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On the Smith normal form of a skew-symmetric D-optimal design of order $n \equiv 2 \pmod{4}$

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Abstract

We show that the Smith normal form of a skew-symmetric D-optimal design of order $n \equiv 2 \pmod{4}$ is determined by its order. Furthermore, we show that the Smith normal form of such a design can be written explicitly in terms of the order n , thereby proving a recent conjecture of Armario. We apply our result to show that certain D-optimal designs of order $n \equiv 2 \pmod{4}$ are not equivalent to any skew-symmetric D-optimal design. We also provide a correction to a result in the literature on the Smith normal form of D-optimal designs.

1 Introduction

The classification of $\{1, -1\}$ -matrices of order n having largest possible determinants is one of the most important problems in design theory. Such matrices are known as **D-optimal designs**. D-optimal designs of order $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$ are, respectively, called Hadamard matrices and EW matrices. For such matrices, the *Smith normal form* is a useful invariant that is

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used to distinguish between those of the same order. The Smith normal form of a skew-symmetric Hadamard matrix is well-known [13, 8]. In this paper we determine the Smith normal form of skew-symmetric EW matrices.

Two $n \times n$ integer matrices M and N are called **equivalent** if there exist $n \times n$ unimodular integer matrices P and Q such that $N = PMQ$.

Let M be an $n \times n$ integer matrix of rank r . Then M is equivalent to a diagonal integer matrix

$$\text{diag}[m_1, m_2, \dots, m_r, m_{r+1}, \dots, m_n],$$

satisfying $m_i \mid m_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $m_{r+1} = 0$. This diagonal matrix is called the **Smith normal form** of M and the values $m_1, \dots, m_n$ are called the **invariant factors** of A .

Throughout, I_n , J_n , and O_n will (respectively) denote the $n \times n$ identity matrix, all-ones matrix, and all-zeros matrix. We omit the subscript when the order is apparent and the matrices denoted by J and O may not necessarily be square matrices. We use $\mathbf{0}_t$ and $\mathbf{1}_t$ to denote the length- t all-zeros and all-ones (column) vectors respectively and we omit the subscript when the length is apparent.

Let X be a $\{1, -1\}$ -matrix of order n . Abusing language, we call X **skew-symmetric** if $X + X^\top = 2I$, i.e., the matrix $X - I$ is skew-symmetric in the usual sense: $X - I + (X - I)^\top = O$.

A famous inequality due to Hadamard [9] is the following:

$$|\det(X)| \leq n^{n/2}.$$

If equality holds then X is called a **Hadamard matrix**. Hadamard's inequality can be improved if we restrict to matrices whose orders are not divisible by 4. Indeed, Ehlich [6] and Wojtas [17] independently showed that for a $\{1, -1\}$ -matrix X of order $n \equiv 2 \pmod{4}$, Hadamard's inequality can be strengthened to

$$|\det(X)| \leq 2(n-1)(n-2)^{(n-2)/2}. \quad (1)$$

Moreover, there exists a $\{1, -1\}$ -matrix achieving equality in (1) if and only if there exists a $\{1, -1\}$ -matrix B such that

$$BB^\top = B^\top B = \begin{pmatrix} (n-2)I + 2J & O_{n/2} \\ O_{n/2} & (n-2)I + 2J \end{pmatrix}. \quad (2)$$

A $\{1, -1\}$ -matrix B of order n is called an **EW matrix** if it satisfies (2). Hence, in particular, an EW matrix of order n has determinant $\pm 2(n-1)(n-2)^{(n-2)/2}$. Working up to equivalence, we may assume that an EW matrix has a positive determinant. Accordingly, throughout this paper, we assume that an EW matrix S of order $4t+2$ has determinant $\det(S) = (4t)^{2t}(8t+2)$. We refer to the *Handbook of Combinatorial Designs* [5] for background on Hadamard and EW matrices.

The Smith normal form of skew-symmetric Hadamard matrices of order $4t$ is uniquely determined [13, 8].

Theorem 1.1. *A skew-symmetric Hadamard matrix of order $4t$ has the Smith normal form*

$$\text{diag}[1, \underbrace{2, \dots, 2}_{2t-1}, \underbrace{2t, \dots, 2t}_{2t-1}, 4t].$$

Armario [3] partially determined invariant factors of a skew-symmetric EW matrix of order $4t + 2$.

Theorem 1.2. [3, Theorem 1.2] *Let $s_1, \dots, s_{4t+2}$ be the invariant factors of a skew-symmetric EW matrix of order $4t+2$. Then $s_1 = 1$ and $s_2 = \dots = s_{2t+1} = 2$.*

Further, Armario [3] conjectured the value of the remaining invariant factors of a skew-symmetric EW matrix. In this note we prove his conjecture. The following is our main theorem.

Theorem 1.3. *A skew-symmetric EW matrix of order $4t+2$ has Smith normal form*

$$\text{diag}[1, \underbrace{2, \dots, 2}_{2t+1}, \underbrace{2t, \dots, 2t}_{2t-1}, 2t(4t+1)].$$

Now we provide an outline of the proof. Let S be a skew-symmetric EW matrix of order $n = 4t + 2$. In particular, we have that

$$SS^\top = \begin{pmatrix} (n-2)I + 2J & O \\ O & (n-2)I + 2J \end{pmatrix}. \quad (3)$$

Lemma 1.4. *Let S be a skew-symmetric EW matrix of order $n = 4t + 2$ and let $s_1, \dots, s_{4t+2}$ be the invariant factors of S .*

$$(i) \quad s_{2t+2} = 2.$$

$$(ii) \quad s_{4t+2} = 2t(4t+1).$$

We will give the proof of Lemma 1.4 in subsequent sections. Using Lemma 1.4 together with Theorem 1.2, we can determine the invariant factors of a skew-symmetric EW matrix. Indeed, we prove Theorem 1.3 as follows.

Proof of Theorem 1.3. Since the other invariant factors have been determined in Theorem 1.2 and Lemma 1.4, it remains to show $s_{2t+3} = \dots = s_{4t+1} = 2t$. Using (2), we see that $\det(S) = s_1 \cdots s_{4t+2} = 2^{2t+1}(2t)^{2t}(4t+1)$. Hence, by Theorem 1.2 and Lemma 1.4, we have

$$s_{2t+3} \cdots s_{4t+1} = (2t)^{2t-1}. \quad (4)$$

Claim 1: s_{4t+1} divides $2t$. Since $s_2 = 2$ divides s_i for each $i \geq 2$, we may set $s_i = 2s'_i$ for $2 \leq i \leq 4t+2$. We have s'_{4t+1} divides $s'_{4t+2} = t(4t+1)$ and $\gcd(t, 4t+1) = 1$. Furthermore, since $4s'_{4t+1}s'_{4t+2} = 4s'_{4t+1}t(4t+1)$ divides $2(4t+1)(4t)^{2t}$ we must have that s'_{4t+1} divides t . Therefore s_{4t+1} is a divisor of $2t$.

Since s_i divides s_{i+1} for each i and $s_{4t+1} \leq 2t$, by Claim 1, we have that $s_i \leq 2t$ for $i = 2t+3, \dots, 4t$, from which it follows that

$$s_{2t+3} \cdots s_{4t+1} \leq (2t)^{2t-1}. \quad (5)$$

By (4), equality must hold in (5). Therefore $s_{2t+3} = \cdots = s_{4t+1} = 2t$. $\square$

Our methods for proving Lemma 1.4 involve a tournament matrix that one associates with an EW matrix, called an *EW tournament matrix* (see Section 2 for its definition). Even though it is not needed for the proof of Lemma 1.4, we also determine the Smith normal form of an EW tournament matrix (see Theorem 8.1).

The organisation of this paper is as follows. In Section 2 we prepare the basic tools for the proof of our main result. We will prove Lemma 1.4 (i) and (ii) in Section 3 and Section 4, respectively. In Section 5, we provide examples of EW matrices with Smith normal forms different from Theorem 1.3. Note that the EW matrix in Example 5.2 is a counterexample to [12, Corollary 2]. We attempt to deal with the counterexample to [12, Corollary 2]. In Section 6 we provide a revised result to [12, Corollary 2], and we pursue the method demonstrated in Section 6 to obtain a result for the case that $4t+1$ is a square of a prime in Section 7. Finally, in Section 8 we determine the Smith normal form of an EW tournament matrix.

2 Preliminaries

In this section we collect some facts about skew-symmetric EW matrices and their invariant factors, which we will use in later sections.

Let X be an EW matrix of order n . It is well-known that $2n-2$ must be a sum of squares [5]. Furthermore, Armario and Frau [2] showed that $2n-3$ must be a square if X is skew-symmetric. Indeed, they showed the following.

Lemma 2.1. [2, Proposition 2.4] *Suppose there exists a skew-symmetric EW matrix of order $4t+2$. Then $\sqrt{8t+1}$ is an odd integer.*

A **tournament matrix** is a $\{0,1\}$ -matrix A such that $A + A^\top = J - I$. A tournament matrix A of order $4t+1$ is called an **EW tournament matrix** if

for some $a \in \mathbb{N} \cup \{0\}$,

$$AA^\top = t(I_{4t+1} + J_{4t+1}) + \begin{pmatrix} -J_t & -J & -J & -J \\ -J & J_t & O & O \\ -J & O & O_a & -J \\ -J & O & -J & O_{2t+1-a} \end{pmatrix}, \quad (6)$$

$$A^\top A = t(I_{4t+1} + J_{4t+1}) + \begin{pmatrix} J_t & -J & O & O \\ -J & -J_t & -J & -J \\ O & -J & J_a & O \\ O & -J & O & -J_{2t+1-a} \end{pmatrix}. \quad (7)$$

Since $A + A^\top = J - I$, we have that

$$(A + I)(A^\top + I) = t(I_{4t+1} + J_{4t+1}) + \begin{pmatrix} O_t & O & O & O \\ O & 2J_t & J & J \\ O & J & J_a & O \\ O & J & O & J_{2t+1-a} \end{pmatrix}, \quad (8)$$

$$(A^\top + I)(A + I) = t(I_{4t+1} + J_{4t+1}) + \begin{pmatrix} 2J_t & O & J & J \\ O & O_t & O & O \\ J & O & 2J_a & J \\ J & O & J & O_{2t+1-a} \end{pmatrix}. \quad (9)$$

A skew-symmetric EW matrix S corresponds to an EW tournament matrix A via the equation [1]:

$$S = \begin{pmatrix} 1 & \mathbf{1}^\top \\ -\mathbf{1} & I + A - A^\top \end{pmatrix}. \quad (10)$$

The invariant factors of S and those of $A + I$ are related as follows.

Lemma 2.2. [3, Lemma 2.2, Corollary 2.3] *Let S be a skew-symmetric EW matrix of order $4t + 2$ and let A be the EW tournament matrix of order $4t + 1$ obtained by (10). Let $s_1, \dots, s_{4t+2}$ be the invariant factors of S and $b_1, \dots, b_{4t+1}$ be the invariant factors of $A + I$. Then*

$$(i) \quad s_{i+1} = 2b_i \text{ for } i = 1, \dots, 4t + 1.$$

$$(ii) \quad \det(A + I) = t^{2t}(4t + 1).$$

Proof. First observe that S is equivalent to $[1] \oplus 2(A + I)$. Indeed, in (10), add the first row to all other rows and subtract the first column from all other columns. Using this equivalence, we see that $s_{i+1} = 2b_i$ for $i = 1, \dots, 4t + 1$ and $\det(A + I) = (1/2^{4t+1}) \det(S) = t^{2t}(4t + 1)$. $\square$

Let M be an integer matrix. The invariant factors of M can be determined from the the greatest common divisor of all minors of M . Indeed, define $d_i(M)$ to be the greatest common divisor of all $i \times i$ minors of M and set $d_0(M) = 1$. Then the next result is standard.

Lemma 2.3 (Corollary 1.20 of [15]). *Let M be an integer matrix of order n and rank r with invariant factors $m_1, \dots, m_n$. Then $m_i = d_i(M)/d_{i-1}(M)$ for $i = 1, \dots, r$.*

Let M be an invertible $n \times n$ matrix with rows and columns indexed by $\{1, \dots, n\}$. For a subset $I \subseteq \{1, \dots, n\}$, define $I' = \{1, \dots, n\} \setminus I$. For nonempty subsets $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J|$, we denote by $M_{I,J}$ the matrix obtained from M by restricting rows to I and columns to J respectively.

Lemma 2.4 (Page 21 of [10]). *Let M be an invertible $n \times n$ matrix with rows and columns indexed by $\{1, \dots, n\}$. Suppose $I, J \subseteq \{1, \dots, n\}$ satisfy $|I| = |J| \neq 0$. Then*

$$\det(M_{I,J}) = \pm \det(M) \det((M^{-1})_{J',I'}).$$

Now we record a result about the determinant of an EW tournament matrix. It is a corollary of [7, Theorem 1.1],

Lemma 2.5. *Let A be an EW tournament matrix of order $4t + 1$. Then $\det(A) = t^{2t}(4t - 1)$.*

Since skew-symmetric EW matrices are in correspondence with EW tournament matrices, the following result implies Lemma 2.1. We will use Theorem 2.6 in Section 8.

Theorem 2.6. *Let A be an EW tournament matrix of order $4t + 1$. Then the integer a in (6) satisfies the quadratic equation $a^2 - (2t + 1)a + t(t - 1) = 0$.*

Proof. Begin with (6), which gives an expression for $AA^\top$. Using elementary row operations and properties of the determinant, we find that

$$\begin{aligned} \det(AA^\top) &= \det \begin{pmatrix} tI_t + (t-1)J_t & (t-1)J & (t-1)J & (t-1)J \\ (t-1)J & tI_t + (t+1)J_t & tJ & tJ \\ (t-1)J & tJ & tI_a + tJ_a & (t-1)J \\ (t-1)J & tJ & (t-1)J & tI_{2t+1-a} + tJ_{2t+1-a} \end{pmatrix} \\ &= \det \begin{pmatrix} t^2 & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top \\ \mathbf{0} & tI_{t-1} & O & O & O \\ -t\mathbf{1} & O & tI_t + 2J_t & J & J \\ -t\mathbf{1} & O & J & tI_a + J_a & O \\ -t\mathbf{1} & O & J & O & tI_{2t+1-a} + J_{2t+1-a} \end{pmatrix} \\ &= t^{t-1} \det \begin{pmatrix} t^2 & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top \\ -t\mathbf{1} & tI_t + 2J_t & J & J \\ -t\mathbf{1} & J & tI_a + J_a & O \\ -t\mathbf{1} & J & O & tI_{2t+1-a} + J_{2t+1-a} \end{pmatrix} \\ &= t^t \det \begin{pmatrix} t & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top & (t-1)\mathbf{1}^\top \\ -\mathbf{1} & tI_t + 2J_t & J & J \\ -\mathbf{1} & J & tI_a + J_a & O \\ -\mathbf{1} & J & O & tI_{2t+1-a} + J_{2t+1-a} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= t^t \det \begin{pmatrix} t & (3t-1)\mathbf{1}^\top & (2t-1)\mathbf{1}^\top & (2t-1)\mathbf{1}^\top \\ -\mathbf{1} & tI_t & O & O \\ -\mathbf{1} & -J & tI_a & -J \\ -\mathbf{1} & -J & -J & tI_{2t+1-a} \end{pmatrix} \\
&= t^t \det \begin{pmatrix} 4t-1 & (3t-1)\mathbf{1}^\top & (2t-1)\mathbf{1}^\top & (2t-1)\mathbf{1}^\top \\ \mathbf{0} & tI_t & O & O \\ (-2)\mathbf{1} & -J & tI_a & -J \\ (-2)\mathbf{1} & -J & -J & tI_{2t+1-a} \end{pmatrix} \\
&= t^{2t} \det \begin{pmatrix} 4t-1 & (2t-1)\mathbf{1}^\top & (2t-1)\mathbf{1}^\top \\ (-2)\mathbf{1} & tI_a & -J \\ (-2)\mathbf{1} & -J & tI_{2t+1-a} \end{pmatrix} \\
&= t^{2t}(4t-1) \det \left(\begin{pmatrix} tI_a & -J \\ -J & tI_{2t+1-a} \end{pmatrix} + \frac{2(2t-1)}{4t-1} J \right) \\
&= t^{2t}(4t-1) \det \begin{pmatrix} tI_a + \frac{2(2t-1)}{4t-1} J & -\frac{1}{4t-1} J \\ -\frac{1}{4t-1} J & tI_{2t+1-a} + \frac{2(2t-1)}{4t-1} J \end{pmatrix}. \quad (11)
\end{aligned}$$

Next, we use the following general equation obtained by the Schur complement [10, (0.8.5.1)]:

$$\det \begin{pmatrix} \alpha I_a + \beta J_a & \gamma J \\ \gamma J & \alpha I_b + \beta J_b \end{pmatrix} = \alpha^{a+b-2} (\alpha^2 + (a+b)\alpha\beta + ab\beta^2 - ab\gamma^2). \quad (12)$$

Then, by (11) and (12), we have

$$\det(AA^\top) = t^{4t-1} ((3-4t)a^2 + (8t^2 - 2t - 3)a + t(12t^2 - t - 2)). \quad (13)$$

By Lemma 2.5, we have $\det(A) = t^{2t}(4t-1)$. Therefore, using the multiplicativity of the determinant we have

$$\det(AA^\top) = t^{4t}(4t-1)^2.$$

Therefore we can simplify (13) to obtain

$$a^2 - (2t+1)a + t(t-1) = 0,$$

as required. $\square$

3 The $(2t+2)$ -nd invariant factor of skew-symmetric EW matrices

In this section, we provide proof of part (i) of Lemma 1.4. For a prime p and an integer matrix M , let $\text{rank}_p(M)$ denote the rank of M over the field of integers of modulo p . The following was shown in [3].

Lemma 3.1. [3, Lemma 2.4] *Let A be an EW tournament matrix of order $4t+1$ and let p be a prime that divides t . Then $2t \leq \text{rank}_p(A+I) \leq 2t+1$.*

Next we provide a strengthening of the above lemma, which is a key lemma to determine the invariant factor s_{2t+2} .

Lemma 3.2. *Let A be an EW tournament matrix of order $4t + 1$ and let p be a prime that divides t . Then $\text{rank}_p(A + I) = 2t + 1$.*

Proof. First we show that $\text{rank}_p(A + I) \geq 2t + 1$. Let $d = \dim(\text{rowsp}_p(A + I) \cap \text{rowsp}_p(A^\top + I))$ be the dimension of the intersection of row spaces over the field of integers modulo p . Then

$$\begin{aligned} 4t &= \text{rank}_p(J + I) = \text{rank}_p((A + I) + (A^\top + I)) \\ &\leq \text{rank}_p(A + I) + \text{rank}_p(A^\top + I) - d \\ &= 2\text{rank}_p(A + I) - d, \end{aligned}$$

that is, we have that $2t + d/2 \leq \text{rank}_p(A + I)$. Hence it suffices to show that $d \geq 1$.

Using the assumption that p divides t together with (8) and (9), it follows that

$$\begin{aligned} \text{span}\{(\mathbf{0}_t^\top, \mathbf{1}_t^\top, \mathbf{1}_a^\top, \mathbf{0}_{2t+1-a}^\top), (\mathbf{0}_t^\top, \mathbf{1}_t^\top, \mathbf{0}_a^\top, \mathbf{1}_{2t+1-a}^\top)\} &\subseteq \text{rowsp}_p(A + I), \\ \text{span}\{(\mathbf{1}_t^\top, \mathbf{0}_t^\top, \mathbf{1}_a^\top, \mathbf{0}_{2t+1-a}^\top), (\mathbf{1}_t^\top, \mathbf{0}_t^\top, \mathbf{0}_a^\top, \mathbf{1}_{2t+1-a}^\top)\} &\subseteq \text{rowsp}_p(A^\top + I). \end{aligned}$$

Therefore

$$\text{span}\{(\mathbf{0}_t^\top, \mathbf{0}_t^\top, \mathbf{1}_a^\top, -\mathbf{1}_{2t+1-a}^\top)\} \subseteq \text{rowsp}_p(A + I) \cap \text{rowsp}_p(A^\top + I),$$

and thus $d \geq 1$.

The inequality $\text{rank}_p(A + I) \leq 2t + 1$ follows from Lemma 3.1. However, for the sake of completeness, we provide a proof. Combine (8) and Sylvester's rank inequality [10, 0.4.5 Rank inequalities (c)] to obtain

$$\begin{aligned} 2\text{rank}_p(A + I) &= \text{rank}_p(A + I) + \text{rank}_p(A^\top + I) \\ &\leq \text{rank}_p((A + I)(A^\top + I)) + 4t + 1 \\ &= 4t + 3. \end{aligned}$$

Thus we have $\text{rank}_p(A + I) \leq 2t + 1$. □

Proposition 3.3. *Let A be an EW tournament matrix of order $4t + 1$ and let $b_1, \dots, b_{4t+1}$ be the invariant factors of $A + I$. Then $b_{2t+1} = 1$.*

Proof. This result follows from the argument¹ in [3, Proof of Theorem 1.2] by replacing the role of b_{2t} with b_{2t+1} . For the sake of the reader we provide a proof.

Assume that there exists a prime p dividing b_{2t+1} . It follows from Lemma 2.3 and Lemma 2.2 that $b_1 \cdots b_{4t+1} = \det(A + I) = t^{2t}(4t + 1)$. Thus p is either

¹In Armario's proof, he derived " p^{2t+1} divides $4t + 1$ " from the assumption on $b_{2t} = 1$. The result obtained from his argument is, in fact, that " p^{2t+2} divides $4t + 1$ ".

- (1) a divisor of t , or
- (2) not a divisor of t and a divisor of $4t + 1$.

For the case (1), Lemma 3.2 contradicts the fact that

$$\text{rank}_p(A + I) = \max\{i \mid p \text{ does not divide } b_i\} \leq 2t.$$

For the case (2), observe that p is odd and hence $p \geq 3$. Furthermore p is a divisor of b_i for $i = 2t + 1, \dots, 4t + 1$. Thus p^{2t+1} divides $4t + 1$. But this is impossible since $p^{2t+1} > 4t + 1$ for all $t \geq 1$. Therefore we must have $b_{2t+1} = 1$. $\square$

Now we can prove part (i) of Lemma 1.4.

Proof of $s_{2t+2} = 2$. By combining Lemma 2.2 and Proposition 3.3 we obtain $s_{2t+2} = 2$. $\square$

Remark 3.4. The equality $b_{2t+1} = 1$ implies that $b_1 = \dots = b_{2t} = 1$. Thus it follows that $s_2 = \dots = s_{2t+1} = 2$, which was already shown in [3, Theorem 1.2].

4 The $(4t+2)$ -nd invariant factor of skew-symmetric EW matrices

In this section, we will show $s_{4t+2} = 2t(4t+1)$ for a skew-symmetric EW matrix of order $4t+2$.

By Lemma 2.4, it is enough to calculate the entries of $\det(S)S^{-1}$ explicitly. By the equation (3) it is shown that

$$S^{-1} = S^\top \left(\frac{1}{4t}I - \frac{1}{4t(4t+1)} \begin{pmatrix} J & O \\ O & J \end{pmatrix} \right). \quad (14)$$

We use the following lemma.

Lemma 4.1. *Set $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ where each S_{ij} is a $(2t+1) \times (2t+1)$ matrix. Then $S_{11}J = S_{22}J = J$ and $S_{12}J = -S_{21}J = \pm\sqrt{8t+1}J$ hold.*

Proof. The eigenvalues of S are $1 \pm \sqrt{-1}(4t-1)$, $1 \pm \sqrt{-8t-1}$, and the eigenvectors corresponding to $1 \pm \sqrt{-8t-1}$ are $\begin{pmatrix} \pm\sqrt{-1}\mathbf{1} \\ \mathbf{1} \end{pmatrix}$ or $\begin{pmatrix} \mp\sqrt{-1}\mathbf{1} \\ \mathbf{1} \end{pmatrix}$, see the proof of [16, Lemma 4.5]. The result now follows from these eigenvalues and eigenvectors. $\square$

Using Lemma 4.1 together with the equation $S + S^\top = 2I$, we calculate the right hand side of the equation (14) as follows:

$$\begin{aligned}
S^{-1} &= (2I - S) \left(\frac{1}{4t}I - \frac{1}{4t(4t+1)} \begin{pmatrix} J & O \\ O & J \end{pmatrix} \right) \\
&= \frac{1}{4t}(2I - S) - \frac{1}{4t(4t+1)} \begin{pmatrix} (2I - S_{11})J & -S_{12}J \\ -S_{21}J & (2I - S_{22})J \end{pmatrix} \\
&= \frac{1}{4t}(2I - S) - \frac{1}{4t(4t+1)} \begin{pmatrix} J & \mp\sqrt{8t+1}J \\ \pm\sqrt{8t+1}J & J \end{pmatrix} \\
&= \frac{1}{4t(4t+1)} \left((4t+1)(2I - S) - \begin{pmatrix} J & \mp\sqrt{8t+1}J \\ \pm\sqrt{8t+1}J & J \end{pmatrix} \right).
\end{aligned}$$

By $\det(S) = (8t+2)(4t)^{2t}$, we obtain

$$\det(S)S^{-1} = 2(4t)^{2t-1} \left((4t+1)(2I - S) - \begin{pmatrix} J & \mp\sqrt{8t+1}J \\ \pm\sqrt{8t+1}J & J \end{pmatrix} \right).$$

Thus the entries of $\det(S)S^{-1}$ are, up to sign,

$$2(4t)^{2t-1} \times 4t, \quad 2(4t)^{2t-1} \times (4t+2), \quad 2(4t)^{2t-1} \times (4t+1 \pm \sqrt{8t+1}).$$

By Lemma 2.1, the expression $\sqrt{8t+1}$ is an odd integer. Hence $4t, 4t+2$, and $4t+1 \pm \sqrt{8t+1}$ are all even. Moreover, the greatest common divisor among $4t, 4t+2, 4t+1 \pm \sqrt{8t+1}$ is two. By Lemma 2.3, we have $d_{4t+1}(S) = 4(4t)^{2t-1}$. Combining Lemma 2.3 and the equalities $d_{4t+2}(S) = \det(S) = (8t+2)(4t)^{2t}$, yields $s_{4t+2} = d_{4t+2}(S)/d_{4t+1}(S) = 2t(4t+1)$.

5 Examples

In this section we provide a couple of examples of EW matrices that have Smith normal forms different from Theorem 1.3.

Example 5.1. [12, Example 1] There exists an EW matrix of order 26 with the Smith normal form different from Theorem 1.3. Let R be the circulant 13×13 $\{1, -1\}$ -matrix with first row

$$(1, 1, 1, 1, -1, 1, -1, -1, 1, 1, 1, -1, 1).$$

Note that R satisfies that $RR^\top = 12I + J$. Then $X = \begin{pmatrix} R & R \\ -R^\top & R^\top \end{pmatrix}$ is an EW matrix of order 26, and its Smith normal form is

$$\text{diag}[1, \underbrace{2, \dots, 2}_{13}, \underbrace{12, \dots, 12}_{10}, 60, 60].$$

Therefore, by Theorem 1.3, the EW matrix X is not equivalent to a skew-symmetric EW matrix of order 26. For a result on the Smith normal form of an EW matrix of order 26 that have a certain block structure, see Section 7.

Example 5.2. The following construction is due to Kharaghani [11]. Let A be the 11×11 circulant $\{0, 1, -1\}$ -matrix with the first row

$$(0, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1).$$

Note that A is skew-symmetric and $AA^\top = 11I - J$. Then $X = \begin{pmatrix} R_1 & R_2 \\ -R_2^\top & R_1^\top \end{pmatrix}$ is an EW matrix of order 66, where

$$\begin{aligned} R_1 &= (A + I_{11}) \otimes (J_3 - I_3) + (J_{11} - 2I_{11}) \otimes I_3, \\ R_2 &= (A + I_{11}) \otimes (J_3 - I_3) + (-A + I_{11}) \otimes I_3, \end{aligned}$$

and its Smith normal form is

$$\text{diag}[1, \underbrace{2, \dots, 2}_{31}, \underbrace{8, \dots, 8}_4, \underbrace{32, \dots, 32}_{29}, 2080].$$

Therefore by Theorem 1.3, X is not equivalent to a skew-symmetric EW matrix of order 66. Note that this EW matrix is a counterexample to [12, Corollary 2]. We provide a revised result of [12, Corollary 2] in Section 6.

6 The Smith normal form of an EW matrix of order $4t + 2$ where $4t + 1$ is square-free

Here we provide a revised result of [12, Corollary 2].

Lemma 6.1. *Let X be an EW matrix of order $4t + 2$. Let $x_1, \dots, x_{4t+2}$ be the invariant factors of X . Then $x_1 = 1$ and $x_2 = 2$.*

Proof. The result for x_1 is trivial, and that for x_2 follows from the fact that some 2×2 minor of X is ± 2 and all 2×2 minors of X are even. $\square$

In the following theorem, we deal with EW matrices that have a block structure. We use the notion $[x]^n$ to mean $\underbrace{x, \dots, x}_{n \text{ times}}$.

Theorem 6.2. *Let X be an EW matrix $X = \begin{pmatrix} R_1 & R_2 \\ -R_2^\top & R_1^\top \end{pmatrix}$ of order $4t + 2$ such that $R_1 J = R_1^\top J = r_1 J$, $R_2 J = R_2^\top J = r_2 J$, and $4t + 1$ is square-free. Let $x_1, \dots, x_{4t+2}$ be the invariant factors of X . Then*

- (i) $x_{4t+2} = 2t(4t + 1)$;
- (ii) $x_{4t+1} = 2t$;
- (iii) if $t = 2^\ell q$ such that q is a square-free odd integer, we have

$$\begin{aligned} &(x_2, \dots, x_{4t}) \\ &= ([2]^{n_1}, \dots, [2^{k-1}]^{n_{k-1}}, [2^k]^{n'_k}, [2^k q]^{n''_k}, [2^{k+1} q]^{n_{k+1}}, \dots, [2^{\ell+1} q]^{n_{\ell+1}}) \end{aligned}$$

for some positive integer k and non-negative integers $n_1, \dots, n_{k-1}, n'_k, n''_k, n_{k+1}, \dots, n_\ell + 1$ such that $1 \leq k \leq \ell + 1$, and

$$\begin{aligned} n_1 + \dots + n_{k-1} + n'_k &= 2t + 1, \\ n''_k + n_{k+1} + \dots + n_{\ell+1} &= 2t - 2, \\ 1 \cdot n_1 + 2 \cdot n_2 + \dots + (\ell + 1) \cdot n_{\ell+1} &= 3 + 2(\ell + 2)(t - 1), \end{aligned}$$

where $n_k = n'_k + n''_k$.

Proof of (i). Since X is an EW matrix, $R_1 R_1^\top + R_2 R_2^\top = 4tI + 2J$. Multiplying all-ones vectors from both sides and dividing by $2t + 1$ yield

$$r_1^2 + r_2^2 = 8t + 2.$$

Considering this equation modulo 4, we have r_1 and r_2 are both odd. Since $4t + 1$ is square-free, $\gcd(r_1, r_2) = 1$.

Next we calculate X^{-1} as follows:

$$\begin{aligned} X^{-1} &= X^\top \left(\frac{1}{4t} I - \frac{1}{4t(4t+1)} \begin{pmatrix} J & O \\ O & J \end{pmatrix} \right) \\ &= \frac{1}{4t} X^\top - \frac{1}{4t(4t+1)} \begin{pmatrix} R_1^\top J & -R_2^\top J \\ R_2^\top J & R_1^\top J \end{pmatrix} \\ &= \frac{1}{4t} X^\top - \frac{1}{4t(4t+1)} \begin{pmatrix} r_1 J & -r_2 J \\ r_2 J & r_1 J \end{pmatrix} \\ &= \frac{1}{4t(4t+1)} \left((4t+1)X^\top - \begin{pmatrix} r_1 J & -r_2 J \\ r_2 J & r_1 J \end{pmatrix} \right). \end{aligned}$$

Since $\det(X) = 2(4t+1)(4t)^{2t}$,

$$\det(X)X^{-1} = 2(4t)^{2t-1} \left((4t+1)X^\top - \begin{pmatrix} r_1 J & -r_2 J \\ r_2 J & r_1 J \end{pmatrix} \right).$$

Therefore the entries of $\det(X)X^{-1}$ are $\pm 2(4t)^{2t-1}(4t+1 \pm x)$ where $x \in \{r_1, r_2\}$. Then the greatest common divisor of $4t+1 \pm x$ where $x \in \{r_1, r_2\}$ is

$$\gcd(4t+1 \pm r_1, 4t+1 \pm r_2) = \gcd(4t+1+r_1, 2r_1, 4t+1+r_2, 2r_2) = 2,$$

where in the last equation we used that r_1 and r_2 are both odd integers and $\gcd(r_1, r_2) = 1$. Therefore $d_{4t+1}(X) = 4(4t)^{2t-1}$, and hence we have $x_{4t+2} = \det(X)/d_{4t+1}(X) = 2t(4t+1)$. $\square$

Since $x_2 = 2$ divides x_i for each $i \geq 2$, we may set $x_i = 2x'_i$ for $2 \leq i \leq 4t+2$.

Proof of (ii). By Lemma 2.4, it is enough to calculate 2×2 minors of X^{-1} . It is straightforward to check that each possible 2×2 minor of $4t(4t+1)X^{-1}$ has the form $\pm 2(4t+1)\alpha$, where α is an element of the set S given by

$$S = \left\{ \begin{array}{ccccc} 0, & 4t, & 2r_1, & 2r_2, & (r_1 \pm 1), \\ (r_2 \pm 1), & (4t+1 \pm r_1), & (4t+1 \pm r_2), & (r_1 \pm r_2), & (4t+2 \pm r_1 \pm r_2) \end{array} \right\}.$$

By the proof of (i), both r_1 and r_2 are odd integers. Hence each element of S is divisible by 2, and therefore each 2×2 minor of $4t(4t+1)X^{-1}$ is divisible by $4(4t+1)$.

Any principal minor induced on the first $(2t+1)$ rows of X is, up to sign, equal to $2(4t+1)(4t+1 \pm r_1)$ or $4(4t+1)r_1$. Also any principal minor induced on a subset of the rows $\{2t+2, \dots, 4t+2\}$ is, up to sign, equal to $2(4t+1)(4t+1 \pm r_2)$ or $4(4t+1)r_2$. Thus the greatest common divisor of the 2×2 minors of $4t(4t+1)S^{-1}$ is $4(4t+1)$. By Lemma 2.4, $d_{4t}(X) = 8(4t)^{2t-2}$. Combining Lemma 2.3 and $d_{4t+1}(X) = 4(4t)^{2t-1}$, we obtain $x_{4t+1} = d_{4t+1}(X)/d_{4t}(X) = 2t$. $\square$

Proof of (iii). By (i), (ii), and the equality $\det(X) = x_1 \cdots x_{4t+2}$, we have that

$$x'_2 \cdots x'_{4t} = t^{2t-2}. \quad (15)$$

Suppose, for a contradiction, that x'_{2t+2} is divisible by an odd integer greater than 1. Let p be any odd prime divisor of x'_{2t+2} . Then $x'_2 \cdots x'_{4t} = t^{2t-2}$ is divisible by p^{2t} . However this is impossible; since $t/2^\ell$ is square-free t^{2t-2} cannot have p^{2t} as a divisor. Therefore x'_{2t+2} is a power of 2.

Next let p' be any prime divisor of q . By the above, we see that $x'_{2t+3} \cdots x'_{4t} = 2^r q^{2t-2}$ for some r . Therefore, for $i \in \{2t+3, \dots, 4t\}$, x_i equals to a power of 2 times q . Thus, the invariant factors x_i must have the form given in part (iii). $\square$

7 The Smith normal form of an EW matrix of order $4t+2$ where $4t+1 = p^2$, for p a prime

Theorem 7.1. *Let X be an EW matrix $X = \begin{pmatrix} R_1 & R_2 \\ -R_2^\top & R_1^\top \end{pmatrix}$ of order $4t+2$ such that $R_1 J = R_1^\top J = r_1 J$, $R_2 J = R_2^\top J = r_2 J$, and $4t+1 = p^2$ where p is a prime. Let $x_1, \dots, x_{4t+2}$ be the invariant factors of X . Then*

- (i) $\gcd(r_1, r_2)$ is 1 or p ;
- (ii) x_{4t+2} is $2t(4t+1)$ or $2tp$;
- (iii) x_{4t+1} is a divisor of $2t$ if $x_{4t+2} = 2t(4t+1)$, and x_{4t+1} is a divisor of $2tp$ and p divides x_{4t+1} if $x_{4t+2} = 2tp$;
- (iv) if t is square-free, then the Smith normal form of X is either

$$\begin{aligned} & \text{diag}[1, \underbrace{2, \dots, 2}_{2t+1}, \underbrace{2t, \dots, 2t}_{2t-1}, 2t(4t+1)], \text{ or} \\ & \text{diag}[1, \underbrace{2, \dots, 2}_{2t+1}, \underbrace{2t, \dots, 2t}_{2t-2}, 2tp, 2tp]. \end{aligned}$$

Proof of (i). By the same argument with Theorem 6.2, we have

$$r_1^2 + r_2^2 = 8t + 2 = 2p^2.$$

Then $\gcd(r_1, r_2)$ divides p , and it follows from the assumption p being prime that $\gcd(r_1, r_2) = 1$ or p . $\square$

Proof of (ii). By the same argument as in the proof of Theorem 6.2, we have

$$\det(X)X^{-1} = 2(4t)^{2t-1} \left((4t+1)X^\top - \begin{pmatrix} r_1 J & -r_2 J \\ r_2 J & r_1 J \end{pmatrix} \right).$$

Therefore the entries of $\det(X)X^{-1}$ are $\pm 2(4t)^{2t-1}(4t+1 \pm x)$ where $x \in \{r_1, r_2\}$. Then the greatest common divisor of $4t+1 \pm x$ where $x \in \{r_1, r_2\}$ is

$$\gcd(4t+1 \pm r_1, 4t+1 \pm r_2) = 2\gcd(4t+1, r_1, r_2) = \begin{cases} 2 & \text{if } \gcd(r_1, r_2) = 1, \\ 2p & \text{if } \gcd(r_1, r_2) = p. \end{cases}$$

Therefore

$$d_{4t+1}(X) = \begin{cases} 4(4t)^{2t-1} & \text{if } \gcd(r_1, r_2) = 1, \\ 4p(4t)^{2t-1} & \text{if } \gcd(r_1, r_2) = p, \end{cases}$$

and thus

$$x_{4t+2} = \det(X)/d_{4t+1}(X) = \begin{cases} 2t(4t+1) & \text{if } \gcd(r_1, r_2) = 1, \\ 2tp & \text{if } \gcd(r_1, r_2) = p. \end{cases} \quad \square$$

Proof of (iii). For the case $x_{4t+2} = 2t(4t+1)$, the proof is same as that of Theorem 6.2(ii).

For the case $x_{4t+2} = 2tp$, x_{4t+1} divides x_{4t+2} . Thus x_{4t+1} divides $2tp$. By $x_{4t+2} = 2tp$ and $\det(X) = x_1 \cdots x_{4t+2}$, we have that

$$x'_2 \cdots x'_{4t+1} = pt^{2t-1}.$$

Since $\gcd(p, t) = 1$, p divides x'_{4t+1} . $\square$

Proof of (iv). For the case $x_{4t+2} = 2t(4t+1)$, the proof is same as that of Theorem 6.2(iii).

For the case $x_{4t+2} = 2tp$, the proof is also similar to that of Theorem 6.2(iii), but we include it here. By (ii) and $\det(X) = x_1 \cdots x_{4t+2}$, we have that $x'_2 \cdots x'_{4t+1} = pt^{2t-1}$. Furthermore we put $x''_{4t+1} = x_{4t+1}/p$, which is integer by (iii). Then $x'_2 \cdots x'_{4t} x''_{4t+1} = t^{2t-1}$. Suppose, for a contradiction, that $x'_{2t+2} > 1$. Let q be any prime divisor of x'_{2t+2} . Then $x'_2 \cdots x'_{4t} x''_{4t+1} = t^{2t-1}$ is divisible by q^{2t} . However this is impossible; since t is square-free t^{2t-1} cannot have q^{2t} as a divisor. Therefore $x'_{2t+2} = 1$. Furthermore

$$x'_{2t+3} \cdots x'_{4t} x''_{4t+1} = t^{2t-1}. \quad (16)$$

By (ii) we have $x''_{4t+1} \leq t$ and thus $x'_i \leq t$ for $i \in \{2t+3, \dots, 4t\}$. Therefore, by (16), we have $x''_{4t+1} = t$ and $x'_i = t$ for all $i \in \{2t+3, \dots, 4t\}$. $\square$

Barba [4] showed that for a $\{1, -1\}$ -matrix X of order $n \equiv 1 \pmod{2}$, Hadamard's inequality can be strengthened to

$$|\det(X)| \leq (2n-1)^{1/2}(n-1)^{(n-1)/2}. \quad (17)$$

Moreover, there exists [5] a $\{1, -1\}$ -matrix achieving equality in (17) if and only if $n \equiv 1 \pmod{4}$ and there exists a $\{1, -1\}$ -matrix B such that

$$BB^\top = B^\top B = (n-1)I + J. \quad (18)$$

A matrix B is called a **Barba matrix** if it satisfies (18). It is known that if R is a Barba matrix of order n , then the matrix $\begin{pmatrix} R & R \\ -R^\top & R^\top \end{pmatrix}$ is an EW matrix of order $2n$. We therefore pose the following question about Smith normal forms of EW matrices constructed from Barba matrices.

Problem 7.2. Does there exist a Barba matrix R such that the Smith normal form of $\begin{pmatrix} R & R \\ -R^\top & R^\top \end{pmatrix}$ is not equal to $\text{diag}[\underbrace{1, \dots, 1}_{2t}, \underbrace{2, \dots, 2}_{2t-1}, 2t\sqrt{8t+1}]$.

8 The Smith normal form of an EW tournament matrix

In this section we determine the Smith normal form of an EW tournament matrix.

Theorem 8.1. *The Smith normal form of an EW tournament matrix of order $4t+1$ is*

$$\text{diag}[\underbrace{1, \dots, 1}_{2t+2}, \underbrace{t, \dots, t}_{2t-2}, t^2(4t-1)].$$

Recall that for an integer matrix M , we denote by $d_i(M)$ the greatest common divisor of all $i \times i$ minors of M and $d_0(M) = 1$. The first lemma we need is elementary.

Lemma 8.2 (page 33 of [14]). *Let M and N be invertible $n \times n$ integer matrices, and let m_i be the invariant factors of M and p_i be the invariant factors of MN . Then m_i divides p_i for each i .*

Now we state and prove two lemmas that we will use to prove Theorem 8.1.

Lemma 8.3. *Let A be an EW tournament matrix of order $4t+1$. Suppose that a prime p divides t . Then $\text{rank}_p(A) = 2t+2$.*

Proof. First we show that $\text{rank}_p(A) \leq 2t+2$. Combine (6) and Sylvester's rank inequality [10, 0.4.5 Rank inequalities (c)] to obtain

$$\begin{aligned} 2\text{rank}_p(A) &= \text{rank}_p(A) + \text{rank}_p(A^\top) \\ &\leq \text{rank}_p(AA^\top) + 4t+1 \\ &= 4t+4. \end{aligned}$$

Thus $\text{rank}_p(A) \leq 2t + 2$.

Next we show that $\text{rank}_p(A) \geq 2t + 2$. Let $d = \dim(\text{rowsp}_p(A) \cap \text{rowsp}_p(A^\top))$ be the dimension of the intersection of row spaces. Then

$$\begin{aligned} 4t &= \text{rank}_p(J - I) = \text{rank}_p(A + A^\top) \\ &\leq \text{rank}_p(A) + \text{rank}_p(A^\top) - d \\ &= 2\text{rank}_p(A) - d, \end{aligned}$$

that is, we have that $2t + d/2 \leq \text{rank}_p(A)$. Hence it suffices to show that $d \geq 3$.

The equations (6) and (7) together with the assumption that p divides t yields

$$W \subseteq \text{rowsp}_p(A) \text{ and } W \subseteq \text{rowsp}_p(A^\top),$$

where

$$W = \text{span}\{(\mathbf{1}_t^\top, \mathbf{1}_t^\top, \mathbf{1}_a^\top, \mathbf{1}_{2t+1-a}^\top), (\mathbf{1}_t^\top, -\mathbf{1}_t^\top, \mathbf{0}_a^\top, \mathbf{0}_{2t+1-a}^\top), (\mathbf{0}_t^\top, \mathbf{0}_t^\top, \mathbf{1}_a^\top, -\mathbf{1}_{2t+1-a}^\top)\}.$$

Therefore

$$W \subseteq \text{rowsp}_p(A) \cap \text{rowsp}_p(A^\top),$$

and thus $d \geq 3$. □

Lemma 8.4. *Let A be an EW tournament matrix of order $4t + 1$ and let $l_1, \dots, l_{4t+1}$ be the invariant factors of $A^2 + A$. Then $l_{4t} = t$ and $l_{4t+1} = t^2(16t^2 - 1)$.*

Proof. Define $\alpha_\pm := t(8t + 3 \pm 2s) + (1 \pm s)/2$, $\beta_\pm := (2t + 1)(2t + (1 \pm s)/2)$, and $\gamma_\pm := t(4t + 1 \pm s)$. It is straightforward to deduce that the matrix $t^2(16t^2 - 1)(A^2 + A)^{-1}$ is equal to

$$-t(16t^2 - 1)I_{4t+1} + \begin{pmatrix} t(8t + 1)J_t & tJ & \gamma_+J & \gamma_-J \\ (5t + 1)J & t(8t + 1)J_t & \beta_+J & \beta_-J \\ \beta_+J & \gamma_+J & \alpha_+J_a & tJ \\ \beta_-J & \gamma_-J & tJ & \alpha_-J_{2t+1-a} \end{pmatrix},$$

where $s = \sqrt{8t + 1}$. Hence, to prove the lemma, it suffices to show that

$$\begin{aligned} d_1(t^2(16t^2 - 1)(A^2 + A)^{-1}) &= 1, \\ d_2(t^2(16t^2 - 1)(A^2 + A)^{-1}) &= t(16t^2 - 1). \end{aligned}$$

It is easy to see that t and $5t + 1$ are (coprime) entries of $t^2(16t^2 - 1)(A^2 + A)^{-1}$, belonging to $(1, 2)$ -block and $(2, 1)$ -block. Thus $d_1(t^2(16t^2 - 1)(A^2 + A)^{-1}) = 1$.

All the possible 2×2 minors of $t^2(16t^2 - 1)(A^2 + A)^{-1}$ have the form $\pm t(16t^2 - 1)\alpha$ where α is an element of the set S given by

$$S = \left\{ 0, t, t+1, 2t, 2t+1, 4t+1, 5t+1, t(8t+1), 8t^2-3t-1, t(16t^2-16t-3), \right. \\ 16t^3-16t^2+t+1, 2t(4t \pm s), t(4t+1 \pm s), t(4t-1 \pm s), \\ 2t(8t^2-8t-1 \pm s), 16t^3-16t^2-7t-1 \pm s(4t+1), \\ \frac{2t+1 \pm s}{2}, \frac{2t-1 \pm s}{2}, \frac{4t+1 \pm s}{2}, \frac{6t+1 \pm s}{2}, \frac{12t+1 \pm s}{2}, \\ \frac{(2t-1)(4t+1 \pm s)}{2}, \frac{(2t+1)(4t-1 \pm s)}{2}, \frac{(2t+1)(4t+1 \pm s)}{2}, \\ \left. \frac{16t^2-4t-1 \pm s}{2}, \frac{16t^2+6t+1 \pm s(4t+1)}{2}, \frac{16t^2-2t-1 \pm s(4t+1)}{2} \right\}.$$

Note that the greatest common divisor of the numbers in the set S is 1. Therefore $d_{4t}(t^2(16t^2 - 1)(A^2 + A)^{-1}) = t(16t^2 - 1)$. $\square$

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Let A be an EW tournament matrix and let $a_1, \dots, a_{4t+1}$ be the invariant factors of A .

Claim 1: $a_{2t+2} = 1$. Assume that there exists a prime p dividing a_{2t+2} . It follows from Lemma 2.3 and Lemma 2.5 that $a_1 \cdots a_{4t+1} = \det(A) = t^{2t}(4t-1)$. Thus p is either

- (1) a divisor of t , or
- (2) not a divisor of t and a divisor of $4t+1$.

For the case (1), Lemma 8.3 contradicts the fact that

$$\text{rank}_p(A) = \max\{i \mid p \text{ does not divide } a_i\} \leq 2t+1.$$

For the case (2), observe that p is odd and hence $p \geq 3$. Furthermore p is a divisor of a_i for $i = 2t+3, \dots, 4t+1$. Thus p^{2t} divides $4t-1$. But this is impossible since $p^{2t} > 4t-1$ for all $t \geq 1$. Therefore we have $a_{2t+2} = 1$. This proves Claim 1.

Claim 2: a_{4t+1} divides $t^2(4t-1)$. By Lemma 8.2, and Lemma 8.4, we have that a_{4t+1} divides $l_{4t+1} = t^2(16t^2-1)$. On the other hand a_{4t+1} divides $\det(A) = t^{2t}(4t-1)$. Therefore a_{4t+1} divides $t^2(4t-1)$. This proves Claim 2.

Claim 3: a_{4t} divides t . By Lemma 8.2, and Lemma 8.4, we have that a_{4t} divides $l_{4t} = t$. Therefore a_{4t} divides t . This proves Claim 3.

Now, by Lemma 2.5, we have

$$a_{2t+3} \cdots a_{4t+1} = t^{2t}(4t-1). \quad (19)$$

Since a_i divides a_{i+1} for each i and $a_{4t} \leq t$, by Claim 3, we have that $a_i \leq t$ for $i = 2t + 3, \dots, 4t - 1$, from which with $a_{4t+1} \leq t^2(4t - 1)$ it follows that

$$a_{2t+3} \cdots a_{4t+1} \leq t^{2t}(4t - 1). \quad (20)$$

By (19), equality must hold in (20). Therefore $a_{2t+3} = \cdots = a_{4t} = t$ and $a_{4t+1} = t^2(4t - 1)$. $\square$

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