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Ratio-of-Distance Rigidity Theory with Application to Similar Formation Control

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Abstract—This paper develops a Ratio-of-Distance (RoD) rigidity theory to study when a framework can be uniquely determined by a set of RoD constraints up to similar transformations (translation, rotation, scaling and reflection). In particular, a framework is specified by a set of RoD constraints (the ratio of distances of a pair of edges joining a common vertex) instead of distance, bearing and angle constraints assumed in existing literature. Its relations to three existing rigidity theories (distance, bearing and angle rigidity theories) are established. The proposed RoD rigidity theory is further applied to the RoD-based similar formation stabilization problem where the desired formation shape is expressed as a set of RoD constraints. Finally, numerical simulations are presented to illustrate the effectiveness of the theoretical results.

Index Terms—Rigidity theory, similar formation control

I. INTRODUCTION

MULTI-AGENT formation control has been studied extensively in the last two decades and existing works can be classified into four categories according to target formation constraints [1]: (1) position-based [2]; (2) displacement-based [3], [4]; (3) distance-based [5]; and (4) bearing-based [6]. The differences in terms of target formation constraint, measurement and convergence are summarized in Table I.

Compared to position-based formation control, displacement-based formation control removes the requirement of absolute coordinates in a global frame, i.e., it adds translational freedom. By developing distance (bearing) rigidity theory, distance-based (bearing-based) formation control further endows the rotational (scaling) freedom to displacement-based formation control. It should be noted that a distance-based formation controller can be implemented without the knowledge of the global frame, however it requires relative position measurements and only local exponential stability can be guaranteed except a few works with range-only measurements [7], triangulated formations [8] and modified controllers [9]. In contrast, bearing-based formation controller ensures almost global convergence and only requires bearing measurements. However, it requires consistent frames or needs to synchronize frames through communications. The relaxed freedoms provided

by distance-based and bearing-based formation control are further leveraged through leader nodes to achieve formation maneuver, such as rotational and translational maneuvering of distance-based rigid formations [10] and scaling and translational maneuvering of bearing-based rigid formations [11]. It is worth noting that none of these four categories can deal with formation with translational, rotational, and scaling invariances simultaneously. In [12]–[14], complex Laplacian was proposed to realize similar formation control in 2-D space, where no global coordinate frame is required and global convergence is guaranteed. In [15], [16], the idea of similar formation control was generalized to 3-D space via signed Laplacian or stress matrix, where an additional shear freedom was introduced and/or maneuvered for collision avoidance. In [17], the similar formation was stabilized through communications and the shear freedom emerged in affine formation was removed by controlling 4 leaders concurrently. Inspired by complex and signed Laplacian matrices, matrix-valued Laplacian constructed from relative rotation and scaling among agents was introduced in [18], [19]¹ to achieve dynamic formation control in arbitrary dimensional spaces, where only the target shape of formation is given in advance. However, absolute coordinate measurements and burdensome communications are required.

Observing that angle-only constraints on target formation can eliminate the implicit assumption on consistent frame in bearing rigidity theory, the authors in [22] developed an angle rigidity theory to explain whether a planar framework can be uniquely determined by a set of angle-only constraints up to similar transformation. Thus, angle-based formation control provides one additional rotational invariance to bearing-based formation control although it uses relative position measurements. However, angle rigidity theory can only deal with planar frameworks and is difficult to be extended to spatial frameworks. In addition, there are several works considering the scale of formation in distance-based formation control. In [23], time-invariant scale or time-varying scale function was treated as an optimization variable to minimize the motion cost for achieving a desired shape. However, the computation of the optimal time-invariant or time-varying function requires global information. A time-varying signal which is known to all agents (i.e., global information) was adopted to precisely control the scaling rate of the distance rigid formation [10]. However, results for determining a framework uniquely up to similar transformations by pure distances (or a function of them) and fully distributed formation stabilization of that

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¹The notion of matrix-weighted Laplacian can also be found in [20], [21], where each entry is a (semi)definite symmetric matrix.

framework are still lacking. On the other hand, observing that a monocular camera cannot measure depth directly but can estimate the ratio of distances from two other identical objects to itself by comparing their projections on the same image, which will be elaborated later.

Motivated by the angle rigidity theory and the above observation about a monocular camera, the first contribution of our work is to propose an RoD rigidity theory to determine whether a set of RoD constraints can uniquely characterize a framework in arbitrary dimensional spaces up to translation, rotation, scaling, and reflection. The role of its 2-D form for distance rigidity theory can be regarded as the counterpart of angle rigidity theory.

Secondly, the equivalence between infinitesimal/global RoD and distance rigidities in arbitrarily dimensional spaces, and furthermore, the equivalence among four kinds of infinitesimal rigidities for planar frameworks are established. Compared to the work [6], the relationships between distance and bearing rigidity matrices, RoD and angle rigidity matrices are generalized to arbitrarily dimensional spaces. Comparing with existing angle rigidity theory which can only deal with 2-D space, we provide two kinds of graphs, namely, trilaterated Laman graphs in 2-D space and quadlaterated Geiringer graphs in 3-D space, whose frameworks can be uniquely determined by a set of RoD constraints and constructed systematically.

Thirdly, based on the proposed RoD rigidity theory, we investigate the distributed RoD-based formation control in both 2-D and 3-D spaces, which extends the flexibility of distance-based formation control by providing one additional scaling freedom and extends the similar formation control to 3-D space without communication requirements and leader nodes. Instead of incorporating the scale in controllers explicitly as in [10], [23], we adopt a set of RoD constraints to represent a desired formation shape, where the scale is an independent variable irrelevant to the shape, as in [13], [18], [19]. In contrast to [17]–[19], where communications and leader nodes are required to achieve the similar formation in 3-D space, a distributed leaderless communication-free RoD-based formation controller is proposed and proven to stabilize the RoD-constrained similar formation locally and exponentially.

The remainder of this paper is structured as follows. Section II introduces some preliminaries and provides two motivating examples. Section III presents the developed RoD rigidity theory and introduces two kinds of frameworks which can be uniquely determined by a set of RoD constraints and constructed systematically. The relations to existing rigidity theories are elaborated in Section IV. Section V studies the RoD-based formation control problem. Simulation results are given in Section VI and the conclusion is finally drawn in Section VII.

II. PRELIMINARIES

A. Notations

Throughout the paper, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ and $\mathbb{Z}_+$ denote the sets of real numbers, positive real numbers, complex numbers and positive integers, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the sets of n -dimensional real vectors and $m \times n$ real matrices,

respectively. Let $\|x\|$, $\text{Span}(x)$, $\text{Card}(x)$ and $x^\perp$ denote the L_2 norm, span, cardinality and perpendicular vector of a given vector $x \in \mathbb{R}^n$, respectively. Given a matrix A , $A^\top$, A^{-1} , $\text{Null}(A)$, $\text{Range}(A)$, $\text{Rank}(A)$, $\det(A)$ are its transpose, inverse, null space, range, rank and determinant, respectively. Let $\text{Diag}(a_1, a_2, \dots, a_n)$ ($\text{Diag}(A_1, A_2, \dots, A_n)$) be the diagonal (block-diagonal) matrix with a_i (A_i) on its i -th diagonal entry (block). Let $I_n \in \mathbb{R}^{n \times n}$ be the n -dimensional identity matrix, $\mathbf{1}_n$ the n -dimensional column vector with all entries of 1 and $\mathbf{0}_n$ the n -dimensional column vector with all entries of 0 or $n \times n$ zero matrix conformed to context. Denote $\mathcal{R} \in SO(d)$ the d -dimensional rotation matrix, where $SO(d) = \{Q \in \mathbb{R}^{d \times d} : Q^\top Q = QQ^\top = I_d, \det(Q) = 1\}$. Let $O(d)$ denote the group of $d \times d$ real orthogonal matrices, i.e., $O(d) = \{Q \in \mathbb{R}^{d \times d} : Q^\top Q = QQ^\top = I_d\}$. For $x \in \mathbb{R}^2$, let $x^\perp = Jx$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $\dim(S)$ be the dimension of space S and $\iota = \sqrt{-1}$ be the imaginary unit. Denote $\Psi \in \mathbb{R}^{d \times d}$ a non-zero skew symmetric matrix, i.e., $\Psi + \Psi^\top = \mathbf{0}_d$. Furthermore, let $\Psi_{(j)}$, $j = 1, 2, \dots, \frac{d(d-1)}{2}$ be a basis of Ψ . For example, $\Psi_{(1)} = J$ for $d = 2$ and $\Psi_{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\Psi_{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $\Psi_{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ for $d = 3$. Let $\otimes$ denote the Kronecker product and $A^\otimes = A \otimes I_d$, $d \geq 2$.

B. Graph theory

A graph is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and edge set, respectively. Agent j is regarded as a neighbor of agent i if $(j, i) \in \mathcal{E}$. A graph $\mathcal{G}$ is undirected if, for all $i, j \in \mathcal{V}$, $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The neighbor set of agent i is denoted as $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}, j \in \mathcal{V}, j \neq i\}$. Consider an undirected graph $\mathcal{G}$ with m edges and N vertices, the orientated incidence matrix $H \in \mathbb{R}^{m \times N}$ is defined by assigning an arbitrary orientation for each edge. The entries of H are defined as $h_{ki} = +1$ if node i is the head node of the k -th edge, or $h_{ki} = -1$ if node i is the tail node of the k -th edge, and $h_{ki} = 0$ otherwise. The adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as follows: $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise, $i, j \in \mathcal{V}$. The Laplacian matrix of $\mathcal{G}$ is then defined as $L = H^\top H \in \mathbb{R}^{N \times N}$. For a connected graph, $\text{Rank}(L) = N - 1$ and $\text{Null}(L) = \text{Null}(H) = \text{Span}(\mathbf{1}_N)$. $\mathcal{K}$ denotes the complete graph.

C. Graph rigidity theory

We refer to a pair $(\mathcal{G}, p)$ as a framework, where $\mathcal{G}$ is a graph and $p = (p_1^\top, \dots, p_N^\top)^\top \in \mathbb{R}^{Nd}$ is called a configuration, $p_i \in \mathbb{R}^d$ is the coordinate of vertex i , $i \in \mathcal{V}$. A configuration p is said to be degenerate if there exist $(d+1)$ coordinates lying on the same $(d-1)$ -dimensional hyperplane. We assume that $p_i \neq p_j$ if $i \neq j$, i.e., no two nodes are at the same position. A smooth rigidity function $r_{\mathcal{G}} : \mathbb{R}^{Nd} \rightarrow \mathbb{R}^z$, $z \in \mathbb{R}_+$

TABLE I: Differences among several existing formation control approaches

C.L.D.	Approaches	Invariances	Refs	Constraints	Mea.	Frames	Conv.	Dim.
Linear	Pos.-based	N/A	[2]	$p_i = p_i^*, i \in \mathcal{V}$	Abs. pos.	GCF	G	$\mathbb{R}^2 \& \mathbb{R}^3$
	Disp.-based	T	[3], [4]	$p_i - p_j = p_i^* - p_j^*, (i, j) \in \mathcal{E}$	Rel. pos.	OALCF	G	$\mathbb{R}^2 \& \mathbb{R}^3$
	Complex Lap.	T&S&R	[13]	$p_i - p_j = s\mathcal{R}(p_i^* - p_j^*), (i, j) \in \mathcal{E}$	Rel. pos.	LCF	G	$\mathbb{R}^2$
	Signed Lap.	T&S&R	[15], [17]	$p_i - p_j = s\mathcal{R}(p_i^* - p_j^*), (i, j) \in \mathcal{E}$	Rel. pos.	LCF	G	$\mathbb{R}^3$
	Bea. Lap.	T&S	[11]	$g_{ij}(p) = g_{ij}^*, (i, j) \in \mathcal{E}$	Rel. pos.	OALCF	G	$\mathbb{R}^2 \& \mathbb{R}^3$
	Matrix-valued Lap.	T&S&R	[18]	$p_i - p_0 = s\mathcal{R}(p_i^* - p_0^*), i \in \mathcal{V}$	Abs. pos.	GCF	G	$\mathbb{R}^2 \& \mathbb{R}^3$
Nonlinear	Dist.-based	T&R	[5]	$\ p_i - p_j\ ^2 = \ p_i^* - p_j^*\ ^2, (i, j) \in \mathcal{E}$	Rel. pos.	LCF	L	$\mathbb{R}^2 \& \mathbb{R}^3$
	Bea.-based	T&S	[6]	$g_{ij}(p) = g_{ij}^*, (i, j) \in \mathcal{E}$	Rel. bea.	OALCF	A.G.	$\mathbb{R}^2 \& \mathbb{R}^3$
	Angle-based	T&S&R	[22]	$g_{ij}^\top(p)g_{ik}(p) = g_{ij}^\top g_{ik}^*, (i, j, k) \in \bar{\mathcal{E}}^*$	Rel. pos.	LCF	L	$\mathbb{R}^2$
	RoD-based	T&S&R	This paper	$\frac{\ e_{ij}(p)\ ^2}{\ e_{ik}(p)\ ^2} = \frac{\ e_{ij}(p^*)\ ^2}{\ e_{ik}(p^*)\ ^2}, (i, j, k) \in \bar{\mathcal{E}}^*$	Rel. pos.	LCF	L	$\mathbb{R}^2 \& \mathbb{R}^3$

C.L.D. = Close Loop Dynamics; Pos. = Position; Disp. = Displacement; Dist. = Distance; Bea. = Bearing; Lap. = Laplacian; T = translational; R = rotational; S = scaling; Mea. = Measurements; Abs. pos. = absolute position; Rel. pos = relative position; Rel. bea. = Relative bearing; Conv. = Convergence; GCF = global coordinate frame; OALCF = orient-aligned local coordinate frame; LCF = local coordinate frame; G = global; L = local; A.G. = almost global; Dim. = Dimension. Other notations refer to Section II.

is introduced to define rigidity of a framework $(\mathcal{G}, p)$. Given a rigidity function $r_{\mathcal{G}}$, the following definitions can be induced.

Given a framework $(\mathcal{G}, p)$, it is said to be rigid if there exists a neighborhood U_p of p such that $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap U_p = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p)) \cap U_p$. $(\mathcal{G}, p)$ is globally rigid if $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p))$. An infinitesimal motion is defined as the motion of vertices that guarantees the invariance of $r_{\mathcal{G}}(p)$, i.e.,

$$\dot{r}_{\mathcal{G}}(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p} \delta_p = \mathbf{0}, \quad (1)$$

where $\delta_p = (\delta_{p_1}^\top, \dots, \delta_{p_N}^\top)^\top$, $\delta_{p_i} = \dot{p}_i$ is the velocity of vertex i . A motion is said to be trivial if it corresponds to translation, rotation and their compositions of the entire framework. A framework is infinitesimally rigid if every infinitesimal motion is trivial. Denote the rigidity matrix $R_r(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p}$. Let d_t be the dimension of the space formed by all trivial motions, then a framework $(\mathcal{G}, p)$ is infinitesimally rigid if and only if $\text{Rank}(R_r(p)) = Nd - d_t$.

In particular, $r_{\mathcal{G}}$ is defined differently in different rigidity theories, thus, corresponding rigidity matrices can be induced. In the distance rigidity theory [5], [24], [25], the distance rigidity function is defined as:

$$D_{\mathcal{G}}(p) = \frac{1}{2}[\dots, \|e_{ij}(p)\|^2, \dots]^\top, (i, j) \in \mathcal{E}, \quad (2)$$

where $e_{ij}(p) = p_i - p_j$. Denote the concatenated form of $e_{ij}(p)$ by $e(p) = H^\otimes p \in \mathbb{R}^{md}$. The distance rigidity matrix is

$$R_D(p) = E^\top H^\otimes, \quad (3)$$

where $E = \text{Diag}(\dots, e_{ij}, \dots) \in \mathbb{R}^{md \times m}$, $(i, j) \in \mathcal{E}$. In the bearing rigidity theory [6], the bearing rigidity function is defined as:

$$B_{\mathcal{G}}(p) = [\dots, g_{ij}^\top(p), \dots]^\top, (i, j) \in \mathcal{E}, \quad (4)$$

where $g_{ij}(p) = \frac{p_i - p_j}{\|p_i - p_j\|}$. The bearing rigidity matrix is

$$R_B(p) = \text{Diag}\left(\frac{P_{g_{ij}}}{\|e_{ij}\|}\right) H^\otimes, \quad (5)$$

where $P_{g_{ij}} = I_d - g_{ij}g_{ij}^\top$ is the orthogonal projection operator matrix. In the angle rigidity theory [22], for a framework

²In [6], $g_{ij}(p) = \frac{p_j - p_i}{\|p_j - p_i\|}$, it is modified in this paper for consistency.

$(\mathcal{G}, p)$ in $\mathbb{R}^2$, $g_{ij}^\top g_{ik}$ is employed as the constraint on the angle between edges e_{ij} and e_{ik} , which is the cosine of this angle. Let $\bar{\mathcal{E}} = \{(i, j, k) \in \mathcal{V}^3 : (i, j), (i, k) \in \mathcal{E}, j < k\}$, then $\{g_{ij}^\top g_{ik} \in [-1, 1]\}_{(i, j, k) \in \bar{\mathcal{E}}}$ is a set of constraints on angles in $(\mathcal{G}, p)$. A framework often has redundant angle information for shape determination. For example, it is sufficient to determine the shape of a triangle with two angles. Whether $(\mathcal{G}, p)$ can be uniquely determined by $A_{\mathcal{G}^*}(p)$ by employing a subset $\bar{\mathcal{E}}^* \subset \bar{\mathcal{E}}$ with $\text{Card}(\bar{\mathcal{E}}^*) = w$ has been explained in the angle rigidity theory, where $\mathcal{G}^* = (\mathcal{V}, \bar{\mathcal{E}}^*)$. The angle rigidity function corresponding to $\mathcal{G}^*$ can be written as

$$A_{\mathcal{G}^*}(p) = [\dots, g_{ij}^\top g_{ik}, \dots]^\top, (i, j, k) \in \bar{\mathcal{E}}^*. \quad (6)$$

The angle rigidity matrix is

$$R_A = \frac{\partial A_{\mathcal{G}^*}(p)}{\partial B_{\mathcal{G}}(p)} \frac{\partial B_{\mathcal{G}}(p)}{\partial p} =: R_{\mathcal{G}}(p) R_B(p), \quad (7)$$

where $R_B(p)$ is given in (5).

For an infinitesimally distance (angle) rigid framework $(\mathcal{K}, p)$ in $\mathbb{R}^d$ ($\mathbb{R}^2$), one has the following level sets:

$$\begin{aligned} D_{\mathcal{K}}^{-1}(D_{\mathcal{K}}(p)) &= \{(I_d \otimes \bar{\mathcal{R}})p + \mathbf{1}_N \otimes \mathbf{t}, \bar{\mathcal{R}} \in O(d), \\ &\quad \mathbf{t} \in \mathbb{R}^d\} =: \mathcal{M}_D, \\ A_{\bar{\mathcal{K}}}^{-1}(A_{\bar{\mathcal{K}}}(p)) &= \{s(I_2 \otimes \bar{\mathcal{R}})p + \mathbf{1}_N \otimes \mathbf{t}, \bar{\mathcal{R}} \in O(2), \\ &\quad s \in \mathbb{R} \setminus \{0\}, \mathbf{t} \in \mathbb{R}^2\} =: \mathcal{M}_A, \end{aligned} \quad (8)$$

where $\bar{\mathcal{K}} = (\mathcal{V}, \bar{\mathcal{E}}_{\mathcal{K}})$ with $\bar{\mathcal{E}}_{\mathcal{K}} = \{(i, j, k) \in \mathcal{V}^3 : i, j, k \in \mathcal{V}, j < k\}$.

D. Two motivating examples

1) *Similarity of two triangles*: Suppose we aim to prove the similarity of two triangles $\triangle_{ijk}(p)$ and $\triangle_{ijk}(q)$ in $\mathbb{R}^2$, see Fig. 1. In [13], the authors consider the following constraint $p_i - p_j = w_c(p_i - p_k)$, where $p_i, p_j, q_i, q_j, w_c \in \mathbb{C}$, and w_c can be regarded as a combination of scaling and rotation from edge e_{ik} to edge e_{ij} , i.e., $w_c = \rho e^{i\theta}$, where ρ and θ are the magnitude and phase angle of w_c , respectively. If two triangles share the same w_c , then we can assert they are similar according to Side-Angle-Side (SAS) Similarity Theorem. In [22], the authors consider that two interior angles $\angle_{ijk}$ and $\angle_{ikj}$ are both equal for two triangles, then the similarity

of them can be proved via Angle-Angle (AA) Similarity Postulate.

In addition to the above-mentioned two methods, we can also conclude that two triangles are similar if

$$\frac{\|e_{ij}(p)\|}{\|e_{ij}(q)\|} = \frac{\|e_{ik}(p)\|}{\|e_{ik}(q)\|} = \frac{\|e_{jk}(p)\|}{\|e_{jk}(q)\|} \quad (9)$$

in the light of Side-Side-Side (SSS) Similarity Theorem. Rewrite (9) as

$$\frac{\|e_{ij}(p)\|}{\|e_{ik}(p)\|} = \frac{\|e_{ij}(q)\|}{\|e_{ik}(q)\|}, \frac{\|e_{jk}(p)\|}{\|e_{ji}(p)\|} = \frac{\|e_{jk}(q)\|}{\|e_{ji}(q)\|}, \quad (10)$$

and term $\frac{\|e_{ij}(p)\|}{\|e_{ik}(p)\|}$ the RoD for node i w.r.t. p , one can easily observe that two triangles are similar if the RoD for node i and j w.r.t. p are equal to that w.r.t. q . Noting that the complex weight w_c contains more information (rotation and scaling) than pure angle or pure RoD, thus it only needs single equation (i.e., node i) to recognize the similarity while the latter two cases require two equations (i.e., both for agent i and j) hold simultaneously.

2) *Extraction of RoD from a single image*: Suppose each agent in $\mathbb{R}^3$ carries a spherical marker of the same yet unknown size (or the agent itself is sphere-like). See Fig. 2, the projection of spherical marker onto the image plane of agent i is an ellipse, whose analytical equation

$$(r_0^2 - o_y^2 - o_z^2)x^2 + (r_0^2 - o_x^2 - o_z^2)y^2 + 2o_x o_y xy + 2o_x o_z f_l x + 2o_y o_z f_l y + (r_0^2 - o_x^2 - o_y^2)f_l^2 = 0 \quad (11)$$

can be obtained given the center $o = [o_x, o_y, o_z]^\top$, the radius of the sphere r_0 and the focal length f_l of the camera [26]. We will operate reversely in order to extract RoD. Given an image from agent i 's camera, we assume that agent i can detect and extract the analytical equations of ellipses projected from two neighboring agents j and k by computer vision algorithms [27]. By assigning suitable radius of sphere and focal length, two pairs of coordinates of centers can be solved from the over-constrained equations formed by coefficients of ellipse parametric equation. Note that these coordinates are not actual positions of agent j and k 's spherical centers. Nevertheless, they are enough to generate the RoD, i.e., $\frac{\|e_{ij}\|}{\|e_{ik}\|}$. Note that the authors in [28] used colored circular targets as markers, where a linear relationship between the estimated position and the radius of the circular pattern was obtained (Eq. (S9) in [28] and Theorem 4 in [29]), which implies the extraction of $\frac{\|e_{ij}\|}{\|e_{ik}\|}$ in this case is feasible. However, this detection strategy may not work well when the circular pattern is almost perpendicular to the image plane of agent. The markers used for the extraction of RoD in $\mathbb{R}^2$ can be simplified to cylinders/ cubes (or the agent is cube-like or cylinder-like) with identical (yet unknown) size since only one rotational variable is involved and line features in images are enough for the extraction.

In this paper, we are interested in generalizing the above-mentioned RoD in triangles to a set of RoD constraints in more general graphs and in arbitrary dimensional spaces. The immediate question is whether this set of RoD constraints can uniquely determine a framework up to translation, rotation,

Fig. 1: Similar triangles.

Fig. 2: Top: an illustration of the projection of sphere from pinhole camera model. Bottom: a simulated image of three identical spheres from pinhole camera model [26]. The yellow lines and red point denote major/minor axis and center of the ellipse. The value adjacent to the ellipse denotes its area w.r.t. known resolution.

scaling and reflection. Therefore, the following sections aim to provide some insights into this question.

III. ROD RIGIDITY

In this section, we extend the RoD for triangles to a set of RoD constraints in a general framework $(\mathcal{G}, p)$. Similar to the three rigidity theories mentioned in the last section, the RoD rigidity function is defined as ³

$$Q_{\mathcal{G}^*}(p) = \frac{1}{2}[\cdots, \frac{\|e_{ij}(p)\|^2}{\|e_{ik}(p)\|^2}, \cdots]^\top, (i, j, k) \in \bar{\mathcal{E}}^*, \quad (12)$$

where the definition of $\bar{\mathcal{E}}^*$ is the same as in (6).

Firstly, the following definitions are presented similar to three existing rigidity theories.

Definition III.1. A framework $(\mathcal{G}, p)$ is said to be RoD rigid if there exists a $\mathcal{G}^*$ and a neighborhood U_p of p such that $Q_{\mathcal{G}^*}^{-1}(Q_{\mathcal{G}^*}(p)) \cap U_p = Q_{\mathcal{K}}^{-1}(Q_{\mathcal{K}}(p)) \cap U_p$.

³The element in the RoD rigidity function can be defined as $\frac{1}{2}(\frac{\|e_{ij}(p)\|}{\|e_{ik}(p)\|})^l$, $l \in \mathbb{Z}_+$. $l = 2$ is used in this paper for consistency with the existing distance rigidity theory.

Fig. 3: (a) A RoD rigid but not globally RoD rigid framework. (b) A globally RoD rigid but not infinitesimally RoD rigid framework. (The red arrow denotes non-trivial infinitesimal RoD motions; both the red and blue arrows denote non-trivial infinitesimal angle motions.)

Definition III.2. A framework $(\mathcal{G}, p)$ is globally RoD rigid if there exists a $\mathcal{G}^*$ such that $Q_{\mathcal{G}^*}^{-1}(Q_{\mathcal{G}^*}(p)) = Q_{\mathcal{K}}^{-1}(Q_{\mathcal{K}}(p))$.

Definition III.3. A framework $(\mathcal{G}, p)$ is minimally RoD rigid if it is RoD rigid, and deletion of any edge will make it not RoD rigid.

By definition, global RoD rigidity implies RoD rigidity. The converse is not true. For example, the framework shown in Fig. 3(a) is RoD rigid but not globally RoD rigid.

Next we present the definition of the infinitesimal RoD rigidity, which is central to RoD rigidity theory. The infinitesimal RoD motion δ_p is defined by the following equation:

$$\begin{aligned} \dot{Q}_{\mathcal{G}^*} &= \frac{\partial Q_{\mathcal{G}^*}}{\partial D_{\mathcal{G}}} \frac{\partial D_{\mathcal{G}}}{\partial e} \frac{\partial e}{\partial p} \delta_p \\ &= R_{D_0}(p) R_D(p) \delta_p \\ &= R(p) \delta_p \end{aligned} \quad (13)$$

where $R_{D_0}(p) = \frac{\partial Q_{\mathcal{G}^*}}{\partial D_{\mathcal{G}}}$ and $R_D(p) = \frac{\partial D_{\mathcal{G}}}{\partial p} = E^\top(p) H^\otimes$ is the distance rigidity matrix. $R(p) = R_{D_0}(p) R_D(p)$ is termed as RoD rigidity matrix.

In this paper, we assume $N \geq d$. As indicated in Section I, in distance rigidity theory, there are 2 kinds of trivial motions in $\mathbb{R}^d$, translation and rotation. The dimension of trivial motion space $d_{tD} = \frac{d(d+1)}{2}$. Similarly, in bearing rigidity theory, there are 2 kinds of trivial motions, translation and scaling. The dimension of trivial motion $d_{tB} = d + 1$. In angle rigidity theory, there are 3 kinds of trivial motions, translation, rotation and scaling. The dimension of trivial motion space $d_{tA} = 4$ (only $d = 2$ was considered in [22]). Intuitively, angle rigidity theory relaxes the requirement of consistent frame where the orientation synchronization can be achieved via inter-agent communication. However, in distance rigidity theory, agents locally exponentially converge to the prescribed distances. Through RoD rigidity theory, we endow one degree-of-freedom to distance rigidity theory, i.e., the distance constraint in (2) can be scaled by a common factor which is not known to any agent. Therefore, one more dimension is added to the trivial motion space of distance rigidity theory, i.e., trivial motion space for RoD rigidity is of dimension $d_{tQ} = \frac{d(d+1)}{2} + 1$. In other words, an infinitesimal RoD motion is called trivial if it corresponds to translation, rotation, scaling and their compositions of the entire framework. Actually, infinitesimal RoD rigidity is equivalent to infinitesimal angle

rigidity in $\mathbb{R}^2$, which we will show later. To characterize the relationship among infinitesimal RoD rigidity, dimension of trivial motion space and rigidity matrix, the following definition is introduced.

Definition III.4. A framework $(\mathcal{G}, p)$ is infinitesimally RoD rigid if there exists a $\mathcal{G}^*$ such that all the infinitesimal RoD motions satisfying $R(p) \delta_p = \mathbf{0}$ are trivial, i.e., $\dim(\text{Null}(R(p))) = \frac{d(d+1)}{2} + 1$.

Lemma III.5. The trivial motion space for RoD rigidity in $\mathbb{R}^d$ is $S = S_t \cup S_r \cup S_s$, where $S_t = \text{Span}\{\mathbf{I}_N^\otimes\}$ is the space formed by translations, $S_r = \{(I_N \otimes \Psi)p\}$ is the space formed by rotations, $S_s = \text{Span}\{p\}$ is the space formed by scalings.

Proof. It can be easily checked that $\text{Null}(H^\otimes) = S_t$, $R_D S_r = E^\top H^\otimes (I_N \otimes \Psi)p = \mathbf{0}$ (The last equality will be used many times in the sequel). It remains to show that $S_s \subset \text{Null}(R(p))$. Let $\nu^\top = \frac{1}{2}(\frac{\partial(\|e_{ij}\|^2/\|e_{ik}\|^2)}{\partial D_{\mathcal{G}}})^\top$ be an arbitrary row of R_{D_0} . It suffices to show $\nu^\top R_D(p)p = 0$. Since $\nu = [\mathbf{0}, \frac{1}{\|e_{ik}\|^2}, \mathbf{0}, -\frac{\|e_{ij}\|^2}{\|e_{ik}\|^4}, \mathbf{0}]^\top$, we have

$$\begin{aligned} \nu^\top R_D(p)p &= [\mathbf{0}, \frac{1}{\|e_{ik}\|^2}, \mathbf{0}, -\frac{\|e_{ij}\|^2}{\|e_{ik}\|^4}, \mathbf{0}] E^\top H^\otimes p \\ &= [\mathbf{0}, \frac{1}{\|e_{ik}\|^2}, \mathbf{0}, -\frac{\|e_{ij}\|^2}{\|e_{ik}\|^4}, \mathbf{0}] E^\top e \\ &= [\mathbf{0}, \frac{1}{\|e_{ik}\|^2}, \mathbf{0}, -\frac{\|e_{ij}\|^2}{\|e_{ik}\|^4}, \mathbf{0}] \\ &\quad [\cdots, \|e_{ij}\|^2, \cdots, \|e_{ik}\|^2, \cdots]^\top \\ &= 0. \end{aligned}$$

Then the conclusion follows. $\blacksquare$

It is worth noting that trivial motion space for the RoD rigidity is the same as the angle rigidity in $\mathbb{R}^2$. The following lemma can be obtained directly.

Lemma III.6. A framework $(\mathcal{G}, p)$ is infinitesimally RoD rigid if and only if $\text{Null}(R(p)) = S$.

In fact, similar to the results in angle rigidity theory, we have the following theorem.

Theorem III.7. If $(\mathcal{K}, p)$ is infinitesimally RoD rigid, then $\mathcal{M}_Q := Q_{\mathcal{K}}^{-1}(Q_{\mathcal{K}}(p)) = \{s(I_d \otimes \bar{\mathcal{R}})p + \mathbf{I}_N \otimes \mathbf{t}, \bar{\mathcal{R}} \in O(d), s \in \mathbb{R} \setminus \{0\}, \mathbf{t} \in \mathbb{R}^d\}$ is a $(\frac{d(d+1)}{2} + 1)$ -dimensional manifold.

The proof of Theorem III.7 needs the equivalence between infinitesimal RoD and distance rigidities (given in Section IV), thus will be provided later. It can be easily checked that $\mathcal{M}_Q = \mathcal{M}_A$ in $\mathbb{R}^2$.

Theorem III.8. If $(\mathcal{G}, p)$ is infinitesimally RoD rigid, then $(\mathcal{G}, p)$ is RoD rigid.

Proof. The proof can follow a similar line of arguments as in that of Theorem 2 in [22]. $\blacksquare$

The converse of Theorem III.8 is not true. For example, a framework $(\mathcal{K}, p)$ with p being a degenerate configuration shown in Fig. 3(b) is globally RoD rigid (implies RoD rigid) but not infinitesimally RoD rigid.

In the following, we present a necessary combinatorial condition for minimal infinitesimal RoD rigidity.

Definition III.9. A graph $\mathcal{G}^* = (\mathcal{V}, \bar{\mathcal{E}}^*)$ is said to be (d, d_t) -tight if $\text{Card}(\bar{\mathcal{E}}^*) = \text{Card}(\mathcal{V})d - d_t$ and every subset of k ($k \geq 2$) vertices spans at most $kd - d_t$ edges.

Lemma III.10. A (d, d_t) -tight graph does not contain nodes of degree $d - 1$.

Proof. The proof follows that of Proposition 6.1 in [30]. ■

Theorem III.11. If a framework $(\mathcal{G}, p)$ is minimally infinitesimally RoD rigid for $\mathcal{G}^*$, then $\mathcal{G}^*$ is (d, d_{tQ}) -tight and $\text{Card}(\mathcal{N}_{0i}) \geq d$ for all $i \in \mathcal{V}$, where $\mathcal{N}_{0i} = \{j \in \mathcal{V} : (i, j, \cdot) \in \bar{\mathcal{E}}^* \text{ or } (i, \cdot, j) \in \bar{\mathcal{E}}^* \text{ or } (j, i, \cdot) \in \bar{\mathcal{E}}^* \text{ or } (j, \cdot, i) \in \bar{\mathcal{E}}^*\}$.

Proof. We prove the theorem by contradiction. Firstly, suppose that a framework $(\mathcal{G}, p)$ is minimally infinitesimally RoD rigid for $\mathcal{G}^*$ but there exists $\mathcal{U} \subset \mathcal{V}$ such that $\mathcal{U}$ spans more than $2\text{Card}(\mathcal{U}) - 4$ edges. Note that the rank of the RoD rigidity matrix of a subgraph induced from $\mathcal{U}$ must not exceed $2\text{Card}(\mathcal{U}) - 4$ by Lemma III.5, which implies that the rows are linearly dependent and thus contradicts the minimality. Secondly, suppose there exists node i such that $\text{Card}(\mathcal{N}_{0i}) < d$. By Lemma III.10, node i is at least of degree d . Then, there always exists a nontrivial motion of node i as $\text{Card}(\mathcal{N}_{0i} \cup \{i\}) \leq d$ and d nodes span at most $(d - 1)$ -dimensional space. By Lemma III.6, this establishes a contradiction. ■

Remark III.12. Although Theorem III.11 only presents a necessary combinatorial condition, we conjecture that the necessary conditions are also sufficient for planar minimal RoD rigidity with generic configurations. Some closely related results can be found in [31], [32]. The difficulty lies in how to prove that the RoD rigidity matrix is full row rank with generic configurations either from generic matrix (generically full rank) of matroids (Lemma 3.5 in [31]) or extension operation on hypergraph (Theorem 3.3 in [32]).

We defer the equivalence proof of infinitesimal/global RoD and distance rigidities to Theorem IV.1 and IV.4. The approaches to constructing infinitesimally/globally distance rigid frameworks can provide some insights on constructing infinitesimally/globally RoD rigid frameworks. In the following, we present two classes of frameworks which can be uniquely determined by RoD constraints and constructed systematically, i.e. 1) trilaterated Laman graphs ($d = 2$); 2) quadlaterated Geiringer graphs ($d = 3$).

A. Triangulated and Trilaterated Laman graphs

In [8], the authors have introduced the triangulated Laman graphs, which are constructed by pure Henneberg vertex-add operations, for stabilizing planar formation globally. Denote by $\mathcal{G}_{L(N)} = (\mathcal{V}_N, \mathcal{E}_{\mathcal{G}_{L(N)}})$, $N \geq 3$ the N -agent triangulated Laman graph. The authors in [22] has constructed minimally infinitesimally and globally angle rigid frameworks based on triangulated Laman graphs. As seen from Fig. 4, there are two triangles $\triangle_{124}$ and $\triangle_{234}$ ($\triangle_{23'4}$) which are globally rigid by themselves. However, attaching them together with common

Fig. 4: (a) A minimally infinitesimally distance/angle rigid framework $(\mathcal{G}_{L(4)}, p)$ in $\mathbb{R}^2$. (b) Another framework $(\mathcal{G}_{L(4)}, p')$ with $D_{\mathcal{G}_{L(4)}}(p) = D_{\mathcal{G}_{L(4)}}(p')$ and $A_{\mathcal{G}_{L(4)}^*}(p) = A_{\mathcal{G}_{L(4)}^*}(p')$. (Black and red lines denote edges contained in $\mathcal{G}_{L(4)}$, additional edges which make (a) and (b) globally distance rigid, respectively. Black and green arcs denote angle formed by $\bar{\mathcal{E}}_{\mathcal{G}_{L(4)}}^*$ and additional angle formed by $\hat{\mathcal{E}}_{\mathcal{G}_{L(4)}^*}$, respectively.)

edge 24 will destroy the global rigidity, i.e., $\triangle_{234}$ can be reflected over edge 24. To ensure the global rigidity, edge 13 (13') is added by distance rigidity theory. In angle rigidity theory [22], $\angle_{123}$ can be added to distinguish the frameworks (a) and (b) in Fig. 4. In our RoD rigidity theory, this new constraint can be $\frac{1}{2} \frac{\|e_{12}\|^2}{\|e_{13}\|^2}$. Since $e_{13} \notin \mathcal{E}_{\mathcal{G}_{L(4)}}$, trilaterated Laman graphs $\mathcal{G}_{L^+(N)} = (\mathcal{V}_N, \mathcal{E}_{\mathcal{G}_{L^+(N)}})$ are constructed, where $\mathcal{E}_{\mathcal{G}_{L^+(N)}} = \mathcal{E}_{\mathcal{G}_{L(N)}} \cup \{(k, l) : k \xrightarrow{1} \mathcal{V}_{l-1} \setminus \{i, j\}, i, j \in \mathcal{N}_l, i < j < l, l = 4, \dots, N\}$ if $N \geq 4$ and $\mathcal{E}_{\mathcal{G}_{L^+(N)}} = \mathcal{E}_{\mathcal{G}_{L(N)}}$ otherwise. $x \xrightarrow{1} \mathcal{X}$ means that x randomly chooses an element in set $\mathcal{X}$ as its value. In other words, an additional edge is added in each vertex-add operation. Then we formally introduce the following result.

Theorem III.13. If $(\mathcal{G}_{L(N)}, p)$ is strongly rigid, i.e., p_i, p_j and p_k are not collinear for any three vertices i, j, k satisfying $(i, j), (j, k), (i, k) \in \mathcal{E}_{\mathcal{G}_{L(N)}}$, then the following statements hold:

1) both $(\mathcal{G}_{L(N)}, p)$ and $(\mathcal{G}_{L^+(N)}, p)$ are minimally infinitesimally RoD rigid for

$$\bar{\mathcal{E}}_{\mathcal{G}_{L(N)}}^* = \{(i, j, k) \in \mathcal{V}_N^3 : (i, j), (j, k), (i, k) \in \mathcal{E}_{\mathcal{G}_{L(N)}}, i, j < k\}; \quad (14)$$

2) if additional edges

$$\hat{\mathcal{E}}_{\mathcal{G}_{L^+(N)}} = \begin{cases} \{(l, i, k) : (i, j), (i, l), (j, l), (k, l) \in \mathcal{E}_{\mathcal{G}_{L^+(l)}}, \\ i < j < l, l = 4, \dots, N\}, \text{ if } N \geq 4; \\ \emptyset, \text{ otherwise,} \end{cases} \quad (15)$$

are added into $\bar{\mathcal{E}}_{\mathcal{G}_{L(N)}}^*$ so that $\check{\bar{\mathcal{E}}}_{\mathcal{G}_{L^+(N)}} = \bar{\mathcal{E}}_{\mathcal{G}_{L(N)}}^* \cup \hat{\mathcal{E}}_{\mathcal{G}_{L^+(N)}}$, $(\mathcal{G}_{L^+(N)}, p)$ is globally RoD rigid for $\check{\bar{\mathcal{E}}}_{\mathcal{G}_{L^+(N)}}$.

3) $Q_{\bar{\mathcal{G}}_{L^+(N)}}^{-1}(Q_{\bar{\mathcal{G}}_{L^+(N)}}(p)) = \mathcal{M}_Q$.

Proof. By Proposition 1 in [8] and Lemma 2 in [33], the additions of edges in 1) and 2) preserve minimal infinitesimal and global distance rigidity, respectively. Then the proof is straightforward by Theorem IV.1 and IV.4 and thus omitted. The same argument can be applied to Theorem III.18. ■

Fig. 5: A framework with a Laman graph.

Remark III.14. Note that in the first motivating example and this theorem, angles and RoDs involved are related to triangles. The necessity of introducing RoDs are: 1) angles and RoDs in triangles are mutually convertible through the Law of Sines, for example, $\frac{\sin(\angle_{ijk})}{\sin(\angle_{jik})} = \frac{\|e_{ik}\|}{\|e_{jk}\|}$ and $\frac{\sin(\angle_{ikj})}{\sin(\angle_{jik})} = \frac{\|e_{ij}\|}{\|e_{jk}\|}$. The computation of two RoDs for agent i and j requires the angle information $\angle_{jik}$, which can only be obtained by communicating with agent k or mutual communication. Thus, more agent or communication will be involved. 2) Laman graphs are generically infinitesimally distance/bearing rigid [34]. Thus, they are generically infinitesimally angle/RoD rigid. However, angles and RoDs in frameworks with Laman graphs may not be mutually convertible since angles and RoDs may not always be in triangles. For example, see Fig. 5, $\frac{\|e_{14}\|}{\|e_{15}\|}$ cannot be obtained through any existing angle measurement.

B. Trilaterated and Quadlaterated Geiringer graphs

Theorem 7 in [22] relies on the equivalence between infinitesimal distance and bearing rigidities in $\mathbb{R}^2$ [6], i.e., strongly rigidity implies infinitesimal distance rigidity [8], which further implies infinitesimal bearing rigidity and infinitesimal angle rigidity [22]. However, this equivalence does not exist in $\mathbb{R}^3$, rendering generalizing angle rigidity theory to $\mathbb{R}^3$ very difficult. On the other hand, the equivalence between infinitesimal distance and RoD rigidities in $\mathbb{R}^d$ will be established in Theorem IV.1, thus we are interested in generalizing Theorem III.13 to $\mathbb{R}^3$. Before that, we make some necessary definitions. Denote by $\mathcal{G}_{G(N)} = (\mathcal{V}_N, \mathcal{E}_{\mathcal{G}_{G(N)}})$, $N \geq 4$ the N -agent trilaterated Geiringer graphs, where we follow [35] to call these graphs Geiringer graphs for honoring Hilda Pollaczek-Geiringer who worked on rigid graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ [36], [37].

Definition III.15. Let $\mathcal{G}_{G(3)} = (\mathcal{V}_3, \mathcal{E}_{\mathcal{G}_{G(3)}})$, where $\mathcal{V}_3 = \{1, 2, 3\}$ and $\mathcal{E}_{\mathcal{G}_{G(3)}} = \{(1, 2), (1, 3), (2, 3)\}$. $\mathcal{G}_{G(N)}$ is a graph obtained by adding a vertex N and three edges (N, i) , (N, j) and (N, k) into graph $\mathcal{G}_{G(N-1)}$ for some $i, j, k \in \mathcal{V}_{N-1}$ satisfying $(i, j), (i, k), (j, k) \in \mathcal{E}_{\mathcal{G}_{G(N-1)}}$.

Definition III.16. A framework $(\mathcal{G}_{G(N)}, p)$ in $\mathbb{R}^3$ is strongly rigid if for any i, j, k and l satisfying $(i, j), (i, k), (i, l), (j, k), (j, l), (k, l) \in \mathcal{E}_{\mathcal{G}_{G(N)}}$, the tetrahedron formed by them is non-degenerate, i.e., $p_i - p_j$, $p_i - p_k$ and $p_i - p_l$ are linearly independent.

Lemma III.17. Given a framework $(\mathcal{G}_{G(N)}, p)$ in $\mathbb{R}^3$, strong rigidity implies infinitesimal distance rigidity.

Proof. The proof follows the proof of Proposition 1 in [8]. ■

The definition of trilaterated Geiringer graphs differs from that of triangulated Laman graphs in that we add three edges instead of two edges in each Henneberg step. Intuitively, the former can be regarded as composing of triangles, while the latter consists of tetrahedrons. Definition III.15 also differs from that of trilaterated Laman graph in that three edges are added to three nodes which are connected with each other. Similar to Subsection III-A, quadlaterated Geiringer graphs $\mathcal{G}_{G^+(N)} = (\mathcal{V}_N, \mathcal{E}_{\mathcal{G}_{G^+(N)}})$ are constructed, where $\mathcal{E}_{\mathcal{G}_{G^+(N)}} = \mathcal{E}_{\mathcal{G}_{G(N)}} \cup \{(r, l) : r \leftarrow \mathcal{V}_{l-1} \setminus \{i, j, k\}, i, j, k \in \mathcal{N}_l, i < j < k < l, l = 5, \dots, N\}$ if $N \geq 5$ and $\mathcal{E}_{\mathcal{G}_{G^+(N)}} = \mathcal{E}_{\mathcal{G}_{G(N)}}$ otherwise. From the equivalence between infinitesimal distance and RoD rigidity in $\mathbb{R}^3$, we have the following result.

Theorem III.18. If $(\mathcal{G}_{G(N)}, p)$ is strongly rigid, then the following statements hold:

1) both $(\mathcal{G}_{G(N)}, p)$ and $(\mathcal{G}_{G^+(N)}, p)$ are minimally infinitesimally RoD rigid for

$$\begin{aligned} \bar{\mathcal{E}}_{\mathcal{G}_{G(N)}}^* &= \{(1, 2, 3), (2, 1, 3)\} \cup \{(i, j, l), (j, k, l), \\ &\quad (k, i, l) \in \mathcal{V}_N^3 : (i, j), (i, k), (i, l), (j, k), \\ &\quad (j, l), (k, l) \in \mathcal{E}_{\mathcal{G}_{G(N)}}, i < j < k < l\}; \end{aligned} \quad (16)$$

2) if additional edges

$$\hat{\mathcal{E}}_{\mathcal{G}_{G^+(N)}} = \begin{cases} \{(l, i, r) : (i, j), (i, k), (i, l), (j, k), (j, l), \\ (k, l), (r, l) \in \mathcal{E}_{\mathcal{G}_{G^+(l)}}, i < j < k < l, \\ l = 5, \dots, N\}, & \text{if } N \geq 5; \\ \emptyset, & \text{otherwise,} \end{cases} \quad (17)$$

are added into $\bar{\mathcal{E}}_{\mathcal{G}_{G(N)}}^*$ so that $\check{\mathcal{E}}_{\mathcal{G}_{G^+(N)}} = \bar{\mathcal{E}}_{\mathcal{G}_{G(N)}}^* \cup \hat{\mathcal{E}}_{\mathcal{G}_{G^+(N)}}$, $(\mathcal{G}_{G^+(N)}, p)$ is globally RoD rigid for $\check{\mathcal{E}}_{\mathcal{G}_{G^+(N)}}$.

3) $Q_{\mathcal{G}_{G^+(N)}}^{-1}(Q_{\check{\mathcal{G}}_{G^+(N)}}(p)) = \mathcal{M}_Q$.

An illustrative example similar to Fig. 4 is provided in Fig. 6. There are two tetrahedrons $\mathfrak{N}_{1234}$ and $\mathfrak{N}_{1235}$ ($\mathfrak{N}_{1235'}$) which are globally rigid by themselves. However, attaching them together with common triangle $\triangle_{123}$ will destroy the global rigidity, i.e., $\mathfrak{N}_{1235}$ can be reflected over triangle $\triangle_{123}$. To ensure the global rigidity, edge 45 (45') is added by distance rigidity theory. In RoD rigidity theory, $\frac{\|e_{41}\|}{\|e_{45}\|}$ can be added to distinguish the frameworks (a) and (b) in Fig. 6.

IV. THE RELATION TO OTHER THREE RIGIDITIES

In this section, we aim to present some results on the relationships among the existing three rigidity theories and our proposed RoD rigidity theory. The equivalence between infinitesimal/global RoD and distance rigidities is firstly established. Then, a direct relation to angle rigidity matrix is deduced. Finally, the equivalence among four infinitesimal rigidities is concluded.

A. The relations to distance rigidity

In this subsection, we will show the equivalence of infinitesimal RoD and distance rigidity, which corresponds to the similar relationship between infinitesimal angle and bearing rigidity.

Fig. 6: (a) A minimally infinitesimally distance/RoD rigid framework $(\mathcal{G}_{G(5)}, p)$ in $\mathbb{R}^3$. (b) Another framework $(\mathcal{G}_{G(5)}, p')$ with $D_{\mathcal{G}_{G(5)}}(p) = D_{\mathcal{G}_{G(5)}}(p')$ and $Q_{\mathcal{G}_{G(5)}}(p) = Q_{\mathcal{G}_{G(5)}}(p')$. (Black and red lines denote edges contained in $\mathcal{G}_{G(5)}$, additional edges which make (a) (b) global distance rigid, respectively. Black and red arcs with arrowheads denote RoD constraints formed by $\bar{\mathcal{E}}_{\mathcal{G}_{G(5)}}^*$ and additional RoD constraints formed by $\hat{\mathcal{E}}_{\mathcal{G}_{G(5)}}^+$, respectively. For example, (i, j, k) is a RoD constraint, then we use an arc with arrowhead pointed from edge (i, j) to edge (i, k) .)

Theorem IV.1. A framework $(\mathcal{G}, p)$ is infinitesimally RoD rigid if and only if it is infinitesimally distance rigid.

Proof. Necessity. The proof of necessity follows from that of Theorem 3 in [22].

Sufficiency. It follows from $R(p) = R_{D_0}(p)R_D(p)$ and $\text{Null}(R_D) = S_t \cup S_r$ that $\text{Null}(R(p)) = S$ holds if for any $\nu \in \text{Null}(R_{D_0}) \setminus \text{Range}(R_D)$, there exists $\mu \in S_s$ such that $\nu = R_D \mu$. Let $\nu = R_D \mu$ and $R_{D_0} \nu = R_{D_0} R_D \mu = \mathbf{0}$ for some $\mu = (\mu_1^\top, \dots, \mu_N^\top)^\top \in \mathbb{R}^{Nd}$. Let $\frac{1}{2} \frac{\|e_{ij}(q)\|^2}{\|e_{ik}(q)\|^2}$ be an arbitrary element of $Q_{\mathcal{G}^*}$, then one has

$$\frac{1}{\|e_{ik}\|^2} e_{ij}^\top (\mu_i - \mu_j) - \frac{\|e_{ij}\|^2}{\|e_{ik}\|^4} e_{ik}^\top (\mu_i - \mu_k) = 0. \quad (18)$$

There must exist $s_{ij}, s_{ik}, c_{ij(j)}, c_{ik(j)} \in \mathbb{R}, j = 1, \dots, d-1$ such that

$$\begin{aligned} \mu_i - \mu_j &= s_{ij} g_{ij} + \sum_{j=1}^{d-1} c_{ij(j)} g_{ij(j)}^\perp, \\ \mu_i - \mu_k &= s_{ik} g_{ik} + \sum_{j=1}^{d-1} c_{ik(j)} g_{ik(j)}^\perp, \end{aligned} \quad (19)$$

where $V_{ij} = [g_{ij}, g_{ij(1)}^\perp, \dots, g_{ij(d-1)}^\perp]$ and $V_{ik} = [g_{ik}, g_{ik(1)}^\perp, \dots, g_{ik(d-1)}^\perp]$ are two orthogonal matrices constructed by Gram-Schmidt Orthogonalization with $g_{ij(j)}^\perp, g_{ik(j)}^\perp \in \mathbb{R}^d, j \in \{1, \dots, d-1\}$.

Substitute (19) into (18), one has

$$\frac{\|e_{ij}\|^2}{\|e_{ik}\|^2} \frac{s_{ij}}{\|e_{ij}\|} - \frac{\|e_{ij}\|^2}{\|e_{ik}\|^2} \frac{s_{ik}}{\|e_{ik}\|} = 0. \quad (20)$$

There must exist some $s_{ijk} \in \mathbb{R} \setminus \{0\}$, such that $\frac{s_{ij}}{\|e_{ij}\|} = \frac{s_{ik}}{\|e_{ik}\|} = s_{ijk}$. Next, we can show that all of the edges share the same s_{ijk} (refer to as s in the sequel) by using the same

analysis in [22]. From the above discussion, we have $\mu_i - \mu_j = s e_{ij} + \sum_{j=1}^{d-1} c_{ij(j)} g_{ij(j)}^\perp, \forall (i, j) \in \mathcal{E}$. Then one has

$$\begin{aligned} \nu &= R_D \mu \\ &= E^\top H^\otimes \mu \\ &= E^\top s H^\otimes p + [\dots, e_{ij}^\top \sum_{j=1}^{d-1} c_{ij(j)} g_{ij(j)}^\perp, \dots]^\top \\ &= R_D(sp), \end{aligned} \quad (21)$$

where the last equality follows from $e_{ij}^\top g_{ij(j)}^\perp = 0, \forall (i, j) \in \mathcal{E}, j \in \{1, \dots, d-1\}$. Clearly, $sp \in S_s$, which completes the proof. ■

Lemma IV.2. Given two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ where $\mathcal{G}$ is connected, $q \in Q_{\mathcal{G}}^{-1}(Q_{\mathcal{G}}(p))$ if and only if $sq \in D_{\mathcal{G}}^{-1}(D_{\mathcal{G}}(p))$, where $s \in \mathbb{R} \setminus \{0\}$.

Proof. Sufficiency. $sq \in D_{\mathcal{G}}^{-1}(D_{\mathcal{G}}(p))$ is equivalent to $D_{\mathcal{G}}(sq) = D_{\mathcal{G}}(p)$. Furthermore, it is equivalent to $s^2 D_{\mathcal{G}}(q) = D_{\mathcal{G}}(p)$. Then we have $s^2 \|e_{ij}(q)\|^2 = \|e_{ij}(p)\|^2, (i, j) \in \mathcal{E}$ and $s^2 \|e_{ik}(q)\|^2 = \|e_{ik}(p)\|^2, (i, k) \in \mathcal{E}$. Let $\frac{1}{2} \frac{\|e_{ij}(q)\|^2}{\|e_{ik}(q)\|^2}$ be an arbitrary element of $Q_{\mathcal{G}}(q)$. Then one has

$$\frac{\|e_{ij}(q)\|^2}{\|e_{ik}(q)\|^2} = \frac{\|e_{ij}(p)\|^2}{\|e_{ik}(p)\|^2}, (i, j, k) \in \bar{\mathcal{E}}, \quad (22)$$

which is equivalent to $Q_{\mathcal{G}}(q) = Q_{\mathcal{G}}(p)$, i.e., $q \in Q_{\mathcal{G}}^{-1}(Q_{\mathcal{G}}(p))$.

Necessity. Since $\mathcal{G}$ is connected, there must exist a spanning tree $\mathcal{G}_{st} = (\mathcal{V}, \mathcal{E}_{st})$, where $\text{Card}(\mathcal{E}_{st}) = N - 1$. In $\mathcal{G}_{st}$, there must exist at least two nodes (denote one of them as j) which have exactly one neighbor (denote it as i , i must have another neighbor k). From $q \in Q_{\mathcal{G}}^{-1}(Q_{\mathcal{G}}(p))$, there holds

$$\frac{\|e_{ij}(q)\|^2}{\|e_{ij}(p)\|^2} = \frac{\|e_{ik}(q)\|^2}{\|e_{ik}(p)\|^2} = \bar{s}^2.$$

Then we cut node j and edge (i, j) from $\mathcal{G}_{st}$ and repeat the above analysis until $\text{Card}(\mathcal{E}_{st}) = 1$, one has

$$\|e_{ij}(q)\|^2 = \bar{s}^2 \|e_{ij}(p)\|^2, (i, j) \in \mathcal{E}_{st}. \quad (23)$$

Since $\mathcal{G}_{st}$ contains all nodes, $\forall (i, j) \in \mathcal{E} \setminus \mathcal{E}_{st}$, there must exist $(i, k) \in \mathcal{E}_{st}$ such that $(i, j, k) \in \mathcal{E}$. From $q \in Q_{\mathcal{G}}^{-1}(Q_{\mathcal{G}}(p))$ (i.e. (22)) and (23), one has

$$\|e_{ij}(q)\|^2 = \bar{s}^2 \|e_{ij}(p)\|^2, (i, j) \in \mathcal{E} \setminus \mathcal{E}_{st}.$$

Let $\bar{s} = 1/s$, the proof is complete. ■

Remark IV.3. Unlike in [22] where $\mathcal{G}$ needs to be both globally angle and globally bearing rigid, a connected graph is sufficient for the equivalence in Lemma IV.2 since the ratio relationships can be ‘transmitted’ from one edge to its neighboring edge. However, the angle constraints cannot be ‘transmitted’ from one edge to its neighboring edge, i.e., $g_{ik}(p)$ is not necessarily equal to $g_{ik}(q)$ even if $g_{ij}(p) = g_{ij}(q)$.

Theorem IV.4. A framework $(\mathcal{G}, p)$ is RoD (resp. globally RoD) rigid if and only if it is distance (resp. globally distance) rigid.

Proof. Necessity. Suppose $(\mathcal{G}, p)$ is RoD rigid, by definition, there exists a neighborhood U_p of p such that $Q_{\mathcal{G}}^{-1}(Q_{\mathcal{G}}(p)) \cap$

$U_p = Q_{\bar{\mathcal{K}}}^{-1}(Q_{\bar{\mathcal{K}}}(p)) \cap U_p$. For any $q \in D_{\mathcal{G}}^{-1}(D_{\mathcal{G}}(p)) \cap U_p$, it follows from $D_{\mathcal{G}}(p) = D_{\mathcal{G}}(q)$ that $Q_{\mathcal{G}^*}(p) = Q_{\mathcal{G}^*}(q)$. Therefore, $Q_{\bar{\mathcal{K}}}(p) = Q_{\bar{\mathcal{K}}}(q)$. As a result, it follows from Lemma IV.2 that $D_{\mathcal{K}}(p) = s^2 D_{\mathcal{K}}(q)$ for some $s \in \mathbb{R} \setminus \{0\}$. Since $D_{\mathcal{G}}(p) = D_{\mathcal{G}}(q)$, we have $s = 1$. Thus $D_{\mathcal{K}}(p) = D_{\mathcal{K}}(q)$, which is equivalent to $q \in D_{\bar{\mathcal{K}}}^{-1}(D_{\bar{\mathcal{K}}}(p))$, i.e., $(\mathcal{G}, p)$ is distance rigid. Sufficiency. It suffices to prove that there always exists a $\mathcal{G}^*$ such that $(\mathcal{G}, p)$ is RoD rigid for $\mathcal{G}^*$ given that $(\mathcal{G}, p)$ is distance rigid. The following construction process can be considered: start from any edge in $\mathcal{G}$, construct an RoD using this edge and its neighboring edge (must exist since each vertex involves at least two edges), then start with this neighboring edge, repeat the above process until each edge in $\mathcal{G}$ has been involved in RoDs. $\mathcal{G}^*$ always exists since it is induced from $\mathcal{G}$. That $(\mathcal{G}, p)$ is RoD rigid for $\mathcal{G}^*$ can be proved by reversing the proof of the necessity. The proof of the equivalence in parentheses is similar except a suppression of U_p in the above arguments. ■

We are now in the position to give the proof of Theorem III.7.

Proof of Theorem III.7. From Theorem IV.1, $(\mathcal{K}, p)$ is infinitesimally distance rigid. Combine $D_{\bar{\mathcal{K}}}^{-1}(D_{\bar{\mathcal{K}}}(p)) = \mathcal{M}_D$ with Lemma IV.2, there must hold $Q_{\bar{\mathcal{K}}}^{-1}(Q_{\bar{\mathcal{K}}}(p)) = \mathcal{M}_Q$.

Next we show that $\mathcal{M}_Q$ is a manifold. For any $q \in Q_{\bar{\mathcal{K}}}^{-1}(Q_{\bar{\mathcal{K}}}(p))$, it is obvious that $q = s((I_N \otimes \bar{\mathcal{R}})p + \mathbf{1}_N \otimes \mathbf{t})$ for some $s \in \mathbb{R} \setminus \{0\}$, $\bar{\mathcal{R}} \in O(d)$ and $\mathbf{t} \in \mathbb{R}^d$. From the chain rule, we have

$$\begin{aligned} \text{Rank}\left(\frac{\partial Q_{\bar{\mathcal{K}}}(q)}{\partial q}\right) &= \text{Rank}\left(\frac{1}{s} \frac{\partial Q_{\bar{\mathcal{K}}}(p)}{\partial p} (I_N \otimes \bar{\mathcal{R}}^\top)\right) \\ &= Nd - \frac{d(d+1)}{2} - 1. \end{aligned} \quad (24)$$

Note that $Q_{\bar{\mathcal{K}}} : \mathbb{R}^{Nd} \rightarrow \mathbb{R}^{\text{Card}(\bar{\mathcal{E}}_{\mathcal{K}})}$ is a smooth map, thus $Q_{\bar{\mathcal{K}}}^{-1}(Q_{\bar{\mathcal{K}}}(p))$ is a properly embedded sub-manifold of dimension $Nd - (Nd - \frac{d(d+1)}{2} - 1) = \frac{d(d+1)}{2} + 1$ according to Theorem 8.8 in [38]. ■

B. Relations to bearing rigidity and angle rigidity

After presenting the relationship between RoD and distance rigidities, it is natural to question the relationships among these four rigidity theories.

Firstly, we present the following Lemma.

Lemma IV.5. *Given $g \in \mathbb{R}^d$, $g \neq \mathbf{0}_d$ and $\|g\| = 1$, $P_g = I_d - gg^\top = \sum_{j=1}^{\frac{d(d-1)}{2}} (\Psi_{(j)} g)(\Psi_{(j)} g)^\top$.*

Proof. Denote $g = [g_1, \dots, g_d]^\top$, one has

$$I - gg^\top = \begin{bmatrix} 1 - g_1^2 & -g_1 g_2 & \cdots & -g_1 g_d \\ -g_2 g_1 & 1 - g_2^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -g_d g_1 & \cdots & \cdots & 1 - g_d^2 \end{bmatrix}.$$

Let (i, j) and (j, i) be the indices of 1 and -1 in $\Psi_{(j)}$ (the mappings $j \rightarrow (i, j)$ are bijective for given d). One can easily obtain that

$$(\Psi_{(j)} g)(\Psi_{(j)} g)^\top = \begin{bmatrix} & i & & j & \\ \ddots & & & & \\ & g_j^2 & & -g_j g_i & \\ & & \ddots & & \\ -g_i g_j & & & g_i^2 & \\ & & & & \ddots \end{bmatrix} \begin{matrix} i \\ \\ j \\ \\ \end{matrix}.$$

Summing them together and combining with $g^\top g = 1$ complete the proof. ■

Lemma IV.6. *Given a framework $(\mathcal{G}, p)$ and a generalized angle rigidity matrix $R_A(p)$, i.e., d is not restricted to 2, it always holds that*

$$R_A(p) = \sum_{j=1}^{\frac{d(d-1)}{2}} \text{Diag}\left(\frac{g_{ik}^\top \Psi_{(j)} g_{ij} \|e_{ik}\|^2}{\|e_{ij}\|^2}\right) R(p) (I_N \otimes \Psi_{(j)}^\top), \quad (25)$$

where $(i, j, k) \in \bar{\mathcal{E}}$.

Proof. Consider an arbitrary row of $R_{D_0}(p)$, i.e., $\nu_\ell^\top = [\mathbf{0}, \frac{1}{\|e_{ik}\|^2}, \mathbf{0}, -\frac{\|e_{ij}\|^2}{\|e_{ik}\|^4}, \mathbf{0}]$, $\ell \in \{1, \dots, w\}$, then its respective row of $R(p)$ can be represented by $\nu_\ell^\top R_D$. According to angle rigidity theory, the same row of $R_A(p)$ can be represented by

$$\begin{aligned} \nu_{A_\ell}^\top R_B &= \nu_{A_\ell}^\top \text{Diag}(P_{g_{ij}} / \|e_{ij}\|) H^\otimes \\ &= \nu_{A_\ell}^\top \text{Diag}\left(\sum_{j=1}^{\frac{d(d-1)}{2}} (\Psi_{(j)} g_{ij})(\Psi_{(j)} g_{ij})^\top / \|e_{ij}\|\right) H^\otimes \\ &= \sum_{j=1}^{\frac{d(d-1)}{2}} \nu_{A_\ell}^\top \text{Diag}\left(\frac{(\Psi_{(j)} g_{ij})(\Psi_{(j)} g_{ij})^\top}{\|e_{ij}\|}\right) H^\otimes \\ &=: \sum_{j=1}^{\frac{d(d-1)}{2}} f_{j1} f_{j2}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} f_{j1} &= \nu_{A_\ell}^\top (I_m \otimes \Psi_{(j)}) \text{Diag}\left(\frac{g_{ij}}{\|e_{ij}\|^2}\right) \\ &= [\mathbf{0}, g_{ik}^\top, \mathbf{0}, g_{ij}^\top, \mathbf{0}] (I_m \otimes \Psi_{(j)}) \text{Diag}\left(\frac{g_{ij}}{\|e_{ij}\|^2}\right) \\ &= [\mathbf{0}, g_{ik}^\top \Psi_{(j)} g_{ij} / \|e_{ij}\|^2, \mathbf{0}, g_{ij}^\top \Psi_{(j)} g_{ik} / \|e_{ik}\|^2, \mathbf{0}] \\ &= \frac{g_{ik}^\top \Psi_{(j)} g_{ij} \|e_{ik}\|^2}{\|e_{ij}\|^2} \nu_l^\top, \end{aligned} \quad (27)$$

and

$$\begin{aligned} f_{j2} &= \text{Diag}(e_{ij}^\top) (I_m \otimes \Psi_{(j)}^\top) (H \otimes I_d) \\ &= E^\top (H \otimes I_d) (I_N \otimes \Psi_{(j)}^\top) \\ &= R_D(I_N \otimes \Psi_{(j)}^\top), \end{aligned} \quad (28)$$

in which $\nu_{A_\ell}^\top$ is a row in R_g with the same row number as $\nu_\ell^\top$ in R_{D_0} . The last equality of (27) comes from $g_{ik}^\top \Psi_{(j)} g_{ij} = g_{ij}^\top \Psi_{(j)}^\top g_{ik} = -g_{ij}^\top \Psi_{(j)} g_{ik}$. Combine (27) and (28), one has

$$\begin{aligned} R_A(p) &= [\nu_{A_1}, \dots, \nu_{A_w}]^\top R_B \\ &= \sum_{j=1}^{\frac{d(d-1)}{2}} \text{Diag}\left(\frac{g_{ik}^\top \Psi_{(j)} g_{ij} \|e_{ik}\|^2}{\|e_{ij}\|^2}\right) R_{D_0} R_D (I_N \otimes \Psi_{(j)}^\top) \\ &= \sum_{j=1}^{\frac{d(d-1)}{2}} \text{Diag}\left(\frac{g_{ik}^\top \Psi_{(j)} g_{ij} \|e_{ik}\|^2}{\|e_{ij}\|^2}\right) R(p) (I_N \otimes \Psi_{(j)}^\top). \end{aligned}$$

The relationship between $\text{Rank}(R(p))$ and $\text{Rank}(R_A(p))$ is difficult to obtain from the equality (25) for $d \geq 3$ since a summation operation is involved. However, we have following results for $d = 2$.

Theorem IV.7. *For any framework $(\mathcal{G}, p)$ in $\mathbb{R}^2$, $\text{Rank}(R(p)) \geq \text{Rank}(R_A(p))$. If the framework is non-degenerate, i.e., p_i, p_j and p_k are not collinear $\forall (i, j, k) \in \bar{\mathcal{E}}$, $\text{Rank}(R(p)) = \text{Rank}(R_A(p))$.*

Proof. For $d = 2$, $\Psi_{(j)} = J$. Denote $g_{ij}^\perp = J g_{ij}$ for $(i, j) \in \mathcal{E}$. From (25), one has

$$R_A(p) = \text{Diag}\left(\frac{g_{ik}^\top g_{ij}^\perp \|e_{ik}\|^2}{\|e_{ij}\|^2}\right) R(p) (I_N \otimes J^\top). \quad (29)$$

Thus, $\text{Rank}(R_A(p)) \leq \min(\text{Rank}(\text{Diag}(\frac{g_{ik}^\top g_{ij}^\perp \|e_{ik}\|^2}{\|e_{ik}\|^2})), \text{Rank}(R(p))$. If $(\mathcal{G}, p)$ is non-degenerate, the square matrix $\text{Diag}(\frac{g_{ik}^\top g_{ij}^\perp \|e_{ik}\|^2}{\|e_{ik}\|^2})$ is invertible, thus we have $\text{Rank}(R(p)) = \text{Rank}(R_A(p))$. ■

A direct consequence from (29) is the following result.

Corollary IV.8. *For any non-degenerate framework $(\mathcal{G}, p)$ in $\mathbb{R}^2$, the vector δ_p is an infinitesimal RoD motion if and only if $(I_N \otimes J)\delta_p$ is an infinitesimal angle motion.*

Remark IV.9. 1) *For a degenerate framework, nontrivial infinitesimal angle motions may have more dimensions than nontrivial infinitesimal RoD motions (See Fig. 3).*

2) *The procedure for decomposing bearing rigidity matrix shown in (27) and (28) can be adopted to establish the connection between distance and bearing rigidity matrices, which has been analyzed in $\mathbb{R}^2$ [6]. It can be obtained that*

$$R_B(p) = \sum_{j=1}^{\frac{d(d-1)}{2}} \text{Diag}\left(\frac{\Psi_{(j)} g_{ij}}{\|e_{ij}\|^2}\right) R_D(p) (I_N \otimes \Psi_{(j)}^\top).$$

However, the existing result on the rank properties of matrix involving summation is still insufficient, rendering the connection in high dimensional space ($d > 2$) difficult to be analyzed.

Theorem IV.10. *For any framework $(\mathcal{G}, p)$ in $\mathbb{R}^2$, the following statements are equivalent:*

- 1) $(\mathcal{G}, p)$ is infinitesimally distance rigid;
- 2) $(\mathcal{G}, p)$ is infinitesimally bearing rigid;

- 3) $(\mathcal{G}, p)$ is infinitesimally angle rigid;
- 4) $(\mathcal{G}, p)$ is infinitesimally RoD rigid.

Proof. The proof of equivalence of 1) and 4) has been provided in Theorem IV.1). The equivalence of 1) and 2) can be found in [6]. The equivalence of 2) and 3) can refer to [22]. Then the proof is complete. ■

V. RoD-BASED SIMILAR FORMATION CONTROL

After answering the question whether a framework can be uniquely determined by a set of RoD constraints up to translation, rotation, scaling and reflection, we now apply the RoD constraints to the formation stabilization problem. Consider a team of agents with dynamics modeled by single-integrator kinematics:

$$\dot{p}_i = u_i, \forall i \in \mathcal{V}, \quad (30)$$

where $p_i, u_i \in \mathbb{R}^d$ ($d = 2, 3$) denote the position and control input (i.e., velocity) of agent i in the global coordinate system, respectively.

For a desired position set $p^* = [p_1^{*\top}, \dots, p_N^{*\top}]^\top$, the desired formation shape is defined as:

$$\begin{aligned} \mathcal{T} &= \{p \in \mathbb{R}^{Nd} : s(I_d \otimes \bar{\mathcal{R}})p^* + \mathbf{1}_N \otimes \mathbf{t}, \bar{\mathcal{R}} \in O(d), \\ &\quad s \in \mathbb{R} \setminus \{0\}, \mathbf{t} \in \mathbb{R}^d\}. \end{aligned} \quad (31)$$

In addition, a desired equilibrium set $\mathcal{T}_e$ is defined as:

$$\mathcal{T}_e = \{p \in \mathbb{R}^{Nd} : \frac{\|e_{ij}\|^2}{\|e_{ik}\|^2} = \frac{\|e_{ij}^*\|^2}{\|e_{ik}^*\|^2}, \forall (i, j, k) \in \bar{\mathcal{E}}^*\}, \quad (32)$$

where $\|e_{ij}\|^2 = \|e_{ij}(p)\|^2$, $\|e_{ij}^*\|^2 = \|e_{ij}(p^*)\|^2$, $\forall (i, j) \in \mathcal{E}$. It is worth mentioning that it always holds $\mathcal{T} \subset \mathcal{T}_e$. $\mathcal{T} = \mathcal{T}_e$ if and only if the framework $(\mathcal{G}, p^*)$ is globally RoD rigid. In bearing rigidity theory, where infinitesimally bearing rigidity implies global bearing rigidity [6], the controller motivated by infinitesimal bearing rigidity can achieve desired formation almost globally. However, similar to distance and angle rigidity theories, infinitesimally RoD rigidity does not imply global RoD rigidity. Therefore, a controller derived from rigidity matrix must not guarantee global convergence to $\mathcal{T}$ while it can provide local stability of $\mathcal{T}$ if $\mathcal{T}_e$ is stable.

Definition V.1. *The RoD constraints $Q_{\mathcal{G}^*}(p) = \{\frac{1}{2} \frac{\|e_{ij}(p^*)\|^2}{\|e_{ik}(p^*)\|^2}\}_{(i,j,k) \in \bar{\mathcal{E}}^*}$ is said to be feasible if there exists some $q \in \mathbb{R}^{Nd}$ such that $\frac{\|e_{ij}(q)\|^2}{\|e_{ik}(q)\|^2} = \frac{\|e_{ij}(p)\|^2}{\|e_{ik}(p)\|^2}$ for all $(i, j, k) \in \bar{\mathcal{E}}^*$.*

The feasible RoD constraints can be computed from the desired position set p^* or arbitrary element which belongs to the desired formation shape set $\mathcal{T}$. Then we formulate the RoD-based formation control problem as follows.

Problem V.2. *Given feasible RoD constraints $Q_{\mathcal{G}^*}(p^*) = \{\frac{1}{2} \frac{\|e_{ij}(p^*)\|^2}{\|e_{ik}(p^*)\|^2}\}_{(i,j,k) \in \bar{\mathcal{E}}^*}$ and the initial position $p(0)$, design a distributed controller u_i , $\forall i \in \mathcal{V}$, where only local relative position measurements $\{p_i^i - p_j^j, j \in \mathcal{N}_i\}$ are available, such that $\frac{\|e_{ij}(p(t))\|^2}{\|e_{ik}(p(t))\|^2} = \frac{\|e_{ij}(p^*)\|^2}{\|e_{ik}(p^*)\|^2}$ as $t \rightarrow +\infty$ for all $(i, j, k) \in \bar{\mathcal{E}}^*$.*

A. Controller design

For the sake of simplicity, we denote $\delta_{ijk} = \frac{1}{2}(\frac{\|e_{ij}\|^2}{\|e_{ik}\|^2} - \frac{\|e_{ji}^*\|^2}{\|e_{ik}^*\|^2})$, $\forall (i, j, k) \in \bar{\mathcal{E}}^*$.

Consider the following potential function:

$$V(p) = \frac{1}{2} \sum_{(i,j,k) \in \bar{\mathcal{E}}^*} \delta_{ijk}^2, \quad (33)$$

a gradient descent control law can be derived as:

$$\begin{aligned} u_i &= -\nabla_{p_i} V(p) \\ &= - \sum_{(j,k) \in \mathcal{N}_{\mathcal{G}_{i1}}} u_{ijk} - \sum_{(j,k) \in \mathcal{N}_{\mathcal{G}_{i2}}} u_{jik} - \sum_{(j,k) \in \mathcal{N}_{\mathcal{G}_{i3}}} u_{jki}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \mathcal{N}_{\mathcal{G}_{i1}} &= \{(j, k) \in \mathcal{V}^2 : (i, j, k) \in \bar{\mathcal{E}}^*\}, \\ \mathcal{N}_{\mathcal{G}_{i2}} &= \{(j, k) \in \mathcal{V}^2 : (j, i, k) \in \bar{\mathcal{E}}^*\}, \\ \mathcal{N}_{\mathcal{G}_{i3}} &= \{(j, k) \in \mathcal{V}^2 : (j, k, i) \in \bar{\mathcal{E}}^*\}, \\ u_{ijk} &= 2\delta_{ijk} \left(\frac{1}{\|e_{ik}\|^2} e_{ij} - \frac{\|e_{ij}\|^2}{\|e_{ik}\|^4} e_{ik} \right), \\ u_{jik} &= 2\delta_{jik} \left(\frac{1}{\|e_{jk}\|^2} e_{ij}, \right. \\ u_{jki} &= 2\delta_{jki} \left(-\frac{\|e_{jk}\|^2}{\|e_{ji}\|^4} e_{ij} \right), \end{aligned}$$

in which $e_{jk} = p_j - p_k = e_{ik} - e_{ij}$ can be computed by agent i locally.

It should be noted that (j, i, k) and (j, k, i) share the same controller term in [22] since $g_{ji}^\top g_{jk} = g_{jk}^\top g_{ji}$. However, $\frac{\|e_{ji}\|^2}{\|e_{jk}\|^2} \neq \frac{\|e_{jk}\|^2}{\|e_{ji}\|^2}$ in general. Therefore, we should consider (j, i, k) and (j, k, i) separately in $\mathcal{N}_{\mathcal{G}_{i2}}$ and $\mathcal{N}_{\mathcal{G}_{i3}}$, respectively.

Not surprisingly, the RoD-based controller (34) inherits the same properties of the distance-based controller. Define the overall linear momentum as $\mathbb{L} = \sum_{i \in \mathcal{V}} \dot{p}_i$ and the overall angular momentum as $\mathbb{A} = \sum_{i \in \mathcal{V}} (J(p_i - p_c))^\top (\dot{p}_i - \dot{p}_c)$ for $d = 2$ and $\mathbb{A} = \sum_{i \in \mathcal{V}} (p_i - p_c) \times (\dot{p}_i - \dot{p}_c)$ for $d = 3$, where $p_c = \sum_{i \in \mathcal{V}} p_i / N$ [25]. Denote $\delta = \{\delta_{ijk}\}_{(i,j,k) \in \bar{\mathcal{E}}^*}$, thus the concatenated form of $p_i, \forall i \in \mathcal{V}$ can be expressed as:

$$\dot{p} = -(\nabla_p V) = -R^\top(p) \delta. \quad (35)$$

Theorem V.3. *Under the control law (34), the following statements hold:*

- 1) *it can be implemented using only its local frame;*
- 2) *the overall linear momentum $\mathbb{L} \equiv \mathbf{0}$ (the centroid of p is invariant) and the overall angular momentum $\mathbb{A} \equiv \mathbf{0}$.*

Proof. The validity of 1) can be easily checked via the method adopted in Lemma 3 of [39].

The proof of 2) is provided as below. From $p_c = (\mathbf{1}_N \otimes I_d)^\top p / N$, one has

$$\begin{aligned} \dot{p}_c &= (\mathbf{1}_N \otimes I_d)^\top \dot{p} / N \\ &= -(\mathbf{1}_N \otimes I_d)^\top R^\top \delta / N \\ &= -(R_{D_0} E^\top (H \otimes I_d) (\mathbf{1}_N \otimes I_d))^\top \delta / N \\ &= \mathbf{0}. \end{aligned} \quad (36)$$

For $d = 2$, the overall angular momentum

$$\begin{aligned} \mathbb{A} &= ((I_N \otimes J)(p - \mathbf{1}_N \otimes p_c))^\top (\dot{p} - \mathbf{1}_N \otimes \dot{p}_c) \\ &= ((I_N \otimes J)(p - \mathbf{1}_N \otimes p_c))^\top R^\top \delta \\ &= (R_{D_0} E^\top (H \otimes I_2) (I_N \otimes J)(p - \mathbf{1}_N \otimes p_c))^\top \delta \\ &= \mathbf{0}. \end{aligned} \quad (37)$$

For $d = 3$, $a \times b = [-(\Psi_{(3)} a)^\top b, (\Psi_{(2)} a)^\top b, -(\Psi_{(1)} a)^\top b]^\top$ for $a, b \in \mathbb{R}^3$, the overall angular momentum

$$\begin{aligned} \mathbb{A} &= \begin{bmatrix} (p - \mathbf{1}_N \otimes p_c)^\top (I_N \otimes \Psi_{(3)}) (\dot{p} - \mathbf{1}_N \otimes \dot{p}_c) \\ -(p - \mathbf{1}_N \otimes p_c)^\top (I_N \otimes \Psi_{(2)}) (\dot{p} - \mathbf{1}_N \otimes \dot{p}_c) \\ (p - \mathbf{1}_N \otimes p_c)^\top (I_N \otimes \Psi_{(1)}) (\dot{p} - \mathbf{1}_N \otimes \dot{p}_c) \end{bmatrix} \\ &= \begin{bmatrix} (p - \mathbf{1}_N \otimes p_c)^\top (I_N \otimes \Psi_{(3)}) \\ -(p - \mathbf{1}_N \otimes p_c)^\top (I_N \otimes \Psi_{(2)}) \\ (p - \mathbf{1}_N \otimes p_c)^\top (I_N \otimes \Psi_{(1)}) \end{bmatrix} R^\top \delta \\ &= \mathbf{0}. \end{aligned} \quad (38)$$

This completes the proof. ■

B. Stability analysis

Firstly, two necessary assumptions are presented.

Assumption V.4. *For any three agents i, j and k such that $(i, j, k) \in \bar{\mathcal{E}}^*$, $p_i^* \neq p_j^* \neq p_k^*$. Moreover, $p_i(t) \neq p_j(t) \neq p_k(t)$ always holds during motion.*

Remark V.5. *Assumption V.4 ensures RoDs in (34) are always well defined during motion. This assumption can be fulfilled by installing inexpensive off-the-shelf proximity sensors or designing low-level control algorithms for collision avoidance.*

Assumption V.6. *The RoD constraints $\{\frac{1}{2} \frac{\|e_{ij}(p^*)\|^2}{\|e_{ik}(p^*)\|^2}\}_{(i,j,k) \in \bar{\mathcal{E}}^*}$ ensure minimal and infinitesimal RoD rigidity.*

Theorem V.7. *Consider a team of agents with dynamics (30) and the RoD-based controller (34). Under Assumption V.6, for any $q \in \mathcal{T}$, there is a neighborhood U_q of q , such that if $p(0) \in U_q$, then $p(t)$ exponentially converges to some $p^\infty \in \mathcal{T}$. Furthermore, the formation scale $s(t) = \|p - \mathbf{1}_N \otimes p_c\| / \sqrt{N}$ is invariant.*

Proof. The proof of the first part is very similar to [5], [22] via center manifold theory [40], thus not provided here.

For the second statement, one has

$$\begin{aligned} \dot{s}(t) &= \frac{(p - \mathbf{1}_N \otimes p_c)^\top \dot{p}}{\sqrt{N} \|p - \mathbf{1}_N \otimes p_c\|} \\ &= \frac{\delta^\top R(p) p}{\sqrt{N} \|p - \mathbf{1}_N \otimes p_c\|} \\ &= 0, \end{aligned} \quad (39)$$

where the last equality follows from Lemma III.6. This completes the proof. ■

Remark V.8. *Noting that if the desired position is infinitesimally angle rigid, it holds that the overall angular momentum is invariant and equals $\mathbf{0}$ for the angle-based formation controller in [22] since R and R_A share the same null space in $\mathbb{R}^2$.*

Fig. 7: The state trajectories of agents in $\mathbb{R}^2$.

Fig. 8: The scale variations of team in $\mathbb{R}^2$.

Fig. 9: The evolution of $V(t)$ in $\mathbb{R}^2$.

Remark V.9. In fact, the relaxed scale freedom from our RoD representation can be controlled actively via adding leader nodes. In particular, 2 leaders and 3 or 4 leaders can be added to control the position, orientation, scale and their compositions of the entire network in 2-D space [22], [41] and 3-D space [17], [39], respectively.

VI. SIMULATION

In this section, we present two simulation examples in $\mathbb{R}^2$ and $\mathbb{R}^3$. Consider a team of $N = 6$ ($N = 7$) agents in $\mathbb{R}^2$ ($\mathbb{R}^3$) with dynamics (30) and distributed controller (34). The desired shapes are set as planar and stereo heart shapes. The edges of the derived graph $\bar{\mathcal{G}}$ from $\mathcal{G}$ are defined by following (14) and (16), respectively. The initial position of the agents is given as $p(0) = p^* + p_{pert}$, where $p_{pert} \in \mathbb{R}^{12}$ ($p_{pert} \in \mathbb{R}^{21}$) is a random vector in box $[0, 2]^{12}$ ($[0, 2]^{21}$). Fig. 7 shows the trajectories of agents under both the RoD-based and distance-based formation controllers. It can be seen that the final configuration is similar to the desired configuration, which supports our theoretical results. Furthermore, the scale variations of the team under both controllers are computed and plotted in Fig. 8. Fig. 9 records the evolution of potential function (33) during agents' motion. It can be seen that $V(t)$ converges to 0 exponentially. The figures of scale variations and evolution of potential function in $\mathbb{R}^3$ resemble Fig. 8 and Fig. 9 and thus omitted.

VII. CONCLUSION AND FUTURE WORK

In this paper, we have introduced a novel RoD rigidity theory to investigate whether a set of RoD constraints can determine a unique shape up to translation, rotation, scaling and reflection. The connections to existing distance, bearing and angle rigidity theories have been analyzed. The equivalence among four kinds of infinitesimal rigidities for planar frameworks have been established through the proven equivalence between distance and RoD rigidity in arbitrary dimensions. The relationships between distance and bearing rigidity matrices, angle and RoD rigidity matrices in arbitrary dimensions have been explored through a decomposition of the orthogonal projection operator matrix. We have introduced two classes of graphs, trilaterated Laman graphs in 2-D space and quadrilaterated Geiringer graphs in 3-D space, whose frameworks can be uniquely determined by a set of RoD constraints and under which the graphical conditions for the minimal infinitesimal RoD rigidity and the global RoD rigidity have been explicitly given. Based on the proposed RoD rigidity theory, a distributed RoD-based formation controller is proposed and proven to address the similar formation problem in both 2-D and 3-D spaces with local exponential stability. Finally, the theoretical analysis has been verified by some simulation examples.

Future works can be on the control of the relaxed scale freedom via leader nodes and global stabilization of RoD-based controller.

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Fig. 10: The state trajectories of agents in $\mathbb{R}^3$.

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