.
- [164] R. Larsson, J. Mediavilla, and M. Fagerström, Dynamic fracture modeling in shell structures based on XFEM. *International Journal for Numerical Methods in Engineering*, 2011. **86**(4-5): 499-527.
- [165] T. Rabczuk and P. Areias, A meshfree thin shell for arbitrary evolving cracks based on an extrinsic basis. *CMES-Computer Modeling in Engineering & Sciences*, 2006. **16**(2): 115-130.
- [166] T. Rabczuk, P.M.A. Areias, and T. Belytschko, A meshfree thin shell method for non-linear dynamic fracture. *International Journal for Numerical Methods in Engineering*, 2007. **72**(5): 524-548.
- [167] D. Qian, T. Eason, S. Li, and W.K. Liu, Meshfree simulation of failure modes in thin cylinders subjected to combined loads of internal pressure and localized heat. *International Journal for Numerical Methods in Engineering*, 2008. **76**(8): 1159-1184.

-
- [168] F. Caleyron, A. Combescure, V. Faucher, and S. Potapov, Dynamic simulation of damage-fracture transition in smoothed particles hydrodynamics shells. *International Journal for Numerical Methods in Engineering*, 2012. **90**(6): 707-738.
- [169] A. Yadav, R. Godara, and G. Bhardwaj, A review on XIGA method for computational fracture mechanics applications. *Engineering Fracture Mechanics*, 2020: 107001.
- [170] S.S. Ghorashi, N. Valizadeh, S. Mohammadi, and T. Rabczuk. *Extended isogeometric analysis of plates with curved cracks*. in *Proceedings of the eighth international conference on engineering computational technology, Civil-Comp Press, Stirlingshire, UK*. 2012.
- [171] H. Yang and C. Dong, Adaptive extended isogeometric analysis based on PHT-splines for thin cracked plates and shells with Kirchhoff–Love theory. *Applied Mathematical Modelling*, 2019. **76**: 759-799.
- [172] H. Zhang, J. Wu, and D. Wang, Free vibration analysis of cracked thin plates by quasi-convex coupled isogeometric-meshfree method. *Frontiers of Structural and Civil Engineering*, 2015. **9**(4): 405-419.
- [173] W. Li, N. Nguyen-Thanh, J. Huang, and K. Zhou, Adaptive analysis of crack propagation in thin-shell structures via an isogeometric-meshfree moving least-squares approach. *Computer Methods in Applied Mechanics and Engineering*, 2020. **358**: 112613.
- [174] G.A. Francfort and J.-J. Marigo, Revisiting brittle fracture as an energy minimization problem. *Journal of the Mechanics and Physics of Solids*, 1998. **46**(8): 1319-1342.
- [175] B. Bourdin, G.A. Francfort, and J.-J. Marigo, Numerical experiments in revisited brittle fracture. *Journal of the Mechanics and Physics of Solids*, 2000. **48**(4): 797-826.
- [176] T. Gerasimov and L. De Lorenzis, A line search assisted monolithic approach for phase-field computing of brittle fracture. *Computer Methods in Applied Mechanics and Engineering*, 2016. **312**: 276-303.
- [177] C. Miehe, F. Welschinger, and M. Hofacker, Thermodynamically consistent phase-field models of fracture: Variational principles and multi-field FE implementations. *International Journal for Numerical Methods in Engineering*, 2010. **83**(10): 1273-1311.
- [178] H. Amor, J.-J. Marigo, and C. Maurini, Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments. *Journal of the Mechanics and Physics of Solids*, 2009. **57**(8): 1209-1229.
- [179] C. Miehe, M. Hofacker, and F. Welschinger, A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on

- operator splits. *Computer Methods in Applied Mechanics and Engineering*, 2010. **199**(45-48): 2765-2778.
- [180] M.J. Borden, Isogeometric Analysis of Phase-Field Models for Dynamic Brittle and Ductile Fracture. 2012.
- [181] H. Ulmer, M. Hofacker, and C. Miehe, Phase field modeling of brittle and ductile fracture. *PAMM*, 2013. **13**(1): 533-536.
- [182] M. Ambati, T. Gerasimov, and L. De Lorenzis, Phase-field modeling of ductile fracture. *Computational Mechanics*, 2015. **55**(5): 1017-1040.
- [183] M.J. Borden, T.J.R. Hughes, C.M. Landis, A. Anvari, and I.J. Lee, A phase-field formulation for fracture in ductile materials: Finite deformation balance law derivation, plastic degradation, and stress triaxiality effects. *Computer Methods in Applied Mechanics and Engineering*, 2016. **312**: 130-166.
- [184] M. Dittmann, F. Aldakheel, J. Schulte, P. Wriggers, and C. Hesch, Variational phase-field formulation of non-linear ductile fracture. *Computer Methods in Applied Mechanics and Engineering*, 2018. **342**: 71-94.
- [185] R. Mahnken, Goal-oriented adaptive refinement for phase field modeling with finite elements. *International Journal for Numerical Methods in Engineering*, 2013. **94**(4): 418-440.
- [186] T. Heister, M.F. Wheeler, and T. Wick, A primal-dual active set method and predictor-corrector mesh adaptivity for computing fracture propagation using a phase-field approach. *Computer Methods in Applied Mechanics and Engineering*, 2015. **290**: 466-495.
- [187] T. Wick, Goal functional evaluations for phase-field fracture using PU-based DWR mesh adaptivity. *Computational Mechanics*, 2016. **57**(6): 1017-1035.
- [188] F. Tian, X. Tang, T. Xu, J. Yang, and L. Li, A hybrid adaptive finite element phase-field method for quasi-static and dynamic brittle fracture. *International Journal for Numerical Methods in Engineering*, 2019. **120**(9): 1108-1125.
- [189] Y. Shao, Q. Duan, and S. Qiu, Adaptive consistent element-free Galerkin method for phase-field model of brittle fracture. *Computational Mechanics*, 2019. **64**(3): 741-767.
- [190] M.J. Borden, C.V. Verhoosel, M.A. Scott, T.J.R. Hughes, and C.M. Landis, A phase-field description of dynamic brittle fracture. *Computer Methods in Applied Mechanics and Engineering*, 2012. **217**: 77-95.
- [191] S. Goswami, C. Anitescu, and T. Rabczuk, Adaptive phase field analysis with dual hierarchical meshes for brittle fracture. *Engineering Fracture Mechanics*, 2019. **218**: 106608.
- [192] T. Belytschko, W.K. Liu, B. Moran, and K. Elkhodary, *Nonlinear finite elements for continua and structures*. 2013: John Wiley & Sons.

-
- [193] T. Rabczuk and T. Belytschko, Adaptivity for structured meshfree particle methods in 2D and 3D. *International Journal for Numerical Methods in Engineering*, 2005. **63**(11): 1559-1582.
- [194] A. Ben-Israel, A Newton-Raphson method for the solution of systems of equations. *Journal of Mathematical Analysis and Applications*, 1966. **15**(2): 243-252.
- [195] T. Belytschko, H. Stolarski, W.K. Liu, N. Carpenter, and J.S. Ong, Stress projection for membrane and shear locking in shell finite elements. *Computer Methods in Applied Mechanics and Engineering*, 1985. **51**(1-3): 221-258.
- [196] K.Y. Sze, X.H. Liu, and S.H. Lo, Popular benchmark problems for geometric nonlinear analysis of shells. *Finite Elements in Analysis and Design*, 2004. **40**(11): 1551-1569.
- [197] T.X. Duong, F. Roohbakhshan, and R.A. Sauer, A new rotation-free isogeometric thin shell formulation and a corresponding continuity constraint for patch boundaries. *Computer Methods in Applied Mechanics and Engineering*, 2017. **316**: 43-83.
- [198] D. Organ, M. Fleming, T. Terry, and T. Belytschko, Continuous meshless approximations for nonconvex bodies by diffraction and transparency. *Computational Mechanics*, 1996. **18**(3): 225-235.
- [199] G. Ventura, J. Xu, and T. Belytschko, A vector level set method and new discontinuity approximations for crack growth by EFG. *International Journal for Numerical Methods in Engineering*, 2002. **54**(6): 923-944.
- [200] H. Tada, P.C. Paris, and G.R. Irwin, *The Stress Analysis of Cracks Handbook*. 2000: ASME Press, New York.
- [201] Y.T. Gu, W. Wang, L.C. Zhang, and X.Q. Feng, An enriched radial point interpolation method (e-RPIM) for analysis of crack tip fields. *Engineering Fracture Mechanics*, 2011. **78**(1): 175-190.
- [202] H.D. Huynh, M.N. Nguyen, G. Cusatis, S. Tanaka, and T.Q. Bui, A polygonal XFEM with new numerical integration for linear elastic fracture mechanics. *Engineering Fracture Mechanics*, 2019. **213**: 241-263.
- [203] Z. Zhu, L. Wang, B. Mohanty, and C. Huang, Stress intensity factor for a cracked specimen under compression. *Engineering Fracture Mechanics*, 2006. **73**(4): 482-489.
- [204] S. Glodež, and Zoran Ren, Modelling of crack growth under cyclic contact loading. *Theoretical and Applied Fracture Mechanics*, 1998. **30**(2): 159-173.
- [205] J. Newman Jr, Y. Yamada, and M. James, Stress-intensity-factor equations for compact specimen subjected to concentrated forces. *Engineering Fracture Mechanics*, 2010. **77**(6): 1025-1029.

-
- [206] T. Belytschko, R. Gracie, and G. Ventura, A review of extended/generalized finite element methods for material modeling. *Modelling and Simulation in Materials Science and Engineering*, 2009. **17**(4): 043001.
- [207] Y. Luo and U. Häussler-Combe, *An adaptivity procedure based on the gradient of strain energy density and its application in meshless methods*, in *Meshfree methods for partial differential equations*. 2003, Springer. 267-279.
- [208] Z. Zhang, G. Liu, T.X. Liu, Q.R. Zeng, and L.Y. Wu, An adaptive meshless computational system for elastoplastic contact problems. *International Journal of Computational Methods*, 2008. **5**(3): 433-447.
- [209] N.S. Huh, D.J. Shim, S. Choi, and K.B. Park, Stress intensity factors and crack opening displacements for slanted axial through-wall cracks in pressurized pipes. *Fatigue & Fracture of Engineering Materials & Structures*, 2008. **31**(6): 428-440.
- [210] Z. Zhuang, Z. Liu, B. Cheng, and J. Liao, *Extended Finite Element Method: Tsinghua University Press Computational Mechanics Series*. 2014: Academic Press.
- [211] A.L. Keesecker, C.G. Dávila, E.R. Johnson, and J.H. Starnes, Crack path bifurcation at a tear strap in a pressurized shell. *Computers & Structures*, 2003. **81**(16): 1633-1642.
- [212] H. Bayesteh and S. Mohammadi, XFEM fracture analysis of shells: The effect of crack tip enrichments. *Computational Materials Science*, 2011. **50**(10): 2793-2813.
- [213] J.D. Clayton and J. Knap, Phase field modeling of directional fracture in anisotropic polycrystals. *Computational Materials Science*, 2015. **98**: 158-169.
- [214] Q. Du, V. Faber, and M. Gunzburger, Centroidal Voronoi tessellations: Applications and algorithms. *SIAM Review*, 1999. **41**(4): 637-676.
- [215] M. Tanemura, T. Ogawa, and N. Ogita, A new algorithm for three-dimensional Voronoi tessellation. *Journal of Computational Physics*, 1983. **51**(2): 191-207.
- [216] S. Fernández-Méndez and A. Huerta, Imposing essential boundary conditions in mesh-free methods. *Computer Methods in Applied Mechanics and Engineering*, 2004. **193**(12-14): 1257-1275.
- [217] F. Zhang, W. Huang, X. Li, and S. Zhang, Moving mesh finite element simulation for phase-field modeling of brittle fracture and convergence of Newton's iteration. *Journal of Computational Physics*, 2018. **356**: 127-149.
- [218] M. Ambati, T. Gerasimov, and L. De Lorenzis, A review on phase-field models of brittle fracture and a new fast hybrid formulation. *Computational Mechanics*, 2014. **55**(2): 383-405.

-
- [219] S. Goswami, C. Anitescu, and T. Rabczuk, Adaptive fourth-order phase field analysis for brittle fracture. *Computer Methods in Applied Mechanics and Engineering*, 2020. **361**: 112808.
- [220] R.U. Patil, B.K. Mishra, and I.V. Singh, An adaptive multiscale phase field method for brittle fracture. *Computer Methods in Applied Mechanics and Engineering*, 2018. **329**: 254-288.
- [221] G. Molnár and A. Gravouil, 2D and 3D Abaqus implementation of a robust staggered phase-field solution for modeling brittle fracture. *Finite Elements in Analysis and Design*, 2017. **130**: 27-38.
- [222] M. Cervera, G. Barbat, and M. Chiumenti, Finite element modeling of quasi-brittle cracks in 2D and 3D with enhanced strain accuracy. *Computational Mechanics*, 2017. **60**(5): 767-796.
- [223] R. Citarella and F.-G. Buchholz, Comparison of crack growth simulation by DBEM and FEM for SEN-specimens undergoing torsion or bending loading. *Engineering Fracture Mechanics*, 2008. **75**(3-4): 489-509.
- [224] C. Bilgen, S. Homberger, and K. Weinberg, Phase-field fracture simulations of the Brazilian splitting test. *International Journal of Fracture*, 2019. **220**(1): 85-98.
- [225] S. Li, W.K. Liu, A.J. Rosakis, T. Belytschko, and W. Hao, Mesh-free Galerkin simulations of dynamic shear band propagation and failure mode transition. *International Journal of Solids and Structures*, 2002. **39**(5): 1213-1240.
- [226] C. Wu, N. Ma, K. Takada, and H. Okada, A meshfree continuous-discontinuous approach for the ductile fracture modeling in explicit dynamics analysis. *Computational Mechanics*, 2016. **58**(3): 391-409.
- [227] S. Loehnert and T. Belytschko, A multiscale projection method for macro/microcrack simulations. *International Journal for Numerical Methods in Engineering*, 2007. **71**(12): 1466-1482.
- [228] M. Malekan and F.B. Barros, Numerical analysis of a main crack interactions with micro-defects/inhomogeneities using two-scale generalized/extended finite element method. *Computational Mechanics*, 2017. **62**(4): 783-801.
- [229] J. Simo, D. Fox, and M. Rifai, On a stress resultant geometrically exact shell model. Part II: The linear theory; computational aspects. *Computer Methods in Applied Mechanics and Engineering*, 1989. **73**(1): 53-92.
- [230] G.-R. Liu and Y.-T. Gu, *An introduction to meshfree methods and their programming*. 2005: Springer Science & Business Media.
- [231] D.D. Wang and H.J. Zhang, A consistently coupled isogeometric-meshfree method. *Computer Methods in Applied Mechanics and Engineering*, 2014. **268**: 843-870.

- [232] A.T. Zehnder and M.J. Viz, Fracture mechanics of thin plates and shells under combined membrane, bending, and twisting loads. *Applied Mechanics Reviews*, 2005. **58**(1): 37-48.
- [233] G.P. Nikishkov and S.N. Atluri, Calculation of fracture mechanics parameters for an arbitrary three-dimensional crack, by the ‘equivalent domain integral’ method. *International Journal for Numerical Methods in Engineering*, 1987. **24**(9): 1801-1821.
- [234] P. Wriggers and G. Zavarise, Computational contact mechanics. *Encyclopedia of Computational Mechanics*, 2004.
- [235] L. De Lorenzis, I. Temizer, P. Wriggers, and G. Zavarise, A large deformation frictional contact formulation using NURBS-based isogeometric analysis. *International Journal for Numerical Methods in Engineering*, 2011. **87**(13): 1278-1300.