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SINGAPORE

**STUDY ON SEVERAL NOVEL TOPOLOGICAL PHASES IN
CLASSICAL WAVE SYSTEMS**

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SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2020

STUDY ON SEVERAL NOVEL TOPOLOGICAL PHASES IN CLASSICAL WAVE SYSTEMS

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SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for the
degree of Doctor of Philosophy

2020

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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I have reviewed the content and presentation style of this thesis and declare it of sufficient grammatical clarity to be examined. To the best of my knowledge, the thesis is free of plagiarism and the research and writing are those of the candidate's except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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Date

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Baile Zhang

Authorship Attribution Statement

This thesis contains material from 5 papers published in the following peer-reviewed journals and 1 paper posted on arXiv in which I am listed as an author.

Section 2.1 in Chapter 2 is published as Fei Gao, Haoran Xue, Zhaoju Yang, Kueifu Lai, Yang Yu, Xiao Lin, Yidong Chong, Gennady Shvets, and Baile Zhang. Topologically protected refraction of robust kink states in valley photonic crystals. *Nat. Phys.* **14**, 140–144 (2018). DOI: 10.1038/nphys4304.

The contributions of the co-authors are as follows:

- Fei Gao, Zhaoju Yang and Baile Zhang conceived the research.
- Fei Gao, Haoran Xue, Zhaoju Yang and Yang Yu performed the simulations.
- Fei Gao and Haoran Xue conducted the experiments with the help from all authors.
- All authors discussed and analyzed the data.
- Baile Zhang, Gennady Shvets, Fei Gao and Yidong Chong wrote the manuscript with the input from all authors.
- Baile Zhang, Gennady Shvets and Yidong Chong supervised and guided the research.

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- Fei Gao, Haoran Xue and Baile Zhang conceived the research.
- Haoran Xue, Fei Gao and Yang Yu performed the simulations.
- Haoran Xue and Fei Gao conducted the experiments.
- All authors discussed and analyzed the data.
- Haoran Xue, Baile Zhang, Fei Gao and Gennady Shvets wrote the manuscript with the input from all authors.

- Baile Zhang, Gennady Shvets and Yidong Chong supervised and guided the research.

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- Haoran Xue designed the structure.
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The contributions of the co-authors are as follows:

- Haoran Xue and Baile Zhang conceived the research.
- Haoran Xue designed the structure and performed numerical simulations.
- Haoran Xue, Yahui Yang and Guigeng Liu conducted the experiments.
- All authors discussed and analyzed the data.
- Haoran Xue, Baile Zhang and Yidong Chong wrote the manuscript with the input from all authors.
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- All authors discussed and analyzed the data.
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- Haoran Xue and Yidong Chong performed the theoretical studies with the input from all authors.
- Haoran Xue and Yidong Chong wrote the manuscript with the input from all authors.
- Yidong Chong and Baile Zhang supervised and guided the research.

25/07/2020

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Abstract

Topological phases of matter have attracted great research interests in the last two decades among various fields, ranging from condensed matter physics to photonics and acoustics. In classical wave systems like photonics and acoustics, the discoveries of topological states offer exciting playgrounds for both new theoretical studies and novel applications. This thesis studies several novel topological phases of matter in classical wave systems which includes experimental demonstrations of photonic valley-Hall phases and acoustic higher-order topological states, and theoretical discovery of non-Hermitian Dirac cones.

In Chapter 1, I briefly review the development history of topological phases of matter in condensed matter physics and classical wave systems, introduce basic knowledge and concepts for topological physics, and introduce several novel topological phases which will be studied in detail in the following chapters.

In Chapter 2, I study photonic topological valley Hall phases. Topologically protected refraction of valley kink states and spin-valley controlled topological phases are experimentally demonstrated in 2D photonic crystals.

In Chapter 3, I study acoustic higher-order topological phases which include experimental demonstrations of a second-order topological insulator on a kagome lattice, a third-order topological insulator on a diamond lattice and an octupole topological insulator in acoustic systems.

In Chapter 4, I present a general theory for non-Hermitian systems to host diabolic points. I show how gain and loss, which is available in various classical wave systems, can interact with diabolic points in unexpected manners.

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List of publications

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4. **Haoran Xue**, Yong Ge, Hong-Xiang Sun, Qiang Wang, Ding Jia, Yi-Jun Guan, Shou-Qi Yuan, Yidong Chong, and Baile Zhang. Observation of an acoustic octupole topological insulator. *Nat. Commun.* **11**, 2442 (2020).
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topological insulator. *Nature* **565**, 622–626 (2019).

9. Yihao Yang, Hong-xiang Sun, Jian-ping Xia, **Haoran Xue**, Zhen Gao, Yong Ge, Ding Jia, Shou-qi Yuan, Yidong Chong, and Baile Zhang. Topological triply degenerate point with double Fermi arcs. *Nat. Phys.* **15**, 645–649 (2019).

10. Gui-Geng Liu, Peiheng Zhou, Yihao Yang, **Haoran Xue**, Xin Ren, Xiao Lin, Hong-xiang Sun, Lei Bi, Yidong Chong, and Baile Zhang. Observation of an unpaired photonic Dirac point. *Nat. Commun.* **11**, 1873 (2020).

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List of abbreviations

BBC: bulk-boundary correspondence

BBH: Benalcazar, Bernevig and Hughes

BZ: Brillouin zone

DOF: degree of freedom

DP: Dirac point

EP: exceptional points

HOTI: higher-order topological insulator

LL: Landau level

NN: nearest-neighbor

NNN: next-nearest-neighbor

PTI: photonic topological insulator

QH: quantum Hall

QSH: quantum spin Hall

QVH: quantum valley Hall

SOC: spin-orbit coupling

TE: transverse electric

TI: topological insulator

TM: transverse magnetic

TRS: time-reversal symmetry

WC: Wannier center

0D: zero-dimensional

1D: one-dimensional

2D: two-dimensional

3D: three-dimensional

Chapter 1 Introduction

In this chapter, I give a brief introduction to several research areas to which the studies presented in this thesis belong. I first give a short introduction to the concept of topological phases of matter. Then I review the developments of topological phases in classical wave systems, especially in photonics and acoustics. Next, I introduce three novel topological phases of matter that are under extensive studies in classical wave systems, which are valley Hall phases, higher-order topological phases and non-Hermitian topological phases. At last, I present our motivation and outlook for the studies in this thesis.

1.1 Topological phases of matter

Advanced mathematical tools and concepts have been playing important roles in the development of physics. One typical example is the field of topological phases of matter [1–5], where the concept of topology is introduced to reveal the richness of phases of matter beyond conventional expectation. Topological materials are characterized by global quantities called topological invariants which are defined by the wavefunctions of energy bands. A topological invariant only takes quantized values and cannot change its value unless a topological phase transition happens. Through the celebrated principle of bulk-boundary correspondence (BBC), topological invariants defined by the bulk give rise to boundary phenomena like topological edge states. The boundary phenomena are robust in the sense that they are tightly linked to the bulk topological invariants which are not affected by perturbations. This topological protection makes topological phases of matter not only fascinating but also useful in practice.

The study on topological phases of matter started with the discovery of the quantum Hall (QH) effect [6]. When two-dimensional (2D) electron gas is subjected to a perpendicular magnetic field, robust Hall conductance quantized to integer values is found. This quantized Hall conductance is associated with robust chiral edge states around the sample, which can be determined by the first Chern number that is calculated from the Bloch states over the entire first Brillouin zone (BZ) and can only take integer values [7]. In the QH effect, the bulk states form Landau levels (LLs) separated by bandgaps. The chiral edge states spread the energy gaps with their number and group velocity locked to the Chern number (Fig. 1.1(a)). These unique features guarantee a strong topological protection: a perturbation that is not strong enough to close the energy gap will not affect the existence of the chiral edge states (since Chern number is not changed) and will not scatter the edge states to the opposite direction or the bulk (since there are no such states at the same energy). Although the first example of the QH phase happens in a system with LLs and external

Figure 1.1. Quantum Hall and quantum spin Hall phases. (a) Illustration of a quantum-Hall type topological insulator. Left panel: dispersion for a semi-infinite lattice where a gapless edge mode is present. Right panel: the chiral edge mode propagates unidirectionally around the edges of a finite sample. (b) Illustration of a quantum-spin-Hall type topological insulator. Left panel: dispersion for a semi-infinite lattice where two gapless edge modes with opposite are present. Right panel: the helical edge modes counter-propagate around the edges of a finite sample.

magnetic fields, they are not tied to a QH phase. The most important ingredient associated with a QH phase is the nonzero Chern number, which is the consequence of time-reversal symmetry (TRS) breaking. By noticing this, in 1988 Haldane proposed a simple two-band lattice model with TRS broken by complex next-nearest-neighbor (NNN) couplings to realize QH effect[8]. This extension from 2D electron gas to periodic lattice is not only useful for theoretical study, but also provides important insight that guides the realization for QH phase in TRS breaking materials [9].

Although TRS breaking is crucial for QH phases, it is not a necessary condition for realizing other kinds of topological phases of matter. A typical example is the quantum spin Hall (QSH) effect which does not require breaking TRS but instead is protected by TRS [10–12]. QSH effect is like two copies of QH effect, one for spin up state and the other one for spin down state. Chern numbers for different spin bands take opposite values so that the total Chern number is zero as required by TRS. In this case, helical edge modes with opposite spin counter-propagate (Fig.

1.1(b)). These two edge modes are protected by TRS, which guarantees the degeneracy of two spins at TRS invariant momenta. Thus, a gap cannot open on the edge under TRS-preserving perturbations.

Nowadays, band topology has been extensively studied and various topological phases of matter have been identified, such as topological crystalline insulators protected by spatial symmetries [13], topological semimetals [14, 15] and topological superconductors [16]. More recently, systematic approaches towards discovering topological materials have been developed [17–19], further paving ways to potential technological applications.

1.2 Topological phases in classical wave systems

Topological phases of matter have been mainly studied in condensed-matter systems. However, the concept of topology is so universal that topological phenomena can almost be found everywhere, including both fermionic systems and bosonic systems. The extension of topological phases and topologically nontrivial bands from condensed-matter systems to classical wave systems like photonic and acoustic systems is kind of straightforward in periodic structures like photonic and phononic crystals. In such cases, Bloch theorem can be applied and the band topology can be analyzed with the Bloch wavefunctions in a similar way to what is well-established in condensed-matter systems. Here I briefly review the main developments in classical wave topological systems.

Similar to the history of discovering topological phases in condensed-matter systems, people first found analogs of QH phase in classical wave systems and then various other topological phases were also developed. The first topological phase in classical wave systems was proposed in 2008 [20–22] and realized in gyromagnetic photonic crystals at microwave frequencies in 2009 [23]. This realization relies on the unusual properties of gyromagnetic materials, whose permeability tensor grows off-diagonal elements under external magnetic field. This effect breaks TRS and leads to the existence of chiral edge mode which is protected without any symmetries. This chiral edge mode propagates unidirectionally and is robust against disorders (Fig. 1.2(a)). This realization is a direct analog of the QH insulator [8]. This phenomenon holds great potential for wave manipulation in various applications. However, the gyromagnetic effect becomes weak at optical frequencies. To overcome this issue, alternative approaches to break TRS are needed. One way to achieve this is to use coupled optical waveguides as shown in Fig. 1.2(b) [24]. The system is regarded as a 2D system with the out-of-plane direction being the effective time axis. The spiral structure of the waveguides then effectively breakings TRS and opens a topological bandgap

Figure 1.2. Examples of topological classical wave systems. (a) Unidirectional electromagnetic modes in magnetic photonic crystals. (b) A photonic Floquet topological insulator based on coupled helical waveguides. (c) A coupled ring resonator system with magnetic flux in each plaquette. (d) Proposal for realizing a photonic topological insulator using bianisotropic metamaterial. (e) Proposal for realizing a photonic QSH phase using photonic/phononic crystals. (f) A valley photonic/phononic crystal with dielectric rods.

where chiral edge mode exists (see the red arrows in Fig. 1.2(b)). In this case, the trick of using a synthetic time axis makes the implementation simple and elegant. However, this system is intrinsically three-dimensional (3D) and cannot be used for on-chip applications. Nevertheless, this system has become an important platform to simulate various kinds of topological phases due to its great flexibility [25–30].

Besides TRS-broken designs, many other designs without breaking TRS have also been pro-

posed in photonics. A large class of these designs [31–36] aims to mimic the QSH [10–12] and QVH [37, 38] effects. Although spin is intrinsically lacking in classical wave systems, pseudo-spin can be engineered in various ways. For example, Hafezi et al. [31] propose a model based on coupled ring resonators to simulate the Hamiltonian for a particle in a magnetic field (Fig. 1.2(c)). Although TRS is preserved for the whole system, for each spin (clockwise mode and counter-clockwise mode) it is broken. This model is totally based on 2D photonic structure and has been realized experimentally [39]. Implementing gain and loss into this lattice, applications like topological lasing [40] and topological light steering [41] have been demonstrated. Pseudo-spin can also be constructed using internal degree of freedoms (DOFs) in Maxwell equation. Khanikaev et al. [32] propose an analog of QSH phase based on bianisotropic metamaterials where two spins are built from in-phase and out-of-phase coupled transverse electric (TE) and transverse magnetic (TM) modes. The bianisotropy χ plays the role of spin-orbit coupling (SOC) and open a topological bandgap inside which counter-propagating edge modes with opposite spin exist (Fig. 1.2(d)). This idea has been realized in various microwave photonic crystals and metamaterials [42–44]. Another widely used structure to mimic QSH is based on Kekulé textured photonic crystals [34]. The structure is based on honeycomb lattice with an enlarged unit cell. Without any deformations, the two Dirac points (DPs) which are originally located at the corners of BZ are now folded at the zone center due to the choice of a large cell. Then a topological bandgap can be created by deforming the lattice as shown in Fig. 1.2(e). The low-energy Hamiltonian around Γ points resembles the Bernevig-Hughes-Zhang Hamiltonian [12] and thus has a Z_2 topology. This model is quite simple and is easy to realize in various platforms, ranging from MHz frequencies [45] and GHz frequencies [46, 47] to optical frequencies [48]. Besides analogs of QSH phase mentioned above, photonic analogs of QVH phase have also been proposed [35, 36] and demonstrated [49–52] in recent years. Different from the analogs of QSH phase which take various forms and are based

on very different mechanisms, analogs of QVH phase are based on similar design principles. As shown in Fig. 1.2(f), a VPC can be constructed from a triangular or honeycomb lattice which hosts a pair of DPs. Then a perturbation that breaks inversion or reflection symmetry of the lattice will lift the degeneracies of DPs and opens a bandgap. Although, due to TRS, the Chern number is always zero, the lack of inversion symmetry indicates that Berry curvature does not have to vanish everywhere in the BZ. In VPCs, the Berry curvature at K and K' points takes opposite values. Thus an interface between two domains with opposite perturbations will host so-called valley kink states, which are solutions to the massive Dirac problem. The kink states can travel through sharp bends without reflection. A more detailed introduction to VPCs will be presented in the next chapter.

Similar to but a little bit later than the development of realizing topological phases in photonics, various models have also been proposed and implemented in acoustic and mechanic systems. The key to realizing QH-like phases in acoustic and mechanic systems is also breaking TRS. In acoustics, one method to break TRS is to introducing flows into the system [53]. Implementing flows in an acoustic lattice can lead to QH phases [54–56]. However, in practice it is challenging to realize low-loss and large scale lattice with flows in it, making experimental demonstrations of this approach very limited [57]. In mechanic systems, it is suggested that gyroscopic components can be used to break TRS [58, 59]. In these designs, breaking of TRS leads to extremely robust boundary phenomena against all kinds of defects and disorders that do not close the topological bandgap. However, breaking TRS is always accompanied with complicated setups and other unwanted effects, making it incompatible with many applications. Like in photonics, topological phases can also be achieved without breaking TRS in acoustics and mechanics by mimicking QSH and QVH effects. This can be done by a direct realization of a topological Hamiltonian. For example, Süsstrunk and Huber constructed a mechanic lattice with coupled pendulums which can be described by a QSH Hamiltonian [60], similar to the coupled ring resonator idea. Besides

constructing a target Hamiltonian with coupled modes, some designs proposed in photonics also work for acoustics and mechanics. One example is the Kekulé textured honeycomb lattice model [34]. Since it only relies on lattice symmetry, the model can also be used in acoustics to create an acoustic TI [61]. Another example is the bianisotropic design. This design strategy can also be used to construct a phononic TI with a coupling between two degenerate modes [33]. I would like to note that in bosonic systems TRS cannot protect a QSH phase. Indeed, most analogs of QSH phase in classical wave systems have fragile topology [62–64]. Nevertheless, the edge states in most cases are still quite robust, as evidenced by various experiments [39, 42–44, 46, 47]. Besides the QSH-type models, QVH phase has also been realized in phononic crystals consisting of triangular scatters [65].

Besides photonic, acoustic and mechanic systems as mentioned above, electric circuits have emerged as another platform for topological physics in recent years [66–68]. The diversity of the classical platforms allows people to demonstrate different topological models and also offers chances to various applications. Here we briefly compare these systems and discuss their advantages and limitations in terms of implementation of topological modes and potential applications. Electric circuits are very powerful in realizing topological lattice models due to the simplicity in design and fabrication. The advantage of choosing circuits becomes especially obvious when dealing with models that are non-Hermitian, nonlinear and higher dimensional. However, so far efforts in topological circuit systems have been made on demonstrating new physics, with potential applications unclear. This is in contrast with other classical systems where many potential applications at different frequencies can be expected. Besides electric circuits, the difficulty levels in implementing topological models in other three systems (photonic, acoustic and mechanic systems) are pretty much similar. The differences mainly come from fabrication and characterization. Acoustic systems are easier to fabricate due to the commercialization of 3D printing technique. Besides,

photonic and mechanic systems at lower frequencies are also easy to construct due to the relatively large system size. As for characterization, system dissipation is an important issue. The intrinsic loss for acoustic systems is relatively large compared with photonic and mechanic systems.

Despite the great success in introducing topological phases into classical wave systems, there are noticeable differences between classical systems, which are bosonic, and condensed matter systems that are fermionic. Perhaps the most important difference is that bosonic systems do not have the concept of Fermi energy. This feature can lead to both advantages and disadvantages. Without the restriction of Fermi energy, one can access the bandstructure at any frequencies (or energies) where the topological phenomena happen. However, due to the lack of a Fermi surface, it is very hard to find robust quantized observables like the Hall conductance in bosonic topological systems. Thus, when talking about classical analog of quantum Hall effect, it mostly refers to edge physics instead of quantized observables. Regardless of the lack of quantized observables, classical wave systems still provide useful platforms to measure a topological induce without directly relating it to an observable. This can be done, for example, by mapping out Berry curvature from anomalous transport [69], measuring edge winding number [70, 71] or measuring fractional mode density [72].

Over the last decade, the studies on topological classical wave systems have grown rapidly [73–77]. Many more topological phases and related exotic phenomena, such as Floquet topological systems [24, 78–82], 3D topological systems [83–89], topological phases in synthetic dimensions [25, 90–92], higher-order topological phases [93–96], fragile topological phases [62–64, 97] and non-Hermitian topological systems [41, 98–100]. These studies not only provide platforms to experimentally study the physics of topological phases of matter which are hard to realize in condensed matter settings, but also offer chances to creating new devices like topological lasers [40, 101].

1.3 Valley contrasting physics and valley photonic crystals

In many 2D materials with a hexagonal BZ like graphene and transition metal dichalcogenides, the two inequivalent points at the corners of BZ are called K and K' (or +K and -K) valleys, as shown in Fig. 1.3(a). These two valleys have contrasting physics that has attracted a lot of attention. Take the tight-binding model for graphene as an example. It is known the two bands touch at the two valleys, forming two massless DPs (see the blue curves in Fig. 1.3(b)). If opposite on-site energy detunings are added to the two sublattices of graphene, then the degeneracies will be lifted and two massive Dirac cones emerge at the two valleys (see the red curves in Fig. 1.3(b)). Although the dispersions at K and K' valleys are identical as required by TRS, the states associated with the two valleys have very different properties. Firstly, the states at K and K' valleys carry opposite orbital magnetic moment. In other words, the Bloch states have a self-rotating form, whose direction is locked to valley DOF. This orbital magnetic moment is an extra DOF for information procession and also wave manipulation. Secondly, although the Chern number, which is the integration of Berry curvature, is zero due to TRS, the Berry curvature itself does not have to be zero in the whole BZ because of the inversion symmetry breaking. In fact, Berry curvature is concentrated at the two valleys but takes opposite values at the two valleys (Fig. 1.3(c)), similar to the orbital magnetic moment. Besides these valley contrasting properties arising in the bulk, valley physics also happens on the edges. In particular, an interface between two valley materials with opposite configurations will host so-call valley kink states. These kinks states are confined at the interface and their propagation directions are linked to the two valleys (Fig. 1.3(d)). These kink states can be seen as the solutions to domain wall massive Dirac equation, where the mass term flips sign across the interface due to the flip of the lattice configuration. Although these kink states do not have a standard topological origin, they are still robust against certain defects, such as a sharp bend. With the discoveries of graphene and many other 2D materials, the studies on the

Figure 1.3. Valley contrasting physics. (a) Illustration of the first Brillouin zone and K/K' points. (b) Dispersion for a graphene model with (blue line) and without (red line) inversion symmetry. (c) Berry curvature distribution for a valley material where blue (red) colors indicate negative (positive) values of Berry curvature. (d) Dispersion for a slab containing two domains without opposite mass. Kink states are found at the domain wall. (d) Example of a valley photonic crystal with dielectric rods.

physics of valley have become an active research direction called valleytronics [102].

As mentioned in the previous section, photonic analogs of valley materials are called VPCs. Most designs of VPCs rely on honeycomb lattices and triangular lattices, which have a hexagonal BZ. An example of a triangular lattice has been given in Fig.1.2(f) [35]. Here I show a honeycomb lattice design (Fig. 1.3(e)) [36]. It is made up of dielectric rods arranged at the sites of a honeycomb lattice. The inversion symmetry breaking is achieved by tuning the radii of rods.

The physics of VPCs is similar to the 2D valley materials. The Bloch states at the two valleys have opposite vortex and Berry curvature at the two valleys take opposite sign. VPCs have been realized in experiments recently and shown several intriguing features such as robust kink states and topological refraction [49–52]. In Chapter 2, related works on one of the first realizations of VPCs [51] and the physics of VPCs interacted with another spin DOF [103] will be introduced.

1.4 Higher-order topological insulators

TIs are usually considered as materials that have insulating bulk but conducting boundaries. Here boundaries usually refer to the $d - 1$ dimensional boundaries, with d being the dimension of the system. Recently, a new class of TIs called higher-order TIs (HOTIs) have been identified. Unlike the conventional case, a HOTI hosts topological states at $d - n$ dimensional boundaries with n larger than one. Here n denotes the order of a TI. For example, a 2D second-order TI does not have protected one-dimensional (1D) edge states. Instead, the non-trivial topology manifests through topologically protected zero-dimensional (0D) corner states (see Fig. 1.4(a)). Similarly, a 3D third-order TI is featured by topological states at the 0D corners (Fig. 1.4(b)) and a 3D second-order TI is featured by topological states at 1D hinges (Fig. 1.4(c)).

The discovery of HOTIs is accompanied by the generalization of modern theory of polarization for dipole moment [104–106] to multipole moments. The first HOTI model is proposed by Benalcazar, Bernevig and Hughes (BBH) [107], which is a HOTI with a quantized quadrupole moment in the bulk and topologically protected zero modes at the corners. Since HOTIs have

Figure 1.4. Higher-order topological insulators. (a) A 2D second-order topological insulator with topological corner states. (b) A 3D third-order topological insulator with topological corner states. (c) A 3D second-order topological insulator with topological hinge states.

gapped $d - 1$ dimensional boundaries, conventional topological invariants like Chern number are not useful to reveal the nontrivial topology. In the initial work [107], the higher-order topology is characterized by a method called nested Wilson loops. This method is a generalization of the standard Wilson loop method and will be introduced in detail in Chapter 3. Since the proposal by BBH, various types of HOTIs have been proposed [108–112]. Although most proposals are based on condensed matter context, most experimental realizations so far are based on classical wave systems [29, 93–96, 113, 114]. In Chapter 3, related works on the experimental realizations of several types of HOTIs in acoustic systems will be presented [95, 115, 116].

1.5 non-Hermitian topological systems

Hermiticity, which is an important assumption for quantum mechanics, guarantees real eigenvalues with orthogonal eigenstates associated with a Hermitian Hamiltonian. However, in many cases especially in classical wave systems with gain and loss, a non-Hermitian description is necessary. Unlike Hermitian systems, non-Hermitian ones are characterized by complex eigenvalues with non-orthogonal eigenstates, and have many intriguing properties such as parity-time (PT) symmetry [117] that are absent in the Hermitian systems. While in the past people mainly focused on small scale non-Hermitian systems and studied phenomena solely from non-Hermiticity, recently growing interests are put into non-Hermitian topological systems where the interplay between band topology and non-Hermiticity leads to unprecedented physics. A brief comparison between Hermitian and non-Hermitian topological systems is listed in Table I. At the most fundamental level, for example, non-Hermiticity significantly enriches the topological classification [118–120] and alters the conventional BBC [121–126]. Even the notion of bandgap is enriched in the presence of non-Hermiticity. A unique type of bandgap, point gap, can emerge in non-Hermitian systems, which lead to new topological invariants and BBC [118, 120]. In prac-

TABLE I: Hermitian and non-Hermitian topological systems

Topological property	Hermitian	Non-Hermitian
AZ class	10	38
Band degeneracy	Diabolic point	Exceptional/diabolic point
Band gap	line gap	line/point gap
Topological invariant	Defined by Bloch wavefunction	Defined by wavefunction/energy dispersion
BBC	Bloch-type	Bloch/Non-Bloch type

tice, non-Hermitian topological systems also have many applications such as topological lasing [40, 101].

In non-Hermitian systems, degeneracies are usually exceptional points (EPs) where both eigenvalues and eigenvectors are degenerate. The physics of EP has been extensively studied over the last few years [127]. However, the other type of degeneracy, diabolic points, is less studied in non-Hermitian systems. Diabolic points play important roles in Hermitian topological systems. They exist in topological semimetals and can give rise to gapped topological phases upon perturbations. Various studies have shown that diabolic points are unstable under non-Hermiticity, transforming into EPs and other exceptional objects [100, 128, 129]. Whether diabolic points like DPs can survive in non-Hermitian systems remains unclear. In Chapter 4, I introduce a general approach to have and control diabolic points in non-Hermitian systems [130].

1.6 Motivation and outlook

In Fig. 1.6, I briefly summarize the topological phases that have been studied in classical wave systems and highlight the ones that are studied in this thesis. As can be seen, starting from QH phases, various topological phases have been introduced in classical wave systems. Our works, which contribute to several important types of topological phases, are mainly driven by the following three motivations. First, I want to transfer the concept of topology and the celebrated examples of topological phases from condensed matter to classical wave systems. By doing this, I want to take advantage of the unique features of topological states and find useful applications in classical wave systems. Second, I want to establish classical wave systems as useful and regular platforms to implement all kinds of novel topological phases of matter which are hard to realize in real materials. In such cases, our studies can push the whole field of topological phases of matter forward. Third, I want to explore new physics for topological phases on the platforms of classical wave systems where new effects such as non-Hermiticity and nonlinearity are available.

Based on the first motivation, I am interested in realizing topological phases that haven't been realized in classical wave systems. According to the previous introduction, both QH and QSH phases have been realized in classical wave systems. However, QVH phase was only realized in acoustic systems when our project on realizing QVH phase in photonic systems was started. In Chapter 2, I present our study on the realization of a VPC, which is one of the first realizations of QVH phase in photonics. My co-workers and I not only observed expected physics for QVH phase, but also found the so-called topologically protected refraction which is unique to classical wave systems. Based on this project, My co-workers and I further included spin DOF and realized the first photonic TI (PTI) with both valley and spin DOF.

Based on the second motivation, I am interested in realizing topological phases that are only theoretically proposed. Thus, My co-workers and I took an active part in the experimental real-

Figure 1.5. Summary of topological phases studied in classical wave systems.

izations of recently discovered higher-order topological phases. In Chapter 3, I introduce our experimental works on several types of higher-order topological phases in acoustic systems, which include one of the first realizations of a Wannier-type HOTI, one of the first realizations of a third-order TI and one of the first realizations of an octupole TI.

Based on the third motivation, I move my interest onto non-Hermitian topological systems. By noticing that lots of research attention has been paid to EPs and break down of BBC, My co-workers and I took a different path and studied the physics of diabolic points and associated topological phases in non-Hermitian systems. My co-workers and I proposed, for the first time, a mechanism for non-Hermitian systems to host diabolic points and showed its power with several striking examples. This study is presented in Chapter 4.

In the future, I would expect the first two motivations will become weak. This is because

many important topological phases have already found their analogs in classical wave systems, and realizing a non-interacting topological lattice model in classical wave systems would be less difficult after all the pioneer works. Instead, the following aspects require more effort. Firstly, on the theoretical level, systematic theory for classical wave topological systems is highly desired. Secondly, on the practical level, efforts on real applications and devices should be made. Thirdly, the third motivation is still very strong. The study on non-Hermitian and nonlinear topological systems is still at its infant age. It is very likely there are treasures to be found along these directions.

Chapter 2 Wave manipulation in valley photonic crystals

In this chapter, I study the properties of VPCs. In the first section, I present the design and demonstration of the kink states in a VPC. I show that the kink states can go through sharp bents and refract into free space without reflection. In the second section, I introduce spin DOF into the VPC and demonstrate spin-valley locked bulk bands and topological phase transition between QSH phase and QVH phase due to the competition between spin DOF and valley DOF. From these studies, I show that VPCs provide a fruitful platform for wave manipulation, which is promising for 2D on-chip nanophotonics and other applications.

2.1 Topologically protected refraction of valley kink states

Recently proposed VPCs [35, 36, 131] mimic many of the unusual properties of 2D gapped valleytronic materials [132–138]. Of the utmost interest to optical communications is their ability to support topologically protected chiral kink states at the internal domain wall between two VPCs with opposite valley-Chern indices. Here I experimentally demonstrate valley-polarized kink states with polarization multiplexing in VPCs, designed from a spin-compatible four-band model. When the valley pseudospin is conserved, I show that the kink states exhibit nearly perfect out-coupling efficiency into directional beams, through the intersection between the internal domain wall and the external edge separating the VPCs from ambient space. The findings presented in this section are published in Ref [51].

2.1.1 Design of the valley photonic crystal

To design a VPC, I first design a photonic crystal with DPs. This can be easily done by arranging metal rods into a triangle lattice and sandwiching them with two parallel metal plates. The two metal plates create an effective 2D system where Maxwell equations can be decoupled into two sets, supporting TE modes and TM modes. If system parameters are randomly chosen, although both TE and TM bands have DPs at the two valleys as required by lattice symmetries, they are in general at different frequencies. To match the frequencies for TE and TM DPs, I further tune the radii and heights of the rods. As I will show throughout this chapter, this matching will offer the possibility to study the rich physics on the interaction of the valley DOF and a spin DOF. The designed structure is shown in Fig. 2.1(a). Note here the rods are suspended in air to prevent other modes entering the frequency window of our interest. The two DPs overlap with each other at around 6 GHz. The physics around this region can be described by a Dirac-type Hamiltonian:

$$H_0 = v(\tau_z s_0 \sigma_x \delta k_x + \tau_0 s_0 \sigma_y \delta k_y). \quad (1)$$

Figure 2.1. Design of a valley photonic crystal. (a) Unit cell of the unperturbed photonic crystal, which consists of metallic rods sandwiched by two parallel metallic plates. The height and radius of the rod are $h = 34.6$ mm and $r = 6.35$ mm, respectively. There is a small gap with size $g = 1.1$ mm between the rod and two plates. The lattice constant is 36.8 mm. (b) Bulk bands for the structure shown in (a). (c) Unit cell of designed valley photonic crystal. The height of the tripod is $h = 34.6$ mm. Other parameters are $r = 3.68$ mm, $l = 7.95$ mm and $w = 2.21$ mm. (d) Bulk bands for the structure shown in (c).

Here v is the group velocity around the DPs (which is taken to be isotropic for simplicity). δk_x and δk_y are the momentum measured from K/K' points. $\tau_{x,y,z}$, $s_{x,y,z}$ and $\sigma_{x,y,z}$ are Pauli matrices correspond to valley, spin and orbital DOFs. τ_0 and s_0 are identity matrices. Starting from this structure, several different perturbations can be introduced into this system to create gapped topological phases. Previously, it has been shown that a reduction of reflection symmetry along z direction can hybridize TE modes and TM modes and create a QSH-like phase [43, 44, 139]. Here I choose a different perturbation to create a QVH phase. As mentioned in the introduction,

this can be done by breaking inversion symmetry. I do this by deforming the rods into tripods, as shown in Fig. 2.1(c). Now the degeneracies are lifted, and a bandgap exists at around 6 GHz (Fig. 2.1(d)). Note this perturbation breaks inversion symmetry but still preserves z -reflection. Thus, TE modes and TM modes are not coupled and can be treated independently. This procedure introduces a mass term into the effective Hamiltonian:

$$H_1 = \Delta_P \tau_0 s_0 \sigma_z, \quad (2)$$

where Δ_1 corresponds to the perturbation strength. For a detailed derivation of this effective Hamiltonian, refer to Ref. [140].

2.1.2 Contrasting physics of valley photonic crystal

With the design ready, let's now proceed to numerically demonstrate the valley physics of our model. I first look at the mode profiles for the TE mode below the bandgap at K valley which is computed by COMSOL Multiphysics. As shown in Fig. 2.2(a), the power flow forms a vortex structure, which indicates this mode has an orbital angular momentum, which can serve as a new DOF and allows for selective manipulation. A similar vortex structure is also found for the TM mode at K valley (Fig. 2.2(b)). The modes at K' valley are in contrast with those at K valley. As can be seen from Fig. 2.2(e)-(f), the circulating direction of the vortex is reversed, indicating the modes at the two valleys carry opposite orbital angular momentum or chirality. This DOF gives rise to the possibilities of selective manipulation of bulk valley modes. However, achieving the selection is not very easy. For example, to selectively excite the states in a certain valley, a source that matches the chirality or momentum is required [141]. To make the access to independent valley easier, we can introduce spin DOF to realize spin-valley locking [103, 131, 142] or break TRS to create a frequency window where only one valley can be excited [143, 144].

Besides the orbital angular momentum, Berry curvature is also an important quantity that has

Figure 2.2. Contrasting physics of the valley photonic crystal. (a) Profile and energy flow (arrows) of the TE mode below the bandgap at K valley. (b) Profile and energy flow (arrows) of the TM mode below the bandgap at K valley. (c) Berry curvature distribution for the TE band below the bandgap around K valley. (d) Berry curvature distribution for the TM band below the bandgap around K valley. (e)-(h) are similar to (a)-(d), but are the results for K' valley.

valley contrasting physics. Berry curvature is a gauge-invariant quantity which plays the role of magnetic field in momentum space. In 2D, it only has z -component which can be calculated from Berry connection:

$$\mathbf{\Omega}(\mathbf{k}) = -\nabla \times \mathbf{A}(\mathbf{k}). \quad (3)$$

Here Berry connection is defined as:

$$\mathbf{A}(\mathbf{k}) = -i\langle u(\mathbf{k}) | \nabla_{\mathbf{k}} | u(\mathbf{k}) \rangle, \quad (4)$$

where $u(\mathbf{k})$ is the periodic part of Bloch wavefunction. I calculate the Berry curvature around the two valleys for the two modes below the band gap following Eqs. (3)-(4) with numerically

obtained eigenmodes. I note that the Bloch wavefunction in COMSOL has the form $\psi(\mathbf{k}) = u(\mathbf{k})e^{-i\mathbf{k}\mathbf{r}}$ rather than $\psi(\mathbf{k}) = u(\mathbf{k})e^{i\mathbf{k}\mathbf{r}}$. I also note for simple structures the wavefunctions given by COMSOL have a smooth gauge. Thus, we can directly use the definition in Eqs. (3)-(4) to calculate the Berry curvature. The calculated Berry curvature in an area of $\frac{0.6\pi}{a} \times \frac{0.6\pi}{a}$ around K valley is plotted in Fig. 2.2(c) (TE mode) and (d) (TM mode). As can be seen, the Berry curvature is concentrated at the valley and takes negative values. In contrast, Berry curvature at K' valley takes positive values (Fig. 2.2(g) and (h)). They will cancel each other in total. However, we can still define the so-called valley Chern number to account for the non-zero Berry curvature. Valley Chern number is the integration of Berry curvature over the region around a valley:

$$C_v = \frac{1}{2\pi} \int_v \Omega(\mathbf{k}) d^2k. \quad (5)$$

Following this definition, we have $C_{K/K'} = \mp \frac{1}{2}$. These half-integer numbers cannot induce gapless edge states since the conventional Chern number is zero. But they play an important role in a domain geometry where the valley Chern numbers flip their signs across the interface. In such case, as we will show later, gapless kink states will be induced due to the nonzero valley Chern numbers.

2.1.3 Valley kink states

Although valley Chern number is a half-integer, the difference between two valley Chern numbers can be an integer and give rise to boundary states. Consider a heterostructure which consists of two domains. The lower domain is the VPC that we just designed, while the upper domain is another VPC which is the inversion of the original one (Fig. 2.3(a)). In the lower domain, the valley Chern numbers are $C_{K/K'} = \mp \frac{1}{2}$. In the upper domain, in contrast, the valley Chern numbers are $C_{K/K'} = \pm \frac{1}{2}$. Thus, the difference of valley Chern numbers between the upper domain and lower domain are $\Delta C_{K/K'} = \pm 1$. As we will see, $\Delta C_{K/K'}$ predicts the number and

Figure 2.3. Valley kink states. (a) Schematic of the heterostructure. In the lower domain, the tripods point upwards. In the upper domain, the tripods point downwards. The domain wall is highlighted by the red dashed line. (b) Dispersion for the heterostructure shown in (a). The kink states are marked in red (TE modes) and blue (TM modes). (c) Mode profile of the kink states at K' valley. Left and right panels are for TE and TM modes, respectively.

the slope of kink states at each valley. For example, $\Delta C_K = 1$ indicates there is a gapless kink state with a positive group velocity at K valley. To verify this, I compute the dispersion for the heterostructure shown in Fig. 2.3(a). We take 12 unit cells along y for each domain and apply periodic condition along x . The results are given in Fig. 2.3(b). As can be seen, the number and the velocity of kink states are in agreement with the valley Chern numbers. This can be regarded as a special type of BBC. To further verify the in-gap states are kink states, I plot their profiles in Fig. 2.3(c), which shows that the electromagnetic waves are indeed localized around the domain wall.

To have a deeper understanding on the existence of the kink states and the BBC, I next show an analytic approach to the kink states based on the low-energy Hamiltonian. For simplicity, let us just consider one mode and one valley. The physics for the other mode and valley is similar. In the low-energy discription, the effective Hamiltonian for a homogeneous lattice reads:

$$\mathcal{H}(\mathbf{k}) = \sigma_x k_x - \sigma_y k_y + m\sigma_z. \quad (6)$$

Here m is the mass term which opens the band gap and determines the valley Chern number via $C_v = \frac{1}{2}\text{sgn}(m)$. For a heterostructure like Fig. 2.3(a), the mass term flips sign across the domain wall and the periodicity along y is no longer preserved. Such a lattice is described by the equation:

$$(\sigma_x k_x + i\sigma_y \frac{\partial}{\partial y} + m(y)\sigma_z)\psi = E\psi, \quad (7)$$

with the function $m(y)$ being:

$$m(y) = \begin{cases} m_v(y < 0) \\ -m_v(y > 0) \end{cases}. \quad (8)$$

We want to find a solution that is localized at the domain and propagating along x . Take the following ansatz:

$$\psi = \begin{pmatrix} U \\ V \end{pmatrix} e^{ik_x x} e^{-\alpha|y|}, \quad (9)$$

with α being a positive real number. Substitute Eq. (9) into Eq. (7):

$$\begin{aligned} (\sigma_x k_x + i\alpha\sigma_y + m_v\sigma_z) \begin{pmatrix} U \\ V \end{pmatrix} &= E \begin{pmatrix} U \\ V \end{pmatrix} \quad (y < 0), \\ (\sigma_x k_x - i\alpha\sigma_y - m_v\sigma_z) \begin{pmatrix} U \\ V \end{pmatrix} &= E \begin{pmatrix} U \\ V \end{pmatrix} \quad (y > 0). \end{aligned} \quad (10)$$

For $y < 0$, we have eigenvalues and eigenvectors as:

$$E = \pm\sqrt{k_x^2 - \alpha^2 + m^2}, \quad \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{E-m_v}{k_x+\alpha} \end{pmatrix}. \quad (11)$$

For $y > 0$, we have:

$$E = \pm\sqrt{k_x^2 - \alpha^2 + m^2}, \quad \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{E+m_v}{k_x-\alpha} \end{pmatrix}. \quad (12)$$

To ensure the continuity of the wavefunction across the domain wall, we can obtain:

$$\begin{aligned}\alpha &= m_v, \\ E &= -k_x.\end{aligned}\tag{13}$$

Thus, we obtain a solution which is localized at the domain wall and has a linear dispersion with a negative slope, consistent with $\Delta C_v = -1$. When the configurations of the two domains are exchanged (i.e., $m = -m_v$ for $y < 0$ and $m = m_v$ for $y > 0$), we can obtain a similar solution but with $E = k_x$. Thus, we can use the valley Chern numbers to predict the kink states.

2.1.4 Topologically protected refraction of kink states

Although the solution given by domain Dirac problem is a chiral mode, the kink states are different from the chiral mode of a Chern insulator. This is because there are two valleys, and the kink states at different valleys counter-propagate. The situation here is also different from the QSH case where two edge modes with opposite spins counter propagate. In the QSH case, backscattering is forbidden by TRS. In contrast, kink states at different valleys can be scattered into each other upon lattice symmetry breakings. This makes the valley kink states less robust than the chiral edge states of Chern insulators and helical edge states of Z_2 TIs. Nevertheless, valley kink states are immune to sharp turns of the path that are compatible with lattice symmetries such as $\pi/3$ and $2\pi/3$ turns. This is in great contrast to conventional waveguides where reflection generally happens at path bends. The transport of valley kink states has been studied both numerically and experimentally [65]. Here I ask if there is any other interesting feature associated with valley kink states. By looking at the transport of the kink states beyond what can happen inside the lattice, an interesting and also important phenomenon was found: protected refraction of the kink states at the interface between the lattice and surrounding free space.

To study the refraction and reflection of the kink states at the interface between the lattice

Figure 2.4. Selective excitation of valley kink states and reflectance measurement. (a) Schematic of the experimental setup. (b) Photo of the sample for selectively excite one of the kink states. (c) Photo of the sample for measuring reflectance from an armchair boundary. (d) Photo of the sample for measuring reflectance from a zigzag boundary.

and the surrounding space, I first introduce an experimental method for selectively exciting the kink state at a certain valley and then measuring the reflectance of the excited kink state from an interface. The schematic of the experimental setup is shown in Fig. 2.4(a). To excite at one of the valleys, we use two linear antennas, denoted as antenna A and antenna B. They are placed at the domain wall between two VPCs with opposite valley Chern numbers to excite the kink states. The experiment can be divided into several steps. Firstly, we need to achieve selective excitation. In this step, we don't want the reflection at the boundaries to cause trouble for measurements. Thus, I place absorbing material at the boundaries (Fig. 2.4(b)). To select one valley, I tune the phase and magnitude of the signal in one of the antennas. The fields excited by the two antennas

Figure 2.5. Robust refraction of valley kink states. (a) Left panel: measured reflectance of TE kink state from armchair termination (purple line) and zigzag (grey line) termination. Error bars represent standard deviation. Right panel: illustration of the refraction process of TE kink state through zigzag termination. (b) Similar to (a) but for TM mode. (c) Upper panel: field pattern of the refraction of TE kink mode through zigzag termination. The left and right parts show the simulation and experimental results, respectively. Lower panel: field pattern of the refraction of TE kink mode through armchair termination. The left and right parts show the simulation and experimental results, respectively. (d) Similar to (c) but for TM mode.

then interfere, and selective excitation can be achieved at a certain phase modulation. This two-antenna source is like a simplified version of a phased array. Since in real experiments the initial relationships of the phase and magnitude between the two antennas are unknown, I continuously

tune the attenuator and the phase modulator until the signal measured at the position marked by the red dot in Fig. 2.4(b) reaches background noise level, which indicates only the right-moving kink state is excited. Next, I remove the absorbing materials placed at the right boundary and study the refraction of the kink state at this boundary. There are two types of terminations that are compatible with the lattice: armchair boundary ((Fig. 2.4(c)) and zigzag boundary ((Fig. 2.4(d)). In the experiment, we can switch between the two boundaries by inserting or removing a movable part of the lattice (see the area indicated by the black dashed lines in (Fig. 2.4(c)). Then the reflectance is collected by a linear antenna (for measuring TM mode) or a loop antenna (for measuring TE mode) which is placed at the position marked by the black dots in Fig. 2.4(c)-(d). Besides, the refracted wave in the empty waveguide region can also be recorded by field scanning measurements.

Now we proceed to the investigation on the refraction of the kink states. I excited the kink mode at K valley using the method described above and measured the reflectance. In general, the reflectance is very low for both types of terminations due to the fact that inter-valley scattering is quite small (Fig. 2.5(a)-(b)). This is especially prominent for the zigzag termination, where the reflection is almost absent (see the grey lines in Fig. 2.5(a)-(b)). To further validate this claim, I simulate the field pattern with selective excitation. The simulated H_z field distribution at mid-gap frequency (6.12 GHz) is shown in Fig. 2.5(c). As can be seen, almost all the energies go to the empty space. I also mapped out the fields in the empty waveguide region experimentally, which agree well with the simulations (Fig. 2.5(c), right panel). The situation for TM mode is slightly different. As shown in Fig. 2.5(d), the reflection is still quite small. But the pattern of the refracted beams is different. In particular, there are two out-coupling beams for TM polarization but only one for TE polarization. This difference comes from the different dispersions for TE polarization and TM polarization in the empty waveguide region. For TE polarization, the dispersion

relationship is:

$$k = \sqrt{(\omega/c)^2 - (\pi/d)^2}. \quad (14)$$

While for TM polarization, the dispersion relationship is:

$$k = \omega/c. \quad (15)$$

Here d is the distance between the two plates and c is the speed of light in air. Due to the different dispersion, the contours for the two polarizations are different. If we apply phase-matching conditions at the interface, then we can find these two dispersions lead to different refraction patterns (see Fig. 2.5(a)-(b)).

2.1.5 Summary and outlook

The topologically protected refraction demonstrated above can be useful in many applications such as directional antenna. The high efficiency of the coupling between a topological mode and free space could enhance the performance of the electromagnetic devices. Besides, the dual-mode nature of our system and the polarization multiplexing at the refraction process can double the data capacity to support high-efficiency and robust wireless and optical data networks [145, 146]. Moreover, topological cavities can be made out of the topological interface where the kink states can exhibit high performance lasing behavior [147].

I also note the phenomenon of topologically protected refraction is solely due to valley conservation. Thus, it should not be limited to electromagnetic systems that are studied here. In fact, our work has inspired the search for topologically protected refraction in other systems such as acoustics [148]. Besides the topologically protected refraction, our work, together with related experimental works during the same period [49, 52], paves the way to studying valley physics in photonic systems. In the future, I hope VPCs can be used in real devices for robust waveguiding,

signal processing and other applications. Besides, the topologically protected refraction demonstrated here is solely due to valley conservation. Thus, I believe similar phenomena should also be found in other classical wave systems.

2.2 Spin-valley-controlled photonic topological insulators

The discovery of PTIs has opened the door to fundamentally new topological states of light. Time-reversal-invariant PTIs generally function as analogs of either QSH effect or QVH effect (as demonstrated in the previous section) in condensed-matter systems, yielding photonic topological edge states that are locked to either a ‘photonic pseudospin’ (abbreviated as ‘spin’) or a valley DOF. In this section, I demonstrate a new class of PTIs whose topological phases are not determined solely by spin or valley, but by the competition between them. These PTIs are realized experimentally using metallic photonic crystals operating at microwave frequencies; both SOC and inversion symmetry breaking are present and can be accurately tuned using different structural parameters. I demonstrate, for the first time, a topological transition between QSH and QVH phases, a phenomenon that is difficult to achieve in condensed-matter systems. This study merges the emerging fields of spintronics and valleytronics in the same photonic platform and offers novel PTIs with reconfigurable topological phases. The findings presented in this section are posted on arXiv as a preprint [103].

2.2.1 Effective Hamiltonian analysis

Analogues of QSH phase and QVH phase both have been proposed and realized in photonic structures [32, 34–36, 42–44, 46, 47, 49–52, 139]. To realize a QSH phase in photonics, one can define photonic spins based on in-phase and out-of-phase superpositions of TE and TM modes [32], and realize photonic SOC by either using bianisotropic metamaterials [42] or breaking the out-of-plane mirror symmetry [43, 44, 139]. To realize the QVH phase, inversion symmetry can be broken using a staggered sublattice potential [36, 50, 52] or triangle-shaped scatterers in each unit cell [35, 49, 51]. To having both spin and valley DOFs in one structure, let us first consider

Figure 2.6. Illustration of the spin-valley-controlled physics. (a) Dispersion for H_0 . (b) Dispersion for $H_0 + H_2$. (c) Dispersion for $H_0 + H_1$. (d) Dispersion for $H_0 + H_1 + H_2$.

the following effective Hamiltonian with both SOC and inversion symmetry breaking:

$$H = H_0 + H_1 + H_2, \quad (16)$$

where H_0 and H_1 are given by Eq. (1) and eq. (2), respectively. $H_2 = \Delta_{soc}\tau_z s_z \sigma_z$ is induced by SOC. Here $\tau_{x,y,z}$, $s_{x,y,z}$ and $\sigma_{x,y,z}$ are Pauli matrices acting on valley, spin and orbital subspaces. This Hamiltonian can be used to describe massive Dirac cones with both SOC and inversion symmetry breaking.

In the absence of both SOC and inversion-symmetry breaking, the gapless Hamiltonian H_0 forms Dirac cones at K and K' valleys, as shown in Fig. 2.6(a). When only SOC is present, the bandgap opened in Fig. 2.6(b) has size $2|\Delta_{soc}|$. When only inversion-symmetry breaking

is present, the bandgap in Fig. 2.6(c) has size $2|\Delta_P|$. In both cases, the bandgap size applies to both spin up and spin down states, and in both the K and K' valleys. However, when SOC and inversion-symmetry breaking are simultaneously present, Eq. (16) predicts that the bandgaps for the two spin states at two valleys are proportional to $|s\Delta_{\text{SOC}} + v\Delta_P|$, where $s = \pm 1$ denote up/down spin states and $v = \pm 1$ denote the K and K' valleys, as shown in Fig. 2.6(d). This results in a spin-valley locking of the bulk bands. Furthermore, the topological phase is determined not solely by $|\Delta_{\text{SOC}}|$ or $|\Delta_P|$, but by their relative size. One can calculate the valley-projected spin Chern numbers as $C_{s,v} = \frac{1}{2}\text{sgn}(s\Delta_{\text{SOC}} + v\Delta_P)$. Thus, when $|\Delta_{\text{SOC}}| > |\Delta_P|$, the spin Chern number is nonzero, $C_s = C_{s,K} + C_{s,K'} = \pm\text{sgn}(\Delta_{\text{SOC}})$; this gives rise to the QSH phase. However, when $|\Delta_{\text{SOC}}| < |\Delta_P|$, the two valley-projected spin Chern numbers $C_{s,K}$ and $C_{s,K'}$ have opposite signs, and summing them gives a zero value for C_s ; in this case, the system is in the QVH phase. By tuning the relative size of $|\Delta_{\text{SOC}}|$ and $|\Delta_P|$, we can thus induce a topological transition between the QSH and QVH phases (which are thus also shown to be distinct topological phases). This behaviour, which connects the two emerging fields of spintronics and valleytronics, would be extremely challenging to demonstrate in condensed-matter systems, due to the need to accurately control the relative strengths of SOC and inversion-symmetry.

2.2.2 Structure design

In this subsection, I show the structure design of the spin-valley-controlled PTI. Our construction is based on previous designs of QSH and QVH PTIs [139, 140]. It is well-known the first term of the Hamiltonian in Eq. (16) can be realized with a photonic crystal shown in Fig. 2.7(a), where a PEC cylinder is suspended between two metal plates and forms a triangle lattice [139, 140]. For such a lattice, there are DPs at the corners of the first BZ for both TE modes and TM modes. By tuning the parameters of the lattice, one can make the DPs for two polarizations (TE and TM)

Figure 2.7. Structure design of spin-valley-controlled photonic topological insulator. (a) The band diagram and the schematic of a unit cell without inversion-symmetry breaking and SOC. (b), Schematic of a unit cell with SOC but without inversion-symmetry breaking. A bandgap emerges in the corresponding band diagram. (c), Schematic of a unit cell with inversion-symmetry breaking but without SOC. The corresponding band diagram contains a bandgap. (d), Schematic of a unit cell with both inversion-symmetry breaking and SOC, together with the corresponding band diagram. In all figures, the lattice constant is 36.8 mm. In (a), the height and radius of the cylinder are 36.12 mm and 6.35 mm, and the air gap is 1.84 mm. In b, the cylinder touches the bottom plate but has a gap of 5.52 mm to the top plate. In (c), the cylinder deforms into a tripod. There are two gaps of 1.1 mm between the tripod and both the upper/lower plates. In (d), the tripod touches the bottom plate and there is a gap of 2.2 mm between the tripod and the top plate.

degenerate to realize a double DP.

Next, I introduce two perturbations to mimic the two perturbation terms in Eq. (16). First, I introduce asymmetry in the z direction shown in Fig. 2.7(b), which plays the role of photonic

SOC. This perturbation can open the bandgap at both valleys, and couple TE and TM modes to construct the photonic pseudo-spins ($TE \pm TM$). The simulated band diagram in the presence of SOC is also plotted in Fig. 2.7(b). It has been shown that such a perturbation will lead to a perturbation term $\Delta_{\text{SOC}} \tau_z s_z \sigma_z$ [139], which is the second term in Eq. (16). Besides, the mass term Δ_{SOC} is proportional to the gap between the cylinder and the upper plate.

Then, I introduce anisotropy to the PEC cylinder shown in Fig. 2.7(c). This inversion-symmetry-breaking anisotropy will lead to the perturbation term: $\Delta_P \tau_0 s_0 \sigma_z$ and open a bandgap for both TE mode and TM mode [140], as demonstrated in the previous section. Here Δ_P is the effective mass induced by inversion-symmetry-breaking of the tripod geometry and is dependent on the orientation of tripods. The simulated band structures are shown in Fig. 2.7(c).

To introduce both perturbations into one lattice, let us consider a unit cell shown in Fig. 2.7(d), where the small gap separating the tripod from the upper plate introduces SOC, and the tripod geometry causes inversion symmetry breaking. The simulated band diagrams around $K(K')$ valley show that the two spin bands are no longer degenerate, which is consistent with the analysis on the effective Hamiltonian.

2.2.3 General physics of spin-valley-controlled photonic topological insulator

With the design at hand, I now show the general physics of the spin-valley-controlled PTI. The photonic crystal unit cell in our design is shown in Fig. 2.8(a). It consists of metallic tripods, arranged in triangular lattice within a parallel-plate waveguide. Each tripod touches the bottom plate of the waveguide but leaves a gap g from the top plate. This breaks the out-of-plane mirror symmetry and hence introduces SOC, which is the fundamental difference between the current design and previously-demonstrated VPCs [49–52]. The SOC strength can be tuned by altering the gap size g . The strength of inversion-symmetry breaking can be adjusted by rotating the

Figure 2.8. Properties of spin-valley-controlled photonic topological insulator. (a) Schematic of the unit cell. The parameters for the tripod are $h = 34.6$ mm, $r = 3.68$ mm and $l = 7.95$ mm. The lattice constant is $a = 36.8$ mm. (b) Phase diagram. The dashed white line indicates where the bandgap vanishes and the color denotes the bandgap size. (c) Band diagram with $g = 2.2$ mm and $\theta = 30^\circ$ (labelled as point "C" in (b)). Red and blue arrows indicate the spin up and spin down states, respectively. (d) Phase difference between E_z and H_z fields at K point for the bands labelled "1" to "4" in (c).

orientation angle θ of the tripod. Numerical simulations show that varying θ and g produces the phase diagram shown in Fig. 2.8(b). There are distinct QSH and QVH phases, separated by a sharp boundary where the bandgap vanishes, which corresponds to the topological QSH-QVH transition. The demonstration of this transition will be shown later.

Figure 2.9. Spin-valley-locked bulk flow. (a) Simulated $|E_z|$ field at 6.35 GHz, for $(g, \theta) = (2.2 \text{ mm}, 30^\circ)$. The white star on the middle top of the lattice indicates the dipole source. The inset shows the photo of the lattice in measurement. (b) Measured $|E_z|$ field distribution along the while solid line in (a). (c) Simulated phase difference between $|E_z|$ and $|H_z|$ fields at 6.35 GHz. (d) Measured phase difference between $|E_z|$ and $|H_z|$ fields at A and B points in (c).

Besides the band topology, spin-valley-locking of the bulk dispersion can also be observed in the current lattice. For example, in Fig. 2.8(b), the point $(\theta, g) = (30^\circ, 2.2 \text{ mm})$, denoted "C", lies in the QVH phase. Its spin-valley-locked band structure is plotted in Fig. 2.8(c). Two spin states can be identified through the phase difference between the E_z and H_z fields, and the simulated phase differences at K for the four bands (labeled "1" to "4" in Fig. 2.8(c)) are shown in Fig. 2.8(d).

2.2.4 Spin-valley-locked bulk bands

I now demonstrate the phenomenon of valley-dependent spin splitting in the bulk bands, which was predicted theoretically in Ref. [131]. Let us consider the structure corresponding to the point C in Fig. 2.8(b) (in the QVH phase). Fig. 2.9(a) shows the simulated $|E_z|$ fields excited by a linear dipole source at 6.35 GHz; note that spin up (down) states can only be excited at the K (K') valleys as a result of the interplay between spin DOF and valley DOF (see Fig. 2.9(c)). Due to the trigonal warping distortion of the eigenfrequency contour [131, 142, 149], the launched Bloch waves split into two beams propagating along the ΓK and $\Gamma K'$ directions (see Fig. 2.9(a)). Together with the phase difference between $|E_z|$ field and $|H_z|$ field as plotted in Fig. 2.9(c), the spin-valley locked bulk flow is further confirmed in simulations. To observe this phenomenon experimentally, we constructed a lattice of 19×15 tripods, surrounded by absorbing materials. A linear dipole whose position is indicated by the white star in Fig. 2.9(a) is used to excite the beams. I experimentally scanned $|E_z|$ field along the white line in Fig. 2.9(a) and observed two split beams in a range of frequencies, as shown in Fig. 2.9(b). To confirm their spin polarization, I measured the phase difference between $|E_z|$ and $|H_z|$ at point A and B in Fig. 2.9(c), and found that the phase difference at point A (B) is approximately π (0) as shown in Fig. 2.9(d), consistent with the simulation results of Fig. 2.9(c).

2.2.5 Topological phase transition

I now demonstrate the topological phase transition between QSH and QVH phases. Through the BBC principle, changes in the bulk topological phase can be observed via interface states [150]. I first take $(\theta, g) = (5^\circ, 2.2 \text{ mm})$, denoted A in Fig. 2.8(b), which lies in the QSH phase. I build a domain wall between two PTI domains with opposite settings, as shown in the inset of Fig. 2.10(a). In the upper domain, the valley-projected spin Chern numbers are $C_{s,K} = \mp 1/2$ and $C_{s,K'} = \mp 1/2$, while in the lower domain, they have opposite values: $C_{s,K} = \pm 1/2$ and $C_{s,K'} = \pm 1/2$.

$C_{s,K'} = \pm 1/2$. Accordingly, the changes of valley-projected spin Chern numbers across the domain wall are $\Delta C_{s,K} = \mp 1$ and $\Delta C_{s,K'} = \mp 1$, implying that there will be one spin-up edge state with negative group velocity and one spin-down edge state with positive group velocity in each of the K and K' valleys. This theoretical analysis is confirmed by the simulated band diagram in Fig. 2.10(a); each spin state occupies both valleys, and the group velocity of topological edge states is determined by spin, not valley.

I then excited the domain wall with a dipole antenna at the left port (marked as a star in Fig. 2.10(b)). Using a loop antenna mounted on a robotic arm to scan the spatial distribution of H_z field, I captured two out-coupled beams inside the empty waveguide region as shown in Fig. 2.10(b). Applying a phase-matching analysis [51], we can determine that the two out-coupled beams correspond to the two different valleys. Then I measured the phase difference between E_z and H_z along the domain wall, whose values are approximately π as shown in Fig. 2.10(c). This shows that the right-moving domain-wall edge states are spin-down polarized. Exchanging the excitation and detection positions, the phase difference between E_z and H_z now becomes approximately 0. In other words, the switch of propagating direction from right-moving to left-moving also flips the spin polarization of the edge state.

I then take $(\theta, g) = (30^\circ, 1.1 \text{ mm})$, denoted by point B in Fig. 2.8(b), which lies in the QVH phase. The domain wall is shown in the inset of Fig. 2.10(d). In this case, the valley-projected spin Chern numbers in the upper domain are $C_{s,K} = 1/2$ and $C_{s,K'} = -1/2$, and in the lower domain, $C_{s,K} = -1/2$ and $C_{s,K'} = 1/2$. Since now the changes of valley-projected spin Chern numbers across the domain wall are $\Delta C_{s,K} = 1$ and $\Delta C_{s,K'} = -1$, the K (K') valley should host spin states both having positive (negative) group velocities; in other words, the group velocity of topological edge states is determined by valley, not spin.

In contrast to the QSH phase, the captured out-coupling TE waveguide mode from a QVH

Figure 2.10. Topological phase transition between QSH and QVH phases. (a) Simulated band diagram of a domain wall between two QSH-phase domains with opposite settings. The inset is a photo of the domain wall. The positive (negative) sign of g means the tripods touch the bottom (top) plate. The positive (negative) sign of θ denotes the right leg of each tripod is below (above) the x axis. The red and blue lines represent spin up and spin down for the edge states, respectively. (b) Outcoupling of the edge states in the QSH phase through a zigzag termination at 6.08 GHz. The parameters and configurations of the domain wall are the same as in (a). The star denotes the location of a dipole source. (c) Measured phase difference between E_z and H_z fields at the end of the domain wall. (d)-(f) Similar to (a)-(c) but for the QVH phase.

domain wall only exhibits a single beam, as shown in Fig. 2.10(e). This means the right-moving edge states are locked to a single valley. Moreover, the measured phase difference between E_z and H_z now varies between 0 and π as shown in Fig. 2.10(f); this phase difference also varies along the domain wall. This is because two nondegenerate spins are excited for the edge states. Therefore, the phase difference between E_z and H_z is not locked to any specific value.

2.2.6 Summary and outlook

These results demonstrate a new class of PTIs in which spin and valley DOFs can be adjusted, in order to realize competing QSH and QVH topological phases. This allows spintronics and valleytronics to be carried out in a single photonic platform, which can in the future be used to study phenomena such as spin-valley-coupled Klein tunnelling [143]. By tuning lattice parameters, one can construct reconfigurable topological domains, allowing for new functionalities such as programmable topological Moiré patterns [151]. Finally, similar to silicene [152], the introduction of TRS breaking (which can be achieved by using gyromagnetic materials) can lead to an even wider variety of topological phases.

The idea together with the platform introduced in this work is useful in realizing reconfigurable wave manipulation. This functionality is especially desired in on-chip optical circuits. Compared to topological systems with only spin or valley DOF, our system possesses more topological phases such as spin-valley-locked edge states. This makes our systems more reconfigurable and can have more potential applications. For example, the different types of edge states (i.e., QSH-type, QVH-type and spin-valley-locked-type) can be used to perform various logic operations [153]. Another possibility is that, when these states (both bulk and edge states) are built in a laser, different functionalities and far field properties can be realized when different states are lasing [154, 155].

Chapter 3 Higher-order topological phases in acoustic systems

In this chapter, I present three experimental studies on higher-order topological phases on the platform of acoustic metamaterials. The three studies realize different types of HOTIs. In the first study, I realize a second-order TI on a 2D kagome lattice which hosts topological corner states. In the second study, I generalize the first study to 3D case and build an acoustic third-order TI which hosts topological corner states in a 3D lattice. While the first two works are based on Wannier-type HOTIs which carry quantized dipole polarization, in the third work, I realize a 3D acoustic HOTI with a quantized octupole moment in the bulk and demonstrate a full hierarchy of multipole moment. These studies not only provide experimental platforms to study the rich physics of various types of HOTIs but also offer possibilities for further applications in classical wave systems ranging from acoustics to photonics.

3.1 Acoustic second-order topological insulator on a kagome lattice

In this section, we present an experimental study on an acoustic second-order TI based on a kagome lattice. This section is organized as follows. First, I study the tight-binding model of the kagome lattice and show the higher-order topology. Second, I design an acoustic lattice which realizes the kagome lattice model and perform numerical simulations on the designed structure. At last, I show the experimental results on the measurements of the designed lattice. The findings presented in this section is published as Ref. [95].

3.1.1 Tight-binding model of kagome lattice

The kagome lattice model is one of the first models for HOTIs [111]. As shown in Fig. 3.1(a), kagome lattice is a triangle lattice with three sublattices in each unit cell. There is a s -orbital on each sublattice and only nearest-neighbor (NN) couplings are allowed. Intra-cell and inter-cell couplings are denoted by t_1 and t_2 , respectively. The lattice vectors are $\mathbf{a}_1 = (a, 0)$ and $\mathbf{a}_2 = (\frac{a}{2}, \frac{\sqrt{3}a}{2})$ where a is the lattice constant. The Bloch Hamiltonian is:

$$H(\mathbf{k}) = \begin{pmatrix} 0 & t_1 + t_2 e^{-ik_x} & t_1 + t_2 e^{-i(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2})} \\ t_1 + t_2 e^{ik_x} & 0 & t_1 + t_2 e^{i(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2})} \\ t_1 + t_2 e^{i(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2})} & t_1 + t_2 e^{-i(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2})} & 0 \end{pmatrix}, \quad (17)$$

where I have chosen the gauge so that there is no phase for t_1 terms and set $a = 1$ without loss of generality.

The eigenenergies and Bloch states are then given by the eigen-problem $H\psi = E\psi$. When there is no dimerization (i.e., $t_1 = t_2$), the dispersion is gapless. The lower two bands touch at the BZ corners, and the upper two bands touch at the BZ center (see Fig. 3.1(b)). When dimerization is present, the degeneracy of the lower two bands is lifted, and a bandgap opens. (see Fig. 3.1(c) for the dispersion when $t_1 = t_2/2$).

Next, I analyze the topology of this system when dimerization is present. Conventionally, to see the first-order topology, we can calculate a Wannier Hamiltonian through a Wilson loop operator. The Wannier Hamiltonian is adiabatically connected to the physical Hamiltonian of the edge. Thus we can determine whether there is a gapless edge state by looking at the eigenvalues of the Wannier Hamiltonian. This approach is more practical than calculating topological invariants using Berry connection since it does not require a smooth gauge (only fixing the gauge for the first and last elements in the Wilson loop is needed). Here a Wilson loop operator along y direction is

Figure 3.1. Higher-order topological insulator on the kagome lattice. (a) Schematic of the kagome lattice. Red and blue bonds indicate couplings with strength t_1 and t_2 , respectively. (b) Bulk bands when $t_1 = t_2$. (c) Bulk bands when $t_1 = t_2/2$. (d) Wannier centers for the lowest band when $t_1 = t_2$. (e) Eigenenergies of a triangular-shaped finite size kagome lattice when $t_1 = t_2/5$. Three corner states (marked in red) are found at zero energy. (f) Sum of propabilities of the three corner states.

defined as:

$$\mathcal{W}_y(\mathbf{k}) = M_y(\mathbf{k} + N_y \delta k_y) \cdots M_y(\mathbf{k} + \delta k_y) M_y(\mathbf{k}), \quad (18)$$

where $M_y(\mathbf{k})$ is a matrix whose elements are $[M_y(\mathbf{k})]^{mn} = \langle u_m(\mathbf{k} + \delta k_y) | u_n(\mathbf{k}) \rangle$. Here $|u(\mathbf{k})\rangle$ is the periodic part of Bloch wavefunction, and m and n run over the occupied bands. $\delta k_y = 2\pi/N_y$ is the numerical spacing the sampling k points. The Wannier Hamiltonian is then expressed by:

$$\mathcal{H}_{\mathcal{W}_y}(\mathbf{k}) = e^{i\mathcal{W}_y(\mathbf{k})}. \quad (19)$$

The Wannier Hamiltonian $\mathcal{H}_{\mathcal{W}_y}(\mathbf{k})$ has eigenvalues $v_y(k_y)$ which are proportional to Wannier centers (WCs). Fig. 3.1(d) shows numerically calculated WCs for the lowest band. In contrast to Chern insulators or Z_2 insulators, the WCs do not wind around the whole unit cell, which indicates there is no nontrivial first-order topology. However, the values of v_y are nonzeros, which indicates there is a mismatch between the WCs and atomic sites. Moreover, this mismatch is symmetry-protected. In the present case, the mirror and C_3 symmetries fix the WCs to be at two possible positions: the center of the triangle formed by three red bonds and the center of the triangle formed by three blue bonds. From now on, I refer to these two cases as trivial phase and topological phase, respectively. In both cases, there is a mismatch between the WCs and atomic sites. Thus, if we consider a finite sample, there will be boundary states if this goes through the WCs. To be more concrete, let us consider a finite lattice of triangular shape. The eigenvalues of this sample at $t_1 = t_2/5$ are plotted in Fig. 3.1(e), where we can see three zero energy states (red dots in Fig. 3.1(e)). They are corner states induced by the mismatch between the WCs and atomic sites (Fig. 3.1(f) for their profile). The mismatch also induces some edge states. The question on whether a corner or an edge will have boundary states will be addressed in more detail later. I would like to note here that the energies of corner states are not pinned to zero since there is no chiral symmetry. A perturbation may shift their energies away from zero. However, the existence

of the corner states is guaranteed by the mirror and C_3 symmetries which quantize the bulk polarization.

3.1.2 Design of an acoustic kagome lattice

Classical wave systems are ideal platforms to realize various tight-binding model. The physics of coupled resonance modes is similar to the problem described by a tight-binding model. In acoustic, coupled acoustic resonator systems have been used to simulated various topological tight-binding models [85, 88, 156–159]. Here I construct an acoustic version of the kagome lattice to realize a second-order TI with topological corner states.

To start with, I first consider a single resonator of cylindrical shape with height $H = 41$ mm and radius $r = 20$ mm. The structure is hollow and is surrounded by hard walls. The media inside the resonator is nothing but air. Such a structure can support various resonance modes at different

Figure 3.2. Acoustic kagome lattice of coupled resonators. (a) Dipole mode of a cylindrical acoustic resonator at 4185.4 Hz. The resonator is a hollow hard-wall cylinder filled with air. (b) Unit cell of the acoustic kagome lattice which consists of three coupled resonator. The height H and the radius r are 41 mm and 20 mm, respectively. The intra-cell coupling waveguides and inter-cell coupling waveguides have radii of r_{c1} and r_{c2} , respectively. The lattice constant is $a = 108$ mm. (c) Bulk bands for the acoustic kagome lattice shown in (b) with $r_{c1} = r_{c2} = 5.2$ mm.

frequencies. Here I take a dipole mode at 4185.4 Hz as shown in Fig. 3.2(a), whose profile is isotropic in the xy plane. This isotropic distribution is important for engineering isotropic couplings in the tight-binding model for the kagome lattice. Next, I couple the resonators to form an acoustic kagome lattice. To do so, I arrange the resonators at the positions that correspond to the atomic sites of the kagome lattice and then connect them with thin cylindrical couplings waveguides. The designed structure is depicted in Fig. 3.2(b), with the lattice constant being $a = 108$ mm. The strength of intra-cell and inter-cell couplings can be easily tuned by changing the radii of the corresponding coupling waveguides (i.e., r_{c1} and r_{c2}). I calculate the bulk bands to check the validity of our design. The result for $r_{c1} = r_{c2} = 5.2$ mm is displayed in Fig. 3.2(c), which is a typical kagome lattice dispersion: the first two bands are dispersive and touch at the corners of BZ, and the third band is flat and touches the second band at the BZ center.

3.1.3 Numerical studies on the acoustic kagome lattice

In order to open a bandgap, the intra-cell couplings and inter-cell couplings must be different. I set the radii of coupling waveguides to be $r_{c1} = 2.08$ mm and $r_{c2} = 8.32$ mm. This corresponds to the topological phase, where the WCs are located at the centers of downwards-pointing triangles (see Fig. 3.3(a) for the positions of WCs). Now I consider a finite sample of triangular shape as indicated by red dashed lines in Fig. 3.3(a). When we cut this finite sample from an infinite lattice, the cutting will go through WCs, leaving boundary states at the corners and edges. To verify this prediction, I calculate the eigenstates of real acoustic structure with the eigensolver in COMSOL Multiphysics. As shown in Fig. 3.3(b), I indeed find three in-gap corner states (see Fig. 3.3(c) for the profile of one of the corner states). In addition, there are also edge states localized on the three edges (Fig. 3.3(d)). The rest are bulk states (Fig. 3.3(e)-(f)). These results match well with the prediction from the tight-binding model.

Figure 3.3. Numerical simulations on the acoustic kagome lattice. (a) Illustration of the Wannier centers in the lattice. Finite lattice with different shapes indicated by dashed lines can be cut from an infinite lattice. (b) Eigen frequencies of a triangular-shaped finite lattice. Grey, blue and red dot represent bulk, edge and corner states. (c) Profile for one of the corner states. (d) Profile for one of the edge states. (e) Profile for one of the lower bulk states. (f) Profile for one of the upper bulk states.

3.1.4 Experimental studies on the acoustic kagome lattice

Let us now proceed to the experimental investigation on the acoustic kagome lattice. The finite lattice structure is fabricated by drilling hole in aluminum plates, as shown in Fig. 3.4(a). The source and probe are inserted into the sample through two small holes at the bottom and top (see the inset to Fig. 3.4(a)). The bulk states and edge states can be detected by measuring

transmissions in the bulk and at the edge, respectively. In the experiment, I measure the bulk transmission by placing the source and probe at positions denoted by "1" and "2" in Fig. 3.4(a). The result is plotted as the black line in Fig. 3.4(b). Two transmission peaks at around 3970 Hz and 4420 Hz are observed, which correspond to the upper and lower bulk states in Fig. 3.4(b).

Figure 3.4. Realization of the acoustic kagome lattice. (a) Photo of the fabricated sample for a triangular-shaped kagome lattice. (b) Upper panel: measured bulk (grey line) and edge (blue line) transmission. Lower panel: measured corner spectra for clean case (red line) and disordered case (green line). (c) Measured acoustic pressure at 4203 Hz. (d) Measured acoustic pressure at 4065 Hz. (e) Measured acoustic pressure at 3961 Hz. (f) Measured acoustic pressure at 4397 Hz.

Edge transmission is measured in a similar way. In the measurement of edge transmission, I place the source and speaker at positions denoted by "3" and "4" in Fig. 3.4(a). The result is plotted as the blue line in Fig. 3.4(b). Again we observe two peaks. The peak at around 4070 Hz corresponds to the edge states, which agrees with the simulation results shown in The result is plotted as the black line in Fig. 3.4(b). While the peak at around 4400Hz is in fact the result of bulk states since the upper bulk states extend to the edges (see Fig. 3.4(f) for eigen profile for upper bulk states).

Next, let us move our attention to the corner states. The corner states manifest themselves as corner localized resonances. To detect the corner states, I place the source and the probe both at the lower-left corner resonator. The measured spectrum is shown as the red line in the lower panel of Fig. 3.4(b). As can be seen, a single resonance peak at the frequency corresponding to the corner states is observed. Since the corner state is highly localized at the corner and is a consequence of nontrivial bulk topology, it should have some sense of robustness. To see this, I randomly perturbed the resonance frequencies of all resonators except for the one at the lower-left corner. In the experiment, this is done by putting metal rods of random heights into the resonators. Although this perturbation breaks the mirror symmetries locally everywhere, the mirror symmetries are preserved on average. Thus, I still obtain a spectrum (see the green line in the lower panel of Fig. 3.4(b)) which is almost the same as the one for the unperturbed lattice. This demonstrates the topological stability of the corner states. In fact, as long as the symmetries are preserved on average and the bulk gap is not closed, the existence of the corner states will be guaranteed as can be seen from the mismatch between the WCs and atomic sites, although the resonance frequency could be shifted.

Besides the evidence on the frequency spectra, I also probe the local density of states as another proof for the existence of the corner states. To do so, I measure the acoustic pressure at a site by exciting and detecting at this site. This procedure is repeated for all the sites of the

lattice. In Fig. 3.4(c)-(f) we plot the measured acoustic pressure at different frequencies. At the frequency corresponding to the corner states, the acoustic energy is concentrated at the three corners (Fig. 3.4(c)). Similarly, at frequencies corresponding to edge states and bulk states, the measured acoustic pressure is large at the edges (Fig. 3.4(d)) and in the bulk (Fig. 3.4(e)-(f)). Together with frequency spectra, these results are all consistent with numerical simulations in Fig. 3.4(b).

As mentioned above, the existence of the corner states (also edge states) is a result of the nontrivial bulk polarization and can be predicted by inspecting the WC distribution. To further prove this, I construct a second sample with a parallelogram shape which has four different corners labeled as A , B_1 , B_2 and C as shown in Fig. 3.5(a). we can see from the green dashed line in Fig. 3.3(a) that only corner A corresponds to separating WCs and will host corner state. Similarly, only the bottom and right edges will host edge states. These predictions will be tested numerically and experimentally. In Fig. 3.5(b), I plot numerically calculated eigenfrequencies of a finite-sized lattice with a parallelogram shape. There is only one corner state found, which is located at corner A as predicted (see inset to Fig. 3.5(b) for the profile of the corner state). In addition, the edge states (whose profiles are not shown) are located on the bottom and right edges. To verify these findings experimentally, I again conduct local acoustic measurements over all sites to probe the local density of states. The results at frequencies corresponding to corner and edge states are shown in the upper and lower panel of Fig. 3.5(c), respectively. As can be seen, the sound is concentrated on corner A at the corner state frequency and distributed over the bottom and right edge at a frequency corresponding to the edge states. I also measure the frequency spectra at the four corners. As shown in Fig. 3.5(d), only the spectrum at corner A (red curve in Fig. 3.5(d)) has one single peak inside the bandgap. The spectra at other three corners have two peaks which correspond to bulk and edge states. One interesting thing is that when intra-cell and inter-cell couplings are

Figure 3.5. Corner state on a finite lattice of parallelogram geometry. (a) Photograph of the fabricated parallelogram-shaped structure. (b) Numerically computed eigenfrequencies of the structure. There is a single non-degenerate corner state, localized at corner A. (c) Top (bottom) panel: measured acoustic pressure distributions at 4170 (4066) Hz. (d) Measured spectra at the four corners of the sample.

exchanged, corner C instead of corner A will host corner state. However, the obtuse-angle corners (corner B_1 and corner B_2) will not host corner state in both situations. This shape dependence property of corner state can be used as a tool for reconfigurable wave trapping.

3.1.5 Summary and outlook

In summary, we have demonstrated for the first time a HOTI with a quantized bulk polarization. Based on the kagome lattice model, I design an acoustic lattice with coupled resonators. I show both numerically and experimentally the existence of corner states and demonstrate how to understand the underlying physics with WCs. This work, together with a similar work that appears at the same time [96], has inspired a number of studies on Wannier-type HOTIs in both acoustic and photonic systems.

I believe the topological corner states can be found useful in applications such as sensing and lasing. For the sensing application, the advantage of using a topological corner state comes from the unique properties of corner states. Being highly localized at the corner site, the corner state is only sensitive to perturbations at the corners, yielding a high sensitivity. Moreover, since the corner state is topologically protected, imperfections of the sensor that are not near the corner will not affect the performance. In the future, it is an important task to investigate the usage of topological corner states in applications related to wave localization.

3.2 Acoustic third-order topological insulator on a diamond lattice

The order of a TI is defined by the difference between the dimension of the system and the dimension of the topological states. By this definition, a third-order TI with topological corner states is the highest order TI that can be realized (without using synthetic dimensions). In this section, I introduce the first realization of a third-order TI based on an acoustic platform. The organization of this section is as follows. First, I review the tight-binding model for an anisotropic diamond lattice which realizes a Wannier-type third-order TI [160]. Second, I design an acoustic lattice to realize this tight-binding model and perform numerical studies. Next, I conduct experiments to characterize the designed structure. At last, I close this section with a short summary. The findings presented in this section are published in Ref. [115].

3.2.1 Tight-binding model for an anisotropic diamond lattice

The isotropic diamond lattice model (s orbital) with only NN coupling is a simple two-band model with nodal lines [161]. When anisotropy is turned on (coupling along one direction can be different from other three directions), the nodal lines evolve and after a critical value the system becomes gapped. This band structure evolution has been known [162] before the concept of HOTI is proposed. After the recognition of higher-order topology, anisotropic diamond lattice is soon identified as a HOTI of Wannier type [160]. Here I briefly review the tight-binding model for anisotropic diamond lattice since our design is fully based on this tight-binding model.

Fig. 3.6(a) shows the schematic of the diamond lattice. Each unit cell contains two sites, with one s orbital per site. Three lattice vectors are $\mathbf{a}_1 = (a/2, a/2, 0)$, $\mathbf{a}_2 = (0, a/2, a/2)$ and $\mathbf{a}_3 = (a/2, 0, a/2)$, where $a/\sqrt{2}$ is the lattice constant. Only NN couplings are allowed and the couplings are anisotropic. Couplings along [111] direction (colored in red in Fig. 3.6(a)) have strength t_2 while those along other directions (colored in blue in Fig. 3.6(a)) have strength t_1 . In

Figure 3.6. Anisotropic diamond lattice. (a) Schematic of the anisotropic diamond lattice. Blue and red bonds denote couplings with strength t_1 and t_2 , respectively. (b) The first Brillouin zone of the anisotropic diamond lattice. (c) Bulk dispersion along high symmetry paths for the anisotropic diamond lattice when $t_1 = t_2$ (red curves) and $t_1 = 0.1t_2$ (blue curves). (d) Illustration of a finite like with a rhombohedron shape. (e) Eigen energies for a rhombohedron-shaped finite lattice as a function of t_1/t_2 . Two-fold degenerate zero energy modes emerge when the system is gapped. (f) Sum of propabilities of the two zero energy modes when $t_1 = 0.1t_2$, identifying they are corner states.

the following calculations, I have taken $a = 4$ and $t_2 = 1$ without loss of generality.

The k space Hamiltonian of this lattice can be written as:

$$H(\mathbf{k}) = \begin{pmatrix} 0 & t_1(e^{i\mathbf{k}\mathbf{d}_2} + e^{i\mathbf{k}\mathbf{d}_3} + e^{i\mathbf{k}\mathbf{d}_4}) + t_2e^{i\mathbf{k}\mathbf{d}_1} \\ t_1(e^{-i\mathbf{k}\mathbf{d}_2} + e^{-i\mathbf{k}\mathbf{d}_3} + e^{-i\mathbf{k}\mathbf{d}_4}) + t_2e^{-i\mathbf{k}\mathbf{d}_1} & 0 \end{pmatrix}, \quad (20)$$

where $\mathbf{d}_1 = (1, 1, 1)$, $\mathbf{d}_2 = (1, -1, -1)$, $\mathbf{d}_3 = (-1, 1, -1)$ and $\mathbf{d}_4 = (-1, -1, 1)$. The bulk

dispersion along high symmetric path (see Fig. 3.6(b) for the corresponding BZ) for $t_1 = t_2$ is plotted as red curve in Fig. 3.6(c), which is gapless with nodal lines. As t_1 decreases, the nodal lines involve and the system becomes gapped when $|t_1/t_2| < 1/3$ (see the blue curve in Fig. 3.6(c) for dispersion at $t_1 = 0.1t_2$). Now I analyse the bulk topology in the gapped phase. The bulk Hamiltonian has mirror symmetries:

$$M_\alpha^{-1}H(k_\alpha)M_\alpha = H(-k_\alpha), \quad (21)$$

where $M_\alpha = i\sigma_\alpha$ with σ_α being Pauli matrices and $\alpha = x, y, z$. The bulk Hamiltonian also has chiral symmetry:

$$\sigma_z^{-1}H(\mathbf{k})\sigma_z = -H(\mathbf{k}). \quad (22)$$

Both symmetries quantize the bulk polarization. To determine whether the bulk polarization is nontrivial, I numerically calculate it using the formula:

$$P_\alpha = -\frac{1}{V} \int_{BZ} A_\alpha d^3k, \quad (23)$$

where $A_\alpha = -i\langle u | \partial k_\alpha | u \rangle$ is the Berry connection for the lower band and V is the volume of first BZ. The bulk polarization takes a quantized value of $(1/2, 1/2, 1/2)$ when the system is gapped. This nontrivial polarization will give rise to topological corner states.

To see the consequence of the nontrivial bulk polarization, let us consider a finite lattice of rhombohedron shape (Fig. 3.6(d)), whose corners correspond to positions where WCs get separated (see details in the next section). I calculate the eigenvalues as function of the ratio t_1/t_2 . Two zero-energy states are found in the region where the system is gapped (Fig. 3.6(e)). They are the corner states induced by the nontrivial bulk polarization (see Fig. 3.6(f) for the profile of the corner states at $t_1 = 0.1t_2$)

3.2.2 Design and numerical studies of an acoustic anisotropic diamond lattice

I now proceed to implement above tight-binding model of the anisotropic diamond lattice with coupled acoustic resonators. The design process is similar to how I construct the acoustic kagome lattice as described in the previous section. Here I also use the dipole mode of cylindrical acoustic resonators with size $r = 20$ mm and $H = 60$ mm as "orbitals" and couple them through thin waveguides. Designed unit cell is plotted in Fig. 3.7(a), with two resonators placed at positions corresponding to the atomic sites of the anisotropic diamond lattice model (i.e., at $(0, 0, 0)$ and $(a/4, a/4, a/4)$ with $a = 175$ mm). The anisotropy of the couplings is enabled by tuning the radii of coupling waveguides. Coupling waveguides with radius r_{c1} (r_{c2}) correspond to the blue (red) bonds in the tight-binding model.

When all couplings waveguides are identical, the computed bandstructure is gapless as expected (see Fig. 3.7(b) for the bandstructure when $r_{c1} = r_{c2} = 4.9$ mm). To open a bandgap, we need to introduce anisotropy in the couplings. To do so, I tune $r_{c1} = 2$ mm and $r_{c2} = 7.8$ mm, which drives the system into the gapped phase. As shown by the blue curve in Fig. 3.7(c), now a bandgap can be clearly seen. I also fit dispersion obtained from full-wave simulation (COMSOL) with the tight-binding dispersion calculated from Eq. (20). The fitting tight-binding dispersion is plotted in Fig. 3.7(c) as the dashed black curve, which agrees well with the simulation results. The obtained ratio of t_1/t_2 is 0.076, which also indicates the system is in the gapped phase.

As revealed by the tight-binding calculations, the gapped phase of the anisotropic diamond lattice is indeed a higher-order topological phase. To see this in our acoustic lattice, I perform full-wave simulation for a finite rhombohedron-shaped acoustic lattice. The calculated eigenfrequencies are plotted in Fig. 3.7(d), where apart from the gapped bulk states (black dots), two in-gap states are found (red dots). These two states are the corner states (see Fig. 3.7(e)-(f) for their profiles) induced by the quantized bulk polarization. I note that, similar to the kagome lattice case, here the existence of the corner states can be well-predicted by looking at the position of

Figure 3.7. Acoustic anisotropic diamond lattice. (a) unit cell of designed acoustic anisotropic diamond lattice, which consists of two cylindrical resonators coupling by thin waveguides. The structural parameter values are given in the main text. (b) Bulk bandstructure for the acoustic anisotropic diamond lattice with $r_{c1} = r_{c2} = 4.9$ mm obtained from COMSOL simulation. (c) Bulk bandstructures for the acoustic anisotropic diamond lattice with $r_{c1} = 2$ mm and $r_{c2} = 7.8$ mm obtained from COMSOL simulation (solid blue line) and tight-binding calculation (dashed black line). (d) Calculated eigenfrequencies for a rhombohedron-shaped finite acoustic lattice. Black and red dots denote bulk and corner states, respectively. (e) and (f) Eigenmode profile of the two corner states in (d). The acoustic energy is highly concentrated at the corners.

WCs. Here the WC is at the center of the coupling waveguides with radius r_{c2} . As can be seen from Fig. 3.7(e)-(f), only the two corner resonators are cut from coupling waveguides with radius r_{c2} while all other resonators are connected to coupling waveguides with radius r_{c2} . Thus, there will be boundary states at these two corners.

3.2.3 Experimental studies on the acoustic anisotropic diamond lattice

Now let us proceed to the experimental demonstration of the phenomena predicted above. A rhombohedron-shaped lattice is fabricated with stereolithography 3D printing. A photo of the sample is shown in Fig. 3.8(a). The resonators and coupling waveguides are hollow structures filled with air (as in previous simulations) and surrounded by 6-mm-thick printing materials acting as hard walls for sound. There are two small holes on the two sides of each resonator, allowing for excitation and detection. They are blocked with plugs when not in use (see Fig. 3.8(a)). All other structures that appear in Fig. 3.8(a) but not in the simulations are frames to support the whole sample.

I first probe the existence of the bandgap by placing the speaker (as source) and microphone (as detector) at two separate resonators inside the bulk of the sample. Fig. 3.8(b) shows the measured bulk transmission spectrum. There are two peaks at around 2800 Hz and 3000 Hz, corresponding to the two branches of bulk states in Fig. 3.8(d). In between these two peaks, the transmission is low, confirming the presence of the bulk bandgap. Then I measure the response of the two corner resonators (labeled "A" and "B" Fig. 3.8(d)) by placing the speaker and microphone at the same corner resonator. In contrast to the bulk spectrum, the measured corner spectra only have one single peak at around 2900 Hz (see Fig. 3.8(c)), which is inside the bulk gap and matches with the eigenfrequency of the corner states. To confirm this peak corresponds to the corner states, I repeat above experiment for all resonators and obtain the acoustic response for each resonator at 2900 Hz. The result, which is proportional to local density of states, is plotted in Fig. 3.8(d). As can be seen, the measured acoustic pressure is much higher at the two corners than in other regions, identifying the existence of the corner states in this 3D sample.

Figure 3.8. Realization of the acoustic anisotropic diamond lattice. (a) A photo of fabricated sample. (b) Measured bulk transmission spectrum. (c) Measured spectra by placing the source and detector at corner A (red curve) and corner B (blue curve). (d) Measured acoustic pressure for all sites at 2900 Hz.

3.2.4 Summary and outlook

In conclusion, I have experimentally demonstrated a third-order TI on an acoustic anisotropic diamond lattice. This work can be seen as an extension of the previous work on kagome lattice. The demonstration reveals the generality of Wannier-type HOTIs and the important role that WC

plays in characterizing HOTIs. Besides, this work also proves the power of coupled acoustic resonator system in realizing tight-binding models and demonstrating topological physics. The third-order topological corner states observed in this work, together with other related works that appear during the same period [163, 164], provide useful tools for robust sound trapping in 3D structures, which may be used in future applications.

3.3 Acoustic octupole topological insulator

The Berry phase [165] associated with energy bands in crystals can lead to quantized observables, such as the quantized electric dipole polarizations [104–106] in 1D TIs [166]. Recent theories [107, 108] have generalized the concept of quantized dipoles to higher multipole moments, resulting in the discovery of multipole TIs, which exhibit a hierarchy of multipole topology: a quantized octupole moment in a 3D bulk induces quantized quadrupole moments on its 2D surfaces, which in turn induce quantized dipole moments on the 1D hinges. 2D quadrupole TIs, which exhibit 0D in-gap corner states, have previously been realized in various systems including classical photonic and acoustic structures [29, 93, 94, 113, 167–169]. In this section, I report on the realization of a quantized octupole TI in a 3D acoustic metamaterial. We directly observe 0D topological corner states, 1D gapped hinge states, 2D gapped surface states, and 3D gapped bulk states, representing the topological hierarchy of quantized octupole, quadrupole and dipole moments. The conditions for forming a nontrivial octupole moment are demonstrated by comparison with two different lattice configurations having trivial octupole moments. This work thus establishes the multipole topology and its full hierarchy in 3D geometries. The findings presented in this section are published in Ref. [116]

3.3.1 Design of an acoustic octupole topological insulator

In this subsection, I present the design process of an acoustic octupole TI, which realizes the tight-binding model for an octupole TI proposed by BBH in Refs. [107, 108]. The basic physics of an octupole TI is illustrated in Fig. 3.9(a), where a bulk octupole moment induces surface quadrupole moments, edge dipole moments and corner charges. The lattice model is shown schematically in Fig. 3.9(b). The intra-cell couplings (left panel) and inter-cell couplings (right panel) have hoppings of γ and λ , while solid and dashed lines denote positive and negative

Figure 3.9. Octupole topological insulator and its acoustic metamaterial realization. (a) Schematic of a bulk octupole moment, which induces surface quadrupole moments, edge dipole moments and corner charges. (b) Tight-binding model of a quantized octupole TI. The left (right) panel illustrates the intra-cell (inter-cell) couplings. Here solid and dashed lines representing positive and negative couplings, respectively. (c) Acoustic metamaterial realization of the model in (b). Here orange and blue colors denote positive and negative couplings, respectively. (d) Implementation of negative coupling. The left panel shows the dipole mode of a single resonator. The right panel shows the eigenmodes of two coupled-resonator systems with coupling waveguides located at different sides of the resonance's nodal line, corresponding to opposite coupling signs.

couplings respectively. The unit cell can be regarded as two coupled unit cells of quadrupole TIs with opposite signs of couplings. Due to the negative couplings, there is a π flux on each facet, which not only opens a spectral gap but also maintains the mirror symmetries (up to a gauge transformation). In the presence of mirror and inversion symmetries, the bulk octupole moment O_{xyz} takes quantized values of 0 or 1/2, depending on the relative strength of γ and λ .

I again use coupled acoustic resonators to implement this model. Each lattice site is a hard-walled cuboid resonator filled with air, of size 80 mm $\times$ 40 mm $\times$ 10 mm; these dimensions are chosen so that each resonator supports a dipole resonance mode (Fig. 1d, left panel) at $f_0 = 2162.5$ Hz. NN couplings are realized by connecting resonators with thin waveguides. The extra design challenge here compared to the previous two works (i.e., kagome and diamond lattice) is the implementation of negative coupling. When two resonators are coupled with a positive coupling coefficient γ , the resulting eigenmodes exhibit symmetric and antisymmetric phase relations with split eigenfrequencies $f_0 + \gamma$ and $f_0 - \gamma$. If the sign of the coupling is reversed, the eigenmodes are exchanged [170]. To achieve this, I simply relocate the connecting waveguide to the other side of the dipole mode's nodal line; the two eigenmodes then switch eigenfrequencies (Fig. 3.9(d), right panel). The amplitude of the coupling is controlled by tuning the width of connecting waveguide. By applying this mechanism, I arrive at the acoustic octupole TI structure depicted in Fig. 3.9(c), where the connecting waveguides corresponding to positive (negative) couplings are marked in orange (blue). Here the sign of a coupling in the lattice can be determined by looking at the configuration of associated resonators and connecting waveguide. There are only two possible in-plane configurations as shown in Fig. 3.9(d) where the connecting waveguides are either located at the upper part (configuration A) or lower part (configuration B). Throughout this paper, I assume positive (negative) coupling is implemented by configuration A (B). Note that one can also assume positive (negative) coupling is implemented by configuration B (A), which corresponds to a gauge

transformation and thus does not alter the topology of the system.

With the preliminary design in hand (Fig. 3.9(c)), I proceed to optimize the system. There are several factors to consider. First, the lattice is anisotropic (for instance, the coupling strengths along z are not equal to those in the xy plane). Second, the coupling waveguides introduce resonance frequency shifts, which may be different at different sites. Third, the coupling waveguides do not only couple the dipole modes on different sites, but also other modes. Our simulation studies indicate that the first issue does not threaten to spoil the quantization of the octupole moment; the other two might if their effects are strong enough. I note these issues generally happen when mapping coupled resonator model to idea tight-binding model. However, in the previous

Figure 3.10. Design of the acoustic octupole topological insulator. (a) Bulk dispersions of the designed acoustic lattice along high symmetry lines calculated from the effective tight-binding model (blue lines) and full-wave simulations (red dots). (b) and (c) Illustrations of the effective tight-binding model extracted from simulations. (b) and (c) show the in-plane configurations for the bottom and top layers, respectively. The dots denote the sites, and the width and color of the lines correspond to the strength and sign of the couplings, respectively. The NN couplings along z have the same strength as those in the xy plane, and the next-nearest couplings along z are negligible.

two studies, the real structures exactly preserve the protecting symmetries. Thus these effects will not alter the quantization of the bulk invariant. In contrast, here due to the negative couplings, the real structure (Fig. 3.9(c)) apparently does not have exact mirror symmetries. So it is crucial to investigate how accurately our acoustic lattice can be mapped to the desired tight-binding model.

To translate between the metamaterial and an equivalent tight-binding model, I use the Schrieffer-Wolff transformation method [93, 171–173]. First, I solve the lowest five eigenmodes of a single resonator numerically (using COMSOL Multiphysics). Then I solve for the lowest 320 modes of a small lattice containing eight unit cells (64 sites, i.e., five modes per site, excluding modes introduced by coupling waveguides). I discretize the simulation data to produce matrices U_j , whose i th column contains the i th eigenmode at site j . This is projected onto the basis of single resonator eigenmodes by taking $P_j = (A^T A)^{-1} A^T U_j$, where the columns of matrix A contain the eigenmodes of a single resonator. Hence I obtain the coupling matrix $V = P D P^{-1} - H_0$, where $P = (P_1, P_2, \dots, P_{64})^T$, D is a diagonal matrix containing the eigenfrequencies of the 8-unit-cell sample, and H_0 is a diagonal matrix containing the eigenfrequencies of the single resonator repeated 64 times. To restrict the effective description of the mode of interest, I perform the Schrieffer-Wolff transformation perturbatively [171, 173]. With the effective tight-binding model in hand, I tune the widths of the coupling waveguides and the distances between the resonators so that the NN couplings along z are almost the same as the couplings in the xy plane, the resonance frequency shifts and long ranges couplings are small, and the frequency gap is large. Moreover, small holes are introduced to the resonators at the boundary sites, in order to shift their resonance frequencies to be the same as those in the bulk. In the final design, the lattice constants along x , y and z directions are 200 mm, 200 mm and 100 mm, respectively, and the widths for intra-cell (inter-cell) coupling waveguides are 1.6 mm (4 mm) in the xy plane and 1.34 mm (3.08 mm) along z direction. The final design has a ratio of $\gamma/\lambda \approx 0.18$ with $\gamma = 9.8 \times (1 \pm 4\%)$ Hz and

$\lambda = 53.3 \times (1 \pm 4\%)$ Hz, and the resonance frequencies of the eight resonators in the unit cell are $f_0 = 2145.1 \times (1 \pm 0.05\%)$ Hz. The dispersion relation calculated from the effective tight-binding model matches full-wave simulation results (Fig. 3.10(a)). Due to the large separation between the target mode and other modes, the next-nearest couplings in the extracted tight-binding model are quite small (Fig. 3.10(b)-(c)). Thus, the designed acoustic lattice deviates almost negligibly from the ideal tight-binding model.

3.3.2 Numerical studies on the acoustic octupole topological insulator

To demonstrate the physics of the octupole TI, three samples are constructed. Nested Wilson loops are used to reveal the topological hierarchy [107, 108]. First, define a Wilson loop along z direction:

$$\mathcal{W}_z(\mathbf{k}) = M_z(\mathbf{k} + N_z \delta k_z) \cdots M_z(\mathbf{k} + \delta k_z) M_z(\mathbf{k}), \quad (24)$$

where $M_z(\mathbf{k})$ is a matrix whose elements are $[M_z(\mathbf{k})]^{mn} = \langle u_m(\mathbf{k} + \delta k_z) | u_n(\mathbf{k}) \rangle$. Here $|u(\mathbf{k})\rangle$ is the periodic part of Bloch wavefunction, and m and n run over the occupied bands. The Wilson loop operator is a 4×4 matrix, whose eigenstates are given by:

$$\mathcal{W}_z(\mathbf{k}) |v_z(\mathbf{k})\rangle = e^{i2\pi v_z(k_x, k_y)} |v_z(\mathbf{k})\rangle. \quad (25)$$

Here the phase of the eigenvalue, v_z , only depends on the transverse momenta k_x and k_y . However, the eigenstate depends on all three momenta. The spectrum of v_z is gapped for the present model, indicating a gapped surface. I denote the two sectors separated by the Wannier gap as $v_z^\pm$. Each of these sectors carries a quadrupole topology. To see this, let us construct the Wannier basis within the v_z^+ sector:

$$|w_z^{+,j}(\mathbf{k})\rangle = \sum_{n=1}^{N_{\text{occ}}} |u_n(\mathbf{k})\rangle [v_z^{+,j}]^n \quad (26)$$

Then we can define a nested Wilson loop operator along y as:

$$\mathcal{W}_y(\mathbf{k}) = M_y(\mathbf{k} + N_y \delta k_y) \cdots M_y(\mathbf{k} + \delta k_y) M_y(\mathbf{k}), \quad (27)$$

where $M_y(\mathbf{k})$ is a matrix whose elements are $[M_y(\mathbf{k})]^{mn} = \langle w_z^{+,m}(\mathbf{k} + \delta k_y) | w_z^{+,n}(\mathbf{k}) \rangle$. Then we again solve the eigen-equation:

$$\mathcal{W}_y(\mathbf{k}) |\kappa_y(\mathbf{k})\rangle = e^{i2\pi v_y(k_x)} |\kappa_y(\mathbf{k})\rangle, \quad (28)$$

which yeilds a gapped 1D Wannier band with two sectors $\kappa_y^\pm$. To examine the topology of these sectors, we again define the following Wannier basis:

$$|w_y^+(\mathbf{k})\rangle = \sum_{n=1}^{N_{\text{occ}}} |u_n(\mathbf{k})\rangle [\kappa_y^+]^n. \quad (29)$$

Then the Wannier sector polarization is given by

$$p_x^+ = -\frac{i}{2\pi} \frac{1}{N_z N_y} \sum_{k_z, k_y} \log[\mathcal{W}_x(\mathbf{k})], \quad (30)$$

where $\mathcal{W}_x(\mathbf{k})$ is the third Wilson loop:

$$\mathcal{W}_x(\mathbf{k}) = M_x(\mathbf{k} + N_x \delta k_x) \cdots M_x(\mathbf{k} + \delta k_x) M_x(\mathbf{k}), \quad (31)$$

with $[M_x(\mathbf{k})]^{mn} = \langle w_y^{+,m}(\mathbf{k} + \delta k_x) | w_y^{+,n}(\mathbf{k}) \rangle$.

I calculate the Wannier sector polarizations and eigenmodes for finite lattices with parameters obtained from simulations. For the first sample, which has intra-cell couplings smaller than inter-cell couplings ($\gamma = 9.8\text{Hz} < \lambda = 53.3\text{Hz}$), the topological indices are close to $\{p_x^\pm, p_y^\pm, p_z^\pm\} = \{1/2, 1/2, 1/2\}$, indicating that the hinges have quantized dipole moments, which are induced by quantized surface quadrupole moments, which in turn arise from a quantized bulk octupole moment. Calculations for a finite $5 \times 5 \times 5$ lattice (which contains 1000 sites) show the physical consequences of the bulk topological properties, in the form of gapped bulk states, gapped surface states, gapped hinge states, and in-gap corner states (Fig. 3.11(a),(d)).

Figure 3.11. Calculations on finite lattices in topological and trivial phases. (a)-(c) Calculated eigenvalues of $5 \times 5 \times 5$ lattices with different coupling configurations. (d)-(f) Sum of probabilities for different types of states in (a)-(c), respectively. In (a), there are four types of states (bulk, surface, hinge, and corner), which are plotted in (d). In (b), all the states are bulk states (e). In (c), there are three types of states (bulk, surface and hinge), which are plotted in (f).

The second sample that we fabricated has intra-cell couplings larger than inter-cell couplings ($\gamma = 53.3\text{Hz} < \lambda = 9.8\text{Hz}$). In this case, $\{p_x^\pm, p_y^\pm, p_z^\pm\} = \{0, 0, 0\}$, meaning that there are only bulk states and no topologically guaranteed corner states, which is consistent with numerical results (Fig. 3.11(b),(e)).

The third sample is more subtle: the intra-cell couplings are smaller than the inter-cell cou-

plings along the x and y directions ($\gamma_{x,y} = 9.8\text{Hz} < \lambda_{x,y} = 53.3\text{Hz}$), but the reverse is true in the z direction ($\gamma_z = 53.3\text{Hz} > \lambda_z = 9.8\text{Hz}$). In this case, the topological indices are $\{p_x^\pm, p_y^\pm, p_z^\pm\} = \{1/2, 1/2, 0\}$. Here, the nontrivial values of $p_x^\pm$ and $p_y^\pm$ suggest each Wannier sector $v_z^\pm$ has a quadrupole topology, which induces at the boundaries of the structure surface states on surfaces normal to x and y directions and hinge states on hinges along z (Fig. 3.11(c),(f)). However, the trivial value of $p_z^\pm$ indicates the induced hinge states are trivial, and thus there is no quantized dipole moment along z direction to induce corner states.

3.3.3 Experimental studies on the acoustic octupole topological insulator

I now probe the above signatures of the octupole TI experimentally. I measure the acoustic response to local excitations at each site of the first sample (Fig. 3.12(a)), which consists of 1000 sites. The left panel of Fig. 3b shows the measurement results at an arbitrarily-chosen frequency of 2000 Hz. I divide the sample into four non-overlapping regions (Fig. 3.12(b), right panel): the “corner” region consists of the 8 corner sites; the “hinge” region consists of the 12 hinges, each containing 8 sites; the “surface” region consists of the 6 surfaces, each containing 64 sites; and the remaining 512 sites constitute the “bulk” region. The measured data are recorded and processed in the following way. In the measurements of site-resolved local response, the source and probe are always located at the same site. At each site i , I obtain the intensity of acoustic field normalized by the intensity of the source over a range of frequencies (1900 Hz – 2400 Hz), denoted as $P_i(w)$. This procedure is repeated over all the 1000 sites. To eliminate the influence of varying excitation efficiencies on different sites, I normalize the data by the sum of the intensities over all frequencies to get normalized spectra: $N_i(w) = P_i(w) / \sum_w P_i(w)$. The average intensity spectra for four different regions (bulk, surface, hinge and corner) are obtained by calculating average normalized intensity within each area respectively: $A_\alpha(w) = \sum_{i \in \alpha} N_i(w) / N_\alpha$, where α

Figure 3.12. Experimental demonstration of an acoustic octupole topological insulator. (a) Photo of the fabricated sample. (b) Left panel: measured sound intensity of all sites at an arbitrarily chosen frequency (2000 Hz). Right panel: illustration of the four regions (corner, hinge, surface and bulk) used to calculate the average intensity spectra. (c) Measured average intensity spectra for a finite lattice in the topological phase. (d)-(g) Spatial distributions of the acoustic response intensity, integrated over different sets of frequencies corresponding to the corner (d), hinge (e), surface (f) and bulk (g) spectra; the frequency ranges are indicated by the corresponding colored regions in (c).

denotes calculated region (i.e., bulk, surface, hinge or corner), the summation is taken over the sites within the calculated region, and N_α is the number of sites within the calculated region.

I plot the average intensity spectra for the four regions (Fig. 3.12(b)). The peak frequencies for these spectra are found to agree well with the respective eigenfrequency ranges of the

Figure 3.13. Demonstration of two trivial phases. (a) Measured average intensity spectra for the sample with $\gamma > \lambda$. (b), Measured average intensity spectra for the sample with $\gamma_{x,y} < \lambda_{x,y}$ and $\gamma_z > \lambda_z$. Spectra with peaks located at similar frequencies are plotted in the same figure. Shaded regions denote the frequency regions for intensity integration. (c)-(e), Spatial distributions of the acoustic response intensity, integrated over the spectral peaks in (b). These results reveal that the peaks in (b) correspond to hinge, surface and bulk states, respectively.

theoretically obtained corner, hinge, surface, and bulk eigenstates. Next, I plot the spatial distributions of the acoustic response integrated over frequency ranges corresponding to each type of spectral peak (the four different integration regions are indicated in Fig. 3.12(c)). The resulting distributions are indeed concentrated at the corners, hinges, surfaces, and bulk of the lattice, re-

spectively (Fig. 3.12(d)-(g)). The corner states are clearly observable (Fig. 3.12(d)); they exist at mid-gap frequencies distinct from the other eigenfrequencies, which are protected by chiral symmetry that approximately holds in the real structure due to the careful design process. The hinge, surface and bulk states are closer in frequency, leading to overlaps in the integrated intensity spectra (Fig. 3.12(e)-(g)). Note only the corner states are topologically protected while the hinge and surface states are not.

Similar measurements were conducted on the second and third samples. For the second sample, the intensity spectra for the four different regions have identical peaks (Fig. 3.13(a)), consistent with the absence of boundary states. For the third sample, I account for the anisotropy of the lattice by treating the hinges and surfaces along different directions separately. The measured spectra in the corner region (Fig. 3.13(b), upper panel) has peaks overlapping the spectra for the z hinge region (Fig. 3.13(c)), consistent with the absence of corner states. The x and y hinge regions (Fig. 3.13(b), middle panel) have spectral peaks coinciding with those of the x and y surface regions (Fig. 3.13(d)). (For brevity, the spectra for the y hinge and y surface are not plotted.) The bulk states are observed to spread into the z surface region (Fig. 3.13(b), lower panel, and Fig. 3.13(e)). These results match the predictions in Fig. 3.11, and show that the second and third samples cannot fulfill the construction of three-level hierarchy of multipole topology.

3.3.4 Summary and outlook

In conclusion, an acoustic octupole TI has been designed and demonstrated through direct local acoustic measurements. The topological in-gap corner states, gapped hinge states, gapped surface states, and gapped bulk states are all clearly distinguishable in the experimental results on the 3D lattice with quantized octupole moment. Contrasting experiments on different lattice configurations revealed a case where there are no boundary states at all, and a case where there

are hinge and surface states but no corner states, in agreement with the topological indices. These results, together with other parallel efforts [174], pave the way toward studying phenomena such as topological multipole moment pumping [107, 108] and the effects of non-Hermiticity [175–177], nonlinearity [178] and disorder [179] on multipole TIs, as well as the possibility of implementing reconfigurable multipole TIs with tunable corner, hinge, or surface states. Some other important yet not answered questions are: what is the physical consequence of the fractional charge associated with the topological corner states in the classical context and how this fractional charge can be useful in classical wave systems.

Chapter 4 non-Hermitian Dirac cones

In the studies presented in previous chapters, Hermiticity is always assumed in the theoretical considerations. However, in reality almost all classical wave systems are non-Hermitian since background losses are inevitable. Moreover, non-Hermiticity can not only come from background losses, but also can be induced by additional gain/absorption and nonreciprocal couplings. In the latter cases, non-Hermiticity can interact nontrivially with band topology. In a Hermitian system, the eigenvalues are always real and the associated eigenvectors are orthogonal. In contrast, the eigenvalue reality and eigenvector orthogonality are not guaranteed anymore in a non-Hermitian system. This contrast is also reflected by different types of degeneracies in the bandstructures of Hermitian and non-Hermitian systems. In the Hermitian case, the degeneracies are diabolic points like DPs and Weyl points. They are basic elements in the topological band theory in the sense that they can be the transition point between topologically distinct gapped phases and can carry topological charge which gives rise to robust boundary states [15]. Moreover, the effective description near a diabolic point matches the equations for fundamental particles. In crystalline materials, this gives rise to elementary quasiparticles like Dirac and Weyl fermions. In non-Hermitian systems, degeneracies are generally found to be EPs where both eigenvalues and eigenvectors coalesce. EPs also have many unusual physical effects with applications ranging from waveguide mode conversion to optical sensing [127]. While the existence of EPs is apparently forbidden in Hermitian systems by eigenvector orthogonality, there are no fundamental constraints preventing the existence of diabolic points in non-Hermitian systems. However, despite the rich properties associated with diabolic points, little attention has been paid to their physics in non-Hermitian systems. In fact, previous works have only demonstrated that diabolic points are generally unstable upon non-Hermitian perturbations and evolve into exceptional degeneracies [100, 128, 129].

In this chapter, I introduce a framework named "non-Hermitian Dirac cone" to obtain diabolic points in non-Hermitian systems. I show two non-Hermitian symmetries, which can not only give rise to real eigenvalues but also pairwise orthogonal eigenvectors. I derive a general matrix form that satisfies proposed symmetries, from which we construct several lattice models. First, I propose a 2D non-Hermitian lattice model with DPs, whose positions in momentum space can be tuned by non-Hermiticity. This unique feature allows us to realize pseudo-magnetic fields via gain and loss engineering. Second, I propose a 2D non-Hermitian lattice model with massive Dirac cones, whose mass term contains a parameter related to gain and loss. Thus, I demonstrate a non-Hermiticity-driven topological phase transition between a Chern insulator and a trivial one. Third, I construct a 3D non-Hermitian lattice model with Weyl points. Our framework, together with the three striking examples, paves the way to studying the physics of diabolic points in non-Hermitian systems. The findings presented in this chapter are published in Ref. [130].

4.1 General framework

In this section, I introduce a set of non-Hermitian symmetries whose combination allows for the existence of diabolic points in non-Hermitian systems. From these symmetries, I derive the general matrix which satisfies the symmetries. The general form will guide our search for lattice models in the next section.

4.1.1 non-Hermitian symmetries

I define the following 4×4 matrices which will be used throughout this section:

$$\Sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \quad (32)$$

with $i = 0, 1, 2, 3$. Here σ_0 is a 2×2 identity matrix and $\sigma_{1,2,3}$ are Pauli matrices. Consider a general 4×4 matrix H which satisfy two symmetries: a pseudo-Hermiticity condition and a anti-PT symmetry. The pseudo-Hermiticity is given by:

$$\Sigma_0 H \Sigma_0 = H^\dagger. \quad (33)$$

The anti-PT symmetry is given by:

$$\{H, \Sigma_3 \Sigma_1 T\} = 0. \quad (34)$$

The pseudo-Hermiticity condition is responsible for the real spectrum of a non-Hermitian matrix [180–182]. Let E_n be the eigenvalues of H , with associated eigenvectors:

$$H|\psi_n^R\rangle = E_n|\psi_n^R\rangle \quad (35)$$

$$H^\dagger|\psi_n^L\rangle = E_n^*|\psi_n^L\rangle \quad (36)$$

From Eq. (36) and Eq. (33) we have:

$$\begin{aligned}
H^\dagger |\psi_n^L\rangle &= E_n^* |\psi_n^L\rangle \\
\Rightarrow \Sigma_0 H^\dagger |\psi_n^L\rangle &= E_n^* \Sigma_0 |\psi_n^L\rangle \\
\Rightarrow H(\Sigma_0 |\psi_n^L\rangle) &= E_n^* (\Sigma_0 |\psi_n^L\rangle)
\end{aligned} \tag{37}$$

Thus, $\Sigma_0 |\psi_n^L\rangle$ is an eigenvector of H with eigenvalue E_n^* . So the implication of Eq. (33) is the eigenvalues of H are either real or come in complex conjugate pairs.

Let us now turn to the anti-PT symmetry given in Eq. (34). Let $|\psi\rangle$ be an eigenvector of H with eigenvalue E :

$$H|\psi\rangle = E|\psi\rangle \tag{38}$$

Using Eq. (34), we have

$$\begin{aligned}
\Sigma_1 \Sigma_3 T H |\psi\rangle &= \Sigma_1 \Sigma_3 T E |\psi\rangle \\
\Rightarrow -\Sigma_3 \Sigma_1 T H |\psi\rangle &= E^* \Sigma_1 \Sigma_3 T |\psi\rangle \\
\Rightarrow H \Sigma_3 \Sigma_1 T |\psi\rangle &= E^* \Sigma_1 \Sigma_3 T |\psi\rangle \\
\Rightarrow H(\Sigma_1 \Sigma_3 T |\psi\rangle) &= -E^* (\Sigma_1 \Sigma_3 T |\psi\rangle)
\end{aligned} \tag{39}$$

Therefore, $|\psi'\rangle = \Sigma_1 \Sigma_3 T |\psi\rangle$ is also an eigenvector of H with eigenvalue $-E^*$. To see that $|\psi\rangle$ and $|\psi'\rangle$ are orthogonal, we further let

$$|\psi\rangle = \begin{pmatrix} |\varphi_+\rangle \\ |\varphi_-\rangle \end{pmatrix}, \tag{40}$$

where $|\varphi_{+,-}\rangle$ are two-component vectors. Then $|\psi'\rangle$ has the following form:

$$|\psi'\rangle = \Sigma_1 \Sigma_3 T |\psi\rangle = -i \begin{pmatrix} \sigma_2 T |\varphi_+\rangle \\ \sigma_2 T |\varphi_-\rangle \end{pmatrix}. \tag{41}$$

Then the overlap between the two eigenvectors are:

$$\langle \psi | \psi' \rangle = -i(\langle \varphi_+ | \sigma_2 T | \varphi_+ \rangle + \langle \varphi_- | \sigma_2 T | \varphi_- \rangle). \tag{42}$$

This must vanish since for any two-component vector $\varphi = (a, b)^T$, we have:

$$\langle \varphi | \sigma_2 T | \varphi \rangle = ia^*b^* - ib^*a^* = 0. \quad (43)$$

Hence, the implication of the anti-PT symmetry given by Eq. (34) is that the eigenvalues come in pairs of E and $-E^*$ and the eigenvectors of each pair are orthogonal.

With the presence of both symmetries, there are only two possible configurations of the eigenvalues. The first possible configuration is that all eigenvalues are complex (the symmetry-broken phase), forming the set $\{E, E^*, -E, -E^*\}$ with E being a nonreal number. The second possibility is that the eigenvalues are all real (the symmetry-unbroken phase), forming the sets $\{E_1, -E_1\}$ and $\{E_2, -E_2\}$ with $E_{1,2}$ being real numbers. In the latter case, the eigenvectors within each set are orthogonal (although eigenvectors from different sets are generally not orthogonal). Thus, the symmetry-unbroken phase provides the necessary conditions (eigenvalue reality and eigenvector orthogonality) for realizing diabolic points.

4.1.2 General form of the non-Hermitian matrix

In this section, I derive the general form of any 4×4 matrix that satisfies the pseudo-Hermiticity symmetry (Eq. (33)) and anti-PT symmetry (Eq. (34)), which can be used to guide the construction of lattice models.

Define a general 4×4 matrix as:

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (44)$$

where A, B, C and D are 2×2 matrices. Using the pseudo-Hermiticity condition (Eq. (33)), we

have:

$$\begin{aligned}
\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} &= \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix}. \\
\Rightarrow \begin{pmatrix} C & D \\ A & B \end{pmatrix} \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} &= \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix}. \\
\Rightarrow \begin{pmatrix} D & C \\ B & A \end{pmatrix} &= \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix}.
\end{aligned} \tag{45}$$

Thus, in order to satisfy Eq. (33), the block-diagonal elements A and D are complex conjugates of each other, and the elements B and C must be Hermitian. Then the general form can be written as:

$$H = \begin{pmatrix} \mathcal{W} & \mathcal{V}_+ \\ \mathcal{V}_- & \mathcal{W}^\dagger \end{pmatrix}, \quad \mathcal{V}_\pm = \mathcal{V}_\pm^\dagger. \tag{46}$$

Now we turn to the anti-PT symmetry. To satisfy Eq. (34), H needs to further satisfy:

$$\begin{aligned}
\Sigma_3 \Sigma_1 T H T^{-1} \Sigma_1 \Sigma_3 &= -H. \\
\Rightarrow \Sigma_3 \Sigma_1 H^* \Sigma_3 \Sigma_1 &= -H.
\end{aligned} \tag{47}$$

Taking the form of H in Eq. (54) into consideration, we have:

$$\begin{aligned}
\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \mathcal{W}^* & \mathcal{V}_+^* \\ \mathcal{V}_-^* & \mathcal{W}^T \end{pmatrix} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} &= - \begin{pmatrix} \mathcal{W} & \mathcal{V}_+ \\ \mathcal{V}_- & \mathcal{W}^\dagger \end{pmatrix}. \\
\Rightarrow \begin{pmatrix} \sigma_2 \mathcal{W}^* \sigma_2 & \sigma_2 \mathcal{V}_+^* \sigma_2 \\ \sigma_2 \mathcal{V}_-^* \sigma_2 & \sigma_2 \mathcal{W}^T \sigma_2 \end{pmatrix} &= - \begin{pmatrix} \mathcal{W} & \mathcal{V}_+ \\ \mathcal{V}_- & \mathcal{W}^\dagger \end{pmatrix}.
\end{aligned} \tag{48}$$

Let

$$\mathcal{W} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{49}$$

Then from $\sigma_2 \mathcal{W}^* \sigma_2 = -\mathcal{W}$ we have:

$$\begin{aligned} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} &= - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \Rightarrow \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} &= - \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \quad (50)$$

From $\sigma_2 \mathcal{W}^T \sigma_2 = -\mathcal{W}^\dagger$ we have:

$$\begin{aligned} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} &= - \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \\ \Rightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &= - \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \end{aligned} \quad (51)$$

From Eq. (50) and Eq. (51), we see that $\mathcal{W}$ has the form: $\begin{pmatrix} a & b \\ b^* & -a^* \end{pmatrix}$. Let (Note $\mathcal{V}_\pm$ is Hermitian required by the pseudo-Hermiticity condition)

$$\mathcal{V}_\pm = \begin{pmatrix} a_\pm & c_\pm \\ c_\pm^* & b_\pm \end{pmatrix} \quad (52)$$

From $\sigma_2 \mathcal{V}_\pm^* \sigma_2 = -\mathcal{V}_\pm$, we have:

$$\begin{pmatrix} b_\pm & -c_\pm \\ -c_\pm^* & a_\pm \end{pmatrix} = \begin{pmatrix} -a_\pm & -c_\pm \\ -c_\pm^* & -b_\pm \end{pmatrix} \quad (53)$$

Thus, $\mathcal{V}_\pm$ has the form: $\begin{pmatrix} \lambda_\pm & c_\pm \\ c_\pm^* & -\lambda_\pm \end{pmatrix}$ with $\lambda_\pm$ being real numerbs.

In summary, a matrix that satisfies Eq. (33) and Eq. (34) has the form

$$H = \begin{pmatrix} \mathcal{W} & \mathcal{V}_+ \\ \mathcal{V}_- & \mathcal{W}^\dagger \end{pmatrix}, \quad \mathcal{V}_\pm = \mathcal{V}_\pm^\dagger, \quad (54)$$

$$\mathcal{W} = \begin{pmatrix} a & b \\ b^* & -a^* \end{pmatrix}, \quad \mathcal{V}_\pm = \begin{pmatrix} \lambda_\pm & c_\pm \\ c_\pm^* & -\lambda_\pm \end{pmatrix}, \quad (55)$$

where $a, b, c_\pm \in \mathbb{C}$ and $\lambda_\pm \in \mathbb{R}$.

4.2 Non-Hermitian honeycomb lattice with Dirac points

In this section, I propose a 2D non-Hermitian honeycomb lattice with on-site gain and loss which hosts a pair of DPs.

4.2.1 Lattice design

The proposed lattice is illustrated in Fig. 4.1(a), where red (blue) sites have an on-site mass $i\gamma$ ($-i\gamma$). This gain and loss distribution is different from conventional PT symmetric honeycomb lattice where two sublattices experience gain and loss, respectively [183]. Here the gain and loss pattern enlarges the unit cell and makes the Hamiltonian a 4×4 matrix. This is important for the Hamiltonian to satisfy the general matrix form given in Eq. (54) and Eq. (55). The lattice vectors are $\mathbf{a}_1 = (a, 0)$ and $\mathbf{a}_2 = (0, \sqrt{3}a)$. The NN coupling is denoted by t . The corresponding bulk Hamiltonian reads

$$H_{\text{Dirac}} = \begin{pmatrix} i\gamma & 2 \cos(\frac{k_x}{2}) e^{i \frac{k_y}{2\sqrt{3}}} & 0 & e^{-i \frac{k_y}{\sqrt{3}}} \\ 2 \cos(\frac{k_x}{2}) e^{-i \frac{k_y}{2\sqrt{3}}} & i\gamma & e^{i \frac{k_y}{\sqrt{3}}} & 0 \\ 0 & e^{-i \frac{k_y}{\sqrt{3}}} & -i\gamma & 2 \cos(\frac{k_x}{2}) e^{i \frac{k_y}{2\sqrt{3}}} \\ e^{i \frac{k_y}{\sqrt{3}}} & 0 & 2 \cos(\frac{k_x}{2}) e^{-i \frac{k_y}{2\sqrt{3}}} & -i\gamma \end{pmatrix}, \quad (56)$$

where I have set $a = 1$ and $t = 1$ without loss of generality. It is straightforward to see that at any momentum Eq. (56) satisfy Eqs. (54)-(55) with

$$\begin{aligned} a &= i\gamma, \\ b &= 2 \cos(\frac{k_x}{2}) e^{i \frac{k_y}{2\sqrt{3}}}, \\ \lambda_{\pm} &= 0, \\ c_{\pm} &= e^{-i \frac{k_y}{\sqrt{3}}}. \end{aligned} \quad (57)$$

4.2.2 Bulk dispersion

Now let us proceed to investigate the bandstructure of this lattice. In the Hermitian limit ($\gamma = 0$), our lattice reduces to the well-known graphene model with two DPs (see the black curve in Fig. 4.1(b)). We are interested in the fate of these DPs when non-Hermiticity is turned on. Before I show the numerical results, let us first do some simple analysis. Since the two bands forming the DPs are forced to be orthogonal by the anti-PT symmetry, the DPs cannot be split

Figure 4.1. Non-Hermitian honeycomb lattice with Dirac points. (a) Schematic of the non-Hermitian honeycomb lattice, where red (blue) sites have an on-site mass $i\gamma$ ($-i\gamma$). (b) Real parts of the bulk dispersions of the lattice shown in (a) along momentum k_x (at $k_y = 0$) for $\gamma = 0$ (black curves), $\gamma = 0.6$ (blue curves) and $\gamma = 0.9$ (red curves). The imaginary parts are zero. (c) and (d) Real (c) and imaginary (d) part of the bulk dispersion over the whole Brillouin zone for $\gamma = 0.6$. Two Dirac points are present.

into EPs as in many previous studies. Thus, it is possible for the non-Hermiticity to enter the effective Dirac Hamiltonian as a Hermitian term. There are two possibilities we can think of. The non-Hermiticity term γ can act as either a gauge field that moves the DPs or a mass term that gaps out the DPs. As we will find out, here the gain and loss move the Dirac cone. In the next section, we will see that non-Hermiticity can also open a bandgap and induce a massive Dirac cone.

The bulk dispersions along k_x (at $k_y = 0$) for $\gamma = 0.6$ and $\gamma = 0.9$ are plotted in Fig. 4.1(b) as red and blue curves, respectively. As can be seen, the zero-energy degeneracies are still present. Compared to the Hermitian case (black curve), the position of the degeneracies along k_x changes as γ varies. To see these degeneracies are indeed DPs, I also plot in Fig. 4.1(c)-(d) the full band-structure at $\gamma = 0.6$. We can clearly see two point degeneracies inside the symmetry-unbroken region. They are associated with real eigenvalues and orthogonal eigenvectors, indicating they are DPs rather than EPs.

To see how the positions of the DPs changes with γ , I also work out the bulk dispersion analytically, which reads

$$\begin{aligned} E_1 &= \pm \sqrt{3 - \gamma^2 + 2 \cos(k_x) - 2 \sqrt{2 \cos^2(k_x/2) (\cos(\sqrt{3}k_y) + 1 - 2\gamma^2)}}, \\ E_2 &= \pm \sqrt{3 - \gamma^2 + 2 \cos(k_1) + 2 \sqrt{2 \cos^2(k_x/2) (\cos(\sqrt{3}k_y) + 1 - 2\gamma^2)}}. \end{aligned} \quad (58)$$

Thus, for $|\gamma| < 1$, the energies are real inside the region

$$|k_y| < \frac{\cos^{-1}(2\gamma^2 - 1)}{\sqrt{3}}, \quad (59)$$

as well as along the curves $k_x = \pm\pi$. Zero-energy band degeneracy points occur at

$$K^\pm = \begin{pmatrix} -2\tau\theta \\ 0 \end{pmatrix}, \quad \text{where} \quad \begin{cases} \cos 2\theta &= -(1 + \gamma^2)/2, \\ \tau &= \pm 1. \end{cases} \quad (60)$$

The τ variable describes which degeneracy point we are referring to, similar to the valley index in graphene. From above numerical and analytic results, I show that our proposed lattice is a 2D non-Hermitian lattice with DPs, whose positions can be tuned by non-Hermiticity.

4.3 Non-Hermitian Haldane model with massive Dirac cones

In this section, I construct a non-Hermitian Haldane model by including real on-site potential and complex NNN couplings into the 2D non-Hermitian honeycomb lattice model introduced in the previous section. Intriguingly, I find the mass term of the low-energy Hamiltonian not only contains terms proportional to the on-site mass and NNN coupling strength but also is related to gain and loss. From that, I demonstrate a topological phase transition between a trivial insulator and a Chern insulator driven by gain and loss rather than Hermitian parameters.

4.3.1 Lattice model

Consider a lattice shown in Fig. 4.2(a). Compared to the lattice given in Fig. 4.1(a), on-site masses and complex NNN couplings are further included, which make this lattice a non-Hermitian version of the Haldane model. Thick (thin) outline circles indicate a real on-site mass term m ($-m$) while red (blue) colors denote a imaginary on-site term $i\gamma$ ($-i\gamma$). The NNN couplings are indicated by the orange arrows. Couplings along (opposite to) the arrow directions are given by it_2 ($-it_2$). Note I have taken the phase of the NNN couplings to be $\pm\frac{\pi}{2}$ in order to satisfy Eqs. (54)-(55). The corresponding Hamiltonian reads

$$H_{\text{Haldane}} = \begin{pmatrix} i\gamma + m + 2t_2 \sin(k_x) & 2 \cos(\frac{k_x}{2}) e^{i\frac{k_y}{2\sqrt{3}}} & -4t_2 \sin(\frac{k_x}{2}) \cos(\frac{\sqrt{3}k_y}{2}) & e^{-i\frac{k_y}{\sqrt{3}}} \\ 2 \cos(\frac{k_x}{2}) e^{-i\frac{k_y}{2\sqrt{3}}} & i\gamma - m - 2t_2 \sin(k_x) & e^{i\frac{k_y}{\sqrt{3}}} & 4t_2 \sin(\frac{k_x}{2}) \cos(\frac{\sqrt{3}k_y}{2}) \\ -4t_2 \sin(\frac{k_x}{2}) \cos(\frac{\sqrt{3}k_y}{2}) & e^{-i\frac{k_y}{\sqrt{3}}} & -i\gamma + m + 2t_2 \sin(k_x) & 2 \cos(\frac{k_x}{2}) e^{i\frac{k_y}{2\sqrt{3}}} \\ e^{i\frac{k_y}{\sqrt{3}}} & 4t_2 \sin(\frac{k_x}{2}) \cos(\frac{\sqrt{3}k_y}{2}) & 2 \cos(\frac{k_x}{2}) e^{-i\frac{k_y}{2\sqrt{3}}} & -i\gamma - m - 2t_2 \sin(k_x) \end{pmatrix}, \quad (61)$$

It is straightforward to see that at any momentum Eq. (61) satisfy Eqs. (54)-(55) with

$$\begin{aligned}
a &= i\gamma + m, \\
b &= 2 \cos\left(\frac{k_x}{2}\right) e^{i\frac{k_y}{2\sqrt{3}}}, \\
\lambda_{\pm} &= -4t_2 \sin\left(\frac{k_x}{2}\right) \cos\left(\frac{\sqrt{3}k_y}{2}\right), \\
c_{\pm} &= e^{-i\frac{k_y}{\sqrt{3}}}.
\end{aligned} \tag{62}$$

Thus, eigenvectors of this Hamiltonian form orthogonal pairs. Then it is natural to ask the fate of the DPs under all the newly introduced terms. As I will show, very similar to the Hermitian case, the NNN couplings together with on-site masses enter the Dirac Hamiltonian as mass terms. To my surprise, the non-Hermitian term γ also enters the Dirac Hamiltonian as mass term, competing with other mass terms and leading to topological phase transition driven by gain and loss.

4.3.2 Low-energy theory

In this section, I derive the effective Hamiltonian for the states near zero energy around the Dirac cones. For $m = t_2 = 0$, we have two DPs located at $-2\tau\theta$ (see Eq.(60)). I expand around these points and linearize the Hamiltonian by taking m , t_2 and E as small quantities. For convenience, we rewrite Eq.(61) as:

$$\begin{aligned}
H &= \begin{bmatrix} \mathcal{W} & \mathcal{V} \\ \mathcal{V} & \mathcal{W}^\dagger \end{bmatrix}, \\
\mathcal{W}(k) &= \begin{bmatrix} m + i\gamma + 2t_2 \sin k_x & 2 \cos(k_x/2) e^{i\sqrt{3}k_y/6} \\ 2 \cos(k_x/2) e^{-i\sqrt{3}k_y/6} & -m + i\gamma - 2t_2 \sin k_x \end{bmatrix}, \\
\mathcal{V}(k) &= \begin{bmatrix} -4t_2 \sin(k_x/2) \cos(\sqrt{3}k_y/2) & e^{-i\sqrt{3}k_y/3} \\ e^{i\sqrt{3}k_y/3} & 4t_2 \sin(k_x/2) \cos(\sqrt{3}k_y/2) \end{bmatrix}.
\end{aligned} \tag{63}$$

Figure 4.2 Non-Hermitian Haldane model with massive Dirac cones. (a) Schematic of the non-Hermitian lattice. (b) Phase diagram. There are two gapped phases as indicated by white (Chern insulator) and orange (trivial insulator) areas. These two phases are separated by the solid black curves which are numerically obtained gap closing points. The system hosts a single Dirac cone at the phase transition boundaries. The dashed blue curves represent the phase transition boundary given by the low-energy Hamiltonian. (c)-(e) Real part of the bulk bandstructure of the lattice given in (a) for $\gamma = 0.446$ (c), $\gamma = 0.846$ (d) and $\gamma = 0.946$ (e). (f)-(h) Real part of the dispersion for a slab geometry with periodic boundary condition along x and 100 sites along y , using the parameters of (c)-(e). The colors denote the imaginary parts of the eigenvalues. In all calculations, we have set $t_2 = 0.1$ and $m = 0.35$.

It is convenient to define

$$k = \begin{pmatrix} -2\tau\theta + q_1 \\ \sqrt{3}\tilde{q}_2 \end{pmatrix}, \quad (64)$$

where $\tilde{q}_2 = q_2/\sqrt{3}$. Consider an eigenstate of the form

$$|\psi\rangle = \begin{bmatrix} |\varphi_+\rangle \\ |\varphi_-\rangle \end{bmatrix}, \quad (65)$$

where $|\varphi_\pm\rangle$ are two-component vectors, with energy E . Plugging this into the Schrödinger equation gives

$$\left[\mathcal{W}_0 + \mathcal{W}_1(q)\right]|\varphi_+\rangle + \left(\sigma_1 + \sigma_2\tilde{q}_2 + 4\tau t_2 \sin\theta\sigma_3\right)|\varphi_-\rangle = E|\varphi_+\rangle \quad (66)$$

$$\left(\sigma_1 + \sigma_2\tilde{q}_2 + 4\tau t_2 \sin\theta\sigma_3\right)|\varphi_+\rangle + \left[\mathcal{W}_0^\dagger + \mathcal{W}_1(q)\right]|\varphi_-\rangle = E|\varphi_-\rangle, \quad (67)$$

where

$$\mathcal{W}_0 = i\gamma\sigma_0 + [m - 2\tau t_2 \sin(2\theta)]\sigma_3 + 2\cos\theta\sigma_1 \quad (68)$$

$$\mathcal{W}_1 = \tau \sin\theta\sigma_1 q_1 - \cos\theta\sigma_2\tilde{q}_2.$$

Multiplying Eq. (66) by $(\sigma_1 + \sigma_2\tilde{q}_2 + 4\tau t_2 \sin\theta\sigma_3)$, and assuming that $t_2 \sim O(q)$, gives

$$|\varphi_-\rangle = (\sigma_1 + \sigma_2\tilde{q}_2 + 4\tau t_2 \sin\theta\sigma_3)(E - \mathcal{W}_0 - \mathcal{W}_1)|\varphi_+\rangle + O(q^2). \quad (69)$$

Plugging this into Eq. (67) gives

$$(\sigma_1 + \sigma_2\tilde{q}_2 + 4\tau t_2 \sin\theta\sigma_3)|\varphi_+\rangle = (E - \mathcal{W}_0^\dagger - \mathcal{W}_1)(\sigma_1 + \sigma_2\tilde{q}_2 + 4\tau t_2 \sin\theta\sigma_3)(E - \mathcal{W}_0 - \mathcal{W}_1)|\varphi_+\rangle + O(q^2). \quad (70)$$

Now I further assume that $E, m \sim O(q)$, and expand the polynomials on the right-hand side to first order. (Note that $\mathcal{W}_0$ is zeroth-order in q and $\mathcal{W}_1$ is first-order.) We obtain

$$\begin{aligned} & \left[(\mathcal{W}_0^\dagger\sigma_1\mathcal{W}_0 - \sigma_1) + (\mathcal{W}_0^\dagger\sigma_1\mathcal{W}_1 + \mathcal{W}_1\sigma_1\mathcal{W}_0) \right. \\ & \left. + (\mathcal{W}_0^\dagger\sigma_2\mathcal{W}_0 - \sigma_2)\tilde{q}_2 + 4\tau t_2 \sin\theta(\mathcal{W}_0^\dagger\sigma_3\mathcal{W}_0 - \sigma_3) \right] |\varphi_+\rangle = E(\sigma_1\mathcal{W}_0 + \mathcal{W}_0^\dagger\sigma_1)|\varphi_+\rangle + O(q^2). \end{aligned} \quad (71)$$

Using Eq. (68), we have

$$\begin{aligned}
\mathcal{W}_0^\dagger \sigma_1 \mathcal{W}_0 - \sigma_1 &= 4 \cos \theta (m - 2\tau t_2 \sin 2\theta) \sigma_3^+ + O(q^2) \\
\mathcal{W}_0^\dagger \sigma_1 \mathcal{W}_1 + \mathcal{W}_1 \sigma_1 \mathcal{W}_0 &= 4 \cos \theta (\tau \sin \theta \sigma_1 q_1 - \cos \theta \sigma_2^+ \tilde{q}_2) + O(q^2) \\
\mathcal{W}_0^\dagger \sigma_2 \mathcal{W}_0 - \sigma_2 &= -8 \cos^2 \theta \sigma_2^+ + O(q) \\
\mathcal{W}_0^\dagger \sigma_3 \mathcal{W}_0 - \sigma_3 &= -8 \cos^2 \theta \sigma_3^+ + O(q) \\
\sigma_1 \mathcal{W}_0 + \mathcal{W}_0^\dagger \sigma_1 &= 4 \cos \theta \sigma_0.
\end{aligned} \tag{72}$$

where we define the γ -dependent modified Pauli matrices

$$\sigma_2^\pm = \sigma_2 \pm \frac{\gamma}{2 \cos \theta} \sigma_3 \tag{73}$$

$$\sigma_3^\pm = \sigma_3 \mp \frac{\gamma}{2 \cos \theta} \sigma_2. \tag{74}$$

Eq. (71) simplifies to

$$(\tau \alpha_1 q_1 + \alpha_2^+ q_2 + \beta_+) |\varphi_+\rangle = E |\varphi_+\rangle + O(q^2), \tag{75}$$

where we have set $q_2 = \sqrt{3} \tilde{q}_2$ and

$$\alpha_1 = \sin \theta \sigma_1 \tag{76}$$

$$\alpha_2^\pm = -\sqrt{3} \cos \theta \sigma_2^\pm \tag{77}$$

$$\beta_\pm = (m - 6\tau t_2 \sin 2\theta) \sigma_3^\pm \tag{78}$$

We can repeat the above procedure, solving for the lower two components φ_- . The resulting linearized Schrödinger equation is similar to Eq. (71), but with $\mathcal{W}_0$ and $\mathcal{W}_0^\dagger$ swapped, which is the same as swapping $\gamma \leftrightarrow -\gamma$. Hence,

$$(\tau \alpha_1 q_1 + \alpha_2^\pm q_2 + \beta_\pm) |\varphi_\pm\rangle = E |\varphi_\pm\rangle + O(q^2). \tag{79}$$

To simplify this further, define the γ -dependent unitary operator

$$U_\pm = \exp \left(\pm \frac{i\phi}{2} \sigma_1 \right) \quad \text{where} \quad \phi = \cos^{-1}(2 \cos \theta) = \sin^{-1} \gamma. \tag{80}$$

Then

$$U_{\pm} \sigma_2^{\pm} U_{\mp} = \frac{1}{2 \cos \theta} \sigma_2 \quad (81)$$

$$U_{\pm} \sigma_3^{\pm} U_{\mp} = \frac{1}{2 \cos \theta} \sigma_3. \quad (82)$$

Now Eq. (79) can be transformed as follows:

$$\frac{\sqrt{3}}{2} \left(\tau \eta_{\gamma} \sigma_1 q_1 - \sigma_2 q_2 + M(\gamma) \sigma_3 \right) U_{\pm} |\varphi_{\pm}\rangle = E U_{\pm} |\varphi_{\pm}\rangle + O(q^2) \quad (83)$$

$$\eta_{\gamma} = 2 \sin \theta / \sqrt{3} \quad (84)$$

$$M(\gamma) = \frac{m - 6\tau t_2 \sin 2\theta}{2 \cos \theta}. \quad (85)$$

As can be seen, this is a massive Dirac Hamiltonian, whose mass term is determined by m , t_2 and γ .

4.3.3 Topological phase transition driven by gain and loss

The effective Hamiltonian given in Eq. (83) tells that the system exhibits two types of gapped phases. If the term proportional to τt_2 in Eq. (83) dominates the term proportional to m , the Dirac cones have opposite mass, and if the reverse is true, the Dirac cones have the same mass. This is similar to the Chern insulator and conventional insulator phases of the Haldane model [8]. The phase transition is predicted to occur at

$$|m| = 3|t_2| \sqrt{(3 + \gamma^2)(1 - \gamma^2)}. \quad (86)$$

At the transition point, there is an unpaired Dirac cone at one of the K^{τ} points, similar to the transition described in the Haldane model[8].

The phase diagram is shown in Fig. 4.2(b). The predicted phase boundaries of Eq. (86) agree with those found by numerically searching for band degeneracies, with the discrepancies diminishing as t_2 is further reduced. The shape of the transition curve raises the interesting possibility of

driving a phase transition entirely via gain and loss. As indicated by the points labeled "A"- "C" in Fig. 4.2(a), we can fix nonzero values of m and t_2 and increase γ from zero, and thereby change the system from a non-Hermitian Chern insulator (which reduces to a Hermitian Chern insulator for $\gamma = 0$) into a non-Hermitian conventional insulator (Fig. 4.2(c)-(e)). The two gapped bulk phases are accompanied by edge state behaviors similar to Hermitian Chern and conventional insulators, as seen in Fig. 4.2(f)-(h). The non-Hermitian Chern insulator hosts edge states spanning the gap, arising from the fact that the non-Hermitian k -space Hamiltonian has effective 2D Dirac solutions to which the standard topological BBC applies. The truncation of the real-space lattice breaks the symmetries (Eqs. (33)-(34)), so that the edge state energies acquire substantial imaginary parts; these values depend on the choice of boundary termination, and in the case of Fig. 4.2(f)-(h) range from around -0.9 to 0.9. For most of the bulk states, the energies are almost exactly real.

4.4 Landau levels induced by gain and loss

The fact that the positions of DPs can be controlled by the gain/loss parameter γ (Eq. (60)) offers an interesting possibility to engineer gauge fields via gain and loss. In the Hermitian case, it is well-known that a proper strain in graphene can induce a gauge which leads to a pseudo-magnetic field and LLs [184]. The basic mechanism is that the positions of the two DPs are related to the NN coupling strength. When the couplings are tuned by strain, the DPs move in momentum space, leading to a position-dependent gauge field in the low-energy equation. Then the position-dependent nature of this gauge field guarantees the formation of an effective magnetic field acting on the states near the DPs. By the same token, we have similar physics here in our non-Hermitian lattice with the role played by strain replaced by gain and loss.

4.4.1 Analytic results

Here I provide an analytic derivation of the LLs induced by gain and loss.

For $m = t_2 = 0$, the band structure simplifies considerably. In this case, the Hamiltonian obeys the symmetries

$$\left\{ H(k), \Sigma_3 \right\} = 0, \quad (87)$$

$$\left[H(k), \Sigma_1 T \right] = 0. \quad (88)$$

These are, respectively, a particle-hole symmetry and a PT symmetry. Eqs. (87) and (88) together imply the anti-PT symmetry $\{H(k), \Sigma_3 \Sigma_1 T\} = 0$, but the reverse is not necessarily true.

The particle-hole symmetry (87) implies that if $H|\psi\rangle = E|\psi\rangle$, then $H\Sigma_3|\psi\rangle = -E\Sigma_3|\psi\rangle$. Moreover, if $|\psi\rangle$ does not spontaneously break the PT symmetry (88), then $E = E^*$ and $\Sigma_1 T|\psi\rangle = \exp(i\phi)|\psi\rangle$ for some ϕ . Now consider an eigenstate of the form (65). If PT is unbroken, and the

$\Sigma_1 T$ eigenvalue is $\exp(i\phi)$, then

$$\sigma_1 T |\varphi_+\rangle = e^{i\phi} |\varphi_-\rangle \quad \text{and} \quad \sigma_1 T |\varphi_-\rangle = e^{i\phi} |\varphi_+\rangle. \quad (89)$$

By setting the global phase DOF for $|\psi\rangle$, we can always choose $\phi = 0$, so that

$$|\varphi_-\rangle = \sigma_1 T |\varphi_+\rangle. \quad (90)$$

Thus, it is really only necessary to keep track of $|\varphi_+\rangle$.

For the ungapped Dirac cones, Eq. (83) simplifies to

$$v \left(\tau \eta_\gamma \sigma_1 q_1 - \sigma_2 q_2 \right) U_+ |\varphi_+\rangle \simeq E U_+ |\varphi_+\rangle \quad (91)$$

where $v = \sqrt{3}/2$, $U_+ = \exp[i(\phi/2)\sigma_1]$, $\phi = \sin^{-1} \gamma$, and

$$\eta_\gamma = 2 \sin \theta / \sqrt{3}. \quad (92)$$

Note that $q = k - K^\tau(\gamma)$ is the k -space displacement relative to the DP at gain/loss parameter γ .

Now consider slow spatial variations in γ . The microscopic wavefunction can be described by the ansatz

$$\psi(r) = \varphi_A(r) u_A(r) + \varphi_B(r) u_B(r), \quad (93)$$

where $u_{A/B}(r)$ are the Bloch wavefunctions at some reference DP, $K_1^\tau(\theta_0) = -2\tau\theta_0$. The slowly-varying envelope fields $\varphi_{A/B}(r)$ form a two-component field; let us define

$$\Phi(r) = U_+ \begin{bmatrix} \varphi_A(r) \\ \varphi_B(r) \end{bmatrix} = \begin{bmatrix} \Phi_A(r) \\ \Phi_B(r) \end{bmatrix}, \quad (94)$$

where U_+ may vary in space via its dependence on γ . Eq. (91) translates into the wave equation

$$v \left[\tau \eta_\gamma \sigma_1 (-i\partial_1 + A_1) + i\sigma_2 \partial_2 \right] \Phi(r) \simeq E \Phi(r), \quad (95)$$

where A_1 is a gauge field that characterizes the spatial variation of γ . There is no x_2 gauge field component since the Dirac cones only move along k_1 when γ is varied. When γ is spatially uniform, we have a plane wave solution

$$\Phi(r) = e^{i\kappa \cdot r} \Phi, \quad v \left[\tau \eta_\gamma \sigma_1 (\kappa_1 + A_1) - \sigma_2 \kappa_2 \right] \Phi \simeq E \Phi, \quad (96)$$

where $\kappa_1 = k - K_1^\tau(0)$ is the k -vector component relative to the reference DP. Comparing to Eq. (91) gives

$$A_1 = -K_1^\tau(\gamma) + K_1^\tau(0) = 2\tau(\theta - \theta_0). \quad (97)$$

Let us now suppose γ varies only along x_2 , so that there is a conserved momentum κ_1 . Moreover, we suppose that the variation of γ is such that $\theta = \theta_0 + \beta x_2$, where $\beta x_2 \ll 1$. From Eq. (92),

$$\eta_\gamma = \frac{2}{\sqrt{3}} \left(\sin \theta_0 + \beta x_2 \cos \theta_0 \right) + O(\beta^2 x_2^2). \quad (98)$$

The first term inside the square parentheses of Eq. (95) becomes

$$\tau \eta_\gamma \sigma_1 [\kappa_1 + 2\tau(\theta - \theta(0))] \approx \frac{\Omega}{\sqrt{2}} (x_2 - \mu) \sigma_1 \quad (99)$$

where

$$\Omega = \sqrt{\frac{8}{3}} (2 \sin \theta_0 + \kappa_1 \tau \cos \theta_0) \beta \quad (100)$$

$$\mu = -\frac{\kappa_1 \tau \sin \theta_0}{\Omega \sqrt{3/8}}. \quad (101)$$

Eq. (95) then reduces to

$$v \left[\frac{\Omega}{\sqrt{2}} (x_2 - \mu) + \partial_2 \right] \Phi_B(x_2) \simeq E \Phi_A(x_2) \quad (102)$$

$$v \left[\frac{\Omega}{\sqrt{2}} (x_2 - \mu) - \partial_2 \right] \Phi_A(x_2) \simeq E \Phi_B(x_2) \quad (103)$$

This has two types of solutions. The first are the zero modes, which arise when Eqs. (102) and (103) are both zero:

$$E = 0, \quad \Phi_{A/B}(x_2) \sim \exp \left[\int^{x_2} dx'_2 \frac{\Omega}{\sqrt{2}}(x'_2 - \mu) \right] \Phi_{A/B}^{(0)}, \quad \Phi_{B/A}(r) = 0. \quad (104)$$

The envelope fields are then given by $\varphi(r) = U_+^\dagger \Phi(r)$ [Eq. (94)]. Note that because of the $U_+^\dagger$ operator, the zero modes are not sublattice polarized.

The second type of solution is the non-zero modes. These are obtained by combining Eqs. (102)–(103):

$$aa^\dagger \Phi_A = \frac{E^2}{\sqrt{2}\Omega v^2} \Phi_A, \quad \text{where} \quad a = \frac{1}{(\sqrt{2}\Omega)^{1/2}} \left[\frac{\Omega}{\sqrt{2}}(x_2 - \mu) + \partial_2 \right]. \quad (105)$$

Note that $[a, a^\dagger] = 1$. The resulting spectrum is

$$E = \pm v \sqrt{\sqrt{2}\Omega n} \quad n = 1, 2, 3 \dots \quad (106)$$

Because Ω depends on κ_1 [see Eq. (100)], these bands are not perfectly flat.

4.4.2 Numerical results

Here I provide numerical calculations to support the analytic results given above. Fig. 4.3(a) shows the lattice under consideration, which is periodic along x and finite in y . Since the k -space displacement of the DP is proportional to θ (Eq. (60)), we vary γ along y so that $\theta(\gamma(y)) = \theta_0 + \beta y$. The computed band diagram by numerically diagonalizing the lattice tight-binding Hamiltonian with 300 sites along y and $\beta = 0.002$ is shown in Fig. 4.3(b), where the zeroth LL is clearly present. I also plot in Fig. 4.3(c) a zoomed-in picture around one of the valleys. The energies of the nonzero bands are also close to theoretical predictions given by Eq. (106) (dashed orange lines). Fig. 4.3(d) plots the wavefunction amplitude distributions for four eigenstates labeled (i)–(iii) in Fig. 4.3(c). The eigenstates at (i) are two-fold degenerate, and consist of a bulk mode and

Figure 4.3. Landau levels induced by gain and loss. (a) Schematic of a non-Hermitian lattice which is periodic along x but finite along y . γ slowly changes along y so that the positions of Dirac points vary linearly with y as $\theta = \theta_0 + \beta y$ (b) Real part of the dispersion for a lattice shown in (a) with 300 sites along y and $\beta = 0.002$. (c) A closer view of the dispersion near one of the valleys (indicated by the orange box in (b)), showing the formation of Landau levels. Dashed orange lines show the lowest few Landau levels obtained analytically from low-energy theory. (d) Wavefunction amplitude profiles for the four eigenstates denoted by (i)-(iii) in (c). The eigenstates at (i) are two-fold degenerate. The number below each plot indicate the corresponding eigenvalue.

an edge mode; the eigenstates at (ii) and (iii) are edge modes connected to the zeroth LL. All these results are consistent with the analytic ones.

I emphasize that this phenomenon is generated by a spatial variation in gain/loss, with no strain engineering [185, 186] or real magnetic field. Thus, this approach would be very useful in controlling the density of states in nanophotonics where gain and loss can be manipulated externally and actively.

4.5 Non-Hermitian lattice with Weyl points

The present framework for generating diabolic points in non-Hermitian systems is not limited to DPs in 2D lattices. In this section, I formulate a non-Hermitian 3D lattice model exhibiting Weyl points, the simplest 3D diabolic points.

The lattice, shown in Fig. 4.4(a), is constructed by stacking copies of the previous 2D lattice with $m = t_2 = 0$. Along z direction, the positive (negative) interlayer couplings are indicated by solid (dashed) lines. The lattice Hamiltonian then satisfy the general matrix form given in Eq. (54) and Eq. (55); for each k_z , the interlayer couplings act as m . For $\gamma = 0$, this is a Weyl semimetal [85]. Upon varying γ , the Weyl points shift in momentum space, raising the possibility of gain and loss induced chiral LLs [187]. Fig. 4.4(b) shows the formation of complex-valued Fermi arcs, whose real parts connect the projections of the real-valued bulk Weyl cones.

Figure 4.4. Non-Hermitian lattice with Weyl points. (a) 3D non-Hermitian lattice hosting Weyl points. Solid (dashed) lines indicate positive (negative) couplings $\pm t_3$. (b) Surface dispersion for a slab with periodic boundary conditions along x and z , and open boundaries along y . The lattice parameters are $t_1 = t_3 = 1$, $t_2 = m = 0$ and $\gamma = 0.6$.

4.6 Summary and outlook

In this chapter, I have shown that diabolic points can occur in non-Hermitian systems by using symmetry constraints to enforce eigenstate orthogonality. This allows for non-Hermitian systems that exhibit the rich phenomenology of diabolic points, as illustrated by the three striking examples presented above—topological transitions in two dimensions, LLs without magnetic fields, and Weyl points. These models are not the only ones that can achieve such outcomes; it would be desirable to find a general description of all such non-Hermitian models. It will also be interesting to study the properties of other non-Hermitian diabolic points, such as quadratic band degeneracies [188], type-II Weyl points [28, 189], higher-order Weyl points [190, 191], and nodal lines [192].

These theoretical ideas can be realized using a variety of experimental platforms. Laser-written optical waveguide arrays [193] are one promising platform; it has been shown that waveguide losses in these arrays can be individually customized (to realize non-hermitian bulk topological transitions [27] or Weyl exceptional rings [100]), while TRS can be effectively broken by twisting the waveguides [24]. Electric circuits, which have been used to demonstrate various topological and non-Hermitian models [68, 194, 195], can also be used to implement the models. Finally, it would be exciting to pursue realizations using active nanophotonic platforms, such as resonator arrays [31, 39, 41, 196, 197] with actively tunable gain or loss. A longstanding obstacle to applying strain engineering ideas [184] to photonics is the impracticality of mechanically deforming photonic devices. The present scheme allows for using active gain or loss to, for example, strongly alter the photonic density of states on demand.

Chapter 5 Conclusion and outlook

In conclusion, I have presented in this thesis experimental and theoretical works on several novel topological phases in classical wave systems.

First, I present studies on VPCs at microwave frequencies. I experimentally demonstrate the existence of valley kink states and show their unique features, including the robustness against sharp bends and topologically protected refraction at zigzag termination. In addition, when spin DOF is introduced, we demonstrate the rich physics resulted from the interplay between valley and spin. I show spin-valley lock bulk flow and topological phase transition between QSH and QVH phases.

Second, I present studies on HOTIs in acoustic systems. I experimentally realize three different types of HOTIs with coupled acoustic resonators: a 2D second-order TI based on a kagome lattice, a 3D third-order TI based on a diamond lattice and a 3D octupole TI based on the BBH model. In the first two works, I demonstrate how to use WCs to predict the existence of corner states. In the third work, I demonstrate the full hierarchy of a bulk octupole moment.

Third, I present a theory for non-Hermitian Hamiltonians to host diabolic points. I develop a general framework for non-Hermitian Hamiltonians to have not only real eigenvalues but also pairwise orthogonal eigenstates by imposing pseudo-Hermiticity and an anti-PT symmetry. I then show the power of our theory by several concrete lattice models, including a 2D non-Hermitian honeycomb lattice model with DPs, a 2D non-Hermitian Haldane model with massive Dirac cones and a 3D model with Weyl points. I also find interesting phenomena like gain and loss induced LLs and topological phases transition due to the interplay between diabolic points and non-Hermiticity.

There are many exciting possibilities to be explored in the future. Here I list some possible directions.

One possible direction is to find suitable applications for the novel topological phases demonstrated in this thesis. I have shown many unique properties of the valley Hall phases and higher-order topological phases. Valley Hall kink states can replace traditional waveguides and offer robust waveguiding, and topological corner states can serve as robust resonances. It would be highly desired to implement them into certain devices. Efforts along this direction should go beyond principle demonstration and target at real applications. Besides, it would also be desired to find applications for other types of topological phases like QH phase, QSH phase and topological semimetal phase.

Another promising direction is to explore the new physics in non-Hermitian and/or nonlinear topological systems. Although the physics for topological phases of matter under Hermitian and linear framework is well-understood, the physics for non-Hermitian and/or nonlinear topological systems is still to be explored. Classical wave systems naturally provide great platforms to study related phenomena where various forms of non-Hermiticity and nonlinearity are available. For example, phenomena predicted in Chapter 4 can be tested in various classical wave systems. I believe there are many other possibilities to be explored both theoretically and experimentally when topology meets non-Hermiticity and/or nonlinearity.

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