A unified approach to Banach-Stone theorem on spaces of differentiable functions under various norms

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**A UNIFIED APPROACH TO
BANACH-STONE THEOREM ON
SPACES OF DIFFERENTIABLE
FUNCTIONS UNDER VARIOUS
NORMS**

NG HONG WAI

**SCHOOL OF PHYSICAL AND MATHEMATICAL
SCIENCES**

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NG HONG WAI

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfillment of the requirements for the
degree of Doctor of Philosophy

2021

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This thesis does not contain any materials from papers published in peer-reviewed journals or from papers accepted at conferences in which I am listed as an author.

17 November 2020

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Ng Hong Wai

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Summary

The classical Banach-Stone Theorem asserts that the isometric structure of the space of real-valued continuous functions determines a compact Hausdorff space up to homeomorphism. There are various extensions and generalizations of the result in different contexts. One of them is to consider vector-valued differentiable function space endowed with different norms. Existing literatures utilized special geometrical properties of norms to obtain a variant of the result. However, their methods are restricted to the considered norms only. In this thesis, we developed a general framework, which provides a sufficient condition to obtain Banach-Stone Theorem for vector-valued differentiable function space. Then we applied the framework on two different norms, which generalized most norms considered in existing literatures. When restricting them to ℓ^p -norms, where $p \in [1, \infty)$, we obtained a characterization of Banach-Stone Theorem.

List of Symbols

Symbols	Page
Q	7
E	7
Λ	7
$C_0^p(Q, E)$	7
E^Λ	7
ρ	7
ρ^*	8
$\ \cdot\ _{D,\rho}$ and $\ \cdot\ _{S,\rho}$	9
K	9
F	9
Γ	9
θ	9
$C_0^q(K, F)$	9
T	10
$x^* \circ \delta_s$	10
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Chapter 1

Introduction

1.1 Literature Review

The classical Banach-Stone Theorem [1, 2] asserts that the isometric structure of the space of real-valued continuous functions $C(X)$ determines a compact Hausdorff space X up to homeomorphism. More precisely, if Y is a compact Hausdorff space and $T : C(X) \rightarrow C(Y)$ is an isometry, then T must be a weighted composition operator, that is, $Tf(t) = V(t)f(\varphi(t))$ for some homeomorphism $\varphi : Y \rightarrow X$ and continuous unimodular function $V(t)$. To construct the homeomorphism φ , Banach and Stone [1, 2] applied peak point method, that is, a map that identifies points where given functions achieve their norms.

Thereafter, mathematicians extended the result to various settings. To name a few, Gelfand and Kolmogoroff [26] considered algebra isomorphisms between $C(X)$ and $C(Y)$. Holsztynski [24] considered linear isometries, not necessarily surjective, on real-valued continuous function spaces. Cambern [25] considered linear isomorphisms, instead of isometries, and showed that the underlying spaces are homeomorphic. Jerison [23] considered continuous vector-valued function spaces. More precisely, if E is a strictly convex Banach space, then any isometry between $C(X, E)$ and $C(Y, E)$ is a weighted composition operator.

The popular method of proof of the Banach-Stone Theorem is the extreme point method, which involves finding extreme points of dual space of function space. This method was first used by Arens and Kelley [21] and then popularized by Dunford

and Schwartz [22, page 441, Lemma 6]. For a comprehensive survey of this topic, we refer the reader to the monographs by Fleming and Jamison [11, 12]. In this thesis, we focused on vector-valued differentiable function spaces. We provided an overview of existing literatures of such spaces in the next paragraph. Prior to that, we need to introduce several notations.

We fix natural numbers $p \geq 1$ and $m \geq 1$. We let X be a bounded regular closed subset of $\mathbb{R}^m$. Recall that a set X is bounded regular closed if $X = \overline{\text{int}(X)}$. Clearly $[0, 1]$ is bounded regular closed. We also let E be a Banach space. We let $\Lambda = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{N}_0^m : \lambda_1 + \dots + \lambda_m \leq p\}$. For each $f \in C^p(X, E)$ and $\lambda \in \Lambda$, we denote $\partial^\lambda f = \frac{\partial^{\lambda_1 + \dots + \lambda_m} f}{\partial s_1^{\lambda_1} \partial s_2^{\lambda_2} \dots \partial s_m^{\lambda_m}}$. Partial derivatives of f at boundary points of X will be taken as one-sided derivative. Let us consider the following norms on $C^p(Q, E)$.

$$\|f\|_C = \max_{s \in X} \sum_{\lambda \in \Lambda} \|\partial^\lambda f(s)\|, \quad (1.1.1)$$

$$\|f\|_\Sigma = \sum_{\lambda \in \Lambda} \|\partial^\lambda f\|_\infty \quad \text{and} \quad (1.1.2)$$

$$\|f\|_M = \max_{\lambda \in \Lambda} \|\partial^\lambda f\|_\infty. \quad (1.1.3)$$

Cambern [3] showed that the surjective linear isometries on $(C^1([0, 1], \mathbb{C}), \|\cdot\|_C)$ are weighted composition operators. Then Pathak [5] extended the result to $(C^p([0, 1], \mathbb{C}), \|\cdot\|_C)$. On the other hand, Jarosz and Pathak [9] characterized onto linear isometries on $(C^1[0, 1], \|\cdot\|_\Sigma)$ and $(C^1[0, 1], \|\cdot\|_M)$. Botelho and Jamison [6] considered a vector-valued continuously differentiable function space $(C^1([0, 1], E), \|\cdot\|_C)$, where E is a finite-dimensional Hilbert space and showed that all onto linear isometries on such function spaces are weighted composition operators. Subsequently, Li and Wang [7] extended their result to vector-valued differentiable function space which vanish at infinity $(C_0^p(Q, E), \|\cdot\|_C)$, where Q is an open subset of $\mathbb{R}^m$ and E is a reflexive and strictly convex Banach space. Recently, Li et al. [8] characterized onto linear isometries on $(C_0^p(Q, E), \|\cdot\|_\Sigma)$ where $Q \subseteq \mathbb{R}$ is open and E is a strictly convex Banach space E with dimension greater than 1. In this thesis, we aim to provide a unified approach for the norms $\|\cdot\|_C$ and $\|\cdot\|_\Sigma$.

1.2 Major Contributions

We let Q be an open subset of $\mathbb{R}^m$ and E be a Banach space. We consider the following general norms on $C_0^p(Q, E)$.

$$\|f\|_{D,\rho} = \sup_{s \in Q} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \quad \text{and} \quad \|f\|_{S,\rho} = \rho[(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda}]$$

where $\rho : \mathbb{R}^{\#\Lambda} \rightarrow \mathbb{R}$ is a normalized and strictly monotone norm (for definition of normalization and strict monotonicity, please refer to Section 2.1). If $\rho = \|\cdot\|_1$, then the norms in (1.1.1) and (1.1.2) become $\|\cdot\|_{D,\rho}$ and $\|\cdot\|_{S,\rho}$ respectively. In this thesis, we provide a unified approach to establish the Banach-Stone Theorem for vector-valued differentiable function spaces with respect to the two norms.

The main results of this thesis are Theorems 3.34 and 5.70, which assert respectively, that all onto linear isometries on $(C_0^p(Q, E), \|\cdot\|_{D,\rho})$ or $(C_0^p(Q, E), \|\cdot\|_{S,\rho})$ are weighted composition operators. Furthermore, if $\rho = \|\cdot\|_s$ where $s \in [1, \infty)$, then we obtain a characterization of Banach-Stone Theorem, given by Theorems 4.51 and 6.17.

In this thesis, we restrict our study on p th time vector-valued differentiable functions on an open set Q of Euclidean space. The difficulty of extending Q to an open set in an infinite-dimensional space lies in the construction of the peak functions (Sections 3.1 and 5.1)

1.3 Outline of the Thesis

We will give an outline of this thesis in this section.

In Section 2.1, we introduce the spaces of interest $C_0^p(Q, E)$ and $C_0^q(K, F)$, and define two norms $\|\cdot\|_{D,\rho}$ and $\|\cdot\|_{S,\rho}$, whose definitions are given by (2.1.3) and (2.1.4). To establish the Banach-Stone Theorem, we need to construct a mapping $\varphi : K \rightarrow Q$ and show that any surjective linear isometry $T : C_0^p(Q, E) \rightarrow C_0^q(K, F)$ is a weighted composition operator. To construct the mapping φ , we employ the extreme point technique, which relies on a characterization of extreme points of the dual unit ball $B[C_0^p(Q, E)^*]$ of $C_0^p(Q, E)$. Propositions 2.2 and 2.5 provide a

‘partial’ characterization for such extreme points. However, Proposition 2.5 requires that the set of ‘good coefficients’, denoted by $\mathcal{G}(\cdot)$, to be nonempty. Thanks to Corollary 2.7, it suffices to show that the set of ‘peakable vectors’, denoted by $\mathcal{P}(\cdot)$, is nonempty, which is the main objective of Section 3.1 and Section 5.1. In Section 2.2.2, Proposition 2.12 is the main result, which asserts that if φ is independent of y^* in some subset N whose closed convex hull is weak-star dense in an ambient space, then φ is independent of any y^* in the ambient space. In the final Section 2.2.3, Theorem 2.20 gives a sufficient condition for our isometry T to be a weighted composition operator. The main ingredient in the proof of the theorem is [13, Theorem 6.2], which asserts that any biseparating map between differentiable function spaces (not necessarily vanishing at infinity) is a weighted composition operator. We would like to point out that all results in Chapter 2 are independent of norms considered on $C_0^p(Q, E)$ and $C_0^q(K, F)$.

In Chapter 3, there are three main tasks. In Section 3.1, we show that the set $\mathcal{P}(\cdot)$ is nonempty, which is given by Corollary 3.11. We achieve this by constructing a peaking function, which depends on constructing a single variable function h that satisfies certain properties. In Section 3.2, we construct a mapping $\varphi : K \rightarrow Q$ that satisfies the extreme point property, which is given by Theorem 3.32. We employ components ‘flipping’ technique in the construction, which is given by Lemma 3.18. After constructing the mapping φ , we show that φ satisfies the condition imposed by Theorem 2.20. Then we conclude that T is a weighted composition operator, which is given by Theorem 3.34.

In the long Chapter 4, we investigate the properties of V and φ obtained from Theorem 3.34. In particular, we show that $V(t)$ is an isometry for all $t \in K$, and both V and φ are locally constant. To do so, in Section 4.1, we establish (4.1.13) connecting f and Tf through a matrix multiplication $\tilde{A}$ which consists of entries from the Jacobian matrix of φ . This is achieved through Lemma 4.1, a Leibniz’s rule for operators. In Section 4.2, we investigate some entries of $\tilde{A}$, and obtain their representation, which is given by Corollary 4.4. In Section 4.3, we assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. Then we show that for most cases, the Jacobian matrix of φ is a generalized permutation matrix. In Section 4.4, we show that if the Jacobian matrix of φ is a generalized permutation matrix and $m \geq 2$, then V locally constant and $s = r$. This result is given by Propositions 4.29 and 4.30. In Section 4.5, we obtain several interesting results. Under certain

conditions on s, r and q , we can show that the Jacobian matrix of φ must be a generalized permutation matrix when the norms on E and F are not induced by an inner product, and if the Jacobian matrix of φ is not necessarily a generalized permutation matrix, then E and F must be Hilbert spaces. This result is given by Proposition 4.45. We conclude the chapter by stating the main result, which is Theorem 4.51. This concludes the investigation for the norm $\|\cdot\|_{D,\rho}$.

We start our investigation on $\|\cdot\|_{S,\rho}$ in Chapter 5. Note that this chapter has a similar structure as Chapter 3. In Section 5.1, we prove that the set $\mathcal{P}_S(\cdot)$ is nonempty, which is achieved through a series of results. Proposition 5.8 is the main result in that section, which asserts that $\mathcal{P}_S(\cdot)$ is nonempty. In Section 5.2, the main result is Theorem 5.53, which asserts the existence of a mapping $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ that satisfies the extreme point property. While constructing the φ , we assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. In Section 5.3, the main result is Theorem 5.70, which asserts that T is a weighted composition operator. To this end, we aim to apply Theorem 2.20. However, unlike in Section 3.3, the φ constructed in Section 5.2 is not applicable directly. Therefore, we establish a series of preliminary results before applying Theorem 2.20. In particular, we show that T takes a certain form, which is given by (5.3.9).

The aim in Chapter 6 is the same as in Chapter 4, that is, we show that $V(t)$ is an isometry for all $t \in K$, and both V and φ are locally constant. We consider two cases, either $m \geq 2$ or $m = 1$. In both cases, we show that $s = r$, the Jacobian matrix of φ is a signed permutation matrix and $V(t)$ is an isometry. The result is given by Corollaries 6.8 and 6.13. Then we proceed to show that V is locally constant, given by Proposition 6.15. Finally, we conclude the chapter with Theorem 6.17.

In the concluding Chapter 7, we provide a list of further questions which are relevant in this research direction.

Chapter 2

General Framework

In this chapter, we will develop a general framework. Prior to this, we start with necessary definitions and notations.

2.1 Definitions and Notations

We refer the readers to [18] for the standard theorems and notation in Banach space theory. In this section, we will introduce the function space $C_0^p(Q, E)$ and relevant definitions. Fix natural numbers $m \geq 1$ and $p \geq 1$. Set

$$\Lambda = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{N}_0^m : \lambda_1 + \dots + \lambda_m \leq p\}.$$

For any $\lambda \in \Lambda$, denote $|\lambda| = \lambda_1 + \dots + \lambda_m$. Let $Q \subseteq \mathbb{R}^m$ be an open subset and E be a Banach space. Let $C_0^p(Q, E)$ the space of all the E -valued continuous functions f on Q whose partial derivatives

$$\partial^\lambda f = \frac{\partial^{|\lambda|} f}{\partial s_1^{\lambda_1} \partial s_2^{\lambda_2} \dots \partial s_m^{\lambda_m}}$$

exist, are continuous for every $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Lambda$ and vanish at infinity, that is, for any $\varepsilon > 0$, the set $\{s \in Q : \|\partial^\lambda f(s)\| \geq \varepsilon\}$ is compact for every $\lambda \in \Lambda$.

Let $\#\Lambda$ be the cardinality of the set Λ and $E^\Lambda = E^{\#\Lambda}$. Let $\rho : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ be a norm. Assume that ρ satisfies the following properties:

1. ρ is *normalised*, that is, for all $\lambda \in \Lambda$, $\rho(0, \dots, 0, \overset{\lambda}{1}, 0, \dots, 0) = 1$.
2. ρ is *monotone*, that is, for all $(\alpha_\lambda)_{\lambda \in \Lambda}$ and $(\alpha'_\lambda)_{\lambda \in \Lambda}$ in $(\mathbb{R}^\Lambda, \rho)$, if $|\alpha_\lambda| \leq |\alpha'_\lambda|$ for all $\lambda \in \Lambda$, then $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] \leq \rho[(\alpha'_\lambda)_{\lambda \in \Lambda}]$. Moreover, ρ is *strictly monotone*, that is, if $|\alpha_\lambda| < |\alpha'_\lambda|$ for some $\lambda \in \Lambda$, then $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] < \rho[(\alpha'_\lambda)_{\lambda \in \Lambda}]$.

An example of ρ is $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = \left(\sum_{\lambda \in \Lambda} |\alpha_\lambda|^s \right)^{\frac{1}{s}}$, where $s \in [1, \infty)$. Let E_ρ^Λ be the vector space E^Λ endowed with the norm

$$\|(x_\lambda)_{\lambda \in \Lambda}\|_\rho = \rho[(\|x_\lambda\|)_{\lambda \in \Lambda}].$$

Denote $B(E_\rho^\Lambda) = \{(x_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda : \|(x_\lambda)_{\lambda \in \Lambda}\|_\rho \leq 1\}$. Let $\rho^* : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ be the dual norm of ρ , that is,

$$\rho^*[(\alpha_\lambda^*)_{\lambda \in \Lambda}] = \sup_{(\alpha_\lambda)_{\lambda \in \Lambda} \in B(\mathbb{R}_\rho^\Lambda)} \left| \sum_{\lambda \in \Lambda} \alpha_\lambda^*(\alpha_\lambda) \right|.$$

From the definition of ρ^* , clearly

$$\left| \sum_{\lambda \in \Lambda} \alpha_\lambda^*(\alpha_\lambda) \right| \leq \rho^*[(\alpha_\lambda^*)_{\lambda \in \Lambda}] \cdot \rho[(\alpha_\lambda)_{\lambda \in \Lambda}] \quad (2.1.1)$$

for all $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathbb{R}_{\rho^*}^\Lambda$ and $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}_\rho^\Lambda$.

Let $(E_\rho^\Lambda)^*$ be the set of all bounded linear functional $x^* = (x_\lambda^*)_{\lambda \in \Lambda}$ on E_ρ^Λ defined by $x^*(x) = \sum_{\lambda \in \Lambda} x_\lambda^*(x_\lambda)$ for all $x = (x_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$. The space $(E_\rho^\Lambda)^*$ is endowed with the norm

$$\|(x_\lambda^*)_{\lambda \in \Lambda}\|_{\rho^*} = \rho^*[(\|x_\lambda^*\|)_{\lambda \in \Lambda}].$$

The following lemma is a simple consequence of strict monotonicity of ρ and (2.1.1).

Lemma 2.1. *Assume that ρ is strictly monotone. Fix any $(x_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$ and $(x_\lambda^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$. If $x_\lambda \neq 0$, $\|(x_\lambda^*)_{\lambda \in \Lambda}\|_{\rho^*} \leq 1$, and*

$$\sum_{\lambda \in \Lambda} x_\lambda^*(x_\lambda) = \|(x_\lambda)_{\lambda \in \Lambda}\|_\rho, \quad (2.1.2)$$

then $x_\lambda^ \neq 0$ for all $\lambda \in \Lambda$.*

Proof. Suppose that $x_{\lambda_0}^* = 0$ for some $\lambda_0 \in \Lambda$. Observe that

$$\begin{aligned}
& \| (x_\lambda)_{\lambda \in \Lambda} \|_\rho \\
&= \sum_{\lambda \in \Lambda} x_\lambda^*(x_\lambda) \quad \text{by (2.1.2)} \\
&\leq \left(\sum_{\lambda \neq \lambda_0} \|x_\lambda^*\| \cdot \|x_\lambda\| \right) + \|x_{\lambda_0}^*\| \cdot 0 \quad \text{since } x_{\lambda_0}^* = 0 \\
&\leq \| (x_\lambda)_{\lambda \in \Lambda} \|_{\rho^*} \cdot \| ((x_\lambda)_{\lambda \neq \lambda_0}, 0) \|_\rho \quad \text{by Inequality 2.1.1} \\
&\leq \| ((x_\lambda)_{\lambda \neq \lambda_0}, 0) \|_\rho \quad \text{since } (x_\lambda^*)_{\lambda \in \Lambda} \in B[(E_\rho^\Lambda)^*] \\
&< \| (x_\lambda)_{\lambda \in \Lambda} \|_\rho \quad \text{since } \|x_{\lambda_0}\| > 0 \text{ and } \rho \text{ is strictly monotone,}
\end{aligned}$$

a contradiction. This concludes a proof of the lemma. $\square$

In this thesis, we considered the following norms on $C_0^p(Q, E)$ defined by

$$\|f\|_{D,\rho} = \sup_{s \in Q} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \quad \text{and} \quad (2.1.3)$$

$$\|f\|_{S,\rho} = \rho[(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda}]. \quad (2.1.4)$$

Observe that

$$\|f\|_{S,\rho} = \sup_{(s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda} \rho[(\|\partial^\lambda f(s_\lambda)\|)_{\lambda \in \Lambda}]. \quad (2.1.5)$$

We call $\|\cdot\|_{D,\rho}$ and $\|\cdot\|_{S,\rho}$ the diagonal and square norms respectively, as the former norm involves evaluating all partial derivative of f at the same point, whereas the latter involves evaluation at a string of possibly different points.

Let Q_∞ be the 1-point compactification of Q . For each $f \in C_0^p(Q, E)$, we extend $\partial^\lambda f : Q \rightarrow E$ to $\partial^\lambda f : Q_\infty \rightarrow E$ continuously by defining $\partial^\lambda f(\infty) = 0$.

From (2.1.3) and (2.1.5), the spaces $L_D = (C_0^p(Q, E), \|\cdot\|_{D,\rho})$ and $L_S = (C_0^p(Q, E), \|\cdot\|_{S,\rho})$ can be embedded isometrically into $C(Q_\infty, E_\rho^\Lambda)$ and $C(Q_\infty^\Lambda, E_\rho^\Lambda)$ respectively, via the mappings

$$\begin{aligned}
f \in (C_0^p(Q, E), \|\cdot\|_{D,\rho}) &\mapsto \hat{f} \in (C(Q_\infty, E_\rho^\Lambda), \|\cdot\|_\infty), \quad \hat{f}(s) = (\partial^\lambda f(s))_{\lambda \in \Lambda}, \\
f \in (C_0^p(Q, E), \|\cdot\|_{S,\rho}) &\mapsto \tilde{f} \in (C(Q_\infty^\Lambda, E_\rho^\Lambda), \|\cdot\|_\infty), \quad \tilde{f}((s_\lambda)_{\lambda \in \Lambda}) = (\partial^\lambda f(s_\lambda))_{\lambda \in \Lambda}.
\end{aligned}$$

Fix natural numbers $n \geq 1$ and $q \geq 1$. Let F be a Banach space and $K \subseteq \mathbb{R}^n$ be an open subset. Set $\Gamma = \{(\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n : \gamma_1 + \dots + \gamma_n \leq q\}$. The function

space $C_0^q(K, F)$ and the norms $\theta : \mathbb{R}^\Gamma \rightarrow \mathbb{R}$ and $\theta^* : \mathbb{R}^\Gamma \rightarrow \mathbb{R}$ are defined similarly as $C_0^p(Q, E)$, ρ and ρ^* respectively. Let $T : C_0^p(Q, E) \rightarrow C_0^q(K, F)$ be a surjective linear isometry.

2.2 Abstract Framework

In this section, we develop an abstract framework and apply it to the isometric embedding from $C_0^p(Q, E)$ into $C(X, E_\rho^\Lambda)$, for a suitable compact Hausdorff space X . We will apply the main results from this section to Chapters 3 and 5, where X is Q_∞ and Q_∞^Λ for diagonal norm and square norm respectively. To this end, we fix a Banach space H and a closed subspace L of $C(X, H)$. In the subsequent sections, we will obtain results for the setting above.

2.2.1 Extreme points of the dual of function space

In this section, we will investigate the form of extreme points $\text{ext } B(L^*)$ of the ball of L^* . Recall that x is an *extreme point* of $B(H)$ if $x = \frac{1}{2}(x_1 + x_2)$ for some $x_1, x_2 \in B(H)$, then $x = x_1 = x_2$. Note that we have an isometric embedding from $C(X, H)$ into $C(X \times B(H^*))$ via the mapping

$$f \mapsto \hat{f} \quad \text{where} \quad \hat{f}(s, x^*) = x^*(f(s)). \quad (2.2.1)$$

For each $s \in X$ and $x^* \in H^*$, let $x^* \circ \delta_s$ be the bounded linear functional on $C(X, H)$ given by $\langle f, x^* \circ \delta_s \rangle = x^*(f(s))$.

We start this section with the following well-known proposition, which asserts that all extreme points of a subspace of $B[C(X, H)^*]$ have the form $x^* \circ \delta_s$ for some $s \in X$ and $x^* \in \text{ext } B(H^*)$.

Proposition 2.2. [11, Corollary 2.3.6] *If L is a subspace of $C(X, H)$, then*

$$\text{ext } B(L^*) \subseteq \{x^* \circ \delta_s : s \in X, x^* \in \text{ext } B(H^*)\}.$$

However, clearly the equality may not be true in general. Nonetheless, if s_0 and x_0^* satisfy properties (1) and (2) in the following proposition, then $x_0^* \circ \delta_{s_0}$ is an extreme point of $B(L^*)$.

Proposition 2.3. *Suppose that $s_0 \in X$ and $x_0^* \in \text{ext } B(H^*)$. Assume that*

- (1) *for any neighborhood $U \subseteq X$ of s_0 and any $\varepsilon_0 > 0$, there exist $0 < \varepsilon < \varepsilon_0$ and $f \in L$ such that $\|f\|_L = 1$, $x_0^*(f(s_0)) > 1 - \varepsilon$ and $\|f(s)\| < 1 - \sqrt{\varepsilon}$ if $s \notin U$, and*
- (2) *for any $x \in H$, there exists $h \in L$ so that $h(s_0) = x$.*

Then $(x_0^ \circ \delta_{s_0})|_L$ is an extreme point of $B(L^*)$.*

Proof. Suppose that

$$(x_0^* \circ \delta_{s_0})|_L = \frac{1}{2}(u_1^* + u_2^*) \quad (2.2.2)$$

for some $u_1^*, u_2^* \in B(L^*)$. Since L is a closed subspace of $C(X, H)$ and $C(X, H)$ can be embedded isometrically into $C[X \times B(H^*)]$ via (2.2.1), so we may view L as a subspace of $C[X \times B(H^*)]$. Fix an $i \in \{1, 2\}$. By Hahn-Banach Theorem, we may extend u_i^* to a bounded linear functional on $C[X \times B(H^*)]$. Now, by applying Riesz-Markov-Kakutani Representation Theorem [28, Theorem 2.14] on the extension of u_i^* , there exist regular Borel signed measures μ_i on $C[X \times B(H^*)]$ with $\|\mu_i\| \leq 1$ and $\mu_i|_L = u_i^*$ such that

$$\langle \hat{f}, \mu_i \rangle = \int_{X \times B(H^*)} \hat{f} d\mu_i \quad \text{for all } \hat{f} \in C[X \times B(H^*)]. \quad (2.2.3)$$

We claim that $|\mu_i|(\{s_0\}^c \times B(H^*)) = 0$ for all $i = 1, 2$. Take any neighbourhood U of s_0 and any $\varepsilon_0 > 0$. Obtain $0 < \varepsilon < \varepsilon_0$ and f as in assumption (1). Then

$$\begin{aligned} 1 - \varepsilon &< x_0^*(f(s_0)) && \text{by assumption (1)} \\ &= \frac{1}{2}(\langle f, u_1^* \rangle + \langle f, u_2^* \rangle) && \text{by (2.2.2).} \end{aligned} \quad (2.2.4)$$

For each $i = 1, 2$, since $f \in L$, so $\langle f, u_i^* \rangle = \langle \hat{f}, \mu_i \rangle$. It follows that

$$\begin{aligned}
 \langle f, u_i^* \rangle &= \int_{U \times B(H^*)} \hat{f} d\mu_i + \int_{U^c \times B(H^*)} \hat{f} d\mu_i && \text{by (2.2.3)} \\
 &\leq |\mu_i|(U \times B(H^*)) + \int_{U^c \times B(H^*)} \hat{f} d\mu_i && \text{since } \|\mu_i\| \leq 1 \\
 &\leq |\mu_i|(U \times B(H^*)) + (1 - \sqrt{\varepsilon})|\mu_i|(U^c \times B(H^*)) && \text{by assumption (1).} \\
 & && (2.2.5)
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 2 - 2\varepsilon &< \langle f, u_1^* \rangle + \langle f, u_2^* \rangle && \text{by (2.2.4)} \\
 &\leq \sum_{i=1}^2 (|\mu_i|(U \times B(H^*)) + (1 - \sqrt{\varepsilon})|\mu_i|(U^c \times B(H^*))) && \text{by (2.2.5)} \\
 &= 2 - \sqrt{\varepsilon} \sum_{i=1}^2 |\mu_i|(U^c \times B(H^*)).
 \end{aligned}$$

Hence, $\sum_{i=1}^2 |\mu_i|(U^c \times B(H^*)) < \sqrt{\varepsilon} < \sqrt{\varepsilon_0}$. Since $\varepsilon_0 > 0$ is arbitrary, so $|\mu_i|(U^c \times B(H^*)) = 0$. As μ_i is regular, it follows that $|\mu_i|(\{s_0\}^c \times B(H^*)) = 0$. This concludes the proof of the claim.

If $x \in H$, the function $g_x : (s, x^*) \mapsto x^*(x)$ belongs to $C(X \times B(H^*))$. Fix any $i = 1, 2$. Define $x_i^* : H \rightarrow \mathbb{R}$ by

$$x_i^*(x) = \int_{\{s_0\} \times B(H^*)} g_x d\mu_i. \quad (2.2.6)$$

We claim that $x_0^* = x_i^*$. Clearly $x_i^* \in B(H^*)$. For any $x \in H$, from property (2), let $h \in L$ be such that $h(s_0) = x$. Then $g_x(s_0, x^*) = \hat{h}(s_0, x^*)$ for any $x^* \in H^*$.

Therefore,

$$\begin{aligned}
x_0^*(x) &= \langle h, x_0^* \circ \delta_{s_0} \rangle && \text{since } h(s_0) = x \\
&= \frac{1}{2} (\langle h, u_1^* \rangle + \langle h, u_2^* \rangle) && \text{by (2.2.2)} \\
&= \frac{1}{2} (\langle h, \mu_1^* \rangle + \langle h, \mu_2^* \rangle) && \text{since } h \in L \\
&= \frac{1}{2} \left(\int_{X \times B(H^*)} \hat{h} d\mu_1 + \int_{X \times B(H^*)} \hat{h} d\mu_2 \right) && \text{by (2.2.3)} \\
&= \frac{1}{2} \left(\int_{\{s_0\} \times B(H^*)} g_x d\mu_1 + \int_{\{s_0\} \times B(H^*)} g_x d\mu_2 \right) && \text{since } |\mu_i|(\{s_0\}^c \times B(H^*)) = 0 \\
&= \frac{1}{2} (x_1^* + x_2^*)(x) && \text{by (2.2.6).}
\end{aligned}$$

Since $x \in H$ is arbitrary, so $x_0^* = \frac{1}{2}(x_1^* + x_2^*)$. Since $x_0^* \in \text{ext } B(H^*)$, so $x_i^* = x_0^*$.

Finally, for any $f \in L$, let $x = f(s_0)$. Then $\hat{f}(s_0, x^*) = g_x(s_0, x^*)$ for any $x^* \in H^*$.

Thus, for any $i = 1, 2$,

$$\begin{aligned}
\langle f, u_i^* \rangle &= \int_{X \times B(H^*)} \hat{f} d\mu_i && \text{by (2.2.3)} \\
&= \int_{\{s_0\} \times B(H^*)} \hat{f} d\mu_i && \text{since } |\mu_i|(\{s_0\}^c \times B(H^*)) = 0 \\
&= \int_{\{s_0\} \times B(H^*)} g_x d\mu_i && \text{since } \hat{f}(s_0, x^*) = g_x(s_0, x^*) \\
&= x_i^*(x) && \text{by (2.2.6)} \\
&= \langle f, x_i^* \circ \delta_{s_0} \rangle && \text{since } x = f(s_0).
\end{aligned}$$

Hence, $u_i^* = (x_i^* \circ \delta_{s_0})|_L = (x_0^* \circ \delta_{s_0})|_L$. Therefore, $x_0^* \circ \delta_{s_0}$ is an extreme point of $B(L^*)$. This concludes the proof of the proposition. $\square$

Proposition 2.3 provides a criterion to identify extreme points of $B(L^*)$. However, conditions (1) and (2) are hard to verify and thus it is difficult to apply Proposition 2.3 directly. Nonetheless, we are only interested in the case where $H = E_\rho^\Lambda$. In this case, we can establish a more useful version of Proposition 2.3. The following lemma characterises the extreme points of $B[(E_\rho^\Lambda)^*]$.

Lemma 2.4.

$$\text{ext } B[(E_\rho^\Lambda)^*] = \{(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} : (\alpha_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B(\mathbb{R}^\Lambda_{\rho^*}), x_\lambda^* \in \text{ext } B(E^*)\}.$$

Proof. Let A be the set

$$\{(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} : x_\lambda^* \in \text{ext } B(E^*), (\alpha_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(\mathbb{R}^\Lambda, \rho^*)], \alpha_\lambda^* \geq 0\}.$$

We wish to prove that $A \subseteq \text{ext } B[(E_\rho^\Lambda)^*]$. Clearly $A \subseteq B[(E_\rho^\Lambda)^*]$. Let $\alpha^* x^* = (\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} \in A$. Suppose that

$$\alpha^* x^* = \frac{1}{2}(u^* + v^*)$$

for some $u^* = (u_\lambda^*)_{\lambda \in \Lambda}$ and $v^* = (v_\lambda^*)_{\lambda \in \Lambda}$ in $B[(E_\rho^\Lambda)^*]$. Then for every $\lambda \in \Lambda$, we have

$$\alpha_\lambda^* x_\lambda^* = \frac{1}{2}(u_\lambda^* + v_\lambda^*). \quad (2.2.7)$$

By triangle inequality of $\|\cdot\|$, we have

$$\alpha_\lambda^* = \|\alpha_\lambda^* x_\lambda^*\| \leq \frac{1}{2}(\|u_\lambda^*\| + \|v_\lambda^*\|).$$

Clearly $(\|u_\lambda^*\|)_{\lambda \in \Lambda}$ and $(\|v_\lambda^*\|)_{\lambda \in \Lambda}$ are in $B[(\mathbb{R}^\Lambda, \rho^*)]$.

We claim that $\alpha_\lambda^* = \frac{1}{2}(\|u_\lambda^*\| + \|v_\lambda^*\|)$ for all $\lambda \in \Lambda$. Observe that

$$(\alpha_\lambda^*)_{\lambda \in \Lambda} = \frac{\left[\left(\frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2}\right)_{\lambda \in \Lambda}\right] + \left[\left(2\alpha_\lambda^* - \frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2}\right)_{\lambda \in \Lambda}\right]}{2}.$$

Since both $(\|u_\lambda^*\|)_{\lambda \in \Lambda}$ and $(\|v_\lambda^*\|)_{\lambda \in \Lambda}$ are in $B[(\mathbb{R}^\Lambda, \rho^*)]$, by triangle inequality of ρ^* ,

$$\rho^* \left[\left(\frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2} \right)_{\lambda \in \Lambda} \right] \leq \frac{1}{2} [\rho^*((\|u_\lambda^*\|)_{\lambda \in \Lambda}) + \rho^*((\|v_\lambda^*\|)_{\lambda \in \Lambda})] \leq 1.$$

Since $\left| 2\alpha_\lambda^* - \frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2} \right| \leq \frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2}$, by monotonicity of ρ^* , we have

$$\rho^* \left[\left(2\alpha_\lambda^* - \frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2} \right)_{\lambda \in \Lambda} \right] \leq \rho^* \left[\left(\frac{\|u_\lambda^*\| + \|v_\lambda^*\|}{2} \right)_{\lambda \in \Lambda} \right] \leq 1.$$

By extremity of $(\alpha_\lambda^*)_{\lambda \in \Lambda}$, it follows that for all $\lambda \in \Lambda$,

$$\alpha_\lambda^* = \frac{1}{2}(\|u_\lambda^*\| + \|v_\lambda^*\|).$$

Now, it follows from $\alpha_\lambda^* = \frac{1}{2}(\|u_\lambda^*\| + \|v_\lambda^*\|)$ for all $\lambda \in \Lambda$ and extremity of $(\alpha_\lambda^*)_{\lambda \in \Lambda}$ that

$$(\alpha_\lambda^*)_{\lambda \in \Lambda} = (\|u_\lambda^*\|)_{\lambda \in \Lambda} = (\|v_\lambda^*\|)_{\lambda \in \Lambda}.$$

In particular, for every $\lambda \in \Lambda$,

$$\alpha_\lambda^* = \|u_\lambda^*\| = \|v_\lambda^*\|.$$

If $\alpha_\lambda^* = 0$, then

$$\alpha_\lambda^* x_\lambda^* = u_\lambda^* = v_\lambda^* = 0.$$

If $\alpha_\lambda^* \neq 0$, from (2.2.7),

$$\begin{aligned} x_\lambda^* &= \frac{1}{2\alpha_\lambda^*} (u_\lambda^* + v_\lambda^*) \\ &= \frac{1}{2} \left(\frac{u_\lambda^*}{\|u_\lambda^*\|} + \frac{v_\lambda^*}{\|v_\lambda^*\|} \right). \end{aligned}$$

For every $\lambda \in \Lambda$, by extremity of x_λ^* , we have

$$x_\lambda^* = \frac{u_\lambda^*}{\|u_\lambda^*\|} = \frac{v_\lambda^*}{\|v_\lambda^*\|}.$$

It follows that $\alpha_\lambda^* x_\lambda^* = u_\lambda^* = v_\lambda^*$ for all $\lambda \in \Lambda$. In either case, we have

$$\alpha^* x^* = u^* = v^*.$$

Therefore, $\alpha^* y^* \in \text{ext } B[(E_\rho^\Lambda)^*]$. Hence, $A \subseteq \text{ext } B[(E_\rho^\Lambda)^*]$.

We wish to prove that $\text{ext } B[(E_\rho^\Lambda)^*] \subseteq A$. Fix any extreme point $z^* = (z_\lambda^*)$ of $B[(E_\rho^\Lambda)^*]$. For each $\lambda \in \Lambda$, if $z_\lambda^* = 0$, let x_λ^* be any extreme point of $B(E^*)$. If $z_\lambda^* \neq 0$, let

$$x_\lambda^* = \frac{z_\lambda^*}{\|z_\lambda^*\|}.$$

Also, let $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in (\mathbb{R}^\Lambda, \rho^*)$ be such that

$$\alpha_\lambda^* = \|z_\lambda^*\|.$$

Clearly $z^* = (\alpha_\lambda x_\lambda^*)_{\lambda \in \Lambda}$, $\alpha_\lambda^* \geq 0$ for all $\lambda \in \Lambda$ and $\rho^*[(\alpha_\lambda^*)_{\lambda \in \Lambda}] = 1$. We claim that x_λ^* is an extreme point of $B(E^*)$ for all $\lambda \in \Lambda$. Fix $\lambda_0 \in \Lambda$. If $z_{\lambda_0}^* = 0$, then by choice, $x_{\lambda_0}^*$ is an extreme point of $B(E^*)$. Assume that $z_{\lambda_0}^* \neq 0$. Clearly $x_{\lambda_0}^*$ has norm 1. Let

$$x_{\lambda_0}^* = \frac{1}{2}(u_{\lambda_0}^* + v_{\lambda_0}^*)$$

for some $u_{\lambda_0}^*$ and $v_{\lambda_0}^*$ in $B(E^*)$. Since $x_{\lambda_0}^*$ has norm 1, so are $u_{\lambda_0}^*$ and $v_{\lambda_0}^*$. Then

$$z_{\lambda_0}^* = \alpha_{\lambda_0}^* x_{\lambda_0}^* = \frac{1}{2}(\alpha_{\lambda_0}^* u_{\lambda_0}^* + \alpha_{\lambda_0}^* v_{\lambda_0}^*).$$

Let

$$u^* = [(z_\lambda)_{\lambda \neq \lambda_0}, \alpha_{\lambda_0}^* u_{\lambda_0}^*] \quad \text{and} \quad v^* = [(z_\lambda)_{\lambda \neq \lambda_0}, \alpha_{\lambda_0}^* v_{\lambda_0}^*].$$

Then

$$z^* = \frac{1}{2}(u^* + v^*)$$

Since $\rho^*[(\alpha_\lambda^*)_{\lambda \in \Lambda}] = 1$, it follows that

$$\rho^*[(\|z_\lambda\|)_{\lambda \neq \lambda_0}, \alpha_{\lambda_0}^* \|u_{\lambda_0}^*\|] = \rho^*[(\alpha_\lambda^*)_{\lambda \in \Lambda}] = 1.$$

Similarly,

$$\rho^*[(\|z_\lambda\|)_{\lambda \neq \lambda_0}, \alpha_{\lambda_0}^* \|v_{\lambda_0}^*\|] = 1.$$

Since both u^* and v^* are in $B[(E_\rho^\Lambda)^*]$, by extremity of z^* , we have $z^* = u^* = v^*$. In particular,

$$x_{\lambda_0}^* = u_{\lambda_0}^* = v_{\lambda_0}^*.$$

Hence, $x_{\lambda_0}^*$ is an extreme point of $B(E^*)$.

It remains to prove that $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(\mathbb{R}^\Lambda, \rho^*)]$. Suppose that

$$(\alpha_\lambda^*)_{\lambda \in \Lambda} = \frac{1}{2}[(\beta_\lambda^*)_{\lambda \in \Lambda} + (\gamma_\lambda^*)_{\lambda \in \Lambda}]$$

for some $(\beta_\lambda^*)_{\lambda \in \Lambda}$ and $(\gamma_\lambda^*)_{\lambda \in \Lambda}$ in $B[(\mathbb{R}^\Lambda, \rho^*)]$. Since $(\alpha_\lambda^*)_{\lambda \in \Lambda}$ has norm 1 in $(\mathbb{R}^\Lambda, \rho^*)$, so are $(\beta_\lambda^*)_{\lambda \in \Lambda}$ and $(\gamma_\lambda^*)_{\lambda \in \Lambda}$. For each $\lambda \in \Lambda$, we have

$$\alpha_\lambda^* = \frac{1}{2}(\beta_\lambda^* + \gamma_\lambda^*).$$

If $\alpha_\lambda^* = 0$, then $\beta_\lambda^* + \gamma_\lambda^* = 0$. If $\alpha_\lambda^* \neq 0$, then equation above implies that

$$1 = \frac{1}{2} \left(\frac{\beta_\lambda^*}{\alpha_\lambda^*} + \frac{\gamma_\lambda^*}{\alpha_\lambda^*} \right). \quad (2.2.8)$$

Let

$$u_\lambda^* = \begin{cases} \frac{\beta_\lambda^*}{\alpha_\lambda^*} z_\lambda^* & \text{if } z_\lambda^* \neq 0 \\ \beta_\lambda^* y_\lambda^* & \text{if } z_\lambda^* = 0 \end{cases}$$

and

$$v_\lambda^* = \begin{cases} \frac{\gamma_\lambda^*}{\alpha_\lambda^*} z_\lambda^* & \text{if } z_\lambda^* \neq 0 \\ \gamma_\lambda^* y_\lambda^* & \text{if } z_\lambda^* = 0 \end{cases}$$

where $y_\lambda^* \in \text{ext } B(E^*)$. Observe that

$$(z_\lambda^*)_{\lambda \in \Lambda} = \frac{1}{2} [(u_\lambda^*)_{\lambda \in \Lambda} + (v_\lambda^*)_{\lambda \in \Lambda}].$$

Indeed, if $z_\lambda^* \neq 0$, then from (2.2.8),

$$z_\lambda^* = \frac{1}{2} \left(\frac{\beta_\lambda^*}{\alpha_\lambda^*} + \frac{\gamma_\lambda^*}{\alpha_\lambda^*} \right) z_\lambda^* = \frac{1}{2} (u_\lambda^* + v_\lambda^*).$$

If $z_\lambda^* = 0$, since $\beta_\lambda^* + \gamma_\lambda^* = 0$, we also have $z_\lambda^* = \frac{1}{2} (u_\lambda^* + v_\lambda^*)$.

Note that

$$\rho^* [(\|u_\lambda^*\|)_{\lambda \in \Lambda}] = \rho^* [(\beta_\lambda^*)_{\lambda \in \Lambda}] = 1.$$

Similarly,

$$\rho^* [(\|v_\lambda^*\|)_{\lambda \in \Lambda}] = 1.$$

By extremity of $(z_\lambda^*)_{\lambda \in \Lambda}$, we have

$$(z_\lambda^*)_{\lambda \in \Lambda} = (u_\lambda^*)_{\lambda \in \Lambda} = (v_\lambda^*)_{\lambda \in \Lambda}.$$

Since $y_\lambda^* \neq 0$, we have

$$\alpha_\lambda^* = \beta_\lambda^* = \gamma_\lambda^* \quad \text{for all } \lambda \in \Lambda.$$

Therefore, $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(\mathbb{R}^\Lambda, \rho^*)]$. Hence, $\text{ext } B[(E_\rho^\Lambda)^*] \subseteq A$. Thus, this concludes the proof of the lemma. $\square$

By Proposition 2.2 and Lemma 2.4, all extreme points of $B(L^*)$ are of the form $(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} \circ \delta_s$. We want to identify those ‘good’ coefficients $(\alpha_\lambda^*)_{\lambda \in \Lambda}$ so that

$(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} \circ \delta_s$ is an extreme point of $B(L^*)$ for any $x_\lambda^* \in \text{ext } B[(E_\rho^\Lambda)^*]$. This motivates the definition of $\mathcal{P}$ and $\mathcal{G}$ sets below.

For any $s_0 \in X$, let $\mathcal{P}(s_0)$ be the subset of $\mathbb{R}^\Lambda$ consisting of all $(\alpha_\lambda)_{\lambda \in \Lambda}$ such that

1. $\alpha_\lambda > 0$ for all $\lambda \in \Lambda$,
2. $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1$ and
3. for any neighborhood U of s_0 , there exists $\delta > 0$ so that if $x_\lambda \in E$, $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$, then there exists $f \in L$ so that $\|f\| = 1$, $f(s_0) = (\alpha_\lambda x_\lambda)_{\lambda \in \Lambda}$ and $\sup_{s \notin U} \|f(s)\| \leq 1 - \delta$.

Let $\mathcal{G}(s_0)$ be the set of all $(\alpha_\lambda^*)_{\lambda \in \Lambda}$ in $\text{ext } B(\mathbb{R}_{\rho^*}^\Lambda)$ with $\alpha_\lambda^* \geq 0$ for all $\lambda \in \Lambda$ so that there exists $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}(s_0)$ with

$$\sum_{\lambda \in \Lambda} \alpha_\lambda^* \alpha_\lambda = 1. \quad (2.2.9)$$

Now, we are ready to prove a version of Proposition 2.3 with $H = E_\rho^\Lambda$.

Proposition 2.5. *Assume that $H = E_\rho^\Lambda$. For any $s_0 \in X$ and $x_0^* = (\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda}$, if $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}(s_0)$ and $x_\lambda^* \in \text{ext } B(E^*)$ for all $\lambda \in \Lambda$, then $(x_0^* \circ \delta_{s_0})|_L$ is an extreme point of $B(L^*)$.*

Proof. We claim that the pair (s_0, x_0^*) satisfy the hypotheses of Proposition 2.3, from which we can conclude that $(x_0^* \circ \delta_{s_0})|_L$ is an extreme point of $B(L^*)$. Let $\alpha = (\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}(s_0)$ be as in the definition of $\mathcal{G}(s_0)$ above with respect to $\alpha^* = (\alpha_\lambda^*)_{\lambda \in \Lambda}$. First observe that $x_0^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ by Lemma 2.4. Now let a neighbourhood U of s_0 and $\varepsilon_0 > 0$ be given. Obtain δ as in property (3) of $\mathcal{P}(s_0)$ above for the point s_0 and neighbourhood U . Choose ε so that $0 < \varepsilon < \varepsilon_0 \wedge \delta^2$. Then let $x_\lambda \in E$, $\|x_\lambda\| = 1$ be such that $x_\lambda^*(x_\lambda) > 1 - \varepsilon$ for all $\lambda \in \Lambda$. By choice of δ , there exists $f \in L$ so that $\|f\| = 1$, $f(s_0) = (\alpha_\lambda x_\lambda)_{\lambda \in \Lambda}$, and $\sup_{s \notin U} \|f(s)\| \leq 1 - \delta$. Then

$$x_0^*(f(s_0)) = \sum_{\lambda \in \Lambda} \alpha_\lambda \alpha_\lambda^* x_\lambda^*(x_\lambda) > 1 - \varepsilon.$$

If $s \notin U$, then

$$\|f(s)\| \leq \sup_{s \notin U} \|f(s)\| \leq 1 - \delta < 1 - \sqrt{\varepsilon}.$$

Hence, conditions (1) and (2) of Proposition 2.3 are satisfied. This concludes the proof of the proposition. $\square$

By Banach-Alaoglu Theorem and Krein-Milman Theorem, $B(E^*)$ has at least one extreme point. Therefore, to apply Proposition 2.5 above, it is sufficient to show that $\mathcal{G}(s_0)$ is nonempty for all $s_0 \in X$. To achieve this, we need the following lemma, which asserts the existence of a norming extreme point.

Lemma 2.6. *For any x in a Banach space, there exists an extreme point x^* in the dual closed unit ball such that $x^*(x) = \|x\|$.*

The following corollary follows immediately from Lemma 2.6.

Corollary 2.7. *For any $s_0 \in X$, if $\mathcal{P}(s_0)$ is nonempty, so is $\mathcal{G}(s_0)$.*

By Corollary 2.7, we see that as long as $\mathcal{P}(s)$ is nonempty, there will be such extreme points. We end this section with Proposition 2.9, which asserts that as long as $\mathcal{P}(s)$ is nonempty, the linear span of such extreme points contains extreme points of $B[(E_\rho^\Lambda)^*]$. To this end, we need a result which ensures that all $(\alpha_\lambda^*)_{\lambda \in \Lambda}$ in $\mathcal{G}(s)$ are nonzero, which follows from the strict monotonicity of ρ .

Corollary 2.8. *If ρ is strictly monotone, then for any $s \in X$, all $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}(s)$ satisfies $\alpha_\lambda^* \neq 0$ for all $\lambda \in \Lambda$.*

Proof. Suppose that $\alpha_{\lambda_0}^* = 0$ for some $\lambda_0 \in \Lambda$. Since $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}(s)$, by definition of $\mathcal{G}(s)$, we have $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in B(\mathbb{R}_{\rho^*}^\Lambda)$ and $\sum_{\lambda \in \Lambda} \alpha_\lambda^* \alpha_\lambda = 1$ for some $\alpha = (\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}(s)$. By the contrapositive of Lemma 2.1, we have $\alpha_{\lambda'} = 0$ for some $\lambda' \in \Lambda$, contradicting $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}(s)$. This concludes the proof of the corollary. $\square$

Proposition 2.9. *Fix any $s \in X$ and assume that $\mathcal{P}(s) \neq \emptyset$. Let*

$$A = \{(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} : (\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}(s), x_\lambda^* \in \text{ext } B(E^*) \text{ for all } \lambda \in \Lambda\}.$$

Then $\text{ext } B[(E_\rho^\Lambda)^] \subseteq \text{span}(A)$. In particular, if $E = \mathbb{R}$, then*

$$\text{ext } B(\mathbb{R}_{\rho^*}^\Lambda) \subseteq \text{span}\{(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} : (\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}(s), |x_\lambda^*| = 1\}.$$

Proof. To prove that $\text{ext } B[(E_\rho^\Lambda)^*] \subseteq \text{span}(A)$, by Lemma 2.4, it suffices to show that for any $\lambda_0 \in \Lambda$ and $x_{\lambda_0}^* \in \text{ext } B(E^*)$, we have $((0)_{\lambda \neq \lambda_0}, x_{\lambda_0}^*) \in \text{span}(A)$. Fix $\lambda_0 \in \Lambda$ and choose $(x_\lambda^*)_{\lambda \in \Lambda}$ where $x_\lambda^* \in \text{ext } B(E^*)$ for all $\lambda \in \Lambda$. Since $\mathcal{P}(s) \neq \emptyset$, let $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}(s)$. By Lemma 2.6, there exists $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B(\mathbb{R}_{\rho^*}^\Lambda)$ such that $\sum_{\lambda \in \Lambda} \alpha_\lambda^* \alpha_\lambda = \|(\alpha_\lambda)_{\lambda \in \Lambda}\|_\rho = 1$. So, we have $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}(s)$. Therefore, we have $(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} \in A$. Since $-x_{\lambda_0}^* \in \text{ext } B(E^*)$, so $((-\alpha_\lambda^* x_\lambda^*)_{\lambda \neq \lambda_0}, \alpha_{\lambda_0}^* x_{\lambda_0}^*) \in A$. Since ρ is strictly monotone, by Corollary 2.8, we have $\alpha_{\lambda_0}^* \neq 0$. It follows that

$$((0)_{\lambda \neq \lambda_0}, x_{\lambda_0}^*) = \frac{1}{2\alpha_{\lambda_0}^*} [(\alpha_\lambda^* x_\lambda^*)_{\lambda \in \Lambda} - (\alpha_{\lambda_0}^* x_{\lambda_0}^*, (-\alpha_\lambda^* x_\lambda^*)_{\lambda \neq \lambda_0})] \in \text{span}(A).$$

This concludes the proof of the proposition. $\square$

By Corollary 2.7, it remains to show that $\mathcal{P}(\cdot)$ is nonempty, which is the main objective of Sections 3.1 and 5.1.

2.2.2 Weak-star extension

In this section, we obtain results that are useful to Section 3.2 and Section 5.2. The main result of this section is Proposition 2.12.

Proposition 2.10 asserts that the weak-star limit of a net $(x_\alpha^* \circ \delta_s)_\alpha$ must have the same form. It plays an important role in the proof of Proposition 2.12.

Recall that L is a closed subspace of $C(X, E_\rho^\Lambda)$. Also, for any $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s \in X$, $x^* \circ \delta_s$ is a functional on L defined by

$$\langle f, x^* \circ \delta_s \rangle = \langle f(s), (x_\lambda^*)_{\lambda \in \Lambda} \rangle \quad \text{for all } f \in L.$$

For any $g \in C(X, \mathbb{R}_\rho^\Lambda)$ and $e \in E$, if $g = (g_\alpha)_{\alpha \in \Lambda}$ where $g_\alpha : X \rightarrow \mathbb{R}$ for all $\alpha \in \Lambda$, then we let $g \otimes e : X \rightarrow E_\rho^\Lambda$ be a continuous function such that $(g \otimes e)(s) = (g_\alpha(s) \cdot e)_{\lambda \in \Lambda}$.

Proposition 2.10. *Let $(x_\alpha^* \circ \delta_s)_\alpha$ be a net in L^* where $x_\alpha^* \in (E_\rho^\Lambda)^*$ and $s \in X$. Assume that for any $\lambda_0 \in \Lambda$, there exists $g_{\lambda_0} \in C(X, \mathbb{R}_\rho^\Lambda)$ such that $g_{\lambda_0}(s) = ((0)_{\lambda \neq \lambda_0}, 1)$ and $g_{\lambda_0} \otimes e \in L$ for any $e \in E$. If $\|x_\alpha^* \circ \delta_s\| \leq 1$ for all α and $(x_\alpha^* \circ \delta_s)_\alpha$ converges in the weak* topology of L^* , then the weak-star limit of $(x_\alpha^* \circ \delta_s)_\alpha$ is $x^* \circ \delta_s$ for some $x^* \in (E_\rho^\Lambda)^*$.*

Proof. For each α , let $x_\alpha^* = (x_{\alpha\lambda}^*)_{\lambda \in \Lambda}$. Fix $\lambda_0 \in \Lambda$. We claim that $(x_{\alpha\lambda_0}^*)_\alpha$ converges to $x_{\lambda_0}^*$ weak-star for some $x_{\lambda_0}^* \in E^*$. Indeed, fix any $e_{\lambda_0} \in E$. Obtain g_{λ_0} from hypothesis and set $f = g_{\lambda_0} \otimes e_{\lambda_0}$. By hypothesis, $f \in L$. Also, we have $f(s) = ((0)_{\lambda \neq \lambda_0}, e_{\lambda_0})$. Since $(x_\alpha^* \circ \delta_s)_\alpha$ has a weak-star limit, so

$$\lim_\alpha \langle f, x_\alpha^* \circ \delta_s \rangle = \lim_\alpha \langle ((0)_{\lambda \neq \lambda_0}, e_{\lambda_0}), (x_{\alpha\lambda}^*)_{\lambda \in \Lambda} \rangle = \lim_\alpha x_{\alpha\lambda_0}^*(e_{\lambda_0}) \quad (2.2.10)$$

exists. Define $x_{\lambda_0}^* : E \rightarrow \mathbb{R}$ by $x_{\lambda_0}^*(e_{\lambda_0}) = \lim_\alpha x_{\alpha\lambda_0}^*(e_{\lambda_0})$. It remains to show that $x_{\lambda_0}^*$ is bounded. From (2.2.10), we have

$$|x_{\lambda_0}^*(e_{\lambda_0})| = \left| \lim_\alpha \langle f, x_\alpha^* \circ \delta_s \rangle \right| \leq \lim_\alpha \|f\|_L \cdot \|x_\alpha^* \circ \delta_s\| \leq \|f\|_L \leq \|g_{\lambda_0}\|_L \cdot \|e_{\lambda_0}\|.$$

So, $x_{\lambda_0}^*$ is bounded. This concludes the proof of the claim.

Now, we let $x^* : E_\rho^\Lambda \rightarrow \mathbb{R}$ given by $x^*[(e_\lambda)_{\lambda \in \Lambda}] = \sum_{\lambda \in \Lambda} x_\lambda^*(e_\lambda)$. Clearly x^* is linear. It remains to show that $x^* \circ \delta_s$ is the weak-star limit of $(x_\alpha^* \circ \delta_s)_\alpha$ in L^* . Fix any $f \in L$ and let $f(s) = (e_\lambda)_{\lambda \in \Lambda}$. Then

$$\begin{aligned} \lim_\alpha \langle f, x_\alpha^* \circ \delta_s \rangle &= \lim_\alpha \langle (e_\lambda)_{\lambda \in \Lambda}, (x_{\alpha\lambda}^*)_{\lambda \in \Lambda} \rangle && \text{by definition of } \langle \cdot, \cdot \rangle \\ &= \lim_\alpha \sum_{\lambda \in \Lambda} x_{\alpha\lambda}^*(e_\lambda) \\ &= \sum_{\lambda \in \Lambda} x_\lambda^*(e_\lambda) && \text{by definition of } x_\lambda^* \\ &= \langle f, x^* \circ \delta_s \rangle. \end{aligned}$$

This concludes the proof of the proposition. □

We will use the following simple lemma in the proof of Proposition 2.12.

Lemma 2.11. *For any $x^* \in (E_\rho^\Lambda)^*$ and $s \in X$, we have $\|x^* \circ \delta_s\| \leq \|x^*\|$*

Proof. Let $x^* = (x_\lambda^*)_{\lambda \in \Lambda}$. Fix any $f \in B(L)$ and set $f(s) = (x_\lambda)_{\lambda \in \Lambda}$. Then

$$\begin{aligned}
|\langle f, x^* \circ \delta_s \rangle| &= \left| \sum_{\lambda \in \Lambda} x_\lambda^*(x_\lambda) \right| \\
&\leq \sum_{\lambda \in \Lambda} \|x_\lambda^*\| \cdot \|x_\lambda\| && \text{by triangle inequality} \\
&\leq \rho^*[(\|x_\lambda^*\|)_{\lambda \in \Lambda}] \cdot \rho[(\|x_\lambda\|)_{\lambda \in \Lambda}] && \text{by Inequality 2.1.1} \\
&\leq \rho^*[(\|x_\lambda^*\|)_{\lambda \in \Lambda}] && \text{since } \|f\|_L \leq 1 \\
&= \|x^*\|.
\end{aligned}$$

Since f is arbitrary, so $\|x^* \circ \delta_s\| \leq \|x^*\|$. This concludes the proof of the lemma. $\square$

For any subset N of $(F_\theta^\Gamma)^*$, let $co(N)$ be the convex hull of N . Let M be a closed subspace of $C(Y, F_\theta^\Gamma)$, where Y is a compact Hausdorff space. For any $y_\alpha^*, y^* \in (F_\theta^\Gamma)^*$, we denote the weak-star convergence of the net $(y_\alpha^*)_\alpha$ to y^* as $y_\alpha^* \xrightarrow{w^*} y^*$. We end this section with the following proposition.

Proposition 2.12. *Let N be a subset of $(F_\theta^\Gamma)^*$ that satisfies $B[(F_\theta^\Gamma)^*] = \overline{co(N)}^{w^*}$. Assume that for any $s \in X$ and $\lambda_0 \in \Lambda$, there exists $g_{\lambda_0} \in C(X, \mathbb{R}_\rho^\Lambda)$ such that $g_{\lambda_0}(s) = ((0)_{\lambda \neq \lambda_0}, 1)$ and $g_{\lambda_0} \otimes e \in L$ for any $e \in E$. For any $t \in Y$, if there exists $s \in X$ such that for all $y^* \in N$, we have*

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_s \quad (2.2.11)$$

for some $x^* \in (E_\rho^\Lambda)^*$, then the same s satisfies (2.2.11) for all $y^* \in (F_\theta^\Gamma)^*$.

Proof. Fix any $t \in Y$ and obtain $s \in X$ that satisfies the hypothesis. By linearity of T^* , we have for any $y^* \in co(N)$, $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in (E_\rho^\Lambda)^*$.

Since $B[(F_\theta^\Gamma)^*] = \overline{co(N)}^{w^*}$, to conclude the proof of the proposition, it suffices to show that for any $y^* \in \overline{co(N)}^{w^*}$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in (E_\rho^\Lambda)^*$. Indeed, since $y^* \in \overline{co(N)}^{w^*}$, there exists a net $(y_\alpha^*)_\alpha$ in $co(N)$ such that $y_\alpha^* \xrightarrow{w^*} y^*$. Clearly $y_\alpha^* \circ \delta_t \xrightarrow{w^*} y^* \circ \delta_t$. Since T^* is weak*-to-weak* continuous, so

$$T^*(y_\alpha^* \circ \delta_t) \xrightarrow{w^*} T^*(y^* \circ \delta_t). \quad (2.2.12)$$

For each α , since $y_\alpha^* \in \text{co}(N)$, by previous paragraph, we have

$$T^*(y_\alpha^* \circ \delta_t) = x_\alpha^* \circ \delta_s. \quad (2.2.13)$$

for some $x_\alpha^* \in (E_\rho^\Lambda)^*$. By (2.2.12) and 2.2.13, we have

$$x_\alpha^* \circ \delta_s \xrightarrow{w^*} T^*(y^* \circ \delta_t). \quad (2.2.14)$$

We claim that $\|x_\alpha^* \circ \delta_s\| \leq 1$ for all α . Indeed,

$$\begin{aligned} \|x_\alpha^* \circ \delta_s\| &= \|T^*(y_\alpha^* \circ \delta_t)\| && \text{by (2.2.13)} \\ &= \|y_\alpha^* \circ \delta_t\| && \text{since } T \text{ is an isometry} \\ &\leq \|y_\alpha^*\| && \text{by Lemma 2.11} \\ &\leq 1 && \text{since } y_\alpha^* \in \text{co}(N) \subseteq B[(F_\theta^\Gamma)^*]. \end{aligned}$$

Therefore, by Proposition 2.10, we have

$$x_\alpha^* \circ \delta_s \xrightarrow{w^*} x^* \circ \delta_s \quad (2.2.15)$$

for some $x^* \in (E_\rho^\Lambda)^*$. From (2.2.14) and (2.2.15), we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$. This concludes the proof of the proposition. $\square$

2.2.3 A sufficient condition for T to be a weighted composition operator

For each $f \in C^p(Q, E)$, let $c(f) = \{s \in Q : f(s) \neq 0\}$. Given a linear map $S : C^p(Q, E) \rightarrow C^q(K, F)$, S is *separating* if $c(Sf) \cap c(Sg) = \emptyset$ whenever $f, g \in C^p(Q, E)$ satisfy $c(f) \cap c(g) = \emptyset$. Recall that S is *biseparating* if it is a bijection and both S and S^{-1} are separating.

Recall that $T : C_0^p(Q, E) \rightarrow C_0^q(K, F)$ is a surjective linear isometry. The main result in this section is Theorem 2.20, which asserts that under certain condition, T is a weighted composition operator. The main ingredient of its proof is [13, Theorem 6.2], which states that any biseparating map between vector-valued differentiable function spaces C^p and C^q (not necessarily vanish at infinity) is a weighted composition operator. Our strategy is to establish that certain extensions of T and

T^{-1} to C^p and C^q respectively, are biseparating. To this end, we need to establish several preparatory results.

Given any mapping $h : K \rightarrow Q$, we say that h satisfies $(\clubsuit(T))$ if for any $t \in K$ and $f \in C_0^p(Q, E)$, if $(\partial^\lambda f(s))_{s=h(t)} = 0$, then $Tf(t) = 0$. The following lemma is a simple consequence of the $(\clubsuit(T))$ property.

Lemma 2.13. *Assume that there exists $h : K \rightarrow Q$ that satisfies $(\clubsuit(T))$. For any $f \in C_0^p(Q, E)$ and $t \in K$, if $f = 0$ on some neighbourhood of $h(t)$, then $Tf(t) = 0$.*

Fix $h : K \rightarrow Q$. Given any $f \in C^p(Q, E)$ and $t \in K$, let $A \subseteq Q$ be a neighbourhood of $h(t)$ and $\tau = \tau_{h(t)} \in C_0^p(Q)$ be such that $0 \leq \tau \leq 1$ and $\tau = 1$ on A . Clearly $\tau \cdot f \in C_0^p(Q, E)$. Define

$$\tilde{T} : C^p(Q, E) \rightarrow C^q(K, F) \quad \text{by} \quad \tilde{T}f(t) = T(\tau \cdot f)(t). \quad (2.2.16)$$

Lemma 2.14. *If $h : K \rightarrow Q$ satisfies $(\clubsuit(T))$, then $\tilde{T}$ from (2.2.16) is well-defined.*

Proof. It suffices to show that $\tilde{T}$ is independent of the function τ and open set A . Fix $f \in C^p(Q, E)$ and $t \in K$. Assume that A_1 and A_2 are open neighbourhoods of $h(t)$ and $\tau_1, \tau_2 \in C_0^p(Q)$ such that $\tau_i = 1$ on A_i for all $i = 1, 2$. Since $h(t) \in A_1 \cap A_2$, so there is an open neighbourhood A_3 of $h(t)$ such that $A_3 \subseteq A_1 \cap A_2$. Since $\tau_1 = \tau_2 = 1$ on A_3 , so $\tau_1 \cdot f = \tau_2 \cdot f$ on A_3 . By Lemma 2.13, $T(\tau_1 \cdot f)(t) = T(\tau_2 \cdot f)(t)$. This concludes the proof of the lemma. $\square$

We need the following lemma in the proof of Proposition 2.16.

Lemma 2.15. *Given any $f, g \in C^p(Q, E)$, if $c(f) \cap c(g) = \emptyset$, then for any $s \in Q$, either $(\partial^\lambda f(s))_{\lambda \in \Lambda} = 0$ or $(\partial^\lambda g(s))_{\lambda \in \Lambda} = 0$.*

Proof. Clearly $c(f) \cap c(g) = \emptyset$ implies that $g = 0$ on $c(f)$. Since $c(f)$ is open, so $(\partial^\lambda g(s))_{\lambda \in \Lambda} = 0$ for all $s \in c(f)$. By continuity of $\partial^\lambda g$, $(\partial^\lambda g(s))_{\lambda \in \Lambda} = 0$ for all $s \in \overline{c(f)} = \text{supp}(f)$. On the other hand, for any $s \notin \text{supp}(f)$, clearly $(\partial^\lambda f(s))_{\lambda \in \Lambda} = 0$. This concludes the proof of the lemma. $\square$

Proposition 2.16. *If $h : K \rightarrow Q$ satisfies $(\clubsuit(T))$, then $\tilde{T}$ defined in (2.2.16) is a separating extension of T .*

Proof. Since T is linear, so is $\tilde{T}$. We first show that $\tilde{T}$ is separating. Let $f, g \in C^p(Q, E)$ be such that $c(f) \cap c(g) = \emptyset$. Fix $t \in K$. Choose $\tau \in C_0^p(Q)$ such that $\tau = 1$ on some neighbourhood of $h(t)$, $\tilde{T}f(t) = T(\tau \cdot f)(t)$ and $\tilde{T}g(t) = T(\tau \cdot g)(t)$. Clearly $c(\tau \cdot f) \cap c(\tau \cdot g) \subseteq c(f) \cap c(g) = \emptyset$. By Lemma 2.15, either $(\partial^\lambda(\tau \cdot f)(h(t)))_{\lambda \in \Lambda} = 0$ or $(\partial^\lambda(\tau \cdot g)(h(t)))_{\lambda \in \Lambda} = 0$. Since $\tau \cdot f$ and $\tau \cdot g$ are elements in $C_0^p(Q, E)$, as h satisfies $(\clubsuit(T))$, so $T(\tau \cdot f)(t) = 0$ or $T(\tau \cdot g)(t) = 0$. Therefore, $\tilde{T}f(t) = 0$ or $\tilde{T}g(t) = 0$. Since $t \in K$ is arbitrary, so $c(\tilde{T}f) \cap c(\tilde{T}g) = \emptyset$. Therefore, $\tilde{T}$ is separating.

It remains to show that $\tilde{T}$ is an extension of T . This follows immediately from Lemma 2.13. Therefore, this concludes the proof of the proposition. $\square$

For any set X , let $Id_X : X \rightarrow X$ be the identity map on X . We will apply Lemmas 2.17 and 2.18 in the proof of Proposition 2.19.

Lemma 2.17. *Let $S : C^p(Q, E) \rightarrow C^p(Q, E)$ be a linear separating extension of the identity map $Id_{C_0^p(Q, E)}$ and $U \subseteq Q$ be an open set. If $f \in C_0^p(Q, E)$ and $f = 0$ on U , then $Sf = 0$ on U .*

Proof. Suppose that $Sf(s_0) \neq 0$ for some $s_0 \in U$. Choose $g \in C_0^p(Q, E)$ such that $\text{supp } g \subseteq U$ and $g(s_0) = Sf(s_0)$. Then $c(g) \cap c(f) = \emptyset$. Since S is separating, so $c(Sg) \cap c(Sf) = \emptyset$. But $g \in C_0^p(Q, E)$, so $Sg(s_0) = g(s_0) = Sf(s_0) \neq 0$, implying that $s_0 \in c(Sg) \cap c(Sf)$, contradicting $c(Sg) \cap c(Sf) = \emptyset$. $\square$

Lemma 2.18. *If $S : C^p(Q, E) \rightarrow C^p(Q, E)$ is a linear separating extension of the identity map $Id_{C_0^p(Q, E)}$, then $S = Id_{C^p(Q, E)}$.*

Proof. Suppose not. Then $Sf(s_0) \neq f(s_0)$ for some $f \in C^p(Q, E)$ and $s_0 \in Q$. Let U be a neighborhood of s_0 with $\bar{U} \subseteq Q$ and $g \in C_0^p(Q, E)$ such that $g = f$ on U . By linearity of S and Lemma 2.17, $Sg = Sf$ on U . Thus, $f(s_0) = g(s_0) = Sg(s_0) = Sf(s_0)$, a contradiction. $\square$

Proposition 2.19. *If there exist $h : K \rightarrow Q$ and $\bar{h} : Q \rightarrow K$ that satisfy $(\clubsuit(T))$ and $(\clubsuit(T^{-1}))$ respectively, then $\tilde{T} : C^p(Q, E) \rightarrow C^q(K, F)$ is a linear biseparating extension of T .*

Proof. By Proposition 2.16, $\tilde{T} : C^p(Q, E) \rightarrow C^q(K, F)$ and $\widetilde{T^{-1}} : C^q(K, F) \rightarrow C^p(Q, E)$ are linear separating extension of $T : C_0^p(Q, E) \rightarrow C_0^q(K, F)$ and $T^{-1} : C_0^q(K, F) \rightarrow C_0^p(Q, E)$ respectively. Since both $\tilde{T}$ and $\widetilde{T^{-1}}$ are separating, so is

$S = \widetilde{T^{-1}} \circ \widetilde{T} : C^p(Q, E) \rightarrow C^p(Q, E)$. Also, the restriction of S to $C_0^p(Q, E)$ is the identity map. Therefore, from Lemma 2.18, we have $S = Id_{C^p(Q, E)}$. Similarly, we have $\widetilde{T} \circ \widetilde{T^{-1}} = Id_{C^q(K, F)}$. Therefore, $\widetilde{T}$ is a bijection with separating inverse $\widetilde{T^{-1}} = \widetilde{T^{-1}}$. Hence, $\widetilde{T}$ is biseparating. This concludes the proof of the proposition. $\square$

Let $L(E, F)$ be the collection of all bounded linear operators from E into F . Suppose that $V : K \rightarrow L(E, F)$ is a continuous map, where $L(E, F)$ is endowed with the norm topology. For each $e \in E$, we define $V_e : K \rightarrow F$ as $V_e(t) = V(t)e$ for every $t \in K$. We say that V is of class $s - C^1$ if the following two statements are satisfied:

1. for every $e \in E$, the map V_e admits all partial derivatives of order 1 in K , and
2. for each $j = 1, 2, \dots, n$, the map $\frac{\partial}{\partial t_j} V : K \rightarrow L(E, F)$ sending each $t \in K$ and each $e \in E$ into $\frac{\partial}{\partial t_j} V_e(t)$ is continuous when $L(E, F)$ is endowed the strong operator topology.

Let $q \geq 2$. A map $V : K \rightarrow L(E, F)$ is said to be of class $s - C^q$ if the following two statements are satisfied:

1. V is of class C^{q-1} (considering $L(E, F)$ as a Banach space) with respect to operator norm.
2. All partial derivatives of order $q - 1$ of V are of class $s - C^1$.

We are now ready to prove the main result of this section.

Theorem 2.20. *Suppose that $T : C_0^p(Q, E) \rightarrow C_0^q(K, F)$ is a linear bijection, where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and both E and F are Banach spaces. If there exist $h : K \rightarrow Q$ and $\bar{h} : Q \rightarrow K$ that satisfy $(\clubsuit(T))$ and $(\clubsuit(T^{-1}))$ respectively, then $p = q$, $m = n$, $h : K \rightarrow Q$ is a diffeomorphism of class C^p and for each $t \in K$, there is a Banach space isomorphism $V(t) : E \rightarrow F$ such that*

$$Tf(t) = V(t)f(h(t)) \quad \text{for every } t \in K \text{ and } f \in C_0^p(Q, E).$$

Furthermore, the mapping $t \mapsto V(t)$ is of class $s - C^p$.

Proof. We obtain $\tilde{T}$ from Proposition 2.19. Since $\tilde{T}$ is a linear biseparating map, by [13, Theorem 6.2], we have $p = q, m = n$, there exist a diffeomorphism $\varepsilon : K \rightarrow Q$ of class C^p and a bounded linear bijection $V(t)$ for every $t \in K$ such that

$$\tilde{T}f(t) = V(t)f(\varepsilon(t)) \quad \text{for all } f \in C^p(Q, E) \text{ and } t \in K.$$

Furthermore, the mapping $t \mapsto V(t)$ is of class $s - C^p$. Since $\tilde{T}$ is an extension of T , so $Tf(t) = V(t)f(\varepsilon(t))$ for every $t \in K$ and $f \in C_0^p(Q, E)$.

It remains to show that $h = \varepsilon$ on K . Suppose that $h(t_0) \neq \varepsilon(t_0)$ for some $t_0 \in K$. Choose $f \in C_0^p(Q, E)$ such that $f = 0$ on some neighbourhood of $h(t_0)$ and $f(\varepsilon(t_0)) \neq 0$. From Lemma 2.13, we have $Tf(t_0) = 0$. However, from the representation of T , we have $0 = Tf(t_0) = V(t_0)(f(\varepsilon(t_0)))$. Since $V(t_0)$ is a linear injection, so $f(\varepsilon(t_0)) = 0$, contradicting to the choice of f . Therefore, $h = \varepsilon$ on K . This concludes the proof of the theorem. $\square$

Chapter 3

Banach-Stone Theorem for Diagonal Norm

3.1 Construction of $\mathcal{P}_D$ Set

We partition the set of multi-indices Λ into $\Lambda = \Lambda_f \cup \Lambda_b$, where

$$\Lambda_b = \{(p, 0, \dots, 0), (0, p, 0, \dots, 0), \dots, (0, \dots, 0, p)\} \quad \text{and} \quad \Lambda_f = \Lambda \setminus \Lambda_b.$$

The subscripts b and f stand for back and front respectively as we place all the pure p th derivatives at the back in our notation. Note that $\#\Lambda_b = m$.

Let $\mathcal{P}_D(s)$ and $\mathcal{G}_D(s)$ be the sets $\mathcal{P}(s)$ and $\mathcal{G}(s)$ that correspond to the function space $(C_0^p(Q, E), \|\cdot\|_{D,\rho})$ defined in Section 2.1. As mentioned at the end of Section 2.2.1, the main result of this section is Corollary 3.11, which asserts that $\mathcal{P}_D(s_0)$ is nonempty for all $s_0 \in Q$.

As the proof of corollary involves lengthy calculations, we will now give a rough idea of the proofs. Without loss of generality, we may assume that $s_0 = 0 \in Q$. We will focus on those $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}_\rho^\Lambda$ that satisfy properties (1), (2) of $\mathcal{P}_D(0)$, in addition, α_λ is 'small' for all $\lambda \in \Lambda_f$ and α_λ is 'large' for all $\lambda \in \Lambda_b$. We only need to show that such $(\alpha_\lambda)_{\lambda \in \Lambda}$ satisfies (3) of $\mathcal{P}_D(0)$. Recall that (3) requires for any neighbourhood U of 0, there exists $\delta > 0$ so that if $x_\lambda \in E$, $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$, then there exists $f \in C_0^p(Q, E)$ so that

- (a) $\|f\|_{D,\rho} = 1$,
- (b) $(\partial^\lambda f(0))_{\lambda \in \Lambda} = (\alpha_\lambda x_\lambda)_{\lambda \in \Lambda}$ and
- (c) $\sup_{s \notin U} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \leq 1 - \delta$.

Note that both (a) and (b) above imply that f attains norm at 0 with respect to the norm $\|\cdot\|_{D,\rho}$. To construct such a function f , for $\lambda \in \Lambda_f$, we will attach a monomial to the $\alpha_\lambda x_\lambda$, and for $\lambda \in \Lambda_b$, we shall attach some single variable function h such that its p th derivative possesses some desired properties. The sum of these function form a preliminary version of f . However, it may not be C_0 . To ensure that a C_0 function is obtained, it is multiplied with a smooth ‘bump’ ϕ which is flat in a neighbourhood of s_0 .

We will now establish several preliminary lemmas and corollaries.

Given $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda$ with $\alpha_\lambda \geq 0$ for all $\lambda \in \Lambda$, let the elements in $\{\alpha_\lambda : \lambda \in \Lambda_b\}$ be $\alpha_1, \dots, \alpha_m$ where $\alpha_{(0, \dots, 0, \overset{i}{p}, 0, \dots, 0)}$ corresponds to α_i for all $1 \leq i \leq m$.

Fix $a \in \left(0, \frac{1}{\rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b})}\right)$. Note that

$$B = \{(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda : \rho((\alpha_\lambda)_{\lambda \in \Lambda}) = 1, \alpha_\lambda \geq a \text{ for all } \lambda \in \Lambda_b\}$$

is a nonempty compact set. For each $1 \leq j_0 \leq m$ and $(\alpha_\lambda)_{\lambda \in \Lambda} \in B$, let $\zeta_{j_0} : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$\zeta_{j_0}(u) = \rho\left((\alpha_\mu)_{\mu \in \Lambda_f}, (\alpha_j)_{j \neq j_0}, \alpha_{j_0} u\right).$$

Since ρ is strictly monotone, so ζ_{j_0} is strictly increasing. Now, let $\xi : [0, 1] \rightarrow \mathbb{R}$ be defined by $\xi(u) = \max_{1 \leq j_0 \leq m} \sup_{(\alpha_\lambda)_{\lambda \in \Lambda} \in B} \zeta_{j_0}(u)$. Clearly

$$\zeta_{j_0}(0) \leq \xi(0) \quad \text{for all } 1 \leq j_0 \leq m. \quad (3.1.1)$$

For any $(\alpha_\lambda)_{\lambda \in \Lambda} \in B$ and $1 \leq j_0 \leq m$, the strict monotonicity of ρ implies that $\zeta_{j_0}(0) < 1$. Therefore, by compactness of B , we have $\xi(0) < 1$.

Assume that $0 \in Q$. For each $s = (s_1, \dots, s_m) \in Q$, denote $|s| = \max_{1 \leq i \leq m} |s_i|$. Choose $r > 0$ such that $\{s \in Q : |s| < 2r\} \subseteq Q$ and

$$r < \frac{1 - \xi(0)}{2(\#\Lambda)^2}. \quad (3.1.2)$$

Note that r is independent of $(\alpha_\lambda)_{\lambda \in \Lambda}$. Choose $\phi \in C_0^\infty(\mathbb{R}^m)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $\{s \in Q : 0 < |s| \leq r\}$ and $\phi = 0$ outside $\{s \in Q : |s| \leq 2r\}$. Let $C = \max_{\lambda \in \Lambda} \|\partial^\lambda \phi\|_\infty < \infty$. Then there exist $0 < \eta < \frac{2[1 - a\rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b})]}{\#\Lambda_f}$ and

$$0 < \varepsilon < r \quad (3.1.3)$$

such that

$$\beta + \gamma < \frac{1 - \xi(0)}{\#\Lambda} \quad (3.1.4)$$

where $\beta = (2r)\#\Lambda\eta + \gamma$ and $\gamma = C[\eta\#\Lambda(p!)^m(1 + \#\Lambda 2r) + m2^p\varepsilon]$. Note that η is independent of $(\alpha_\lambda)_{\lambda \in \Lambda}$. Now, fix $(\alpha_\lambda)_{\lambda \in \Lambda} \in B$ with $0 \leq \alpha_\lambda < \eta$ for all $\lambda \in \Lambda_f$. From Inequalities 3.1.2 and 3.1.3, we have $\xi(0) < 1 - \#\Lambda(\#\Lambda r + (m+1)\varepsilon)$. Therefore, for all $u \in (0, r)$ and $1 \leq j_0 \leq m$,

$$1 - \#\Lambda(\#\Lambda u + (m+1)(u \wedge \varepsilon)) \geq 1 - \#\Lambda(\#\Lambda r + (m+1)\varepsilon) > \xi(0) \geq \zeta_{j_0}(0).$$

Hence, $\zeta_{j_0}^{-1}[1 - \#\Lambda(\#\Lambda u + (m+1)(u \wedge \varepsilon))] > 0$ for all $u \in (0, r]$ and $1 \leq j_0 \leq m$. Thus, there exists $h \in C^p(\mathbb{R})$ such that $h^{(p)}(0) = 1$, $h^{(p)}$ is an even function, $0 \leq h^{(p)} \leq \chi_{(-\varepsilon, \varepsilon)}$,

$$h^{(p)}(u) < \zeta_{j_0}^{-1}[1 - \#\Lambda(\#\Lambda u + (m+1)(u \wedge \varepsilon))] \quad \text{for all } u \in (0, r] \text{ and } 1 \leq j_0 \leq m, \quad (3.1.5)$$

and for all $0 \leq k \leq p-1$,

$$h^{(k)}(u) = \int_0^u h^{(k+1)}(\sigma) d\sigma. \quad (3.1.6)$$

From (3.1.6), clearly $h^{(k)}(0) = 0$ for all $0 \leq k \leq p-1$. Also, for each $k \in \{0, 1, \dots, p-1\}$, we have $h^{(k)} \geq 0$ on $(-\infty, 0]$ if and only if $h^{(k)}$ is even and $h^{(k)} \leq 0$ on $(-\infty, 0]$ if and only if $h^{(k)}$ is odd. Therefore,

$$|h^{(k)}(u)| \leq \int_0^{|u|} h^{(k+1)}(\sigma) d\sigma. \quad (3.1.7)$$

The following lemma provides an upper bound for $|h^{(k)}|$.

Lemma 3.1. *For any $0 \leq k \leq p-1$ and $u \in [-1, 1]$, we have $|h^{(k)}(u)| \leq |u| \wedge \varepsilon$.*

Proof. We apply backward induction on k . When $k = p - 1$, we have for all $u \in \mathbb{R}$,

$$\begin{aligned}
 |h^{(p-1)}(u)| &\leq \int_0^{|u|} h^{(p)}(\sigma) d\sigma && \text{by Inequality 3.1.7} \\
 &\leq \begin{cases} \int_0^{|u|} d\sigma & \text{if } |u| \leq \epsilon, \\ \int_0^\epsilon d\sigma & \text{if } |u| \geq \epsilon, \end{cases} && \text{since } 0 \leq h^{(p)} \leq \chi_{(-\epsilon, \epsilon)} \\
 &= |u| \wedge \epsilon.
 \end{aligned}$$

So, the base case is proven. Suppose that the inequality holds for some k . We want to prove that it holds for $k - 1$. For all $u \in \mathbb{R}$, we have

$$\begin{aligned}
 |h^{(k-1)}(u)| &\leq \int_0^{|u|} h^{(k)}(\sigma) d\sigma && \text{by Inequality 3.1.7} \\
 &\leq \int_0^{|u|} |\sigma| \wedge \epsilon d\sigma && \text{by inductive hypothesis} \\
 &\leq \int_0^{|u|} |\sigma| d\sigma \wedge \int_0^{|u|} \epsilon d\sigma \\
 &\leq \int_0^{|u|} |\sigma| d\sigma \wedge \int_0^1 \epsilon d\sigma && \text{since } u \in [-1, 1] \\
 &\leq |u| \wedge \epsilon.
 \end{aligned}$$

This concludes the proof of the lemma. $\square$

For any $s = (s_1, \dots, s_m) \in Q$ and $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda$, we denote

$$\frac{s^\lambda}{\lambda!} = \prod_{i=1}^m \frac{s_i^{\lambda_i}}{\lambda_i!}.$$

Let $(x_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$ be given. If $\lambda = (0, \dots, 0, \overset{i}{p}, 0, \dots, 0) \in \Lambda_b$, denote x_λ as x_i . If $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$, define $P : Q \rightarrow E$ and $R : Q \rightarrow E$ by

$$P(s) = \sum_{\lambda \in \Lambda_f} \alpha_\lambda x_\lambda \frac{s^\lambda}{\lambda!} \quad \text{and} \quad R(s) = \sum_{i=1}^m \alpha_i x_i h(s_i). \quad (3.1.8)$$

Let $\tilde{P} = P\phi$ and $\tilde{R} = R\phi$. Denote

$$f = \tilde{P} + \tilde{R}. \quad (3.1.9)$$

Clearly $f \in C_0^p(Q, E)$. We define the product order $\leq$ on Λ such that for any $\mu = (\mu_i)_{i=1}^m$ and $\lambda = (\lambda_i)_{i=1}^m$ in Λ , $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq m$. The following simple results deal with the norms of the summands of f .

Lemma 3.2. *For any $\mu, \lambda \in \Lambda$ and $s \in Q$, we have*

$$\partial^\mu \left(\frac{s^\lambda}{\lambda!} \right) = \begin{cases} \frac{1}{(\lambda - \mu)!} s^{\lambda - \mu} & \text{if } \mu \leq \lambda, \\ 0 & \text{if } \mu \not\leq \lambda. \end{cases}$$

Corollary 3.3. *For $\mu \in \Lambda_f$ and $|s| < 2r < 1$,*

$$\|\partial^\mu P(s)\| \leq \sum_{\mu \leq \lambda} \alpha_\lambda \frac{1}{(\lambda - \mu)!} s^{\lambda - \mu} \leq \alpha_\mu + |s| \sum_{\lambda \in \Lambda_f} \alpha_\lambda \leq \alpha_\mu + \#\Lambda_f |s|.$$

Corollary 3.4. *For $\mu \in \Lambda_b$, $\|\partial^\mu P(s)\| = 0$.*

Lemma 3.5. *For any $|s| < 2r < 1$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m) \in \Lambda$, if $\mu_i \neq 0$ for at least two i 's, then $\partial^\mu R(s) = 0$. If $\mu = (0, \dots, 0, \mu_j, 0, \dots, 0)$ for some $1 \leq j \leq m$, then from Lemma 3.1 and the fact that R is a sum of functions of one variable,*

$$\|\partial^\mu R(s)\| \leq \begin{cases} m(|s| \wedge \varepsilon) & \text{if } \mu_j = 0, \\ \alpha_j (|s| \wedge \varepsilon) & \text{if } \mu_j = 1, 2, \dots, p-1, \\ \alpha_j \cdot h^{(p)}(s_j) & \text{if } \mu_j = p. \end{cases}$$

Recall that we need to show that $\|f\|_{D,\rho} = 1$. This will be done by computing the norm at different regions of its support. First, we deal with ‘flat’ part of ϕ , i.e. when $0 < |s| \leq r$.

Corollary 3.6. *For any $0 < |s| \leq r$, we have $\rho \left((\|\partial^\mu f(s)\|)_{\mu \in \Lambda} \right) < 1$.*

Proof. Let $1 \leq j_0 \leq m$ be such that $|s| = |s_{j_0}|$. Since $0 < |s| \leq r$, so $f(s) = P(s) + R(s)$. By the triangle inequality, we have

$$\rho \left((\|\partial^\mu f(s)\|)_{\mu \in \Lambda} \right) \leq \rho \left((\|\partial^\mu P(s)\|)_{\mu \in \Lambda} + (\|\partial^\mu R(s)\|)_{\mu \in \Lambda} \right).$$

By Corollaries 3.3, 3.4 and Lemma 3.5, we have

$$\begin{aligned} & \rho \left[(\|\partial^\mu P(s)\|)_{\mu \in \Lambda} + (\|\partial^\mu R(s)\|)_{\mu \in \Lambda} \right] \\ & \leq \rho \left[((\alpha_\mu + \#\Lambda_f |s|)_{\mu \in \Lambda_f}, (0)_{\mu \in \Lambda_b}) + ((m(|s| \wedge \varepsilon))_{\mu=0}, (|s| \wedge \varepsilon)_{\mu \in \Lambda_f \setminus \{0\}}, (\alpha_j \cdot h^{(p)}(s_j))_{j=1}^m) \right] \end{aligned}$$

Observe that

$$\begin{aligned}
& \rho \left[((\alpha_\mu + \#\Lambda\eta|s|)_{\mu \in \Lambda_f}, (0)_{\mu \in \Lambda_b}) + ((m(|s| \wedge \varepsilon))_{\mu=0}, (|s| \wedge \varepsilon)_{\mu \in \Lambda_f \setminus \{0\}}, (\alpha_j \cdot h^{(p)}(s_j))_{j=1}^m) \right] \\
& \leq \rho \left((\alpha_\mu)_{\mu \in \Lambda_f}, (\alpha_j \cdot h^{(p)}(s_j))_{j=1}^m \right) + \rho \left((\#\Lambda\eta|s|)_{\mu \in \Lambda_f}, 0 \right) \\
& + \rho \left((m(|s| \wedge \varepsilon))_{\mu=0}, (|s| \wedge \varepsilon)_{\mu \in \Lambda_f \setminus \{0\}}, 0 \right) \\
& \leq \rho \left((\alpha_\mu)_{\mu \in \Lambda_f}, (\alpha_j)_{j \neq j_0}, \alpha_{j_0} h^{(p)}(|s|) \right) + \#\Lambda (\#\Lambda|s| + (m+1)(|s| \wedge \varepsilon)) \\
& = \zeta_{j_0} (h^{(p)}(|s|)) + \#\Lambda (\#\Lambda|s| + (m+1)(|s| \wedge \varepsilon)), \tag{3.1.10}
\end{aligned}$$

where the last equality follows from the definition of ζ_{j_0} . Since $|s| \in (0, r]$, Inequality 3.1.5 implies that $\zeta_{j_0} (h^{(p)}(|s|)) + \#\Lambda (\#\Lambda|s| + (m+1)(|s| \wedge \varepsilon)) < 1$. By combining all inequalities above, the corollary follows. $\square$

For any $\mu = (\mu_1, \dots, \mu_m)$ and $\lambda = (\lambda_1, \dots, \lambda_m)$ in Λ , denote $\binom{\mu}{\lambda} = \prod_{i=1}^m \binom{\mu_i}{\lambda_i}$. Now, we want to show that $\rho \left((\|\partial^\mu f(s)\|)_{\mu \in \Lambda} \right) < 1$ for all $r < |s| \leq 2r$. As above, we deal with the norms of the summands of f separately. Recall that $C = \max_{\lambda \in \Lambda} \|\partial^\lambda \phi\|_\infty$.

Lemma 3.7. *For $r \leq |s| \leq 2r$ and $\mu \in \Lambda$,*

$$\left\| \partial^\mu \tilde{P}(s) \right\| \leq \begin{cases} \alpha_\mu + \#\Lambda\eta|s| + C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda\eta|s|) & \text{if } \mu \in \Lambda_f, \\ C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda\eta|s|) & \text{if } \mu \in \Lambda_b. \end{cases}$$

Proof. For any $\mu \in \Lambda$, we have

$$\begin{aligned}
\left\| \partial^\mu \tilde{P}(s) \right\| &= \left\| \partial^\mu P(s) + \sum_{\lambda < \mu} \binom{\mu}{\lambda} \partial^\lambda P(s) \partial^{\mu-\lambda} \phi(s) \right\| && \text{by Leibniz's rule} \\
&\leq \|\partial^\mu P(s)\| + C \sum_{\lambda < \mu} \binom{\mu}{\lambda} \|\partial^\lambda P(s)\| \\
&\leq \|\partial^\mu P(s)\| + C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda\eta|s|) && \text{by Corollary 3.3.}
\end{aligned}$$

If $\mu \in \Lambda_f$, then by Corollary 3.3 again, we have

$$\left\| \partial^\mu \tilde{P}(s) \right\| \leq \alpha_\mu + \#\Lambda\eta|s| + C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda\eta|s|).$$

If $\mu \in \Lambda_b$, then $\partial^\mu P(s) = 0$. So, $\|\partial^\mu \tilde{P}(s)\| \leq C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda \eta |s|)$. This concludes the proof of the lemma. $\square$

For any $(s_1, \dots, s_m) \in Q$, $1 \leq i \leq p$ and $1 \leq j \leq m$, we denote $\partial_j^i R(s_1, \dots, s_m) = \frac{\partial^i R}{\partial s_j^i}(s_1, \dots, s_m)$.

Lemma 3.8. *For $r \leq |s| \leq 2r$ and $\mu = (\mu_1, \dots, \mu_m) \in \Lambda$, we have*

$$\|\partial^\mu \tilde{R}(s)\| \leq \begin{cases} Cm2^p \varepsilon & \text{if } \mu \in \Lambda_f, \\ \alpha_j |h^{(p)}(s_j)| + Cm2^p \varepsilon & \text{if } \mu_j = p. \end{cases}$$

Proof. If $\mu \in \Lambda_f$ with $\mu = (\mu_1, \mu_2, \dots, \mu_m)$, then

$$\begin{aligned} \|\partial^\mu \tilde{R}(s)\| &= \left\| \sum_{\lambda \leq \mu} \binom{\mu}{\lambda} \partial^\lambda R(s) \partial^{\mu-\lambda} \phi(s) \right\| && \text{by Leibniz's rule} \\ &\leq C \sum_{j=1}^m \sum_{i=0}^{\mu_j} \binom{\mu_j}{i} \|\partial_j^i R(s)\| && \text{by Lemma 3.5} \\ &\leq Cm(|s| \wedge \varepsilon) \sum_{j=1}^m 2^{\mu_j} && \text{by Lemma 3.5} \\ &\leq Cm\varepsilon 2^p && \text{by } \varepsilon < r < |s|, \end{aligned}$$

where the first inequality follows from the fact that R is a sum of functions of single variable and the Lemma 3.5. If $\mu \in \Lambda_b$, say $\mu = (0, \dots, 0, \mu_j, 0, \dots, 0)$ with $\mu_j = p$, then

$$\begin{aligned} \|\partial^\mu \tilde{R}(s)\| &\leq \|\partial_j^p R(s)\| + C \sum_{i=0}^{p-1} \binom{p}{i} \|\partial_j^i R(s)\| && \text{by Leibniz's rule} \\ &\leq \alpha_j |h^{(p)}(s_j)| + Cm(2^p(|s| \wedge \varepsilon)) && \text{by Lemma 3.5} \\ &= \alpha_j |h^{(p)}(s_j)| + Cm2^p \varepsilon && \text{by } \varepsilon < r < |s|. \end{aligned}$$

This concludes the proof of the lemma. $\square$

Corollary 3.9. *For any $r \leq |s| \leq 2r$, we have $\rho[(\|\partial^\mu f(s)\|)_{\mu \in \Lambda}] < 1$.*

Proof. By Lemmas 3.7 and 3.8, if $\mu \in \Lambda_f$, then

$$\begin{aligned} \|\partial^\mu f(s)\| &\leq \|\partial^\mu \tilde{P}(s)\| + \|\partial^\mu \tilde{R}(s)\| \\ &\leq \alpha_\mu + \#\Lambda\eta |s| + C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda\eta |s|) + Cm2^p\varepsilon \\ &\leq \alpha_\mu + \beta \end{aligned}$$

where $\beta = (2r)\#\Lambda\eta + C[\eta\#\Lambda(p!)^m(1 + \#\Lambda 2r) + m2^p\varepsilon]$ since $\alpha_\lambda < \eta$ for all $\lambda \in \Lambda_f$,

If $\mu \in \Lambda_b$, say $\mu = (0, \dots, 0, \mu_j, 0, \dots, 0)$ with $\mu_j = p$, then by Lemmas 3.7 and 3.8 again,

$$\begin{aligned} \|\partial^\mu f(s)\| &\leq \|\partial^\mu \tilde{P}(s)\| + \|\partial^\mu \tilde{R}(s)\| \\ &\leq C \sum_{\lambda < \mu} \binom{\mu}{\lambda} (\alpha_\lambda + \#\Lambda\eta |s|) + \alpha_j |h^{(p)}(s_j)| + Cm2^p\varepsilon \\ &\leq \gamma + \alpha_j |h^{(p)}(s_j)| \end{aligned}$$

where $\gamma = C[\eta\#\Lambda(p!)^m(1 + \#\Lambda 2r) + m2^p\varepsilon]$. Let $1 \leq j_0 \leq m$ be such that $|s| = |s_{j_0}|$. Hence,

$$\begin{aligned} &\rho\left((\|\partial^\mu f(s)\|)_{\mu \in \Lambda}\right) \\ &\leq \rho\left((\alpha_\mu)_{\mu \in \Lambda_f}, (\alpha_j |h^{(p)}(s_j)|)_{j=1}^m\right) + \rho\left((\beta)_{\lambda \in \Lambda_f}, (0)_{\lambda \in \Lambda_b}\right) + \rho\left((0)_{\lambda \in \Lambda_f}, (\gamma)_{\lambda \in \Lambda_b}\right) \\ &\leq \rho\left((\alpha_\mu)_{\mu \in \Lambda_f}, (\alpha_j)_{j \neq j_0}, \alpha_{j_0} h^{(p)}(|s|)\right) + \#\Lambda(\beta + \gamma) \\ &= \zeta_{j_0}(h^{(p)}(|s|)) + \#\Lambda(\beta + \gamma). \end{aligned} \tag{3.1.11}$$

Lastly, since $|s| \geq r$, so $h^{(p)}(|s|) = 0$. So, Inequalities 3.1.1 and 3.1.4 imply that

$$\zeta_{j_0}(h^{(p)}(|s|)) + \#\Lambda(\beta + \gamma) < 1. \tag{3.1.12}$$

This concludes the proof of the lemma. $\square$

Now, let us recall the definition of $\mathcal{P}_D(s_0)$. If $s_0 \in X$, then $\mathcal{P}(s_0)$ is the subset of $\mathbb{R}^\Lambda$ consisting of all $(\alpha_\lambda)_{\lambda \in \Lambda}$ such that

1. $\alpha_\lambda > 0$ for all $\lambda \in \Lambda$,

2. $\rho((\alpha_\lambda)_{\lambda \in \Lambda}) = 1$.

3. For any neighborhood U of s_0 , there exists $\delta > 0$ so that if $x_\lambda \in E$, $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$, then there exists $f \in C_0^p(Q, E)$ so that $\|f\|_{D, \rho} = 1$, $(\partial^\lambda f(s_0))_{\lambda \in \Lambda} = (\alpha_\lambda x_\lambda)_{\lambda \in \Lambda}$ and $\sup_{s \notin U} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \leq 1 - \delta$.

Proposition 3.10. *For any $s_0 \in Q$ and $a \in \left(0, \frac{1}{\rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b})}\right)$, there exists $0 < \eta < \frac{2[1 - a\rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b})]}{\#\Lambda_f}$ such that for any $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}_\rho^\Lambda$, if $\rho((\alpha_\lambda)_{\lambda \in \Lambda}) = 1$, $0 \leq \alpha_\lambda \leq \eta$ for all $\lambda \in \Lambda_f$ and $\alpha_\lambda \geq a$ for all $\lambda \in \Lambda_b$, then 3. of $\mathcal{P}(s_0)$ above holds.*

Proof. Without loss of generality, we may assume that $s_0 = 0 \in Q$. Fix any $a \in \left(0, \frac{1}{\rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b})}\right)$. Choose $\eta > 0$ that satisfies (3.1.4) above. Pick any $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda$ that satisfies assumptions given in this proposition.

Fix any neighbourhood U of 0. For any $s \in Q$, let $1 \leq j_0 \leq m$ such that $|s_{j_0}| = |s|$. Observe that

$$\begin{aligned} g_1(s) &= \zeta_{j_0}(h^{(p)}(|s|)) + \#\Lambda(\#\Lambda|s| + |s| \wedge \epsilon) \quad \text{and} \\ g_2(s) &= \zeta_{j_0}(h^{(p)}(|s|)) + \#\Lambda(\beta + \gamma) \end{aligned}$$

are continuous on the compact sets $\{s \in Q : |s| \leq r\} \setminus U$ and $\{s \in Q : r \leq |s| \leq 2r\} \setminus U$ respectively. By Inequality 3.1.5, we have $g_1(s) < 1$ for all $0 < |s| \leq r$. Also, by Inequalities 3.1.12, we have $g_2(s) < 1$ for all $r \leq |s| \leq 2r$. It follows that there exist $\delta_1, \delta_2 > 0$ such that

$$\sup_{s \in \{s \in Q : |s| \leq r\} \setminus U} g_1(s) \leq 1 - \delta_1 \quad \text{and} \quad \sup_{s \in \{s \in Q : r \leq |s| \leq 2r\} \setminus U} g_2(s) \leq 1 - \delta_2.$$

Let $\delta = \delta_1 \wedge \delta_2 > 0$.

For any $(x_\lambda)_{\lambda \in \Lambda}$ in E_ρ^Λ with $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$, recall that $f : Q \rightarrow E$ is given by (3.1.9) above. Clearly $\text{supp}(f) \subseteq \{s \in Q : |s| \leq 2r\}$, $f \in C_0^p(Q, E)$ and $\rho[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}] = \rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1$. By Corollaries 3.6 and 3.9, we have $\|f\|_{D, \rho} = 1$.

Lastly, by Inequalities 3.1.10 and 3.1.11, we have

$$\begin{aligned}
& \sup_{s \notin U} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \\
&= \sup_{s \in \{s \in Q: |s| \leq r\} \setminus U} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \vee \sup_{s \in \{s \in Q: r \leq |s| \leq 2r\} \setminus U} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \\
&\leq \sup_{s \in \{s \in Q: |s| \leq r\} \setminus U} g_1(s) \vee \sup_{s \in \{s \in Q: r \leq |s| \leq 2r\} \setminus U} g_2(s) \\
&\leq 1 - \delta.
\end{aligned}$$

Therefore, we conclude that $(\alpha_\lambda)_{\lambda \in \Lambda}$ satisfies 3. of $\mathcal{P}_D(0)$. $\square$

We are now ready to prove that $\mathcal{P}_D(s_0)$ is nonempty for all $s_0 \in Q$.

Corollary 3.11. *For any $s_0 \in Q$, we have $\mathcal{P}_D(s_0) \neq \emptyset$.*

Proof. Fix any $s_0 \in Q$ and $a \in \left(0, \frac{1}{\rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b})}\right)$. Obtain $\eta > 0$ from Proposition 3.10. To conclude that $\mathcal{P}_D(s_0) \neq \emptyset$, it suffices to show the existence of $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}_\rho^\Lambda$ that satisfies the assumption of Proposition 3.10, with $\alpha_\lambda \neq 0$ for all $\lambda \in \Lambda_f$. Consider $g : [0, \infty) \rightarrow [0, \infty)$ defined by $g(u) = \rho\left[\left(\frac{\eta}{2}\right)_{\lambda \in \Lambda_f}, (u)_{\lambda \in \Lambda_b}\right]$. Since $g(0) = \rho\left[\left(\frac{\eta}{2}\right)_{\lambda \in \Lambda_f}, (0)_{\lambda \in \Lambda_b}\right] \leq \frac{\eta}{2}(\#\Lambda_f) < 1$ and $\lim_{u \rightarrow \infty} g(u) = \infty$, so there exists $u_0 \in [0, \infty)$ such that $g(u_0) = 1$. We claim that $u_0 \geq a$. If $u_0 < a$, then

$$1 = g(u_0) \leq \frac{\eta}{2}(\#\Lambda_f) + u_0 \rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b}) < \frac{\eta}{2}(\#\Lambda_f) + a \rho((0)_{\lambda \in \Lambda_f}, (1)_{\lambda \in \Lambda_b}) < 1,$$

a contradiction. Therefore, $\left[\left(\frac{\eta}{2}\right)_{\lambda \in \Lambda_f}, (u_0)_{\lambda \in \Lambda_b}\right] \in \mathcal{P}_D(s_0)$. $\square$

We conclude this section with the following corollary, which follows easily from Corollaries 3.11 and 2.7.

Corollary 3.12. *For any $s_0 \in Q$, $\mathcal{G}_D(s_0)$ is nonempty.*

3.2 Construction of φ

Recall that for any $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s \in Q$, $x^* \circ \delta_s$ is a functional on $C_0^p(Q, E)$ defined by

$$\langle f, x^* \circ \delta_s \rangle = \sum_{\lambda \in \Lambda} x_\lambda^*(\partial^\lambda f(s)) \quad \text{for all } f \in C_0^p(Q, E).$$

Also, T^* is the adjoint operator of T . The main result of this section is Theorem 3.32, which we state now.

Theorem. *Let $T : (C_0^p(Q, E), \|\cdot\|_{D,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,\theta})$ be a surjective linear isometry where Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. Then there exists a mapping $\varphi : K \rightarrow Q$ such that for all $t \in K$ and $y^* \in (F_\theta^\Gamma)^*$, we have*

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)} \quad \text{for some } x^* \in (E_\rho^\Lambda)^*. \quad (3.2.1)$$

The following proposition asserts the existence of φ that satisfies (3.2.1), which is an immediate consequence of Propositions 2.2, 2.5 and Corollary 3.12.

Proposition 3.13. *For any $t \in K$, there exists $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} \in (F_\theta^\Gamma)^*$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$ such that $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q$.*

Proof. Fix $t \in K$. By Corollary 3.12, the set $\mathcal{G}_D(t)$ is nonempty, say $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$. Pick any $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$ and denote $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$. By Proposition 2.5, $y^* \circ \delta_t \in \text{ext } B[C_0^q(K, F)^*]$. Since T^* is an isometry, so $T^*(y^* \circ \delta_t) \in \text{ext } B[C_0^p(Q, E)^*]$. Recall that $X = Q_\infty$, where X is obtained from Section 2.2.1. So, by Proposition 2.2, there exist $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q_\infty$ such that $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$.

It remains to show that $s \neq \infty$. If $s = \infty$, then $\langle f, T^*(y^* \circ \delta_t) \rangle = \langle f, x^* \circ \delta_s \rangle = 0$ for all $f \in C_0^p(Q, E)$, which implies that $y^* \circ \delta_t = 0$, a contradiction. This concludes the proof of the proposition. $\square$

In fact, we can use similar proof in Proposition 3.13 above to obtain the following corollary.

Corollary 3.14. *For any $t \in K$ and $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$, there exists $s \in Q$ such that*

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_s \quad \text{for some } x^* \in \text{ext } B[(E_\rho^\Lambda)^*] \quad (3.2.2)$$

By Corollary 3.14, it remains to show that the point s in (3.2.2) is independent of y^* . By Proposition 2.9, if s in (3.2.2) is independent of those y^* with ‘good’ coefficients, then we can extend (3.2.2) to all y^* in $(F_\theta^\Gamma)^*$. Proposition 3.15 below recorded this fact, which follows from Propositions 2.12 and 2.9.

Proposition 3.15. *For any $t \in K$, if there exists $s \in Q$ such that for all $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$, then the above conclusion holds for all $y^* \in (F_\theta^\Gamma)^*$.*

Proof. Fix any $t \in K$ and obtain s from the hypothesis. Let $N = \text{ext } B[(F_\theta^\Gamma)^*]$ and

$$A(t) = \{(\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} : (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t), y_\gamma^* \in \text{ext } B(F^*) \text{ for all } \gamma \in \Gamma\}.$$

By choice of s , s is independent of $y^* \in A(t)$. By linearity of T^* , s is also independent of $y^* \in \text{span}(A(t))$. From Proposition 2.9, we have $N \subseteq \text{span}(A(t))$. Therefore, s is independent of $y^* \in N$. According to Proposition 2.12, it remains to show that $B[(F_\theta^\Gamma)^*] = \overline{\text{co}(\text{ext } B[(F_\theta^\Gamma)^*])}^{w^*}$. However, the equality follows immediately from the Krein-Milman theorem. This concludes the proof of the proposition. $\square$

According to Proposition 3.15, to obtain a proof of Theorem 3.32, it remains to show that the assumption of Proposition 3.15 holds. Fix any $t \in K$ and let $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ be given by Proposition 3.13. From the choice of y^* , we have

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_s \quad (3.2.3)$$

for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q$. Define a map $\Phi_{y^*} : K \rightarrow Q$ by $\Phi_{y^*}(t) = s$.

We want to show that $\Phi_{y^*}(t)$ is independent of y^* . To this end, we employ components ‘flipping’ technique on y^* . More precisely, fix $\gamma_0 \in \Gamma$. Let $[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (\alpha_{\gamma_0}^* y_{\gamma_0}^*)]$

be a functional $(z_\gamma^*)_{\gamma \in \Gamma}$ on F_θ^Γ such that $z_\gamma^* = \beta_\gamma^* y_\gamma^*$ for all $\gamma \neq \gamma_0$ and $z_{\gamma_0}^* = \alpha_{\gamma_0}^* y_{\gamma_0}^*$. Since $-y_{\gamma_0}^* \in \text{ext } B(F^*)$, by Corollary 3.14, we have

$$T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (-\beta_{\gamma_0}^* y_{\gamma_0}^*)] \circ \delta_t = x_{\gamma_0}^* \circ \delta_{s_{\gamma_0}} \quad (3.2.4)$$

for some $x_{\gamma_0}^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s_{\gamma_0} \in Q$. Thus, $\Phi_{[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (-\beta_{\gamma_0}^* y_{\gamma_0}^*)]}(t) = s_{\gamma_0}$. Define $\varphi_{y^*} : K \rightarrow 2^Q$ by

$$\begin{aligned} \varphi_{y^*}(t) &= \{\Phi_{y^*}(t)\} \cup \left\{ \Phi_{[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (-\beta_{\gamma_0}^* y_{\gamma_0}^*)]}(t) : \gamma_0 \in \Gamma \right\} \\ &= \{s\} \cup \{s_{\gamma_0} : \gamma_0 \in \Gamma\}. \end{aligned} \quad (3.2.5)$$

We first prove that φ_{y^*} is independent of y^* as a set-valued mapping, which is given by Lemma 3.18. To this end, we need to establish the following lemma.

For any $f \in C_0^p(Q, E)$, we denote $\text{supp}(f) = \overline{\{s \in Q : f(s) \neq 0\}}$.

Lemma 3.16. *For any $\ell \geq 1$, let $x^* \in (E_\rho^\Lambda)^*$ and $s \in Q$. If $x^* \neq 0$ and*

$$x^* \circ \delta_s = x_1^* \circ \delta_{s^1} + \cdots + x_\ell^* \circ \delta_{s^\ell} \quad (3.2.6)$$

for some $x_1^, \dots, x_\ell^* \in (E_\rho^\Lambda)^*$ and $s^1, \dots, s^\ell \in Q$, then $s \in \{s^k : 1 \leq k \leq \ell\}$. If, in addition, (3.2.6) is irreducible (i.e., there does not exist a nonempty $B \subseteq \{1, \dots, \ell\}$ such that $\sum_{i \in B} x_i^* \circ \delta_{s^i} = 0$), then $s = s^1 = \cdots = s^\ell$.*

Proof. We prove the first assertion by contradiction. Suppose that $s \notin \{s^k : 1 \leq k \leq \ell\}$. Since $x^* \neq 0$, let $x = (x_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$ be such that

$$x^*(x) \neq 0 \quad (3.2.7)$$

Choose $\phi \in C_0^\infty(Q)$ such that $\phi \equiv 1$ in some neighbourhood of s and $\text{supp}(\phi) \cap \{s^k : 1 \leq k \leq \ell\} = \emptyset$. Define $f : Q \rightarrow E$ by $f(u) = \phi(u) \sum_{\mu \in \Lambda} x_\mu \frac{(u - s)^\mu}{\mu!}$. Clearly $f \in C_0^p(Q, E)$,

$$\text{supp}(f) \cap \{s^k : 1 \leq k \leq \ell\} = \emptyset \quad \text{and} \quad (3.2.8)$$

$$(\partial^\lambda f(s))_{\lambda \in \Lambda} = x. \quad (3.2.9)$$

Observe that

$$\begin{aligned}
0 &\neq x^*(x) && \text{by (3.2.7)} \\
&= \langle f, x^* \circ \delta_s \rangle && \text{by (3.2.9)} \\
&= \langle f, x_1^* \circ \delta_{s^1} + \cdots + x_\ell^* \circ \delta_{s^\ell} \rangle && \text{by (3.2.6)} \\
&= 0 && \text{by (3.2.8),}
\end{aligned}$$

a contradiction. Therefore, $s \in \{s^k : 1 \leq k \leq \ell\}$.

We prove the second assertion by induction on ℓ . If $\ell = 1$, then it follows from the first assertion. Suppose that the second assertion holds for all $k < \ell$ and assume that (3.2.6) holds. Let $A = \{1 \leq k \leq \ell : s^k = s\}$. Suppose that $\#A < \ell$. Then A^c is nonempty. Let $z^* = x^* - \sum_{k \in A} x_k^*$. From (3.2.6), we have $z^* \circ \delta_s = \sum_{k \in A^c} x_k^* \circ \delta_{s^k}$. If $z^* \neq 0$, then by the inductive hypothesis, $s = s^k$ for all $k \in A^c$, contradicting the definition of A^c . Therefore, $z^* = 0$. However, from the equation above, we have $\sum_{k \in A^c} x_k^* \circ \delta_{s^k} = 0$. However, A^c is nonempty, contradicting with irreducibility. Therefore, $\#A = \ell$. This completes the induction. Thus, the lemma follows. $\square$

Remark 3.17. If $\ell = 2$ and all x^* , x_1^* and x_2^* are nonzero, then (3.2.6) is always irreducible.

Lemma 3.18. Fix $t \in K$. Then for any $z^* = (\alpha_\gamma^* z_\gamma^*)_{\gamma \in \Gamma}$ where $(\alpha_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $z_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$, we have $\varphi_{z^*}(t) = \varphi_{y^*}(t)$.

Proof. Fix $\gamma_0 \in \Gamma$. Adding (3.2.3) and 3.2.4 above gives

$$T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (0)] \circ \delta_t = \frac{1}{2} (x^* \circ \delta_s + x_{\gamma_0}^* \circ \delta_{s_{\gamma_0}}). \quad (3.2.10)$$

Since $-z_{\gamma_0}^* \in \text{ext } B(F^*)$, by Proposition 2.5,

$$[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (\beta_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t \quad \text{and} \quad [(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (-\beta_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t$$

are extreme points of $B[C_0^q(K, F)^*]$. So there exist $v^*, v_{\gamma_0}^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $r, r_{\gamma_0} \in Q$ such that

$$\begin{aligned}
T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (\beta_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t &= v^* \circ \delta_r \quad \text{and} \\
T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (-\beta_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t &= v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}.
\end{aligned}$$

Adding and subtracting two equations above yield

$$T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (0)] \circ \delta_t = \frac{1}{2}(v^* \circ \delta_r + v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}) \quad \text{and} \quad (3.2.11)$$

$$T^*[(0)_{\gamma \neq \gamma_0}, (\beta_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t = \frac{1}{2}(v^* \circ \delta_r - v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}). \quad (3.2.12)$$

By (3.2.10) and 3.2.11, we have

$$x^* \circ \delta_s + x_{\gamma_0}^* \circ \delta_{s_{\gamma_0}} = v^* \circ \delta_r + v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}.$$

We claim that $\{r, r_{\gamma_0}\} = \{s, s_{\gamma_0}\}$. Suppose that $r \notin \{s, s_{\gamma_0}\}$. Since $v^* \neq 0$, by Lemma 3.16, we have $r \in \{s, s_{\gamma_0}, r_{\gamma_0}\}$. Since $r \notin \{s, s_{\gamma_0}\}$, so $r = r_{\gamma_0}$. Note that by Corollary 2.8, $\beta_\gamma^* \neq 0$ for all $\gamma \in \Gamma$. So, from (3.2.11), we have $v^* + v_{\gamma_0}^* \neq 0$. So, by Lemma 3.16 again, we have $r \in \{s, s_{\gamma_0}\}$, a contradiction. Therefore, $r \in \{s, s_{\gamma_0}\}$. Since $v_{\gamma_0}^* \neq 0$ as well, by Lemma 3.16 again, we have $r_{\gamma_0} \in \{r, s, s_{\gamma_0}\} = \{s, s_{\gamma_0}\}$. Therefore, $\{r, r_{\gamma_0}\} \subseteq \{s, s_{\gamma_0}\}$. Since $x^*, x_{\gamma_0}^* \neq 0$, by symmetry, we can show that $\{s, s_{\gamma_0}\} \subseteq \{r, r_{\gamma_0}\}$. This concludes the proof of the claim.

Since $\gamma_0 \in \Gamma$ is arbitrary, therefore $\{r\} \cup \{r_{\gamma_0} : \gamma_0 \in \Gamma\} = \{s\} \cup \{s_{\gamma_0} : \gamma_0 \in \Gamma\}$.

Since $\beta_{\gamma_0}^* \neq 0$, (3.2.12) above implies that

$$T^*[(0)_{\gamma \neq \gamma_0}, (\alpha_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t = \frac{\alpha_{\gamma_0}^*}{2\beta_{\gamma_0}^*}(v^* \circ \delta_r - v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}). \quad (3.2.13)$$

Suppose that for any $\gamma_0 \in \Gamma$,

$$T^*(z^* \circ \delta_t) = w^* \circ \delta_{\Phi_{z^*}(t)} \quad \text{and} \\ T^*[(\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, (-\alpha_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t = w_{\gamma_0}^* \circ \delta_{\Phi_{[-\alpha_{\gamma_0}^* z_{\gamma_0}^*], (\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}]}(t)}.$$

By (3.2.13), we have

$$w^* \circ \delta_{\Phi_{z^*}(t)} = T^*(z^* \circ \delta_t) = \sum_{\gamma_0 \in \Gamma} \frac{\alpha_{\gamma_0}^*}{2\beta_{\gamma_0}^*}(v^* \circ \delta_r - v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}).$$

Since $w^* \neq 0$, by Lemma 3.16, we have

$$\Phi_{z^*}(t) \in \{r\} \cup \{r_{\gamma_0} : \gamma_0 \in \Gamma\} = \{s\} \cup \{s_{\gamma_0} : \gamma_0 \in \Gamma\} = \varphi_{y^*}(t).$$

Similarly, for each $\gamma_0 \in \Gamma$, by (3.2.13), we have

$$\begin{aligned} & w_{\gamma_0}^* \circ \delta_{\Phi[(\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, (-\alpha_{\gamma_0}^* z_{\gamma_0}^*)]}(t) \\ &= T^*[(\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, (-\alpha_{\gamma_0}^* z_{\gamma_0}^*)] \circ \delta_t \\ &= \sum_{\mu \neq \gamma_0} \frac{\alpha_\mu^*}{2\beta_\mu^*} (v^* \circ \delta_r - v_\mu^* \circ \delta_{r_\mu}) - \frac{\alpha_{\gamma_0}^*}{2\beta_{\gamma_0}^*} (v^* \circ \delta_r - v_{\gamma_0}^* \circ \delta_{r_{\gamma_0}}). \end{aligned}$$

Since $w_{\gamma_0}^* \neq 0$, by Lemma 3.16, we have

$$\Phi[(\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, (-\alpha_{\gamma_0}^* z_{\gamma_0}^*)](t) \in \{r\} \cup \{r_{\gamma_0} : \gamma_0 \in \Gamma\} = \{s\} \cup \{s_{\gamma_0} : \gamma_0 \in \Gamma\} = \varphi_{y^*}(t).$$

Therefore,

$$\varphi_{z^*}(t) = \{\Phi_{z^*}(t)\} \cup \left\{ \Phi[(\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, (-\alpha_{\gamma_0}^* z_{\gamma_0}^*)](t) : \gamma_0 \in \Gamma \right\} \subseteq \varphi_{y^*}(t).$$

By symmetry, we can also show that $\varphi_{y^*}(t) \subseteq \varphi_{z^*}(t)$. Thus, we conclude that $\varphi_{z^*}(t) = \varphi_{y^*}(t)$. $\square$

The following useful corollary follows immediately from Lemma 3.18.

Corollary 3.19. *Fix $t \in K$. If $\varphi_{y^*}(t) = \{s\}$ for some $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ and $s \in Q$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$, then $\varphi_{z^*}(t) = \{s\}$ for all $z^* = (\alpha_\gamma^* z_\gamma^*)_{\gamma \in \Gamma}$ where $(\alpha_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $z_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$.*

Remark 3.20. *If $\varphi_{y^*}(t) = \{s\}$ for some $y^* = (\beta_\gamma^* y_\gamma^*)$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$, then we define $\varphi : K \rightarrow Q$ given by $\varphi(t) = s$. By Corollary 3.19, φ satisfies (3.2.1) and is independent of such y^* . Therefore, it follows from Proposition 3.15 that (3.2.1) holds for all bounded linear functionals on F_θ^Γ .*

Thus, for the remaining of this section, it suffices to show that the assumption of Corollary 3.19 holds. To this end, we split the proof into 4 cases.

Case 1 : $\#\Gamma \geq 3$,

Case 2 : $\#\Gamma = 2$ and $F \neq \mathbb{R}$,

Case 3 : $\#\Gamma = 2, F = \mathbb{R}$ and $(\#\Lambda \geq 3 \text{ or } E \neq \mathbb{R})$ and

Case 4 : $\#\Gamma = 2, F = \mathbb{R}, \#\Lambda = 2$ and $E = \mathbb{R}$.

For convenience, proofs of Case 1, 2, 3, 4 are shown at Propositions 3.21, 3.24, 3.30 and 3.31 respectively.

3.2.1 Case 1: $\#\Gamma \geq 3$

The following proposition is the main result of this section. Recall that φ_{y^*} is given by (3.2.5).

Proposition 3.21. *Assume that $\#\Gamma \geq 3$. For any $t \in K$, we have $\varphi_{y^*}(t) = \{s\}$ for some $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ and $s \in Q$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$.*

Proof. Fix $t \in K$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$. Choose $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$ from Corollary 3.12. For all $\gamma_0 \in \Gamma$, there exist $x^*, x_{\gamma_0}^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s, s_{\gamma_0} \in Q$ such that

$$T^*(\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} \circ \delta_t = x^* \circ \delta_s, \quad (3.2.14)$$

$$T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, (-\beta_{\gamma_0}^* y_{\gamma_0}^*)] \circ \delta_t = x_{\gamma_0}^* \circ \delta_{s_{\gamma_0}}. \quad (3.2.15)$$

We would like to show that $s = s_{\gamma_0}$ for all $\gamma_0 \in \Gamma$. Since $\#\Gamma \geq 3$, adding (3.2.15) over all $\gamma_0 \in \Gamma$ gives

$$(\#\Gamma - 2)T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} \circ \delta_t] = \sum_{\gamma_0 \in \Gamma} x_{\gamma_0}^* \circ \delta_{s_{\gamma_0}}.$$

So, from (3.2.14), we have

$$(\#\Gamma - 2)x^* \circ \delta_s = \sum_{\gamma_0 \in \Gamma} x_{\gamma_0}^* \circ \delta_{s_{\gamma_0}}. \quad (3.2.16)$$

We claim that there does not exist a nonempty $B \subseteq \Gamma$ such that $\sum_{\mu \in B} x_\mu^* \circ \delta_{s_\mu} = 0$. Suppose not. Summing (3.2.15) over all $\mu \in B$ gives

$$\sum_{\mu \in B} T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \mu}, (-\beta_\mu^* y_\mu^*)] \circ \delta_t = \sum_{\mu \in B} x_\mu^* \circ \delta_{s_\mu} = 0.$$

Since T^* is a linear injection, so

$$\sum_{\mu \in B} [(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \mu}, (-\beta_\mu^* y_\mu^*)] = 0. \quad (3.2.17)$$

It follows that there exists $\mu_0 \in \Gamma$ such that $\beta_{\mu_0}^* y_{\mu_0}^* = 0$. But $\beta_\gamma^* \neq 0$ and $y_\gamma^* \neq 0$ for all $\gamma \in \Gamma$, a contradiction. Therefore, (3.2.16) is irreducible. Hence, by Lemma 3.16, we have $s = s_{\gamma_0}$ for all $\gamma_0 \in \Gamma$. This concludes the proof of the proposition. $\square$

3.2.2 Case 2: $\#\Gamma = 2$ and $F \neq \mathbb{R}$

Fix $t \in K$. Since $\#\Gamma = 2$, so for each $y^* = (\beta_1^* y_1^*, \beta_2^* y_2^*)$ with $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ and $y_1^*, y_2^* \in \text{ext } B(F^*)$, we have $\Phi_{(-\beta_1^* y_1^*, \beta_2^* y_2^*)}(t) = \Phi_{(\beta_1^* y_1^*, -\beta_2^* y_2^*)}(t)$. Therefore,

$$\varphi_{y^*}(t) = \{\Phi_{y^*}(t), \Phi_{(-\beta_1^* y_1^*, \beta_2^* y_2^*)}(t), \Phi_{(\beta_1^* y_1^*, -\beta_2^* y_2^*)}(t)\} = \{\Phi_{y^*}(t), \Phi_{(-\beta_1^* y_1^*, \beta_2^* y_2^*)}(t)\}. \quad (3.2.18)$$

Lemma 3.22. *Assume that $\#\Gamma = 2$. For any $t \in K$ and $y^* = (\beta_1^* y_1^*, \beta_2^* y_2^*)$ where $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ and $y_1^*, y_2^* \in \text{ext } B(F^*)$, if $y_1^* \neq \pm y_2^*$, then*

$$\begin{aligned} \Phi_{(\beta_1^* y_1^*, \beta_2^* y_2^*)}(t) &= \Phi_{(\beta_1^* y_1^*, \beta_2^* y_1^*)}(t) = \Phi_{(\beta_1^* y_1^*, -\beta_2^* y_1^*)}(t) = \Phi_{(\beta_1^* y_2^*, \beta_2^* y_2^*)}(t) \\ &= \Phi_{(-\beta_1^* y_2^*, \beta_2^* y_2^*)}(t). \end{aligned}$$

Proof. Fix $t \in K$ and $y^* = (\beta_1^* y_1^*, \beta_2^* y_2^*)$ as given by the assumption of the lemma. Then there exist $x^*, x_1^*, x_2^*, x_3^*, x_4^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s, s_1, s_2, s_3, s_4 \in Q$ such that

$$\begin{aligned} T^*(\beta_1^* y_1^*, \beta_2^* y_2^*) \circ \delta_t &= x^* \circ \delta_s, \\ T^*(\beta_1^* y_1^*, \beta_2^* y_1^*) \circ \delta_t &= x_1^* \circ \delta_{s_1}, \\ T^*(\beta_1^* y_1^*, -\beta_2^* y_1^*) \circ \delta_t &= x_2^* \circ \delta_{s_2}, \\ T^*(\beta_1^* y_2^*, \beta_2^* y_2^*) \circ \delta_t &= x_3^* \circ \delta_{s_3}, \text{ and} \\ T^*(-\beta_1^* y_2^*, \beta_2^* y_2^*) \circ \delta_t &= x_4^* \circ \delta_{s_4}. \end{aligned} \quad (3.2.19)$$

We wish to show that $s = s_1 = s_2 = s_3 = s_4$. Adding the second, third, fourth and fifth equations above gives us that

$$2T^*(\beta_1^* y_1^*, \beta_2^* y_2^*) \circ \delta_t = (x_1^* \circ \delta_{s_1} + x_2^* \circ \delta_{s_2} + x_3^* \circ \delta_{s_3} + x_4^* \circ \delta_{s_4}).$$

By (3.2.19), we have

$$2x^* \circ \delta_s = x_1^* \circ \delta_{s_1} + x_2^* \circ \delta_{s_2} + x_3^* \circ \delta_{s_3} + x_4^* \circ \delta_{s_4}. \quad (3.2.20)$$

By Corollary 2.8, $\beta_1^* \neq 0$ and $\beta_2^* \neq 0$. Since $y_1^* \neq \pm y_2^*$ and T^* is a linear injection, so there does not exist $B \subseteq \{1, 2, 3, 4\}$ such that $\sum_{i \in B} x_i^* \circ \delta_{s_i} = 0$. By Lemma 3.16, $s = s_1 = s_2 = s_3 = s_4 = s_5$. This concludes the proof of the lemma. $\square$

Lemma 3.23. *If $F \neq \mathbb{R}$, then for any $y^* \in \text{ext } B(F^*)$, there exists $w^* \in \text{ext } B(F^*)$ such that $w^* \neq \pm y^*$.*

Proof. Since $y^* : F \rightarrow \mathbb{R}$ is a nonzero linear functional and $F \neq \mathbb{R}$, there exists $y \in F$ such that $\|y\| = 1$ and $y^*(y) = 0$. By Lemma 2.6, there exists $w^* \in \text{ext } B(F^*)$ such that $w^*(y) = 1$. Clearly $w^* \neq \pm y^*$. Otherwise, $1 = w^*(y) = \pm y^*(y) = 0$, a contradiction. This concludes the proof of the lemma. $\square$

The following proposition is the main result of this subsection.

Proposition 3.24. *Assume that $\#\Gamma = 2$ and $F \neq \mathbb{R}$. For any $t \in K$, we have $\varphi_{y^*}(t) = \{s\}$ for some $y^* = (\beta_1^* y_1^*, \beta_2^* y_2^*)$ and $s \in Q$ where $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ and $y_1^*, y_2^* \in \text{ext } B(F^*)$.*

Proof. Fix $t \in K$ and $y_1^*, y_2^* \in \text{ext } B(F^*)$. Choose $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ from Corollary 3.12. By (3.2.18), it suffices to show that $\Phi_{(\beta_1^* y_1^*, \beta_2^* y_2^*)}(t) = \Phi_{(-\beta_1^* y_1^*, \beta_2^* y_2^*)}(t)$. If $y_1^* \neq \pm y_2^*$, then by Lemma 3.22, we have $\Phi_{(\beta_1^* y_1^*, \beta_2^* y_2^*)}(t) = \Phi_{(\beta_1^* y_2^*, \beta_2^* y_2^*)}(t) = \Phi_{(-\beta_1^* y_1^*, \beta_2^* y_2^*)}(t)$. Assume that $y_1^* = \pm y_2^*$. Since $\dim(F) \geq 2$, choose $w^* \in \text{ext } B(F^*) \setminus \{\pm y_1^*\}$ from Lemma 3.23. By Lemma 3.22, we have $\Phi_{(\beta_1^* y_1^*, \beta_2^* w^*)}(t) = \Phi_{(\beta_1^* y_1^*, \pm \beta_2^* y_1^*)}(t)$ and $\Phi_{(\beta_1^* y_1^*, \beta_2^* w^*)}(t) = \Phi_{(-\beta_1^* y_1^*, \pm \beta_2^* y_1^*)}(t)$. Therefore,

$$\Phi_{(\beta_1^* y_1^*, \beta_2^* y_2^*)}(t) = \Phi_{(\beta_1^* y_1^*, \pm \beta_2^* y_1^*)}(t) = \Phi_{(\beta_1^* y_1^*, \beta_2^* w^*)}(t) = \Phi_{(-\beta_1^* y_1^*, \beta_2^* y_2^*)}(t).$$

This concludes the proof of the proposition. $\square$

3.2.3 Case 3: $\#\Gamma = 2$, $F = \mathbb{R}$ and $(\#\Lambda \geq 3 \text{ or } E \neq \mathbb{R})$

Lemma 3.26 plays an important role in the remaining sections. We will use Lemma 3.25 in the proof of Lemma 3.26. Let $S(\mathbb{R}_{\theta^*}^2) = \{(\beta_1^*, \beta_2^*) \in \mathbb{R}^2 : \theta^*(\beta_1^*, \beta_2^*) = 1\}$.

Lemma 3.25. *If $\{(1, 0), (0, 1), (1, 1), (1, -1)\} \subseteq S(\mathbb{R}_{\theta^*}^2)$, then $\theta^* = \|\cdot\|_\infty$.*

Proof. Fix any $(\beta_1^*, \beta_2^*) \in \mathbb{R}^2$. Observe that

$$\begin{aligned} \theta^*(\beta_1^*, \beta_2^*) &\geq \theta^*(\beta_1^*, 0) \vee \theta^*(0, \beta_2^*) && \text{by monotonicity of } \theta^* \\ &= |\beta_1^*| \theta^*(1, 0) \vee |\beta_2^*| \theta^*(0, 1) && \text{since } \theta^* \text{ is a norm} \\ &= |\beta_1^*| \vee |\beta_2^*| && \text{since } (1, 0), (0, 1) \in S(\mathbb{R}_{\theta^*}^2). \end{aligned}$$

Therefore, $\theta^* \geq \|\cdot\|_\infty$. On the other hand, observe that

$$(\beta_1^*, \beta_2^*) = \frac{\beta_1^* + \beta_2^*}{2}(1, 1) + \frac{\beta_1^* - \beta_2^*}{2}(1, -1). \quad (3.2.21)$$

Therefore,

$$\begin{aligned} \theta^*(\beta_1^*, \beta_2^*) &\leq \frac{|\beta_1^* + \beta_2^*|}{2} \theta^*(1, 1) + \frac{|\beta_1^* - \beta_2^*|}{2} \theta^*(1, -1) && \text{by (3.2.21)} \\ &= \frac{|\beta_1^* + \beta_2^*|}{2} + \frac{|\beta_1^* - \beta_2^*|}{2} && \text{since } (1, 1), (1, -1) \in S(\mathbb{R}_{\theta^*}^2) \\ &= |\beta_1^*| \vee |\beta_2^*|. \end{aligned}$$

Hence, $\theta^* \leq \|\cdot\|_\infty$. This concludes the proof of the lemma. $\square$

Lemma 3.26. *Assume that $\#\Gamma = 2$ and $F = \mathbb{R}$. For any $t \in K$, if there exists $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ such that*

$$T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = u^* \circ \delta_r \quad \text{and} \quad T^*[(-\beta_1^*, \beta_2^*) \circ \delta_t] = v^* \circ \delta_s \quad (3.2.22)$$

for some $u^*, v^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $r, s \in Q$ where $r \neq s$, then there do not exist functions f_1 and f_2 in $C_0^p(Q, E)$ such that $\text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset$, $\|f_1\|_{D, \rho} = \|f_2\|_{D, \rho} = 1$ and $\langle f_1, u^* \circ \delta_r \rangle = \langle f_2, v^* \circ \delta_s \rangle = 1$.

Proof. Assume that such functions f_1 and f_2 exist. Since β_1^* and β_2^* are nonzero, subtracting and adding the two equations in (3.2.22) yield

$$T^*[(1, 0) \circ \delta_t] = \frac{1}{2\beta_1^*}(u^* \circ \delta_r - v^* \circ \delta_s) \quad \text{and} \quad (3.2.23)$$

$$T^*[(0, 1) \circ \delta_t] = \frac{1}{2\beta_2^*}(u^* \circ \delta_r + v^* \circ \delta_s). \quad (3.2.24)$$

Since T is an isometry and f_1 and f_2 have disjoint support, so $\|T(f_1 - f_2)\|_{D,\rho} = \|f_1 - f_2\|_{D,\rho} = 1$. Also, $\langle f_2, u^* \circ \delta_r \rangle = \langle f_1, v^* \circ \delta_s \rangle = 0$. Therefore,

$$\begin{aligned} [T(f_1 - f_2)](t) &= \langle f_1 - f_2, T^*[(1, 0) \circ \delta_t] \rangle && \text{by definition of } \langle \cdot, \cdot \rangle \\ &= \frac{1}{2\beta_1^*} [\langle f_1, u^* \circ \delta_r \rangle - \langle -f_2, v^* \circ \delta_s \rangle] && \text{by (3.2.23)} \\ &= \frac{1}{\beta_1^*} && \text{by assumption.} \end{aligned}$$

Since $\|T(f_1 - f_2)\|_{D,\theta} = 1$, so $\frac{1}{\beta_1^*} = |[T(f_1 - f_2)](t)| \leq 1$, that is, $\beta_1^* \geq 1$. Note that $\beta_1^* \leq \theta^*(\beta_1^*, \beta_2^*) = 1$, Consequently, $\beta_1^* = 1$. Similarly, by applying $f_1 + f_2$ to (3.2.24) above, we can show that $\beta_2^* = 1$. As a result, we have proven that $(1, 1), (-1, 1), (1, -1), (-1, -1) \in \text{ext } B(\mathbb{R}_{\theta^*}^2)$. By Lemma 3.25, we have $\theta^* = \|\cdot\|_\infty$. Therefore, $\theta = \|\cdot\|_1$.

Let $g = T[(f_1 - f_2)] \in C_0^1(K)$. From the paragraph above, we have $\|g\|_{D,\theta} = 1$, $g(t) = 1$ and $g'(t) = 0$. Let t_* be the largest t' such that g' is not identically zero on $[t, t']$. Since $g \in C_0(K)$ and $g(t) = 1$, such t_* exists and $t_* \in K$. Note that $g(t_*) = 1$ and $g'(t_*) = 0$. Choose $u \in K$ such that $u \in (t_*, t_* + 1)$ and $[t_*, u] \subseteq K$. Since g' is continuous on a compact interval $[t_*, u]$, it attains minimum at some $v \in (t_*, u]$. Note that $v \in (t_*, t_* + 1)$. By Mean Value Theorem on $[t_*, v]$, we have

$$g(v) \geq g(t_*) + g'(v)(v - t_*) = 1 + g'(v)(v - t_*). \quad (3.2.25)$$

If $g'(v) > 0$, since $v - t_* > 0$, from (3.2.25), we have $g(v) > 1$. However,

$$1 = \|g\|_{D,\theta} \geq \theta(|g(v)|, |g'(v)|) \geq \theta(|g(v)|, 0) = |g(v)| > 1,$$

a contradiction. If $g'(v) = 0$, by minimality of g' at v , we must have $g' = 0$ on $[t_*, v]$, contradicting the choice of t_* . If $g'(v) < 0$, since $0 < v - t_* < 1$, from (3.2.25), we have

$$g(v) \geq 1 + g'(v)(v - t_*) > 1 + g'(v).$$

Hence, $g(v) - g'(v) > 1$. However, $1 = \|g\|_{D,\theta} \geq |g(v)| + |g'(v)| \geq g(v) - g'(v) > 1$, a contradiction. Therefore, such f_1 and f_2 do not exist. This concludes the proof of the lemma. $\square$

Lemma 3.27. *Assume that*

$$(c_1^*, c_2^*) = \frac{1}{2} \left(\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) (\beta_1^*, \beta_2^*) + \frac{1}{2} \left(-\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) (-\beta_1^*, \beta_2^*) \quad (3.2.26)$$

for some $(c_1^*, c_2^*), (\beta_1^*, \beta_2^*) \in \text{ext } B(\mathbb{R}_{\theta^*}^2)$, where β_1^* and β_2^* are nonzero. If either $\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} = 0$ or $-\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} = 0$, then $|c_1^*| = |\beta_1^*|$ and $|c_2^*| = |\beta_2^*|$.

Proof. Suppose that $-\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} = 0$. Then $\frac{1}{2} \left(\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) = \frac{c_2^*}{\beta_2^*} = \frac{c_1^*}{\beta_1^*}$. Consequently, (3.2.26) implies that

$$(c_1^*, c_2^*) = \frac{c_2^*}{\beta_2^*} (\beta_1^*, \beta_2^*).$$

Since both (c_1^*, c_2^*) and (β_1^*, β_2^*) have norms 1, equation above implies that $1 = \theta^*(c_1^*, c_2^*) = \frac{|c_2^*|}{|\beta_2^*|} \theta^*(\beta_1^*, \beta_2^*) = \frac{|c_2^*|}{|\beta_2^*|}$, which leads to $|c_2^*| = |\beta_2^*|$. Since $\frac{c_2^*}{\beta_2^*} = \frac{c_1^*}{\beta_1^*}$, we conclude that $|c_2^*| = |\beta_2^*|$ and $|c_1^*| = |\beta_1^*|$. By using similar arguments as above, we can also obtain that if $\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} = 0$, then $|c_1^*| = |\beta_1^*|$ and $|c_2^*| = |\beta_2^*|$. $\square$

Lemma 3.28. *Assume that $\#\Gamma = 2$ and $F = \mathbb{R}$. Suppose that for any $t \in K$, there exist $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ and $(c_1^*, c_2^*) \in \text{ext } B(\mathbb{R}_{\theta^*}^2)$ such that*

$$T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = u^* \circ \delta_r,$$

$$T^*[(-\beta_1^*, \beta_2^*) \circ \delta_t] = v^* \circ \delta_s \quad \text{and} \quad (3.2.27)$$

$$T^*[(c_1^*, c_2^*) \circ \delta_t] = w^* \circ \delta_{r'}. \quad (3.2.28)$$

for some $u^*, v^*, w^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $r, s, r' \in Q$. If $r \neq s$, then $|c_1^*| = |\beta_1^*|$ and $|c_2^*| = |\beta_2^*|$. If, in addition, $r' = r$, then either $w^* = u^*$ or $w^* = -u^*$.

Proof. Assume that $r \neq s$. We claim that $|c_1^*| = |\beta_1^*|$ and $|c_2^*| = |\beta_2^*|$. Suppose not. Since $\beta_1^* \neq 0$ and $\beta_2^* \neq 0$, so

$$(c_1^*, c_2^*) = \frac{1}{2} \left(\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) (\beta_1^*, \beta_2^*) + \frac{1}{2} \left(-\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) (-\beta_1^*, \beta_2^*).$$

From the contrapositive of Lemma 3.27, we have $-\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \neq 0$ and $\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \neq 0$. Now, by (3.2.28), (3.2.26) and linearity of T^* , we have

$$\begin{aligned} w^* \circ \delta_{r'} &= T^*[(c_1^*, c_2^*) \circ \delta_t] \\ &= \frac{1}{2} \left(\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) u^* \circ \delta_r + \frac{1}{2} \left(-\frac{c_1^*}{\beta_1^*} + \frac{c_2^*}{\beta_2^*} \right) v^* \circ \delta_s. \end{aligned}$$

By Lemma 3.16, $r' = r = s$, contradicting with $r \neq s$. Hence, we must have $|c_1^*| = \beta_1^*$ and $|c_2^*| = \beta_2^*$. This concludes the proof of the first assertion.

Now, assume that $r' = r$. Then (3.2.28) becomes $T^*[(c_1^*, c_2^*) \circ \delta_t] = w^* \circ \delta_r$. Suppose that $c_1^* = -\beta_1^*$ and $c_2^* = \beta_2^*$. Then $T^*[(-\beta_1^*, \beta_2^*) \circ \delta_t] = w^* \circ \delta_r$. However, (3.2.27) implies that $v^* \circ \delta_s = T^*[(-\beta_1^*, \beta_2^*) \circ \delta_t] = w^* \circ \delta_r$. Since $v^* \neq 0$, by Lemma 3.16, we have $r = s$, contradicting $r \neq s$. We obtain similar contradiction if $c_1^* = \beta_1^*$ and $c_2^* = -\beta_2^*$. In conclusion, we must have either $c_1^* = \beta_1^*$ and $c_2^* = \beta_2^*$ or $c_1^* = -\beta_1^*$ and $c_2^* = -\beta_2^*$. In the former, $w^* \circ \delta_r = T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = u^* \circ \delta_r$, implying that $w^* = u^*$. On the other hand, if $c_1^* = -\beta_1^*$ and $c_2^* = -\beta_2^*$, then we have $w^* \circ \delta_r = T^*[(-\beta_1^*, -\beta_2^*) \circ \delta_t] = -u^* \circ \delta_r$, implying that $w^* = -u^*$. This concludes the proof of the lemma. $\square$

The following lemma asserts that if the mapping $\psi : Q \rightarrow K$ corresponding to $(T^*)^{-1}$ exists, then the mapping $\varphi : K \rightarrow Q$ corresponding to T^* exists as well.

Lemma 3.29. *Assume that $\#\Gamma = 2$ and $F = \mathbb{R}$. Suppose that for any $t \in K$, there exists $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ such that*

$$T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = u^* \circ \delta_r \quad \text{and} \quad (3.2.29)$$

$$T^*[(-\beta_1^*, \beta_2^*) \circ \delta_t] = v^* \circ \delta_s \quad (3.2.30)$$

for some $u^*, v^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $r, s \in Q$. Assume that for any $s' \in Q$, there exists $t' \in K$ such that for any $x^* = (\varepsilon_\lambda^* x_\lambda^*)_{\lambda \in \Lambda}$ where $(\varepsilon_\lambda^*)_{\lambda \in \Lambda} \in \mathcal{G}_D(s')$ and $x_\lambda^* \in \text{ext } B(E^*)$ for all $\lambda \in \Lambda$, we have $(T^*)^{-1}(x^* \circ \delta_{s'}) = b^* \circ \delta_{t'}$ for some $b^* \in \text{ext } B(\mathbb{R}_{\theta^*}^2)$. Then $r = s$.

Proof. Assume that $r \neq s$. Let U and V be disjoint neighbourhoods of r and s respectively. Let $u^* = (u_\lambda^*)_{\lambda \in \Lambda}$. Choose $(u'_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$ such that $u'_\lambda \neq 0$ and $u_\lambda^*(u'_\lambda) \geq 0$ for all $\lambda \in \Lambda$. Let $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}_D(r)$. Consider $u = (u_\lambda) \in E_\rho^\Lambda$ where $u_\lambda = \alpha_\lambda \frac{u'_\lambda}{\|u'_\lambda\|}$. Since $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}_D(r)$, there exists $f_1 \in C_0^p(Q, E)$ such that $\|f_1\|_{D, \rho} = 1$, $s \notin \text{supp}(f_1)$ and $(\partial^\lambda f_1(r))_{\lambda \in \Lambda} = (u_\lambda)_{\lambda \in \Lambda}$. By choice of α_λ and u'_λ , we have

$$\langle f_1, u^* \circ \delta_r \rangle = \sum_{\lambda \in \Lambda} u_\lambda^*(u_\lambda) = \sum_{\lambda \in \Lambda} \frac{\alpha_\lambda}{\|u'_\lambda\|} u_\lambda^*(u'_\lambda) \geq 0.$$

By Lemma 2.6, there exist $(\alpha_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B(\mathbb{R}_{\rho^*}^\Lambda)$ and $w_\lambda^* \in \text{ext } B(E^*)$ for all $\lambda \in \Lambda$ such that

$$\sum_{\lambda \in \Lambda} \alpha_\lambda^*(\alpha_\lambda) = \rho((\alpha_\lambda)_{\lambda \in \Lambda}) = 1 \quad \text{and} \quad w_\lambda^* \left(\frac{u'_\lambda}{\|u'_\lambda\|} \right) = \left\| \frac{u'_\lambda}{\|u'_\lambda\|} \right\| = 1.$$

Denote $w_1^* = (\alpha_\lambda^* w_\lambda^*)_{\lambda \in \Lambda}$. By Corollary 3.14, we have

$$(T^*)^{-1}(w_1^* \circ \delta_r) = (c_1^*, c_2^*) \circ \delta_{t'} \quad (3.2.31)$$

for some $(c_1^*, c_2^*) \in \text{ext } B(\mathbb{R}_{\theta^*}^2)$ and $t' \in K$. From assumption, (3.2.29) and (3.2.31), we have $t = t'$. By Lemma 3.28, either $w_1^* = u^*$ or $w_1^* = -u^*$. Since $\langle f_1, u^* \circ \delta_r \rangle \geq 0$ and $\langle f_1, w_1^* \circ \delta_r \rangle = 1 > 0$, we must have $w_1^* = u^*$. As a result, (3.2.29) becomes $T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = w_1^* \circ \delta_r$.

In conclusion, we have shown that there exist a function $f_1 \in C_0^p(Q, E)$ and $w_1^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ such that $\|f_1\|_{D,\rho} = 1$, $\text{supp}(f_1) \subseteq U$, $\langle f_1, w_1^* \circ \delta_r \rangle = 1$ and $T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = w_1^* \circ \delta_r$. By symmetry, one can show that there exist a function $f_2 \in C_0^p(Q, E)$ and $w_2^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ such that $\|f_2\|_{D,\rho} = 1$, $\text{supp}(f_2) \subseteq V$, $\langle f_2, w_2^* \circ \delta_s \rangle = 1$ and $T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = w_2^* \circ \delta_s$. However, this contradicts with Lemma 3.26. Therefore, $r = s$. This concludes the proof of the lemma. $\square$

Now, we will use Lemma 3.29 to show that the assumption of Corollary 3.19 holds. The following proposition is the main result of this subsection.

Proposition 3.30. *Assume that $\#\Gamma = 2$ and $F = \mathbb{R}$. If either $\#\Lambda \geq 3$ or $E \neq \mathbb{R}$, then for any $t \in K$, we have $\varphi_{y^*}(t) = \{s\}$ for some $y^* = (\beta_1^*, \beta_2^*)$ and $s \in Q$ where $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$.*

Proof. Fix any $t \in K$ and choose $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ from Corollary 3.12. By Corollary 3.14, we have $T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = u^* \circ \delta_r$ and $T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] = v^* \circ \delta_s$ for some $u^*, v^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $r, s \in Q$. We would like to show that $r = s$. By replacing Γ with Λ and F with E in Propositions 3.21 and 3.24, the assumption of Lemma 3.29 is satisfied. By Lemma 3.29, we have $r = s$. This concludes the proof of the proposition. $\square$

3.2.4 Case 4: $\#\Gamma = 2$, $F = \mathbb{R}$, $\#\Lambda = 2$ and $E = \mathbb{R}$

Proposition 3.31. *Assume that $\#\Gamma = 2$, $F = \mathbb{R}$, $\#\Lambda = 2$ and $E = \mathbb{R}$. For any $t \in K$, we have $\varphi_{y^*}(t) = \{s\}$ for some $y^* = (\beta_1^*, \beta_2^*)$ and $s \in Q$ where $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$.*

Proof. Fix $t \in K$ and choose $(\beta_1^*, \beta_2^*) \in \mathcal{G}_D(t)$ from Corollary 3.12. By Corollary 3.14, we have

$$\begin{aligned} T^*[(\beta_1^*, \beta_2^*) \circ \delta_t] &= (b_1^*, b_2^*) \circ \delta_r \quad \text{and} \\ T^*[(-\beta_1^*, \beta_2^*) \circ \delta_t] &= (d_1^*, d_2^*) \circ \delta_s \end{aligned} \quad (3.2.32)$$

for some $(b_1^*, b_2^*), (d_1^*, d_2^*) \in \text{ext } B(\mathbb{R}_{\rho^*}^2)$ and $r, s \in Q$. We would like to show that $r = s$. To this end, we consider two cases.

Case 1: $(|b_1^*|, |b_2^*|) \notin \mathcal{G}_D(r)$ or $(|d_1^*|, |d_2^*|) \notin \mathcal{G}_D(s)$.

If $(|b_1^*|, |b_2^*|) \notin \mathcal{G}_D(r)$, by Corollary 3.12, there exists $(e_1^*, e_2^*) \in \mathcal{G}_D(r)$ such that $e_1^* \neq |b_1^*|$ or $e_2^* \neq |b_2^*|$. By Corollary 3.14, we have

$$(T^*)^{-1}[(e_1^*, e_2^*) \circ \delta_r] = x^* \circ \delta_{t'} \quad \text{and} \quad (3.2.33)$$

$$(T^*)^{-1}[(-e_1^*, e_2^*) \circ \delta_r] = z^* \circ \delta_{t''} \quad (3.2.34)$$

for some $x^*, z^* \in \text{ext } B(\mathbb{R}_{\theta^*}^2)$ and $t', t'' \in K$. By (3.2.33), (3.2.34), (3.2.32) and Lemma 3.28, $t' = t''$. By Corollary 3.19, there exists a mapping $\psi : Q \rightarrow K$ that satisfies the assumption of Lemma 3.29. Therefore, by Lemma 3.29, we have $r = s$. Similarly, if $(|d_1^*|, |d_2^*|) \notin \mathcal{G}_D(s)$, arguments above imply that $r = s$.

Case 2: $(|b_1^*|, |b_2^*|) \in \mathcal{G}_D(r)$ and $(|d_1^*|, |d_2^*|) \in \mathcal{G}_D(s)$.

Suppose that $r \neq s$. Then there exist two disjoint open neighbourhoods U and V of r and s respectively. Since $(|b_1^*|, |b_2^*|) \in \mathcal{G}_D(r)$, so $|b_1^*|(b_1) + |b_2^*|(b_2) = 1$ for some $(b_1, b_2) \in \mathcal{P}_D(r)$. Then there exists $f_1 \in C_0^1(Q)$ such that $\text{supp}(f_1) \subseteq U$, $\|f_1\|_{D, \rho} = 1$ and $(f_1(r), f_1'(r)) = (\text{sgn}(b_1^*)b_1, \text{sgn}(b_2^*)b_2)$. It follows that $\langle f_1, (b_1^*, b_2^*) \circ \delta_r \rangle = 1$. Similarly, since $(|d_1^*|, |d_2^*|) \in \mathcal{G}_D(s)$, there exists $f_2 \in C_0^1(Q)$ such that $\text{supp}(f_2) \subseteq V$, $\|f_2\|_{D, \rho} = 1$ and $\langle f_2, (d_1^*, d_2^*) \circ \delta_s \rangle = 1$. However, existence of f_1 and f_2 contradicts with Lemma 3.26. Therefore, $r = s$. This concludes the proof of the proposition. $\square$

We are now ready to state the main result of this section, which follows immediately from Propositions 3.21, 3.24, 3.30, 3.31, 3.15, Corollary 3.19 and Remark 3.20.

Theorem 3.32. *Let $T : (C_0^p(Q, E), \|\cdot\|_{D,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,\theta})$ be a surjective linear isometry where $p, q, m, n \geq 1$, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. Then there exists $\varphi : K \rightarrow Q$ such that for all $t \in K$ and $y^* \in (F_\theta^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_\rho^\Lambda)^*$.*

3.3 T is a weighted composition operator

Henceforth, we let $\varphi : K \rightarrow Q$ and $\psi : Q \rightarrow K$ be induced by T^* and $(T^*)^{-1}$ according to Theorem 3.32.

The main result of this concluding section is Theorem 3.34, which asserts that T is a weighted composition operator. In virtue of Theorem 2.20, it suffices to show that $\varphi : K \rightarrow Q$ and $\psi : Q \rightarrow K$ satisfy $(\clubsuit(T))$ and $(\clubsuit(T^{-1}))$ respectively. Let us recall the definition of $(\clubsuit(T))$. We say that $\varphi : K \rightarrow Q$ satisfies $(\clubsuit(T))$ if for any $t \in K$ and $f \in C_0^p(Q, E)$, we have $Tf(t) = 0$ whenever $(\partial^\lambda f(\varphi(t)))_{\lambda \in \Lambda} = 0$.

Lemma 3.33. *For any $t \in K$ and any $f \in C_0^p(Q, E)$, if $(\partial^\lambda f(\varphi(t)))_{\lambda \in \Lambda} = 0$, then $(\partial^\gamma Tf(t))_{\gamma \in \Gamma} = 0$. In particular, φ satisfies $(\clubsuit(T))$.*

Proof. For each $\gamma \in \Gamma$, from Lemma 2.6, there exists $y_\gamma^* \in \text{ext } B(F^*)$ such that $y_\gamma^*(\partial^\gamma Tf(t)) = \|\partial^\gamma Tf(t)\|$. By Corollary 3.12, pick any $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_D(t)$. Denote $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$. By Proposition 2.5, $y^* \circ \delta_t \in \text{ext } B[C_0^q(K, F)^*]$. By Theorem 3.32, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$. Since $(\partial^\lambda f(\varphi(t)))_{\lambda \in \Lambda} = 0$, so $\langle f, x^* \circ \delta_{\varphi(t)} \rangle = 0$. Therefore,

$$\sum_{\gamma \in \Gamma} \beta_\gamma^* \|\partial^\gamma Tf(t)\| = \sum_{\gamma \in \Gamma} \beta_\gamma^* y_\gamma^*(\partial^\gamma Tf(t)) = \langle Tf, y^* \circ \delta_t \rangle = \langle f, x^* \circ \delta_{\varphi(t)} \rangle = 0.$$

From Corollary 2.8, we have $\beta_\gamma^* > 0$ for all $\gamma \in \Gamma$. Therefore, $(\partial^\gamma Tf(t))_{\gamma \in \Gamma} = 0$. This concludes the proof of the lemma. $\square$

By symmetry, we can also show that ψ satisfies $(\clubsuit(T^{-1}))$. We are now ready to state the main result of this chapter, the Banach-Stone Theorem for diagonal norm, which follows immediately from Theorem 2.20.

Theorem 3.34. *Let $T : (C_0^p(Q, E), \|\cdot\|_{D,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,\theta})$ be a linear isometry where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and both E, F are Banach spaces. Then T is surjective if and only if $p = q$, $m = n$, $\varphi : K \rightarrow Q$ is a diffeomorphism of class C^p and for each $t \in K$, there is a Banach space isomorphism $V(t) : E \rightarrow F$ such that*

$$Tf(t) = V(t)f(\varphi(t)) \quad \text{for every } t \in K \text{ and } f \in C_0^p(Q, E).$$

Furthermore, the mapping $t \mapsto V(t)$ is of class $s - C^p$.

We will investigate further properties of V and φ in the next chapter. In particular, we show that the mapping $V(t)$ is an isometry for all $t \in K$ and both V and the Jacobian matrix of φ are constants on each connected component.

Chapter 4

Further Properties of Isometry under Diagonal Norm

We let $T : (C_0^p(Q, E), \|\cdot\|_{D,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,\theta})$ be a surjective linear isometry. By Theorem 3.34, we have $p = q$, $m = n$, and T takes the form $Tf(t) = V(t)f(\varphi(t))$ for some class C^q diffeomorphism $\varphi : K \rightarrow Q$ and Banach space isomorphisms $V(t) : E \rightarrow F$ for all $t \in K$. Assume that 0 is in both Q and K and $\varphi(0) = 0$.

In this chapter, we would like to obtain some properties for V and φ in the spirit of Botelho and Jamison [6]. In particular, we discovered that V is locally constant and the Jacobian matrix of φ is orthogonal, and in most cases, this will result in φ being locally constant.

4.1 Jacobian matrix of φ

For each $t \in K$, denote $\varphi(t) = (\varphi_1(t), \dots, \varphi_m(t))$. We let A denote the Jacobian matrix of φ , i.e.,

$$A = \begin{bmatrix} \frac{\partial \varphi_1}{\partial t_1} & \frac{\partial \varphi_1}{\partial t_2} & \dots & \frac{\partial \varphi_1}{\partial t_m} \\ \frac{\partial \varphi_2}{\partial t_1} & \frac{\partial \varphi_2}{\partial t_2} & \dots & \frac{\partial \varphi_2}{\partial t_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m}{\partial t_1} & \frac{\partial \varphi_m}{\partial t_2} & \dots & \frac{\partial \varphi_m}{\partial t_m} \end{bmatrix}.$$

It is clear that the higher derivatives of Tf will involve matrices much more complex than A . To study the problem with clarity, we consider a small set of functions which will be enough to highlight the essential features of T .

From [13, Theorem 6.2], $t \mapsto V(t)$ is of class $s - C^q$ (see definition below). In particular, it is of class C^{q-1} with respect to the operator norm.

We let $L(E, F)$ be the collection of all bounded linear operators from E into F . Suppose that $V : K \rightarrow L(E, F)$ is a continuous map, where $L(E, F)$ is endowed with the norm topology. For each $e \in E$, we define $V_e : K \rightarrow F$ as $V_e(t) = V(t)e$ for every $t \in K$. We say that V is of class $s - C^1$ if the following two statements are satisfied:

1. for every $e \in E$, the map V_e admits all partial derivatives of order 1 in K , and
2. for each $j = 1, 2, \dots, n$, the map $\frac{\partial}{\partial t_j} V : K \rightarrow L(E, F)$ sending each $t \in K$ and each $e \in E$ into $\frac{\partial}{\partial t_j} V_e(t)$ is continuous when $L(E, F)$ is endowed the strong operator topology.

We let $q \geq 2$. A map $V : K \rightarrow L(E, F)$ is said to be of class $s - C^q$ if the following two statements are satisfied:

1. V is of class C^{q-1} (considering $L(E, F)$ as a Banach space) with respect to operator norm.
2. All partial derivatives of order $q - 1$ of V are of class $s - C^1$.

We recall that for any $\mu = (\mu_1, \dots, \mu_m)$ and $\lambda = (\lambda_1, \dots, \lambda_m)$ in Λ , we denote

$$\binom{\mu}{\lambda} = \prod_{i=1}^m \binom{\mu_i}{\lambda_i}$$

and the ordering on Λ is defined by $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq m$. For any $f \in C_0^q(Q, E)$, $(s_1, \dots, s_m) \in Q$, $1 \leq i \leq q$ and $1 \leq j \leq m$, we denote

$$\partial_j^i f(s_1, \dots, s_m) = \frac{\partial^i f}{\partial s_j^i}(s_1, \dots, s_m),$$

the i th derivative of f in the direction of s_j component. The following lemma asserts that the partial derivatives of $V(t)g(t)$ with respect to t behave exactly as given by Leibniz's rule.

Lemma 4.1. *For any $q \geq 0$, if $g \in C_0^q(K, E)$ and $\gamma \in \Gamma$ with $|\gamma| = q$, then*

$$\partial^\gamma[V(t)g(t)] = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \partial^\mu g(t) \quad \text{for all } t \in K. \quad (4.1.1)$$

Proof. We prove by induction on q . Clearly (4.1.1) holds for $q = 0$.

Assume that (4.1.1) holds for all $\gamma \in \Gamma$ with $|\gamma| \leq q - 1$. Suppose that $|\gamma| = q$. Let $\gamma = (\gamma_1, \dots, \gamma_m)$. Since $q \geq 1$, without loss of generality, assume that $\gamma_1 \geq 1$. Let $\gamma^* = (\gamma_1 - 1, \gamma_2, \dots, \gamma_m)$. For simplicity, we let $t = (0, \dots, 0)$ and $t' = (t_1, 0, \dots, 0)$. Observe that

$$\partial^\gamma V(0)g(0) = \partial_1 \partial^{\gamma^*} V(0)g(0) = \lim_{t_1 \rightarrow 0} \frac{\partial^{\gamma^*} V(t')g(t') - \partial^{\gamma^*} V(0)g(0)}{t_1} \quad (4.1.2)$$

where the limit is taken in the norm topology. Since $|\gamma^*| = q - 1$ and $g \in C_0^{q-1}(K, E)$, by inductive hypothesis, we have

$$\partial^{\gamma^*}[V(t)g(t)] = \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \partial^{\gamma^*-\mu} V(t) \partial^\mu g(t) \quad \text{for all } t \in K. \quad (4.1.3)$$

It follows from (4.1.2) and (4.1.3) that

$$\begin{aligned} \partial^\gamma V(0)g(0) &= \lim_{t_1 \rightarrow 0} \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \frac{\partial^{\gamma^*-\mu} V(t') \partial^\mu g(t') - \partial^{\gamma^*-\mu} V(0) \partial^\mu g(0)}{t_1} \\ &= \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \lim_{t_1 \rightarrow 0} \left[\frac{\partial^{\gamma^*-\mu} V(t') \partial^\mu g(t') - \partial^{\gamma^*-\mu} V(0) \partial^\mu g(0)}{t_1} \right]. \end{aligned} \quad (4.1.4)$$

We claim that for all $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ with $\mu \leq \gamma^*$, the limit in (4.1.4) above converges to $\partial^{\gamma^*-\mu} V(0) \partial^{\mu^\#} g(0) + \partial^{\gamma-\mu} V(0) \partial^\mu g(0)$ in norm where $\mu^\# = (\mu_1 + 1, \mu_2, \dots, \mu_m)$. Note that

$$\begin{aligned} &\frac{\partial^{\gamma^*-\mu} V(t') \partial^\mu g(t') - \partial^{\gamma^*-\mu} V(0) \partial^\mu g(0)}{t_1} \\ &= \partial^{\gamma^*-\mu} V(t') \frac{\partial^\mu g(t') - \partial^\mu g(0)}{t_1} + \frac{\partial^{\gamma^*-\mu} V(t') - \partial^{\gamma^*-\mu} V(0)}{t_1} \partial^\mu g(0). \end{aligned} \quad (4.1.5)$$

Since $\partial^{\gamma^*-\mu}V$ is of class $s - C^1$, taking limits for the second term gives

$$\begin{aligned} \lim_{t_1 \rightarrow 0} \frac{\partial^{\gamma^*-\mu}V(t') - \partial^{\gamma^*-\mu}V(0)}{t_1} \partial^\mu g(0) &= \partial_1 \partial^{\gamma^*-\mu}V(0) \partial^\mu g(0) \\ &= \partial^{\gamma-\mu}V(0) \partial^\mu g(0). \end{aligned} \quad (4.1.6)$$

For the first term, $\partial^{\gamma^*-\mu}V(t') \frac{\partial^\mu g(t') - \partial^\mu g(0)}{t_1}$, let $W(t') = \partial^{\gamma^*-\mu}V(t')$ and $h(t') = \frac{\partial^\mu g(t') - \partial^\mu g(0)}{t_1}$. Since $|\mu^\#| \leq q$, so $\lim_{t_1 \rightarrow 0} h(t') = \partial_1 \partial^\mu g(0) = \partial^{\mu^\#} g(0)$. Then we have

$$\begin{aligned} &\left\| W(t') h(t') - W(0) \partial^{\mu^\#} g(0) \right\| \\ &\leq \|W(t')\| \left\| h(t') - \partial^{\mu^\#} g(0) \right\| + \left\| [W(t') - W(0)] \partial^{\mu^\#} g(0) \right\|. \end{aligned}$$

Since W is continuous with respect to the strong operator topology, $\|W(t')\|$ is uniformly bounded as $t' \rightarrow 0$. Since h is continuous, so $\|W(t')\| \left\| h(t') - \partial^{\mu^\#} g(0) \right\| \rightarrow 0$ as $t' \rightarrow 0$. Since $W(t') \rightarrow W(0)$ in the strong operator topology as $t' \rightarrow 0$, so $\left\| [W(t') - W(0)] \partial^{\mu^\#} g(0) \right\| \rightarrow 0$. Therefore,

$$\begin{aligned} \lim_{t_1 \rightarrow 0} \partial^{\gamma^*-\mu}V(t') \frac{\partial^\mu g(t') - \partial^\mu g(0)}{t_1} &= \lim_{t_1 \rightarrow 0} W(t') h(t') = W(0) \partial^{\mu^\#} g(0) \\ &= \partial^{\gamma^*-\mu}V(0) \partial^{\mu^\#} g(0). \end{aligned} \quad (4.1.7)$$

This concludes the proof of the claim.

Therefore, (4.1.4), (4.1.5), (4.1.6) and (4.1.7) imply that

$$\partial^\gamma V(0)g(0) = \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \left[\partial^{\gamma^*-\mu}V(0) \partial^{\mu^\#} g(0) + \partial^{\gamma-\mu}V(0) \partial^\mu g(0) \right].$$

Clearly $\partial^{\gamma^*-\mu}V(0) = \partial^{\gamma-\mu^\#}V(0)$. Thus,

$$\begin{aligned} \partial^\gamma V(0)g(0) &= \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \left[\partial^{\gamma-\mu^\#}V(0) \partial^{\mu^\#} g(0) + \partial^{\gamma-\mu}V(0) \partial^\mu g(0) \right] \\ &= \sum_{\substack{\mu \leq \gamma \\ \mu_1 \geq 1}} \binom{\gamma^*}{\mu^*} \partial^{\gamma-\mu}V(0) \partial^\mu g(0) + \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \partial^{\gamma-\mu}V(0) \partial^\mu g(0). \end{aligned} \quad (4.1.8)$$

Observe that the first term of (4.1.8) above can be expressed as

$$\begin{aligned}
& \sum_{\substack{\mu \leq \gamma \\ \mu_1 \geq 1}} \binom{\gamma^*}{\mu^*} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) \\
&= \sum_{\substack{\mu \leq \gamma^* \\ \mu_1 \geq 1}} \binom{\gamma^*}{\mu^*} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) + \sum_{\substack{\mu \leq \gamma \\ \mu_1 = \gamma_1}} \binom{\gamma^*}{\mu^*} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) \\
&= \sum_{\substack{\mu \leq \gamma^* \\ \mu_1 \geq 1}} \binom{\gamma^*}{\mu^*} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) + \sum_{\substack{\mu \leq \gamma \\ \mu_1 = \gamma_1}} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0). \tag{4.1.9}
\end{aligned}$$

Also, the second term of (4.1.8) above can be expressed as

$$\begin{aligned}
& \sum_{\mu \leq \gamma^*} \binom{\gamma^*}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) \\
&= \sum_{\substack{\mu \leq \gamma^* \\ \mu_1 \geq 1}} \binom{\gamma^*}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) + \sum_{\substack{\mu \leq \gamma^* \\ \mu_1 = 0}} \binom{\gamma^*}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) \\
&= \sum_{\substack{\mu \leq \gamma^* \\ \mu_1 \geq 1}} \binom{\gamma^*}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) + \sum_{\substack{\mu \leq \gamma \\ \mu_1 = 0}} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0). \tag{4.1.10}
\end{aligned}$$

Since $\binom{\gamma^*}{\mu^*} + \binom{\gamma^*}{\mu} = \binom{\gamma}{\mu}$, adding (4.1.9) and (4.1.10) above and compare it with (4.1.8) give

$$\partial^\gamma V(0)g(0) = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0).$$

This concludes the proof of the lemma. $\square$

The following corollary follows immediately from Lemma 4.1.

Corollary 4.2. *For any $t \in K$ and $g \in C_0^q(K, E)$, if $\partial^\gamma g(t) = 0$ for all $\gamma \in \Gamma$ with $|\gamma| < q$, then $\partial^\gamma V(t)g(t) = V(t)\partial^\gamma g(t)$ for all $\gamma \in \Gamma$ with $|\gamma| = q$.*

Fix any $t \in K$. Choose $f \in C_0^q(Q, E)$ such that $\partial^\lambda f(\varphi(t)) = 0$ for all $\lambda \in \Lambda$ with $|\lambda| < q$ and let $g = f \circ \varphi$. By Constantine and Savits [17, Theorem 2.1], we have $\partial^\gamma g(t) = 0$ for all $\gamma \in \Gamma$ with $|\gamma| < q$. Since $Tf(t) = V(t)g(t)$, it follows from the

choice of f and Corollary 4.2 that

$$\partial^\gamma T f(t) = \begin{cases} 0 & \text{if } |\gamma| < q, \\ V(t) \partial^\gamma g(t) & \text{if } |\gamma| = q. \end{cases} \quad (4.1.11)$$

Let $N = \#\{\gamma \in \Gamma : |\gamma| = q\} = \binom{q+m-1}{q}$. Since $\partial^\lambda f(\varphi(t)) = 0$ for all $|\lambda| < q$, by Constantine and Savits [17, Theorem 2.1] again, there exists an $N \times N$ square matrix $\tilde{A}(t) = (c_{\gamma\lambda}(t))_{\gamma,\lambda}$ of scalars such that

$$(\partial^\gamma g(t))_{|\gamma|=q} = \tilde{A}(t) (\partial^\lambda f(t))_{|\lambda|=q}. \quad (4.1.12)$$

Note that rows and columns of $\tilde{A}(t)$ are indexed by γ and λ respectively. Also, if $m = 1$, then $\tilde{A}(t) = (\varphi'(t))^q$. We denote the operator $V(t) \cdot \tilde{A}(t) : E^N \rightarrow F^N$ by

$$V(t) \cdot \tilde{A}(t) \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix} = \tilde{A}(t) \begin{pmatrix} V(t)e_1 \\ \vdots \\ V(t)e_N \end{pmatrix}.$$

So, we have

$$\begin{aligned} (\partial^\gamma T f(t))_{|\gamma|=q} &= (V(t) \partial^\gamma g(t))_{|\gamma|=q} && \text{by (4.1.11)} \\ &= V(t) \cdot \tilde{A}(t) (\partial^\lambda f(t))_{|\lambda|=q} && \text{by (4.1.12)}. \end{aligned} \quad (4.1.13)$$

If we view $(\partial^\gamma g(t))_{|\gamma|=q}$ and $(\partial^\lambda f(t))_{|\lambda|=q}$ as

$$\begin{pmatrix} (\partial^\gamma g(t))_{|\gamma|=q, \gamma \in \Gamma_f} \\ (\partial^\gamma g(t))_{|\gamma|=q, \gamma \in \Gamma_b} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \quad \text{respectively,}$$

then from (4.1.12), we may write

$$\tilde{A}(t) = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}$$

where $A_1(t)$, $A_3(t)$ act on $(\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_f}$ and $A_2(t)$, $A_4(t)$ act on $(\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_b}$. Therefore, we have

$$\begin{aligned} (\partial^\gamma g(t))_{|\gamma|=q, \gamma \in \Gamma_f} &= \begin{pmatrix} A_1(t) & A_2(t) \end{pmatrix} \begin{pmatrix} (\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \quad \text{and} \\ (\partial^\gamma g(t))_{|\gamma|=q, \gamma \in \Gamma_b} &= \begin{pmatrix} A_3(t) & A_4(t) \end{pmatrix} \begin{pmatrix} (\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\partial^\lambda f(t))_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix}. \end{aligned}$$

To study the Jacobian matrix A of φ , we will need to determine the matrices A_2 , A_3 , and A_4 . A_1 seems to be unimportant in our study.

4.2 The entries in A_2 , A_3 and A_4

Let $g = f \circ \varphi$ for some $f \in C_0^q(Q, E)$. If $|\gamma| = q$, then by Constantine and Savits [17, Theorem 2.1], we can express

$$\partial^\gamma g = \sum_{\{\lambda \in \Lambda: 0 \neq |\lambda| \leq q\}} c_{\gamma\lambda} \cdot \partial^\lambda f \circ \varphi \quad (4.2.1)$$

for some $c_{\gamma\lambda} \in C(K)$.

We want to compute a representation formula of $c_{\gamma\lambda}$ for $\gamma \in \Gamma$ and $\lambda \in \Lambda$ satisfying $|\gamma| = |\lambda| = q$ and either γ or λ is a pure term, i.e.,

$$\gamma = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0) \quad \text{or} \quad \lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0).$$

This is achieved by the following lemma. Note that for any $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda$, we denote

$$\binom{q}{\lambda} = \frac{q!}{\lambda_1! \cdots \lambda_m!}.$$

Lemma 4.3. *If $\lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$ and $|\gamma| = |(\gamma_1, \dots, \gamma_m)| = q$, then*

$$c_{\gamma\lambda} = \prod_{i=1}^m (\partial_i \varphi_j)^{\gamma_i}.$$

If $\gamma = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0)$ and $|\lambda| = |(\lambda_1, \dots, \lambda_m)| = q$, then

$$c_{\gamma\lambda} = \binom{q}{\lambda} \prod_{j=1}^m (\partial_i \varphi_j)^{\lambda_j}.$$

Proof. We prove both assertions by induction on q . If $q = 1$, then for all $1 \leq i \leq m$, Chain rule implies that

$$\partial_i g = \sum_{j=1}^m \partial_i \varphi_j \cdot \partial_j f \circ \varphi.$$

Both parts of the lemma follows from this equation.

Assume that the first assertion holds for some q . Let $\gamma = (\gamma_1, \dots, \gamma_m)$ be of order $q + 1$ and $\lambda = (0, \dots, 0, \overset{j}{q} + 1, 0, \dots, 0)$. Choose i_0 so that $\gamma_{i_0} \neq 0$. Set $\gamma^* = (\gamma_1, \dots, \gamma_{i_0-1}, \gamma_{i_0} - 1, \gamma_{i_0+1}, \dots, \gamma_m)$. Then

$$\begin{aligned} \partial^\gamma g &= \partial_{i_0}(\partial^{\gamma^*} g) \\ &= \partial_{i_0} \left(\sum_{0 \neq |\bar{\lambda}| \leq q} c_{\gamma^* \bar{\lambda}} \cdot \partial^{\bar{\lambda}} f \circ \varphi \right) && \text{by (4.2.1)} \\ &= \sum_{0 \neq |\bar{\lambda}| \leq q} \partial_{i_0} c_{\gamma^* \bar{\lambda}} \cdot \partial^{\bar{\lambda}} f \circ \varphi + \sum_{0 \neq |\bar{\lambda}| \leq q} c_{\gamma^* \bar{\lambda}} \sum_{j'=1}^m \partial_{i_0} \varphi_{j'} \cdot \partial_{j'} \partial^{\bar{\lambda}} f \circ \varphi && \text{by product rule.} \end{aligned}$$

Set $\lambda_0 = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$. Clearly first summand in the above equation does not contain any $c_{\gamma\lambda}$ whereas the second summand in the equation above contains $c_{\gamma\lambda}$ only when $\bar{\lambda} = \lambda_0$ and $j' = j$, that is, $c_{\gamma\lambda} = c_{\gamma^* \lambda_0} \partial_{i_0} \varphi_j$. By the inductive hypothesis on $c_{\gamma^* \lambda_0}$, we have

$$c_{\gamma^* \lambda_0} = (\partial_{i_0} \varphi_j)^{\gamma_{i_0}-1} \prod_{i \neq i_0} (\partial_i \varphi_j)^{\gamma_i}.$$

Therefore, $c_{\gamma\lambda} = \prod_{i=1}^m (\partial_i \varphi_j)^{\gamma_i}$. This concludes the proof of the first assertion.

Assume that the second assertion holds for some q . Let $\gamma = (0, \dots, 0, \overset{i}{q} + 1, 0, \dots, 0)$ and $\lambda = (\lambda_1, \dots, \lambda_m)$ be of order $q + 1$. Set $\gamma^* = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0)$. By replacing i_0 with i in the expression for $\partial^\gamma g$ in the equation above, we have

$$\partial^\gamma g = \sum_{0 \neq |\bar{\lambda}| \leq q} \partial_i c_{\gamma^* \bar{\lambda}} \cdot \partial^{\bar{\lambda}} f \circ \varphi + \sum_{0 \neq |\bar{\lambda}| \leq q} c_{\gamma^* \bar{\lambda}} \sum_{j=1}^m \partial_i \varphi_j \cdot \partial_j \partial^{\bar{\lambda}} f \circ \varphi.$$

Let $J = \{j : \lambda_j > 0\}$. Since $|\lambda| = q + 1 \geq 1$, so $J \neq \emptyset$. In order for $\partial_j \partial^{\bar{\lambda}} = \partial^{\lambda}$, we must have both $j \in J$ and

$$\bar{\lambda} = (\lambda_1, \dots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \dots, \lambda_m) \stackrel{\text{def}}{=} \pi_j.$$

Hence, we have

$$\begin{aligned} c_{\gamma\lambda} &= \sum_{j \in J} c_{\gamma^* \pi_j} \cdot \partial_i \varphi_j \\ &= \sum_{j \in J} \left[\binom{q}{\pi_j} \cdot (\partial_i \varphi_j)^{\lambda_j - 1} \prod_{j' \neq j} (\partial_i \varphi_{j'})^{\lambda_{j'}} \right] \cdot \partial_i \varphi_j \quad \text{by inductive hypothesis} \\ &= \prod_{j'=1}^m (\partial_i \varphi_{j'})^{\lambda_{j'}} \cdot \sum_{j \in J} \binom{q}{\pi_j}. \end{aligned}$$

Since $\sum_{j \in J} \lambda_j = \sum_{j=1}^m \lambda_j = q + 1$, so

$$\sum_{j \in J} \binom{q}{\pi_j} = \sum_{j \in J} \frac{q! \lambda_j}{\lambda_1! \cdots \lambda_m!} = \binom{q}{\lambda} \sum_{j \in J} \lambda_j = \binom{q+1}{\lambda}.$$

Therefore, the inductive step is proven. This concludes the proof of lemma. $\square$

For those $\gamma = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0)$, we label $c_{\gamma\lambda}$ as $c_{i\lambda}$. Similarly, if $\lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$, we label $c_{\gamma\lambda}$ as $c_{\gamma j}$.

Let $a_{ij} = \frac{\partial \varphi_i}{\partial t_j}$. Note that $A_4 = (c_{\gamma\lambda})_{\lambda \in \Lambda_b, \gamma \in \Gamma_b}$. Since

$$\Lambda_b = \left\{ (0, \dots, 0, \overset{j}{q}, 0, \dots, 0) : j = 1, \dots, m \right\}$$

and $\Gamma_b = \left\{ (0, \dots, 0, \overset{i}{q}, 0, \dots, 0) : i = 1, \dots, m \right\}$, we may write $c_{\gamma\lambda} = (\partial_i \varphi_j)^q = a_{ji}^q$. So $A_4 = (a_{ji}^q)_{i,j}$, where the (i, j) entry corresponds to the multi-indices $\lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$ and $\gamma = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0)$.

Since $A_2 = (c_{\gamma\lambda})_{\lambda \in \Lambda_b, \gamma \in \Gamma_f}$, so $A_2 = (c_{\gamma j})_{j, \gamma \in \Gamma_f}$ where

$$c_{\gamma j} = \prod_{i=1}^m (\partial_i \varphi_j)^{\gamma_i} = \prod_{i=1}^m (a_{ji})^{\gamma_i}.$$

So, we have $A_2 = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \end{pmatrix}$, where $c_j = (\prod_{i=1}^m (a_{ji})^{\gamma_i})_{\gamma \in \Gamma_f, |\gamma|=q}$ is a column vector.

Similarly, since $A_3 = (c_{\gamma\lambda})_{\lambda \in \Lambda_f, \gamma \in \Gamma_b}$, so $A_3 = (c_{i\lambda})_{i, \lambda \in \Lambda_f}$ where

$$c_{i\lambda} = \binom{q}{\lambda} \prod_{j=1}^m (\partial_i \varphi_j)^{\lambda_j} = \binom{q}{\lambda} \prod_{j=1}^m (a_{ji})^{\lambda_j}.$$

So, we have $A_3 = \begin{pmatrix} r_1^T & r_2^T & \cdots & r_m^T \end{pmatrix}^T$, where $r_i = ((\binom{q}{\lambda} \prod_{j=1}^m (a_{ji})^{\lambda_j})_{\lambda \in \Lambda_f, |\lambda|=q})$ is a row vector. Therefore, we have the following corollary:

Corollary 4.4. *Let $a_{ij} = \frac{\partial \varphi_i}{\partial t_j}$ for all $1 \leq i, j \leq m$. Then*

$$A_2 = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \end{pmatrix}, \quad A_3 = \begin{pmatrix} r_1^T & r_2^T & \cdots & r_m^T \end{pmatrix}^T \quad \text{and}$$

$$A_4 = \begin{pmatrix} a_{11}^q & a_{21}^q & \cdots & a_{m1}^q \\ a_{12}^q & a_{22}^q & \cdots & a_{m2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m}^q & a_{2m}^q & \cdots & a_{mm}^q \end{pmatrix}$$

where $r_i = ((\binom{q}{\lambda} \prod_{j=1}^m (a_{ji})^{\lambda_j})_{\lambda \in \Lambda_f, |\lambda|=q})$ is a row vector and $c_j = (\prod_{i=1}^m (a_{ji})^{\gamma_i})_{\gamma \in \Gamma_f, |\gamma|=q}$ is a column vector for all $1 \leq i, j \leq m$.

By Corollary 4.4, since A_4 is the transpose and q th power of each entry of the Jacobian matrix A of φ , we would like to show that A_4 takes certain form. In particular, we can show that A_4 is a sparse matrix in most cases.

4.3 Jacobian matrix of φ is a generalized permutation matrix

The main result of this section is Corollary 4.12, which asserts that if any two columns in $A_4(t)$ are disjoint (see definition below), then $A(t)$ contains a lot of zeros. To this end, we need to establish several preliminaries results.

Recall that $\varphi(0) = 0$. The following lemma will be used repeatedly in this chapter. It asserts that if f attains its norm at $\varphi(t)$ for some $t \in K$ with respect to $\|\cdot\|_{D,\rho}$, then Tf will also attain its norm at t with respect to $\|\cdot\|_{D,\theta}$.

Lemma 4.5. *For any $f \in C_0^q(Q, E)$ and $t \in K$, if $\|f\|_{D,\rho} = \rho[(\|\partial^\lambda f(\varphi(t))\|)_{\lambda \in \Lambda}]$, then $\|Tf\|_{D,\theta} = \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}]$ and $\rho[(\|\partial^\lambda f(\varphi(t))\|)_{\lambda \in \Lambda}] = \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}]$.*

Proof. Without loss of generality, we may assume that $t = 0$. Suppose that there exists $f \in C_0^q(Q, E)$ with $\|f\|_{D,\rho} = 1$ such that

$$\rho[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}] = 1 \quad \text{but} \quad \theta[(\|\partial^\gamma Tf(0)\|)_{\gamma \in \Gamma}] < 1.$$

Then $\theta[(\|\partial^\gamma Tf(0)\|)_{\gamma \in \Gamma}] + \varepsilon < 1$ for small enough $\varepsilon > 0$. By continuity, there exists an open neighbourhood $B \subseteq K$ of 0 such that

$$\theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}] + \varepsilon < 1 \quad \text{for all } t \in B. \quad (4.3.1)$$

Clearly $0 \in \varphi(B)$. Choose a function $g \in C_0^q(Q, E)$ such that $\text{supp}(g) \subseteq \varphi(B)$, $\|g\|_{D,\rho} \leq \varepsilon$ and $\partial^\lambda g(0) = \varepsilon' \partial^\lambda f(0)$ for all $\lambda \in \Lambda$ for sufficiently small $0 < \varepsilon' < \varepsilon$. Note that

$$\|f + g\|_{D,\rho} \geq \rho[(\|\partial^\lambda (f + g)(0)\|)_{\lambda \in \Lambda}] = (1 + \varepsilon') \rho[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}] > 1.$$

To obtain a contradiction, we will show that $\|Tf + Tg\|_{D,\theta} \leq 1$. Indeed, for all $t \notin B$, we have $\varphi(t) \notin \text{supp}(g)$. So, $g = 0$ on some neighbourhood of $\varphi(t)$. Therefore, $g \circ \varphi = 0$ on some neighbourhood of t . Since $Tg(t) = V(t)g(\varphi(t))$, so $Tg = 0$ on some neighbourhood of t . It follows that $(\partial^\gamma Tg(t))_{\gamma \in \Gamma} = 0$. Therefore,

$$\theta[(\|\partial^\gamma (Tf + Tg)(t)\|)_{\gamma \in \Gamma}] = \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}] \leq 1.$$

On the other hand, for all $t \in B$, triangle inequality and (4.3.1) imply that

$$\begin{aligned} \theta[(\|\partial^\gamma (Tf + Tg)(t)\|)_{\gamma \in \Gamma}] &\leq \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}] + \theta[(\|\partial^\gamma Tg(t)\|)_{\gamma \in \Gamma}] \\ &\leq \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}] + \|Tg\|_{D,\theta} \\ &= \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}] + \|g\|_{D,\rho} \\ &\leq \theta[(\|\partial^\gamma Tf(t)\|)_{\gamma \in \Gamma}] + \varepsilon \\ &< 1. \end{aligned}$$

Hence, $\|Tf + Tg\|_{D,\theta} \leq 1$. But it contradicts with T is an isometry. $\square$

For any $s \in [1, \infty)$, the s -norm $\|\cdot\|_s$ on $\mathbb{R}^\Lambda$ is defined by

$$\|(\alpha_\lambda)_{\lambda \in \Lambda}\|_s = \left(\sum_{\lambda \in \Lambda} |\alpha_\lambda|^s \right)^{\frac{1}{s}} \quad \text{for all } (\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda.$$

Henceforth, we assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $1 \leq s, r < \infty$. We denote $\|\cdot\|_{D,s} = \|\cdot\|_{D,\|\cdot\|_s}$ and $\|\cdot\|_{D,r} = \|\cdot\|_{D,\|\cdot\|_r}$. The following proposition shows that the operator $V(t) \cdot \tilde{A}(t)$ is an isometry on a set of ‘special’ vectors.

Proposition 4.6. *For any $t \in K$ and $a \in (0, \frac{1}{m^{1/s}})$, there exists $\eta > 0$ such that for any $(x_\lambda)_{\lambda \in \Lambda, |\lambda|=q} \in E^N$, if $0 \leq \|x_\lambda\| \leq \eta$ for all $\lambda \in \Lambda_f$, $\|x_\lambda\| \geq a$ for all $\lambda \in \Lambda_b$ and $\|(x_\lambda)_{\lambda \in \Lambda, |\lambda|=q}\|_s = 1$, then*

$$\left\| V(t) \cdot \tilde{A}(t) \begin{pmatrix} (x_\lambda)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (x_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_r = 1.$$

Proof. Without loss of generality, we may assume that $t = 0$. Fix $a \in (0, \frac{1}{m^{1/s}})$. Choose $\eta > 0$ from Corollary 3.10. Let $y = (y_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$ be such that $y_\lambda = x_\lambda$ if $|\lambda| = q$ and $y_\lambda = 0$ if $|\lambda| < q$. Clearly y satisfies the assumption of Corollary 3.10. So, there exists $f \in C_0^q(Q, E)$ such that $\|f\|_{D,s} = \|[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}]\|_s = 1$ and $(\partial^\lambda f(0))_{\lambda \in \Lambda} = y$. By Lemma 4.5, we have

$$1 = \|[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}]\|_s = \|(\|\partial^\gamma T f(0)\|)_{\gamma \in \Gamma}\|_r. \quad (4.3.2)$$

Observe that

$$\begin{aligned} & \|(\|\partial^\gamma T f(0)\|)_{\gamma \in \Gamma}\|_r \\ &= \|(\|V(0) \partial^\gamma g(0)\|)_{|\gamma|=q}\|_r && \text{by (4.1.11)} \\ &= \|V(0) \cdot \tilde{A}(0) (\partial^\lambda f(0))_{|\lambda|=q}\|_r && \text{by (4.1.13)} \\ &= \left\| V(0) \cdot \tilde{A}(0) \begin{pmatrix} (x_\lambda)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (x_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix}^T \right\|_r \end{aligned} \quad (4.3.3)$$

From (4.3.2) and 4.3.3, we obtained the desired equation. This concludes the proof of the proposition. $\square$

Corollary 4.7. *For any $t \in K$ and $x_\lambda \in E$ for all $\lambda \in \Lambda_b$, we have*

$$\left\| V(t) \cdot \tilde{A}(t) \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (x_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_r = \left\| \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (x_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_s. \quad (4.3.4)$$

Hence, for any $t \in K$, $e \in E$ and $(\alpha_\lambda)_{\lambda \in \Lambda_b} \in \mathbb{R}^m$, we have

$$\left\| \|V(t)e\| \cdot \tilde{A}(t) \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\alpha_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_r = \left\| \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\alpha_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_s \cdot \|e\|.$$

Proof. Fix any $t \in K$. Since (4.3.4) is homogeneous, we may assume that $\|(0)_{|\lambda|=q}, (\|x_\lambda\|)_{|\lambda|=q, \lambda \in \Lambda_b}\|_s = 1$.

1. Assume that $x_\lambda \neq 0$ for all $\lambda \in \Lambda_b$. Let $0 < a < \min_{\lambda \in \Lambda_b} \|x_\lambda\|$. We claim that $a < \frac{1}{m^{1/s}}$. If $a \geq \frac{1}{m^{1/s}}$, then

$$1 = \sum_{\lambda \in \Lambda_b} \|x_\lambda\|^s > \sum_{\lambda \in \Lambda_b} a^s \geq \sum_{\lambda \in \Lambda_b} \frac{1}{m} = 1,$$

a contradiction. So, $a \in (0, \frac{1}{m^{1/s}})$. Now, from Proposition 4.6, (4.3.4) holds for all $x_\lambda \neq 0$ for all $\lambda \in \Lambda_b$. By continuity, (4.3.4) holds for all elements in E^m . This concludes the proof of the first assertion.

By letting $x_\lambda = \alpha_\lambda e$ for all $\lambda \in \Lambda_b$, the second assertion follows from the first assertion. This concludes the proof of the corollary. $\square$

The following corollary is a consequence of Corollary 4.7.

Corollary 4.8. *For any $t \in K$, we have $V(t) = C(t)J(t)$ for some isometry $J(t) : E \rightarrow F$ where $C(t) = \|V(t)\|$. Hence, for any $t \in K$ and $(\alpha_\lambda)_{\lambda \in \Lambda_b} \in \mathbb{R}^m$,*

$$\left\| C(t) \cdot \tilde{A}(t) \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\alpha_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_r = \left\| \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\alpha_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_s.$$

Proof. Fix any $t \in K$. Fix any $e \in E \setminus \{0\}$ and $(\alpha_\lambda)_{\lambda \in \Lambda_b} \in \mathbb{R}^m$. By Corollary 4.7, we have

$$\|V(t)e\| \cdot \left\| \tilde{A}(t) \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\alpha_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_r = \left\| \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (\alpha_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_s \cdot \|e\|.$$

Therefore, $\|V(t)e\| = C(t)\|e\|$ for some positive constant $C(t)$ which is independent of e . Hence, $\frac{V(t)}{C(t)}$ is an isometry and thus, $\|V(t)\| = C(t)$. This proves the first

assertion. Clearly the second assertion follows from the first assertion and Corollary 4.7. This concludes the proof of the corollary. $\square$

For any $t \in K$, let $c(t) = \|V(t)\|^{1/q} = [C(t)]^{1/q} \neq 0$. A square matrix of scalars is a *generalized permutation matrix* if each row and each column contain exactly one nonzero entry. A *signed permutation matrix* is a generalized permutation matrix where nonzero entries are ± 1 .

Recall that if $m = 1$, then $\tilde{A}(t) = (\varphi'(t))^q$ for any $t \in K$. The following corollary follows immediately from Corollary 4.8.

Corollary 4.9. *If $m = 1$, then $V(t) = \|V(t)\|J(t)$ and $c(t)|\varphi'(t)| = 1$ for all $t \in K$. In particular, $c(t)A(t)$ is a signed permutation matrix for all $t \in K$.*

It remains to consider the case $m \geq 2$.

Corollary 4.10. *If $m \geq 2$, then for any $t \in K$, $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $1 \leq i \neq j \leq m$, we have*

$$C(t)\|\varepsilon_1 \mathbf{a}_i + \varepsilon_2 \mathbf{a}_j\|_r = \|(\varepsilon_1, \varepsilon_2)\|_s \quad (4.3.5)$$

where $\mathbf{a}_i, \mathbf{a}_j \in \mathbb{R}^N$ are the i^{th} and j^{th} column of $\begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$ that correspond to $\lambda = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0)$ and $\lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$ respectively.

Proof. Fix any $t \in K$, $e \in E$ with $\|e\| = 1$, $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $1 \leq i \neq j \leq m$. Denote $\lambda' = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0)$ and $\lambda'' = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$. Let $x_{\lambda'} = \varepsilon_1 e$, $x_{\lambda''} = \varepsilon_2 e$ and $x_{\lambda} = 0$ for all $\lambda \in \Lambda_b \setminus \{\lambda', \lambda''\}$. It follows from Corollary 4.7 that $C(t)\|\varepsilon_1 \mathbf{a}_i + \varepsilon_2 \mathbf{a}_j\|_r = \|(\varepsilon_1, \varepsilon_2)\|_s$ where $\mathbf{a}_i, \mathbf{a}_j \in \mathbb{R}^N$ are the i^{th} and j^{th} column of $\begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$. This concludes the proof of the corollary. $\square$

Two vectors $\mathbf{a} = (a_i)_{i=1}^m$ and $\mathbf{b} = (b_i)_{i=1}^m$ in $\mathbb{R}^m$ are *disjoint* if either $a_i = 0$ or $b_i = 0$ for all $1 \leq i \leq m$.

The following simple lemma provides a sufficient condition for $c(t)A(t)$ to be a signed permutation matrix for every $t \in K$.

Lemma 4.11. *Assume that $m \geq 2$. For any $t \in K$, if any two columns in $A_4(t)$ are disjoint, then $A_4(t)$ is a generalized permutation matrix. In addition, if all nonzero entries of $c(t)^q A_4(t)$ are ± 1 , then $c(t)A(t)$ is a signed permutation matrix.*

Proof. Fix any $t \in K$. By disjointness, every row of $A_4(t)$ contains at most one nonzero entry. It follows from the invertibility of the Jacobian matrix $A(t)$ of φ and Corollary 4.4 that every row and column of $A_4(t)$ must have exactly one nonzero entry. So, $A_4(t)$ is a generalized permutation matrix. This proves the first assertion. Second assertion follows immediately from Corollary 4.4. This concludes the proof of the lemma. $\square$

Corollary 4.12. *Assume that $m \geq 2$. For any $t \in K$, if any two columns in $A_4(t)$ are disjoint, then $c(t)A(t)$ is a signed permutation matrix.*

Proof. Fix any $t \in K$. We claim that $A_2(t) = 0$. Indeed, from Lemma 4.11, $A_4(t)$ is a generalized permutation matrix. Also, for each $\lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0) \in \Lambda_b$ and $\gamma = (\gamma_i)_{i=0}^m \in \Gamma_f$ with $|\gamma| = q$, from Corollary 4.4, the (γ, λ) -entry of $A_2(t)$ is $\prod_{i=1}^m (a_{ji})^{\gamma_i}$. Note that such γ must have at least two nonzero components. Since $A_4(t)$ is a generalized permutation matrix, so all entries of $A_2(t)$ are zero, that is, $A_2(t) = 0$. Now, from Lemma 4.11 again, it remains to show that all nonzero entries of $c(t)^q A_4(t)$ are ± 1 . Fix $1 \leq i \leq m$, $e \in E$ with $\|e\| = 1$ and denote $\lambda_0 = (0, \dots, 0, \overset{i}{q}, 0, \dots, 0) \in \Lambda_b$. By letting $\alpha_{\lambda_0} = 1$ and $\alpha_\lambda = 0$ for all $\lambda \in \Lambda_b \setminus \{\lambda_0\}$, Corollary 4.7 implies that the i th column of $c(0)^q \begin{pmatrix} A_2(t) \\ A_4(t) \end{pmatrix}$ has r -norm 1. Since $A_2(t) = 0$ and $A_4(t)$ is a generalized permutation matrix, so all nonzero entries of $c(t)^q A_4(t)$ must be ± 1 . This concludes the proof of the corollary. $\square$

In subsequent sections, we will show that the matrix $c(t)A(t)$ is either a signed permutation matrix or in certain special case, it is an orthogonal matrix, which is not necessarily a generalized permutation matrix. For the reader's convenience, we give a summary below. $c(t)A(t)$ is a signed permutation matrix if $m \geq 2$ and $\{s, r\} \neq \{2\}$ (Section 4.3.1), or $m \geq 2$, $s = r = 2$ and $q > 1$ (Section 4.3.2). We consider $m \geq 2$, $s = r = 2$ and $q = 1$ in Section 4.5.1, where we prove that $c(t)A(t)$ is an orthogonal matrix.

4.3.1 $m \geq 2$ and $\{s, r\} \neq \{2\}$

The following lemma asserts that if the vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ from Corollary 4.10 are non-disjoint, then either both s and r must be 1 or s must be 2.

Lemma 4.13. *Let $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Assume that there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$C\|\mathbf{a} + \varepsilon\mathbf{b}\|_r = \|(1, \varepsilon)\|_s \quad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0). \quad (4.3.6)$$

If $\mathbf{a}$ and $\mathbf{b}$ are non-disjoint, then $s = r = 1$ or $s = 2$.

Proof. Assume that $s = 1$. Since the RHS of (4.3.6) is not differentiable at $\varepsilon = 0$, so $r = 1$. Conversely, if $r = 1$, similar argument leads to $s = 1$.

Assume that $s > 1$. Then the previous paragraph implies that $r > 1$. Since $\mathbf{a} = (a_i)_{i=1}^m$ and $\mathbf{b} = (b_i)_{i=1}^m$ are not disjoint, without loss of generality, assume that $a_1 b_1 \neq 0$. Then for all $0 \leq \varepsilon < \varepsilon_0$, (4.3.6) gives

$$C^r \sum_{k=1}^m |a_k + \varepsilon b_k|^r = (1 + \varepsilon^s)^{r/s}.$$

Note that the LHS of the equation above can be expressed as

$$C^r \sum_{k=1}^m |a_k + \varepsilon b_k|^r = C^r \varepsilon^r \sum_{a_k=0} |b_k|^r + C^r \sum_{a_k \neq 0} |a_k|^r \left(1 + \frac{b_k}{a_k} \varepsilon\right)^r$$

where we choose $\varepsilon > 0$ such that $\left(1 + \frac{b_k}{a_k} \varepsilon\right)$ is positive for all k with $a_k \neq 0$. It follows that

$$C^r \varepsilon^r \sum_{a_k=0} |b_k|^r + C^r \sum_{i=0}^{\infty} \binom{r}{i} \varepsilon^i \sum_{a_k \neq 0} |a_k|^r \left(\frac{b_k}{a_k}\right)^i = \sum_{i=0}^{\infty} \binom{\frac{r}{s}}{i} \varepsilon^{si}. \quad (4.3.7)$$

Now, we are going to compare the least positive power of ε in both sides of (4.3.7). Clearly the least positive power of RHS is s . Since $r > 1$, we must have $\sum_{a_k \neq 0} |a_k|^r \left(\frac{b_k}{a_k}\right)^i = 0$. Otherwise, the least positive power of ε on the LHS is 1, implying that $s = 1$, a contradiction. Thus the smallest positive power of ε in

$$C^r \sum_{i=0}^{\infty} \binom{r}{i} \varepsilon^i \sum_{a_k \neq 0} |a_k|^r \left(\frac{b_k}{a_k}\right)^i$$

is 2. If $\sum_{a_k=0} |b_k|^r = 0$, then $s = 2$. If $\sum_{a_k=0} |b_k|^r \neq 0$, then $\min\{r, 2\} = s$. If $1 < r < 2$, then $s = r \notin \mathbb{Z}$. This is not possible as the LHS of (4.3.7) has an ε^2 term and its RHS does not have a ε^2 . Hence, $r \geq 2$ and thus $s = 2$. This concludes the proof of the lemma. $\square$

The following proposition is the main result of this subsection.

Proposition 4.14. *Assume that $m \geq 2$. If $\{s, r\} \neq \{1\}$ and $\{s, r\} \neq \{2\}$, then $c(t)A(t)$ is a signed permutation matrix for any $t \in K$.*

Proof. Fix any $t \in K$. By assumption, we have $\{s, r\} \neq \{1\}$ and either $s \neq 2$ or $r \neq 2$. Suppose that the latter holds, i.e., $\{s, r\} \neq \{1\}$ and $s \neq 2$. From Corollary 4.10 and the contrapositive of Lemma 4.13, any two columns in $A_4(t)$ are disjoint. It follows from Corollary 4.12 that $c(t)A(t)$ is a signed permutation matrix.

Suppose now that $\{s, r\} \neq \{1\}$ and $r \neq 2$. Then we consider the inverse mapping T^{-1} which is also an isometry and the matrix $\tilde{B}$ corresponding to the inverse diffeomorphism $\psi = \varphi^{-1}$. By the same argument as above, $\frac{B(t)}{c(t)}$ is a signed permutation matrix. It follows that $c(t)A(t) = \left(\frac{B(t)}{c(t)}\right)^{-1}$ is a signed permutation matrix. This concludes the proof of the theorem. $\square$

It remains to consider the cases $s = r = 1$ in this subsection.

Lemma 4.15. *If $\mathbf{u} = (u_i)_{i=1}^m$ and $\mathbf{v} = (v_i)_{i=1}^m$ are column vectors in $\mathbb{R}^m$ such that $\|a\mathbf{u} + b\mathbf{v}\|_1 = |a| + |b|$ for all $a, b \in \mathbb{R}$, then for all $1 \leq i \leq m$, either $u_i = 0$ or $v_i = 0$.*

Proof. Let $I = \{i : u_i v_i > 0\}$ and $J = \{i : u_i v_i < 0\}$. We want to show that $I = J = \emptyset$. Assume that $J \neq \emptyset$. Let $a = b = 1$. Since I and J are disjoint sets, so

$$2 = \|\mathbf{u} + \mathbf{v}\|_1 = \sum_{i \notin I \cup J} |u_i + v_i| + \sum_{i \in I} |u_i + v_i| + \sum_{i \in J} |u_i + v_i|.$$

Observe that

$$\sum_{i \in J} |u_i + v_i| = \sum_{i \in J} ||u_i| - |v_i|| \leq \sum_{i \in J} |u_i - v_i| \leq \sum_{i \in J} (|u_i| + |v_i|).$$

Since $\|\mathbf{u}\|_1 = \|\mathbf{v}\|_1 = 1$, so

$$2 = \sum_{i \notin I \cup J} |u_i + v_i| + \sum_{i \in I} |u_i + v_i| + \sum_{i \in J} |u_i + v_i| \leq \sum_{i=1}^m |u_i| + \sum_{i=1}^m |v_i| = 2.$$

Hence, we must have

$$\sum_{i \in J} ||u_i| - |v_i|| = \sum_{i \in J} |u_i - v_i|.$$

It follows that $||u_i| - |v_i|| = |u_i - v_i|$ for all $i \in J$, which implies that either $u_i = 0$ or $v_i = 0$. However, both u_i and v_i are nonzero for all $i \in J$, a contradiction. So, $J = \emptyset$. Similarly, by computing $\|\mathbf{u} - \mathbf{v}\|_1$, we can conclude that $I = \emptyset$. This concludes the proof of the lemma. $\square$

The following proposition is immediate from Corollaries 4.7, 4.12 and Lemma 4.15.

Proposition 4.16. *Assume that $m \geq 2$. If $s = r = 1$, then $c(t)A(t)$ is a signed permutation matrix for all $t \in K$.*

The following is the main result of this subsection, which follows immediately from Propositions 4.14 and 4.16.

Proposition 4.17. *If $m \geq 2$ and $\{s, r\} \neq \{2\}$, then $c(t)A(t)$ is a signed permutation matrix for all $t \in K$.*

4.3.2 $m \geq 2$, $s = r = 2$ and $q > 1$

In this section, the inner product induced by the ℓ^2 norm and orthogonality of vectors play an important role.

Lemma 4.18. *If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are column vectors in $\mathbb{R}^m$ such that*

$$\|a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_m\mathbf{v}_m\|_2^2 = \sum_{i=1}^m a_i^2 \quad \text{for all } (a_i)_{i=1}^m \in \mathbb{R}^m, \quad (4.3.8)$$

then $\{\mathbf{v}_i : 1 \leq i \leq m\}$ is orthonormal.

Proof. Clearly (4.3.8) implies that $\|\mathbf{v}_i\| = 1$ for all $1 \leq i \leq m$. Also, by (4.3.8) and bilinearity of inner product induced by $\|\cdot\|_2$, we have

$$\sum_{i=1}^m a_i^2 = \|a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_m\mathbf{v}_m\|_2^2 = \sum_{i=1}^m a_i^2 + \sum_{i \neq j} a_i a_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

for any $(a_i)_{i=1}^m \in \mathbb{R}^m$, so $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$. This concludes the proof of the lemma. $\square$

Recall that $N = \#\{\gamma \in \Gamma : |\gamma| = q\} = \binom{q+m-1}{q}$.

Lemma 4.19. *Assume that $m \geq 2$ and $q > 1$. Let $\{\mathbf{v}_i : 1 \leq i \leq m\}$ be an orthonormal set of column vectors in $\mathbb{R}^N$ and $\{\mathbf{v}_i : m+1 \leq i \leq N\}$ be a set of column vectors in $\mathbb{R}^N$. If there exists $\varepsilon_0 > 0$ such that*

$$\left\| \sum_{i=1}^N a_i \mathbf{v}_i \right\|_2^2 = \sum_{i=1}^N a_i^2 \quad (4.3.9)$$

whenever $(a_i) \in \mathbb{R}^N$ such that

$$|a_i| \begin{cases} \geq 1 & \text{if } 1 \leq i \leq m \\ < \varepsilon_0 & \text{if } m+1 \leq i \leq N \end{cases}, \quad (4.3.10)$$

then $\{\mathbf{v}_i : 1 \leq i \leq N\}$ is orthonormal.

Proof. Consider $\{\mathbf{v}_i : 1 \leq i \leq m+k\}$. We prove the lemma by induction on k . If $k = 0$, then by assumption, the set is orthonormal.

Assume that $\{\mathbf{v}_i : 1 \leq i \leq m+k\}$ is orthonormal for some $k \in \mathbb{N} \cup \{0\}$. We want to show that $\mathbf{v}_{m+k+1}$ is orthogonal to $\mathbf{v}_i$ for all $1 \leq i \leq m+k$ and $\mathbf{v}_{m+k+1}$ has norm 1. Let

$$|a_i| = \begin{cases} 1 & \text{if } 1 \leq i \leq m \\ \varepsilon_i & \text{if } m+1 \leq i \leq m+k+1, 0 < \varepsilon_i < \varepsilon_0. \\ 0 & \text{otherwise} \end{cases}$$

By (4.3.9), we have

$$\left\| \sum_{i=1}^{m+k+1} a_i \mathbf{v}_i \right\|_2^2 = m + \sum_{i=m+1}^{m+k+1} \varepsilon_i^2. \quad (4.3.11)$$

On the other hand, by bilinearity of inner product induced by $\|\cdot\|_2$, we have

$$\left\| \sum_{i=1}^{m+k+1} a_i \mathbf{v}_i \right\|_2^2 = \sum_{i=1}^{m+k+1} a_i^2 \|\mathbf{v}_i\|^2 + 2 \sum_{1 \leq i < j \leq m+k+1} a_i a_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle. \quad (4.3.12)$$

By choice of a_i , we have

$$\sum_{i=1}^{m+k+1} a_i^2 \|\mathbf{v}_i\|^2 = m + \sum_{i=m+1}^{m+k} \varepsilon_i^2 + \varepsilon_{m+k+1}^2 \|\mathbf{v}_{m+k+1}\|^2. \quad (4.3.13)$$

Also, by inductive hypothesis, we have

$$2 \sum_{1 \leq i < j \leq m+k+1} a_i a_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 2 \sum_{i=1}^{m+k} a_i a_{m+k+1} \langle \mathbf{v}_i, \mathbf{v}_{m+k+1} \rangle. \quad (4.3.14)$$

Therefore, comparing (4.3.11), 4.3.12, 4.3.13 and 4.3.14 gives

$$2 \sum_{i=1}^{m+k} a_i a_{m+k+1} \langle \mathbf{v}_i, \mathbf{v}_{m+k+1} \rangle = \varepsilon_{m+k+1}^2 (1 - \|\mathbf{v}_{m+k+1}\|^2). \quad (4.3.15)$$

Fix $1 \leq j \leq m+k$ and let $a'_i = \begin{cases} a_i & \text{if } i \neq j \\ -a_i & \text{if } i = j \end{cases}$. Then we also have

$$2 \sum_{i=1}^{m+k} a'_i a_{m+k+1} \langle \mathbf{v}_i, \mathbf{v}_{m+k+1} \rangle = \varepsilon_{m+k+1}^2 (1 - \|\mathbf{v}_{m+k+1}\|^2). \quad (4.3.16)$$

It follows from (4.3.15) and 4.3.16 that $\langle \mathbf{v}_j, \mathbf{v}_{m+k+1} \rangle = 0$ for all $1 \leq j \leq m+k$ and $\|\mathbf{v}_{m+k+1}\| = 1$. Therefore, the set $\{\mathbf{v}_i : 1 \leq i \leq m+k+1\}$ is orthonormal. This concludes the proof of the lemma. $\square$

Recall that $C(t) = \|V(t)\|$.

Lemma 4.20. *If $s = r = 2$, then the matrix $C(t)\tilde{A}(t)$ is orthogonal for any $t \in K$.*

Proof. Fix any $t \in K$. If $q = 1$, then $\{\lambda \in \Lambda_f : |\lambda| = 1\} = \emptyset$. Since $s = r = 2$, from Corollary 4.8, $C(t)\tilde{A}(t)$ is orthogonal.

Now, assume that $q \geq 2$. Let $\{\mathbf{v}_\lambda : \lambda \in \Lambda_b\}$ be the set of columns of $\begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$. Fix $e \in E$ with $\|e\| = 1$ and $(a_\lambda)_{\lambda \in \Lambda_b} \in \mathbb{R}^m$. Then we have

$$\begin{aligned} \left\| \sum_{\lambda \in \Lambda} a_\lambda (C(t)\mathbf{v}_\lambda) \right\|_2 &= \left\| C(t) \cdot \tilde{A}(t) \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (a_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_2 \\ &= \left\| \begin{pmatrix} (0)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (a_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_2 && \text{by Corollary 4.7} \\ &= \|(a_\lambda)_{\lambda \in \Lambda_b}\|_2. \end{aligned}$$

Therefore, by Lemma 4.18, $\{C(t)\mathbf{v}_\lambda : \lambda \in \Lambda_b\}$ is an orthonormal set.

Now, let $\{\mathbf{v}_\lambda : |\lambda| = q, \lambda \in \Lambda_f\}$ be columns of $\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}$. Let $(a_\lambda)_{|\lambda|=q} \in \mathbb{R}^N$ be given such that

$$|a_\lambda| \begin{cases} \geq 1 & \text{if } \lambda \in \Lambda_b \\ < \varepsilon_0 & \text{if } |\lambda| = q, \lambda \in \Lambda_f \end{cases}.$$

Clearly $(a_\lambda)_{|\lambda|=q}$ satisfies (4.3.10). Fix $e \in E$ with $\|e\| = 1$. Then we have

$$\begin{aligned} \left\| \sum_{|\lambda|=q} a_\lambda (C(t)\mathbf{v}_\lambda) \right\|_2^2 &= \left\| C(t) \cdot \tilde{A}(t) \begin{pmatrix} (a_\lambda)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (a_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_2^2 \\ &= \left\| \begin{pmatrix} (a_\lambda)_{|\lambda|=q, \lambda \in \Lambda_f} \\ (a_\lambda)_{|\lambda|=q, \lambda \in \Lambda_b} \end{pmatrix} \right\|_2^2 && \text{by Corollary 4.7} \\ &= \sum_{|\lambda|=q} a_\lambda^2. \end{aligned}$$

By Lemma 4.19, $\{C(t)\mathbf{v}_\lambda : |\lambda| = q\}$ is orthonormal. Therefore, $C(t)\tilde{A}(t)$ is orthogonal. This concludes the proof of the proposition. $\square$

In the case $q > 1$, we can show that $c(t)A(t)$ is a signed permutation matrix.

Proposition 4.21. *If $m \geq 2$, $s = r = 2$ and $q > 1$, then $c(t)A(t)$ is a signed permutation matrix for all $t \in K$.*

Proof. Without loss of generality, we may assume that $t = 0$. Let $\tilde{B}$ be the corresponding matrix associated with T^{-1} . By Lemma 4.20, $C(0)\tilde{A}(0)$ is orthogonal. So, $\tilde{B}(0) = \tilde{A}^{-1}(0) = C^2(0)\tilde{A}^T(0)$. It follows that $B_4(0) = C^2(0)A_4^T(0)$. Thus, for all $1 \leq i, j \leq m$, we have $|b_{ij}^q(0)| = C^2(0)|a_{ji}^q(0)|$, which implies that $|b_{ij}(0)| = C^{\frac{2}{q}}(0)|a_{ji}(0)|$.

Consider now the entries in $B_3(0)$. Fix any $1 \leq j, k, \ell \leq m$ with $k \neq \ell$. Let $\lambda = (0, \dots, 0, \overset{j}{q}, 0, \dots, 0)$ and pick $\gamma = (0, \dots, \overset{k}{1}, \dots, \overset{\ell}{q-1}, \dots, 0)$. By Lemma 4.3, the (λ, γ) -entry is of $B_3(0)$ is $\binom{q}{1, q-1} b_{kj}(0) b_{\ell j}^{q-1}(0)$. Since $B_3(0) = C^2(0)A_2^T(0)$, so

$$\left| \binom{q}{1, q-1} b_{kj}(0) b_{\ell j}^{q-1}(0) \right| = C^2(0) |a_{jk}(0) a_{j\ell}^{q-1}(0)|.$$

It follows from $|b_{ij}(0)| = C^{\frac{2}{q}}(0) |a_{ji}(0)|$ that $q |a_{jk}(0)a_{j\ell}^{q-1}(0)| = |a_{jk}(0)a_{j\ell}^{q-1}(0)|$. Since $q > 1$, we must have $a_{jk}(0)a_{j\ell}(0) = 0$. Therefore, any two columns of $A_4(0)$ are disjoint. By Corollary 4.12, $c(0)A(0)$ is a signed permutation matrix. This concludes the proof of the proposition. $\square$

Remark 4.22. If $q = 1$, then $\tilde{A}(t) = A_4(t) = A^T(t)$ where the second equality follows from Corollary 4.4. Also, $\{\lambda \in \Lambda_f : |\lambda| = 1\} = \emptyset$. Therefore, Corollaries 4.8 imply that for all $(\alpha_\lambda)_{\lambda \in \Lambda_b} \in \mathbb{R}^m$, we have $\|C(t)A^T(t)(\alpha_\lambda)_{\lambda \in \Lambda_b}\|_2 = \|(\alpha_\lambda)_{\lambda \in \Lambda_b}\|_2$. So, $C(t)A^T(t)$ is an orthogonal matrix. However, as shown by the following example, A is not necessarily a generalized permutation matrix.

Example 4.23. Let U be any $m \times m$ orthogonal matrix. Define $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $\varphi(t) = Ut$. If $T : C_0^1(\mathbb{R}^m)_2 \rightarrow C_0^1(\mathbb{R}^m)_2$ is given by $Tf(t) = f(\varphi(t))$, then T is a surjective linear isometry and the Jacobian matrix of φ is U .

Proof. Let $Tf(t) = f(Ut) = f(s_1, s_2, \dots, s_m)$ where $s_i = \sum_{j=1}^m u_{ij}t_j$ and $U = (u_{ij})_{1 \leq i, j \leq m}$. Then for any $f \in C_0^1(\mathbb{R}^m)_2$ and $1 \leq j \leq m$, by Chain rule, we have

$$\partial_j Tf = \sum_{i=1}^m \partial_i f \cdot \partial_j s_i = \sum_{i=1}^m \partial_i f \cdot u_{ij}. \quad (4.3.17)$$

Since U is orthogonal, so is

$$X = \begin{pmatrix} 1 & 0 \\ 0 & U^T \end{pmatrix},$$

where X is an $(m+1) \times (m+1)$ square matrix. So, (4.3.17) implies that

$$\begin{aligned} \|Tf\|_2^2 &= \sup_{t \in \mathbb{R}^m} (|Tf(t)|^2 + |\partial_1 Tf(t)|^2 + \dots + |\partial_m Tf(t)|^2) \\ &= \sup_{s \in \mathbb{R}^m} \left\| X(f(s), (\partial_i f(s))_{i=1}^m)^T \right\|_2^2 \\ &= \sup_{s \in \mathbb{R}^m} \left\| (f(s), (\partial_i f(s))_{i=1}^m)^T \right\|_2^2 \\ &= \|f\|_2^2. \end{aligned}$$

Therefore, T is an isometry. Clearly the Jacobian matrix of φ is U , which is not necessarily a generalized permutation matrix. $\square$

In fact, in subsection 4.5.2, we shall show that if $s = r = 2$, $q > 1$ and $m \geq 2$, then the Jacobian matrix A of φ can be an orthogonal matrix but not necessarily

a generalized permutation matrix. In this case, E and F must be Hilbert spaces. However, for now, we shall show that V is locally constant for those cases where A is a generalized permutation matrix in the next section.

We end this section with the following proposition which summarises all results that we have obtained so far. The proposition follows immediately from Corollaries 4.8, 4.9 and Propositions 4.17 and 4.21.

Proposition 4.24. *Let $T : (C_0^q(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,r})$ be a surjective linear isometry given by $Tf(t) = V(t)f(\varphi(t))$, where $q, m \geq 1$ are natural numbers, Q and K are respectively open subsets of $\mathbb{R}^m$, both E and F are Banach spaces, $s, r \in [1, \infty)$, $V(t) : E \rightarrow F$ is a Banach space isomorphism for all $t \in K$ and $\varphi : K \rightarrow Q$ is a C^q -diffeomorphism. Then $V(t) = c^q(t)J(t)$ where $J(t)$ is an isometry for all $t \in K$. In addition, if either $\{s, r\} \neq \{2\}$ or $q > 1$, then $c(t)A(t)$ is a signed permutation matrix for all $t \in K$ where $c(t) = \|V(t)\|^{1/q}$.*

4.4 V is locally constant and $s = r$

For each $t \in K$, recall that $c(t) = \|V(t)\|^{1/q} \neq 0$. In this section, we will show that if $c(t)A(t)$ is a signed permutation matrix, then $s = r$, $A(t)$ is a signed permutation matrix and V is a locally constant surjective linear isometry. The result is given by Theorem 4.41. To this end, we consider two cases: $m \geq 2$ or $m = 1$.

4.4.1 $m \geq 2$

Lemma 4.25. *Assume that $m \geq 2$. If $c(t)A(t)$ is a signed permutation matrix for all $t \in K$, then on each connected component U of K , there exists a positive scalar c such that $c(t) = c$ and $\varphi(t) = \frac{1}{c}(\varepsilon_i t_{\pi(i)})$ where $\varepsilon_i = \pm 1$ and $\pi : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ is a bijection for all $t = (t_1, \dots, t_m) \in U$.*

Proof. Assume that K is connected. Fix $t_0 \in K$. Since $c(t)A(t)$ is a signed permutation matrix, by composing φ with permutations and reflections, without loss of generality, we may assume that $c(t_0)A(t_0) = I_{m \times m}$ where $I_{m \times m}$ is the $m \times m$ identity matrix. Note that $t \mapsto c(t)A(t)$ is a continuous function. Fix i . Then $\{t : a_{ij}(t) \neq 0\}, j = 1, 2, \dots, m\}$ is a partition of K where each element is a clopen

set. Since $\{t : a_{ii}(t) \neq 0\} \neq \emptyset$, so $K = \{t : a_{ii}(t) \neq 0\}$. Therefore, we can conclude that $c(t)A(t) = I_{m \times m}$ for all $t \in K$. It follows that for any $t = (t_1, \dots, t_n) \in K$, we have $\frac{\partial \varphi_i}{\partial t_j}(t) = 0$ if $i \neq j$. So, for all $1 \leq i \leq m$, $\varphi_i(t)$ is independent of t_j for $j \neq i$ and therefore, we may treat $\varphi_i(t)$ as a function of single variable t_i , i.e. $\varphi_i(t) = \varphi_i(t_i)$. Then $\frac{\partial \varphi_i}{\partial t_i} = \varphi'_i(t_i)$. Thus,

$$\text{diag}(1, \dots, 1) = I_{m \times m} = c(t) \text{diag}(\varphi'_1(t_1), \dots, \varphi'_m(t_m)). \quad (4.4.1)$$

It follows that for any $t \in K$, we have $\varphi'_1(t_1) = \dots = \varphi'_m(t_m)$. Therefore, fixing $m - 1$ variables and varying the remaining variable of t , we conclude that

$$\varphi'_1(t) = \dots = \varphi'_m(t) = \text{constant} \quad \text{for all } t \in K.$$

Now, for any $t, t' \in K$ with $t \neq t'$, by path-connectedness of K , there exists a path $\sigma : [0, 1] \rightarrow K$ such that it remains constant at all but one coordinate and $\sigma(0) = t$ and $\sigma(1) = t'$. It follows that $c(t) = c(t')$. Therefore, $c(t)$ is a constant function. Hence, $\varphi(t) = \frac{t}{c}$ for all $t \in K$. To conclude the lemma, we simply compose $\varphi(t) = \frac{t}{c}$ with inverse of permutations and reflections taken at the beginning. If K is not connected, then we apply the same arguments as above to each connected component to obtain the desired result. $\square$

Remark 4.26. We would like to point out that the proof of Lemma 4.25 does not depend on the norm $\|\cdot\|_{D,\rho}$ on $C_0^p(Q, E)$ and $C_0^q(K, F)$. In particular, we can apply Lemma 4.25 to $\|\cdot\|_{S,\rho}$.

For simplicity, we may assume that $\varphi(t) = \frac{t}{c}$. Recall that the ordering $\leq$ on Λ is defined by $(\mu_i)_{i=1}^m \leq (\lambda_i)_{i=1}^m$ if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq m$. The following lemma follows immediately from Lemma 4.1.

Lemma 4.27. If $Tf(t) = V(t)f(\frac{t}{c})$, then for any $\gamma \in \Gamma$ and $t \in K$, we have

$$\partial^\gamma Tf(t) = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \partial^\mu f\left(\frac{t}{c}\right) \left(\frac{1}{c}\right)^{|\mu|}.$$

Lemma 4.28. For any $a \in (0, m^{-1/s})$ and $\eta \in \left[0, \frac{2(1-am^{1/s})}{\#\Lambda_f}\right)$, there exists $b \in \mathbb{R}$ such that for any $\lambda_0 \in \Lambda_f$, we have $\rho\left[(0)_{\lambda \in \Lambda_f \setminus \{\lambda_0\}}, \frac{\eta}{2}, (b)_{\lambda \in \Lambda_b}\right] = 1$ and $b \geq a$. In particular, if $\eta = 0$, then $b = \frac{1}{m^{1/s}}$.

Proof. Let $\xi(t) = \rho[(0)_{\lambda \in \Lambda_f \setminus \{\lambda_0\}}, \frac{\eta}{2}, (t)_{\lambda \in \Lambda_b}]$. Since $\xi(0) < 1$ and $\xi(1) > 1$, by Intermediate Value Theorem, there exists $b \in (0, 1)$ such that $\xi(b) = 1$. If $b < a$, then

$$1 = \xi(b) \leq \frac{\eta}{2} + bm^{1/s} < \frac{\eta}{2} + am^{1/s} < \frac{1 - am^{1/s}}{\#\Lambda_f} + am^{1/s} \leq 1,$$

a contradiction. This concludes the proof of the lemma. $\square$

We recall that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$. The following proposition is one of the main results of this section, which asserts that V is locally constant.

Proposition 4.29. *If $\varphi(t) = \frac{t}{c}$ where $c = \|V(t)\|^{1/q}$, then $\partial_j V(t)u = 0$ for all $t \in K$, $u \in E$ and $1 \leq j \leq m$.*

Proof. Suppose not. Without loss of generality, we may assume that $\partial_1 V(0)u \neq 0$ for some $u \in E \setminus \{0\}$. We let $a \in (0, \frac{1}{m^{1/s}})$ and obtain $\eta \in (0, \frac{2(1-am^{1/s})}{\#\Lambda_f})$ from Proposition 3.10. We obtain b from Lemma 4.28. We let $\tilde{u} = \frac{\eta u}{2\|u\|}$. Since $V(0)$ is onto, there exists $e \in E$ with $\|e\| = b$ such that $V(0)e$ is a positive multiple of $\partial_1 V(0)\tilde{u}$. From Proposition 3.10, we choose $f \in C_0^q(Q, E)$ such that $\|f\|_{D,s} = \|[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}]\|_s = 1$, $\partial_1^{q-1} f(0) = \tilde{u}$, $\partial_i^q f(0) = e$ for all $1 \leq i \leq m$ and $\partial^\mu f(0) = 0$ for all other μ . By Lemma 4.5, we have $1 = \|[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}]\|_s = \|[(\|\partial^\gamma T f(0)\|)_{\gamma \in \Gamma}]\|_r$. Therefore,

$$1 = \|[(\|\partial^\gamma T f(0)\|)_{\gamma \in \Gamma}]\|_r = \left\| \left[(\|\partial^\gamma T f(0)\|)_{\gamma \neq (q,0,\dots,0)}, \|\partial_1^q T f(0)\| \right] \right\|_r. \quad (4.4.2)$$

Observe that

$$\begin{aligned} \|\partial_1^q T f(0)\| &= \left\| \sum_{k=0}^q \binom{q}{k} \partial_1^{q-k} V(0) \partial_1^k f(0) c^{-k} \right\| && \text{by Lemma 4.27} \\ &= \|q \partial_1 V(0) \tilde{u} c^{-(q-1)} + V(0) e c^{-q}\| && \text{by choice of } f \\ &= \|q \partial_1 V(0) \tilde{u} c^{-(q-1)}\| + \|V(0) e c^{-q}\| && \text{from the choice of } e. \end{aligned}$$

It follows from (4.4.2) that

$$1 = \left\| \left[(\|\partial^\gamma T f(0)\|)_{\gamma \neq (q,0,\dots,0)}, \|q \partial_1 V(0) \tilde{u} c^{-(q-1)}\| + \|V(0) e c^{-q}\| \right] \right\|_r. \quad (4.4.3)$$

On the other hand, by Proposition 3.10, there exists $g \in C_0^q(Q, E)$ such that $\|g\|_{D,s} = \|[(\|\partial^\lambda g(0)\|)_{\lambda \in \Lambda}]\|_s$, $\partial^\lambda g(0) = \partial^\lambda f(0)$ for all $\lambda \neq (q, 0, \dots, 0)$ and $\partial_1^q g(0) =$

$-\partial_1^q f(0) = -e$. Replacing f with g in (4.4.2), we have

$$1 = \left\| \left[(\|\partial^\gamma Tg(0)\|)_{\gamma \neq (q,0,\dots,0)}, \|\partial_1^q Tg(0)\| \right] \right\|_r. \quad (4.4.4)$$

Fix $\gamma \neq (q, 0, \dots, 0)$. Then for all μ with $\mu \leq \gamma$, we have $\mu \neq (q, 0, \dots, 0)$. So,

$$\partial^\mu g(0) = \partial^\mu f(0) \quad \text{for all } \mu \leq \gamma. \quad (4.4.5)$$

It follows that for all $\gamma \neq (q, 0, \dots, 0)$, we have

$$\begin{aligned} \partial^\gamma Tg(0) &= \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu g(0) c^{-|\mu|} && \text{by Lemma 4.27} \\ &= \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu f(0) c^{-|\mu|} && \text{by (4.4.5)} \\ &= \partial^\gamma Tf(0) && \text{by Lemma 4.27.} \end{aligned} \quad (4.4.6)$$

Also, observe that

$$\begin{aligned} \|\partial_1^q Tg(0)\| &= \left\| \sum_{k=0}^q \binom{q}{k} \partial_1^{q-k} V(0) \partial_1^k g(0) c^{-k} \right\| && \text{by Lemma 4.27} \\ &= \|q\partial_1 V(0) \tilde{u} c^{-(q-1)} - V(0) e c^{-q}\| && \text{by choice of } g \\ &= \left| \|q\partial_1 V(0) \tilde{u} c^{-(q-1)}\| - \|V(0) e c^{-q}\| \right| && \text{from the choice of } e. \end{aligned} \quad (4.4.7)$$

By (4.4.4), 4.4.6 and 4.4.7, we have

$$1 = \left\| \left[(\|\partial^\gamma Tf(0)\|)_{\gamma \neq (q,0,\dots,0)}, \left| \|q\partial_1 V(0) \tilde{u} c^{-(q-1)}\| - \|V(0) e c^{-q}\| \right| \right] \right\|_r. \quad (4.4.8)$$

Comparing (4.4.3) and 4.4.8 gives

$$\left| \|q\partial_1 V(0) \tilde{u} c^{-(q-1)}\| + \|V(0) e c^{-q}\| \right| = \left| \|q\partial_1 V(0) \tilde{u} c^{-(q-1)}\| - \|V(0) e c^{-q}\| \right|$$

It follows that either $\partial_1 V(0) \tilde{u} = 0$ or $V(0) e = 0$, a contradiction. Therefore, $\partial_1 V(0) = 0$. This concludes the proof of the proposition. $\square$

The following proposition is the second main result of this section.

Proposition 4.30. *If $\varphi(t) = \frac{t}{c}$ where $|c| = \|V(t)\|^{1/q}$, then $|c| = 1$ and $s = r$.*

Proof. Without loss of generality, we may assume that $s \geq r$. Let $a \in (0, \frac{1}{m^{1/s}})$ and obtain $\eta \in (0, \frac{2(1-am^{1/s})}{\#\Lambda_f})$ from Proposition 3.10. It suffices to show that for all $u \in E$ with $\|u\| \in [0, \eta)$, we have

$$\frac{(1 - \|u\|^s)^{1/s}}{m^{1/s}} = \frac{(1 - |c|^r \|u\|^r)^{1/r}}{m^{1/r}}. \quad (4.4.9)$$

Indeed, differentiating (4.4.9) with respect to $\|u\|$ gives

$$\frac{1}{m^{1/s}}(1 - \|u\|^s)^{\frac{1}{s}-1}(\|u\|^{s-1}) = \frac{1}{m^{1/r}}(1 - |c|^r \|u\|^r)^{\frac{1}{r}-1}(|c|^r \|u\|^{r-1}).$$

From (4.4.9), we have $\|u\|^{s-r} = |c|^r$ for all $\|u\| \in (0, \eta)$. Therefore, we must have $s = r$ and $|c| = 1$.

It remains to prove (4.4.9). We pick any $u \in E$ with $\|u\| \in [0, \eta)$. We obtain b from Lemma 4.28. We choose $e \in E$ with $\|e\| = b$. From Proposition 3.10, there exists $f \in C_0^q(Q, E)$ such that $\|f\|_{D,s} = \|[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}]\|_s = 1$, $\partial^\lambda f(0) = e$ for all $\lambda \in \Lambda_b$, $\partial_1^{q-1} f(0) = u$ and $\partial^\lambda f(0) = 0$ for all other λ . Note that $\|u\|^s + mb^s = 1$. So, $b = \frac{(1-\|u\|^s)^{1/s}}{m^{1/s}}$. Also,

$$\begin{aligned} 1 &= \|[(\|\partial^\lambda f(0)\|)_{\lambda \in \Lambda}]\|_s && \text{by choice of } f \\ &= \|[(\|\partial^\gamma T f(0)\|)_{\gamma \in \Gamma}]\|_r && \text{by Lemma 4.5.} \end{aligned} \quad (4.4.10)$$

For all $\gamma \in \Gamma$, note that

$$\begin{aligned} \partial^\gamma T f(0) &= \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(0) \partial^\mu f(0) c^{-|\mu|} && \text{by Lemma 4.27} \\ &= V(0) \partial^\gamma f(0) c^{-|\gamma|} && \text{by Proposition 4.29} \\ &= \begin{cases} V(0)u \cdot c^{-(q-1)} & \text{if } \gamma = (q-1, 0, \dots, 0), \\ V(0)e \cdot c^{-q} & \text{if } \gamma \in \Gamma_b, \\ 0 & \text{otherwise.} \end{cases} && \text{by choice of } f. \end{aligned} \quad (4.4.11)$$

Therefore,

$$\begin{aligned}
1 &= \|[(\|\partial^\gamma T f(0)\|)_{\gamma \in \Gamma}]\|_r && \text{by (4.4.10)} \\
&= \left\| \left[\|V(0)u\| \cdot c^{-(q-1)}, (\|V(0)e\| \cdot c^{-q})_{i=1}^m \right] \right\|_r && \text{by (4.4.11)} \\
&= \| [c\|u\|, (b)_{i=1}^m] \|_r && \text{by Corollary 4.8.}
\end{aligned}$$

So, the equation above implies that $1 = |c|^r \|u\|^r + mb^r$. Since $b = \frac{(1-\|u\|^s)^{1/s}}{m^{1/s}}$, so (4.4.9) holds and this concludes the proof of the proposition. $\square$

Remark 4.31. Proposition 4.30 is independent of m .

We collect all results in this section for $m \geq 2$ in the following proposition, which follows immediately from Lemma 4.25, Propositions 4.29, 4.30, Recall that $c(t) = \|V(t)\|^{1/q}$ and $A(t)$ is the Jacobian matrix of φ .

Proposition 4.32. *Let $T : (C_0^p(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,r})$ be a surjective linear isometry, where $q, m \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$, both E and F are Banach spaces and $s, r \in [1, \infty)$. If $m \geq 2$ and $c(t)A(t)$ is a signed permutation matrix for all $t \in K$, then $s = r$, $A(t)$ is a signed permutation matrix for all $t \in K$, and for each connected component of K , $V : E \rightarrow F$ is a surjective linear isometry,*

4.4.2 $m = 1$

Lemma 4.33. *For any $f \in C_0^q(Q, E)$ and $t \in K$, if $f^{(i)}(\varphi(t)) = 0$ for all $i = 0, 1, \dots, q-2$, then $\frac{d^q}{dt^q}(f \circ \varphi)(t) = f^{(q-1)}(\varphi(t)) \left[\frac{q(q-1)}{2} \cdot (\varphi'(t))^{q-2} \varphi''(t) \right] + f^{(q)}(\varphi(t)) \cdot (\varphi'(t))^q$.*

Proof. Fix any $f \in C_0^q(Q, E)$ and $t \in K$. If $q = 1$, then the lemma holds from Chain rule. Assume that $q \geq 2$. Let $a_k = (f^{(k)} \circ \varphi)^{(q-k)}(t)$. For each $0 \leq k \leq q-1$,

by product rule and the choice of f , we have

$$\begin{aligned}
 a_k &= (f^{(k+1)} \circ \varphi \cdot \varphi')^{(q-k-1)}(t) \\
 &= \sum_{i=0}^{q-k-1} \binom{q-k-1}{i} (f^{(k+1)} \circ \varphi)^{(i)}(t) \cdot (\varphi')^{(q-k-1-i)}(t) \\
 &= (q-k-1)(f^{(k+1)} \circ \varphi)^{(q-k-2)}(t) \cdot \varphi''(t) + (f^{(k+1)} \circ \varphi)^{(q-k-1)}(t) \cdot \varphi'(t) \\
 &= (q-k-1)f^{(q-1)} \circ \varphi(t) \cdot (\varphi'(t))^{q-k-2} \cdot \varphi''(t) + a_{k+1} \cdot \varphi'(t).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 a_0 &= f^{(q-1)} \circ \varphi(t) \cdot (\varphi'(t))^{q-2} \cdot \varphi''(t) [(q-1) + (q-2) + \cdots + 2 + 1] + a_q \cdot (\varphi'(t))^q \\
 &= f^{(q-1)} \circ \varphi(t) \cdot (\varphi'(t))^{q-2} \cdot \varphi''(t) \left[\frac{q(q-1)}{2} \right] + (f^{(q)} \circ \varphi)(t) \cdot (\varphi'(t))^q.
 \end{aligned}$$

This concludes the proof of the lemma. $\square$

Lemma 4.34. For any $f \in C_0^q(Q, E)$ and $t \in K$, if $f^{(i)}(\varphi(t)) = 0$ for all $i = 0, 1, \dots, q-2$, then $Tf^{(q)}(t) = qV'(t)f^{(q-1)}(\varphi(t)) \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)f^{(q-1)}(\varphi(t)) \cdot (\varphi'(t))^{q-2}\varphi''(t) + V(t)f^{(q)}(\varphi(t)) \cdot (\varphi'(t))^q$

Proof. Since $Tf(t) = V(t)f(\varphi(t))$, from Lemmas 4.1 and 4.33, we have

$$\begin{aligned}
 Tf^{(q)}(t) &= \sum_{k=0}^q \binom{q}{k} V^{(q-k)}(t) (f(\varphi(t)))^{(k)} \\
 &= \binom{q}{q-1} V'(t) f^{(q-1)}(\varphi(t)) \cdot (\varphi'(t))^{q-1} + V(t) (f(\varphi(t)))^{(q)} \\
 &= qV'(t) f^{(q-1)}(\varphi(t)) \cdot (\varphi'(t))^{q-1} \\
 &\quad + V(t) \left[\frac{q(q-1)}{2} f^{(q-1)}(\varphi(t)) \cdot (\varphi'(t))^{q-2} \varphi''(t) + f^{(q)}(\varphi(t)) \cdot (\varphi'(t))^q \right].
 \end{aligned}$$

This concludes the proof of the lemma. $\square$

Lemma 4.35. $qV'(t)u \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)u \cdot (\varphi'(t))^{q-2}\varphi''(t) = 0$ for all $t \in K$ and $u \in E$.

Proof. Suppose that $qV'(t)u \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)u \cdot (\varphi'(t))^{q-2}\varphi''(t) \neq 0$ for some $t \in K$ and $u \neq 0$. Let $a \in (0, 1)$. Obtain $\eta \in \left(0, \frac{2(1-a)}{q}\right)$ from Proposition 3.10.

Obtain b from Lemma 4.28. Let $\tilde{u} = \frac{\eta u}{2\|u\|}$. From surjectivity of $V(t)$, there exists $e \in E$ with $\|e\| = 1$ such that $V(t)e$ is a positive multiple of $qV'(t)\tilde{u} \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)\tilde{u} \cdot (\varphi'(t))^{q-2}\varphi''(t)$. From Proposition 3.10, choose $f \in C_0^q(Q, E)$ such that

$$\|f\|_{D,\rho} = \rho \left[(\|f^{(i)}(\varphi(t))\|)_{i=0}^q \right] = \rho(0, \dots, 0, \frac{\eta}{2}, b) \quad (4.4.12)$$

and $f^{(i)}(\varphi(t)) = 0$ for all $0 \leq i \leq q-2$, $f^{(q-1)}(\varphi(t)) = \tilde{u}$ and $f^{(q)}(\varphi(t)) = eb$. From (4.4.12) and Lemma 4.5, we have $\rho(0, \dots, 0, \frac{\eta}{2}, b) = \theta \left[(\|Tf^{(j)}(t)\|)_{j=0}^q \right]$. Now, from Proposition 3.10 again, choose $g \in C_0^q(Q, E)$ such that $\|g\|_{D,\rho} = \rho \left[(\|g^{(i)}(\varphi(t))\|)_{i=0}^q \right] = \rho(0, \dots, 0, \frac{\eta}{2}, b)$ and $g^{(i)}(\varphi(t)) = f^{(i)}(\varphi(t))$ for all $0 \leq i \leq q-1$, and $g^{(q)}(\varphi(t)) = -eb$. Similar reasoning above yields $\rho(0, \dots, 0, \frac{\eta}{2}, b) = \theta \left[(\|Tg^{(j)}(t)\|)_{j=0}^q \right]$. Therefore, we have $\theta \left[(\|Tf^{(j)}(t)\|)_{j=0}^q \right] = \theta \left[(\|Tg^{(j)}(t)\|)_{j=0}^q \right]$. Since $f^{(i)}(\varphi(t)) = g^{(i)}(\varphi(t))$ for all $i = 0, 1, \dots, q-1$, so $Tf^{(j)}(t) = Tg^{(j)}(t)$ for all $j = 0, 1, \dots, q-1$. From strict monotonicity of θ , we have $\|Tf^{(q)}(t)\| = \|Tg^{(q)}(t)\|$. From Lemma 4.34 and the choice of e , we have

$$\begin{aligned} & \|Tf^{(q)}(t)\| \\ &= \|qV'(t)\tilde{u} \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)\tilde{u} \cdot (\varphi'(t))^{q-2}\varphi''(t) + V(t)eb \cdot (\varphi'(t))^q\| \\ &= \|qV'(t)\tilde{u} \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)\tilde{u} \cdot (\varphi'(t))^{q-2}\varphi''(t)\| + \|V(t)eb \cdot (\varphi'(t))^q\|, \end{aligned}$$

and

$$\begin{aligned} & \|Tg^{(q)}(t)\| \\ &= \|qV'(t)\tilde{u} \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)\tilde{u} \cdot (\varphi'(t))^{q-2}\varphi''(t) - V(t)eb \cdot (\varphi'(t))^q\| \\ &= \left\| qV'(t)\tilde{u} \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)\tilde{u} \cdot (\varphi'(t))^{q-2}\varphi''(t) \right\| - \|V(t)eb \cdot (\varphi'(t))^q\|, \end{aligned}$$

where the last equality follows from the choice of e . Since $\|Tf^{(q)}(t)\| = \|Tg^{(q)}(t)\|$, it follows that either $qV'(t)\tilde{u} \cdot (\varphi'(t))^{q-1} + \frac{q(q-1)}{2}V(t)\tilde{u} \cdot (\varphi'(t))^{q-2}\varphi''(t) = 0$ or $V(t)eb \cdot (\varphi'(t))^q = 0$. By supposition, we must have $V(t)eb \cdot (\varphi'(t))^q = 0$. However, this contradicts with the fact that all $b, \varphi'(t)$ and $V(t)e$ are nonzero. This concludes the proof of the lemma. $\square$

Lemma 4.36. $|\varphi'|$ is constant on K .

Proof. Fix any $t \in K$ and $u \in E \setminus \{0\}$. Let $a \in (0, 1)$. Obtain $\eta \in \left(0, \frac{2(1-a)}{q}\right)$ from Proposition 3.10. Obtain b from Lemma 4.28. Let $\tilde{u} = \frac{\eta u}{2\|u\|}$. Fix any $e \in E$ with $\|e\| = 1$. From Proposition 3.10, choose $f \in C_0^q(Q, E)$ such that $\|f\|_{D,\rho} = \rho\left[\left(\|f^{(i)}(\varphi(t))\|\right)_{i=0}^q\right] = \rho\left(0, \dots, 0, \frac{\eta}{2}, b\right)$ and $f^{(i)}(\varphi(t)) = 0$ for all $0 \leq i \leq q-2$, $f^{(q-1)}(\varphi(t)) = \tilde{u}$ and $f^{(q)}(\varphi(t)) = eb$. Clearly $Tf^{(j)}(t) = 0$ for all $j = 0, 1, \dots, q-2$. Observe that

$$\begin{aligned} & Tf^{(q-1)}(t) \\ &= \sum_{k=0}^{q-1} \binom{q-1}{k} V^{(q-1-k)}(t) (f(\varphi(t)))^{(k)} \\ &= V(t) f^{(q-1)}(\varphi(t)) \cdot (\varphi'(t))^{q-1} \quad \text{since } f^{(i)}(\varphi(t)) = 0 \text{ for all } i = 0, \dots, q-2 \\ &= V(t) \tilde{u} \cdot (\varphi'(t))^{q-1}. \end{aligned}$$

Also, from Lemmas 4.34 and 4.35, we have $Tf^{(q)}(t) = V(t)eb \cdot (\varphi'(t))^q$. Therefore,

$$\begin{aligned} & \rho\left(0, \dots, 0, \frac{\eta}{2}, b\right) \\ &= \theta \left[\left(\|Tf^{(j)}(t)\| \right)_{j=0}^q \right] \\ &= \theta \left[(0)_{j=0}^{q-2}, \|V(t)\tilde{u} \cdot (\varphi'(t))^{q-1}\|, \|V(t)eb \cdot (\varphi'(t))^q\| \right] \\ &= \theta \left[(0)_{j=0}^{q-2}, \|V(t)\| \cdot \|\tilde{u}\| \cdot |\varphi'(t)|^{q-1}, \|V(t)\| \cdot b \cdot |\varphi'(t)|^q \right] \quad \text{by Corollary 4.8} \\ &= \theta \left[(0)_{j=0}^{q-2}, \frac{\|\tilde{u}\|}{|\varphi'(t)|}, b \right] \quad \text{by Corollary 4.9.} \end{aligned}$$

Since the equation above holds for all $t \in K$ and its LHS is independent of $t \in K$, from strictly monotonicity of θ , we have $|\varphi'(t)|$ is constant. This concludes the proof of the lemma. $\square$

The following corollary follows immediately from Lemmas 4.35 and 4.36

Corollary 4.37. $V'(t)u = 0$ for all $t \in K$ and $u \in E$.

From $\varphi(0) = 0$, Lemma 4.36 and Corollaries 4.37, 4.9, we have $\varphi(t) = \frac{t}{c}$ where $|c| = \|V(t)\|^{1/q}$. From Proposition 4.30 and Remark 4.31, we have $|c| = 1$ and $s = r$. The following proposition collects all results in this section for $m = 1$.

Proposition 4.38. Let $T : (C_0^q(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,r})$ be a surjective linear isometry, where $q, m \geq 1$ are natural numbers, Q and K are open

subsets of $\mathbb{R}^m$, both E and F are Banach spaces and $s, r \in [1, \infty)$. If $m = 1$, then $s = r$, $A(t)$ is a signed permutation matrix for all $t \in K$, and for each connected component of U of K , there exists an surjective linear isometry $V : E \rightarrow F$ such that we have $Tf(t) = Vf(\varphi(t))$ for all $f \in C_0^q(Q, E)$ and $t \in U$.

Lemma 4.39. Assume that $p = q$ and $m = n$. Fix any $f \in C_0^p(Q, E)$ and $t \in K$. If the Jacobian matrix $A(t) = (\partial_j \varphi_i(t))_{1 \leq i, j \leq m}$ of φ is a signed permutation matrix, then for any $\gamma \in \Gamma$, there exists a unique $\lambda \in \Lambda$ such that $\|\partial^\gamma(f \circ \varphi)(t)\| = \|\partial^\lambda f(\varphi(t))\|$.

Proof. Fix any $f \in C_0^p(Q, E)$ and $t \in K$. For each $i = 1, 2, \dots, m$, let $\pi(i)$ be an integer such that $|\partial_{\pi(i)} \varphi_i(t)| = 1$. Since $A(t)$ is a signed permutation matrix, so π is a bijection on $\{1, 2, \dots, m\}$. Fix any $\gamma = (\gamma_j)_{j=1}^m \in \Gamma$ and let $\lambda = (\lambda_i)_{i=1}^m$ where $\lambda_i = \gamma_{\pi(i)}$. Clearly $\lambda \in \Lambda$. Fix any $j = 1, 2, \dots, m$ and assume that $\gamma_j \neq 0$. Let $\varepsilon_j \in \{-1, 1\}$. From Chain rule, we have

$$\begin{aligned} \partial_j^{\gamma_j}(f \circ \varphi)(t) &= \partial_j^{\gamma_j-1} \left[\sum_{i=1}^m (\partial_i f \circ \varphi)(t) \cdot \partial_j \varphi_i(t) \right] = \varepsilon_j \partial_j^{\gamma_j-1} (\partial_{\pi^{-1}(j)} f \circ \varphi)(t) \\ &= \dots = \varepsilon_j^{\gamma_j} (\partial_{\pi^{-1}(j)}^{\gamma_j} f \circ \varphi)(t). \end{aligned}$$

Therefore,

$$\begin{aligned} \|\partial^\gamma(f \circ \varphi)(t)\| &= \|\partial_1^{\gamma_1} \dots \partial_m^{\gamma_m}(f \circ \varphi)(t)\| = \|\varepsilon_1^{\gamma_1} \dots \varepsilon_m^{\gamma_m} (\partial_{\pi^{-1}(1)}^{\gamma_1} \dots \partial_{\pi^{-1}(m)}^{\gamma_m} f \circ \varphi)(t)\| \\ &= \|(\partial_{\pi^{-1}(1)}^{\gamma_1} \dots \partial_{\pi^{-1}(m)}^{\gamma_m} f \circ \varphi)(t)\| = \|\partial^\lambda f(\varphi(t))\|, \end{aligned}$$

where the last equality follows from the choice of λ . This concludes the proof of the lemma. $\square$

The following proposition provides one direction of the characterization of the Banach-Stone Theorem.

Proposition 4.40. Let $T : (C_0^q(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,s})$ be a linear bijection given by $Tf(t) = Vf(\varphi(t))$ where Q and K are open subsets of $\mathbb{R}^m$, $V : E \rightarrow F$ is an isometry and $\varphi : K \rightarrow Q$ is a C^q -diffeomorphism. If the Jacobian matrix $A(t) = (\partial_j \varphi_i(t))_{1 \leq i, j \leq m}$ of φ is a signed permutation matrix for all $t \in K$, then T is an isometry.

Proof. Pick any $f \in C_0^p(Q, E)$. Note that

$$\begin{aligned}
\|Tf\|_{D,s} &= \sup_{t \in K} \|(\|\partial^\gamma T f(t)\|)_{\gamma \in \Gamma}\|_s && \text{by definition of } \|\cdot\|_{D,s} \\
&= \sup_{t \in K} \|(\|V \partial^\gamma (f \circ \varphi)(t)\|)_{\gamma \in \Gamma}\|_s && \text{by the formula of } T f \\
&= \sup_{t \in K} \|(\|\partial^\gamma (f \circ \varphi)(t)\|)_{\gamma \in \Gamma}\|_s && \text{since } V \text{ is an isometry.}
\end{aligned}$$

Fix any $t_0 \in K$. From Lemma 4.39, there exists a bijection $\tau : \Gamma \rightarrow \Lambda$ such that for each $\gamma \in \Gamma$, we have $\|\partial^\gamma (f \circ \varphi)(t_0)\| = \|\partial^{\tau(\gamma)} f(s_0)\|$ where $s_0 = \varphi(t_0)$. So,

$$\|(\|\partial^\gamma (f \circ \varphi)(t_0)\|)_{\gamma \in \Gamma}\|_s = \|(\|\partial^{\tau(\gamma)} f(s_0)\|)_{\gamma \in \Gamma}\|_s \leq \sup_{s \in Q} \|(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}\|_s = \|f\|_{D,s}.$$

Therefore, $\|Tf\|_{D,s} \leq \|f\|_{D,s}$. By symmetry, one can also show that $\|Tf\|_{D,s} \geq \|f\|_{D,s}$. Thus, T is an isometry. This concludes the proof of the proposition. $\square$

The following is the main result of this section, which follows immediately from Propositions 4.40, 4.24 4.32 and 4.38.

Theorem 4.41. *Let $T : (C_0^p(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,r})$ be a linear bijection, where $p, q, m, n \geq 1$ are natural numbers, Q and K are respectively open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, both E and F are Banach spaces and $s, r \in [1, \infty)$. If $\{s, r\} \neq \{2\}$ or $q > 1$ or $m = 1$, then T is an isometry if and only if $p = q$, $m = n$, $s = r$ and there exists a C^q -diffeomorphism $\varphi : K \rightarrow Q$ and for each connected component U of K , there exists a surjective linear isometry $V : E \rightarrow F$ such that $Tf(t) = Vf(\varphi(t))$ for all $f \in C_0^p(Q, E)$ and $t \in U$, and the Jacobian matrix $A(t)$ of φ is a signed permutation matrix for all $t \in K$.*

4.5 Jacobian matrix of φ is orthogonal

As mentioned at the end of Section 4.3.2, we return to the case where $s = r = 2$ and $q = 1$ in this section. We unearth many interesting phenomena. For instance, the Jacobian matrix A of φ must be a generalized permutation matrix when E and F are not Hilbert spaces, and if A is not necessarily a generalized permutation matrix, then E and F must be Hilbert spaces. Many interesting tools are being

employed here including Lorch's Theorem on classification of inner product spaces [14] and Liouville's Theorem [15], [16].

4.5.1 $m \geq 2$, $s = r = 2$ and $q = 1$

For any $m \times m$ matrix A , we let $L_A : (\mathbb{R}^m, \|\cdot\|_2) \rightarrow (\mathbb{R}^m, \|\cdot\|_2)$ be defined by $L_A(z) = Az$. For each $t \in K$, we denote $c(t) = \|V(t)\|$. The following proposition follows immediately from Remark 4.22, Corollaries 4.7 and 4.8.

Proposition 4.42. *If $T : (C_0^1(Q, E), \|\cdot\|_{D,2}) \rightarrow (C_0^1(K, F), \|\cdot\|_{D,2})$ is a surjective linear isometry given by $Tf(t) = V(t)f(\varphi(t))$, then for all $t \in K$, we have both $V(t) \cdot A(t) : E_2^m \rightarrow F_2^m$ and $L_{c(t)A(t)}$ are isometries (and hence $c(t)A(t)$ is orthogonal) and $V(t) = c(t)J(t)$ where $J(t) : E \rightarrow F$ is an isometry.*

Proposition 4.43. *Let $T : (C_0^1(Q, E), \|\cdot\|_{D,2}) \rightarrow (C_0^1(K, F), \|\cdot\|_{D,2})$ be a linear bijection given by $Tf(t) = Vf(\varphi(t))$ where Q and K are open subsets of $\mathbb{R}^m$, $V : E \rightarrow F$ is a surjective linear isometry and $\varphi : K \rightarrow Q$ is a C^1 -diffeomorphism. if the Jacobian matrix $A(t) = (\partial_j \varphi_i(t))_{1 \leq i, j \leq m}$ of φ is orthogonal for all $t \in K$ and E is a Hilbert space, then T is an isometry.*

Proof. Given any $f \in (C_0^1(Q, E), \|\cdot\|_{D,2})$, assume that f attains norm at 0. Let $f(0) = u$ and $\partial_i f(0) = e_i$ for all $1 \leq i \leq m$. Then

$$\|f\|_{D,2}^2 = \|(\|f(0)\|, \|\partial_1 f(0)\|, \dots, \|\partial_m f(0)\|)\|_2^2 = \|u\|^2 + \sum_{i=1}^m \|e_i\|^2. \quad (4.5.1)$$

On the other hand, for each $1 \leq j \leq m$, since V is an isometry and independent of t , we have $\|\partial_j(Tf)(0)\|^2 = \|\partial_j(f \circ \varphi)(0)\|^2$. Let $a_{ij}(0) = \partial_j \varphi_i(0)$. By Chain rule, we have

$$\|\partial_j(f \circ \varphi)(0)\|^2 = \left\| \sum_{i=1}^m \partial_i f(0) \cdot \partial_j \varphi_i(0) \right\|^2 = \left\| \sum_{i=1}^m a_{ij}(0) e_i \right\|^2. \quad (4.5.2)$$

Since $f(0) = u$ and V is an isometry, so $\|Tf(0)\| = \|Vf(0)\| = \|u\|$. It follows that

$$\begin{aligned} \|Tf\|_{D,2}^2 &\geq \|(\|Tf(0)\|, \|\partial_1(Tf)(0)\|, \dots, \|\partial_m(Tf)(0)\|)\|_2^2 \quad \text{by definition of } \|\cdot\|_{D,2} \\ &= \|u\|^2 + \sum_{j=1}^m \left\| \sum_{i=1}^m a_{ij}(0)e_i \right\|^2 \quad \text{by (4.5.2)} \end{aligned} \quad (4.5.3)$$

Note that for each $1 \leq i, k \leq m$, since $A(0)$ is orthogonal, so

$$\sum_{j=1}^m a_{ij}(0)a_{kj}(0) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases} \quad (4.5.4)$$

It follows that

$$\begin{aligned} \sum_{j=1}^m \left\| \sum_{i=1}^m a_{ij}(0)e_i \right\|^2 &= \sum_{j=1}^m \left\langle \sum_{i=1}^m a_{ij}(0)e_i, \sum_{k=1}^m a_{kj}(0)e_k \right\rangle \quad \text{since } E \text{ is a Hilbert space} \\ &= \sum_{i=1}^m \sum_{k=1}^m \sum_{j=1}^m a_{ij}(0)a_{kj}(0) \langle e_i, e_k \rangle \quad \text{by bilinearity of } \langle \cdot, \cdot \rangle \\ &= \sum_{i=1}^m \langle e_i, e_i \rangle \quad \text{by (4.5.4)} \\ &= \sum_{i=1}^m \|e_i\|^2. \end{aligned} \quad (4.5.5)$$

Therefore, (4.5.1), 4.5.3 and 4.5.5 imply that $\|Tf\|_{D,2} \geq \|f\|_{D,2}$. Since $V : E \rightarrow F$ is an isometry and E is a Hilbert space, so F is also a Hilbert space. Since $T^{-1}g(s) = V^{-1}g(\varphi^{-1}(s))$, applying the same argument to T^{-1} gives $\|Tf\|_{D,2} \leq \|f\|_{D,2}$. Hence, T is an isometry. This concludes the proof of the proposition. $\square$

A natural question to ask is if $T : (C_0^1(Q, E), \|\cdot\|_{D,2}) \rightarrow (C_0^1(K, F), \|\cdot\|_{D,2})$ is a surjective linear isometry given by $Tf(t) = V(t)f(\varphi(t))$ where $\varphi : K \rightarrow Q$ is a C^1 -diffeomorphism and $V(t) : E \rightarrow F$ is a Banach space isomorphism for all $t \in K$,

1. must E and F be Hilbert spaces?
2. Is $V : K \rightarrow L(E, F)$ locally constant?

We answered the first question in Proposition 4.45, which asserts that in some cases, both E and F must be Hilbert spaces. To do so, we used the Lorch's characterization of inner product spaces, which is given by Lemma 4.44.

We answered the second question in Theorem 4.49.

Lemma 4.44. [14] *If there exists $d \in \mathbb{R} \setminus \{-1, 0, 1\}$ such that for all $x, y \in E$ with $\|x\| = \|y\|$, we have $\|dy + x\| = \|dx + y\|$, then E is an inner product space.*

Proposition 4.45. *Suppose that $T : (C_0^1(Q, E), \|\cdot\|_{D,2}) \rightarrow (C_0^1(K, F), \|\cdot\|_{D,2})$ is a surjective linear isometry given by $Tf(t) = V(t)f(\varphi(t))$ where $\varphi : K \rightarrow Q$ is a C^1 -diffeomorphism and $V(t) : E \rightarrow F$ is a Banach space isomorphism for all $t \in K$. If there exists $t \in K$ such that $c(t)A(t)$ is not a signed permutation matrix, then E and F are Hilbert spaces.*

Proof. Assume that $t = 0$. In this proof, we will denote $a_{ij} = a_{ij}(0)$ for all $1 \leq i, j \leq m$ and $c = c(0)$. Also, for each $1 \leq i, j \leq m$, we let $r_i = (a_{ij})_{1 \leq j \leq m}$ and $c_j = (a_{ij})_{1 \leq i \leq m}$ be the i th row and j th column of A respectively. For each $1 \leq i \leq m$, let $e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)^T \in \mathbb{R}^m$. By Proposition 4.42, we have

$$V(0) = cJ(0) \quad \text{for some isometry} \quad J(0) : E \rightarrow F. \quad (4.5.6)$$

So, it suffices to show that E is a Hilbert space. To do this, we consider two cases.

Case 1: There exist $1 \leq i_0, j_0 \leq m$ such that $ca_{i_0j_0} \notin \{0, \pm 1, \pm \frac{1}{\sqrt{2}}\}$.

We would like to apply Lemma 4.44 to show that E is a Hilbert space. Fix any $x, y \in E$ with $\|x\| = \|y\|$. Let $z = c \cdot y \cdot r_{i_0}^T + c \cdot a_{i_0j_0}(x - y)e_{j_0}$. Then by (4.5.6), we have

$$V(0) \cdot Az = J(0) \cdot A [c^2 \cdot yr_{i_0}^T + c^2 \cdot a_{i_0j_0}(x - y)e_{j_0}]. \quad (4.5.7)$$

By Proposition 4.42, cA is orthogonal. So, $A(c^2 \cdot yr_{i_0}^T) = y(cA)(cr_{i_0}^T) = ye_{i_0}$. Also, clearly $Ae_{j_0} = c_{j_0}$. It follows that the RHS of (4.5.7) becomes

$$J(0) \cdot A [c^2 \cdot yr_{i_0}^T + c^2 \cdot a_{i_0j_0}(x - y)e_{j_0}] = J(0) [ye_{i_0} + c^2 \cdot a_{i_0j_0}(x - y)c_{j_0}]. \quad (4.5.8)$$

Hence, by (4.5.7) and 4.5.8, we have

$$V(0) \cdot Az = J(0) [ye_{i_0} + c^2 a_{i_0j_0}(x - y)c_{j_0}]. \quad (4.5.9)$$

Observe that

$$\begin{aligned}
\|z\|_2^2 &= \|V(0) \cdot Az\|_2^2 && \text{by Proposition 4.42} \\
&= \|J(0) [ye_{i_0} + c^2 a_{i_0 j_0} (x - y) c_{j_0}]\|_2^2 && \text{by (4.5.9)} \\
&= \|ye_{i_0} + c^2 a_{i_0 j_0} (x - y) c_{j_0}\|_2^2 && \text{since } J(0) \text{ is a linear isometry.}
\end{aligned} \tag{4.5.10}$$

Now, let $z' = c \cdot x \cdot r_{i_0}^T + c \cdot a_{i_0 j_0} (y - x) e_{j_0}$, that is, we interchange x and y in the definition of z above. By similar arguments as above, we have

$$\|z'\|_2^2 = \|xe_{i_0} + c^2 a_{i_0 j_0} (y - x) c_{j_0}\|_2^2. \tag{4.5.11}$$

Observe that

$$\begin{aligned}
\|z\|_2^2 &= \|c \cdot y \cdot r_{i_0}^T + c \cdot a_{i_0 j_0} (x - y) e_{j_0}\|_2^2 && \text{by definition of } z \\
&= \left(\sum_{j \neq j_0} c^2 \cdot \|y\|^2 \cdot a_{i_0 j}^2 \right) + c^2 \cdot a_{i_0 j_0} \|x\|^2 && \text{by definition of } \|\cdot\|_2^2 \\
&= \left(\sum_{j \neq j_0} c^2 \cdot \|x\|^2 \cdot a_{i_0 j}^2 \right) + c^2 \cdot a_{i_0 j_0} \|y\|^2 && \text{since } \|x\| = \|y\| \\
&= \|z'\|_2^2 && \text{by definition of } z'.
\end{aligned} \tag{4.5.12}$$

It follows from (4.5.10), 4.5.11 and 4.5.12 that

$$\|ye_{i_0} + c^2 a_{i_0 j_0} (x - y) c_{j_0}\|_2^2 = \|xe_{i_0} + c^2 a_{i_0 j_0} (y - x) c_{j_0}\|_2^2.$$

Note that

$$\begin{aligned}
\|ye_{i_0} + c^2 a_{i_0 j_0} (x - y) c_{j_0}\|_2^2 &= \|y + c^2 a_{i_0 j_0}^2 (x - y)\|^2 + \sum_{i \neq i_0} c^4 a_{i_0 j_0}^2 \|x - y\|^2 a_{ij_0}^2 \quad \text{and} \\
\|xe_{i_0} + c^2 a_{i_0 j_0} (y - x) c_{j_0}\|_2^2 &= \|x + c^2 a_{i_0 j_0}^2 (y - x)\|^2 + \sum_{i \neq i_0} c^4 a_{i_0 j_0}^2 \|y - x\|^2 a_{ij_0}^2.
\end{aligned}$$

It follows that

$$\|y + c^2 a_{i_0 j_0}^2 (x - y)\|^2 = \|x + c^2 a_{i_0 j_0}^2 (y - x)\|^2. \tag{4.5.13}$$

Let $d = \frac{1 - c^2 a_{i_0 j_0}^2}{c^2 a_{i_0 j_0}^2} \in \mathbb{R}$. So, (4.5.13) is reduced to $\|dy + x\| = \|dx + y\|$. Since

$ca_{i_0j_0} \notin \{0, \pm 1, \pm \frac{1}{\sqrt{2}}\}$, so $d \notin \{-1, 0, 1\}$. By Lemma 4.44, we conclude that E is a Hilbert space.

Case 2 : For any $1 \leq i, j \leq m$, we have $ca_{ij} \in \{0, \pm 1, \pm \frac{1}{\sqrt{2}}\}$.

By Proposition 4.42, L_{cA^T} is an isometry. So, $\|cr_i\|_2 = 1$ for all $1 \leq i \leq m$. Since the matrix cA is not a signed permutation matrix, so we must have $|ca_{i_0j_0}| = \frac{1}{\sqrt{2}}$ for some $1 \leq i_0, j_0 \leq m$. Since $\|cr_{i_0}\|_2 = 1$ and $ca_{ij} \in \{0, \pm 1, \pm \frac{1}{\sqrt{2}}\}$ for all $1 \leq i, j \leq m$, there exists $j_1 \neq j_0$ such that $|ca_{i_0j_1}| = \frac{1}{\sqrt{2}}$ and $a_{i_0j} = 0$ for all $j \notin \{j_0, j_1\}$. Similarly, since $\|cc_{j_0}\|_2 = 1$, there exists $i_1 \neq i_0$ such that $|ca_{i_1j_0}| = \frac{1}{\sqrt{2}}$ and $a_{ij_0} = 0$ for all $i \notin \{i_0, i_1\}$. By Proposition 4.42, $cA(0)$ is orthogonal. So,

$$\langle cr_{i_0}, cr_{i_1} \rangle = c^2 a_{i_0j_0} a_{i_1j_0} + c^2 a_{i_0j_1} a_{i_1j_1} = 0. \quad (4.5.14)$$

Therefore, $a_{i_1j_1} \neq 0$ and it follows that $|ca_{i_1j_1}| = \frac{1}{\sqrt{2}}$. For all $i \notin \{i_0, i_1\}$, we have $a_{ij_1} = 0$. Also, by (4.5.14), we have either

$$(a_{i_0j_0} = a_{i_0j_1} \quad \text{and} \quad a_{i_1j_0} = -a_{i_1j_1}) \quad \text{or} \quad (a_{i_0j_0} = -a_{i_0j_1} \quad \text{and} \quad a_{i_1j_0} = a_{i_1j_1}). \quad (4.5.15)$$

Now, fix any $x, y \in E$. We would like to show that the parallelogram law holds for x and y . Let $z = (z_i)_{i=1}^m \in E^m$ be such that $z_{i_0} = x$, $z_{i_1} = y$ and $z_i = 0$ for all $i \notin \{i_0, i_1\}$. By Proposition 4.42, $V(0) \cdot A$ is an isometry. So, we have

$$\|V(0) \cdot Az\|_2^2 = \|z\|_2^2 = \|x\|^2 + \|y\|^2. \quad (4.5.16)$$

Observe that

$$\begin{aligned} & \|V(0) \cdot Az\|_2^2 \\ &= c^2 \left\| \begin{pmatrix} a_{i_0j_0} J(0)x + a_{i_0j_1} J(0)y \\ a_{i_1j_0} J(0)x + a_{i_1j_1} J(0)y \end{pmatrix} \right\|_2^2 && \text{by (4.5.6)} \\ &= c^2 \|a_{i_0j_0}x + a_{i_0j_1}y\|^2 + c^2 \|a_{i_1j_0}x + a_{i_1j_1}y\|^2 && \text{because } J(0) \text{ is an isometry} \\ &= \frac{1}{2} \left\| x + \frac{a_{i_0j_1}}{a_{i_0j_0}} y \right\|^2 + \frac{1}{2} \left\| x + \frac{a_{i_1j_1}}{a_{i_1j_0}} y \right\|^2 && \text{by } |ca_{i_0j_0}| = |ca_{i_1j_0}| = \frac{1}{\sqrt{2}} \\ &= \frac{1}{2} \|x + y\|^2 + \frac{1}{2} \|x - y\|^2 && \text{by 4.5.15.} \end{aligned}$$

Therefore, (4.5.16) implies that the parallelogram law $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ holds. Hence, E is a Hilbert space. $\square$

If $c(t) \cdot A(t)$ is a signed permutation matrix for all $t \in K$, then by Proposition 4.29, we have V is locally constant. Henceforth, we assume that for some $t \in K$, $c(t) \cdot A(t)$ is not a signed permutation matrix. By Proposition 4.45, E and F must be Hilbert spaces.

For each $1 \leq i \leq m$, let $H_i = \{(b_j)_{j=1}^m \in \mathbb{R}^m : b_i = 0\}$ and $H = \bigcup_{i=1}^m H_i$.

Lemma 4.46. *For any $m \times m$ invertible matrix M , there exists $(d_i)_{i=1}^m \in \mathbb{R}^m$ with $d_1 > 0$ such that $M(d_1, \dots, d_m)^T \notin H$ and $M(-d_1, d_2, \dots, d_m)^T \notin H$.*

Proof. Consider the set $\Pi = \{(a_1, \dots, a_m) \in \mathbb{R}^m : a_1 > 0\}$. Clearly Π is an open set. Since M is invertible, so $M\Pi$ is also an open set. Since H is closed, so $M\Pi \not\subseteq H$. Therefore, there exists $v = (v_j)_{j=1}^m \in M\Pi \setminus H$, that is, $v = Md$ for some $d = (d_i)_{i=1}^m \in \Pi$ and $v \notin H$. By continuity, there exists a neighbourhood O of d such that $O \subseteq \Pi$ and $MO \cap H = \emptyset$. Let $d' = (-d_1, d_2, \dots, d_m)^T$ and $O' = \{(-d_1, d_2, \dots, d_m) \in \mathbb{R}^m : (d_1, \dots, d_m) \in O\}$. Clearly O' is a neighbourhood of d' . Since O' is open, so $MO' \not\subseteq H$. So there exists $d'' = (d''_i)_{i=1}^m \in O'$ such that $Md'' \notin H$. Note that $d''_1 < 0$. Let $d''' = (-d''_1, d''_2, \dots, d''_m)^T$. Then $Md''' \notin H$. Thus, d''' is the required vector. This concludes the proof of the lemma. $\square$

The following proposition asserts that V is locally constant.

Proposition 4.47. *For any $t \in K$, $u \in E$ and $1 \leq j \leq m$, we have $\partial_j V(t)u = 0$.*

Proof. Suppose not. Without loss of generality, we may assume that $\partial_1 V(0)u \neq 0$ for some $u \in E \setminus \{0\}$. Since $V(0)$ is onto, there exists $e \in E$ with $\|e\| = 1$ such that $V(0)e$ is a positive multiple of $\partial_1 V(0)u$. Therefore, for any $d > 0$, we have

$$\|V(0)de + \partial_1 V(0)u\| = \|V(0)de\| + \|\partial_1 V(0)u\| \quad \text{and} \quad (4.5.17)$$

$$\| -V(0)de + \partial_1 V(0)u\| = \|\partial_1 V(0)u\| - \|V(0)de\|. \quad (4.5.18)$$

Since $c^2(0)A(0)$ is invertible, by Lemma 4.46, there exists $(d_i)_{i=1}^m \in \mathbb{R}^m$ with $d_1 > 0$ such that both $c^2(0)A(0)(d_1, \dots, d_m)^T$ and $c^2(0)A(0)(-d_1, d_2, \dots, d_m)^T$ have no zero

component. Let $(v_i)_{i=1}^m, (w_i)_{i=1}^m \subseteq E^m$ be such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = ec^2(0)A(0) \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = ec^2(0)A(0) \begin{pmatrix} -d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}. \quad (4.5.19)$$

Observe that

$$\begin{aligned} \|(v_i)_{i=1}^m\|_2 &= \|e\|c(0) \cdot \|c(0)A(0)(d_1, \dots, d_m)^T\|_2 && \text{by (4.5.19)} \\ &= \|e\|c(0) \cdot \|(d_1, \dots, d_m)^T\|_2 && \text{by Proposition 4.42} \\ &= \|e\|c(0) \cdot \|(-d_1, d_2, \dots, d_m)^T\|_2 \\ &= \|e\|c(0) \cdot \|c(0)A(0)(-d_1, d_2, \dots, d_m)^T\|_2 && \text{by Proposition 4.42} \\ &= \|(w_i)_{i=1}^m\|_2 && \text{by (4.5.19)} \end{aligned} \quad (4.5.20)$$

Also, by Proposition 4.42, the matrix $c(0)A(0)$ is orthogonal. So,

$$A^T(0) \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = e \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} \quad \text{and} \quad A^T(0) \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = e \begin{pmatrix} -d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}. \quad (4.5.21)$$

Let $a = \left(\bigwedge_{1 \leq i \leq m} \|v_i\| \right) \wedge \left(\bigwedge_{1 \leq i \leq m} \|w_i\| \right) \wedge \frac{1}{\sqrt{m+1}}$. Note that $a \in \left(0, \frac{1}{\sqrt{m}}\right)$. Obtain $\eta > 0$ from Proposition 3.10. Let $\tilde{u} = \frac{\eta u}{2\|u\|}$. Clearly $\|\tilde{u}\| < \eta$. Then by Proposition 3.10, there exist functions f and $g \in C_0^1(Q, E)_2$ such that

$$\|f\|_D = \|(\|f(0)\|, \|\partial_1 f(0)\|, \dots, \|\partial_m f(0)\|)\|_2, \quad (4.5.22)$$

$$f(0) = \tilde{u}, \quad \partial_i f(0) = v_i \quad \text{for all } i = 1, \dots, m, \quad (4.5.23)$$

$$\|g\|_D = \|(\|g(0)\|, \|\partial_1 g(0)\|, \dots, \|\partial_m g(0)\|)\|_2, \quad (4.5.24)$$

$$g(0) = \tilde{u}, \quad \partial_i g(0) = w_i \quad \text{for all } i = 1, \dots, m. \quad (4.5.25)$$

By (4.5.20), 4.5.22 and 4.5.24, we have $\|f\|_{D,2} = \|g\|_{D,2}$. Since $Tf(t) = V(t)f(\varphi(t))$, so for all $1 \leq j \leq m$, we have

$$\begin{aligned}
 \partial_j Tf(0) &= V(0) \sum_{i=1}^m \partial_i f(0) \cdot a_{ij}(0) + \partial_j V(0) \tilde{u} && \text{by Lemma 4.1} \\
 &= V(0) \sum_{i=1}^m a_{ij}(0) v_i + \partial_j V(0) \tilde{u} && \text{by (4.5.23)} \\
 &= V(0) d_j e + \partial_j V(0) \tilde{u} && \text{by (4.5.21).} \quad (4.5.26)
 \end{aligned}$$

Observe that

$$\begin{aligned}
 \|f\|_{D,2} &= \|Tf\|_{D,2} \\
 &= \left\| \left[\|Tf(0)\|, \|\partial_1 Tf(0)\|, (\|\partial_j Tf(0)\|)_{j=2}^m \right] \right\|_2 && \text{by Lemma 4.5} \\
 &= \left\| \left[\|Tf(0)\|, \|V(0) d_1 e + \partial_1 V(0) \tilde{u}\|, (\|\partial_j Tf(0)\|)_{j=2}^m \right] \right\|_2 && \text{by (4.5.26)} \\
 &= \left\| \left[\|Tf(0)\|, \|V(0) d_1 e\| + \|\partial_1 V(0) \tilde{u}\|, (\|\partial_j Tf(0)\|)_{j=2}^m \right] \right\|_2 && \text{by (4.5.17).} \\
 & && (4.5.27)
 \end{aligned}$$

Similarly, since $Tg(t) = V(t)g(\varphi(t))$, so for all $1 \leq j \leq m$, we have

$$\begin{aligned}
 \partial_j Tg(0) &= V(0) \sum_{i=1}^m \partial_i g(0) \cdot a_{ij}(0) + \partial_j V(0) \tilde{u} && \text{by Lemma 4.1} \\
 &= V(0) \sum_{i=1}^m a_{ij}(0) w_i + \partial_j V(0) \tilde{u} && \text{by (4.5.25)} \\
 &= \begin{cases} -V(0) d_1 e + \partial_1 V(0) \tilde{u} & \text{if } j = 1, \\ V(0) d_j e + \partial_j V(0) \tilde{u} = \partial_j Tf(0) & \text{otherwise,} \end{cases} && (4.5.28)
 \end{aligned}$$

where the last equality above follows from (4.5.21) and 4.5.26. Therefore,

$$\begin{aligned}
 \|g\|_{D,2} &= \|Tg\|_{D,2} \\
 &= \left\| \left[\|Tg(0)\|, \|\partial_1 Tg(0)\|, (\|\partial_j Tg(0)\|)_{j=2}^m \right] \right\|_2 && \text{by Lemma 4.5} \\
 &= \left\| \left[\|Tg(0)\|, \|-V(0) d_1 e + \partial_1 V(0) \tilde{u}\|, (\|\partial_j Tg(0)\|)_{j=2}^m \right] \right\|_2 && \text{by (4.5.18)} \\
 &= \left\| \left[\|Tf(0)\|, \|\partial_1 V(0) \tilde{u}\| - \|V(0) d_1 e\|, (\|\partial_j Tf(0)\|)_{j=2}^m \right] \right\|_2 && \text{by (4.5.28).} \\
 & && (4.5.29)
 \end{aligned}$$

Since $\|g\|_{D,2} = \|f\|_{D,2}$, comparing (4.5.27) and 4.5.29 gives

$$\|V(0) d_1 e\| + \|\partial_1 V(0) \tilde{u}\| = \|\partial_1 V(0) \tilde{u}\| - \|V(0) d_1 e\|.$$

Thus, either $\partial_1 V(0)\tilde{u} = 0$ or $V(0)d_1 e = 0$, a contradiction. This concludes the proof of the proposition. $\square$

For each $t \in K$, let $V(t) = c(t)J(t)$ where $J(t) : E \rightarrow F$ is an isometry.

Lemma 4.48. $c(t) = 1$ and $V(t)$ is an isometry for all $t \in K$.

Proof. Fix any $t \in K$. It suffices to show that $c(t) = 1$. Choose $f \in C_0^1(Q, E)$ such that $\|f\|_{D,2} = \|(\|f(\varphi(t))\|, (\|\partial_i f(\varphi(t))\|)_{i=1}^m)\|_2$. Since $q = 1$, so $\tilde{A}(t) = A^T(t)$. So, we have

$$(\partial_j T f(t))_{j=0}^m = V(t) \cdot \tilde{A}(t)(\partial_i f(\varphi(t)))_{i=0}^m = c(t)J(t) \cdot A^T(t)(\partial_i f(\varphi(t)))_{i=0}^m.$$

Since $L_{c(t)A^T(t)}$ and $J(t)$ are isometries, so equation above implies that

$$\|(\partial_j T f(t))_{j=0}^m\|_2 = \|(\partial_i f(\varphi(t)))_{i=0}^m\|_2. \quad (4.5.30)$$

Note that $\|T f(t)\| = \|V(t)f(\varphi(t))\| = c(t)\|f(\varphi(t))\|$. It follows that

$$\begin{aligned} & \|(\|f(\varphi(t))\|, (\|\partial_i f(\varphi(t))\|)_{i=1}^m)\|_2^2 \\ &= \|(\|T f(t)\|, (\|\partial_j T f(t)\|)_{j=1}^m)\|_2^2 && \text{by Lemma 4.5} \\ &= \|(c(t)\|f(\varphi(t))\|, (\|\partial_i f(\varphi(t))\|)_{i=1}^m)\|_2^2 && \text{by (4.5.30).} \end{aligned}$$

Hence, $c(t) = 1$. This concludes the proof of the lemma. $\square$

The following theorem follows from Propositions 4.42, 4.45, 4.47 and Lemma 4.48 immediately.

Theorem 4.49. Let $T : (C_0^q(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,r})$ be a surjective linear isometry given by $T f(t) = V(t)f(\varphi(t))$ where $Q, K \subseteq \mathbb{R}^m$ are open subsets, $V(t) : E \rightarrow F$ is a Banach space isomorphism and $\varphi : K \rightarrow Q$ is a C^1 -diffeomorphism. If $m \geq 2$, $s = r = 2$ and $q = 1$, then $V : K \rightarrow L(E, F)$ is constant on each connected component of K and $V(t) : E \rightarrow F$ is an isometry and the Jacobian matrix $A(t)$ of φ is orthogonal for all $t \in K$. In addition, either the Jacobian matrix $A(t)$ of φ is a signed permutation matrix for all $t \in K$ or both E and F are Hilbert spaces.

4.5.2 Jacobian matrix A of φ is locally independent of t

When the Jacobian matrix $A(t)$ of φ is a signed permutation matrix, by continuity, it is constant on connected component of K . We shall consider the case that $A(t)$ is not a signed permutation matrix. As a result of Proposition 4.38, we have $m \geq 2$. In this section, we will prove that the Jacobian matrix A of φ is locally independent of t . To do this, we will employ the Liouville's Theorem in its proof. To this end, we need to recall the following definitions. Let

$$W_{\text{loc}}^{1,m}(K, \mathbb{R}^m) = \{h \in L^m(K, \mathbb{R}^m) : \partial^\alpha h \in L_{\text{loc}}^m(K, \mathbb{R}^m) \text{ (weakly) for all } |\alpha| \leq 1\}.$$

We say that $h \in W_{\text{loc}}^{1,m}(K, \mathbb{R}^m)$ is 1-*quasiconformal* if h is a homeomorphism and the Jacobian matrix Jh of h satisfies the following; $Jh^T(t)Jh(t) = |\det(Jh(t))|^{\frac{2}{m}} \cdot I_{m \times m}$ almost everywhere and either $\det Jh(t) \geq 0$ almost everywhere or $\det Jh(t) \leq 0$ almost everywhere.

Lemma 4.50. *If $m \geq 2$, then the Jacobian matrix $A(t) = (a_{ij}(t))_{1 \leq i,j \leq m}$ of φ is locally independent of t .*

Proof. Let Ω be a bounded open connected component of K . Since φ is C^1 , so $\varphi \in W_{\text{loc}}^{1,m}(\Omega, \mathbb{R}^m)$. By Theorem 4.49, we have $A^T(t)A(t) = I$ for all $t \in \Omega$. Thus, φ is 1-quasiconformal on Ω . Suppose that $m \geq 3$. By Liouville's Theorem, φ is a mobius transformation (See for e.g., [15, Theorem 2.1], [16].). Hence,

$$\varphi(t) = b + \frac{\alpha R(t-a)}{\|t-a\|^\varepsilon} \quad \text{for all } t \in \Omega$$

for some rotation matrix R , $a, b \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ and $\varepsilon \in \{0, 2\}$. It follows from the orthogonality of $A(t)$ that $\varepsilon = 0$ and $|\alpha| = 1$. So, $\varphi(t) = b \pm R(t-a)$. Therefore $A(t) = R$ or $A(t) = -R$. Thus, $A(t)$ is independent of t in Ω .

For $m = 2$, write $\varphi(t_1, t_2) = (u(t_1, t_2), v(t_1, t_2))$. Define $\tilde{\varphi}(t_1, t_2) = u(t_1, t_2) + iv(t_1, t_2)$. Since $\varphi \in C^1$, so are u and v . Since $A(t)$ is orthogonal, the Cauchy-Riemann conditions are satisfied by u and v . Therefore, $\tilde{\varphi}$ is analytic. Moreover, the orthogonality of the Jacobian $A(t)$ yields that $|u_{t_1}(t_1, t_2)|^2 + |v_{t_1}(t_1, t_2)|^2 = 1$. Thus, $|\tilde{\varphi}'| = 1$ on Ω . By the open mapping theorem, $\tilde{\varphi}'$ is constant on Ω . Hence, $A(t) = (a_{ij}(t))$ is locally independent of t . $\square$

Now, we are ready to state the main result of this chapter.

Theorem 4.51. *Let $T : (C_0^p(Q, E), \|\cdot\|_{D,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,r})$ be a linear bijection, where $p, q, m, n \geq 1$ are natural numbers, Q and K are respectively open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, both E and F are Banach spaces and $s, r \in [1, \infty)$. Then T is an isometry if and only if $p = q$, $m = n$, $s = r$ and there exists a C^q -diffeomorphism $\varphi : K \rightarrow Q$ and for each connected component U of K , there exists a surjective linear isometry $V : E \rightarrow F$ such that*

$$Tf(t) = Vf(\varphi(t)) \quad \text{for all } f \in C_0^p(Q, E) \quad \text{and } t \in U,$$

and the Jacobian matrix $A(t)$ of φ is orthogonal for all $t \in K$. In addition, if $q > 1$ or $s \neq 2$ or $m = 1$, then on each connected component of K , the Jacobian matrix of φ is a signed permutation matrix. Otherwise, if $q = 1$, $s = 2$ and $m > 1$, then the Jacobian matrix of φ is an orthogonal matrix. In this case, if E or F is not Hilbert space, then the Jacobian matrix has to be a signed permutation matrix.

Proof. If $\{s, r\} \neq \{2\}$ or $q > 1$ or $m = 1$, then the desired result follows from Theorem 4.41. Assume that $s = r = 2$, $q = 1$ and $m > 1$. Suppose that T is an isometry. One direction of the characterization follows from Theorem 4.49. Conversely, by Propositions 4.40 and 4.43, T is an isometry. This concludes the proof of the theorem. $\square$

We would like to state the following result of Botelho and Jamison [6] to contrast with Theorem 4.51 with $p = q = 1$, $m = n = 1$ and $s = r = 1$.

Theorem 4.52. [6, Corollary 3.7] *If E is a finite-dimensional Hilbert space, then $T : (C^1([0, 1], E), \|\cdot\|_{D,1}) \rightarrow (C^1([0, 1], E), \|\cdot\|_{D,1})$ is a surjective linear isometry if and only if there exists a surjective linear isometry $V : E \rightarrow E$ such that for every $f \in C^1([0, 1], E)$,*

$$Tf(t) = V(f(\varphi(t))) \quad \text{for all } t \in [0, 1]$$

with $\varphi = Id_{[0,1]}$ or $\varphi = 1 - Id_{[0,1]}$.

In fact, our methods would yield Botelho and Jamison result if we consider the embedding $C^1[0, 1] \rightarrow C^1([0, 1], E^2)$ given by $f \mapsto (f, f')$ and proceed with the constructions as in Chapter 3.

Let $m = n = 1$ and $s = r = 1$. Then Theorem 4.51 generalises result by Lei Li et al. [8].

Corollary 4.53. *[8, Theorem 2.13] Let $T : (C_0^p(Q, E), \|\cdot\|_{D,1}) \rightarrow (C_0^q(K, F), \|\cdot\|_{D,1})$ be a surjective linear isometry where Q and K are open subsets in $\mathbb{R}$ and $p, q \in \mathbb{N}$ and E and F are Banach spaces. If E and F are strictly convex with $\dim E, \dim F > 1$, then $p = q$, and there exists a C^p -diffeomorphism $\varphi : K \rightarrow Q$ and onto linear isomorphisms $V(t) : E \rightarrow F$, $t \in K$ such that*

$$Tf(t) = V(t)f(\varphi(t)) \quad \text{for all } f \in C_0^p(Q, E), \quad t \in K.$$

In Chapter 5, we will establish the Banach-Stone Theorem for the norm $\|\cdot\|_{S,\rho}$.

Chapter 5

Banach-Stone Theorem for Square Norm

For any $f \in C_0^p(Q, E)$, recall that $\|f\|_{S,\rho} = \rho [(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda}]$. In this chapter, we would like to show that T is a weighted composition operator when both $C_0^p(Q, E)$ and $C_0^q(K, F)$ are endowed with $\|\cdot\|_{S,\rho}$ and $\|\cdot\|_{S,\theta}$ respectively. To this end, we will proceed with the same structure as in Chapter 3, that is, construction of $\mathcal{P}_S$ set, construction of φ and lastly show that T is a weighted composition operator.

5.1 Construction of $\mathcal{P}_S$ Set

As mentioned in Section 2.2.1, we aim to show that $\mathcal{P}_S(\cdot)$ is nonempty in this section, which is given by Proposition 5.8. Recall that the functions h play an important role in the construction of $\mathcal{P}_D(\cdot)$ in Section 3.1. In the case of $\mathcal{P}_S(\cdot)$, since $\|\cdot\|_{S,\rho}$ has different properties than $\|\cdot\|_{D,\rho}$, we need to construct a different set of bump functions.

Fix any $f \in C_0^p(Q, E)$. Recall that $\text{supp}(f) = \overline{\{s \in Q : f(s) \neq 0\}}$. For any $s_0 \in Q$, we say that f *peaks* at s_0 if $\|f\|_\infty = \|f(s_0)\|$ and $\|f(s_0)\| > \|f(s)\|$ for all $s \neq s_0$. The following lemma serves as our foundation to build a set of bump functions.

Proposition 5.1. *For any $a \in (0, \frac{1}{2})$ and $m \geq 0$, there exists $h_m \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(h_m) \subseteq [-a, a]$, $h_m^{(m)}(0) = 1$, $\|h_m^{(m)}\|_\infty = 1$, $h_m^{(m)}$ peaks at 0, $h_m^{(i)}(0) = 0$ and $\|h_m^{(i)}\|_\infty \leq a$ for any $0 \leq i < m$.*

Proof. Fix $a \in (0, \frac{1}{2})$. We prove by induction on m . Suppose $m = 0$. Define

$$h_0(x) = e^{1 - \frac{a^2}{a^2 - x^2}} \chi_{(-a, a)}(x).$$

Clearly $h_0 \in C_c^\infty(\mathbb{R})$, $\text{supp}(h_0) = [-a, a]$, $h_0(0) = 1$, $\|h_0\|_\infty = 1$ and h_0 peaks at 0.

Suppose that the proposition holds for some $m \geq 0$, that is, there exists $h_m \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(h_m) \subseteq [-a, a]$, $h_m^{(m)}(0) = 1$, $\|h_m^{(m)}\|_\infty = 1$ and $h_m^{(m)}$ peaks at 0, $h_m^{(i)}(0) = 0$ and $\|h_m^{(i)}\|_\infty \leq a$ for any $0 \leq i < m$. Define

$$h_{m+1}(x) = \frac{1}{3^{m+1}} \int_{-\infty}^{3x} h_m(t) - \frac{1}{2} h_m(t - 2a) - \frac{1}{2} h_m(t + 2a) dt.$$

Since $\text{supp}(h_m^{(m)}) \subseteq [-a, a]$, so the supports of $h_m^{(m)}(\cdot)$, $h_m^{(m)}(\cdot - 2a)$ and $h_m^{(m)}(\cdot + 2a)$ are mutually disjoint. It follows that $\text{supp}(h_{m+1}) \subseteq [-a, a]$ and

$$h_{m+1}^{(m+1)}(x) = h_m^{(m)}(3x) - \frac{1}{2} h_m^{(m)}(3x - 2a) - \frac{1}{2} h_m^{(m)}(3x + 2a).$$

So, $h_{m+1}^{(m+1)}(0) = 1$ and $\|h_{m+1}^{(m+1)}\|_\infty = 1$ by disjointness. Since $h_m^{(m)}$ peaks at 0, so is $h_{m+1}^{(m+1)}$. Lastly, for all $0 < i < m + 1$, we have $h_{m+1}^{(i)}(0) = \frac{1}{3^{m+1-i}} h_m^{(i-1)}(0) = 0$. Also, by disjointness, we have $\|h_{m+1}^{(i)}\|_\infty = \frac{1}{3^{m+1-i}} \|h_m^{(i-1)}\|_\infty \leq \|h_m^{(i-1)}\|_\infty \leq a$. Lastly, for any $x \in [-a, a]$, by disjointness, we have

$$\begin{aligned} & |h_{m+1}(x)| \\ & \leq \frac{1}{3^{m+1}} \int_{-3a}^{3x} |h_m(t)| \vee \frac{1}{2} |h_m(t - 2a)| \vee \frac{1}{2} |h_m(t + 2a)| dt \\ & \leq \frac{1}{3^{m+1}} \int_{-3a}^{3x} a dt && \text{by inductive hypothesis} \\ & \leq \frac{6a^2}{3^{m+1}} \leq a, && \text{since } a < \frac{1}{2} \text{ and } m \geq 1. \end{aligned}$$

This conclude the inductive step and thus the proposition is proven. $\square$

For each $\varepsilon > 0$, we denote by $B(\mathbf{0}, \varepsilon)$ the open ball in $(\mathbb{R}^m, \|\cdot\|_2)$ centered at $\mathbf{0}$ with radius ε , that is, $B(\mathbf{0}, \varepsilon) = \{x \in \mathbb{R}^m : \|x\|_2 < \varepsilon\}$.

Let $\leq$ be the lexicographical ordering on Λ . For any $\mu = (\mu_1, \dots, \mu_m)$ and $\lambda = (\lambda_1, \dots, \lambda_m)$ in Λ , we let $<$ be an ordering on Λ defined by $\mu < \lambda$ if $\mu \leq \lambda$ and $\mu \neq \lambda$. In other words, if $\mu < \lambda$, then there exists $1 \leq k \leq m$ such that $\mu_k < \lambda_k$.

We are going to use Proposition 5.1 to build a set multivariable bump functions.

Lemma 5.2. *For any $\varepsilon \in (0, \frac{1}{2})$ and $\lambda \in \Lambda$, there exists $H_\lambda \in C_c^\infty(\mathbb{R}^m)$ such that $\text{supp}(H_\lambda) \subseteq B(\mathbf{0}, \varepsilon)$, $\partial^\lambda H_\lambda(\mathbf{0}) = 1$, $\|\partial^\lambda H_\lambda\|_\infty = 1$ and $\partial^\lambda H_\lambda$ peaks at the origin. Furthermore, for all $\mu \in \Lambda$ with $\mu < \lambda$, $\partial^\mu H_\lambda(\mathbf{0}) = 0$ and $\|\partial^\mu H_\lambda\|_\infty \leq \varepsilon$.*

Proof. Pick any $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Lambda$ and any $\varepsilon \in (0, \frac{1}{2})$. We choose the functions $h_{\lambda_m}, h_{\lambda_{m-1}}, \dots, h_{\lambda_2}, h_{\lambda_1}$ inductively: We choose a_m such that

$$a_m = \varepsilon \wedge \sqrt{\frac{\varepsilon}{m}}.$$

By Proposition 5.1, there exists $h_{\lambda_m} \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(h_{\lambda_m}) \subseteq [-a_m, a_m]$, $h_{\lambda_m}^{(\lambda_m)}(0) = 1$, $\|h_{\lambda_m}^{(\lambda_m)}\|_\infty = 1$, $h_{\lambda_m}^{(\lambda_m)}$ peaks at 0 and $\|h_{\lambda_m}^{(i)}\|_\infty \leq a_m \leq \varepsilon$ for all $0 \leq i < \lambda_m$. Assume that $h_{\lambda_m}, h_{\lambda_{m-1}}, \dots, h_{\lambda_{k+1}}$ have been chosen. Let

$$a_k = \left(\prod_{j=k+1}^m \bigwedge_{i=0}^p \frac{\varepsilon}{\|h_{\lambda_j}^{(i)}\|_\infty} \right) \wedge \sqrt{\frac{\varepsilon}{m}} \wedge \varepsilon.$$

By Proposition 5.1, there exists $h_{\lambda_k} \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(h_{\lambda_k}) \subseteq [-a_k, a_k]$, $h_{\lambda_k}^{(\lambda_k)}(0) = 1$, $\|h_{\lambda_k}^{(\lambda_k)}\|_\infty = 1$, $h_{\lambda_k}^{(\lambda_k)}$ peaks at 0, $h_{\lambda_k}^{(\ell)}(0) = 0$ and

$$\|h_{\lambda_k}^{(\ell)}\|_\infty \leq a_k \leq \prod_{j=k+1}^m \bigwedge_{i=0}^p \frac{\varepsilon}{\|h_{\lambda_j}^{(i)}\|_\infty} \quad (5.1.1)$$

for all $0 \leq \ell < \lambda_k$. We define $H_\lambda : K \rightarrow \mathbb{R}$ by

$$H_\lambda(s_1, \dots, s_m) = \prod_{k=1}^m h_{\lambda_k}(s_k).$$

Note that each $\text{supp}(h_{\lambda_k}) \subseteq [-a_k, a_k]$. So $\text{supp}(H_\lambda) \subseteq \prod_{k=1}^m [-a_k, a_k]$. Since $a_k^2 \leq \varepsilon/m$ for all $1 \leq k \leq m$, therefore $\prod_{k=1}^m [-a_k, a_k] \subseteq B(\mathbf{0}, \varepsilon)$. We conclude that $\text{supp}(H_\lambda) \subseteq B(\mathbf{0}, \varepsilon)$.

Since each $\|h_{\lambda_k}^{(\lambda_k)}\|_\infty = 1$ and $h_{\lambda_k}^{(\lambda_k)}(0) = 1$, so

$$\|\partial^\lambda H_\lambda\|_\infty = \left\| \prod_{k=1}^m h_{\lambda_k}^{(\lambda_k)} \right\|_\infty \leq \prod_{k=1}^m \|h_{\lambda_k}^{(\lambda_k)}\|_\infty = 1$$

and $\partial^\lambda H_\lambda(\mathbf{0}) = 1$. Therefore, $\|\partial^\lambda H_\lambda\|_\infty = 1$. Since each $h_{\lambda_k}^{(\lambda_k)}$ peaks at 0, so $\partial^\lambda H_\lambda$ peaks at $\mathbf{0}$.

It remains to show that for all $\mu \in \Lambda$ with $\mu < \lambda$, we have $\partial^\mu H_\lambda(\mathbf{0}) = 0$ and $\|\partial^\mu H_\lambda\|_\infty \leq \varepsilon$. Assume that $\lambda \neq \mathbf{0}$ and pick any $\mu \in \Lambda$ with $\mu < \lambda$. Then $\mu_k < \lambda_k$ for some $1 \leq k \leq m$. Let $k_0 = \min\{k : \mu_k < \lambda_k\}$. Since $h_{\lambda_{k_0}}^{(\mu_{k_0})}(\mathbf{0}) = 0$, so $\partial^\mu H_\lambda(\mathbf{0}) = \prod_{k=1}^m h_{\lambda_k}^{(\mu_k)}(\mathbf{0}) = 0$. Also, since

$$\begin{aligned} \left\| h_{\lambda_{k_0}}^{(\mu_{k_0})} \right\|_\infty &\leq \prod_{k=k_0+1}^m \bigwedge_{i=0}^p \frac{\varepsilon}{\|h_{\lambda_k}^{(i)}\|_\infty} && \text{by (5.1.1)} \\ &\leq \varepsilon \prod_{k=k_0+1}^m \frac{1}{\|h_{\lambda_k}^{(\mu_k)}\|_\infty}, && (5.1.2) \end{aligned}$$

therefore we have

$$\begin{aligned} \|\partial^\mu H_\lambda\|_\infty &\leq \prod_{k=1}^m \|h_{\lambda_k}^{(\mu_k)}\|_\infty \\ &= \prod_{k=k_0}^m \|h_{\lambda_k}^{(\mu_k)}\|_\infty && \text{by minimality of } k_0 \\ &\leq \varepsilon && \text{by Inequality 5.1.2.} \end{aligned}$$

□

The following proposition plays a crucial role in constructing a peaking function.

Proposition 5.3. *For any $\varepsilon \in (0, \frac{1}{2})$, there exist $(H_\lambda)_{\lambda \in \Lambda}$ and $(\zeta_\lambda)_{\lambda \in \Lambda}, (\eta_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}_\rho^\Lambda$ with $0 < \zeta_\lambda < \eta_\lambda$ such that each H_λ satisfies all properties given in Lemma 5.2, and, in addition,*

$$\eta_\lambda \cdot \|\partial^\mu H_\lambda\|_\infty < \zeta_\mu \quad \text{for all } \mu, \lambda \in \Lambda \quad \text{with } \mu \neq \lambda. \quad (5.1.3)$$

Proof. Fix any $\varepsilon \in (0, \frac{1}{2})$. We construct $(H_\lambda)_{\lambda \in \Lambda}$ and $(\zeta_\lambda)_{\lambda \in \Lambda}, (\eta_\lambda)_{\lambda \in \Lambda}$ inductively. If $\lambda = \mathbf{0}$, we choose $H_0 \in C_c^\infty(\mathbb{R}^m)$ from Lemma 5.2 and positive constants ζ_0, η_0 arbitrarily with $\zeta_0 < \eta_0$.

Let $\lambda' \in \Lambda$ be given and assume that $(H_\lambda)_{\lambda < \lambda'}$ and $(\zeta_\lambda)_{\lambda < \lambda'}, (\eta_\lambda)_{\lambda < \lambda'}$ have been chosen. By Lemma 5.2, there exists $H_{\lambda'} \in C_c^\infty(\mathbb{R}^m)$ such that

$$\bigvee_{\xi < \lambda'} \|\partial^\xi H_{\lambda'}\|_\infty < \bigwedge_{\substack{\lambda, \mu < \lambda' \\ \lambda \neq \mu}} \frac{\zeta_\mu}{\eta_\lambda \cdot \|\partial^{\lambda'} H_\lambda\|_\infty}.$$

Clearly the inequality above implies that

$$\bigvee_{\lambda < \lambda'} \eta_\lambda \cdot \|\partial^{\lambda'} H_\lambda\|_\infty < \bigwedge_{\mu, \xi < \lambda'} \frac{\zeta_\mu}{\|\partial^\xi H_{\lambda'}\|_\infty}.$$

Since the inequality above is strict, there exist constants $\zeta_{\lambda'}$ and $\eta_{\lambda'}$ such that

$$\bigvee_{\lambda < \lambda'} \eta_\lambda \cdot \|\partial^{\lambda'} H_\lambda\|_\infty < \zeta_{\lambda'} < \eta_{\lambda'} < \bigwedge_{\mu, \xi < \lambda'} \frac{\zeta_\mu}{\|\partial^\xi H_{\lambda'}\|_\infty}.$$

It follows that for all $\lambda, \mu \in \Lambda$ with $\lambda < \lambda'$ and $\mu < \lambda'$, we have

$$\eta_\lambda \cdot \|\partial^{\lambda'} H_\lambda\|_\infty < \zeta_{\lambda'} \quad \text{and} \quad \eta_{\lambda'} \cdot \|\partial^\mu H_{\lambda'}\|_\infty < \zeta_\mu.$$

This concludes inductive step. Thus, the proposition is proven. $\square$

Remark 5.4. $(\zeta_\lambda)_{\lambda \in \Lambda}$ and $(\eta_\lambda)_{\lambda \in \Lambda}$ from proposition above depend on $\|\partial^\mu H_\lambda\|_\infty$ only.

The following corollary follows immediate from Proposition 5.3.

Corollary 5.5. For all $\varepsilon \in (0, \frac{1}{2})$, if $(H_\lambda)_{\lambda \in \Lambda}$ and $(\zeta_\lambda)_{\lambda \in \Lambda}, (\eta_\lambda)_{\lambda \in \Lambda}$ are given by Proposition 5.3, then for all $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda$ with $(\alpha_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} (\zeta_\lambda, \eta_\lambda)$, we have

$$\alpha_\lambda \cdot \|\partial^\mu H_\lambda\|_\infty < \alpha_\mu \quad \text{for all} \quad \mu \neq \lambda. \quad (5.1.4)$$

Remark 5.6. Since the Inequality 5.1.4 is homogeneous in $(\alpha_\lambda)_{\lambda \in \Lambda}$, so we can assume additionally that $\rho[(\zeta_\lambda)_{\lambda \in \Lambda}] < 1$ and $\rho[(\eta_\lambda)_{\lambda \in \Lambda}] > 1$ by scaling. In particular, for every $(\alpha_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} (\zeta_\lambda, \eta_\lambda)$, we have $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1$ and Inequality 5.1.4 holds.

Let $D = \{(s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda : s_\lambda = s_{\lambda'} \text{ for some } \lambda \neq \lambda'\}$. In other words, D contains vectors in Q^Λ with at least one repeated entry. The following corollary is needed in the proof of Proposition 5.8.

Corollary 5.7. *For any $(s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda \setminus D$, let $\varepsilon = \frac{1}{4} \min\{|s_\lambda - s_\mu| : \lambda \neq \mu\} \wedge \frac{1}{4} \wedge \varepsilon_0$, $(H_\lambda)_{\lambda \in \Lambda}$ and $(\alpha_\lambda)_{\lambda \in \Lambda}$ be given by Corollary 5.5 with $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1$ where $\varepsilon_0 > 0$ satisfies $B(s_\lambda, 3\varepsilon_0) \subseteq Q$ for all $\lambda \in \Lambda$. Define $g : Q^\Lambda \rightarrow \mathbb{R}$ by*

$$g[(u_\lambda)_{\lambda \in \Lambda}] = \rho \left[\left(\bigvee_{\lambda \in \Lambda} \alpha_\lambda \cdot |\partial^\mu H_\lambda(u_\lambda - s_\lambda)| \right)_{\mu \in \Lambda} \right]. \quad (5.1.5)$$

Then for all neighbourhood U of $(s_\lambda)_{\lambda \in \Lambda}$, we have $\sup_{(u_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda \setminus U} g[(u_\lambda)_{\lambda \in \Lambda}] < 1$.

Proof. Given any neighbourhood U of $(s_\lambda)_{\lambda \in \Lambda}$, since $Q_\infty^\Lambda \setminus U$ is compact, it suffices to show that $g[(u_\lambda)_{\lambda \in \Lambda}] < 1$ for all $(u_\lambda)_{\lambda \in \Lambda} \neq (s_\lambda)_{\lambda \in \Lambda}$. Note that $u_{\lambda'} \neq s_{\lambda'}$ for some $\lambda' \in \Lambda$. Then for all $\mu \neq \lambda'$, by Inequality 5.1.4, we have

$$\bigvee_{\lambda \in \Lambda} \alpha_\lambda \cdot |\partial^\mu H_\lambda(u_\lambda - s_\lambda)| \leq \bigvee_{\lambda \in \Lambda} \alpha_\lambda \cdot \|\partial^\mu H_\lambda\|_\infty \leq \alpha_\mu. \quad (5.1.6)$$

On the other hand, since $\partial^{\lambda'} H_{\lambda'}$ peaks at 0, so $\alpha_{\lambda'} \cdot |\partial^{\lambda'} H_{\lambda'}(u_{\lambda'} - s_{\lambda'})| < \alpha_{\lambda'}$. Also, for all $\lambda \neq \lambda'$, by Inequality 5.1.4, we have $\alpha_\lambda \cdot |\partial^{\lambda'} H_\lambda(u_\lambda - s_\lambda)| \leq \alpha_\lambda \cdot \|\partial^{\lambda'} H_\lambda\|_\infty < \alpha_{\lambda'}$. Therefore, we have

$$\bigvee_{\lambda \in \Lambda} \alpha_\lambda \cdot |\partial^{\lambda'} H_\lambda(u_\lambda - s_\lambda)| < \alpha_{\lambda'}. \quad (5.1.7)$$

Since ρ is strictly monotone, by (5.1.5) and Inequalities 5.1.6, 5.1.7, we have $g[(u_\lambda)_{\lambda \in \Lambda}] < \rho[(\alpha_\mu)_{\mu \in \Lambda}] = 1$. This concludes the proof of the claim and thus the lemma. $\square$

Recall that if $s_0 = (s_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda$, let $\mathcal{P}_S(s_0)$ be the subset of $\mathbb{R}^\Lambda$ consisting of all $(\alpha_\lambda)_{\lambda \in \Lambda}$ such that

1. $\alpha_\lambda > 0$ for all $\lambda \in \Lambda$,
2. $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1$.
3. For any neighborhood U of s_0 , there exists $\delta > 0$ so that if $x_\lambda \in E$, $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$, then there exists $f \in C_0^p(Q, E)$ so that $\|f\|_{S, \rho} = 1$, $(\partial^\lambda f(s_\lambda))_{\lambda \in \Lambda} = (\alpha_\lambda x_\lambda)_{\lambda \in \Lambda}$ and $\sup_{(s_\lambda)_{\lambda \in \Lambda} \notin U} \rho[(\|\partial^\lambda f(s_\lambda)\|)_{\lambda \in \Lambda}] \leq 1 - \delta$.

Denote $S(\mathbb{R}_\rho^\Lambda) = \{(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda : \rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1\}$. We are now ready to prove the main result of this section.

Proposition 5.8. *For any $s \in Q^\Lambda \setminus D$, we have $\mathcal{P}_S(s) \neq \emptyset$. Moreover, there exist $(\zeta_\lambda)_{\lambda \in \Lambda}$ and $(\eta_\lambda)_{\lambda \in \Lambda}$ with $0 < \zeta_\lambda < \eta_\lambda$ in $\mathbb{R}_\rho^\Lambda$ such that $\prod_{\lambda \in \Lambda} (\zeta_\lambda, \eta_\lambda) \cap S(\mathbb{R}_\rho^\Lambda) \subseteq \mathcal{P}_S(s)$.*

Proof. Pick any $s = (s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda \setminus D$ and choose $\varepsilon_0 > 0$ such that $B(s_\lambda, 3\varepsilon_0) \subseteq Q$ for all $\lambda \in \Lambda$. Let

$$\varepsilon = \frac{1}{4} \min\{|s_\lambda - s_\mu| : \lambda \neq \mu\} \wedge \frac{1}{4} \wedge \varepsilon_0.$$

Since $s \notin D$, so $\varepsilon > 0$. Choose $(H_\lambda)_{\lambda \in \Lambda}$ and $(\alpha_\lambda)_{\lambda \in \Lambda}$ with $\rho[(\alpha_\lambda)_{\lambda \in \Lambda}] = 1$ that correspond to ε as in Corollary 5.5 and Remark 5.6. By the choice of ε , supports of all $(H_\lambda(\cdot - s_\lambda))_{\lambda \in \Lambda}$ are mutually disjoint. Since all ζ are positive, so are all α_λ .

Fix a neighbourhood U of s . Let

$$\delta = 1 - \sup_{(u_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda \setminus U} g[(u_\lambda)_{\lambda \in \Lambda}]$$

where g is given by (5.1.5). It follows from Corollary 5.7 that $\delta > 0$. Pick any $(x_\lambda)_{\lambda \in \Lambda} \in E_\rho^\Lambda$ where $\|x_\lambda\| = 1$ for all $\lambda \in \Lambda$. Define $f : Q \rightarrow E$ by

$$f(u) = \sum_{\lambda \in \Lambda} \alpha_\lambda x_\lambda H_\lambda(u - s_\lambda).$$

Since each $H_\lambda \in C_0^\infty(\mathbb{R}^m)$, so $f \in C_0^p(Q, E)$. For all $\mu \in \Lambda$, we have

$$\partial^\mu f(u) = \sum_{\lambda \in \Lambda} \alpha_\lambda x_\lambda \partial^\mu H_\lambda(u - s_\lambda). \quad (5.1.8)$$

Since $\partial^\lambda H_\lambda(\mathbf{0}) = 1$, so we have

$$[\partial^\mu f(s_\mu)]_{\mu \in \Lambda} = (\alpha_\mu x_\mu)_{\mu \in \Lambda}. \quad (5.1.9)$$

For any $\mu \in \Lambda$, we have

$$\begin{aligned} \|\partial^\mu f\|_\infty &= \bigvee_{\lambda \in \Lambda} \alpha_\lambda \cdot \|\partial^\mu H_\lambda\|_\infty && \text{by disjointness of } H_\lambda(\cdot - s_\lambda) \\ &\leq \alpha_\mu && \text{by Inequality 5.1.4.} \end{aligned} \quad (5.1.10)$$

Therefore, we have

$$\begin{aligned}
\|f\|_{S,\rho} &= \rho[(\|\partial^\mu f\|_\infty)_{\mu \in \Lambda}] && \text{by definition of } \|\cdot\|_S \\
&\leq \rho[(\alpha_\mu)_{\mu \in \Lambda}] && \text{by Inequality 5.1.10} \\
&= \rho[(\|\partial^\mu f(s_\mu)\|)_{\mu \in \Lambda}] && \text{by (5.1.9)} \\
&\leq \|f\|_{S,\rho} && \text{by definition of } \|\cdot\|_{S,\rho}.
\end{aligned}$$

So, we have $\|f\| = \rho[(\|\partial^\mu f(s_\mu)\|)_{\mu \in \Lambda}] = 1$. Lastly, by disjointness, we have

$$\begin{aligned}
\sup_{(u_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda \setminus U} \rho[(\|\partial^\mu f(u_\lambda)\|)_{\mu \in \Lambda}] &\leq \sup_{(u_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda \setminus U} \rho \left[\left(\bigvee_{\lambda \in \Lambda} \alpha_\lambda \cdot |\partial^\mu H_\lambda(u_\lambda - s_\lambda)| \right)_{\mu \in \Lambda} \right] \\
&= \sup_{(u_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda \setminus U} g[(u_\lambda)_{\lambda \in \Lambda}] \\
&= 1 - \delta.
\end{aligned}$$

Hence, $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathcal{P}_S(s)$. This concludes the proof of the proposition. $\square$

We end this section with the following corollary, which follows easily from Proposition 5.8 and Corollary 2.7.

Corollary 5.9. *For all $s \in Q^\Lambda \setminus D$, the set $\mathcal{G}_S(s)$ is nonempty.*

5.2 Construction of φ

Recall that $T : (C_0^p(Q, E), \|\cdot\|_{S,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,r})$ is a surjective linear isometry, and for any $f \in C_0^p(Q, E)$, $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s = (s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$, we let

$$\langle f, x^* \circ \delta_s \rangle = \sum_{\lambda \in \Lambda} x_\lambda(\partial^\lambda f(s_\lambda)).$$

In this section, we would like to show that there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for any $y^* \in (F_\theta^\Gamma)^*$, we have

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)} \tag{5.2.1}$$

for some $x^* \in (E_\rho^\Lambda)^*$. In particular, $\varphi(t)$ is independent of y^* . We shall use the phrase ‘ φ is well-defined’ to emphasize the independence of y^* in the definition.

Similarly, the phrase ' ψ is well-defined' means that there exists $\psi : Q^\Lambda \setminus D \rightarrow K^\Gamma$ such that for any $x^* \in (E_\rho^\Lambda)^*$, we have $(T^*)^{-1}(x^* \circ \delta_s) = y^* \circ \delta_{\psi(s)}$ for some $y^* \in (F_\theta^\Gamma)^*$. In particular, $\psi(s)$ is independent of x^* .

We would like to highlight that the function $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ in this section is different from the $\varphi : K \rightarrow Q$ in Section 3.2. Therefore, the methods employed here are different compared to that section. We assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. We also let $\|\cdot\|_{S,\rho} = \|\cdot\|_{S,s}$ and $\|\cdot\|_{S,\theta} = \|\cdot\|_{S,r}$.

Given a Banach space F , we let $NExt(F^*)$ be the norming subset of $\text{ext } B(F^*)$, i.e. $y^* \in NExt(F^*)$ means that $y^* \in \text{ext } B(F^*)$ and $y^*(y) = \|y\|$ for some $y \in F \setminus \{0\}$. We say that a Banach space F^* satisfies $(*)$ if for any $x^*, y^* \in NExt(F^*)$ with $x^* \neq \pm y^*$, there exists $z^* \in \text{ext } B(F^*) \cap \text{span}\{x^*, y^*\}$ with $z^* \notin \{\pm x^*, \pm y^*\}$. The motivation of considering such Banach spaces is given after Example 5.19. It is clear that any strictly convex dual Banach spaces satisfy $(*)$. Note that $\mathbb{R}$ satisfies $(*)$ trivially.

Note that there exists a Banach space that satisfies $(*)$ but not strictly convex. Indeed, we equip $\mathbb{R}^2$ with the norm whose closed unit ball centered at origin is the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, |y| \leq \frac{1}{2}\}$. Clearly such space satisfies $(*)$ but it is not strictly convex.

The main result of this section is Theorem 5.53, which we state now.

Theorem. Let $T : (C_0^p(Q, E), \|\cdot\|_{S,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,r})$ be a surjective linear isometry where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, both E and F are Banach spaces and $s, r \in [1, \infty)$. If F^* satisfies $(*)$, then there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_r^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_s^\Lambda)^*$.

To prove the theorem above, we begin with some preparatory results. The following proposition asserts the existence of φ that satisfies (5.2.1).

Proposition 5.10. For any $t \in K^\Gamma \setminus D$, there exists $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} \in (F_\theta^\Gamma)^*$ with $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $y_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$ such that

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_s \quad (5.2.2)$$

for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q_\infty^\Lambda$.

Proof. Fix any $t \in K^\Gamma \setminus D$. By Corollary 5.9, the set $\mathcal{G}_S(t)$ is nonempty, say $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$. For each $\gamma \in \Gamma$, choose $y_\gamma^* \in \text{ext } B(F^*)$. Denote $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$. By Proposition 2.5, $y^* \circ \delta_t \in \text{ext } B[C_0^q(K, F)^*]$. Since T^* is an isometry, so $T^*(y^* \circ \delta_t) \in \text{ext } B[C_0^p(Q, E)^*]$. By Proposition 2.2, there exist $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q_\infty^\Lambda$ such that $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$. This concludes the proof of the proposition. $\square$

In fact, similar proof in Proposition 5.10 shows that all such y^* gives rise to (5.2.2). We record this fact in Proposition 5.11

Proposition 5.11. *If $t \in K^\Gamma \setminus D$ and $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$, where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and each $y_\gamma^* \in \text{ext } B(F^*)$, then $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q_\infty^\Lambda$.*

Now, it remains to show that the s in (5.2.2) is independent of $y^* \in (F_r^\Gamma)^*$. In virtue of Corollary 5.13, it suffices to show that s in (5.2.2) is independent of y^* in a weak-star dense subset of $B[(F_\theta^\Gamma)^*]$. To this end, we need to establish a preliminary result.

Proposition 5.12. *The convex hull of the set*

$$N = \{(\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} : (\beta_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma), y_\gamma^* \in N\text{Ext}(F^*) \text{ for all } \gamma \in \Gamma\}.$$

is a weak-star dense subset of $B[(F_\theta^\Gamma)^]$.*

Proof. By Lemma 2.4 and Krein-Milman theorem, clearly $\overline{\text{co}(N)}^{w^*} \subseteq B[(F_\theta^\Gamma)^*]$. Assume that there exists $y_0^* \in B[(F_\theta^\Gamma)^*]$ but $y_0^* \notin \overline{\text{co}(N)}^{w^*}$. By the Hahn-Banach separation theorem, there exists $y_0 = (y_\gamma)_{\gamma \in \Gamma} \in F_\theta^\Gamma$ such that

$$y_0^*(y_0) > \sup_{y^* \in \overline{\text{co}(N)}^{w^*}} |y^*(y_0)|. \quad (5.2.3)$$

By taking small perturbations if necessary, we may assume that $y_\gamma \neq 0$ for all $\gamma \in \Gamma$. By Lemma 2.6, there exists $z^* \in \text{ext } B[(F_\theta^\Gamma)^*]$ such that

$$z^*(y_0) = \|y_0\|_\theta. \quad (5.2.4)$$

Since $z^* \in \text{ext } B[(F_\theta^\Gamma)^*]$, by Lemma 2.4, we have $z^* = (\beta_\gamma^* z_\gamma^*)_{\gamma \in \Gamma}$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma)$ and $z_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$. For each $\gamma \in \Gamma$, since $y_\gamma \neq 0$,

$z^* \in B[(F_\theta^\Gamma)^*]$ and z^* satisfies (5.2.4), so by Lemma 2.1, we have $\beta_\gamma^* z_\gamma^* \neq 0$. But $z_\gamma^* \neq 0$. Therefore, $\beta_\gamma^* \neq 0$ for all $\gamma \in \Gamma$.

We claim that $z_\gamma^* \in NExt(F^*)$ for all $\gamma \in \Gamma$. Let $z_\gamma = \frac{y_\gamma}{\|y_0\|_\theta}$ for all $\gamma \in \Gamma$ and $z = (z_\gamma)_{\gamma \in \Gamma}$. Then

$$\begin{aligned}
1 &= z^*(z) && \text{by (5.2.4)} \\
&= \sum_{\gamma \in \Gamma} \beta_\gamma^* z_\gamma^*(z_\gamma) && \text{by definition of } z^* \\
&\leq \sum_{\gamma \in \Gamma} |\beta_\gamma^* z_\gamma^*(z_\gamma)| \\
&\leq \sum_{\gamma \in \Gamma} |\beta_\gamma^*| \cdot \|z_\gamma\| && \text{since } z_\gamma^* \in \text{ext } B(F^*) \\
&\leq \|(\beta_\gamma^*)_{\gamma \in \Gamma}\|_{\theta^*} \cdot \|(z_\gamma)_{\gamma \in \Gamma}\|_\theta && \text{by Inequality 2.1.1} \\
&= 1 && \text{since } (\beta_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma) \text{ and } z_\gamma = \frac{y_\gamma}{\|y_0\|_\theta}.
\end{aligned}$$

Therefore, $|\beta_\gamma^* z_\gamma^*(z_\gamma)| = |\beta_\gamma^*| \cdot \|z_\gamma\|$ for all $\gamma \in \Gamma$. Since $\beta_\gamma^* \neq 0$, so $|z_\gamma^*(z_\gamma)| = \|z_\gamma\|$ for all $\gamma \in \Gamma$. Hence, $z_\gamma^* \in NExt(F^*)$ for all $\gamma \in \Gamma$. This concludes the proof of the claim.

By claim above, we have $z^* \in N$. However,

$$\begin{aligned}
\|y_0\|_\theta &\geq \|y_0^*\|_{\theta^*} \cdot \|y_0\|_\theta && \text{since } y_0^* \in B[(F_\theta^\Gamma)^*] \\
&\geq y_0^*(y_0) && \text{by Inequality 2.1.1} \\
&> \sup_{y^* \in \overline{co(N)}^{w^*}} |y^*(y_0)| && \text{by Inequality 5.2.3} \\
&\geq |z^*(y_0)| && \text{since } z^* \in N \\
&= \|y_0\|_\theta && \text{by (5.2.4),}
\end{aligned}$$

a contradiction. This concludes the proof of proposition. $\square$

Corollary 5.13. For each $t \in K^\Gamma \setminus D$, let

$$N(t) = \left\{ (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} : (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t), (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*) \right\}.$$

For any $t \in K^\Gamma \setminus D$, if there exists $s \in Q^\Lambda$ such that for all $y^* \in N(t)$, we have

$$T^*(y^* \circ \delta_t) = x^* \circ \delta_s \quad (5.2.5)$$

for some $x^* \in (E_\rho^\Lambda)^*$, then the above conclusion also holds for all $y^* \in (F_\theta^\Gamma)^*$.

Proof. Fix any $t \in K^\Gamma \setminus D$ and obtain $s \in Q^\Lambda$ from hypothesis. Let

$$N = \{(\alpha_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} : (\alpha_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma), y_\gamma^* \in N\text{Ext}(F^*) \text{ for all } \gamma \in \Gamma\}.$$

From Proposition 5.12, we have $\overline{co(N)}^{w^*} = B[(F_\theta^\Gamma)^*]$. According to Proposition 2.12, it remains to show that for all $y^* \in N$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in (E_\rho^\Lambda)^*$. Let $y^* = (\alpha_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ where $(\alpha_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma)$ and $(y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} N\text{Ext}(F^*)$. Since $(\alpha_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma)$, by Proposition 2.9, we have

$$(\alpha_\gamma^*)_{\gamma \in \Gamma} = \sum_{k \in I_{\alpha^*}} c_k (\beta_{k\gamma}^* x_{k\gamma}^*)_{\gamma \in \Gamma} \quad (5.2.6)$$

for some $I_{\alpha^*} \subseteq \mathbb{N}$ with $\#I_{\alpha^*} < \infty$, $c_k \in \mathbb{R}$, $(\beta_{k\gamma}^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $|x_{k\gamma}^*| = 1$ for all $\gamma \in \Gamma$. From (5.2.6), we have

$$y^* = (\alpha_\gamma^* y_\gamma^*)_{\gamma \in \Gamma} = \sum_{k \in I_{\alpha^*}} c_k (\beta_{k\gamma}^* y_\gamma^* x_{k\gamma}^*)_{\gamma \in \Gamma}. \quad (5.2.7)$$

For each $k \in I_{\alpha^*}$, since $y_\gamma^* \in N\text{Ext}(F^*)$ and $|x_{k\gamma}^*| = 1$ for all $\gamma \in \Gamma$, so $(y_\gamma^* x_{k\gamma}^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} N\text{Ext}(F^*)$. Therefore, $(\beta_{k\gamma}^* y_\gamma^* x_{k\gamma}^*)_{\gamma \in \Gamma} \in N(t)$. By the choice of s , we have

$$T^*[(\beta_{k\gamma}^* y_\gamma^* x_{k\gamma}^*)_{\gamma \in \Gamma} \circ \delta_t] = x_k^* \circ \delta_s \quad (5.2.8)$$

for some $x_k^* \in (E_\rho^\Lambda)^*$. Hence,

$$\begin{aligned} T^*(y^* \circ \delta_t) &= T^* \left(\sum_{k \in I_{\alpha^*}} c_k (\beta_{k\gamma}^* y_\gamma^* x_{k\gamma}^*)_{\gamma \in \Gamma} \circ \delta_t \right) && \text{by (5.2.7)} \\ &= \sum_{k \in I_{\alpha^*}} c_k T^*[(\beta_{k\gamma}^* y_\gamma^* x_{k\gamma}^*)_{\gamma \in \Gamma} \circ \delta_t] && \text{by linearity of } T^* \\ &= \sum_{k \in I_{\alpha^*}} c_k x_k^* \circ \delta_s && \text{by (5.2.8).} \end{aligned}$$

Let $x^* = \sum_{k \in I_{\alpha^*}} c_k x_k^*$. Then we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$. This concludes the proof of the corollary. $\square$

Recall that a Banach space F^* satisfies $(*)$ if for any $x^*, y^* \in NExt(F^*)$ with $x^* \neq \pm y^*$, there exists $z^* \in \text{ext } B(F^*) \cap \text{span}\{x^*, y^*\}$ with $z^* \notin \{\pm x^*, \pm y^*\}$.

Now, it remains to prove that the assumption of Corollary 5.13 holds. To this end, we consider the following cases.

1. $F \neq \mathbb{R}$ and F^* satisfies $(*)$,
2. $F = \mathbb{R}$ and $r > 1$,
3. $F = \mathbb{R}$, $E \neq \mathbb{R}$, E^* satisfies $(*)$ and $r = 1$,
4. $E = F = \mathbb{R}$, $r = 1$, $s = 1$.

The four cases above exhaust all possibilities. Indeed, after Case 3, it remains to consider $E = F = \mathbb{R}$ and $r = 1$. From Lemma 5.40, we must have $s = 1$.

We end this section with several useful results. Proposition 5.14 asserts that if y_γ^* is a norming extreme point of $B(F^*)$ for all $\gamma \in \Gamma$, then all components of x^* are nonzero and s are not infinity. Recall that $NExt(F^*)$ is the norming subset of $\text{ext } B(F^*)$, i.e. $y^* \in NExt(F^*)$ means that $y^*(y) = \|y\|$ for some $y \in F \setminus \{0\}$. Also, $(y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$ means that $y_\gamma^* \in NExt(F^*)$ for all $\gamma \in \Gamma$.

Proposition 5.14. *For any $t \in K^\Gamma \setminus D$ and $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$, if $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $(y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, then $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s = (s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$ where $x_\lambda^* \neq 0$ and $s_\lambda \neq \infty$ for all $\lambda \in \Lambda$.*

Proof. For each $\gamma \in \Gamma$, since $y_\gamma^* \in NExt(F^*)$, so

$$y_\gamma^*(y_\gamma) = 1 \quad \text{for some } y_\gamma \in F. \quad (5.2.9)$$

Suppose that either $x_{\lambda_0}^* = 0$ or $s_{\lambda_0} = \infty$ for some $\lambda_0 \in \Lambda$. Since $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$,

$$\sum_{\gamma \in \Gamma} \beta_\gamma^* \beta_\gamma = 1 \quad \text{for some } (\beta_\gamma)_{\gamma \in \Gamma} \in \mathcal{P}_S(t). \quad (5.2.10)$$

Denote $t = (t_\gamma)_{\gamma \in \Gamma}$. Since $(\beta_\gamma)_{\gamma \in \Gamma} \in \mathcal{P}_S(t)$, there exists $g \in C_0^q(K, F)$ such that

$$\|g\|_{S, \theta} = 1 \quad \text{and} \quad (5.2.11)$$

$$(\partial^\gamma g(t_\gamma))_{\gamma \in \Gamma} = (\beta_\gamma y_\gamma)_{\gamma \in \Gamma}. \quad (5.2.12)$$

Let $f = T^{-1}g$. Observe that

$$\begin{aligned} \langle f, T^*(y^* \circ \delta_t) \rangle &= \sum_{\gamma \in \Gamma} \beta_\gamma^* y_\gamma^* (\partial^\gamma T f(t_\gamma)) && \text{by definitions of } \langle \cdot, \cdot \rangle \text{ and } y^* \\ &= \sum_{\gamma \in \Gamma} \beta_\gamma^* y_\gamma^* (\partial^\gamma g(t_\gamma)) && \text{since } g = T f \\ &= \sum_{\gamma \in \Gamma} \beta_\gamma^* \beta_\gamma y_\gamma^*(y_\gamma) && \text{by (5.2.12)} \\ &= 1 && \text{by (5.2.10) and 5.2.9} \\ &= \|f\|_{S, \rho} && \text{by (5.2.11) and since } T \text{ is an isometry.} \end{aligned} \quad (5.2.13)$$

On the other hand,

$$\begin{aligned} \langle f, x^* \circ \delta_s \rangle &= \sum_{\lambda \in \Lambda} x_\lambda^* (\partial^\lambda f(s_\lambda)) && \text{by definition of } \langle \cdot, \cdot \rangle \\ &= \sum_{\lambda \neq \lambda_0} x_\lambda^* (\partial^\lambda f(s_\lambda)) && \text{since either } x_{\lambda_0}^* = 0 \text{ or } s_{\lambda_0} = \infty \\ &\leq \rho^* [(\|x_\lambda^*\|)_{\lambda \in \Lambda}] \cdot \rho \left[(\|\partial^\lambda f(s_\lambda)\|)_{\lambda \neq \lambda_0}, 0 \right] && \text{by Inequality 2.1.1} \\ &< 1 \cdot \rho \left[(\|\partial^\lambda f\|_\infty)_{\lambda \neq \lambda_0}, \|\partial^{\lambda_0} f\|_\infty \right] && \text{by strict monotonicity of } \rho \\ &= \|f\|_{S, \rho} && \text{by definition of } \|\cdot\|_{S, \rho}. \end{aligned} \quad (5.2.14)$$

By (5.2.2), (5.2.13) and (5.2.14), we have $\|f\|_{S, \rho} < \|f\|_{S, \rho}$, a contradiction. This concludes the proof of the proposition. $\square$

For any $s = (s_1, \dots, s_m) \in Q$ and $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda$, recall that $\frac{s^\lambda}{\lambda!} = \prod_{i=1}^m \frac{s_i^{\lambda_i}}{\lambda_i!}$.

Lemma 5.15. *For any $\gamma_0 \in \Gamma$, $a_{\gamma_0} \in F \setminus \{0\}$, $u_{\gamma_0} \in K$ and a neighbourhood U of u_{γ_0} , there exists $g \in C_0^q(K, F)$ such that $\text{supp}(g) \subseteq U$, $\partial^{\gamma_0} g(u_{\gamma_0}) = a_{\gamma_0}$ and $\partial^\gamma g(u_{\gamma_0}) = 0$ for all $\gamma \neq \gamma_0$.*

Proof. Choose $\phi \in C_0^\infty(K)$ such that $\phi \equiv 1$ on some neighbourhood V of u_{γ_0} , $\text{supp}(\phi) \subseteq U$ and $\overline{V} \subseteq U$. Define $g : K \rightarrow F$ by

$$g(t) = a_{\gamma_0} \phi(t) \frac{(t - u_{\gamma_0})^{\gamma_0}}{\gamma_0!}.$$

Since $\text{supp}(\phi) \subseteq U$, so is $\text{supp}(g)$. Also, since $\phi \equiv 1$ on a neighbourhood of u_{γ_0} , so $\partial^{\gamma_0} g(u_{\gamma_0}) = a_{\gamma_0}$. This concludes the proof of the lemma. $\square$

Lemma 5.16. Fix any $a^* = (a_\gamma^*)_{\gamma \in \Gamma} \in (F_\theta^\Gamma)^*$ and $(u_\gamma)_{\gamma \in \Gamma} \in K_\infty^\Gamma$. Assume that

$$a^* \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} = \sum_{k=1}^n a_k^* \circ \delta_{(u_\gamma^k)_{\gamma \in \Gamma}} \quad (5.2.15)$$

for some $n \geq 1$, $a_1^*, \dots, a_n^* \in (F_\theta^\Gamma)^*$ and $(u_\gamma^1)_{\gamma \in \Gamma}, \dots, (u_\gamma^n)_{\gamma \in \Gamma} \in K_\infty^\Gamma$. If $a_{\gamma_0}^* \neq 0$ and $u_{\gamma_0} \neq \infty$ for some $\gamma_0 \in \Gamma$, then $u_{\gamma_0} \in \{u_{\gamma_0}^k : 1 \leq k \leq n\}$.

Proof. Suppose that $u_{\gamma_0} \notin \{u_{\gamma_0}^k : 1 \leq k \leq n\}$. Then there exists a neighbourhood $U \subseteq K$ of u_{γ_0} such that

$$u_{\gamma_0}^k \notin U \quad \text{for all } 1 \leq k \leq n. \quad (5.2.16)$$

By shrinking U if necessary, we may assume that for all $u \in \{u_\gamma^k : \gamma \neq \gamma_0, 1 \leq k \leq n\} \cup \{u_\gamma : \gamma \neq \gamma_0\}$,

$$\text{either } u = u_{\gamma_0} \quad \text{or} \quad u \notin U. \quad (5.2.17)$$

Since $a_{\gamma_0}^* \neq 0$, so

$$a_{\gamma_0}^*(a_{\gamma_0}) \neq 0 \quad \text{for some } a_{\gamma_0} \in F. \quad (5.2.18)$$

By Lemma 5.15, there exists $g \in C_0^q(K, F)$ such that $\text{supp}(g) \subseteq U$, $\partial^{\gamma_0} g(u_{\gamma_0}) = a_{\gamma_0}$ and $\partial^\gamma g(u_{\gamma_0}) = 0$ for all $\gamma \neq \gamma_0$. By (5.2.17) and (5.2.16), we have

$$\partial^\gamma g(u_\gamma) = 0 \quad \text{for all } \gamma \neq \gamma_0 \text{ and} \quad (5.2.19)$$

$$\partial^\gamma g(u_\gamma^k) = 0 \quad \text{for all } \gamma \in \Gamma \text{ and } 1 \leq k \leq n. \quad (5.2.20)$$

It follows that

$$\begin{aligned}
0 &\neq a_{\gamma_0}^*(a_{\gamma_0}) && \text{by (5.2.18)} \\
&= \langle g, a^* \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} \rangle && \text{by (5.2.19)} \\
&= \left\langle g, \sum_{k=1}^n a_k^* \circ \delta_{(u_\gamma^k)_{\gamma \in \Gamma}} \right\rangle && \text{by (5.2.15)} \\
&= 0 && \text{by (5.2.20),} \tag{5.2.21}
\end{aligned}$$

a contradiction. Therefore, $u_{\gamma_0} \in \{u_{\gamma_0}^k : 1 \leq k \leq n\}$. This concludes the proof of the lemma. $\square$

The following corollary follows from Lemma 5.16.

Corollary 5.17. *Assume that $(a_\gamma^*)_{\gamma \in \Gamma} \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} = (b_\gamma^*)_{\gamma \in \Gamma} \circ \delta_{(v_\gamma)_{\gamma \in \Gamma}}$. If $a_{\gamma_0}^* \neq 0$ and $u_{\gamma_0} \neq \infty$ for some $\gamma_0 \in \Gamma$, then $b_{\gamma_0}^* = a_{\gamma_0}^*$ and $v_{\gamma_0} = u_{\gamma_0}$. In particular, $b_{\gamma_0}^* \neq 0$ and $v_{\gamma_0} \neq \infty$.*

Proof. Since $a_{\gamma_0}^* \neq 0$ and $u_{\gamma_0} \neq \infty$, by Lemma 5.16, we have $v_{\gamma_0} = u_{\gamma_0}$. Suppose that $b_{\gamma_0}^* \neq a_{\gamma_0}^*$. Then $b_{\gamma_0}^*(a_{\gamma_0}) \neq a_{\gamma_0}^*(a_{\gamma_0})$ for some $a_{\gamma_0} \in F \setminus \{0\}$. Choose $g \in C_0^q(K, F)$ such that

$$\partial^{\gamma_0} g(u_{\gamma_0}) = a_{\gamma_0}, \quad \partial^\gamma g(u_\gamma) = 0 \quad \text{for all } \gamma \neq \gamma_0, \tag{5.2.22}$$

$$\partial^{\gamma_0} g(v_{\gamma_0}) = a_{\gamma_0}, \quad \partial^\gamma g(v_\gamma) = 0 \quad \text{for all } \gamma \neq \gamma_0. \tag{5.2.23}$$

Then

$$\begin{aligned}
a_{\gamma_0}^*(a_{\gamma_0}) &= \langle g, (a_\gamma^*)_{\gamma \in \Gamma} \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} \rangle && \text{by (5.2.22)} \\
&= \langle g, (b_\gamma^*)_{\gamma \in \Gamma} \circ \delta_{(v_\gamma)_{\gamma \in \Gamma}} \rangle && \text{by assumption} \\
&= b_{\gamma_0}^*(a_{\gamma_0}) && \text{by (5.2.23),}
\end{aligned}$$

contradicting $b_{\gamma_0}^*(a_{\gamma_0}) \neq a_{\gamma_0}^*(a_{\gamma_0})$. This concludes the proof of the corollary. $\square$

Corollary 5.18. *Fix any $a^* = (a_\gamma^*)_{\gamma \in \Gamma} \in (F_\theta^\Gamma)^*$ and $(u_\gamma)_{\gamma \in \Gamma} \in K_\infty^\Gamma$. Assume that*

$$a^* \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} = a_1^* \circ \delta_{(u_\gamma^1)_{\gamma \in \Gamma}} + a_2^* \circ \delta_{(u_\gamma^2)_{\gamma \in \Gamma}}$$

for some $a_1^*, a_2^* \in (F_\theta^\Gamma)^*$ and $(u_\gamma^1)_{\gamma \in \Gamma}, (u_\gamma^2)_{\gamma \in \Gamma} \in K_\infty^\Gamma$. If there exists $\gamma_0 \in \Gamma$ such that γ_0 component of a_1^*, a_2^* are nonzero and $u_{\gamma_0}^1, u_{\gamma_0}^2 \neq \infty$, then $u_{\gamma_0}^1 = u_{\gamma_0}^2$. If, in addition, $a_{\gamma_0}^* \neq 0$ and $u_{\gamma_0} \neq \infty$, then $u_{\gamma_0} = u_{\gamma_0}^1 = u_{\gamma_0}^2$.

Proof. Suppose that $u_{\gamma_0}^1 \neq u_{\gamma_0}^2$. Note that

$$a_1^* \circ \delta_{(u_\gamma^1)_{\gamma \in \Gamma}} = a^* \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} - a_2^* \circ \delta_{(u_\gamma^2)_{\gamma \in \Gamma}}.$$

Since γ_0 component of a_1^* is nonzero and $u_{\gamma_0}^1 \neq \infty$, by Lemma 5.16, we have $u_{\gamma_0}^1 \in \{u_{\gamma_0}, u_{\gamma_0}^2\}$. Since $u_{\gamma_0}^1 \neq u_{\gamma_0}^2$, so $u_{\gamma_0}^1 = u_{\gamma_0}$. Now, note that

$$a_2^* \circ \delta_{(u_\gamma^2)_{\gamma \in \Gamma}} = a^* \circ \delta_{(u_\gamma)_{\gamma \in \Gamma}} - a_1^* \circ \delta_{(u_\gamma^1)_{\gamma \in \Gamma}}.$$

By similar arguments above, we have $u_{\gamma_0}^2 \in \{u_{\gamma_0}, u_{\gamma_0}^1\} = \{u_{\gamma_0}^1\}$, a contradiction. The second assertion follows immediately from Lemma 5.16. This concludes the proof of the lemma. $\square$

5.2.1 Case 1: $F \neq \mathbb{R}$ and F^* satisfies $(*)$

The main result of this section is Corollary 5.27, which says that if $F \neq \mathbb{R}$ and F^* satisfies $(*)$, then φ is well-defined. The example below shows that we need to impose certain condition on the Banach space F .

Example 5.19. Assume that $\rho = \theta$, $Q = (-1, 0) \cup (0, 1)$, $K = (0, 1)$ and $E = \mathbb{R}$, $F = \ell^\infty(2)$. Let $T : (C_0^1((-1, 0) \cup (0, 1)), \|\cdot\|_{S, \rho}) \rightarrow (C_0^1((0, 1), \ell^\infty(2)), \|\cdot\|_{S, \rho})$ be defined by

$$Tf(t) = (f(-t), f(t)). \quad (5.2.24)$$

Then T is a surjective linear isometry but Q and K are not homeomorphic, E and F are not isomorphic. Therefore, the Banach-Stone Theorem fails in this example.

Proof. It suffices to show that T is an onto isometry. We first claim that T is an isometry. For all $t \in (0, 1)$ and $f \in C(Q, E)$, note that

$$\sup_{t \in (0, 1)} (|f(-t)| \vee |f(t)|) = \sup_{t \in (-1, 0) \cup (0, 1)} |f(t)|. \quad (5.2.25)$$

Then

$$\begin{aligned}
\|Tf\|_\infty &= \sup_{t \in (0,1)} \|Tf(t)\| && \text{by definition of } \|\cdot\|_\infty \\
&= \sup_{t \in (0,1)} (|f(-t)| \vee |f(t)|) && \text{since } Tf(t) \in \ell^\infty(2) \\
&= \sup_{t \in (-1,0) \cup (0,1)} |f(t)| && \text{by (5.2.25)} \\
&= \|f\|_\infty && \text{by definition of } \|\cdot\|_\infty. \tag{5.2.26}
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|(Tf)'\|_\infty &= \sup_{t \in (0,1)} \|(Tf)'(t)\| && \text{by definition of } \|\cdot\|_\infty \\
&= \sup_{t \in (0,1)} (|-f'(-t)| \vee |f'(t)|) && \text{by (5.2.24)} \\
&= \sup_{t \in (0,1)} (|f'(-t)| \vee |f'(t)|) \\
&= \sup_{t \in (-1,0) \cup (0,1)} |f'(t)| && \text{by (5.2.25)} \\
&= \|f'\|_\infty && \text{by definition of } \|\cdot\|_\infty. \tag{5.2.27}
\end{aligned}$$

Thus,

$$\begin{aligned}
\|Tf\|_{S,\rho} &= \rho(\|Tf\|_\infty, \|(Tf)'\|_\infty) && \text{by definition of } \|\cdot\|_{S,\rho} \\
&= \rho(\|f\|_\infty, \|f'\|_\infty) && \text{by (5.2.26) and 5.2.27} \\
&= \|f\|_{S,\rho} && \text{by definition of } \|\cdot\|_{S,\rho}.
\end{aligned}$$

Therefore, T is an isometry.

It remains to show that T is onto. For any $g \in C_0^1((0,1), \ell^\infty(2))$, let $g = (g_1(t), g_2(t)) \in \ell^\infty(2)$. Define

$$f(t) = \begin{cases} g_2(t) & \text{if } t \in (0,1) \\ g_1(-t) & \text{if } t \in (-1,0) \end{cases}$$

Then $Tf(t) = (f(-t), f(t)) = (g_1(t), g_2(t)) = g(t)$. Therefore, T is onto. This concludes the proof of the example. $\square$

We note that in the above example, there are very few extreme points in $F^* = \ell^1(2)$,

which are essential in the construction of the homeomorphism between K and Q . This motivates us to restrict our consideration to the type of Banach spaces that satisfy $(*)$.

Recall that a Banach space F^* satisfies $(*)$ if for any $x^*, y^* \in NExt(F^*)$ with $x^* \neq \pm y^*$, there exists $z^* \in \text{ext } B(F^*) \cap \text{span}\{x^*, y^*\}$ with $z^* \notin \{\pm x^*, \pm y^*\}$. Clearly any strictly convex Banach space satisfies $(*)$.

For any $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathbb{R}_r^\Gamma$, and $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in (F_r^\Gamma)^*$, we denote $\beta^* \cdot y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$.

Next, we introduce the notion of bad coordinates, which is essential in constructing a well-defined φ . Fix $t \in K^\Gamma \setminus D$. We say that $\lambda_0 \in \Lambda$ is a *bad coordinate* if there exist $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$, $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, $\gamma_0 \in \Gamma$ and $(s_\lambda)_{\lambda \in \Lambda}$, $(s'_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$ so that if

$$T^*(\beta^* \cdot y^* \circ \delta_t) = x^* \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}} \quad \text{and} \quad T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t = u^* \circ \delta_{(s'_\lambda)_{\lambda \in \Lambda}} \quad (5.2.28)$$

for some $x^*, u^* \in \text{ext } B[(E_\rho^\Lambda)^*]$, then $s_{\lambda_0} \neq s'_{\lambda_0}$. Note that by Proposition 5.14, all components of x^* and u^* are nonzero and $s_\lambda, s'_\lambda \neq \infty$ for all $\lambda \in \Lambda$.

Let Λ_{bad} be the subset of Λ consisting of all bad coordinates and $\Lambda_{good} = \Lambda \setminus \Lambda_{bad}$. Therefore, if $\lambda_0 \in \Lambda_{good}$, then for any $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$, $y^* \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, $\gamma_0 \in \Gamma$ and $(s_\lambda)_{\lambda \in \Lambda}$, $(s'_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$ so that if (5.2.28) hold, then $s_{\lambda_0} = s'_{\lambda_0}$. In other words, a good coordinate λ_0 ensures that the λ_0 component of s is independent of ‘flipping’ any component of $\beta^* \cdot y^*$.

An important consequence of Λ_{good} is given by Corollary 5.22, which asserts that if $\Lambda = \Lambda_{good}$, then φ is well-defined. Proposition 5.21 plays an important role in establishing Corollary 5.22. Before that, we need to establish a preliminary result.

Lemma 5.20. *Fix any $t \in K^\Gamma \setminus D$. If $\lambda_0 \in \Lambda_{good}$, then for all $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, there exists $\gamma_0 \in \Gamma$ such that if*

$$T^*(\beta^* \cdot y^* \circ \delta_t) = (x_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}} \quad \text{and} \\ T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t = (u_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s'_\lambda)_{\lambda \in \Lambda}}$$

for some $(x_\lambda^*)_{\lambda \in \Lambda}, (u_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $(s_\lambda)_{\lambda \in \Lambda}, (s'_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$, then $x_{\lambda_0}^* \neq u_{\lambda_0}^*$.

Proof. Suppose not, that is, there exist $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$ so that for any $\gamma_0 \in \Gamma$, we have

$$T^*(\beta^* \cdot y^* \circ \delta_t) = (x_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}} \quad \text{and} \quad (5.2.29)$$

$$T^*[(\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*] \circ \delta_t = (x_{\gamma_0 \lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^{\gamma_0})_{\lambda \in \Lambda}} \quad (5.2.30)$$

where $x_{\lambda_0}^* = x_{\gamma_0 \lambda_0}^*$. We claim that $\#\Gamma \geq 3$. If $\#\Gamma = 2$, then from (5.2.30), we have

$$T^*(-\beta_0^* y_0^*, \beta_1^* y_1^*) \circ \delta_t = (x_{1\lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^1)_{\lambda \in \Lambda}} \quad \text{and}$$

$$T^*(\beta_0^* y_0^*, -\beta_1^* y_1^*) \circ \delta_t = (x_{2\lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^2)_{\lambda \in \Lambda}}$$

where $x_{1\lambda_0}^* = x_{2\lambda_0}^*$. From equations above, we have $-(x_{1\lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^1)_{\lambda \in \Lambda}} = (x_{2\lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^2)_{\lambda \in \Lambda}}$. Since $-x_{1\lambda_0}^* \neq 0$ and $s_{\lambda_0}^1 \neq \infty$, by Corollary 5.17, we have $-x_{1\lambda_0}^* = x_{2\lambda_0}^*$, implying that $x_{1\lambda_0}^* = 0$. However, since $\beta^* \in \mathcal{G}_S(t)$ and $y^* \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, by Proposition 5.14, we have $x_{1\lambda_0}^* \neq 0$, a contradiction. Therefore, $\#\Gamma \geq 3$.

Now, from (5.2.29) and 5.2.30, since $\#\Gamma \geq 3$, so we have

$$(\#\Gamma - 2)(x_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}} = \sum_{\gamma \in \Gamma} (x_{\gamma\lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^\gamma)_{\lambda \in \Lambda}}. \quad (5.2.31)$$

Since $\lambda_0 \in \Lambda_{good}$, so $s_{\lambda_0} = s_{\lambda_0}^{\gamma_0}$ for all $\gamma_0 \in \Gamma$. Since $\beta^* \in \mathcal{G}_S(t)$ and $y^* \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, by Proposition 5.14, we have $x_{\lambda_0}^* \neq 0$. Then $x_{\lambda_0}^*(x) \neq 0$ for some $x \in E$. By shrinking if necessary, there exists a neighbourhood $U \subseteq K$ of s_{λ_0} such that for all $u \in \{s_\lambda : \lambda \neq \lambda_0\} \cup \{s_\lambda^\gamma : \lambda \neq \lambda_0, \gamma \in \Gamma\}$,

$$\text{either } u = s_{\lambda_0} \quad \text{or} \quad u \notin U. \quad (5.2.32)$$

By Lemma 5.15, there exists $f \in C_0^p(Q, E)$ such that $\text{supp}(f) \subseteq U$, $\partial^{\lambda_0} f(s_{\lambda_0}) = x$ and $\partial^\lambda f(s_{\lambda_0}) = 0$ for all $\lambda \neq \lambda_0$. By 5.2.32, we have

$$\partial^{\lambda_0} f(s_{\lambda_0}) = x, \quad \partial^\lambda f(s_\lambda) = 0 \quad \text{for all } \lambda \neq \lambda_0, \quad (5.2.33)$$

$$\partial^{\lambda_0} f(s_{\lambda_0}^\gamma) = x, \quad \partial^\lambda f(s_\lambda^\gamma) = 0 \quad \text{for all } \lambda \neq \lambda_0, \gamma \in \Gamma. \quad (5.2.34)$$

Then we have

$$\begin{aligned}
\left\langle f, \sum_{\gamma \in \Gamma} x_{\gamma}^* \circ \delta_{s\gamma} \right\rangle &= \sum_{\gamma \in \Gamma} \langle f, x_{\gamma}^* \circ \delta_{s\gamma} \rangle && \text{by linearity of summation} \\
&= \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} x_{\gamma\lambda}^* (\partial^{\lambda} f(s_{\lambda})) && \text{by definition of } \langle \cdot, \cdot \rangle \\
&= \sum_{\gamma \in \Gamma} x_{\gamma\lambda_0}^* (x) && \text{by (5.2.34)} \\
&= \sum_{\gamma \in \Gamma} x_{\lambda_0}^* (x) && \text{since } x_{\gamma\lambda_0}^* = x_{\lambda_0}^* \text{ for all } \gamma \in \Gamma \\
&= (\#\Gamma) x_{\lambda_0}^* (x). && (5.2.35)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\#\Gamma - 2) \langle f, x^* \circ \delta_s \rangle &= (\#\Gamma - 2) \sum_{\lambda \in \Lambda} x_{\lambda}^* (\partial^{\lambda} f(s_{\lambda})) && \text{by definition of } \langle \cdot, \cdot \rangle \\
&= (\#\Gamma - 2) x_{\lambda_0}^* (x) && \text{by (5.2.33).} \quad (5.2.36)
\end{aligned}$$

By (5.2.31), (5.2.35) and (5.2.36), we have $(\#\Gamma) x_{\lambda_0}^* (x) = (\#\Gamma - 2) x_{\lambda_0}^* (x)$. However, $x_{\lambda_0}^* (x) \neq 0$, a contradiction. This concludes the proof of the lemma. $\square$

Henceforth, for each $\lambda_0 \in \Lambda$, $@\lambda_0$ refers to the λ_0 coordinate. For example, $(x_{\lambda}^*)_{\lambda \in \Lambda} \circ \delta_{(s_{\lambda})_{\lambda \in \Lambda}} @\lambda_0 = (x_{\lambda_0}^*, s_{\lambda_0})$.

Proposition 5.21. *Assume that F^* satisfies $(*)$. Fix $t \in K^{\Gamma} \setminus D$. For any $\lambda_0 \in \Lambda_{\text{good}}$, there exists $s_0 \in Q$ such that for all $\beta^* = (\beta_{\gamma}^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $y^* \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, if $T^*(\beta^* \cdot y^* \circ \delta_t) = x^* \circ \delta_{(s_{\lambda})_{\lambda \in \Lambda}}$ for some $x^* \in \text{ext } B[(E_s^{\Lambda})^*]$ and $(s_{\lambda})_{\lambda \in \Lambda} \in Q^{\Lambda}$, then $s_{\lambda_0} = s_0$.*

Proof. Fix $t \in K^{\Gamma} \setminus D$ and $\lambda_0 \in \Lambda_{\text{good}}$. Take any $\beta^* = (\beta_{\gamma}^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $y^* \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$. Choose $\gamma_0 \in \Gamma$ as in Lemma 5.20. By choice of γ_0 , we have

$$\begin{aligned}
T^*(\beta^* \cdot y^* \circ \delta_t) @\lambda_0 &= (x_{\lambda_0}^*, s_{\lambda_0}) \\
T^*[(\beta_{\gamma}^* y_{\gamma}^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*] \circ \delta_t @\lambda_0 &= (u_{\lambda_0}^*, s_{\lambda_0}),
\end{aligned}$$

with $x_{\lambda_0}^* \neq u_{\lambda_0}^*$. Adding and subtracting equations above give

$$\begin{aligned} T^*[((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t] @ \lambda_0 &= \left(\frac{x_{\lambda_0}^* + u_{\lambda_0}^*}{2}, s_{\lambda_0} \right) \\ T^*[((0)_{\gamma \neq \gamma_0}, y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 &= \left(\frac{x_{\lambda_0}^* - u_{\lambda_0}^*}{2\beta_{\gamma_0}^*}, s_{\lambda_0} \right). \end{aligned} \quad (5.2.37)$$

Also, take any $\alpha^* = (\alpha_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $z^* = (z_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} \text{Next}(F^*)$. Then

$$\begin{aligned} T^*[\alpha^* \cdot z^* \circ \delta_t] @ \lambda_0 &= (c_{\lambda_0}^*, s_{\lambda_0}'') \\ T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, -\alpha_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 &= (d_{\lambda_0}^*, s_{\lambda_0}'') \end{aligned}$$

for some $c_{\lambda_0}^*, d_{\lambda_0}^* \in E^* \setminus \{0\}$ and $s_{\lambda_0}'' \in Q$. We claim that $s_{\lambda_0}'' = s_{\lambda_0}$. Adding and subtracting equations above give

$$T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t] @ \lambda_0 = \left(\frac{c_{\lambda_0}^* + d_{\lambda_0}^*}{2}, s_{\lambda_0}'' \right) \quad (5.2.38)$$

$$T^*[((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 = \left(\frac{c_{\lambda_0}^* - d_{\lambda_0}^*}{2}, s_{\lambda_0}'' \right). \quad (5.2.39)$$

On the other hand, since $((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* y_{\gamma_0}^*)$ and $((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, -\alpha_{\gamma_0}^* y_{\gamma_0}^*)$ satisfy assumption of Proposition 5.14, so

$$\begin{aligned} T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 &= (a_{\lambda_0}^*, s_{\lambda_0}') \\ T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, -\alpha_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 &= (b_{\lambda_0}^*, s_{\lambda_0}') \end{aligned}$$

for some $a_{\lambda_0}^*, b_{\lambda_0}^* \in E^*$ and $s_{\lambda_0}' \in Q$. Adding and subtracting equations above give

$$T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t] @ \lambda_0 = \left(\frac{a_{\lambda_0}^* + b_{\lambda_0}^*}{2}, s_{\lambda_0}' \right) \quad (5.2.40)$$

$$T^*[((0)_{\gamma \neq \gamma_0}, y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 = \left(\frac{a_{\lambda_0}^* - b_{\lambda_0}^*}{2\alpha_{\gamma_0}^*}, s_{\lambda_0}' \right). \quad (5.2.41)$$

From (5.2.37) and 5.2.41, since $x_{\lambda_0}^* - u_{\lambda_0}^* \neq 0$, by Corollary 5.17, we have

$$s_{\lambda_0}' = s_{\lambda_0} \quad \text{and} \quad a_{\lambda_0}^* - b_{\lambda_0}^* \neq 0.$$

Now, we consider two cases, either $a_{\lambda_0}^* + b_{\lambda_0}^* \neq 0$ or $a_{\lambda_0}^* + b_{\lambda_0}^* = 0$.

Assume that $a_{\lambda_0}^* + b_{\lambda_0}^* \neq 0$. Then from (5.2.40) and 5.2.38, Corollary 5.17 implies that $s_{\lambda_0}'' = s_{\lambda_0}' = s_{\lambda_0}$.

Next, consider the case where $a_{\lambda_0}^* + b_{\lambda_0}^* = 0$. It follows from (5.2.40) and $s_{\lambda_0}' = s_{\lambda_0}$ that

$$T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t] @ \lambda_0 = (0, s_{\lambda_0}). \quad (5.2.42)$$

Consequently, (5.2.38) and 5.2.42 imply that $c_{\lambda_0}^* + d_{\lambda_0}^* = 0$. So, $c_{\lambda_0}^* - d_{\lambda_0}^* \neq 0$. Assume that $z_{\gamma_0}^* = \pm y_{\gamma_0}^*$. Since $s_{\lambda_0} = s_{\lambda_0}'$, from (5.2.41),

$$T^*[((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 = T^*[((0)_{\gamma \neq \gamma_0}, \pm \alpha_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 = \pm \left(\frac{a_{\lambda_0}^* - b_{\lambda_0}^*}{2}, s_{\lambda_0}' \right), \quad (5.2.43)$$

Since $c_{\lambda_0}^* - d_{\lambda_0}^* \neq 0$, by Corollary 5.17, comparing (5.2.43) and 5.2.39 gives $s_{\lambda_0}'' = s_{\lambda_0}' = s_{\lambda_0}$. Otherwise, assume that $z_{\gamma_0}^* \neq \pm y_{\gamma_0}^*$. Since $y_{\gamma_0}^*, z_{\gamma_0}^* \in NExt(F^*)$ and F^* satisfies (*), there exists $w_{\gamma_0}^* \in \text{ext } B(F^*)$ such that $w_{\gamma_0}^* = \alpha y_{\gamma_0}^* + \beta z_{\gamma_0}^*$ where $\alpha, \beta \neq 0$. By Proposition 5.11, we have

$$T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* w_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 = (f_{\lambda_0}^*, s_{\lambda_0}''') \quad (5.2.44)$$

for some $f_{\lambda_0}^* \in E^*$ and $s_{\lambda_0}''' \in Q_\infty$. Observe that

$$\begin{aligned} & T^*[((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* w_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 \\ &= T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* w_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 \\ &= T^*[((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t] @ \lambda_0 \quad \text{by linearity of } T^* \\ &= (f_{\lambda_0}^*, s_{\lambda_0}''') - (0, s_{\lambda_0}) \quad \text{by (5.2.44) and 5.2.42} \\ &= (f_{\lambda_0}^*, s_{\lambda_0}'''). \end{aligned} \quad (5.2.45)$$

Then

$$\begin{aligned} & (f_{\lambda_0}^*, s_{\lambda_0}''') \\ &= T^*[((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* (\alpha y_{\gamma_0}^* + \beta z_{\gamma_0}^*)) \circ \delta_t] @ \lambda_0 \quad \text{by (5.2.45)} \\ &= \alpha T^*[((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 \\ &+ \beta T^*[((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 \quad \text{by linearity of } T^* \\ &= \alpha \left(\frac{a_{\lambda_0}^* - b_{\lambda_0}^*}{2}, s_{\lambda_0} \right) + \beta \left(\frac{c_{\lambda_0}^* - d_{\lambda_0}^*}{2}, s_{\lambda_0}'' \right). \quad \text{by (5.2.41) and 5.2.39} \end{aligned}$$

Since $a_{\lambda_0}^* - b_{\lambda_0}^*, c_{\lambda_0}^* - d_{\lambda_0}^*$, α and β are nonzero, and $s_{\lambda_0}, s_{\lambda_0}''$ are not infinity, by Corollary 5.18, we have $s_{\lambda_0} = s_{\lambda_0}''$. This concludes the proof of the proposition. $\square$

The following corollary follows immediately from Corollary 5.13 and Proposition 5.21.

Corollary 5.22. *Assume that F^* satisfies $(*)$. For any $t \in K^\Gamma \setminus D$, if $\Lambda_{bad} = \emptyset$, then there exists $s \in Q^\Lambda$ such that for any $y^* \in (F_\theta^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in (E_\rho^\Lambda)^*$.*

By Corollary 5.22, it remains to show that $\Lambda_{bad} = \emptyset$ for all $t \in K^\Gamma \setminus D$, which is asserted by Proposition 5.26. To do so, we need to establish several preparatory results.

Lemma 5.23. *Assume that $a_k^* = (a_{k\lambda}^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s^k = (s_\lambda^k)_{\lambda \in \Lambda} \in Q_\infty^\Lambda$ for all $k = 1, 2, 3, 4$. Suppose that*

$$a_1^* \circ \delta_{s^1} + a_2^* \circ \delta_{s^2} = a_3^* \circ \delta_{s^3} + a_4^* \circ \delta_{s^4}. \quad (5.2.46)$$

If there exists $\lambda_0 \in \Lambda$ such that $a_{1\lambda_0}^, a_{2\lambda_0}^*$ are nonzero, $s_{\lambda_0}^1, s_{\lambda_0}^2 \neq \infty$ and $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$, then either*

1. $(a_{1\lambda_0}^*, s_{\lambda_0}^1) = (a_{3\lambda_0}^*, s_{\lambda_0}^3)$ and $(a_{2\lambda_0}^*, s_{\lambda_0}^2) = (a_{4\lambda_0}^*, s_{\lambda_0}^4)$ or
2. $(a_{1\lambda_0}^*, s_{\lambda_0}^1) = (a_{4\lambda_0}^*, s_{\lambda_0}^4)$ and $(a_{2\lambda_0}^*, s_{\lambda_0}^2) = (a_{3\lambda_0}^*, s_{\lambda_0}^3)$.

Proof. Note that (5.2.46) becomes $a_1^* \circ \delta_{s^1} = -a_2^* \circ \delta_{s^2} + a_3^* \circ \delta_{s^3} + a_4^* \circ \delta_{s^4}$. Since $a_{1\lambda_0}^* \neq 0$ and $s_{\lambda_0}^1 \neq \infty$, by Lemma 5.16, we have $s_{\lambda_0}^1 \in \{s_{\lambda_0}^2, s_{\lambda_0}^3, s_{\lambda_0}^4\}$. Since $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$, so $s_{\lambda_0}^1 \in \{s_{\lambda_0}^3, s_{\lambda_0}^4\}$. Without loss of generality, we may assume that $s_{\lambda_0}^1 = s_{\lambda_0}^3$.

Note that (5.2.46) becomes $a_2^* \circ \delta_{s^2} = -a_1^* \circ \delta_{s^1} + a_3^* \circ \delta_{s^3} + a_4^* \circ \delta_{s^4}$. Since $a_{2\lambda_0}^* \neq 0$ and $s_{\lambda_0}^2 \neq \infty$, by Lemma 5.16 again, we have $s_{\lambda_0}^2 \in \{s_{\lambda_0}^1, s_{\lambda_0}^3, s_{\lambda_0}^4\}$. But $s_{\lambda_0}^1 = s_{\lambda_0}^3$ and $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$. Therefore, $s_{\lambda_0}^2 = s_{\lambda_0}^4$.

Now, we claim that $a_{1\lambda_0}^* = a_{3\lambda_0}^*$. Suppose not. Then $a_{1\lambda_0}^*(a) \neq a_{3\lambda_0}^*(a)$ for some $a \in E$. Choose $f \in C_0^p(Q, E)$ such that

$$\begin{aligned} \partial^{\lambda_0} f(s_{\lambda_0}^1) &= a, \quad \partial^\lambda f(s_\lambda^1) = 0 = \partial^\lambda f(s_\lambda^3) \quad \text{for all } \lambda \neq \lambda_0 \quad \text{and} \\ \partial^\lambda f(s_\lambda^2) &= 0 = \partial^\lambda f(s_\lambda^4) \quad \text{for all } \lambda \in \Lambda. \end{aligned}$$

However, $\langle f, a_1^* \circ \delta_{s^1} \rangle + \langle f, a_2^* \circ \delta_{s^2} \rangle = a_{1\lambda_0}^*(a)$ and $\langle f, a_3^* \circ \delta_{s^3} \rangle + \langle f, a_4^* \circ \delta_{s^4} \rangle = a_{3\lambda_0}^*(a) \neq a_{1\lambda_0}^*(a)$, contradicting (5.2.46). Therefore, $a_{1\lambda_0}^* = a_{3\lambda_0}^*$. Similarly, one can show that $a_{2\lambda_0}^* = a_{4\lambda_0}^*$. This concludes the proof of the lemma. $\square$

Lemma 5.24. Fix any $t \in K^\Gamma \setminus D$ and $\lambda_0 \in \Lambda_0$. Suppose that there exist $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$, $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, $\gamma_0 \in \Gamma$ and $s^1 = (s_\lambda^1)_{\lambda \in \Lambda}$, $s^2 = (s_\lambda^2)_{\lambda \in \Lambda} \in Q^\Lambda$ such that

$$T^*(\beta^* \cdot y^* \circ \delta_t) = x_1^* \circ \delta_{s^1}, \quad \text{and} \quad T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t = x_2^* \circ \delta_{s^2} \quad (5.2.47)$$

for some $x_1^* = (x_{1\lambda}^*)_{\lambda \in \Lambda}$, $x_2^* = (x_{2\lambda}^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ with $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$ and $x_{1\lambda_0}^*, x_{2\lambda_0}^* \neq 0$. Then for any $z^* \in \text{ext } B(F^*)$, we have

$$T^*((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* z^*) \circ \delta_t @ \lambda_0 = \pm \frac{1}{2}[(x_{1\lambda_0}^*, s_{\lambda_0}^1) - (x_{2\lambda_0}^*, s_{\lambda_0}^2)]. \quad (5.2.48)$$

Proof. Fix any $z^* \in \text{ext } B(F^*)$. By Proposition 5.11, we have

$$T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* z^*) \circ \delta_t = u^* \circ \delta_{s^3} \quad \text{and} \quad T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* z^*) \circ \delta_t = v^* \circ \delta_{s^4}$$

for some $u^* = (u_\lambda^*)_{\lambda \in \Lambda}$, $v^* = (v_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s^3 = (s_\lambda^3)_{\lambda \in \Lambda}$, $s^4 = (s_\lambda^4)_{\lambda \in \Lambda} \in Q_\infty^\Lambda$. Adding and subtracting equations above give us

$$T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t = \frac{1}{2}(u^* \circ \delta_{s^3} + v^* \circ \delta_{s^4}) \quad \text{and} \quad (5.2.49)$$

$$T^*((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* z^*) \circ \delta_t = \frac{1}{2}(u^* \circ \delta_{s^3} - v^* \circ \delta_{s^4}). \quad (5.2.50)$$

Also, adding (5.2.47) gives

$$T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, 0) \circ \delta_t = \frac{1}{2}(x_1^* \circ \delta_{s^1} + x_2^* \circ \delta_{s^2}). \quad (5.2.51)$$

By (5.2.49) and (5.2.51), we have $x_1^* \circ \delta_{s^1} + x_2^* \circ \delta_{s^2} = u^* \circ \delta_{s^3} + v^* \circ \delta_{s^4}$. Since $x_{1\lambda_0}^*, x_{2\lambda_0}^*$ are nonzero, $s_{\lambda_0}^1, s_{\lambda_0}^2 \neq \infty$ and $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$, by Lemma 5.23, either

1. $(u_{\lambda_0}^*, s_{\lambda_0}^3) = (x_{1\lambda_0}^*, s_{\lambda_0}^1)$ and $(v_{\lambda_0}^*, s_{\lambda_0}^4) = (x_{2\lambda_0}^*, s_{\lambda_0}^2)$ or
2. $(u_{\lambda_0}^*, s_{\lambda_0}^3) = (x_{2\lambda_0}^*, s_{\lambda_0}^2)$ and $(v_{\lambda_0}^*, s_{\lambda_0}^4) = (x_{1\lambda_0}^*, s_{\lambda_0}^1)$.

In either case, (5.2.50) implies that Equation 5.2.48 holds. This concludes the proof of the lemma. $\square$

The following lemma justifies the assumption $F \neq \mathbb{R}$ in this section. We will use it in the proof of Proposition 5.26.

Lemma 5.25. *Assume that $F \neq \mathbb{R}$. Then for any nonzero linear functional $y^* : F \rightarrow \mathbb{R}$, there exist $z^* \in \text{NExt}(F^*)$ and $y \in F$ with $y \neq 0$ such that $z^*(y) \neq 0$ but $y^*(y) = 0$. In particular, $z^* \neq \pm y^*$.*

Proof. Since $y^* : F \rightarrow \mathbb{R}$ is a nonzero linear functional and $F \neq \mathbb{R}$, there exists $y \in F$ with $y \neq 0$ such that $y^*(y) = 0$. By Lemma 2.6, there exists $z^* \in \text{ext } B(F^*)$ such that $z^*(y) = \|y\|$. Clearly $z^* \neq \pm y^*$. Otherwise, $\|y\| = z^*(y) = \pm y^*(y) = 0$, a contradiction. This concludes the proof of the lemma. $\square$

Proposition 5.26. *If $F \neq \mathbb{R}$ and F^* satisfies $(*)$, then $\Lambda_{\text{bad}} = \emptyset$ for all $t \in K^\Gamma \setminus D$.*

Proof. Fix $t \in K^\Gamma \setminus D$ and suppose that $\lambda_0 \in \Lambda_{\text{bad}}$. Then there exist $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$, $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} \text{NExt}(F^*)$, $\gamma_0 \in \Gamma$ and $s = (s_\lambda)_{\lambda \in \Lambda}$, $s' = (s'_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$ so that

$$T^*(\beta^* \cdot y^* \circ \delta_t) = x^* \circ \delta_s, \quad \text{and} \quad (5.2.52)$$

$$T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t = u^* \circ \delta_{s'} \quad (5.2.53)$$

for some $x^* = (x_\lambda^*)_{\lambda \in \Lambda}$, $u^* = (u_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ with $x_{\lambda_0}^*, u_{\lambda_0}^* \neq 0$ and $s_{\lambda_0} \neq s'_{\lambda_0}$. Subtracting (5.2.53) from (5.2.52) above gives

$$T^*[((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 = \frac{1}{2}[(x_{\lambda_0}^*, s_{\lambda_0}) - (u_{\lambda_0}^*, s'_{\lambda_0})]. \quad (5.2.54)$$

Since $F \neq \mathbb{R}$ and $y_{\gamma_0}^*$ is a nonzero linear functional on F , by Lemma 5.25, there exists $z^* \in \text{NExt}(F^*) \setminus \{\pm y_{\gamma_0}^*\}$. Since F^* satisfies $(*)$, there exists $w^* \in \text{ext } B(F^*)$ such that

$$w^* = \alpha y_{\gamma_0}^* + \beta z^* \quad \text{where} \quad \alpha, \beta \neq 0. \quad (5.2.55)$$

Also, since $w^*, z^* \in \text{ext } B(F^*)$, by Lemma 5.24, we have

$$T^*[((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* z^*) \circ \delta_t] @ \lambda_0 = \pm \frac{1}{2}[(x_{\lambda_0}^*, s_{\lambda_0}) - (u_{\lambda_0}^*, s'_{\lambda_0})] \quad \text{and} \quad (5.2.56)$$

$$T^*[((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* w^*) \circ \delta_t] @ \lambda_0 = \pm \frac{1}{2}[(x_{\lambda_0}^*, s_{\lambda_0}) - (u_{\lambda_0}^*, s'_{\lambda_0})]. \quad (5.2.57)$$

Observe that

$$\begin{aligned}
& \pm \frac{1}{2}[(x_{\lambda_0}^*, s_{\lambda_0}) - (u_{\lambda_0}^*, s'_{\lambda_0})] \\
&= T^*[((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* w^*) \circ \delta_t] @ \lambda_0 && \text{by (5.2.57)} \\
&= \alpha T^*[((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t] @ \lambda_0 \\
&+ \beta T^*[((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* z^*) \circ \delta_t] @ \lambda_0 && \text{by (5.2.55)} \\
&= \frac{1}{2}(\alpha \pm \beta)[(x_{\lambda_0}^*, s_{\lambda_0}) - (u_{\lambda_0}^*, s'_{\lambda_0})] && \text{by (5.2.54) and (5.2.56).}
\end{aligned}$$

Therefore, $\alpha \pm \beta = \pm 1$. Assume that $\alpha > 0$ and $\beta > 0$. We consider the 4 cases.

Case 1: $\alpha + \beta = 1$: Since $\alpha, \beta > 0$, by (5.2.55) and extremity of w^* , we have $w^* = y_{\gamma_0}^* = z^*$, contradicting $z^* \neq \pm y_{\gamma_0}^*$.

Case 2: $\alpha - \beta = 1$: Since $\alpha \neq 0$, so $\beta \neq -1$. Then (5.2.55) implies that $y_{\gamma_0}^* = \frac{1}{\beta+1}w^* + \frac{\beta}{\beta+1}(-z^*)$. Since $\beta > 0$ and $y_{\gamma_0}^*$ is an extreme point, so $y_{\gamma_0}^* = w^* = -z^*$, contradicting $z^* \neq \pm y_{\gamma_0}^*$.

Case 3: $\alpha + \beta = -1$: This is impossible as $\alpha, \beta > 0$.

Case 4: $\alpha - \beta = -1$: If $\beta > 1$, then (5.2.55) implies that $z^* = \frac{1}{\beta}w^* + \frac{\beta-1}{\beta}(-y_{\gamma_0}^*)$. Since $\beta > 1$ and z^* is an extreme point, so $z^* = w^* = -y_{\gamma_0}^*$, contradicting $z^* \neq \pm y_{\gamma_0}^*$. Since $\alpha \neq 0$, so $\beta \neq 1$. If $0 < \beta < 1$, then $w^* = (1-\beta)(-y_{\gamma_0}^*) + \beta z^*$. By similar reasonings above, $w^* = -y_{\gamma_0}^* = z^*$, contradicting $z^* \neq \pm y_{\gamma_0}^*$.

If $\alpha < 0$ and $\beta < 0$, then replacing α and β with $-\alpha$ and $-\beta$ respectively in arguments above, we obtain a contradiction. Assume that $\alpha > 0$ and $\beta < 0$. Replacing β with $-\beta$ in arguments above, we also obtain a contradiction. This concludes the proof of the proposition. $\square$

The following corollary is the main result of this section, which follows immediately from Corollary 5.22 and Proposition 5.26.

Corollary 5.27. *Let $T : (C_0^p(Q, E), \|\cdot\|_{S,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,\theta})$ be a surjective linear isometry, where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. If $F \neq \mathbb{R}$ and F^* satisfies (*), then there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_\theta^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_\rho^\Lambda)^*$.*

5.2.2 Case 2: $F = \mathbb{R}$, θ is smooth and strictly convex

The main result of this section is Proposition 5.32, which asserts that if θ is smooth and strictly convex and $F = \mathbb{R}$, then φ is well-defined. In particular, this applies to $\theta = \|\cdot\|_r$ where $r \in (1, \infty)$. Recall that a Banach space X is *smooth* if for each $x \in X$ with $x \neq 0$, there exists a unique element $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Also, X is *strictly convex* if for any $x, y \in X$, we have $\|x + y\| < \|x\| + \|y\|$ whenever x and y are not collinear.

From Proposition 5.8, for any $t \in K^\Gamma \setminus D$, there exist $(\zeta_\gamma)_{\gamma \in \Gamma}$ and $(\eta_\gamma)_{\gamma \in \Gamma}$ in $\mathbb{R}^\Gamma$ with $0 < \zeta_\gamma < \eta_\gamma$ such that

$$U = \prod_{\gamma \in \Gamma} (\zeta_\gamma, \eta_\gamma) \cap S(\mathbb{R}_\theta^\Gamma) \quad (5.2.58)$$

is a nonempty subset of $\mathcal{P}_S(t)$. Let $h : \mathbb{R}_\theta^\Gamma \setminus \{0\} \rightarrow \mathbb{R}_{\theta^*}^\Gamma \setminus \{0\}$ be the duality mapping, that is,

$$h(x) = x^* \quad (5.2.59)$$

where $x^*(x) = \|x\|_\theta$. From smoothness of θ , h is single-valued. Since $\mathbb{R}_\theta^\Gamma$ is reflexive, smooth and strictly convex, so is $\mathbb{R}_{\theta^*}^\Gamma$ [19, page 184, Corollary 2]. Therefore, h is a norm-to-weak* continuous bijection [19, page 177, Proposition 1]. Since $\mathbb{R}_\theta^\Gamma$ is smooth and reflexive, so h^{-1} is continuous. Clearly $h[S(\mathbb{R}_\theta^\Gamma)] \subseteq S(\mathbb{R}_{\theta^*}^\Gamma)$.

Lemma 5.28. *Assume that θ is smooth and strictly convex. For any $t \in K^\Gamma \setminus D$, we have $h(\mathcal{P}_S(t)) \subseteq \mathcal{G}_S(t)$ where h is given by (5.2.59) above.*

Proof. Fix any $t \in K^\Gamma \setminus D$ and $\alpha = (\alpha_\gamma)_{\gamma \in \Gamma} \in \mathcal{P}_S(t)$. Since $\alpha \in S(\mathbb{R}_\theta^\Gamma)$ and $h[S(\mathbb{R}_\theta^\Gamma)] \subseteq S(\mathbb{R}_{\theta^*}^\Gamma)$, so $h(\alpha) \in S(\mathbb{R}_{\theta^*}^\Gamma)$. Since θ^* is strict convexity, so $h(\alpha) \in \text{ext } B(\mathbb{R}_{\theta^*}^\Gamma)$. By definition of h , clearly $\langle \alpha, h(\alpha) \rangle = 1$. Since $\alpha \in \mathcal{P}_S(t)$, so $h(\alpha) \in \mathcal{G}_S(t)$. This concludes the proof of the lemma. $\square$

Lemma 5.29. *Assume that θ is smooth, strictly convex and $F = \mathbb{R}$. Let $\alpha \in U$ and $\alpha^* = h(\alpha) = (\alpha_\gamma^*)_{\gamma \in \Gamma}$ where U is given by (5.2.58). Pick any two distinct γ_1 and γ_2 in Γ and set*

$$A = \{\beta^* = (\beta_\gamma^*) \in S(\mathbb{R}_{\theta^*}^\Gamma) : \beta_\gamma^* = \alpha_\gamma^* \text{ if } \gamma \neq \gamma_1, \gamma_2\}. \quad (5.2.60)$$

Then for any $t \in K^\Gamma \setminus D$, there exist $s \in Q^\Lambda$ and a neighbourhood V of α^ such that for any $\beta^* \in V \cap A$, we have $T^*(\beta^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$.*

Proof. Let $g : \mathbb{R}_{\theta^*}^\Gamma \setminus \{-\alpha^*\} \rightarrow S(\mathbb{R}_{\theta^*}^\Gamma)$ be defined by $g(\beta^*) = \frac{\alpha^* + \beta^*}{\|\alpha^* + \beta^*\|}$. Note that g is continuous. Since $g(\alpha^*) = \alpha^* = h(\alpha) \in h(U)$ and $h(U)$ is open, by continuity of g , there exists a neighbourhood V of α^* with $V \subseteq h(U)$ such that $g(V) \subseteq h(U)$. Choose $\beta^* \in V \cap A$ with $\beta^* \neq \alpha^*$. By Lemma 5.28, we have $\alpha^*, \beta^*, g(\beta^*) \in h(U) \subseteq \mathcal{G}_S(t)$. Since α^* and β^* are in $\mathcal{G}_S(t)$, so by Proposition 5.14, we have

$$T^*(\alpha^* \circ \delta_t) = x^* \circ \delta_s \quad \text{and} \quad T^*(\beta^* \circ \delta_t) = z^* \circ \delta_u, \quad (5.2.61)$$

for some extreme points $x^*, z^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s, u \in Q^\Lambda$ where all components of x^* and z^* are nonzero. It remains to show that $s = u$. Observe that

$$\begin{aligned} & T^*(g(\beta^*) \circ \delta_t) \\ &= \frac{1}{\|\alpha^* + \beta^*\|} T^*[(\alpha^* + \beta^*) \circ \delta_t] \quad \text{by definition of } g \\ &= \frac{1}{\|\alpha^* + \beta^*\|} (x^* \circ \delta_s + z^* \circ \delta_u) \quad \text{by (5.2.61).} \end{aligned} \quad (5.2.62)$$

Since $g(\beta^*) \in \mathcal{G}_S(t)$, so by Proposition 5.14, we have

$$T^*(g(\beta^*) \circ \delta_t) = \bar{x}^* \circ \delta_{\bar{s}} \quad (5.2.63)$$

for some $\bar{x}^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $\bar{s} \in Q^\Lambda$ where all components of $\bar{x}^*$ are nonzero and $\bar{s}$ are not infinity. It follows from (5.2.62) and (5.2.63) that

$$\bar{x}^* \circ \delta_{\bar{s}} = \frac{1}{\|\alpha^* + \beta^*\|} (x^* \circ \delta_s + z^* \circ \delta_u).$$

Since all components of $\bar{x}^*, x^*$ and z^* are nonzero and $\bar{s}, s$ and u are not infinity, so it follows from Corollary 5.18 that $\bar{s} = s = u$. This concludes the proof of the lemma. $\square$

Proposition 5.30. *Assume that θ is smooth and strictly convex and $F = \mathbb{R}$. For any $t \in K^\Gamma \setminus D$, there exists $s \in Q^\Lambda$ such that for any $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathbb{R}_{\theta^*}^\Gamma$, we have $T^*(\beta^* \circ \delta_t) = x^* \circ \delta_s$ for some $x^* \in (E_\rho^\Lambda)^*$.*

Proof. Fix any $t \in K^\Gamma \setminus D$. Pick any $\alpha \in U$ and denote $\alpha^* = h(\alpha) = (\alpha_\gamma^*)_{\gamma \in \Gamma}$. Fix any distinct γ_1 and γ_2 in Γ . By Lemma 5.29, there exist $s \in Q^\Lambda$ and a

neighbourhood V of α^* such that for any $\beta^* \in V \cap A$,

$$T^*(\beta^* \circ \delta_t) = x^* \circ \delta_s \quad \text{for some } x^* \in \text{ext } B[(E_s^\Lambda)^*]. \quad (5.2.64)$$

where the set A is given by (5.2.60). Pick any $(a, b) \in \mathbb{R}^2$ where

$$[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (a, b)] \in V \cap A \quad \text{and} \quad \theta^*[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (a, b)] = \theta^*[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (\alpha_{\gamma_1}^*, \alpha_{\gamma_2}^*)].$$

Then (5.2.64) implies that

$$T^* \left(\left[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (a, b) \right] \circ \delta_t \right) = x_1^* \circ \delta_s \quad (5.2.65)$$

for some $x_1^* \in \text{ext } B[(E_s^\Gamma)^*]$. Clearly $\alpha^* \in V \cap A$. By (5.2.64) again,

$$T^* \left(\left[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (\alpha_{\gamma_1}^*, \alpha_{\gamma_2}^*) \right] \circ \delta_t \right) = x_2^* \circ \delta_s \quad (5.2.66)$$

for some $x_2^* \in \text{ext } B[(E_s^\Gamma)^*]$. Subtracting (5.2.66) from (5.2.65) gives

$$T^* \left(\left[(0)_{\gamma \neq \gamma_1, \gamma_2}, (a - \alpha_{\gamma_1}^*, b - \alpha_{\gamma_2}^*) \right] \circ \delta_t \right) = (x_1^* - x_2^*) \circ \delta_s. \quad (5.2.67)$$

Observe that there exists $(a', b') \in \mathbb{R}^2$ with $(a', b') \neq (a, b)$ such that

$$\theta^*[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (a', b')] = \theta^*[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (\alpha_{\gamma_1}^*, \alpha_{\gamma_2}^*)], \quad [(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (a', b')] \in V \cap A$$

and $\{(a - \alpha_{\gamma_1}^*, b - \alpha_{\gamma_2}^*), (a' - \alpha_{\gamma_1}^*, b' - \alpha_{\gamma_2}^*)\}$ is linearly independent in $\mathbb{R}^2$. Since $\theta^*[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (a', b')] = \theta^*[(\alpha_\gamma^*)_{\gamma \neq \gamma_1, \gamma_2}, (\alpha_{\gamma_1}^*, \alpha_{\gamma_2}^*)]$, so replacing (a, b) above with (a', b') gives

$$T^* \left(\left[(0)_{\gamma \neq \gamma_1, \gamma_2}, (a' - \alpha_{\gamma_1}^*, b' - \alpha_{\gamma_2}^*) \right] \circ \delta_t \right) = w^* \circ \delta_s. \quad (5.2.68)$$

for some $w^* \in (E_s^\Lambda)^*$. Also, since $\{(a - \alpha_{\gamma_1}^*, b - \alpha_{\gamma_2}^*), (a' - \alpha_{\gamma_1}^*, b' - \alpha_{\gamma_2}^*)\}$ is linearly independent, so

$$(1, 0), (0, 1) \in \text{span}\{(a - \alpha_{\gamma_1}^*, b - \alpha_{\gamma_2}^*), (a' - \alpha_{\gamma_1}^*, b' - \alpha_{\gamma_2}^*)\}. \quad (5.2.69)$$

It follows from (5.2.67), (5.2.68) and (5.2.69) that

$$\begin{aligned} T^* [((0)_{\gamma \neq \gamma_1, \gamma_2}, (1, 0)) \circ \delta_t] &= w_1^* \circ \delta_s \quad \text{and} \\ T^* [((0)_{\gamma \neq \gamma_1, \gamma_2}, (0, 1)) \circ \delta_t] &= w_2^* \circ \delta_s. \end{aligned}$$

for some $w_1^*, w_2^* \in (E_\rho^\Lambda)^*$. Since $\gamma_1, \gamma_2 \in \Gamma$ are distinct and arbitrary, this concludes the proof of the proposition. $\square$

The following proposition is the main result of this section, which follows immediately from Corollary 5.13 and Proposition 5.30.

Proposition 5.31. *Let $T : (C_0^p(Q, E), \|\cdot\|_{S,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,\theta})$ be a surjective linear isometry, where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. If $F = \mathbb{R}$ and θ is smooth and strictly convex, then there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in \mathbb{R}_{\theta^*}^\Gamma$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_\rho^\Lambda)^*$.*

If $\theta = \|\cdot\|_r$ where $r \in (1, \infty)$, then the following corollary is a special case of proposition above.

Corollary 5.32. *Let $T : (C_0^p(Q, E), \|\cdot\|_{S,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,\theta})$ be a surjective linear isometry, where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. If $F = \mathbb{R}$ and $\theta = \|\cdot\|_r$ where $r \in (1, \infty)$, then there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in \mathbb{R}_{\theta^*}^\Gamma$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_\rho^\Lambda)^*$.*

5.2.3 Case 3: $F = \mathbb{R}$, $E \neq \mathbb{R}$ and E^* satisfies $(*)$

The main result of this section is Corollary 5.39, which asserts that if $F = \mathbb{R}$, $E \neq \mathbb{R}$ and E^* satisfies $(*)$, then φ is well-defined.

Recall that $\leq$ is an ordering on Λ that satisfies that for any $\mu = (\mu_i)_{i=1}^m$ and $\lambda = (\lambda_i)_{i=1}^m$ in Λ , $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq m$. We denote $\mu > \lambda$ if $\mu \not\leq \lambda$. The following Lemma 5.33 and Proposition 5.34 do not require any assumption on E and F .

Lemma 5.33. *Assume that $x_n^* = (x_{n\lambda}^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s_n = (s_{n\lambda})_{\lambda \in \Lambda} \in Q^\Lambda$ for all $n \geq 1$. Suppose that $(x_n^* \circ \delta_{s_n})_{n \geq 1}$ has a weak-star limit and $(s_n)_{n \geq 1}$ is convergent in Q_∞^Λ . Then for each $\lambda \in \Lambda$, if $\lim_{n \rightarrow \infty} s_{n\lambda} \neq \infty$, then $\sup_{n \geq 1} \|x_{n\lambda}^*\| < \infty$.*

Proof. Assume that $s_n \rightarrow \bar{s} \in Q_\infty^\Lambda$ where $\bar{s} = (\bar{s}_\lambda)_{\lambda \in \Lambda}$. Suppose the lemma fails. Let λ_1 be a minimal element such that $\bar{s}_{\lambda_1} \neq \infty$ and $\sup_{n \geq 1} \|x_{n\lambda_1}^*\| = \infty$. Pick any $x \in E$ and let U be a neighbourhood of $\bar{s}_{\lambda_1}$. Denote $\Lambda_1 = \{\lambda \in \Lambda : \bar{s}_\lambda = \bar{s}_{\lambda_1}\}$. Clearly $\lambda_1 \in \Lambda_1$. For all $\lambda \notin \Lambda_1$, since $s_{n\lambda} \rightarrow \bar{s}_\lambda$, we have $s_{n\lambda}, \bar{s}_\lambda \notin U$ for large n . Let V be an open subset of U such that $\bar{V} \subseteq U$ and $s_{n\lambda} \in V$ for large n and all $\lambda \in \Lambda_1$. Choose $\phi \in C_0^\infty(Q)$ such that $\phi \equiv 1$ on V and $\text{supp}(\phi) \subseteq U$. Define $f : Q \rightarrow E$ by $f(u) = x\phi(u)(u - \bar{s}_{\lambda_1})^{\lambda_1}$. Clearly $f \in C_0^p(Q, E)$ and $\text{supp}(f) \subseteq U$. Then for large n ,

$$\partial^\lambda f(s_{n\lambda}) = \begin{cases} \lambda!(s_{n\lambda} - \bar{s}_{\lambda_1})^{\lambda_1 - \lambda} x & \text{if } \lambda \in \Lambda_1 \text{ and } \lambda \leq \lambda_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then for large n , we have

$$\begin{aligned} \langle f, x_n^* \circ \delta_{s_n} \rangle &= \sum_{\lambda \in \Lambda} x_{n\lambda}^*(\partial^\lambda f(s_{n\lambda})) \\ &= \sum_{\lambda \in \Lambda_1, \lambda < \lambda_1} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - \bar{s}_{\lambda_1})^{\lambda_1 - \lambda} + \lambda_1! x_{n\lambda_1}^*(x). \end{aligned} \quad (5.2.70)$$

We claim that $\left(\sum_{\lambda \in \Lambda_1, \lambda < \lambda_1} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - \bar{s}_{\lambda_1})^{\lambda_1 - \lambda} \right)_{n \geq 1}$ converges to 0. Indeed, if $\lambda \in \Lambda_1$ and $\lambda < \lambda_1$, by minimality of λ_1 , then $\sup_{n \geq 1} \|x_{n\lambda}^*\| < \infty$. So, for large n , since $|s_{n\lambda} - \bar{s}_{\lambda_1}|^{\lambda_1 - \lambda} \rightarrow 0$, so

$$\begin{aligned} \left| \sum_{\lambda \in \Lambda_1, \lambda < \lambda_1} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - \bar{s}_{\lambda_1})^{\lambda_1 - \lambda} \right| &\leq \sum_{\lambda \in \Lambda_1, \lambda < \lambda_1} \lambda! \|x_{n\lambda}^*\| \cdot \|x\| |s_{n\lambda} - \bar{s}_{\lambda_1}|^{\lambda_1 - \lambda} \\ &\leq \sum_{\lambda \in \Lambda_1, \lambda < \lambda_1} \lambda! \sup_{n \geq 1} \|x_{n\lambda}^*\| \cdot \|x\| |s_{n\lambda} - \bar{s}_{\lambda_1}|^{\lambda_1 - \lambda} \rightarrow 0. \end{aligned}$$

This concludes the proof of the claim.

Since $(\langle f, x_n^* \circ \delta_{s_n} \rangle)_{n \geq 1}$ is convergent, from (5.2.70) and claim above, the sequence $(\lambda_1! x_{n\lambda_1}^*(x))_{n \geq 1}$ is convergent as well. Therefore, $\sup_{n \geq 1} |x_{n\lambda_1}^*(x)| < \infty$. Since $x \in E$ is arbitrary, from the uniform boundedness principle, we deduce that

$\sup_{n \geq 1} \|x_{n\lambda_1}^*\| < \infty$, contradicting with the choice of λ_1 . This concludes the proof of the lemma. $\square$

Proposition 5.34. *Assume that $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s = (s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda$. Suppose that $x_n^* \circ \delta_{s_n} \xrightarrow{w^*} x^* \circ \delta_s$ for some $x_n^* = (x_{n\lambda}^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s_n \in Q^\Lambda$ where $s_n \rightarrow \bar{s}$ with $\bar{s} = (\bar{s}_\lambda)_{\lambda \in \Lambda} \in Q_\infty^\Lambda$. If $x_\lambda^* \neq 0$ for some $\lambda \in \Lambda$, then $\bar{s}_\lambda = s_\lambda$ and $x_{n\lambda}^* \xrightarrow{w^*} x_\lambda^*$.*

Proof. We first prove that $\bar{s}_\lambda = s_\lambda$. Suppose that $x_{\lambda_0}^* \neq 0$ and $\bar{s}_{\lambda_0} \neq s_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Then $x_{\lambda_0}^*(x) \neq 0$ for some $x \in E$. Let U be a neighbourhood of s_{λ_0} . Set $\Lambda_0 = \{\lambda \in \Lambda : s_\lambda = s_{\lambda_0}\}$ and $\bar{\Lambda}_0 = \{\lambda \in \Lambda : \bar{s}_\lambda = s_{\lambda_0}\}$. Clearly $\lambda_0 \notin \bar{\Lambda}_0$. Observe that $s_{n\lambda}, \bar{s}_\lambda \notin U$ for large n and all $\lambda \notin \bar{\Lambda}_0$. If $\lambda \notin \Lambda_0$, then we may assume that $s_\lambda \notin U$. Let V be an open subset of U such that $\bar{V} \subseteq U$ and $s_{n\lambda} \in V$ for large n and all $\lambda \in \bar{\Lambda}_0$. Choose $\phi \in C_0^\infty(Q)$ such that $\phi \equiv 1$ on V and $\text{supp}(\phi) \subseteq U$. Define $f : Q \rightarrow E$ by $f(u) = x\phi(u)(u - s_{\lambda_0})^{\lambda_0}$. Clearly $f \in C_0^p(Q, E)$ and $\text{supp}(f) \subseteq U$. For large n , since $\lambda_0 \notin \bar{\Lambda}_0$, we have

$$\partial^\lambda f(s_{n\lambda}) = \begin{cases} \lambda!(s_{n\lambda} - s_{\lambda_0})^{\lambda_0 - \lambda} x & \text{if } \lambda \in \bar{\Lambda}_0 \text{ and } \lambda < \lambda_0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.71)$$

and

$$\partial^\lambda f(s_\lambda) = \begin{cases} \lambda_0! x & \text{if } \lambda = \lambda_0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.72)$$

It follows that for large n , we have

$$\begin{aligned} & |\langle f, x_n^* \circ \delta_{s_n} \rangle| \\ &= \left| \sum_{\lambda \in \bar{\Lambda}_0, \lambda < \lambda_0} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - s_{\lambda_0})^{\lambda_0 - \lambda} \right| && \text{by (5.2.71)} \\ &\leq \sum_{\lambda \in \bar{\Lambda}_0, \lambda < \lambda_0} \lambda! \|x_{n\lambda}^*\| \cdot \|x\| |s_{n\lambda} - s_{\lambda_0}|^{\lambda_0 - \lambda} \\ &\leq \sum_{\lambda \in \bar{\Lambda}_0, \lambda < \lambda_0} \lambda! \sup_{n \geq 1} \|x_{n\lambda}^*\| \cdot \|x\| |s_{n\lambda} - s_{\lambda_0}|^{\lambda_0 - \lambda} && \text{by Lemma 5.33} \\ &\rightarrow 0 && \text{by definition of } \bar{\Lambda}_0. \end{aligned}$$

However, from (5.2.72),

$$\langle f, x^* \circ \delta_s \rangle = \sum_{\lambda \in \Lambda} x_\lambda^*(\partial^\lambda f(s_\lambda)) = \lambda_0! x_{\lambda_0}^*(x) \neq 0,$$

contradicting $x_n^* \circ \delta_{s_n} \xrightarrow{w^*} x^* \circ \delta_s$. This concludes the proof of the first assertion.

It remains to show that $x_{n\lambda}^* \xrightarrow{w^*} x_\lambda^*$. Suppose not. Let λ_0 be a minimal element such that $x_{\lambda_0}^* \neq 0$ and $(x_{n\lambda_0}^*)_{n \geq 1}$ does not converge to $x_{\lambda_0}^*$ in the weak* topology. Then $(x_{n\lambda_0}^*(x))_{n \geq 1}$ does not converge to $x_{\lambda_0}^*(x)$ for some $x \in E$. Also, by minimality of λ_0 , $(x_{n\lambda}^*)_{n \geq 1}$ converges to x_λ^* in the weak* topology for all $\lambda < \lambda_0$. Therefore, $\sup_{n \geq 1} \|x_{n\lambda}^*\| < \infty$ for all $\lambda < \lambda_0$. Let U be a neighbourhood of s_{λ_0} . Set $\Lambda_0 = \{\lambda \in \Lambda : s_\lambda = s_{\lambda_0}\}$ and $\bar{\Lambda}_0 = \{\lambda \in \Lambda : \bar{s}_\lambda = s_{\lambda_0}\}$. From the first assertion, clearly $\lambda_0 \in \bar{\Lambda}_0$. Observe that $s_{n\lambda}$, $\bar{s}_\lambda \notin U$ for large n and all $\lambda \notin \bar{\Lambda}_0$. If $\lambda \notin \Lambda_0$, then we may assume that $s_\lambda \notin U$. Let V be an open subset of U such that $\bar{V} \subseteq U$ and $s_{n\lambda} \in V$ for large n and all $\lambda \in \bar{\Lambda}_0$. Choose $\phi \in C_0^\infty(Q)$ such that $\phi \equiv 1$ on V and $\text{supp}(\phi) \subseteq U$. Define $f : Q \rightarrow E$ by $f(u) = x\phi(u)(u - s_{\lambda_0})^{\lambda_0}$. Clearly $f \in C_0^p(Q, E)$ and $\text{supp}(f) \subseteq U$. For large n , observe that

$$\partial^\lambda f(s_{n\lambda}) = \begin{cases} \lambda!(s_{n\lambda} - s_{\lambda_0})^{\lambda_0 - \lambda} x & \text{if } \lambda \in \bar{\Lambda}_0 \text{ and } \lambda \leq \lambda_0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.73)$$

and

$$\partial^\lambda f(s_\lambda) = \begin{cases} \lambda_0! x & \text{if } \lambda = \lambda_0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.74)$$

It follows that

$$\begin{aligned} & \langle f, x_n^* \circ \delta_{s_n} \rangle \\ &= \sum_{\lambda \in \bar{\Lambda}_0, \lambda \leq \lambda_0} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - s_{\lambda_0})^{\lambda_0 - \lambda} && \text{by (5.2.73)} \\ &= \sum_{\lambda \in \bar{\Lambda}_0, \lambda < \lambda_0} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - s_{\lambda_0})^{\lambda_0 - \lambda} + \lambda_0! x_{n\lambda_0}^*(x). \end{aligned}$$

For all $\lambda < \lambda_0$, since $\sup_{n \geq 1} \|x_{n\lambda}^*\| < \infty$, so

$$\left(\lim_{n \rightarrow \infty} \sum_{\lambda \in \bar{\Lambda}_0, \lambda < \lambda_0} \lambda! x_{n\lambda}^*(x) (s_{n\lambda} - s_{\lambda_0})^{\lambda_0 - \lambda} \right)_{n \geq 1}$$

converges to 0. Since $(\langle f, x_n^* \circ \delta_{s_n} \rangle)_{n \geq 1}$ is convergent, so is $(\lambda_0! x_{n\lambda_0}^*(x))_{n \geq 1}$. Therefore, we have

$$\lim_{n \rightarrow \infty} \lambda_0! x_{n\lambda_0}^*(x) = \lim_{n \rightarrow \infty} \langle f, x_n^* \circ \delta_{s_n} \rangle.$$

On the other hand, from (5.2.74), we have

$$\langle f, x^* \circ \delta_s \rangle = \sum_{\lambda \in \Lambda} x_\lambda^*(\partial^\lambda f(s_\lambda)) = \lambda_0! x_{\lambda_0}^*(x).$$

Since $\lim_{n \rightarrow \infty} \langle f, x_n^* \circ \delta_{s_n} \rangle = \langle f, x^* \circ \delta_s \rangle$, so $\lim_{n \rightarrow \infty} x_{n\lambda_0}^*(x) = x_{\lambda_0}^*(x)$, contradicting the choice of λ_0 . This concludes the proof of the proposition. $\square$

Lemma 5.35. Fix any $t \in K^\Gamma \setminus D$, $\beta^* \in \mathcal{G}_S(t)$ and $y^* \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$. Assume that

$$T^*(\beta^* \cdot y^* \circ \delta_t) = x^* \circ \delta_s \quad (5.2.75)$$

for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s \in Q^\Lambda$. If there exists $\psi : Q^\Lambda \setminus D \rightarrow K^\Gamma$ such that for all $s'' \in Q^\Lambda \setminus D$ and $a^* \in (E_\rho^\Lambda)^*$, we have $(T^*)^{-1}(a^* \circ \delta_{s''}) = b^* \circ \delta_{\psi(s'')}$ for some $b^* \in (F_\theta^\Gamma)^*$, then for any $v^* \in (E_\rho^\Lambda)^*$, there exists $w^* \in (F_\theta^\Gamma)^*$ such that $(T^*)^{-1}(v^* \circ \delta_s) = w^* \circ \delta_t$ where s is obtained from (5.2.75).

Proof. Fix any $v^* = (v_\gamma^*)_{\gamma \in \Gamma} \in (E_\rho^\Lambda)^*$. If $s \in Q^\Lambda \setminus D$, then by definition of ψ , the desired conclusion follows immediately. Now, assume that $s \in D$. Choose a sequence $(s_n)_{n=1}^\infty$ in $Q^\Lambda \setminus D$ that converges to s . Consider x^* from (5.2.75). Then $x^* \circ \delta_{s_n} \xrightarrow{w^*} x^* \circ \delta_s$. Since $(T^*)^{-1}$ is weak*-to-weak* continuous, from (5.2.75), we have $(T^*)^{-1}(x^* \circ \delta_{s_n}) \xrightarrow{w^*} (T^*)^{-1}(x^* \circ \delta_s) = \beta^* \cdot y^* \circ \delta_t$. For each $n \geq 1$, since $s_n \notin D$, let $t_n = \psi(s_n)$. By definition of ψ , we have $(T^*)^{-1}(x^* \circ \delta_{s_n}) = y_n^* \circ \delta_{t_n}$ for some $y_n^* \in (F_\theta^\Gamma)^*$. So, we have $y_n^* \circ \delta_{t_n} \xrightarrow{w^*} \beta^* \cdot y^* \circ \delta_t$. Also, $(t_n)_{n \geq 1}$ lies in a compact set K_∞^Γ , take subsequence if necessary, we may assume that $(t_n)_{n \geq 1}$ is convergent. Since $\beta_\gamma^* y_\gamma^* \neq 0$ for all $\gamma \in \Gamma$, by Proposition 5.34, we have $t_n \rightarrow t$.

On the other hand, since $s_n \rightarrow s$, so $v^* \circ \delta_{s_n} \xrightarrow{w^*} v^* \circ \delta_s$. Since $(T^*)^{-1}$ is weak*-to-weak* continuous, so $(T^*)^{-1}(v^* \circ \delta_{s_n}) \xrightarrow{w^*} (T^*)^{-1}(v^* \circ \delta_s)$. By definition of ψ , we have $(T^*)^{-1}(v^* \circ \delta_{s_n}) = w_n^* \circ \delta_{t_n}$ for some $w_n^* \in (F_\theta^\Gamma)^*$. Therefore, $(w_n^* \circ \delta_{t_n})_{n \geq 1}$ has a weak-star limit. Since $t_n \rightarrow t$ and all components of t are not infinity, so by Lemma 5.33, we have $\sup_{n \geq 1} \|w_n^*\| < \infty$. So, there exists a subnet $(w_{n_\alpha}^*)$ of (w_n^*) that converges to some functional $w^* \in (F_\theta^\Gamma)^*$ in the weak* topology. Clearly $w_{n_\alpha}^* \circ \delta_{t_{n_\alpha}} \xrightarrow{w^*} w^* \circ \delta_t$. Since $w_n^* \circ \delta_{t_n} \xrightarrow{w^*} (T^*)^{-1}(v^* \circ \delta_s)$, we must have $w^* \circ \delta_t = (T^*)^{-1}(v^* \circ \delta_s)$. This concludes the proof of the lemma. $\square$

Lemma 5.36. Fix any $t \in K^\Gamma \setminus D$ and $\lambda_0 \in \Lambda$. Suppose that there exist $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$, $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, $\gamma_0 \in \Gamma$ and $s^1 = (s_\lambda^1)_{\lambda \in \Lambda}$, $s^2 = (s_\lambda^2)_{\lambda \in \Lambda} \in Q^\Lambda$ such that $T^*(\beta^* \cdot y^* \circ \delta_t) = x_1^* \circ \delta_{s^1}$, and $T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t = x_2^* \circ \delta_{s^2}$ for some $x_1^* = (x_{1\lambda}^*)_{\lambda \in \Lambda}$, $x_2^* = (x_{2\lambda}^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ with $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$ and $x_{1\lambda_0}^*, x_{2\lambda_0}^* \neq 0$. Then for any $(\alpha_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $(z_\gamma^*)_{\gamma \in \Gamma} \in (F_\theta^\Gamma)^*$ with $z_\gamma^* \in \text{ext } B(F^*)$ for all $\gamma \in \Gamma$, we have

$$T^*[(\alpha_\gamma^* z_\gamma^*)_{\gamma \in \Gamma} \circ \delta_t] @ \lambda_0 \in \left\{ \left(\pm \frac{\alpha_{\gamma_0}^*}{\beta_{\gamma_0}^*} x_{1\lambda_0}^*, s_{\lambda_0}^1 \right), \left(\pm \frac{\alpha_{\gamma_0}^*}{\beta_{\gamma_0}^*} x_{2\lambda_0}^*, s_{\lambda_0}^2 \right) \right\}.$$

Proof. From Proposition 5.11, we have

$$T^*[(\alpha_\gamma^* z_\gamma^*)_{\gamma \in \Gamma} \circ \delta_t] = \bar{u}^* \circ \delta_{s^5} \quad \text{and} \quad T^*((\alpha_\gamma^* z_\gamma^*)_{\gamma \neq \gamma_0}, -\alpha_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t = \bar{v}^* \circ \delta_{s^6}$$

for some $\bar{u}^* = (\bar{u}_\lambda^*)_{\lambda \in \Lambda}$, $\bar{v}^* = (\bar{v}_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ and $s^5 = (s_\lambda^5)_{\lambda \in \Lambda}$, $s^6 = (s_\lambda^6)_{\lambda \in \Lambda} \in Q_\infty^\Lambda$. Subtracting equations above gives

$$T^*((0)_{\gamma \neq \gamma_0}, \alpha_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t = \frac{1}{2}(\bar{u}^* \circ \delta_{s^5} - \bar{v}^* \circ \delta_{s^6}) \quad (5.2.76)$$

Since $z_{\gamma_0}^* \in \text{ext } B(F^*)$, by Lemma 5.24, we have

$$T^*((0)_{\gamma \neq \gamma_0}, \beta_{\gamma_0}^* z_{\gamma_0}^*) \circ \delta_t @ \lambda_0 = \pm \frac{1}{2}[(x_{1\lambda_0}^*, s_{\lambda_0}^1) - (x_{2\lambda_0}^*, s_{\lambda_0}^2)]. \quad (5.2.77)$$

From (5.2.76) and (5.2.77), we have

$$\frac{1}{\alpha_{\gamma_0}^*} [(\bar{u}_{\lambda_0}^*, s_{\lambda_0}^5) - (\bar{v}_{\lambda_0}^*, s_{\lambda_0}^6)] = \pm \frac{1}{\beta_{\gamma_0}^*} [(x_{1\lambda_0}^*, s_{\lambda_0}^1) - (x_{2\lambda_0}^*, s_{\lambda_0}^2)].$$

Since $x_{1\lambda_0}^*, x_{2\lambda_0}^*$ are nonzero and $s_{\lambda_0}^1 \neq s_{\lambda_0}^2$, by Lemma 5.23, either

1. $\left(\frac{\bar{u}_{\lambda_0}^*}{\alpha_{\gamma_0}^*}, s_{\lambda_0}^5 \right) = \left(\pm \frac{x_{1\lambda_0}^*}{\beta_{\gamma_0}^*}, s_{\lambda_0}^1 \right)$ and $\left(\frac{\bar{v}_{\lambda_0}^*}{\alpha_{\gamma_0}^*}, s_{\lambda_0}^6 \right) = \left(\pm \frac{x_{2\lambda_0}^*}{\beta_{\gamma_0}^*}, s_{\lambda_0}^2 \right)$ or
2. $\left(\frac{\bar{u}_{\lambda_0}^*}{\alpha_{\gamma_0}^*}, s_{\lambda_0}^5 \right) = \left(\pm \frac{x_{2\lambda_0}^*}{\beta_{\gamma_0}^*}, s_{\lambda_0}^2 \right)$ and $\left(\frac{\bar{v}_{\lambda_0}^*}{\alpha_{\gamma_0}^*}, s_{\lambda_0}^6 \right) = \left(\pm \frac{x_{1\lambda_0}^*}{\beta_{\gamma_0}^*}, s_{\lambda_0}^1 \right)$.

This concludes the proof of the lemma. $\square$

Proposition 5.37. If $\dim F = 1$, $\dim E > 1$ and there exists $\psi : Q^\Lambda \setminus D \rightarrow K^\Gamma$ such that for all $s'' \in Q^\Lambda \setminus D$ and $a^* \in (E_\rho^\Lambda)^*$, we have $(T^*)^{-1}(a^* \circ \delta_{s''}) = b^* \circ \delta_{\psi(s'')}$ for some $b^* \in (F_\theta^\Gamma)^*$, then $\Lambda_{bad} = \emptyset$ for all $t \in K^\Gamma \setminus D$.

Proof. Fix $t \in K^\Gamma \setminus D$ and suppose that $\lambda_0 \in \Lambda_{bad}$. Then there exist $\beta^* = (\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$, $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} NExt(F^*)$, $\gamma_0 \in \Gamma$ and $s, s' \in Q^\Lambda$ such that

$$T^*(\beta^* \cdot y^* \circ \delta_t) = x^* \circ \delta_s, \quad \text{and} \quad T^*((\beta_\gamma^* y_\gamma^*)_{\gamma \neq \gamma_0}, -\beta_{\gamma_0}^* y_{\gamma_0}^*) \circ \delta_t = u^* \circ \delta_{s'} \quad (5.2.78)$$

with $s_{\lambda_0} \neq s'_{\lambda_0}$. Let ψ be given in the assumption. By Lemma 5.35, we have for any $v^* \in (E_\rho^\Lambda)^*$, there exists $w^* \in (F_\theta^\Gamma)^*$ such that

$$(T^*)^{-1}(v^* \circ \delta_s) = w^* \circ \delta_t. \quad (5.2.79)$$

Since $\dim E > 1$, by Lemma 5.25, there exist $v_{\lambda_0}^* \in \text{ext } B(E^*)$ and $e \in E$ such that

$$v_{\lambda_0}^*(e) \neq 0 = x_{\lambda_0}^*(e). \quad (5.2.80)$$

For all $\lambda \neq \lambda_0$, let $v_\lambda^* = x_\lambda^*$. Set $v^* = (v_\lambda^*)_{\lambda \in \Lambda}$. From (5.2.79), we have $(T^*)^{-1}(v^* \circ \delta_s) = w^* \circ \delta_t$ for some $w^* \in (F_\theta^\Gamma)^*$. Let $\hat{w}^* = \frac{w^*}{\|w^*\|}$. Then

$$T^*(\hat{w}^* \circ \delta_t) = \frac{v^*}{\|w^*\|} \circ \delta_s. \quad (5.2.81)$$

On the other hand, by Minkowski–Caratheodory Theorem [27, Theorem 8.11], we have $B(\mathbb{R}_{\theta^*}^\Gamma) = \text{co ext } B(\mathbb{R}_{\theta^*}^\Gamma)$. Since $\hat{w}^* \in B[(F_\theta^\Gamma)^*]$ and $\dim F = 1$, so

$$\hat{w}^* = \sum_{k=1}^n c_k \cdot w_k^* \quad (5.2.82)$$

for some $n \in \mathbb{N}$, $c_k \in [0, 1]$ with $\sum_{k=1}^n c_k = 1$ and $w_k^* \in \text{ext } B[(F_\theta^\Gamma)^*]$. Fix any $1 \leq k \leq n$. From Proposition 2.9, we have $\text{ext } B(\mathbb{R}_{\theta^*}^\Gamma) \subseteq \text{span}\{(\alpha_\gamma^* z_\gamma^*)_{\gamma \in \Gamma} : (\alpha_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t), |z_\gamma^*| = 1\}$. So,

$$w_k^* = \sum_{i=1}^{m_k} b_{ki} (\alpha_{ki\gamma}^* z_{ki\gamma}^*)_{\gamma \in \Gamma} \quad (5.2.83)$$

for some $m_k \in \mathbb{N}$, $b_{ki} \in \mathbb{R}$, $(\alpha_{ki\gamma}^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t)$ and $|z_{ki\gamma}^*| = 1$. For each $1 \leq i \leq m_k$, from Lemma 5.36, we have

$$T^*[(\alpha_{ki\gamma}^* z_{ki\gamma}^*)_{\gamma \in \Gamma} \circ \delta_t] = v_{ki}^* \circ \delta_{s^{ki}} \quad (5.2.84)$$

for some $v_{ki}^* = (v_{ki\lambda}^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ and $s^{ki} = (s_\lambda^{ki})_{\lambda \in \Lambda} \in Q^\Lambda$ where

$$(v_{ki\lambda_0}^*, s_{\lambda_0}^{ki}) \in \left\{ \left(\pm \frac{\alpha_{\gamma_0}^*}{\beta_{\gamma_0}^*} x_{\lambda_0}^*, s_{\lambda_0} \right), \left(\pm \frac{\alpha_{\gamma_0}^*}{\beta_{\gamma_0}^*} u_{\lambda_0}^*, s'_{\lambda_0} \right) \right\}. \quad (5.2.85)$$

Therefore,

$$\begin{aligned} \frac{v^*}{\|w^*\|} \circ \delta_s &= T^*(\hat{w}^* \circ \delta_t) && \text{by (5.2.81)} \\ &= \sum_{k=1}^n c_k \cdot T^*(w_k^* \circ \delta_t) && \text{by (5.2.82)} \\ &= \sum_{k=1}^n \sum_{i=1}^{m_k} c_k b_{ki} T^*[(\alpha_{ki\gamma}^* z_{ki\gamma}^*)_{\gamma \in \Gamma} \circ \delta_t] && \text{by (5.2.83)} \\ &= \sum_{k=1}^n \sum_{i=1}^{m_k} c_k b_{ki} \cdot v_{ki}^* \circ \delta_{s^{ki}} && \text{by (5.2.84).} \end{aligned} \quad (5.2.86)$$

Since $v_{\lambda_0}^* \neq 0$ and $s_{\lambda_0} \neq \infty$, by Lemma 5.16, we have $s_{\lambda_0} = s_{\lambda_0}^{k_0 i_0}$ for some k_0 and i_0 . Choose $f \in C_0^p(Q, E)$ such that $\partial^\lambda f(s_\lambda) = \partial^\lambda f(s_\lambda^{ki}) = 0$ for all $1 \leq k \leq n$, $1 \leq i \leq m_k$, $\lambda \neq \lambda_0$, $\partial^{\lambda_0} f(s_{\lambda_0}) = e$ and $\partial^{\lambda_0} f(s'_{\lambda_0}) = 0$ where e satisfies (5.2.80). Then

$$\begin{aligned} 0 &\neq v_{\lambda_0}^*(e) && \text{by (5.2.80)} \\ &= \left\langle f, \frac{v^*}{\|w^*\|} \circ \delta_s \right\rangle \\ &= \sum_{k=1}^n \sum_{i=1}^{m_k} c_k b_{ki} \cdot \langle f, v_{ki}^* \circ \delta_{s^{ki}} \rangle && \text{by (5.2.86)} \\ &= \sum_{k=1}^n \sum_{i=1}^{m_k} c_k b_{ki} \cdot v_{ki\lambda_0}^*(\partial^{\lambda_0} f(s_{\lambda_0}^{ki})) && \text{by choice of } f \\ &= \sum_{k=1}^n \sum_{i=1}^{m_k} \pm c_k b_{ki} \frac{\alpha_{\gamma_0}^*}{\beta_{\gamma_0}^*} x_{\lambda_0}^*(e) && \text{by (5.2.85)} \\ &= 0 && \text{by (5.2.80),} \end{aligned}$$

a contradiction. This concludes the proof of the proposition. $\square$

The following corollary is immediate from Corollary 5.22 and Proposition 5.37.

Corollary 5.38. *Let $T : (C_0^p(Q, E), \|\cdot\|_{s,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{s,\theta})$ be a surjective linear isometry, where $p, q, m, n \geq 1$ are natural numbers, Q and K are open*

subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. If $F = \mathbb{R}$, $E \neq \mathbb{R}$ and E^* satisfies $(*)$, then there exists a mapping $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_\theta^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$.

The following corollary is a special case of corollary above.

Corollary 5.39. *Let $T : (C_0^p(Q, E), \|\cdot\|_{s,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{s,\theta})$ be a surjective linear isometry, where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and both E and F are Banach spaces. If $\theta = \|\cdot\|_1$, $F = \mathbb{R}$, $E \neq \mathbb{R}$ and E^* satisfies $(*)$, then there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_1^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in \text{ext } B[(E_\rho^\Lambda)^*]$.*

Lemma 5.40. *Assume that $E = F = \mathbb{R}$, $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. If $r = 1$, so is s .*

Proof. Suppose that $s > 1$. Fix any $u \in Q^\Lambda \setminus D$ and $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} N\text{Ext}(E^*)$. Recall that $h : S(\mathbb{R}_\rho^\Lambda) \rightarrow S(\mathbb{R}_{\rho^*}^\Lambda)$ is the duality map, given by (5.2.59). Since ρ is strictly convex and smooth, so h is injective. Since $E = \mathbb{R}$, ρ is strictly convex and smooth, by Proposition 5.32, there exists a well-defined mapping $\psi : Q^\Lambda \setminus D \rightarrow K^\Gamma$. Recall that $U = \prod_{\lambda \in \Lambda} (\zeta_\lambda, \eta_\lambda) \cap S(\mathbb{R}_\rho^\Lambda)$. By Lemma 5.28, we have $h(U) \subseteq \mathcal{G}_S(t)$ for all $t \in K^\Gamma \setminus D$. Therefore, for any $\alpha \in U$, we have $(T^*)^{-1}[h(\alpha) \cdot x^*] \circ \delta_u = y^* \circ \delta_{\psi(u)}$ for some $y^* \in \text{ext } B[(F_\theta^\Gamma)^*]$. It follows that T^* induces a map $\alpha \mapsto y^*$, which is injective as $(T^*)^{-1}$ is injective. Since U is a nonempty open set, so it is an infinite set. Therefore, $\text{ext } B[(F_\theta^\Gamma)^*]$ is an infinite set as well. Since $F = \mathbb{R}$, so $r > 1$, a contradiction. This concludes the proof of the lemma. $\square$

5.2.4 Case 4: $r = s = 1$ and $E = F = \mathbb{R}$

The main result of this section is Corollary 5.52, which asserts that if $r = s = 1$ and $E = F = \mathbb{R}$, then φ is well-defined. Let $e = (1)_{\alpha \in \Gamma}$ and $e^\gamma = (e_\alpha^\gamma)_{\alpha \in \Gamma}$, where

$$e_\alpha^\gamma = \begin{cases} -1 & \text{if } \alpha = \gamma, \\ 1 & \text{if } \alpha \neq \gamma. \end{cases}$$

Clearly $\{e\} = \mathcal{G}_S(t)$ for any $t \in K^\Gamma \setminus D$. So, by Proposition 5.14, we have

$$T^*(e \circ \delta_t) = (a_\lambda(t))_{\lambda \in \Lambda} \circ \delta_{(s_\lambda(t))_{\lambda \in \Lambda}} \quad \text{and} \quad (5.2.87)$$

$$T^*(e^\gamma \circ \delta_t) = (a_\lambda^\gamma(t))_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^\gamma(t))_{\lambda \in \Lambda}} \quad (5.2.88)$$

for some $(s_\lambda^\gamma(t))_{\lambda \in \Lambda}, (s_\lambda(t))_{\lambda \in \Lambda} \in Q^\Lambda$ and $a_\lambda(t), a_\lambda^\gamma(t) \in \{1, -1\}$. We begin this section with the following simple lemma. Since a_λ and a_λ^γ are nonzero, the following lemma follows immediately from Proposition 5.34.

Lemma 5.41. *For all $\gamma \in \Gamma$, the functions $(a_\lambda(t))_{\lambda \in \Lambda}$, $(a_\lambda^\gamma(t))_{\lambda \in \Lambda}$, $(s_\lambda(t))_{\lambda \in \Lambda}$ and $(s_\lambda^\gamma(t))_{\lambda \in \Lambda}$ are continuous.*

Remark 5.42. *For any neighbourhood U of t , from Lemma 5.41, by shrinking U if necessary, we may assume that both a_λ and a_λ^γ are constant functions on U .*

Lemma 5.43. *Let $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$ and $\lambda_0 \in \Lambda$. For all nonempty open set $U \subseteq K^\Gamma \setminus D$, there exists a nonempty open set $V \subseteq U$ such that either*

(1) $s_{\lambda_0}^{\gamma_1} = s_{\lambda_0}^{\gamma_2}$ on V and either (a) $a_{\lambda_0}^{\gamma_1} + a_{\lambda_0}^{\gamma_2} = 0$ or (b) $s_{\lambda_0}^{\gamma_1}(t)$ and $s_{\lambda_0}^{\gamma_2}(t)$, $t = (t_\alpha)$, are independent of t_{γ_1} and t_{γ_2} on V ; or

(2) $s_{\lambda_0}^{\gamma_1} \neq s_{\lambda_0}^{\gamma_2}$ at some point in V , $s_{\lambda_0}^{\gamma_1}(t)$ and $s_{\lambda_0}^{\gamma_2}(t)$, $t = (t_\alpha)$, are independent of t_{γ_1} and t_{γ_2} on V .

Proof. Fix $\lambda_0 \in \Lambda$, $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$ and nonempty open set $U \subseteq K^\Gamma \setminus D$. For any $t = (t_\alpha) \in U$ and $f \in C_0^p(Q)$, adding (5.2.88) for $\gamma = \gamma_1$ and $\gamma = \gamma_2$ and applying it to f give

$$2 \sum_{\alpha \in \Gamma \setminus \{\gamma_1, \gamma_2\}} \partial^\alpha (Tf)(t_\alpha) = \sum_{\lambda \in \Lambda} a_\lambda^{\gamma_1} \partial^\lambda f(s_\lambda^{\gamma_1}(t)) + a_\lambda^{\gamma_2} \partial^\lambda f(s_\lambda^{\gamma_2}(t)). \quad (5.2.89)$$

Since LHS of (5.2.89) is independent of t_{γ_1} and t_{γ_2} , so for any $t = (t_\alpha)$, $t' = (t'_\alpha) \in U$ where $t'_\alpha = t_\alpha$ for all $\alpha \neq \gamma_1, \gamma_2$, we have

$$\sum_{\lambda \in \Lambda} a_\lambda^{\gamma_1} \partial^\lambda f(s_\lambda^{\gamma_1}(t)) + a_\lambda^{\gamma_2} \partial^\lambda f(s_\lambda^{\gamma_2}(t)) = \sum_{\lambda \in \Lambda} a_\lambda^{\gamma_1} \partial^\lambda f(s_\lambda^{\gamma_1}(t')) + a_\lambda^{\gamma_2} \partial^\lambda f(s_\lambda^{\gamma_2}(t')) \quad (5.2.90)$$

We consider two cases: either $s_{\lambda_0}^{\gamma_1} = s_{\lambda_0}^{\gamma_2}$ on U or $s_{\lambda_0}^{\gamma_1}(\bar{t}) \neq s_{\lambda_0}^{\gamma_2}(\bar{t})$ for some $\bar{t}$ in U .

Case 1 $s_{\lambda_0}^{\gamma_1} = s_{\lambda_0}^{\gamma_2}$ on U .

Let $V = U$. Suppose that both (a) and (b) fail. Then $a_{\lambda_0}^{\gamma_1} + a_{\lambda_0}^{\gamma_2} \neq 0$ and without loss of generality, we may assume that $s_{\lambda_0}^{\gamma_1}$ is not independent of t_{γ_1} and t_{γ_2} on U . Then there exist $t = (t_\alpha)$, $t' = (t'_\alpha) \in U$ where $t'_\alpha = t_\alpha$ for all $\alpha \neq \gamma_1, \gamma_2$ such that $s_{\lambda_0}^{\gamma_1}(t) \neq s_{\lambda_0}^{\gamma_1}(t')$. Choose $f \in C_0^p(Q)$ such that

$$\partial^\lambda f = 0 \quad \text{on} \quad \{s_{\lambda_0}^{\gamma_1}(t), s_{\lambda_0}^{\gamma_2}(t), s_{\lambda_0}^{\gamma_1}(t'), s_{\lambda_0}^{\gamma_2}(t')\}$$

for all $\lambda \neq \lambda_0$, $\partial^{\lambda_0} f(s_{\lambda_0}^{\gamma_1}(t)) = 1$ and $\partial^{\lambda_0} f(s_{\lambda_0}^{\gamma_1}(t')) = 0$. From (5.2.90), we have $a_{\lambda_0}^{\gamma_1} + a_{\lambda_0}^{\gamma_2} = 0$, a contradiction. Therefore, either (a) or (b) holds.

Case 2 $s_{\lambda_0}^{\gamma_1}(\bar{t}) \neq s_{\lambda_0}^{\gamma_2}(\bar{t})$ for some $\bar{t}$ in U .

By Lemma 5.41, there is a neighborhood $V \subseteq U$ of $\bar{t}$ such that $s_{\lambda_0}^{\gamma_1}(t) \neq s_{\lambda_0}^{\gamma_2}(t')$ for all $t, t' \in V$. Suppose that either $s_{\lambda_0}^{\gamma_1}$ or $s_{\lambda_0}^{\gamma_2}$ is not independent of $t_{\gamma_1}, t_{\gamma_2}$ on V . Without loss of generality, assume that the former holds. Then there exist $t = (t_\alpha)$, $t' = (t'_\alpha) \in V$ where $t_\alpha = t'_\alpha$ for all $\alpha \neq \gamma_1, \gamma_2$ such that $s_{\lambda_0}^{\gamma_1}(t) \neq s_{\lambda_0}^{\gamma_1}(t')$. So,

$$s_{\lambda_0}^{\gamma_1}(t) \notin \{s_{\lambda_0}^{\gamma_1}(t'), s_{\lambda_0}^{\gamma_2}(t), s_{\lambda_0}^{\gamma_2}(t')\}.$$

Choose $f \in C_0^p(Q)$ such that

$$\partial^\lambda f = 0 \quad \text{on} \quad \{s_{\lambda_0}^{\gamma_1}(t), s_{\lambda_0}^{\gamma_2}(t), s_{\lambda_0}^{\gamma_1}(t'), s_{\lambda_0}^{\gamma_2}(t')\}$$

for all $\lambda \neq \lambda_0$, $\partial^{\lambda_0} f = 0$ on $\{s_{\lambda_0}^{\gamma_1}(t'), s_{\lambda_0}^{\gamma_2}(t), s_{\lambda_0}^{\gamma_2}(t')\}$ and $\partial^{\lambda_0} f(s_{\lambda_0}^{\gamma_1}(t)) = 1$. Then (5.2.90) implies that $a_{\lambda_0}^{\gamma_1} = 0$, a contradiction. Therefore, $s_{\lambda_0}^{\gamma_1}$ is independent of t_{γ_1} and t_{γ_2} . This concludes the proof of the lemma. $\square$

Remark 5.44. Observe that if $\#\Gamma = 2$, then the statements (1) and (a) in Lemma 5.43 always hold.

Lemma 5.45. Let $\gamma_0 \in \Gamma$, $\lambda_0 \in \Lambda$. For all nonempty open set $U \subseteq K^\Gamma \setminus D$, there exists a nonempty open set $V \subseteq U$ such that either

- (1) $s_{\lambda_0} = s_{\lambda_0}^{\gamma_0}$ on V and either (a) $a_{\lambda_0} - a_{\lambda_0}^{\gamma_0} = 0$ or (b) $s_{\lambda_0}(t)$ and $s_{\lambda_0}^{\gamma_0}(t)$, $t = (t_\alpha)$, are independent of t_γ for all $\gamma \neq \gamma_0$ on V ; or
- (2) $s_{\lambda_0} \neq s_{\lambda_0}^{\gamma_0}$ at some point in V , $s_{\lambda_0}(t)$ and $s_{\lambda_0}^{\gamma_0}(t)$, $t = (t_\alpha)$, are independent of t_γ for all $\gamma \neq \gamma_0$ on V .

Proof. Fix $\gamma_0 \in \Gamma$, $\lambda_0 \in \Lambda$ and a nonempty open set $U \subseteq K^\Gamma \setminus D$. For any $t = (t_\alpha) \in U$ and $f \in C_0^p(Q)$, subtracting (5.2.88) from (5.2.87) and applying it to f give

$$2\partial^{\gamma_0}(Tf)(t_{\gamma_0}) = \sum_{\lambda \in \Lambda} [a_\lambda \partial^\lambda f(s_\lambda(t)) - a_\lambda^{\gamma_0} \partial^\lambda f(s_\lambda^{\gamma_0}(t))]. \quad (5.2.91)$$

Since the LHS of (5.2.91) is independent of $t_{\gamma'}$ for all $\gamma \neq \gamma_0$, for any $t = (t_\alpha)$, $t' = (t'_\alpha)$ in U with $t'_{\gamma_0} = t_{\gamma_0}$, we have

$$\sum_{\lambda \in \Lambda} [a_\lambda \partial^\lambda f(s_\lambda(t)) - a_\lambda^{\gamma_0} \partial^\lambda f(s_\lambda^{\gamma_0}(t))] = \sum_{\lambda \in \Lambda} [a_\lambda \partial^\lambda f(s_\lambda(t')) - a_\lambda^{\gamma_0} \partial^\lambda f(s_\lambda^{\gamma_0}(t'))] \quad (5.2.92)$$

Now, we consider two cases: either $s_{\lambda_0}(t) = s_{\lambda_0}^{\gamma_0}(t)$ for all $t \in U$ or $s_\lambda(\bar{t}) \neq s_\lambda^{\gamma_0}(\bar{t})$ at some point $\bar{t}$ in U .

Case 1 $s_{\lambda_0} = s_{\lambda_0}^{\gamma_0}$ on U .

Let $V = U$. Suppose that both (a) and (b) fail. Then $a_{\lambda_0} - a_{\lambda_0}^{\gamma_0} \neq 0$ and s_{λ_0} is not independent of t_γ for all $\gamma \neq \gamma_0$ on U . Then there exist $t = (t_\alpha)$, $t' = (t'_\alpha)$ in U with $t'_{\gamma_0} = t_{\gamma_0}$ such that $s_{\lambda_0}(t) \neq s_{\lambda_0}(t')$. Choose $f \in C_0^p(Q)$ such that

$$\partial^\lambda f = 0 \quad \text{on} \quad \{s_\lambda(t), s_\lambda^{\gamma_0}(t), s_\lambda(t'), s_\lambda^{\gamma_0}(t')\}$$

for all $\lambda \neq \lambda_0$, $\partial^{\lambda_0} f(s_{\lambda_0}(t)) = 1$, and $\partial^{\lambda_0} f(s_{\lambda_0}(t')) = 0$. Then (5.2.92) implies that $a_{\lambda_0} - a_{\lambda_0}^{\gamma_0} = 0$, a contradiction. Therefore, either (a) or (b) holds.

Case 2 $s_\lambda(\bar{t}) \neq s_\lambda^{\gamma_0}(\bar{t})$ at some point $\bar{t}$ in U .

By Lemma 5.41, there is a neighborhood $V \subseteq U$ of $\bar{t}$ such that $s_{\lambda_0}(t) \neq s_{\lambda_0}^{\gamma_0}(t')$ for all $t, t' \in V$. Suppose that s_{λ_0} and $s_{\lambda_0}^{\gamma_0}$ are not both independent of t_γ for all $\gamma \neq \gamma_0$ on V . Assume that the former holds. Then there are $t = (t_\alpha)$, $t' = (t'_\alpha) \in V$ with $t'_{\gamma_0} = t_{\gamma_0}$ such that $s_{\lambda_0}(t) \neq s_{\lambda_0}(t')$. Hence,

$$s_{\lambda_0}(t) \notin \{s_{\lambda_0}(t'), s_{\lambda_0}^{\gamma_0}(t), s_{\lambda_0}^{\gamma_0}(t')\}.$$

Choose $f \in C_0^p(Q)$ such that

$$\partial^\lambda f = 0 \quad \text{on} \quad \{s_{\lambda_0}(t), s_{\lambda_0}^{\gamma_0}(t), s_{\lambda_0}(t'), s_{\lambda_0}^{\gamma_0}(t')\}$$

for all $\lambda \neq \lambda_0$, $\partial^{\lambda_0} f = 0$ on $\{s_{\lambda_0}(t'), s_{\lambda_0}^{\gamma_0}(t), s_{\lambda_0}^{\gamma_0}(t')\}$ and $\partial^{\lambda_0} f(s_{\lambda_0}(t)) = 1$. Then (5.2.92) implies that $a_{\lambda_0} = 0$, a contradiction. Therefore, s_{λ_0} is independent of t_γ for all $\gamma \neq \gamma_0$ on V . This concludes the proof of the lemma. $\square$

Applying Lemmas 5.43 and 5.45 repeatedly, we obtain the following propositions.

Proposition 5.46. *For all nonempty open set $U \subseteq K^\Gamma \setminus D$, there exists a nonempty open set $V \subseteq U$ such that for all $\gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2, \lambda_0 \in \Lambda$, either*

(1) $s_{\lambda_0}^{\gamma_1} = s_{\lambda_0}^{\gamma_2}$ on V and either (a) $a_{\lambda_0}^{\gamma_1} + a_{\lambda_0}^{\gamma_2} = 0$ or (b) $s_{\lambda_0}^{\gamma_1}(t)$ and $s_{\lambda_0}^{\gamma_2}(t)$, $t = (t_\alpha)$, are independent of t_{γ_1} and t_{γ_2} on V ; or

(2) $s_{\lambda_0}^{\gamma_1} \neq s_{\lambda_0}^{\gamma_2}$ at some point in V and $s_{\lambda_0}^{\gamma_1}(t)$ and $s_{\lambda_0}^{\gamma_2}(t)$, $t = (t_\alpha)$, are independent of t_{γ_1} and t_{γ_2} on V .

Proposition 5.47. *For all nonempty open set $U \subseteq K^\Gamma \setminus D$, there exists a nonempty open set $V \subseteq U$ such that for all $\gamma_0 \in \Gamma, \lambda_0 \in \Lambda$, either*

(1) $s_{\lambda_0} = s_{\lambda_0}^{\gamma_0}$ on V and either (a) $a_{\lambda_0} - a_{\lambda_0}^{\gamma_0} = 0$ or (b) $s_{\lambda_0}(t)$ and $s_{\lambda_0}^{\gamma_0}(t)$, $t = (t_\alpha)$ are independent of t_γ for all $\gamma \neq \gamma_0$ on V ; or

(2) $s_{\lambda_0} \neq s_{\lambda_0}^{\gamma_0}$ at some point in V and $s_{\lambda_0}(t)$ and $s_{\lambda_0}^{\gamma_0}(t)$, $t = (t_\alpha)$, are independent of t_γ for all $\gamma \neq \gamma_0$ on V .

Proposition 5.48. *For all nonempty open set $U \subseteq K^\Gamma \setminus D$, there exists a nonempty open set $V \subseteq U$ such that for all $\lambda \in \Lambda$ either*

(1) $s_\lambda = s_\lambda^\gamma$ on V for all $\gamma \in \Gamma$, or

(2) s_λ and s_λ^γ are constant on V for all $\gamma \in \Gamma$.

Proof. Let V be an open subset of U obtained from Propositions 5.46 and 5.47. Fix $\lambda \in \Lambda$. Assume that (1) fails, that is, there exist $t_0 \in V$ and $\gamma_0 \in \Gamma$ such that $s_\lambda(t_0) \neq s_\lambda^{\gamma_0}(t_0)$. For each $t \in V$, let $J_t = \{\gamma : s_\lambda(t) \neq s_\lambda^\gamma(t)\}$. Note that $\gamma_0 \in J_{t_0}$.

Assume that there exists $t \in V$ such that $|J_t| \geq 2$. Let $\gamma_1, \gamma_2 \in J_t$ be such that $\gamma_1 \neq \gamma_2$. Then $s_\lambda(t) \neq s_\lambda^{\gamma_i}(t)$ for $i = 1, 2$. By (2) in Proposition 5.47, s_λ and $s_\lambda^{\gamma_1}$ are independent of t_γ for all $\gamma \neq \gamma_1$ on V . By the same proposition, s_λ and $s_\lambda^{\gamma_2}$ are independent of t_γ for all $\gamma \neq \gamma_2$ on V . Therefore, s_λ is constant on V . If $s_\lambda^{\gamma_1} = s_\lambda^{\gamma_2}$ on V , then $s_\lambda^{\gamma_1}$ and $s_\lambda^{\gamma_2}$ are also constant on V . Otherwise, if $s_\lambda^{\gamma_1} \neq s_\lambda^{\gamma_2}$ at some point

in V , by (2) in Proposition 5.46, $s_\lambda^{\gamma_1}$ and $s_\lambda^{\gamma_2}$ are independent of t_{γ_1} and t_{γ_2} on V . Since $s_\lambda^{\gamma_1}$ is also independent of t_γ for all $\gamma \neq \gamma_1$ on V , so $s_\lambda^{\gamma_1}$ is constant on V . Similarly, we conclude that $s_\lambda^{\gamma_2}$ is also constant on V . Therefore, we have shown that s_λ and s_λ^γ are constant on V for all $\gamma \in J_t$. For any $\gamma \notin J_t$, either $s_\lambda^\gamma = s_\lambda$ on V or $s_\lambda^\gamma \neq s_\lambda$ at some point in V . If the former holds, then since s_λ is constant on V , so is s_λ^γ . If the latter holds, then by (2) in Proposition 5.47, both s_λ^γ and s_λ are constant on V . In conclusion, we have shown that s_λ and s_λ^γ are constant on V for all $\gamma \in \Gamma$.

Now, assume that for all $t \in V$, $|J_t| \leq 1$. Then $\#\Gamma \geq 3$. Let $W = \{t \in V : s_\lambda(t) \neq s_\lambda^{\gamma_0}(t)\}$. Since s_λ and $s_\lambda^{\gamma_0}$ are continuous, so W is open. Clearly $t_0 \in W$. Since $J_{t_0} = \{\gamma_0\}$, by assumption, for all $t \in W$, $J_t = \{\gamma_0\}$. Note that for all $\gamma_1 \in \Gamma \setminus \{\gamma_0\}$, $s_\lambda^{\gamma_1} = s_\lambda$ on W . Therefore, $s_\lambda^{\gamma_0}(t) \neq s_\lambda^{\gamma_1}(t)$ for all $t \in W$. By (2) in Proposition 5.46, $s_\lambda^{\gamma_0}$ and $s_\lambda^{\gamma_1}$ are independent of t_{γ_0} and t_{γ_1} on W . On the other hand, since $s_\lambda(t) \neq s_\lambda^{\gamma_0}(t)$ for all $t \in W$, by (2) in Proposition 5.47, s_λ and $s_\lambda^{\gamma_0}$ are independent of t_γ for all $\gamma \neq \gamma_0$. It follows that s_λ and $s_\lambda^{\gamma_0}$ are constants on W . Since $s_\lambda^{\gamma_1} = s_\lambda$ on W , so s_λ and s_λ^γ are constants on W for all $\gamma \in \Gamma$. This concludes the proof of the proposition. $\square$

Corollary 5.49. *For all nonempty open set $U \subseteq K^\Gamma \setminus D$, there exist a nonempty open set $V \subseteq U$ such that for all $\lambda \in \Lambda$, either $s_\lambda, s_\lambda^\gamma$ are constant on V for all $\gamma \in \Gamma$ or $s_\lambda = s_\lambda^\gamma$ on V for all $\gamma \in \Gamma$ and they are dependent on t_{γ_λ} only on V for some $\gamma_\lambda \in \Gamma$.*

Proof. Let V be the intersection of those open sets as given in Proposition 5.46, 5.47, and 5.48 and $\lambda \in \Lambda$ be fixed. Suppose that the assertion $s_\lambda, s_\lambda^\gamma$ are constant on V for all $\gamma \in \Gamma$ fails. By Proposition 5.48, we have $s_\lambda = s_\lambda^\gamma$ on V for all $\gamma \in \Gamma$. Then by (1) in Proposition 5.47, for each $\gamma \in \Gamma$, either (a) $a_\lambda - a_\lambda^\gamma = 0$ or (b) $s_\lambda(t)$ and $s_\lambda^\gamma(t), t = (t_\alpha)$ are independent of $t_{\gamma'}$ for all $\gamma' \neq \gamma$ on V .

Consider the set $L = \{\gamma : a_\lambda - a_\lambda^\gamma \neq 0\}$. If $|L| \geq 2$, let $\gamma_1, \gamma_2 \in L, \gamma_1 \neq \gamma_2$. By (b) above, s_λ and $s_\lambda^{\gamma_i}$ are independent of $t_{\gamma'}$ for all $\gamma' \neq \gamma_i$ on V for $i = 1, 2$. Since $\gamma_1 \neq \gamma_2$, so s_λ is constant on V . Since $s_\lambda = s_\lambda^\gamma$ on V , so s_λ^γ are constant on V for all $\gamma \in \Gamma$, a contradiction.

If $L = \emptyset$, then for any $\gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$, we have $a_\lambda^{\gamma_1} = a_\lambda^{\gamma_2} = a_\lambda$. Thus, $a_\lambda^{\gamma_1} + a_\lambda^{\gamma_2} \neq 0$. So, by (1)(b) in Proposition 5.46, $s_\lambda^{\gamma_1}$ and $s_\lambda^{\gamma_2}$ are independent of t_{γ_1} and t_{γ_2} on

V . Since this is true for all $\gamma_1, \gamma_2 \in \Gamma$ and $s_\lambda^\gamma = s_\lambda$ for all $\gamma \in \Gamma$ on V , so $s_\lambda, s_\lambda^\gamma$ are constant for all $\gamma \in \Gamma$, a contradiction.

Therefore, we must have $|L| = 1$, say $L = \{\gamma_\lambda\}$. By Proposition 5.47, s_λ and $s_\lambda^{\gamma_\lambda}$ are independent of $t_{\gamma'}$ for all $\gamma' \neq \gamma_\lambda$. Since $s_\lambda = s_\lambda^\gamma$ on V for all $\gamma \in \Gamma$, therefore $s_\lambda(t), s_\lambda^\gamma(t), t = (t_\alpha)$ depend on at most t_{γ_λ} on V . $\square$

Proposition 5.50. *If $\#\Gamma \geq \#\Lambda$, then $s_\lambda = s_\lambda^\gamma$ on a dense subset of K^Γ for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$.*

Proof. Let U be any nonempty open set in $K^\Gamma \setminus D$ and V be the nonempty open sets as given in Corollary 5.49. So, for each $\lambda \in \Lambda$, either $s_\lambda, s_\lambda^\gamma$ are constant on V or $s_\lambda = s_\lambda^\gamma$ for all $\gamma \in \Gamma$ and they are dependent on t_{γ_λ} only on V for some $\gamma_\lambda \in \Gamma$. Let $\Lambda_0 \subseteq \Lambda$ be the set of all λ where the latter holds.

We claim that $\Lambda_0 = \Lambda$. If not, since $\#\Gamma \geq \#\Lambda$, then $\#\Gamma > \#\Lambda_0$. So, there exists $\gamma_0 \in \Gamma \setminus \{\gamma_\lambda : \lambda \in \Lambda_0\}$. If $\#\Gamma \geq 3$, since $\{e^\gamma : \gamma \in \Gamma\}$ is a spanning set of $\mathbb{R}^\Gamma$, we have $((0)_{\gamma \neq \gamma_0}, 1) = \sum_{\gamma \in \Gamma} \beta_\gamma e^\gamma$ for some $(\beta_\gamma) \in \mathbb{R}^\Gamma$. Then for any $t \in V$, we have

$$T^*[((0)_{\gamma \neq \gamma_0}, 1) \circ \delta_t] = T^* \left(\sum_{\gamma \in \Gamma} \beta_\gamma e^\gamma \right) \circ \delta_t = \sum_{\gamma \in \Gamma} \beta_\gamma (a_\lambda^\gamma)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^\gamma(t))_{\lambda \in \Lambda}}.$$

So, for any $f \in C_0^p(Q)$,

$$\begin{aligned} \partial^{\gamma_0} T f(t_{\gamma_0}) &= \sum_{\gamma \in \Gamma} \beta_\gamma \left\langle f, (a_\lambda^\gamma)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^\gamma(t))_{\lambda \in \Lambda}} \right\rangle \\ &= \sum_{\gamma \in \Gamma} \beta_\gamma \sum_{\lambda \in \Lambda} a_\lambda^\gamma \partial^\lambda f(s_\lambda^\gamma(t)) \\ &= \sum_{\gamma \in \Gamma} \beta_\gamma \sum_{\lambda \in \Lambda_0} a_\lambda^\gamma \partial^\lambda f(s_\lambda^\gamma(t)) + \sum_{\gamma \in \Gamma} \beta_\gamma \sum_{\lambda \notin \Lambda_0} a_\lambda^\gamma \partial^\lambda f(s_\lambda^\gamma) \\ &= \sum_{\gamma \in \Gamma} \beta_\gamma \sum_{\lambda \in \Lambda_0} a_\lambda^\gamma \partial^\lambda f(s_\lambda(t_{\gamma_\lambda})) + \sum_{\gamma \in \Gamma} \beta_\gamma \sum_{\lambda \notin \Lambda_0} a_\lambda^\gamma \partial^\lambda f(s_\lambda^\gamma). \end{aligned} \quad (5.2.93)$$

Note that the second summation is constant on V . Since $\gamma_0 \neq \gamma_\lambda$ for all $\lambda \in \Lambda_0$, so the RHS of (5.2.93) is independent of t_{γ_0} . Therefore, for all $g \in C_0^q(K)$, (t_α) and (t'_α) in V with $t_\alpha = t'_\alpha$ for all $\alpha \neq \gamma_0$, we have $\partial^{\gamma_0} g(t_{\gamma_0}) = \partial^{\gamma_0} g(t'_{\gamma_0})$. Since V is open, there exist $(t_\alpha) \in V$ and $(t'_\alpha) \in V$ with $t'_\gamma = t_\gamma$ for all $\gamma \neq \gamma_0$ and $t'_{\gamma_0} \neq t_{\gamma_0}$. Choose $g \in C_0^q(K)$ such that $\partial^{\gamma_0} g(t_{\gamma_0}) \neq \partial^{\gamma_0} g(t'_{\gamma_0})$, a contradiction. Similarly, if $\#\Gamma = 2$, then $\{e = (1, 1), e^1 = (-1, 1)\}$ spans $\mathbb{R}^2$. Assume that $\gamma_0 = 0$. Then

$(1, 0) = \beta_1 e + \beta_2 e^1$. So, for any $t \in V$, we have

$$\begin{aligned} T^*(1, 0) \circ \delta_t &= \beta_0 T^*(e \circ \delta_t) + \beta_1 T^*(e^1 \circ \delta_t) \\ &= \beta_0 (a_\lambda)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda(t))_{\lambda \in \Lambda}} + \beta_1 (a_\lambda^1)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^1(t))_{\lambda \in \Lambda}} \end{aligned}$$

Therefore, for any $f \in C_0^p(Q)$, similar calculations in (5.2.93) imply that $Tf(t_0)$ is independent of t_0 , a contradiction. Similarly, if $\gamma_0 = 1$, by replacing $(1, 0)$ with $(0, 1)$, we can obtain a contradiction. Therefore, $\Lambda_0 = \Lambda$. This concludes the proof of the proposition. $\square$

The following proposition asserts that if $\#\Gamma \geq \#\Lambda$, then φ is well-defined.

Proposition 5.51. *Let $T : (C_0^p(Q, E), \|\cdot\|_{s,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{s,r})$ where Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, both E and F are Banach spaces and $s, r \in [1, \infty)$. If $r = s = 1$, $E = F = \mathbb{R}$ and $\#\Gamma \geq \#\Lambda$, then there exists a mapping $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_1^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_1^\Lambda)^*$.*

Proof. For all $t \in K^\Gamma \setminus D$, recall that for all $\gamma \in \Gamma$, we have $T^*(e \circ \delta_t) = (a_\lambda(t))_{\lambda \in \Lambda} \circ \delta_{(s_\lambda(t))_{\lambda \in \Lambda}}$ and $T^*(e^\gamma \circ \delta_t) = (a_\lambda^\gamma(t))_{\lambda \in \Lambda} \circ \delta_{(s_\lambda^\gamma(t))_{\lambda \in \Lambda}}$ for some $s_\lambda^\gamma = s_\lambda^\gamma(t)$, $s_\lambda = s_\lambda(t) \in Q^\Lambda$ and $a_\lambda(t), a_\lambda^\gamma(t) \in \{1, -1\}$. By Corollary 5.13, it suffices to show that $s_\lambda = s_\lambda^\gamma$ for all $\lambda \in \Lambda$ and $\gamma \in \Gamma$. This is achieved by Proposition 5.50 and continuity of a_λ and a_λ^γ . This concludes the proof of the proposition. $\square$

The following corollary removes the assumption of $\#\Gamma \geq \#\Lambda$ from Proposition 5.51.

Corollary 5.52. *Let $T : (C_0^p(Q, E), \|\cdot\|_{s,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{s,r})$ where Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, both E and F are Banach spaces and $s, r \in [1, \infty)$. If $r = s = 1$ and $E = F = \mathbb{R}$, then there exists a mapping $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_1^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_1^\Lambda)^*$.*

Proof. If $\#\Gamma \geq \#\Lambda$, then from Proposition 5.51, the desired conclusion holds. Assume that $\#\Gamma < \#\Lambda$. By Proposition 5.51, there exists a well-defined $\psi : Q^\Lambda \setminus D \rightarrow K^\Gamma$. Fix any $s \in Q^\Lambda \setminus D$ and set $t = \psi(s)$. Clearly $\{(1)_{\gamma \in \Gamma}\} = \mathcal{G}_s(s)$. So, from Proposition 5.14, for any $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in \mathbb{R}_\infty^\Lambda$ where $|x_\lambda| = 1$ for all $\lambda \in \Lambda$,

we have $(T^{-1})^*(x^* \circ \delta_s) = y^* \circ \delta_t$ for some $y^* = (y_\gamma^*)_{\gamma \in \Gamma} \in \text{ext } B(\mathbb{R}_\infty^\Gamma)$. Clearly $x^* \in \text{ext } B(\mathbb{R}_\infty^\Lambda)$. Therefore, $(T^{-1})^*$ induces a map $x^* \mapsto y^*$, which is injective as $(T^{-1})^*$ is injective. It follows that $2^{\#\Lambda} = \#\text{ext } B(\mathbb{R}_\infty^\Lambda) \leq \#\text{ext } B(\mathbb{R}_\infty^\Gamma) = 2^{\#\Gamma}$, which implies that $\#\Lambda \leq \#\Gamma$, contradicting $\#\Gamma < \#\Lambda$. This concludes the proof of the proposition. $\square$

Now, we are ready to prove the main result of this section, which follows immediately from Corollaries 5.27, 5.32, 5.39 and 5.52.

Theorem 5.53. *Let $T : (C_0^p(Q, E), \|\cdot\|_{S,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,r})$ be a surjective linear isometry where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, both E and F are Banach spaces and $s, r \in [1, \infty)$. If E^* and F^* satisfy $(*)$, then there exists $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ such that for all $t \in K^\Gamma \setminus D$ and $y^* \in (F_r^\Gamma)^*$, we have $T^*(y^* \circ \delta_t) = x^* \circ \delta_{\varphi(t)}$ for some $x^* \in (E_s^\Lambda)^*$.*

5.3 T is a weighted composition operator

Henceforth, let $\varphi : K^\Gamma \setminus D \rightarrow Q^\Lambda$ and $\psi : Q^\Lambda \setminus D \rightarrow K^\Gamma$ be induced by T^* and $(T^*)^{-1}$ according to Theorem 5.53. In this concluding section, the main result is Theorem 5.70, which says that the isometry T is a weighted composition operator. To this end, we will utilize Theorem 2.20.

Lemma 5.54. *φ is continuous on $K^\Gamma \setminus D$.*

Proof. Fix any $t_0 \in K^\Gamma \setminus D$ and suppose that $t_n \rightarrow t_0$ for some $(t_n)_{n \geq 1}$ in $K^\Gamma \setminus D$. Let $y^* = (\beta_\gamma^* y_\gamma^*)_{\gamma \in \Gamma}$ where $(\beta_\gamma^*)_{\gamma \in \Gamma} \in \mathcal{G}_S(t_0)$ and $(y_\gamma^*)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} N\text{Ext}(F^*)$. Since T^* is weak*-to-weak* continuous, so $T^*(y^* \circ \delta_{t_n}) \xrightarrow{w^*} T^*(y^* \circ \delta_{t_0})$. From the choice of y^* and Proposition 5.14, we have $T^*(y^* \circ \delta_{t_0}) = x^* \circ \delta_{\varphi(t_0)}$ for some $x^* = (x_\lambda^*)_{\lambda \in \Lambda} \in \text{ext } B[(E_\rho^\Lambda)^*]$ where $x_\lambda^* \neq 0$ for all $\lambda \in \Lambda$. Also, by definition of φ , for each $n \geq 1$, we have $T^*(y^* \circ \delta_{t_n}) = x_n^* \circ \delta_{\varphi(t_n)}$ for some $x_n^* \in (E_\rho^\Lambda)^*$. So, we have $x_n^* \circ \delta_{\varphi(t_n)} \xrightarrow{w^*} x^* \circ \delta_{\varphi(t_0)}$. From Proposition 5.34, $\varphi(t_n) \rightarrow \varphi(t_0)$. Therefore, φ is continuous at t_0 . This concludes the proof of the lemma. $\square$

Lemma 5.55. *For each $t \in K^\Gamma \setminus D$, define $S(t) : (F_\theta^\Gamma)^* \rightarrow (E_\rho^\Lambda)^*$ by*

$$T^*(y^* \circ \delta_t) = (S(t)y^*) \circ \delta_{\varphi(t)}. \quad (5.3.1)$$

Then $S(t)$ is well-defined, bounded, linear and weak*-to-weak* continuous.

Proof. Fix $t \in K^\Gamma \setminus D$ and let $\varphi(t) = (s_\lambda)_{\lambda \in \Lambda}$. We first prove that $S(t)$ is well-defined. Pick any $(x_\lambda^*)_{\lambda \in \Lambda}$ and $(z_\lambda^*)_{\lambda \in \Lambda} \in (E_\rho^\Lambda)^*$ that satisfy $(x_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}} = (z_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}}$. Clearly $(x_\lambda^* - z_\lambda^*)_{\lambda \in \Lambda} \circ \delta_{(s_\lambda)_{\lambda \in \Lambda}} = 0$. Since $s_\lambda \neq \infty$ for all $\lambda \in \Lambda$, by Corollary 5.17, we have $x_\lambda^* = z_\lambda^*$ for all $\lambda \in \Lambda$. This shows that $S(t)$ is well-defined.

Clearly $S(t)$ is linear. We claim that $S(t)$ is weak*-to-weak* continuous. Pick any net $(y_\alpha^*)_\alpha$ in $(F_\theta^\Gamma)^*$ such that $y_\alpha^* \xrightarrow{w^*} y^*$. Clearly $y_\alpha^* \circ \delta_t \xrightarrow{w^*} y^* \circ \delta_t$. Since T^* is weak*-to-weak* continuous, so $T^*(y_\alpha^* \circ \delta_t) \xrightarrow{w^*} T^*(y^* \circ \delta_t)$. By (5.3.1), we have $S(t)y_\alpha^* \circ \delta_{\varphi(t)} \xrightarrow{w^*} S(t)y^* \circ \delta_{\varphi(t)}$. Clearly $S(t)y_\alpha^* \xrightarrow{w^*} S(t)y^*$. Hence, $S(t)$ is weak*-to-weak* continuous.

It remains to show that $S(t)$ is bounded. Since $S(t)$ is weak*-to-weak* continuous and $B[(F_\theta^\Gamma)^*]$ is weak*-compact, so $S(t)B[(F_\theta^\Gamma)^*]$ is also weak*-compact. Therefore, it is a bounded set. Hence, $S(t)$ is bounded. This concludes the proof of the lemma. $\square$

Fix $t \in K^\Gamma \setminus D$. Since $S(t)$ is weak*-to-weak* continuous, there exists a bounded linear operator $A(t) : E_\rho^\Lambda \rightarrow F_\theta^\Gamma$ such that $S(t) = A(t)^*$. We represent $A(t)$ as $(a_{\gamma\lambda}(t))_{(\gamma,\lambda) \in \Gamma \times \Lambda}$, where each $a_{\gamma\lambda}(t) : E \rightarrow F$ is a bounded linear operator. Then

$$A(t)(x_\lambda)_{\lambda \in \Lambda} = (a_{\gamma\lambda}(t))(x_\lambda)_{\lambda \in \Lambda} = \left(\sum_{\lambda \in \Lambda} a_{\gamma\lambda}(t)x_\lambda \right)_{\gamma \in \Gamma}. \quad (5.3.2)$$

Let $\varphi = (\varphi_\lambda)_{\lambda \in \Lambda}$ where $\varphi_\lambda : K^\Gamma \setminus D \rightarrow Q$ for all $\lambda \in \Lambda$. The following lemma expresses partial derivatives of Tf in terms of the operators $a_{\gamma\lambda}$.

Lemma 5.56. For any $t = (t_\gamma)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$ and $\gamma \in \Gamma$,

$$\partial^\gamma Tf(t_\gamma) = \sum_{\lambda \in \Lambda} a_{\gamma\lambda}(t) \partial^\lambda f(\varphi_\lambda(t)) \text{ for all } f \in C_0^p(Q, E) \quad (5.3.3)$$

where $a_{\gamma\lambda}(t) : E \rightarrow F$ is a bounded linear operator for each $\gamma \in \Gamma$ and $\lambda \in \Lambda$.

Proof. For all $f \in C_0^p(Q, E)$ and $y^* \in (F_\theta^\Gamma)^*$, observe that

$$\begin{aligned}
 \left\langle (\partial^\gamma T f(t_\gamma))_{\gamma \in \Gamma}, y^* \right\rangle &= \langle f, T^*(y^* \circ \delta_t) \rangle \\
 &= \langle f, (S(t) y^*) \circ \delta_{\varphi(t)} \rangle && \text{by (5.3.1)} \\
 &= \left\langle (\partial^\lambda f(\varphi_\lambda(t)))_{\lambda \in \Lambda}, S(t) y^* \right\rangle \\
 &= \left\langle A(t) (\partial^\lambda f(\varphi_\lambda(t)))_{\lambda \in \Lambda}, y^* \right\rangle && \text{since } S(t) = A(t)^* \\
 &= \left\langle \left(\sum_{\lambda \in \Lambda} a_{\gamma\lambda}(t) \partial^\lambda f(\varphi_\lambda(t)) \right)_{\gamma \in \Gamma}, y^* \right\rangle && \text{by (5.3.2).}
 \end{aligned}$$

Since $y^* \in (F_\theta^\Gamma)^*$ is arbitrary, so $\partial^\gamma T f(t_\gamma) = \sum_{\lambda \in \Lambda} a_{\gamma\lambda}(t) \partial^\lambda f(\varphi_\lambda(t))$ for all $\gamma \in \Gamma$. This concludes the proof of the lemma. $\square$

The following lemma is a consequence of Lemma 5.56, which asserts that $a_{\gamma_0\lambda_0}$ is independent of t_γ for all $\gamma \neq \gamma_0$. Also, if $a_{\gamma_0\lambda_0}(t) \neq 0$, then φ_{λ_0} is independent of t_γ for all $\gamma \neq \gamma_0$.

Lemma 5.57. *For all $\gamma_0 \in \Gamma, \lambda_0 \in \Lambda, t = (t_\gamma)_{\gamma \in \Gamma}, \bar{t} = (\bar{t}_\gamma)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$, if $t_{\gamma_0} = \bar{t}_{\gamma_0}$, then the following hold.*

(a) *If $a_{\gamma_0\lambda_0}(t) \neq 0$, then $\varphi_{\lambda_0}(t) = \varphi_{\lambda_0}(\bar{t})$.*

(b) *$a_{\gamma_0\lambda_0}(t) = a_{\gamma_0\lambda_0}(\bar{t})$.*

Proof. (a) Suppose that $a_{\gamma_0\lambda_0}(t)e \neq 0$ for some $e \in E$. If $\varphi_{\lambda_0}(t) \neq \varphi_{\lambda_0}(\bar{t})$, choose $f \in C_0^p(Q, E)$ such that $\partial^\lambda f(\varphi_\lambda(t)) = \partial^\lambda f(\varphi_\lambda(\bar{t})) = 0$ for all $\lambda \neq \lambda_0$, $\partial^{\lambda_0} f(\varphi_{\lambda_0}(t)) = e$ and $\partial^{\lambda_0} f(\varphi_{\lambda_0}(\bar{t})) = 0$. By Lemma 5.56, we have $a_{\gamma_0\lambda_0}(t)e = \partial^{\gamma_0} T f(t_{\gamma_0}) = \partial^{\gamma_0} T f(\bar{t}_{\gamma_0}) = 0$, a contradiction. This concludes the proof of (a).

(b) If $a_{\gamma_0\lambda_0}(t) \neq a_{\gamma_0\lambda_0}(\bar{t})$, then $a_{\gamma_0\lambda_0}(t)e \neq a_{\gamma_0\lambda_0}(\bar{t})e$ for some $e \in E$. Without loss of generality, we may assume that $a_{\gamma_0\lambda_0}(t)e \neq 0$. From (a), we have $\varphi_{\lambda_0}(t) = \varphi_{\lambda_0}(\bar{t})$. Then there exists $f \in C_0^p(Q, E)$ such that $\partial^\lambda f(\varphi_\lambda(t)) = \partial^\lambda f(\varphi_\lambda(\bar{t})) = 0$ for all $\lambda \neq \lambda_0$ and $\partial^{\lambda_0} f(\varphi_{\lambda_0}(t)) = \partial^{\lambda_0} f(\varphi_{\lambda_0}(\bar{t})) = e$. Since $t_{\gamma_0} = \bar{t}_{\gamma_0}$, by Lemma 5.56, we have $a_{\gamma_0\lambda_0}(t)e = \partial^{\gamma_0} T f(t_{\gamma_0}) = \partial^{\gamma_0} T f(\bar{t}_{\gamma_0}) = a_{\gamma_0\lambda_0}(\bar{t})e$, contradicting the choice of e . This concludes the proof of (b). $\square$

For each $t \in K^\Gamma \setminus D$, clearly $S(t) = (a_{\lambda\gamma}^*(t))_{\gamma,\lambda}$ where each $a_{\lambda\gamma}^*(t) : F^* \rightarrow E^*$ is the adjoint operator $a_{\gamma\lambda}(t)$. The following lemma asserts that all $a_{\gamma\lambda}$ is either zero or completely not zero on an open subset of $K^\Gamma \setminus D$.

Lemma 5.58. *For every nonempty open set $U \subseteq K^\Gamma \setminus D$, there exists a nonempty open set $V \subseteq U$ such that for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$, either $a_{\gamma\lambda} \equiv 0$ on V or $a_{\gamma\lambda}(t) \neq 0$ for all $t \in V$.*

Proof. Pick a nonempty open set $U \subseteq K^\Gamma \setminus D$. Since Γ and Λ are finite sets, it suffices to show that for any $\gamma_0 \in \Gamma$ and $\lambda_0 \in \Lambda$, there exists $V \subseteq U$ such that either $a_{\gamma\lambda} \equiv 0$ on V or $a_{\gamma\lambda}(t) \neq 0$ for all $t \in V$. Observe that either $a_{\gamma_0\lambda_0} \equiv 0$ on U or $a_{\gamma_0\lambda_0}(t_0) \neq 0$ at some $t_0 \in U$. Assume the latter holds. Then $a_{\lambda_0\gamma_0}^*(t_0) \neq 0$ as well. Hence, $a_{\lambda_0\gamma_0}^*(t_0)y_{\gamma_0}^* \neq 0$ for some $y_{\gamma_0}^* \in F^*$. Let $y^* = ((0)_{\gamma \neq \gamma_0}, y_{\gamma_0}^*)$. Set $S(t_0)y^* = (x_\lambda^*(t_0))_{\lambda \in \Lambda}$. Since $S(t_0)y^* = (a_{\lambda\gamma_0}^*(t_0)y_{\gamma_0}^*)_{\lambda \in \Lambda}$, so $x_{\lambda_0}^*(t_0) = a_{\lambda_0\gamma_0}^*(t_0)y_{\gamma_0}^* \neq 0$.

We claim that $t \mapsto x_{\lambda_0}^*(t)$ is continuous at t_0 . Indeed, assume that $t_n \rightarrow t_0$ for some $(t_n)_{n \geq 1}$ in $K^\Gamma \setminus D$. Since T^* is weak*-to-weak* continuous, so $T^*(y^* \circ \delta_{t_n}) \xrightarrow{w^*} T^*(y^* \circ \delta_{t_0})$. By (5.3.1), we have $S(t_n)y^* \circ \delta_{\varphi(t_n)} \xrightarrow{w^*} S(t_0)y^* \circ \delta_{\varphi(t_0)}$. For each $n \geq 1$, let $S(t_n)y^* = (x_\lambda^*(t_n))_{\lambda \in \Lambda}$ and $\varphi(t_n) = s_n$. Since $(s_n)_{n \geq 1}$ is a sequence in the compact set Q_∞^Λ , by taking subsequence if necessary, we may assume that $s_n \rightarrow \bar{s}$ for some $\bar{s} \in Q_\infty^\Lambda$. Since $x_{\lambda_0}^*(t_0) \neq 0$, by Proposition 5.34, we have $x_{\lambda_0}^*(t_n) \xrightarrow{w^*} x_{\lambda_0}^*(t_0)$. This concludes the proof of the claim.

Since $a_{\lambda_0\gamma_0}^*(t_0)y_{\gamma_0}^* \neq 0$ and $t \mapsto x_{\lambda_0}^*(t) = a_{\lambda_0\gamma_0}^*(t)y_{\gamma_0}^*$ is continuous at t_0 , there is a neighbourhood V of t_0 such that $a_{\lambda_0\gamma_0}^*(t)y_{\gamma_0}^* \neq 0$ for all $t \in V$. Thus, $a_{\lambda_0\gamma_0}^*(t) \neq 0$ for all $t \in V$ and thus $a_{\gamma_0\lambda_0}(t) \neq 0$ for all $t \in V$. $\square$

Remark 5.59. *We may assume $V = \prod_{\gamma \in \Gamma} B(t_\gamma, r)$ for some $(t_\gamma)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$ and $r > 0$.*

As a consequence of Lemmas 5.57 and 5.58, we obtain the following lemma.

Lemma 5.60. *If $a_{\gamma_1\lambda_0}(\bar{t}) \neq 0$ and $a_{\gamma_2\lambda_0}(\bar{t}) \neq 0$ for some $\bar{t} \in V$, $\lambda_0 \in \Lambda$, $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$, then φ_{λ_0} is constant on V .*

Proof. Let $\bar{t} = (\bar{t}_\gamma)_{\gamma \in \Gamma}$ and fix any $t = (t_\gamma)_{\gamma \in \Gamma} \in V$. Set $t' = (t'_\gamma)_{\gamma \in \Gamma}$ where $t'_{\gamma_1} = \bar{t}_{\gamma_1}$ and $t'_\gamma = t_\gamma$ for $\gamma \neq \gamma_1$. Since both $\bar{t}$ and t are in V , so is $t' \in V$. Also,

since $a_{\gamma_0\lambda_0}(\bar{t}) \neq 0$, from Lemma 5.58, so is $a_{\gamma_0\lambda_0}(t')$. Therefore, by Lemma 5.57(a), we have

$$\begin{aligned}\varphi_{\lambda_0}(t) &= \varphi_{\lambda_0}(t') \quad \text{since } a_{\gamma_2\lambda_0}(t') \neq 0 \text{ and } t_{\gamma_2} = t'_{\gamma_2} \\ &= \varphi_{\lambda_0}(\bar{t}) \quad \text{since } a_{\gamma_1\lambda_0}(\bar{t}) \neq 0 \text{ and } t'_{\gamma_1} = \bar{t}_{\gamma_1}.\end{aligned}$$

Since $t \in V$ is arbitrary, therefore φ_{λ_0} is constant on V . $\square$

Obtain V from Lemma 5.58 and consider the matrix $(a_{\gamma\lambda}(t))_{\gamma,\lambda}$. Let

$$J = \{\lambda_0 \in \Lambda : a_{\gamma_1\lambda_0}(\bar{t}) \neq 0 \text{ and } a_{\gamma_2\lambda_0}(\bar{t}) \neq 0 \text{ for some } \bar{t} \in V \text{ and } \gamma_1 \neq \gamma_2\}.$$

From Lemma 5.60, if $\lambda \in J$, then φ_λ is constant on V , say $\varphi_\lambda(t) = s_\lambda$ for all $t \in V$. On the other hand, if $\lambda \notin J$, then by choice of V , either $a_{\gamma_1\lambda_0} = 0$ on V or $a_{\gamma_2\lambda_0} = 0$ on V for all $\gamma_1 \neq \gamma_2$. In other words, for each $\lambda \in J^c$, there exists at most one $\gamma_\lambda \in \Gamma$ such that $a_{\gamma_\lambda\lambda}(t) \neq 0$ for all $t \in V$ and $a_{\gamma\lambda} = 0$ on V if $\gamma \neq \gamma_\lambda$. Note that by choice of V , γ_λ is independent of $t \in V$.

Lemma 5.61. $\#\Gamma \leq \#J^c$

Proof. Suppose that $\#\Gamma > \#J^c$. Then there exists $\gamma_0 \in \Gamma$ such that $\gamma_0 \neq \gamma_\lambda$ for all $\lambda \in J^c$. Then $a_{\gamma_0\lambda}(t) = 0$ for all $t \in V$ and $\lambda \in J^c$. From Lemma 5.56, definition of J and remark above, we have

$$\partial^{\gamma_0} T f(t_{\gamma_0}) = \sum_{\lambda \in J} a_{\gamma_0\lambda}(t) \partial^\lambda f(s_\lambda) \quad \text{for all } t = (t_\gamma)_{\gamma \in \Gamma} \in V, f \in C_0^p(Q, E)$$

for some $s_\lambda \in Q$. For each $\lambda \in J$, let $s^\lambda = (s_\mu^\lambda)_{\mu \in \Lambda} \in Q^\Lambda \setminus D$ be such that $s_\lambda^\lambda = s_\lambda$. Then by Lemma 5.56, for all $g \in C_0^q(K, F)$, we have

$$\partial^\lambda f(s_\lambda) = \sum_{\gamma \in \Gamma} b_{\lambda\gamma}(s^\lambda) \partial^\gamma g(\psi_\gamma(s^\lambda)) \quad \text{for all } f = T^{-1}g,$$

for some bounded linear operators $b_{\lambda\gamma}(s^\lambda) : E \rightarrow F$ and some mappings $\psi_\gamma : Q^\Lambda \setminus D \rightarrow K$. Pick any $f \in C_0^p(Q, E)$ and denote $g = T f$. Then for all $t = (t_\gamma)_{\gamma \in \Gamma} \in V$,

$$\partial^{\gamma_0} g(t_{\gamma_0}) = \sum_{\lambda \in J} a_{\gamma_0\lambda}(t) \partial^\lambda f(s_\lambda) = \sum_{\lambda \in J} a_{\gamma_0\lambda}(t) \left[\sum_{\gamma \in \Gamma} b_{\lambda\gamma}(s^\lambda) \partial^\gamma g(\psi_\gamma(s^\lambda)) \right]. \quad (5.3.4)$$

Since V is an open set, so it contains infinitely many points. Choose $t_0 = (t_\gamma)_{\gamma \in \Gamma} \in V$ such that $t_{\gamma_0} \notin \{\psi_\gamma(s^\lambda) : \lambda \in J, \gamma \in \Gamma\}$. Then there exists $g \in C_0^q(K, F)$ such that $\partial^{\gamma_0} g(t_{\gamma_0}) \neq 0$ and $\partial^\gamma g(\psi_\gamma(s^\lambda)) = 0$ for all $\lambda \in J$ and $\gamma \in \Gamma$. However, (5.3.4) implies that

$$0 \neq \partial^{\gamma_0} g(t_{\gamma_0}) = \sum_{\lambda \in J} a_{\gamma_0 \lambda}(t_0) \left[\sum_{\gamma \in \Gamma} b_{\lambda \gamma}(s^\lambda) 0 \right] = 0,$$

a contradiction. This concludes the proof of the lemma. $\square$

Corollary 5.62. *If $\#\Gamma \geq \#\Lambda$, then $\#\Gamma = \#\Lambda$ and $J = \emptyset$.*

Proof. If $\#\Gamma > \#\Lambda$, then from Lemma 5.61, we have $\#J^c \geq \#\Gamma > \#\Lambda$, contradicting to the fact that J is a subset of Λ . Now, if $J \neq \emptyset$, then $\#J^c < \#\Lambda \leq \#\Gamma$, contradicting Lemma 5.61. $\square$

Corollary 5.63. *$\#\Gamma = \#\Lambda$ and thus $J = \emptyset$.*

Proof. If $\#\Gamma \geq \#\Lambda$, then Corollary 5.62 implies that $\#\Gamma = \#\Lambda$. If $\#\Gamma \leq \#\Lambda$, then applying the arguments to T^{-1} of Corollary 5.62 gives $\#\Gamma = \#\Lambda$. In any case, we have $\#\Gamma = \#\Lambda$. $J = \emptyset$ follows from Corollary 5.62. $\square$

Proposition 5.64. *For any nonempty open set $U \subseteq K^\Gamma \setminus D$, there exist a nonempty open set $V \subseteq U$ such that for all $\gamma \in \Gamma$, there exists the unique $\lambda_\gamma \in \Lambda$ such that for all $t = (t_\gamma)_{\gamma \in \Gamma} \in V$, $f \in C_0^p(Q, E)$, we have $\partial^\gamma T f(t_\gamma) = a_{\gamma \lambda_\gamma}(t) \partial^{\lambda_\gamma} f(\varphi_{\lambda_\gamma}(t))$ where $a_{\gamma \lambda_\gamma}(t) \neq 0$ for all $t \in V$ and φ_{λ_γ} depends only on t_γ .*

Proof. Fix any nonempty open set $U \subseteq K^\Gamma \setminus D$ and obtain V from Lemma 5.58. We claim that the mapping $\lambda \mapsto \gamma_\lambda$ from Λ into Γ , where $a_{\gamma_\lambda \lambda}(t) \neq 0$ for all $t \in V$, is a bijection. Indeed, from Corollary 5.63, we have $J = \emptyset$. So, for each $\lambda \in \Lambda$, there exists at most one $\gamma_\lambda \in \Gamma$ such that $a_{\gamma_\lambda \lambda}(t) \neq 0$ for all $t \in V$ and $a_{\gamma \lambda} = 0$ on V if $\gamma \neq \gamma_\lambda$. Clearly, there is no $\gamma \in \Gamma$ such that $a_{\gamma \lambda}(t) = 0$ for all $t \in V$ and $\lambda \in \Lambda$. Otherwise, from Lemma 5.56, $\partial^\gamma T f \equiv 0$ on V for all $f \in C_0^p(Q, E)$, a contradiction. It follows from the preceding two observations and $\#\Gamma = \#\Lambda$ that for all $\lambda \in \Lambda$, there exists exactly one $\gamma_\lambda \in \Gamma$ such that $a_{\gamma_\lambda \lambda}(t) \neq 0$ for all $t \in V$. Hence $\lambda \mapsto \gamma_\lambda$ is a bijection from Λ onto Γ . This concludes the proof of the claim.

Denote the inverse bijection by $\gamma \mapsto \lambda_\gamma$. Then for all $\lambda \in \Lambda$, $t = (t_\gamma)_{\gamma \in \Gamma} \in V$,

$$\partial^\gamma T f(t_\gamma) = a_{\gamma \lambda_\gamma}(t) \partial^{\lambda_\gamma} f(\varphi_{\lambda_\gamma}(t)) \quad \text{for all } f \in C_0^p(Q, E). \quad (5.3.5)$$

From Lemma 5.57(a), we deduce that φ_{λ_γ} depends only on the component t_γ . This concludes the proof of the proposition. $\square$

The following corollary follows immediately from Proposition 5.64, which asserts that T is a weighted composition operator on an open dense subset of $K^\Gamma \setminus D$.

Corollary 5.65. *There exists an open dense set $V \subseteq K^\Gamma \setminus D$ such that for all $t = (t_\gamma)_{\gamma \in \Gamma} \in V$, there exists $\lambda_t \in \Lambda$ such that $T f(t_0) = a_{0\lambda_t}(t) \partial^{\lambda_t} f(\varphi_{\lambda_t}(t))$ for all $f \in C_0^p(Q, E)$ where $a_{0\lambda_t}(t) \neq 0$ and φ_{λ_t} depends only on t_0 .*

Lemma 5.66. *Fix a sequence $(t^k)_{k=1}^\infty$ in $K^\Gamma \setminus D$ where $t^k = (t_\gamma^k)_{\gamma \in \Gamma}$ and $\lambda_0 \in \Lambda$. If $\sup_{k \geq 1} \|T f(t_0^k)\| < \infty$, $(\varphi_{\lambda_0}(t^k))_{k=1}^\infty$ converges to $s_0 \in Q$ and $T f(t_0^k) = a_{0\lambda_0}(t^k) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k))$ for all $k \geq 1$, then $\sup_{k \geq 1} \|a_{0\lambda_0}(t^k)\| < \infty$.*

Proof. Pick any $e \in E$. Since $(\varphi_{\lambda_0}(t^k))_{k=1}^\infty$ converges to $s_0 \in Q$, there exists $f \in C_0^p(Q, E)$ such that $\partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k)) = e$ for large k . Then $T f(t_0^k) = a_{0\lambda_0}(t^k) e$ for large k . So, we have $\sup_{k \geq 1} \|a_{0\lambda_0}(t^k) e\| = \sup_{k \geq 1} \|T f(t_0^k)\| < \infty$, where the final inequality follows from assumption. Since $e \in E$ is arbitrary, by the uniform boundedness principle, we have $\sup_{k \geq 1} \|a_{0\lambda_0}(t^k)\| < \infty$. This concludes the proof of the lemma. $\square$

Lemma 5.67. *Let $t^k = (t_\gamma^k)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$ for all $k \geq 1$ and $t^0 = (t_\gamma^0)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$. Assume that there exists $\lambda_0 \in \Lambda$ such that $T f(t_0^k) = a_{0\lambda_0}(t^k) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k))$ for all $k \geq 1$ and $T f(t_0^0) = \sum_{\lambda \in \Lambda} a_{0\lambda}(t^0) \partial^\lambda f(\varphi_\lambda(t^0))$ for all $f \in C_0^p(Q, E)$. If $t^k \rightarrow t^0$, then $a_{0\lambda_0}(t^0) \neq 0$ and $a_{0\lambda}(t^0) = 0$ for all $\lambda \neq \lambda_0$.*

Proof. First, note that $a_{0\lambda}(t^0) \neq 0$ for some $\lambda \in \Lambda$. Otherwise, $T f(t_0^0) = 0$ for all $f \in C_0^p(Q, E)$, contradicting with surjectivity of T .

Now, to obtain a contradiction, suppose that either $a_{0\lambda_0}(t^0) = 0$ or $a_{0\lambda'}(t^0) \neq 0$ for some $\lambda' \neq \lambda_0$. If the former holds, then from the claim above, we have $a_{0\lambda'}(t^0) \neq 0$ for some $\lambda' \neq \lambda_0$. In either case, we have $a_{0\lambda'}(t^0) \neq 0$ for some $\lambda' \neq \lambda_0$. So, $a_{0\lambda'}(t^0) e \neq 0$ for some $e \in E$. Since $t^k \rightarrow t^0$ and φ_{λ_0} is continuous on $K^\Gamma \setminus D$, so $\varphi_{\lambda_0}(t^k) \rightarrow \varphi_{\lambda_0}(t^0)$. Let $U, V \subseteq Q$ be bounded open neighbourhoods of $\varphi_{\lambda'}(t^0)$ such

that $\bar{V} \subseteq U$ and for each $\lambda \in \Lambda$, either $\varphi_\lambda(t^0) = \varphi_{\lambda'}(t^0)$ or $\varphi_\lambda(t^0) \notin U$. Choose $\phi \in C_0^\infty(Q)$ such that $\phi \equiv 1$ on V and $\text{supp}(\phi) \subseteq U$. Let $f : Q \rightarrow E$ be given by

$$f(s) = e\phi(s) \frac{(s - \varphi_{\lambda'}(t^0))^{\lambda'}}{\lambda'!}.$$

Clearly $f \in C_0^p(Q, E)$ and $\text{supp}(f) \subseteq U$. We claim that

$$Tf(t_0^0) = a_{0\lambda'}(t^0)e \neq 0. \quad (5.3.6)$$

Indeed, for each $\lambda \neq \lambda'$, if $\varphi_\lambda(t^0) = \varphi_{\lambda'}(t^0)$, then $\partial^\lambda f(\varphi_\lambda(t^0)) = \partial^\lambda f(\varphi_{\lambda'}(t^0)) = 0$. Otherwise, if $\varphi_\lambda(t^0) \notin U$, then clearly $\partial^\lambda f(\varphi_\lambda(t^0)) = 0$. In either case, we have $Tf(t_0^0) = \sum_{\lambda \in \Lambda} a_{0\lambda}(t^0) \partial^\lambda f(\varphi_\lambda(t^0)) = a_{0\lambda'}(t^0)e$.

We claim that $\varphi_{\lambda_0}(t^0) = \varphi_{\lambda'}(t^0)$. If $\varphi_{\lambda_0}(t^0) \neq \varphi_{\lambda'}(t^0)$, since $\varphi_{\lambda_0}(t^k)$ converges to $\varphi_{\lambda_0}(t^0)$, so $\varphi_{\lambda_0}(t^k) \notin U$ for large k . Since $\text{supp}(f) \subseteq U$, clearly $\partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k)) = 0$ for large k . Therefore, $Tf(t_0^k) = a_{0\lambda_0}(t^k) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k)) = 0$ for large k . Since $t_0^k \rightarrow t_0^0$, by continuity of Tf , we have $Tf(t_0^0) = 0$, contradicting to (5.3.6). This concludes the proof of the claim.

From the claim above, we have $\varphi_{\lambda_0}(t^k)$ converges to $\varphi_{\lambda'}(t^0) \in Q$. Since $(t_0^k)_{k \geq 1}$ is convergent, so is $(Tf(t_0^k))_{k \geq 1}$. So, $\sup_{k \geq 1} \|Tf(t_0^k)\| < \infty$. By Lemma 5.66, we have $\sup_{k \geq 1} \|a_{0\lambda_0}(t^k)\| < \infty$. Since $\partial^{\lambda_0} f$ is continuous and $\lambda_0 \neq \lambda'$, so $\partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k))$ converges to $\partial^{\lambda_0} f(\varphi_{\lambda'}(t^0)) = 0$. Therefore, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \|Tf(t_0^k)\| &= \lim_{k \rightarrow \infty} \|a_{0\lambda_0}(t^k) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k))\| \\ &\leq \sup_{k \geq 1} \|a_{0\lambda_0}(t^k)\| \cdot \lim_{k \rightarrow \infty} \|\partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k))\| = 0. \end{aligned}$$

Since $t_0^k \rightarrow t_0^0$, by continuity of Tf , we have $Tf(t_0^0) = 0$, contradicting (5.3.6). Therefore, we have $a_{0\lambda_0}(t^0) \neq 0$ and $a_{0\lambda}(t^0) = 0$ for all $\lambda \neq \lambda_0$. This concludes the proof of the lemma. $\square$

Let $t_0 \in K$ and $\gamma_0 = (0, \dots, 0) \in \Gamma$. Define $\mathcal{K}(t_0) = \{(t_\gamma)_{\gamma \in \Gamma} \in K^\Gamma \setminus D : t_{\gamma_0} = t_0\}$.

Proposition 5.68. *For all $t_0 \in K$, there exists $\lambda_{t_0} \in \Lambda$ such that*

$$Tf(t_0) = a_{0\lambda_{t_0}}(t) \partial^{\lambda_{t_0}} f(\varphi_{\lambda_{t_0}}(t)) \quad \text{and} \quad a_{0\lambda_{t_0}}(t) \neq 0, \quad (5.3.7)$$

for all $f \in C_0^p(Q, E)$ and $t \in \mathcal{K}(t_0)$, where $\varphi_{\lambda_{t_0}}$ depends only on t_0 .

Proof. Pick any $t_0 \in K$ and let $t = (t_\gamma)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$. Obtain V from Corollary 5.65. Since V is dense in $K^\Gamma \setminus D$, so there exists $(t^k)_{k \geq 1}$ where $t^k = (t_\gamma^k)_{\gamma \in \Gamma} \in V$ such that $t^k \rightarrow t$. Clearly $t_0^k \rightarrow t_0$. By Corollary 5.65, we have $Tf(t_0^k) = a_{0\lambda_{t^k}}(t^k) \partial^{\lambda_{t^k}} f(\varphi_{\lambda_{t^k}}(t^k))$ for all $k \geq 1$. Since $\#\Lambda$ is finite, by taking a subsequence if necessary, we may assume that $\lambda_{t^k} = \lambda_0$ for all $k \geq 1$. So, $Tf(t_0^k) = a_{0\lambda_0}(t^k) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^k))$ for all $k \geq 1$. On the other hand, since $t \in K^\Gamma \setminus D$, from Lemma 5.56, we have $Tf(t_0) = \sum_{\lambda \in \Lambda} a_{0\lambda}(t) \partial^\lambda f(\varphi_\lambda(t))$ for all $f \in C_0^p(Q, E)$. From Lemma 5.67, we have $Tf(t_0) = a_{0\lambda_0}(t) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t))$. This concludes the proof of the lemma. $\square$

Observe that Proposition 5.68 gives a well-defined map $t_0 \mapsto \lambda_{t_0}$ from K into Λ where λ_{t_0} satisfies (5.3.7).

Lemma 5.69. *The map $t_0 \mapsto \lambda_{t_0}$ is continuous.*

Proof. Suppose that $t_0^k \rightarrow t_0^0$ in K . Take a subsequence if necessary, we may assume that $\lambda_{t_0^k} = \lambda'$ for all $k \geq 1$. From Proposition 5.68,

$$Tf(t_0^k) = a_{0\lambda'}(t^k) \partial^{\lambda'} f(\varphi_{\lambda'}(t^k)) \quad \text{for all } f \in C_0^p(Q, E),$$

where $t^k = (t_\gamma^k)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$. Similarly, set $\lambda_{t_0^0} = \lambda_0$. Then from Proposition 5.68,

$$Tf(t_0^0) = a_{0\lambda_0}(t^0) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^0)) \quad \text{for all } f \in C_0^p(Q, E), \quad (5.3.8)$$

where $t^0 = (t_\gamma^0)_{\gamma \in \Gamma} \in K^\Gamma \setminus D$. We claim that $\lambda' = \lambda_0$. Indeed, from Lemma 5.56,

$$Tf(t_0^0) = \sum_{\lambda \in \Lambda} a_{0\lambda}(t^0) \partial^\lambda f(\varphi_\lambda(t^0)) \quad \text{for all } f \in C_0^p(Q, E).$$

From Lemma 5.67, we have $Tf(t_0^0) = a_{0\lambda'}(t^0) \partial^{\lambda'} f(\varphi_{\lambda'}(t^0))$. Therefore, from (5.3.8), we have

$$a_{0\lambda'}(t^0) \partial^{\lambda'} f(\varphi_{\lambda'}(t^0)) = a_{0\lambda_0}(t^0) \partial^{\lambda_0} f(\varphi_{\lambda_0}(t^0)) \quad \text{for all } f \in C_0^p(Q, E).$$

Since $a_{0\lambda'}(t^0) \neq 0$, so $a_{0\lambda'}(t^0)e \neq 0$ for some $e \in E$. If $\lambda' \neq \lambda_0$, then there exists $f \in C_0^p(Q, E)$ such that $\partial^{\lambda'} f(\varphi_{\lambda'}(t^0)) = e$ and $\partial^{\lambda_0} f(\varphi_{\lambda_0}(t^0)) = 0$. Then $a_{0\lambda'}(t^0)e = 0$, contradicting with the choice of e . Therefore, $\lambda' = \lambda_0$ and so $t_0 \mapsto \lambda_{t_0}$ is continuous. $\square$

Since the mapping $t_0 \mapsto \lambda_{t_0}$ is continuous and it takes a finite number of values, K can be partitioned into clopen sets $K = \cup K_\lambda$ so that $\lambda_{t_0} = \lambda$ for all $t_0 \in K_\lambda$. So, for all $t_0 \in K_\lambda$,

$$Tf(t_0) = a_{0\lambda}(t) \partial^\lambda f(\varphi_\lambda(t)) \quad \text{where} \quad t = (t_\gamma)_\gamma \in K^\Gamma \setminus D \quad (5.3.9)$$

where φ_λ is continuous on K_λ .

Recall that a mapping $h : K \rightarrow Q$ satisfies $(\clubsuit(T))$ if for any $t \in K$ and $f \in C_0^p(Q, E)$, if $(\partial^\lambda f(s)|_{s=h(t)})_{\lambda \in \Lambda} = 0$, then $Tf(t) = 0$. For each $t \in K$, define $h(t) = \varphi_\lambda(t)$ where $t \in K_\lambda$. Since K is partitioned into clopen sets $K = \cup K_\lambda$, so h is well-defined. From (5.3.9), clearly h satisfies $(\clubsuit(T))$.

We are now ready to state the main result of this chapter, the Banach-Stone Theorem for square norm, which follows immediately from Theorem 2.20.

Theorem 5.70. *Let $T : (C_0^p(Q, E), \|\cdot\|_{S,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,\theta})$ be a linear isometry where $p, q, m, n \geq 1$ are natural numbers, $Q \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^n$ are open and E, F are Banach spaces. If T is onto, E^* and F^* satisfy $(*)$ and $\rho = \|\cdot\|_s, \theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$, then $p = q, m = n$ and there exist a diffeomorphism $\phi : K \rightarrow Q$ of class C^p and for each $t \in K$, there is an Banach space isomorphism $V(t) : E \rightarrow F$ such that*

$$Tf(t) = V(t)f(\phi(t)) \quad \text{for every } t \in K, f \in C_0^p(Q, E).$$

Chapter 6

Further Properties of Isometry under Square Norm

Let $T : (C_0^p(Q, E), \|\cdot\|_{S,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,r})$ be a surjective linear isometry. By Theorem 5.70, we have $p = q$, $m = n$, and T takes the form $Tf(t) = V(t)f(\phi(t))$ for some class C^q diffeomorphism $\phi : K \rightarrow Q$ and Banach space isomorphisms $V(t) : E \rightarrow F$ for all $t \in K$. In this chapter, we investigated properties of V and ϕ . The main result is Theorem 6.17, which asserts that V and the Jacobian matrix of ϕ are locally constants and $V(t)$ is an isometry for all $t \in K$.

Denote $\phi = (\phi_1, \dots, \phi_m)$ where each $\phi_i : K \rightarrow \mathbb{R}$. The following proposition is crucial in the proof of Corollaries 6.4 and 6.5.

Proposition 6.1. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. Suppose that $T : (C_0^q(Q, E), \|\cdot\|_{S,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,r})$ is a surjective linear isometry that takes the form $Tf(t) = V(t)f(\phi(t))$ for some class C^q diffeomorphism $\phi : K \rightarrow Q$ and Banach space isomorphisms $V(t) : E \rightarrow F$ for all $t \in K$. For all $u_1, \dots, u_m \in E$ and $a_1, \dots, a_m \in K$, if $a_i \neq a_j$ for all $i \neq j$, then we have*

$$\begin{aligned} & \rho(\|u_1\|, \|u_2\|, \dots, \|u_m\|) \\ &= \theta \left[\left(\bigvee_{k=1}^m \|V(a_k) u_k\| |\partial_1 \phi_k(a_k)^{\gamma_1} \partial_2 \phi_k(a_k)^{\gamma_2} \dots \partial_m \phi_k(a_k)^{\gamma_m}| \right)_{|\gamma|=q} \right]. \end{aligned}$$

Proof. Pick any $u_1, \dots, u_m \in E \setminus \{0\}$ and $a_1, a_2, \dots, a_m \in K$ where $a_i \neq a_j$ for all $i \neq j$. Choose ε such that

$$\varepsilon \in \left(0, \bigwedge_{i,j=1}^m \frac{\|u_i\|}{\|u_j\|}\right). \quad (6.0.1)$$

Since ϕ is an injection, so $\phi(a_1), \dots, \phi(a_m)$ are distinct. Let $W_1, \dots, W_m$ be pairwise disjoint compact neighbourhoods of $a_1, \dots, a_m$ respectively where $W_k \subseteq B(a_k, \varepsilon)$ for all $1 \leq k \leq m$. Since $\phi(a_k) \in \phi(W_k)$ for each $1 \leq k \leq m$, so there exists $r_k > 0$ such that

$$\prod_{i=1}^m [\phi_i(a_k) - r_k, \phi_i(a_k) + r_k] \subseteq \phi(W_k). \quad (6.0.2)$$

Let

$$r = \frac{1}{4} \min_{1 \leq k \leq m} \{|\phi_k(a_i) - \phi_k(a_j)| : i \neq j\} \wedge \bigwedge_{k=1}^m r_k \wedge \varepsilon \wedge \frac{1}{2}.$$

Clearly $r \in (0, \frac{1}{2})$. By Proposition 5.1, there exist $h_0, h_q \in C^\infty(\mathbb{R})$ such that the supports of h_0 and h_q are subsets of $[-r, r]$, for each $k \in \{0, q\}$, we have $|h_k^{(k)}(t)| < 1$ for all $t \neq 0$, $\|h_k^{(k)}\|_\infty = |h_k^{(k)}(0)| = 1$, and

$$\bigvee_{i=0}^{q-1} \|h_q^{(i)}\|_\infty < \bigwedge_{j=0}^q \frac{\varepsilon}{\left(\|h_0^{(j)}\|_\infty\right)^{m-1}}. \quad (6.0.3)$$

So, for any $\lambda = (\lambda_i)_{i=1}^m \in \Lambda_f$ and $1 \leq k \leq m$, we have

$$\|h_q^{(\lambda_k)}\|_\infty \prod_{i=1, i \neq k}^m \|h_0^{(\lambda_i)}\|_\infty < \varepsilon. \quad (6.0.4)$$

Define $f : Q \rightarrow E$ by

$$\begin{aligned} f(s_1, s_2, \dots, s_m) &= h_q(s_1 - \phi_1(a_1)) h_0(s_2 - \phi_2(a_1)) \cdots h_0(s_m - \phi_m(a_1)) u_1 + \\ &\quad h_0(s_1 - \phi_1(a_2)) h_q(s_2 - \phi_2(a_2)) \cdots h_0(s_m - \phi_m(a_2)) u_2 + \\ &\quad \vdots \\ &\quad h_0(s_1 - \phi_1(a_m)) h_0(s_2 - \phi_2(a_m)) \cdots h_q(s_m - \phi_m(a_m)) u_m \\ &= \sum_{k=1}^m u_k h_q(s_k - \phi_k(a_k)) \prod_{i=1, i \neq k}^m h_0(s_i - \phi_i(a_k)). \end{aligned}$$

By $\text{supp}(h_0), \text{supp}(h_q) \subseteq [-r, r]$ and 6.0.2, we have $\text{supp}(h_q(\cdot - \phi_k(a_k)) \prod_{i \neq k} h_0(\cdot - \phi_i(a_k))) \subseteq \phi(W_k)$ for each $1 \leq k \leq m$. So, $\text{supp}(f) \subseteq \bigcup_{k=1}^m \phi(W_k)$. Clearly

$f \in C_0^p(Q, E)$. Pick $\lambda \in \Lambda_b$ with $\lambda = (0, \dots, 0, \overset{i_0}{q}, 0, \dots, 0)$ for some $1 \leq i_0 \leq m$. By the choice of r , we have $\|\partial_{i_0}^q f(\phi(a_{i_0}))\| = \|u_{i_0}\|$. On the other hand,

$$\begin{aligned} \|\partial_{i_0}^q f\|_\infty &\leq \left(\|u_{i_0}\| \cdot \|h_q^{(q)}\|_\infty \prod_{i \neq i_0} \|h_0\|_\infty \right) \vee \left(\bigvee_{k \neq i_0} \|u_k\| \cdot \|h_0^{(q)}\|_\infty \cdot \|h_q\|_\infty \prod_{i \notin \{k, i_0\}} \|h_0\|_\infty \right) \\ &\leq \|u_{i_0}\| \vee \left(\varepsilon \bigvee_{k \neq i_0} \|u_k\| \right) \\ &\leq \|u_{i_0}\| \end{aligned}$$

where the second and last inequalities above follow from 6.0.3 and 6.0.1 respectively.

Hence, $\|\partial_{i_0}^q f\|_\infty = \|u_{i_0}\|$ for all $\lambda \in \Lambda_b$ with $\lambda = (0, \dots, 0, \overset{i_0}{q}, 0, \dots, 0)$.

For any $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_f$, by disjointness and Inequality 6.0.4, we have

$$\begin{aligned} \|\partial^\lambda f\|_\infty &= \bigvee_{k=1}^m \sup_{s \in \phi(W_k)} \|\partial^\lambda f(s)\| \leq \bigvee_{k=1}^m \sup_{s \in \phi(W_k)} \|u_k\| \cdot \|h_q^{(\lambda_k)}\|_\infty \prod_{i=1, i \neq k}^m \|h_0^{(\lambda_i)}\|_\infty \\ &\leq \bigvee_{k=1}^m \|u_k\| \cdot \varepsilon. \end{aligned}$$

Hence,

$$\begin{aligned} \rho(\|u_1\|, \|u_2\|, \dots, \|u_m\|) &\leq \|f\|_{S, \rho} = \rho[(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda_f}, (\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda_b}] \\ &\leq (\#\Lambda_f)\varepsilon \bigvee_{k=1}^m \|u_k\| + \rho(\|u_1\|, \|u_2\|, \dots, \|u_m\|). \quad (6.0.5) \end{aligned}$$

Next, we calculate $\|Tf\|_{S, \theta}$. Recall that $Tf(t) = V(t)f(\phi(t))$ where V is of class C^{q-1} with respect to operator norm. Let $g = f \circ \phi$. So, for $|\gamma| < q$, we have

$$\partial^\gamma Tf(t) = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \partial^\mu g(t) \quad \text{for any } t \in K.$$

By Lemma 4.1, the equation above holds for $|\gamma| = q$ as well. By Constantine and Savits [17, Theorem 2.1], we can express

$$\partial^\mu g = \sum_{1 \leq |\lambda| \leq |\mu|} c_{\mu\lambda} \cdot \partial^\lambda f \circ \phi$$

for some $c_{\mu\lambda} \in C(K)$. Therefore,

$$\partial^\gamma T f(t) = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \sum_{1 \leq |\lambda| \leq |\mu|} c_{\mu\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \quad \text{for any } t \in K. \quad (6.0.6)$$

Fix any $|\gamma| < q$. From (6.0.6) and disjointness, we have

$$\begin{aligned} & \|\partial^\gamma T f\|_\infty \\ &= \bigvee_{k=1}^m \sup_{t \in W_k} \left\| \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \sum_{\substack{1 \leq |\lambda| \leq |\mu| \\ \lambda = (\lambda_i)_{i=1}^m}} c_{\mu\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| \\ &= \bigvee_{k=1}^m \sup_{t \in W_k} \left\| \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \sum_{\substack{1 \leq |\lambda| \leq |\mu| \\ \lambda = (\lambda_i)_{i=1}^m}} c_{\mu\lambda}(t) \cdot u_k h_q^{(\lambda_k)}(\phi_k(t) - \phi_k(a_k)) \prod_{i \neq k} h_0^{(\lambda_i)}(\phi_i(t) - \phi_i(a_k)) \right\| \\ &\leq \bigvee_{k=1}^m \sup_{t \in W_k} \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \|\partial^{\gamma-\mu} V(t) u_k\| \sum_{\substack{1 \leq |\lambda| \leq |\mu| \\ \lambda = (\lambda_i)_{i=1}^m}} |c_{\mu\lambda}(t)| \cdot \varepsilon \\ &\leq \varepsilon D. \end{aligned} \quad (6.0.7)$$

where $D = (q!)^m \bigvee_{k=1}^m \sum_{\mu \leq \gamma} \sup_{t \in W_k} \|\partial^{\gamma-\mu} V(t) u_k\| \sum_{1 \leq |\lambda| \leq |\mu|} \sup_{t \in W_k} |c_{\mu\lambda}(t)| < \infty$ and the second last inequality follows from Inequality 6.0.4.

Now, fix $|\gamma| = q$ and $1 \leq k \leq m$. Let $\lambda_0 = (0, \dots, 0, \overset{k}{q}, 0, \dots, 0)$. Then

$$\partial^\lambda f(\phi(a_k)) = \begin{cases} u_k & \text{if } \lambda = \lambda_0, \\ 0 & \text{if } \lambda \neq \lambda_0. \end{cases} \quad (6.0.8)$$

Therefore, we have

$$\begin{aligned} \partial^\gamma T f(a_k) &= \sum_{\mu \leq \gamma} \partial^{\gamma-\mu} V(a_k) \sum_{1 \leq |\lambda| \leq |\mu|} c_{\mu\lambda}(a_k) \cdot \partial^\lambda f(\phi(a_k)) && \text{by (6.0.6)} \\ &= V(a_k) u_k \cdot c_{\gamma\lambda_0}(a_k) && \text{by (6.0.8)} \\ &= V(a_k) u_k \cdot \partial_1 \phi_k(a_k)^{\gamma_1} \cdot \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m} && \text{by Lemma 4.3.} \end{aligned}$$

Hence, for any $|\gamma| = q$ we have

$$\begin{aligned} \|\partial^\gamma T f\|_\infty &\geq \bigvee_{k=1}^m \|\partial^\gamma T f(a_k)\| \\ &= \bigvee_{k=1}^m \|V(a_k)u_k\| \cdot |\partial_1 \phi_k(a_k)^{\gamma_1} \cdot \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m}|. \end{aligned} \quad (6.0.9)$$

On the other hand, for each $1 \leq k \leq m$, since $|\gamma| = q$, so

$$\begin{aligned} \|\partial^\gamma T f\|_\infty &= \bigvee_{k=1}^m \sup_{t \in W_k} \|\partial^\gamma T f(t)\| \\ &= \bigvee_{k=1}^m \sup_{t \in W_k} \left\| \sum_{\mu < \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \sum_{1 \leq |\lambda| \leq |\mu|} c_{\mu\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| \\ &\leq \bigvee_{k=1}^m \sup_{t \in W_k} \left\| V(t) \sum_{1 \leq |\lambda| \leq |\gamma|} c_{\gamma\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| \\ &\quad + \bigvee_{k=1}^m \sup_{t \in W_k} \left\| \sum_{\mu < \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) \sum_{1 \leq |\lambda| \leq |\mu|} c_{\mu\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| \\ &\leq \bigvee_{k=1}^m \sup_{t \in W_k} \left\| V(t) \sum_{1 \leq |\lambda| \leq |\gamma|} c_{\gamma\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| + \varepsilon D \quad \text{by Inequality 6.0.7.} \end{aligned}$$

Fix $1 \leq k \leq m$ and let $\lambda_0 = (0, \dots, 0, \overset{k}{q}, 0, \dots, 0)$. Then for all $t \in W_k$, from Lemma 4.3, we have

$$\begin{aligned} &\left\| V(t) \sum_{1 \leq |\lambda| \leq |\gamma|} c_{\gamma\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| \\ &\leq \|V(t)c_{\gamma\lambda_0}(t)\partial^{\lambda_0} f(\phi(t))\| + \left\| V(t) \sum_{\substack{1 \leq |\lambda| \leq |\gamma| \\ \lambda \neq \lambda_0}} c_{\gamma\lambda}(t) \cdot \partial^\lambda f(\phi(t)) \right\| \\ &\leq \|V(t)u_k \cdot \partial_1 \phi_k(t)^{\gamma_1} \partial_2 \phi_k(t)^{\gamma_2} \cdots \partial_m \phi_k(t)^{\gamma_m}\| + \varepsilon D. \end{aligned}$$

Therefore, for any $|\gamma| = q$, we have

$$\|\partial^\gamma T f\|_\infty \leq \left(\bigvee_{k=1}^m \sup_{t \in W_k} \|V(t)u_k \cdot \partial_1 \phi_k(t)^{\gamma_1} \partial_2 \phi_k(t)^{\gamma_2} \cdots \partial_m \phi_k(t)^{\gamma_m}\| \right) + 2\varepsilon D. \quad (6.0.10)$$

Hence, for all $0 < \varepsilon < \bigwedge_{i,j=1}^m \frac{\|u_i\|}{\|u_j\|}$, from Inequalities 6.0.7, 6.0.9 and 6.0.10 and monotonicity of θ , we have

$$\begin{aligned} & 3(\#\Gamma)\varepsilon D + \theta \left[\left(\bigvee_{k=1}^m \sup_{t \in W_k} \|V(t)u_k \partial_1 \phi_k(t)^{\gamma_1} \partial_2 \phi_k(t)^{\gamma_2} \cdots \partial_m \phi_k(t)^{\gamma_m}\| \right)_{|\gamma|=q} \right] \\ & \geq \|Tf\|_{S,\theta} \geq \theta \left[(0)_{|\gamma|<q}, (\|\partial^\gamma Tf\|_\infty)_{|\gamma|=q} \right] \\ & \geq \theta \left[\left(\bigvee_{k=1}^m \|V(a_k)u_k\| |\partial_1 \phi_k(a_k)^{\gamma_1} \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m}| \right)_{|\gamma|=q} \right]. \quad (6.0.11) \end{aligned}$$

Since T is an isometry, from Inequalities 6.0.5 and 6.0.11, we have

$$\begin{aligned} \rho(\|u_1\|, \|u_2\|, \dots, \|u_m\|) & \leq \|f\|_{S,\rho} = \|Tf\|_{S,\theta} \\ & \leq 3(\#\Gamma)\varepsilon D + \theta \left[\left(\bigvee_{k=1}^m \sup_{t \in W_k} \|V(t)u_k \partial_1 \phi_k(t)^{\gamma_1} \partial_2 \phi_k(t)^{\gamma_2} \cdots \partial_m \phi_k(t)^{\gamma_m}\| \right)_{|\gamma|=q} \right]. \end{aligned}$$

and

$$\begin{aligned} & \theta \left[\left(\bigvee_{k=1}^m \|V(a_k)u_k\| |\partial_1 \phi_k(a_k)^{\gamma_1} \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m}| \right)_{|\gamma|=q} \right] \\ & \leq \|Tf\|_{S,\theta} = \|f\|_{S,\rho} \leq (\#\Lambda_f)\varepsilon \bigvee_{k=1}^m \|u_k\| + \rho(\|u_1\|, \|u_2\|, \dots, \|u_m\|). \end{aligned}$$

Taking $\varepsilon \downarrow 0$, we have

$$\begin{aligned} & \rho(\|u_1\|, \|u_2\|, \dots, \|u_m\|) \\ & = \theta \left[\left(\bigvee_{k=1}^m \|V(a_k)u_k\| |\partial_1 \phi_k(a_k)^{\gamma_1} \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m}| \right)_{|\gamma|=q} \right] \end{aligned}$$

for all $u_1, \dots, u_m \in E \setminus \{0\}$. By continuity, the equation above also holds for all $u_1, \dots, u_m \in E$. This concludes the proof of the proposition. $\square$

6.1 $s = r$, $A(t)$ is a signed permutation matrix and $V(t)$ is an isometry

Recall that $A(t)$ is the Jacobian matrix of ϕ . The main result of this section is Proposition 6.14, which asserts that $s = r$, $A(t)$ is a signed permutation matrix and $V(t) : E \rightarrow F$ is an isometry for all $t \in K$. To do so, we consider two cases, either $m \geq 2$ or $m = 1$.

6.1.1 $m \geq 2$

We start this section with the following corollary, which follows immediately from Proposition 6.1.

Corollary 6.2. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. Fix any $a_1, \dots, a_m \in K$ with $a_i \neq a_j$ for all $i \neq j$ and $u \in E$ with $\|u\| = 1$. For any $b_1, \dots, b_m \geq 0$, we have*

$$\rho(b_1, \dots, b_m) = \theta \left[\left(\bigvee_{k=1}^m b_k \|V(a_k)u\| |\partial_1 \phi_k(a_k)^{\gamma_1} \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m}| \right)_{|\gamma|=q} \right].$$

Fix $a_1, \dots, a_m \in K$ with $a_i \neq a_j$ for all $i \neq j$ and $u \in E$ with $\|u\| = 1$, we let

$$x_{k\gamma}(u) = \|V(a_k)u\| |\partial_1 \phi_k(a_k)^{\gamma_1} \partial_2 \phi_k(a_k)^{\gamma_2} \cdots \partial_m \phi_k(a_k)^{\gamma_m}| \quad \text{for all } |\gamma| = q. \quad (6.1.1)$$

Clearly $x_{k\gamma}(u) \geq 0$. We denote $\mathbf{x}_k = \mathbf{x}_k(u) = (x_{k\gamma}(u))_{|\gamma|=q}$ for each $1 \leq k \leq m$. It follows from Corollary 6.2 that

$$\rho(b_1, \dots, b_m) = \theta \left[\left(\bigvee_{k=1}^m b_k x_{k\gamma}(u) \right)_{|\gamma|=q} \right]. \quad (6.1.2)$$

Corollary 6.3. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$. If $T : C_0^q(Q, E) \rightarrow C_0^q(K, F)$ is a surjective linear isometry and $m \geq 2$, then $s = r$.*

Proof. Fix any $a_1, \dots, a_m \in K$ with $a_i \neq a_j$ for all $i \neq j$ and $u \in E$ with $\|u\| = 1$. Fix $1 \leq k \leq m$ and let $b_i = 0$ if $i \neq k$ and $b_k = 1$. From (6.1.2), we have

$$1 = \theta \left[(x_{k\gamma}(u))_{|\gamma|=q} \right] = \|\mathbf{x}_k\|_r. \quad (6.1.3)$$

Since $m \geq 2$, consider $b_1 = b_2 = 1$ and $b_i = 0$ for all $3 \leq i \leq m$. Then (6.1.2) and 6.1.3 imply that

$$\begin{aligned} 2^{\frac{r}{s}} &= \rho(1, 1)^r = \theta \left[(x_{1\gamma}(u) \vee x_{2\gamma}(u))_{|\gamma|=q} \right]^r = \sum_{|\gamma|=q} |x_{1\gamma}(u) \vee x_{2\gamma}(u)|^r \\ &= \sum_{|\gamma|=q} x_{1\gamma}(u)^r \vee x_{2\gamma}(u)^r = \sum_{|\gamma|=q} (x_{1\gamma}(u)^r + x_{2\gamma}(u)^r - x_{1\gamma}(u)^r \wedge x_{2\gamma}(u)^r) \\ &= 2 - \sum_{|\gamma|=q} x_{1\gamma}(u)^r \wedge x_{2\gamma}(u)^r \leq 2. \end{aligned} \quad (6.1.4)$$

It follows that $r \leq s$. By symmetry, we can also conclude that $s \leq r$. Hence, $s = r$. This concludes the proof of the corollary. $\square$

Recall that two vectors $\mathbf{c} = (c_i)_{i=1}^m$ and $\mathbf{d} = (d_i)_{i=1}^m$ in $\mathbb{R}^m$ are *disjoint* if either $c_i = 0$ or $d_i = 0$ for all $1 \leq i \leq m$.

Corollary 6.4. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$. Fix any $a_1, \dots, a_m \in K$ with $a_i \neq a_j$ for all $i \neq j$ and $u \in E$ with $\|u\| = 1$. If $m \geq 2$, then $\|\mathbf{x}_k\|_r = 1$ for all $1 \leq k \leq m$ and $(\mathbf{x}_k)_{k=1}^m$ are pairwise disjoint vectors.*

Proof. Fix any $a_1, \dots, a_m \in K$ with $a_i \neq a_j$ for all $i \neq j$ and $u \in E$ with $\|u\| = 1$. From (6.1.3) in Corollary 6.3, we have $\|\mathbf{x}_k\|_r = 1$ for all $1 \leq k \leq m$. Fix $1 \leq i \neq j \leq m$. Choose $b_i = b_j = 1$ and $b_k = 0$ for all $k \in \{1, 2, \dots, m\} \setminus \{i, j\}$. By similar calculations in 6.1.4 in Corollary 6.3, we have $\sum_{|\gamma|=q} x_{i\gamma}(u)^r \wedge x_{j\gamma}(u)^r = 0$. So, for all $|\gamma| = q$, we have either $x_{i\gamma}(u) = 0$ or $x_{j\gamma}(u) = 0$. Therefore, $(\mathbf{x}_k)_{k=1}^m$ are disjoint vectors. This concludes the proof of the corollary. $\square$

For each $t \in K$, let $c(t) = \|V(t)\|^{1/q}$ and $A(t) = (\partial_j \phi_i(t))_{1 \leq i, j \leq m}$ be the Jacobian matrix of ϕ . A square matrix of scalars is a *generalized permutation matrix* if each row and each column contain exactly one nonzero entry. A *signed permutation matrix* is a generalized permutation matrix where nonzero entries are ± 1 . Recall that from Corollary 4.4, we have $A_4 = [(\partial_i \phi_j)^q]_{1 \leq i, j \leq m}$.

Corollary 6.5. *If $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$, then $\|V(t)u\| = c(t)^q \cdot \|u\|$ for all $t \in K$ and $u \in E$. Furthermore, if $m \geq 2$, then $c(t)A(t)$ is a signed permutation matrix for all $t \in K$.*

Proof. Fix any $t \in K$ and $u \in E$. Consider $a_2, \dots, a_m \in K$ where all elements in $\{t, a_2, \dots, a_m\}$ are mutually distinct. Let $b_1 = 1$ and $b_i = 0$ for all $2 \leq i \leq m$. From Corollary 6.2, we have

$$\|u\| = \|V(t)u\| \cdot \theta \left[(|\partial_1 \phi_1(t)|^{\gamma_1} \cdots |\partial_m \phi_m(t)|^{\gamma_m})_{|\gamma|=q} \right].$$

Therefore, $\|V(t)u\| = C(t)\|u\|$ for some positive constant $C(t)$ which is independent of u . Clearly $C(t) = \|V(t)\| = c(t)^q$. This proves the first assertion of the corollary. Assume that $m \geq 2$. We claim that for any $a_1, \dots, a_m \in K$ where $a_i \neq a_j$ for all $i \neq j$, any two columns of the matrix

$$\begin{pmatrix} c(a_1)^q \partial_1 \phi_1(a_1)^q & \cdots & c(a_m)^q \partial_1 \phi_m(a_m)^q \\ \vdots & \ddots & \vdots \\ c(a_1)^q \partial_m \phi_1(a_1)^q & \cdots & c(a_m)^q \partial_m \phi_m(a_m)^q \end{pmatrix} \quad (6.1.5)$$

are disjoint. Indeed, fix $1 \leq k_0 \neq k_1 \leq m$ and $u \in E$ with $\|u\| = 1$. From Corollary 6.4, $\mathbf{x}_{k_0}$ and $\mathbf{x}_{k_1}$ are disjoint, that is, $x_{k_0\gamma}(u) = 0$ or $x_{k_1\gamma}(u) = 0$ for all $|\gamma| = q$. In particular, for all $1 \leq j_0 \leq m$, if $\gamma = (\gamma_j)_{j=1}^m$ where $\gamma_{j_0} = q$ and $\gamma_j = 0$ for all $j \neq j_0$, then from (6.1.1) and disjointness, either $c(a_{k_0})^q \partial_{j_0} \phi_{k_0}(a_{k_0})^q = 0$ or $c(a_{k_1})^q \partial_{j_0} \phi_{k_1}(a_{k_1})^q = 0$. Since k_0 and k_1 are arbitrary, this concludes the proof of the claim.

From the claim above and continuity, we can deduce that for all $t \in K$, any two columns of $c(t)^q A_4(t)$ are disjoint. Therefore, $c(t)^q A_4(t)$ is a generalized permutation matrix. Hence, for each $1 \leq k \leq m$, we have $x_{k\gamma}(u) = 0$ for all $\gamma \in \Gamma_f$ with $|\gamma| = q$. From Corollary 6.4, $\|\mathbf{x}_k\|_r = 1$ for all $1 \leq k \leq m$. Therefore, all nonzero entries of $c(t)^q A_4(t)$ are ± 1 . Hence, from Lemma 4.11, $c(t)A(t)$ is a signed permutation matrix. This concludes the proof of the corollary. $\square$

The following lemma follows immediately from Corollary 6.5, Lemma 4.25 and Remark 4.26.

Lemma 6.6. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$. If $m \geq 2$, then on each connected component U of K , there exists a positive scalar c such that*

$c(t) = c$ and $\phi(t) = \frac{1}{c}(\varepsilon_i t_{\pi(i)})$ where $\varepsilon_i = \pm 1$ and $\pi : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ is a bijection for all $t = (t_1, \dots, t_m) \in U$.

After permutation and flipping, without loss of generality, take $\phi(t) = (\frac{t_1}{c}, \dots, \frac{t_m}{c}) = \frac{t}{c}$. So, we have

$$Tf(t) = V(t)f\left(\frac{t}{c}\right). \quad (6.1.6)$$

From definition of c and Lemma 6.6, we have $\|V(t)u\| = c^q$ for all u with $\|u\| = 1$.

Lemma 6.7. Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$. If $m \geq 2$, then $c = 1$ and $V(t)$ is an isometry for all $t \in K$.

Proof. From the definition of c , it suffices to prove that $c = 1$. Pick any $(s_\lambda)_{\lambda \in \Lambda} \in Q^\Lambda \setminus D$ and denote $s_\lambda = \phi(t_\lambda) = \frac{t_\lambda}{c}$. By Corollary 5.5, there exist a sequence of positive scalars $(\beta_\lambda)_{\lambda \in \Lambda}$ and a sequence of functions $(H_\lambda)_{\lambda \in \Lambda}$ in $C_0^p(Q)$ such that $(\text{supp}(H_\lambda(\cdot - s_\lambda)))_{\lambda \in \Lambda}$ is mutually disjoint and for each $\lambda \in \Lambda$, $\partial^\lambda H_\lambda(\mathbf{0}) = 1$, $\|\partial^\lambda H_\lambda\|_\infty = 1$ and $\partial^\lambda H_\lambda$ peaks at the origin only, for all $\mu \in \Lambda$ with $\mu < \lambda$, $\partial^\mu H_\lambda(\mathbf{0}) = 0$ and

$$\beta_\lambda \cdot \|\partial^\mu H_\lambda\|_\infty < \beta_\mu \quad \text{for all } \lambda \neq \mu. \quad (6.1.7)$$

Choose a sequence of norm one vectors $(e_\lambda)_{\lambda \in \Lambda}$ in E . Define $f : Q \rightarrow E$ by

$$f(s) = \sum_{\lambda \in \Lambda} \beta_\lambda e_\lambda H_\lambda(s - s_\lambda).$$

We claim that $\|\partial^\lambda f\|_\infty = \beta_\lambda$ for all $\lambda \in \Lambda$. Indeed, $\|\partial^\lambda f\|_\infty \geq \|\partial^\lambda f(s_\lambda)\| = \beta_\lambda$. For any $s \in Q$, by disjointness and Inequality 6.1.7,

$$\|\partial^\lambda f(s)\| \leq \bigvee_{\mu \in \Lambda} \beta_\mu \|\partial^\lambda H_\mu\|_\infty \leq \beta_\lambda.$$

This concludes the proof of the claim. Since $r = s$ from Corollary 6.3, it follows that

$$\begin{aligned} \rho[(\beta_\lambda)_{\lambda \in \Lambda}] &= \rho[(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda}] = \|f\|_{s,\rho} = \|Tf\|_{s,\rho} = \rho[(\|\partial^\lambda Tf\|_\infty)_{\lambda \in \Lambda}] \\ &\geq \rho[(\|\partial^\lambda Tf(t_\lambda)\|)_{\lambda \in \Lambda}] \stackrel{*}{=} \rho \left[\left(\frac{\beta_\lambda \|V(t_\lambda)e_\lambda\|}{c^{|\lambda|}} \right)_{\lambda \in \Lambda} \right] = \rho \left[(\beta_\lambda c^{q-|\lambda|})_{\lambda \in \Lambda} \right]. \end{aligned}$$

where the $(*)$ equality follows from (6.1.6) and disjointness of $H_\lambda(\cdot - s_\lambda)$. Therefore, we must have $c \leq 1$. By symmetry, we also have $c \geq 1$. Thus, $c = 1$. This concludes the proof of the lemma. $\square$

The following corollary follows immediately from Corollaries 6.3, 6.5 and Lemma 6.7.

Corollary 6.8. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$. If $m \geq 2$, then $s = r$, $A(t)$ is a signed permutation matrix and $V(t)$ is an isometry for all $t \in K$.*

6.1.2 $m = 1$

In this section, we start with the proof of $s = r$. To this end, we establish the following preliminary result.

Lemma 6.9. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. If T is a surjective linear isometry, then there exist scalars $(b_i)_{0 \leq i \leq q}$ where $b_i > 0$ and an open subset O of $\mathbb{R}^{q+1}$ such that for all $(\alpha_i)_{i=0}^q \in O$, we have $\rho[(\alpha_i)_{i=0}^q] = \theta[(\alpha_i \cdot b_i)_{i=0}^q]$.*

Proof. Pick any $(s_i)_{i=0}^q \in Q^{q+1} \setminus D$ and $e_0, e_1, \dots, e_q \in E$ with $\|e_i\| = 1$ for all $0 \leq i \leq q$. Let $\varepsilon = \frac{1}{2} \min\{|s_i - s_j| : i \neq j\} \wedge \frac{1}{2}$. Clearly $\varepsilon \in (0, \frac{1}{2})$. From Corollary 5.5, there exist $(h_i)_{i=0}^q$ in $C_0^q(Q)$ and an open set $O \subseteq \mathbb{R}^{q+1}$ such that for all $0 \leq i \leq q$, $\|h_i^{(i)}\|_\infty = |h_i^{(i)}(0)| = 1$, and for all $(\alpha_i)_{i=0}^q \in O$, we have

$$\alpha_i \cdot \|h_i^{(j)}\| < \alpha_j \quad \text{for all } i \neq j. \quad (6.1.8)$$

Fix $(\alpha_i)_{i=0}^q \in O$ and define $f : Q \rightarrow E$ by $f(s) = \sum_{i=0}^q \alpha_i e_i h_i(s - s_i)$. Since $(\text{supp}(h_i(\cdot - s_i)))_{i=0}^q$ are mutually disjoint and $\|h_i^{(i)}\|_\infty = |h_i^{(i)}(0)| = 1$, from Inequality 6.1.8, we have $\|f^{(i)}\|_\infty = \alpha_i$ for all $0 \leq i \leq q$. Therefore, $\|f\|_{s,\rho} = \rho((\alpha_i)_{i=0}^q)$. For each $0 \leq i \leq q$, let $g_i = T(h_i(\cdot - s_i)e_i)$. Clearly g_i is a nonzero function and independent of f . Since T is separating, so g_i are mutually disjoint. From linearity of T , we have $Tf = \sum_{i=0}^q \alpha_i g_i$. Therefore,

$$\|Tf^{(j)}\|_\infty = \bigvee_{i=0}^q \alpha_i \cdot \|g_i^{(j)}\|_\infty \quad \text{for each } 0 \leq j \leq q.$$

For each $0 \leq i, j, k \leq q$, denote

$$S_{i,j}^k = \{(\alpha_0, \dots, \alpha_q) \in \mathbb{R}^{q+1} : \alpha_i \cdot \|g_i^{(k)}\|_\infty = \alpha_j \cdot \|g_j^{(k)}\|_\infty\}.$$

Let $S = \bigcup_{0 \leq i,j,k \leq q} S_{i,j}^k$. Since $g_i \in C_0(K, F)$ and $g_i \neq 0$, so $\|g_i^{(k)}\|_\infty \neq 0$ for all $0 \leq i, k \leq q$. Therefore, S is a nonempty, closed and nowhere dense subset in $\mathbb{R}^{q+1}$. Hence, $O \setminus S$ is an open subset of $\mathbb{R}^{q+1}$. By replacing O with $O \setminus S$, we may assume that for all $(\alpha_i)_{i=0}^q \in O$ and $0 \leq j \leq q$, there exists a unique $i_j \in \{0, 1, \dots, q\}$ and $f \in C_0^p(Q, E)$ such that $\|f\|_{s,\rho} = \rho((\alpha_i)_{i=0}^q)$ and $\|Tf^{(j)}\|_\infty = \alpha_{i_j} \cdot \|g_{i_j}^{(j)}\|_\infty$.

Fix any $0 \leq j \leq q$ and $(\alpha_i)_{i=0}^q \in O$. Choose $i_j \in \{0, 1, \dots, q\}$ and $f \in C_0^p(Q, E)$ such that $\|Tf^{(j)}\|_\infty = \alpha_{i_j} \cdot \|g_{i_j}^{(j)}\|_\infty$. Therefore, $\alpha_{i_j} \cdot \|g_{i_j}^{(j)}\|_\infty > \alpha_i \cdot \|g_i^{(j)}\|_\infty$ for all $i \neq i_j$. Then there exists an open neighbourhood $A_j \subseteq O$ of $(\alpha_i)_{i=0}^q$ such that for all $(\alpha'_i)_{i=0}^q \in A_j$, we have $\|Tf_1^{(j)}\|_\infty = \alpha'_{i_j} \cdot \|g_{i_j}^{(j)}\|_\infty$ for some $f_1 \in C_0^p(Q, E)$. By replacing O with $\bigcap_{0 \leq j \leq q} A_j$, we may assume that for any $0 \leq j \leq q$, there exists a unique $i_j \in \{0, 1, \dots, q\}$ such that for any $(\alpha_i)_{i=0}^q \in O$, there exists $f \in C_0^p(Q, E)$ such that $\|f\|_{s,\rho} = \rho((\alpha_i)_{i=0}^q)$ and $\|Tf^{(j)}\|_\infty = \alpha_{i_j} \cdot \|g_{i_j}^{(j)}\|_\infty$. Since T is an isometry, so for any $(\alpha_i)_{i=0}^q \in O$, we have

$$\rho((\alpha_i)_{i=0}^q) = \theta((\alpha_{i_j} \|g_{i_j}^{(j)}\|_\infty)_{j=0}^q). \quad (6.1.9)$$

Since the LHS of (6.1.9) is dependent on all α_i 's, so is the RHS. Therefore, $(i_j)_{j=0}^q$ must be a permutation of $\{0, 1, \dots, q\}$. Since $\theta = \|\cdot\|_r$ is symmetric, by rearranging, we have $\rho((\alpha_i)_{i=0}^q) = \theta((\alpha_i \|g_i^{(j_i)}\|_\infty)_{i=0}^q)$ for all $(\alpha_i)_{i=0}^q \in O$, where j_i is the inverse permutation of i_j . This concludes the proof of the lemma. $\square$

Corollary 6.10. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. If T is a surjective linear isometry, then $s = r$.*

Proof. Obtain the scalars $(b_j)_{0 \leq j \leq q}$ and an open subset O of $\mathbb{R}^{q+1}$ from Lemma 6.9. Then for any $(\alpha_i)_{i=0}^q \in O$, we have

$$\left(\sum_{i=0}^q \alpha_i^s \right)^{1/s} = \left(\sum_{j=0}^q (\alpha_{i_j} b_j)^r \right)^{1/r}. \quad (6.1.10)$$

Fix $\alpha_0, \dots, \alpha_{q-1}$ and vary α_q . Differentiating both sides of the equation above with respect to α_q gives

$$\frac{1}{s} \left(\sum_{i=0}^q \alpha_i^s \right)^{\frac{1}{s}-1} s \cdot \alpha_q^{s-1} = \frac{1}{r} \left(\sum_{j=0}^q (\alpha_j b_j)^r \right)^{\frac{1}{r}-1} b_q^r \cdot r \cdot \alpha_q^{r-1}.$$

From (6.1.10), we have

$$\left(\sum_{i=0}^q \alpha_i^s \right)^{-1} \alpha_q^{s-1} = \left(\sum_{j=0}^q (\alpha_j b_j)^r \right)^{-1} b_q^r \cdot \alpha_q^{r-1}.$$

It follows that $\alpha_q^{s-r} = \frac{b_q^r (\sum_{i=0}^{q-1} \alpha_i^s)}{\sum_{j=0}^{q-1} (\alpha_j \cdot b_j)^r}$. Since RHS is independent of α_q in the neighborhood, so is LHS. Therefore $s - r = 0$. This concludes the proof of the corollary. $\square$

The following corollary follows immediately from Proposition 6.1.

Corollary 6.11. *If $m = 1$, then $|\phi'(t)|^q \cdot V(t) : E \rightarrow F$ is an isometry for all $t \in K$.*

Lemma 6.12. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $s, r \in [1, \infty)$. If $m = 1$, then $|\phi'(t)| = 1$ and $V(t) : E \rightarrow F$ is an isometry for all $t \in K$.*

Proof. From Corollary 6.11, it suffices to show that $|\phi'(t)| = 1$ for all $t \in K$. We claim that $|\phi'(t)| \geq 1$ for all $t \in K$. Suppose not. Then $|\phi'(t')| < 1$ for some $t' \in K$. Since $|\phi'|$ is continuous, there exists a neighbourhood U of t' such that $|\phi'(t)| < 1$ for all $t \in U$. Clearly U is an infinite set. Pick any $t_0, \dots, t_q \in U$ such that $t_i \neq t_j$ for all $0 \leq i \neq j \leq q$. Denote $s_i = \phi(t_i)$ for all $0 \leq i \leq q$. By Corollary 5.5, there exist a sequence of positive scalars $(\alpha_i)_{i=0}^q$ and a sequence of functions $(h_i)_{i=0}^q$ in $C_0^q(Q)$ such that $(\text{supp}(h_i(\cdot - s_i)))_{i=0}^q$ is mutually disjoint and for each $0 \leq i \leq q$, $h_i^{(i)}(0) = 1$, $\|h_i^{(i)}\|_\infty = 1$ and $h_i^{(i)}$ peaks at the 0 only, for all $k < i$, $h_i^{(k)}(0) = 0$ and

$$\alpha_i \cdot \|h_i^{(j)}\|_\infty < \alpha_j \quad \text{for all } i \neq j. \quad (6.1.11)$$

Pick any $e_0, \dots, e_q \in S(E)$ and consider $f : Q \rightarrow E$ defined by $f(s) = \sum_{k=0}^q \alpha_k e_k h_k(s - s_k)$. From (6.1.11) and disjointness of h_i , we have $\|f^{(i)}\|_\infty = \|f^{(i)}(s_i)\| = \alpha_i$ for all

$0 \leq i \leq q$. Observe that for each $0 \leq j \leq q$, if $i < j$, then by disjointness, we have $f^{(i)}(s_j) = \alpha_j e_j h_j^{(i)}(0) = 0$. Therefore,

$$Tf^{(j)}(t_j) = V(t_j)f^{(j)}(\phi(t_j)) \cdot (\phi'(t_j))^j$$

Note that from Corollary 6.5, we have $|\phi'(t)|^q \cdot V(t)$ is an isometry. Also, from Corollary 6.10, we have $\rho = \theta$. Therefore,

$$\begin{aligned} \rho((\alpha_i)_{i=0}^q) &= \rho((\|f^{(i)}\|_\infty)_{i=0}^q) = \|f\|_{S,\rho} = \|Tf\|_{S,\rho} \geq \rho((\|Tf^{(j)}(t_j)\|)_{j=0}^q) \\ &= \rho((\|V(t_j)f^{(j)}(\phi(t_j)) \cdot (\phi'(t_j))^j\|)_{j=0}^q) \\ &= \rho\left[\left(\frac{\alpha_j}{|\phi'(t_j)|^{q-j}}\right)_{j=0}^q\right]. \end{aligned}$$

For all $0 \leq j \leq q$, since $t_j \in U$, so $|\phi'(t_j)| < 1$. Therefore, by strict monotonicity of ρ , we have

$$\rho((\alpha_i)_{i=0}^q) \geq \rho\left[\left(\frac{\alpha_j}{|\phi'(t_j)|^{q-j}}\right)_{j=0}^q\right] > \rho[(\alpha_j)_{j=0}^q],$$

a contradiction. Hence, $|\phi'(t)| \geq 1$ for all $t \in K$.

By symmetry, $|\phi'(t)| \leq 1$ for all $t \in K$. Thus, $|\phi'(t)| = 1$ for all $t \in K$. This concludes the proof of the lemma. $\square$

The following corollary is immediate from Corollaries 6.10, 6.11 and Lemma 6.12

Corollary 6.13. *Assume that $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$. If $m = 1$, then $s = r$, $|\phi'(t)| = 1$ and $V(t)$ is an isometry for all $t \in K$.*

The following proposition summarises all results in this section, which follows immediately from Corollaries 6.8 and 6.13

Proposition 6.14. *If $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$, then $s = r$, the Jacobian matrix $A(t)$ of ϕ is a signed permutation matrix and $V(t) : E \rightarrow F$ is a surjective linear isometry for all $t \in K$.*

6.2 V is locally constant

The following lemma asserts that V is locally constant. Assume that $\phi(0) = 0$.

Proposition 6.15. *If $\rho = \|\cdot\|_s$ and $\theta = \|\cdot\|_r$ where $r, s \in [1, \infty)$, then $\partial_j V(t)u = 0$ for all $t \in K$, $u \in E$ and $1 \leq j \leq m$.*

Proof. Suppose that $\partial_1 V(0)e_{\mathbf{0}} \neq 0$ for some $e_{\mathbf{0}} \neq 0$. Pick any $(\tilde{t}_\gamma)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)}$ in K such that they are pairwise distinct and nonzero. Also, choose any norm one $(e_\lambda)_{\lambda \neq \mathbf{0}, \lambda \neq (1,0,\dots,0)} \subseteq E$. Let $s_\lambda = \phi(\tilde{t}_\lambda)$ for all $\lambda \in \Lambda \setminus \{\mathbf{0}, (1,0,\dots,0)\}$ and $s_{(1,0,\dots,0)} = 0$. Note that s_λ 's are pairwise distinct for all $\lambda \neq \mathbf{0}$.

Since $V(0)$ is onto, so $\partial_1 V(0)e_{\mathbf{0}} = bV(0)e_{(1,0,\dots,0)}$ for some $e_{(1,0,\dots,0)} \in S(E)$ and $b > 0$. By Corollary 5.5, there exist a sequence of positive scalars $(\alpha_\lambda)_{\lambda \neq \mathbf{0}}$ and a sequence of functions $(H_\lambda)_{\lambda \neq \mathbf{0}}$ in $C_0^p(Q)$ such that supports of $H_\lambda(\cdot - s_\lambda)$ are mutually disjoint and for each $\lambda \neq \mathbf{0}$, $\partial^\lambda H_\lambda(\mathbf{0}) = 1$, $\|\partial^\lambda H_\lambda\|_\infty = 1$ and $\partial^\lambda H_\lambda$ peaks at the origin only, for all $\mu \in \Lambda$ with $\mu < \lambda$, $\partial^\mu H_\lambda(\mathbf{0}) = 0$ and

$$\alpha_\lambda \cdot \|\partial^\mu H_\lambda\|_\infty < \alpha_\mu \text{ for all } \lambda \neq \mu. \quad (6.2.1)$$

For each $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda$, recall that

$$H_{(\lambda_1, \lambda_2, \dots, \lambda_m)}(s_1, \dots, s_m) = h_{\lambda_1}(s_1) \cdot h_{\lambda_2}(s_2) \cdots h_{\lambda_m}(s_m)$$

where each h_{λ_i} is obtained from Proposition 5.1. Choose $k_0 \in C_0^p(Q)$ such that $\text{supp}(k_0) \subseteq \text{supp}(h_1)$ and $k_0 \equiv 1$ on $\{|h'_1| > \frac{1}{2}\}$. Define $H_{(0,\dots,0)} : Q \rightarrow \mathbb{R}$ by

$$H_{(0,\dots,0)}(s_1, \dots, s_m) = k_0(s_1) \cdot h_0(s_2) \cdots h_0(s_m).$$

Since $\text{supp}(k_0) \subseteq \text{supp}(h_1)$, so $\text{supp}(H_{(0,\dots,0)}) \subseteq \text{supp}(H_{(1,0,\dots,0)})$. Define $f : Q \rightarrow E$ given by

$$f(s) = \alpha_{\mathbf{0}} e_{\mathbf{0}} H_{(0,\dots,0)}(s) + \sum_{\lambda \neq \mathbf{0}} \alpha_\lambda e_\lambda H_\lambda(s - s_\lambda)$$

where $\alpha_{\mathbf{0}} > 0$ satisfies

$$\alpha_{\mathbf{0}} < \left(\bigwedge_{\lambda \neq (1,0,\dots,0)} \frac{\alpha_\lambda - \alpha_{(1,0,\dots,0)} \|\partial^\lambda H_{(1,0,\dots,0)}\|_\infty}{\|k_0^{(\lambda_1)}\|_\infty \|h_0^{(\lambda_2)}\|_\infty \cdots \|h_0^{(\lambda_m)}\|_\infty} \right) \wedge \left(\frac{\alpha_{(1,0,\dots,0)}}{2\|e_{\mathbf{0}}\| \|k'_0\|_\infty} \right). \quad (6.2.2)$$

We claim that $\|\partial^\lambda f\|_\infty = \alpha_\lambda$ for all $\lambda \neq \mathbf{0}$. Indeed, if $\lambda \neq \mathbf{0}$ and $\lambda \neq (1,0,\dots,0)$, by disjointness of H_λ , then $\|\partial^\lambda f\|_\infty \geq \|\partial^\lambda f(s_\lambda)\| = \alpha_\lambda$. For any $s \in K$, from

Inequality 6.2.1, we have

$$\begin{aligned}
& \|\partial^\lambda f(s)\| \\
& \leq \|\alpha_{\mathbf{0}} e_{\mathbf{0}} \partial^\lambda H_{\mathbf{0}}(s) + \alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)} \partial^\lambda H_{(1,0,\dots,0)}(s)\| \vee \bigvee_{\mu \neq \mathbf{0}, \mu \neq (1,0,\dots,0)} \alpha_\mu \|\partial^\lambda H_\mu\|_\infty \\
& \leq \left(\alpha_{\mathbf{0}} \|k_0^{(\lambda_1)}\|_\infty \|h_0^{(\lambda_2)}\|_\infty \cdots \|h_0^{(\lambda_m)}\|_\infty + \alpha_{(1,0,\dots,0)} \|\partial^\lambda H_{(1,0,\dots,0)}\|_\infty \right) \vee \alpha_\lambda \\
& \leq \alpha_\lambda,
\end{aligned}$$

where the last inequality follows from Inequality 6.2.2. Therefore, if $\lambda \neq (0, 0, \dots, 0)$ and $\lambda \neq (1, 0, \dots, 0)$, then $\|\partial^\lambda f\|_\infty = \alpha_\lambda$.

Now, let $\lambda = (1, 0, \dots, 0)$. Since $|h'_1(0)| = 1$ and $k_0 \equiv 1$ on $\{|h'_1| > \frac{1}{2}\}$, so

$$\|\partial_1 f\|_\infty \geq \|\partial_1 f(s_{(1,0,\dots,0)})\| = \alpha_{(1,0,\dots,0)}.$$

Pick any $s = (s_1, \dots, s_m) \in \text{supp } H_{(1,0,\dots,0)}$. If $s_1 \in \{|h'_1| > \frac{1}{2}\}$, then $k'_0(s_1) = 0$. So,

$$\begin{aligned}
& \|\partial_1 f(s)\| \\
& = \|\alpha_{\mathbf{0}} e_{\mathbf{0}} \partial_1 H_{(0,0,\dots,0)}(s) + \alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)} \partial_1 H_{(1,0,\dots,0)}(s)\| \\
& = \|\alpha_{\mathbf{0}} e_{\mathbf{0}} k'_0(s_1) h_0(s_2) \cdots h_0(s_m) + \alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)} h'_1(s_1) h_0(s_2) \cdots h_0(s_m)\| \\
& = \|0 + \alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)} h'_1(s_1) h_0(s_2) \cdots h_0(s_m)\| \\
& \leq \alpha_{(1,0,\dots,0)}.
\end{aligned}$$

If $s_1 \notin \{|h'_1| > \frac{1}{2}\}$, then

$$\begin{aligned}
& \|\partial_1 f(s)\| \\
& = \|\alpha_{\mathbf{0}} e_{\mathbf{0}} \partial_1 H_{(0,0,\dots,0)}(s) + \alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)} \partial_1 H_{(1,0,\dots,0)}(s)\| \\
& \leq \|\alpha_{\mathbf{0}} e_{\mathbf{0}} k'_0(s_1) h_0(s_2) \cdots h_0(s_m)\| + \|\alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)} h'_1(s_1) h_0(s_2) \cdots h_0(s_m)\| \\
& \leq \alpha_{\mathbf{0}} \|e_{\mathbf{0}}\| \|k'_0\|_\infty + \alpha_{(1,0,\dots,0)} \cdot \frac{1}{2} \\
& \leq \alpha_{(1,0,\dots,0)}
\end{aligned}$$

where the last inequality follows from Inequality 6.2.2. Now, if $s \in \text{supp } H_\mu$ for some $\mu \in \Lambda \setminus \{\mathbf{0}, (1, 0, \dots, 0)\}$, then $\|\partial_1 f(s)\| = \alpha_\mu \|\partial_1 H_\mu(s)\| \leq \alpha_\mu \|\partial_1 H_\mu\|_\infty < \alpha_{(1,0,\dots,0)}$. Therefore, $\|\partial_1 f\|_\infty = \alpha_{(1,0,\dots,0)}$. This concludes the proof of the claim.

Now, we calculate $\|Tf\|_{S,\rho}$. Note that

$$\begin{aligned}
& \rho \left(\|f\|_\infty, (\alpha_\lambda)_{\lambda \neq \mathbf{0}} \right) \\
&= \rho \left(\|f\|_\infty, (\|\partial^\lambda f\|_\infty)_{\lambda \neq \mathbf{0}} \right) \\
&= \|f\|_{S,\rho} = \|Tf\|_{S,\rho} \\
&= \rho \left(\|Tf\|_\infty, (\|\partial^\gamma Tf\|_\infty)_{\gamma \neq \mathbf{0}} \right) \\
&\geq \rho \left(\|Tf\|_\infty, \|\partial_1 Tf(0)\|, (\|\partial^\gamma Tf(\tilde{t}_\gamma)\|)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right) \\
&= \rho \left(\|f\|_\infty, \|\partial_1 V(0)f(0) + V(0)\partial_1 f(0)\|, (\|\partial^\gamma Tf(\tilde{t}_\gamma)\|)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right).
\end{aligned}$$

Since $V(t) : E \rightarrow F$ is an isometry from Proposition 6.14, so for all $\gamma \in \Gamma \setminus \{0, (1, 0, \dots, 0)\}$, we have

$$\begin{aligned}
\|\partial^\gamma Tf(\tilde{t}_\gamma)\| &= \|\partial^\gamma [V(t)f(\phi(t))]\|_{t=\tilde{t}_\gamma} \\
&= \|\alpha_\gamma \partial^\gamma [V(t)e_\gamma H_\gamma(\phi(t) - \phi(\tilde{t}_\gamma))]\|_{t=\tilde{t}_\gamma} \\
&= \alpha_\gamma \left\| \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \partial^{\gamma-\mu} V(t) e_\gamma \partial^\mu H_\gamma(\phi(t) - \phi(\tilde{t}_\gamma)) \right\|_{t=\tilde{t}_\gamma} \\
&= \alpha_\gamma \|V(\tilde{t}_\gamma)e_\gamma\| = \alpha_\gamma, \quad \text{since } \partial^\mu H_\gamma(0) = 0 \text{ for all } \mu < \gamma.
\end{aligned}$$

Therefore, since $\alpha_0 b > 0$, so

$$\begin{aligned}
& \rho \left(\|f\|_\infty, (\alpha_\lambda)_{\lambda \neq \mathbf{0}} \right) \\
&\geq \rho \left(\|f\|_\infty, \|\partial_1 V(0)f(0) + V(0)\partial_1 f(0)\|, (\|\partial^\gamma Tf(\tilde{t}_\gamma)\|)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right) \\
&= \rho \left(\|f\|_\infty, \|\partial_1 V(0)f(0) + V(0)\partial_1 f(0)\|, (\alpha_\gamma)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right) \\
&= \rho \left(\|f\|_\infty, \|\partial_1 V(0)\alpha_0 e_0 + V(0)\alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)}\|, (\alpha_\gamma)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right) \\
&= \rho \left(\|f\|_\infty, \|\alpha_0 b V(0)e_{(1,0,\dots,0)} + V(0)\alpha_{(1,0,\dots,0)} e_{(1,0,\dots,0)}\|, (\alpha_\gamma)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right) \\
&= \rho \left(\|f\|_\infty, \alpha_0 b + \alpha_{(1,0,\dots,0)}, (\alpha_\gamma)_{\gamma \neq \mathbf{0}, \gamma \neq (1,0,\dots,0)} \right) > \rho \left(\|f\|_\infty, (\alpha_\lambda)_{\lambda \neq \mathbf{0}} \right),
\end{aligned}$$

a contradiction. This concludes the proof of the lemma. $\square$

Recall that $A(t) = (a_{ij}(t))_{1 \leq i,j \leq m}$ where $a_{ij}(t) = \partial_j \phi_i(t)$. From Proposition 6.14, $A(t)$ is a signed permutation matrix on K . So, $|a_{ij}(t)| \in \{0, 1\}$. By continuity,

$a_{ij}(t)$ is constant on connected components of K . The next result says that $|a_{ij}(t)|$ is constant on K for all i, j . Denote $|A|(t) = (|a_{ij}(t)|)$.

Proposition 6.16. *If $T : (C_0^p(Q, E), \|\cdot\|_{S,S}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,S})$ is a surjective linear isometry, then $|A|(t)$ is constant on K .*

Proof. Suppose that $|A|(t)$ is not constant on K . Then there are two connected components of K such that $|A|(t)$ is of different values on them. Without loss of generality, we may assume that

$$\phi(s_1, s_2, \dots, s_m) = \begin{cases} (s_1, s_2, \dots, s_m) & \text{if } (s_1, s_2, \dots, s_m) \in U_1, \\ (s_{\pi(1)}, s_{\pi(2)}, \dots, s_{\pi(m)}) & \text{if } (s_1, s_2, \dots, s_m) \in U_2, \end{cases} \quad (6.2.3)$$

where U_1, U_2 are two disjoint connected components of K and π is a nontrivial permutation of $\{1, 2, \dots, m\}$. We may assume that $\pi(1) \neq 1$. Let $i_0 = \pi^{-1}(1)$. Note that $i_0 \neq 1$. Let $a \in (0, \frac{1}{2})$ so that each of $\phi(U_1)$ and $\phi(U_2)$ contains a ball of radius a . By Proposition 5.1, there exists $h_1 \in C_c^\infty(\mathbb{R})$ such that $\text{supp } h_1 \subseteq [-a, a]$, $\|h_1'\|_\infty = 1$, and $\|h_1\|_\infty \leq a$. Let $c = \frac{\|h_1'\|_\infty}{\|h_1\|_\infty} \geq \frac{1}{a} > 2$. Choose $b \in (0, \frac{1}{c}) \subseteq (0, \frac{1}{2})$. Then by Proposition 5.1 again, there exists $h_{i_0} \in C_c^\infty(\mathbb{R})$ such that $\text{supp } h_{i_0} \subseteq [-b, b]$, $\|h_{i_0}'\|_\infty = 1$, and $\|h_{i_0}\|_\infty \leq b$. Then

$$\frac{\|h_{i_0}'\|_\infty}{\|h_{i_0}\|_\infty} \geq \frac{1}{b} > c = \frac{\|h_1'\|_\infty}{\|h_1\|_\infty}. \quad (6.2.4)$$

For $i \notin \{1, i_0\}$, let $h_i \in C_c^\infty(\mathbb{R})$ such that $\text{supp } h_i \subseteq [-a, a]$. Define $f_0 \in C_0^p(Q, E)$ by

$$f_0(s_1, s_2, \dots, s_m) = \prod_{i=1}^m h_i(s_i) \cdot e,$$

where $e \in E$ is a fixed unit vector. Let f_1 and f_2 be translations of f_0 so that $\text{supp } f_i \subseteq \phi(U_i)$. Note that f_1 and f_2 have disjoint supports.

We claim that $\|\partial^\gamma T f_1\|_\infty = \prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty$ and $\|\partial^\gamma T f_2\|_\infty = \prod_{i=1}^m \|h_i^{(\gamma_{\pi(i)})}\|_\infty$ for all $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma$. Since $Tf(t) = Vf(\phi(t))$, where $V : E \rightarrow F$ is an isometry, from (6.2.3), we have $\|\partial^\gamma T f_1\|_\infty = \|\partial^\gamma f_1\|_\infty$. Since f_1 is a translate of f_0 , we have

$$\|\partial^\gamma f_1\|_\infty = \|\partial^\gamma f_0\|_\infty = \prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty.$$

On the other hand, if $(s_1, s_2, \dots, s_m) \in U_2$, then from (6.2.3),

$$\begin{aligned} \partial^\gamma T f_2(s_1, s_2, \dots, s_m) &= V \partial_1^{\gamma_1} \partial_2^{\gamma_2} \dots \partial_m^{\gamma_m} f \circ \phi(s_1, s_2, \dots, s_m) \\ &= V \partial_{\pi^{-1}(1)}^{\gamma_1} \partial_{\pi^{-1}(2)}^{\gamma_2} \dots \partial_{\pi^{-1}(m)}^{\gamma_m} f_2(s_{\pi(1)}, s_{\pi(2)}, \dots, s_{\pi(m)}) \\ &= V \partial_1^{\gamma_{\pi(1)}} \partial_2^{\gamma_{\pi(2)}} \dots \partial_m^{\gamma_{\pi(m)}} f_2(s_{\pi(1)}, s_{\pi(2)}, \dots, s_{\pi(m)}). \end{aligned}$$

Since π is a bijection and f_2 is a translate of f_0 , we have

$$\|\partial^\gamma T f_2\|_\infty = \|\partial^\gamma f_2\|_\infty = \left\| \partial^{(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \dots, \gamma_{\pi(m)})} f_2 \right\|_\infty = \prod_{i=1}^m \left\| h_i^{(\gamma_{\pi(i)})} \right\|_\infty.$$

This conclude the proof of the claim.

Set

$$f(t) = \begin{cases} f_1(t) & \text{if } t \in \phi(U_1), \\ f_2(t) & \text{if } t \in \phi(U_2), \\ 0 & \text{otherwise.} \end{cases}$$

From disjointness of f_1 and f_2 , we have $f(t) = f_1(t) \vee f_2(t)$. Since $Tf(t) = Vf(\phi(t))$, where $V : E \rightarrow F$ is an isometry, from (6.2.3), we have

$$\begin{aligned} Tf(s_1, s_2, \dots, s_m) &= \begin{cases} Tf_1(s_1, s_2, \dots, s_m) & \text{if } (s_1, s_2, \dots, s_m) \in U_1, \\ Tf_2(s_1, s_2, \dots, s_m) & \text{if } (s_1, s_2, \dots, s_m) \in U_2, \\ 0 & \text{otherwise} \end{cases} \\ &= V \begin{cases} f(s_1, s_2, \dots, s_m) & \text{if } (s_1, s_2, \dots, s_m) \in U_1, \\ f(s_{\pi(1)}, s_{\pi(2)}, \dots, s_{\pi(m)}) & \text{if } (s_1, s_2, \dots, s_m) \in U_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, it follows from disjointness of U_1 and U_2 and claim above that

$$\|\partial^\gamma Tf\|_\infty = \|\partial^\gamma Tf_1\|_\infty \vee \|\partial^\gamma Tf_2\|_\infty = \prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty \vee \prod_{i=1}^m \|h_i^{(\gamma_{\pi(i)})}\|_\infty.$$

Next, we claim that if $\gamma = (1, 0, \dots, 0)$, then

$$\prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty \vee \prod_{i=1}^m \|h_i^{(\gamma_{\pi(i)})}\|_\infty > \|h_1'\|_\infty \|h_{i_0}\|_\infty \prod_{i \notin \{1, i_0\}} \|h_i\|_\infty.$$

Indeed, since $\pi^{-1}(1) = i_0 \neq 1$, we have

$$\begin{aligned} \prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty \vee \prod_{i=1}^m \|h_i^{(\gamma_{\pi(i)})}\|_\infty &\geq (\|h'_1\|_\infty \|h_{i_0}\|_\infty \vee \|h'_{i_0}\|_\infty \|h_1\|_\infty) \prod_{i \notin \{1, i_0\}} \|h_i\|_\infty \\ &> \|h'_1\|_\infty \|h_{i_0}\|_\infty \prod_{i \notin \{1, i_0\}} \|h_i\|_\infty, \end{aligned}$$

where the last inequality above follows from Inequality 6.2.4. Hence, from strict monotonicity of ρ_s , we have

$$\begin{aligned} \|Tf\|_{S,s} &= \rho_s \left[(\|\partial^\gamma T f\|_\infty)_{\gamma \in \Gamma} \right] \\ &= \rho_s \left[\left(\prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty \vee \prod_{i=1}^m \|h_i^{(\gamma_{\pi(i)})}\|_\infty \right)_{\gamma \in \Gamma} \right] \\ &> \rho_s \left[\left(\|h'_1\|_\infty \|h_{i_0}\|_\infty \prod_{i \neq 1, i_0} \|h_i\|_\infty, \left(\prod_{i=1}^m \|h_i^{(\gamma_i)}\|_\infty \right)_{\gamma \neq (1, 0, \dots, 0)} \right) \right] \\ &\geq \rho_s \left(\|\partial^{(1, 0, \dots, 0)} f\|_\infty, (\|\partial^\gamma f\|_\infty)_{\gamma \neq (1, 0, \dots, 0)} \right) \\ &= \|f\|_{S,s}, \end{aligned}$$

contradicting the fact that T is an isometry. This concludes the proof of the proposition. $\square$

Now, we are ready to state the main result of this chapter.

Theorem 6.17. *Let $T : (C_0^p(Q, E), \|\cdot\|_{S,s}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,r})$ be a linear bijection where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, both E and F are Banach spaces and $s, r \in [1, \infty)$. If both E^* and F^* satisfy $(*)$, then T is an isometry if and only if $p = q$, $m = n$, $s = r$ and there exist a diffeomorphism $\phi : K \rightarrow Q$ of class C^p and on each connected component U of K , there exists a surjective linear isometry $V : E \rightarrow F$ such that $Tf(t) = Vf(\phi(t))$ for all $f \in C_0^p(Q, E)$ and $t \in U$, the Jacobian matrix $A(t)$ of ϕ is a signed permutation matrix and $|A|(t)$ is constant for all $t \in K$.*

Proof. If T is an isometry, then the theorem follows from Propositions 6.14, 6.15 and 6.16. Now, to show that T is an isometry, fix any $f \in C_0^p(Q, E)$. Then for any

$\gamma \in \Gamma$, we have

$$\|\partial^\gamma T f\|_\infty = \sup_{t \in K} \|\partial^\gamma T f(t)\| = \sup_{t \in K} \|\partial^\gamma (f \circ \phi)(t)\| = \sup_{s \in Q} \|\partial^\lambda f(s)\| = \|\partial^\lambda f\|_\infty,$$

where the second last equality follows from the fact that $|A|(t)$ is constant on K and $A(t)$ is a signed permutation matrix. Therefore,

$$\|T f\|_{S,r} = \|(\|\partial^\gamma T f\|_\infty)_{\gamma \in \Gamma}\|_r = \|(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda}\|_r = \|f\|_{S,r}.$$

Hence, T is an isometry. □

Chapter 7

Conclusion and Future Work

All Banach-Stone Theorems deal with onto linear isometry on function spaces endowed with the same norm. This raises the following question. Recall that

$$\|f\|_{D,\rho} = \sup_{s \in Q} \rho[(\|\partial^\lambda f(s)\|)_{\lambda \in \Lambda}] \quad \text{and} \quad \|f\|_{S,\rho} = \rho[(\|\partial^\lambda f\|_\infty)_{\lambda \in \Lambda}].$$

Question 7.1. *Consider an onto linear isometry*

$$T : (C_0^p(Q, E), \|\cdot\|_{D,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{S,\theta}),$$

where $p, q, m, n \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and both E and F are Banach spaces. Is T a weighted composition operator?

Koshimizu [10] considered the following norms on the function space $C^p([0, 1])$.

$$\begin{aligned} \|f\|_m &= \max\{|f(0)|, |f'(0)|, \dots, |f^{(p-1)}(0)|, \|f^{(p)}\|_\infty\} \quad \text{and} \\ \|f\|_\sigma &= \sum_{k=0}^{p-1} |f^{(k)}(0)| + \|f^{(p)}\|_\infty. \end{aligned}$$

The two norms above motivate the definition of ‘vertical’ norm, which we define now. Assume that $Q \subseteq \mathbb{R}$. For any $s_0, s_1, \dots, s_{p-1} \in Q$ and $f \in C_0^p(Q, E)$, define

$$\|f\|_{V,\rho} = \rho(\|f(s_0)\|, \|f'(s_1)\|, \dots, \|f^{(p-1)}(s_{p-1})\|, \|f^{(p)}\|_\infty).$$

Question 7.2. *If $T : (C_0^p(Q, E), \|\cdot\|_{V,\rho}) \rightarrow (C_0^q(K, F), \|\cdot\|_{V,\theta})$ is an onto linear isometry, where $p, q \geq 1$ are natural numbers, Q and K are open subsets of $\mathbb{R}$ and both E and F are Banach spaces, must T be a weighted composition operator?*

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