Some modules of symmetric groups

Jiang, Yu

2021

Jiang, Y. (2021). Some modules of symmetric groups. Doctoral thesis, Nanyang Technological University, Singapore. <https://hdl.handle.net/10356/148775>

<https://hdl.handle.net/10356/148775>

<https://doi.org/10.32657/10356/148775>

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License (CC BY-NC 4.0).

Downloaded on 26 Apr 2023 18:16:17 SGT

**NANYANG
TECHNOLOGICAL
UNIVERSITY**

SINGAPORE

SOME MODULES OF SYMMETRIC GROUPS

JIANG YU

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2021

SOME MODULES OF SYMMETRIC GROUPS

JIANG YU

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for the
degree of Doctor of Philosophy

2021

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

01-04-2021

江宇

.....

Date

.....

Jiang Yu

Supervisor Declaration Statement

I have reviewed the content and presentation style of this thesis and declare it of sufficient grammatical clarity to be examined. To the best of my knowledge, the thesis is free of plagiarism and the research and writing are those of the candidate's except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

1 Apr. 2021

.....
Date

.....
Lim Kay Jin

Authorship Attribution Statement

Please select one of the following; *delete as appropriate:

(A) This thesis **does not** contain any materials from papers published in peer-reviewed journals or from papers accepted at conferences in which I am listed as an author.

*(B) This thesis contains material from 3 paper(s) published in the following peer-reviewed journal(s) / from papers accepted at conferences in which I am listed as an author.

Chapter 4 is published as Y. Jiang. On the complexities of some simple modules of symmetric groups. Contributions to Algebra and Geometry **60**, 599-625 (2019). DOI: <https://doi.org/10.1007/s13366-019-00441-2>.

Chapter 5 is published as Y. Jiang. On some trivial source Specht modules. Journal of Algebra **556**, 1073-1100 (2020). DOI: <https://doi.org/10.1016/j.jalgebra.2020.03.028>.

Chapter 6 is published as Y. Jiang. On the symmetric and exterior powers of Young permutation modules. Journal of Algebra **567**, 660-704 (2021). DOI: <https://doi.org/10.1016/j.jalgebra.2020.09.036>.

01-04-2021

江宇

.....

Date

.....

Jiang Yu

To my supervisor, my parents, many professors...

Abstract

In this thesis, we focus on the representation theory of symmetric groups. Especially, we are very interested in some classical modules of a given symmetric group. Let $\mathbb{F}$ be an algebraically closed field of positive characteristic p and $\mathfrak{S}_n$ be the symmetric group acting on n letters. We first compute the complexities of some simple $\mathbb{F}\mathfrak{S}_n$ -modules labelled by two-part partitions. We then consider the classification of trivial source $\mathbb{F}\mathfrak{S}_n$ -Specht modules. For this project, the trivial source Specht modules labelled by hook partitions are completely classified. If $p > 2$, the trivial source Specht modules labelled by two-part partitions are also classified. Moreover, if $p = 2$, a result for the classification of trivial source Specht modules labelled by partitions with 2-weight 2 is obtained. As an application of this result, a conjecture of [33] is justified. Finally, we generalize a result of Benson and Lim, which motivates us to study the symmetric and exterior powers of Young permutation modules. Along the way, we obtain many results for symmetric and exterior powers of $\mathbb{F}\mathfrak{S}_n$ -Young permutation modules. Throughout this thesis, most of the presented results are from [40], [41], [42].

Acknowledgements

As the author of this thesis, I would like to thank many people for their care and help. Without their encouragement and help, my four-year postgraduate experience should be more painful.

I first need to thank my supervisor Professor Lim Kay Jin for accepting me as his first Ph.D. student. I gratefully thank him for many meaningful and helpful conversations. In these conversations, he always listened to my work reports carefully and pointed out countless mistakes and flaws. I gratefully thank him for training me the skills of revising papers, which lets me be able to submit and publish papers by myself. I also gratefully thank him for preparing the recommendation letters for my applications of postdoctoral positions. Moreover, he always provides his own suggestions to me when I am in trouble. In my heart, he looks like one of my elder cousin brothers rather than just an academic supervisor.

I then need to thank Professor Tan Kai Meng for his constant encouragement and help. During the period of pursuing my Ph.D. degree, Professor Tan organized many interesting and meaningful representation theory seminars. In these seminars, I learned a lot of knowledge about Hecke algebras, G -algebras (interior G -algebras) and KLR-algebras. Though I always work on the representation theory of symmetric groups, this knowledge lets me be more familiar with the techniques of representation theory of finite-dimensional algebras. I also gratefully thank Professor Tan for his care and encouragement. His enthusiasm and attitude for mathematics impress me deeply. Moreover, I would like to thank him for preparing the recommendation letters for me.

My appreciation then goes to Professor Xi Changchang and Professor Wang Li-Lian. I met Professor Xi in a conference held in Singapore. His research spirit for mathematics impresses me deeply. Apart from their care and encouragement, I also gratefully thank Professor Xi and Professor Wang for preparing the recommendation letters for me.

I also want to express my gratitude to Professor Jin Ping and Professor Chen Gang. My final year programme report was completed under the guidance of Professor Jin. Though I have left Shanxi university for seven years, he always gives me constant encouragement. I gratefully thank Professor Chen for suggesting me to go abroad to pursue my Ph.D. degree. As my former master degree supervisor, he introduced the representation theory of finite groups to me, which lets me fall in love with this interesting and mysterious theory. Apart from constant support, he also encourages me to learn the representation theory of finite association schemes. I also gratefully thank him for preparing the recommendation letters for me.

Of course, I should thank Professor Chan Song Heng, Professor Wu Guohua, Professor Ku Cheng Yeaw, Professor Le Hai Khoi, Professor Hu Xueqin. Some of them provide necessary help to me while others give me joyous conversations. I also would like to take this chance to thank my thesis advisory committee members for attending to my oral defense.

Finally, my parents deserve to be gratefully thanked. When I feel hopeless and painful, their comfort and support always give me courage and determination to overcome all the difficulties. Their love for their unique son gives me confidence of facing challenges from both mathematics and life.

The completion of my Ph.D. thesis is only a new beginning for me. I hope that I can know more knowledge and have better publications in the future. I will pursue this goal forever.

Contents

Abstract	7
1 Introduction	10
2 Basic Notation and Preliminaries	13
2.1 General Conventions	13
2.2 Vertices and Sources of Indecomposable Modules	13
2.3 Decompositions of Finite-Dimensional Algebras	14
2.4 p -Permutation Modules and Scott Modules	14
2.5 Green Correspondences of Indecomposable Modules	15
2.6 Brauer Quotients of Modules	15
2.7 Rank Varieties and Complexities of Modules	16
2.8 Generic Jordan Types of Modules	17
2.9 Symmetric and Exterior Powers of Modules	18
2.10 Combinatorics	19
2.11 Modules of Symmetric Groups	21
3 Green-Klein Theorem and Some Computational Results	24
3.1 Green-Klein Theorem	24
3.2 Some Computational Results	26
4 The Complexities of Some Simple Modules of Symmetric Groups	30
4.1 Main Results and Some Known Results	30
4.2 Proof of Theorem 4.1.2	32
4.3 Proof of Theorem 4.1.3	40
5 Some Trivial Source Specht Modules	47
5.1 Main Results and Some Known Results	47
5.2 Fixed-Point Functors of Symmetric Groups	48
5.3 The Specht Modules Labelled by Hook Partitions	50
5.4 The Specht Modules Labelled by Two-Part Partitions	57
5.5 Verification of a Conjecture of Hudson	60
6 Symmetric and Exterior Powers of Young Permutation Modules	63
6.1 Main Results and Some Known Results	63
6.2 Proof of Theorem 6.1.3	65
6.3 Classification of Indecomposable Exterior Powers of Young Permutation Modules	68
6.4 Proof of Theorem 6.1.6: Symmetric Powers of Young Permutation Modules . . .	70
6.5 Proof of Theorem 6.1.6: Exterior Squares of Young Permutation Modules . . .	74
6.6 Some Special Cases of Theorem 6.1.6	78
6.7 A Result of Some p -Subgroups of $\mathfrak{S}_n$	81
6.8 A Similar Result of Theorem 6.1.7	87
Summary	92
Bibliography	93

Chapter 1

Introduction

The representation theory of finite groups, first introduced by Frobenius in the late nineteenth century, is now widely identified as a mainstream research field of algebra. As the famous Burnside's $p^a q^b$ -theorem suggests, the initial motivation for introducing this theory is to provide powerful tools for group theorists to study the structure of a given finite group. Surprisingly, mathematicians gradually realize that many deep questions from representations of finite groups deserve to be thoroughly studied. Therefore, this theory has been an active research topic since the last century.

According to the well-known Cayley's theorem, we may regard a given finite group as a subgroup of a finite symmetric group. Therefore, to understand the representations of finite groups better, it is necessary to study the representation theory of finite symmetric groups. The representation theory of finite symmetric groups mainly studies the group algebras of a given finite symmetric group and their module categories. One of the main goals of studying this theory is to understand all the finitely generated modules of a finite symmetric group completely. This theory has been investigated intensively since the last century and is closely related to many research areas of mathematics, including number theory, graph theory, algebraic combinatorics, algebraic geometry.

The ordinary representation theory of finite symmetric groups has been well developed through almost one century. In particular, there are many nice combinatorial results for the complex characters of a given finite symmetric group. For example, let $\mathfrak{S}_n$ be the symmetric group on n letters. Recall that the Murnaghan-Nakayama rule provides a recursive formula for computing the values of irreducible complex characters of $\mathfrak{S}_n$. Additionally, the Young rule and the Littlewood-Richardson rule focus on the decompositions of some complex characters of $\mathfrak{S}_n$. The discovery of these rules illustrates that $\mathfrak{S}_n$ is a fertile field of beautiful results.

If the base field has positive characteristic, the representation theory of finite symmetric groups usually become more interesting and challenging. Compared with the characteristic zero case, very little is known for the positive characteristic case. For example, let $\mathbb{F}$ be an algebraically closed field of positive characteristic p . Unlike the characteristic zero case, the $\mathbb{F}$ -dimensions of all the simple $\mathbb{F}\mathfrak{S}_n$ -modules have not yet been determined, which implies that the decomposition matrix of $\mathbb{F}\mathfrak{S}_n$ is still not determined completely. Meanwhile, fortunately, many deep advancements for $\mathbb{F}\mathfrak{S}_n$ -modules have been obtained. Especially, the Alperin's weight conjecture and the Broué's abelian defect group conjecture are justified for $\mathfrak{S}_n$.

The aim of this thesis is to contribute to the modular representation theory of symmetric groups. Especially, in this thesis, we study some typical $\mathbb{F}\mathfrak{S}_n$ -modules. These $\mathbb{F}\mathfrak{S}_n$ -modules include simple modules, Specht modules, symmetric and exterior powers of Young permutation modules. We describe our core questions and results as follows.

Let G be a finite group and M be an $\mathbb{F}G$ -module. The complexity of M , defined by Alperin and Evens in [3] and denoted by $c_G(M)$, is the smallest non-negative integer c such that

$$\lim_{\ell \rightarrow \infty} \frac{\dim_{\mathbb{F}} P_{\ell}}{\ell^c} = 0,$$

where $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M$ is a minimal projective resolution of M .

According to the definition of $c_G(M)$, observe that $c_G(M)$ describes the growth rate of a minimal projective resolution of M . Therefore, $c_G(M)$ may provide some information for the extensions of modules. However, this invariant is usually very difficult to compute explicitly and is still poorly understood even in the case of $\mathbb{F}\mathfrak{S}_n$ -modules.

Though some advancements have been obtained in computing the complexities of Young permutation modules, Young modules, and some Specht modules (see [28], [46], [47] and [48]), very little is known for computing the complexities of simple $\mathbb{F}\mathfrak{S}_n$ -modules. According to our knowledge, the non-trivial cases that were computed mainly focus on the completely splittable representations, some simple modules labelled by hook partitions and some simple modules lying in the p -blocks with abelian defect groups (see [28], [47] and [50]).

The aim of Chapter 4 is to compute the complexities of simple $\mathbb{F}\mathfrak{S}_n$ -modules contained in the p -blocks with non-abelian defect groups. Let $\langle g \rangle$ be a cyclic group of order p . Using some results about the short exact sequences of $\mathbb{F}\langle g \rangle$ -modules, rank varieties of modules and generic Jordan types of modules, we obtain the complexities of some simple $\mathbb{F}\mathfrak{S}_n$ -modules labelled by two-part partitions. Most of these simple modules are contained in the p -blocks with non-abelian defect groups.

We now describe another question considered in this thesis. Recall that G is a finite group. The trivial source $\mathbb{F}G$ -modules, defined as indecomposable $\mathbb{F}G$ -modules with trivial sources, are important objects in the representation theory of $\mathbb{F}G$. It is well-known that trivial source $\mathbb{F}G$ -modules enjoy a lot of good properties. For example, given the Green ring spanned by all the indecomposable $\mathbb{F}G$ -modules, the complex linear subspace spanned by all the trivial source $\mathbb{F}G$ -modules is in fact a subring of this Green ring.

Though the trivial source $\mathbb{F}G$ -modules are known to have plenty of properties, it is usually very difficult to classify explicitly all the trivial source $\mathbb{F}G$ -modules. Even in the situation of $\mathbb{F}\mathfrak{S}_n$ -modules, it is extremely difficult to explicitly classify all the trivial source $\mathbb{F}\mathfrak{S}_n$ -modules. Let us first recall some known results for the classification of trivial source $\mathbb{F}\mathfrak{S}_n$ -modules. It is already known that trivial source simple $\mathbb{F}\mathfrak{S}_n$ -modules are exactly the simple $\mathbb{F}\mathfrak{S}_n$ -Specht modules (see [13, Proposition 4.6]). Moreover, the signed Young modules are also trivial source modules (see [15, Theorem 3.8 (b)]). However, very little is known for the classification of trivial source Specht modules. According to our knowledge, the project of classifying all trivial source Specht modules begins in the thesis of Hudson (see [33]). For a short summary of her results, one can refer to Section 5.1.

Due to our viewpoint, another motivation for classifying the trivial source Specht modules is to check a conjecture of Hemmer. According to the representation theory of a given Schur algebra, it is known that tilting modules are self-dual with respect to the usual contravariant duality. Moreover, the indecomposable modules with both a Δ -filtration and a ∇ -filtration are precisely the tilting modules. Motivated by these facts, Hemmer conjectures that any self-dual indecomposable $\mathbb{F}\mathfrak{S}_n$ -module is a trivial source module if this module has a Specht filtration. If the classification of trivial source Specht modules was completely known, we may check his conjecture for the self-dual indecomposable Specht modules.

The main results of Chapter 5 are about the classification of trivial source Specht modules. We completely classify the trivial source Specht modules labelled by hook partitions. In the odd characteristic case, we completely classify the trivial source Specht modules labelled by two-part partitions. Moreover, a conjecture of Hudson is also justified. To obtain all these results, we heavily depend on the fixed-point functors of symmetric groups and Broué correspondence of trivial source modules. The analysis of stable generic Jordan types of modules is also involved.

In the final part of this thesis, we work on the symmetric and exterior powers of Young permutation modules. The motivation of this work also comes from the computation of the complexities of modules. Let us first state this motivation. In [7, Lemma 1 (i) and (ii)], Benson and Lim gave an optimal upper bound to the complexities of symmetric and exterior powers of the regular $\mathbb{F}G$ -module. To generalize their results, we shall consider the symmetric and exterior powers of any non-zero projective $\mathbb{F}G$ -module and determine their complexities. As the regular $\mathbb{F}\mathfrak{S}_n$ -module is in fact a Young permutation module, this generalization motivates us to study the symmetric and exterior powers of Young permutation modules. We want to generalize known results of Young permutation modules to their symmetric and exterior powers.

The progress of this project is reported in Chapter 6. Apart from the generalization of the result of Benson and Lim, the other main results of Chapter 6 are now described as follows.

We first explicitly classify all the symmetric and exterior powers of a Young permutation module that are projective. Moreover, though all the indecomposable symmetric powers of $\mathbb{F}\mathfrak{S}_n$ -Young permutation modules are all isomorphic to the trivial module, there is a different story for the exterior powers of Young permutation modules. We also explicitly classify all the indecomposable exterior powers of Young permutation modules, which generalizes a result of Gill (see [25, Theorem 2]).

To describe our remaining results, we define $\mathcal{P}(n)$ to be the set of partitions of n and let $\lambda, \mu \in \mathcal{P}(n)$. Denote by M^λ and Y^λ the $\mathbb{F}\mathfrak{S}_n$ -Young permutation module and the $\mathbb{F}\mathfrak{S}_n$ -Young module respectively. For the decomposition of M^λ into a direct sum of Young modules, write $Y^\mu \mid M^\lambda$ to imply that Y^μ is isomorphic to a direct summand of M^λ . We can explicitly determine the set

$$\{\nu \in \mathcal{P}(n) : Y^\nu \mid M^\lambda \text{ and } c_{\mathfrak{S}_n}(Y^\nu) = c_{\mathfrak{S}_n}(M^\lambda)\}$$

by [17, (3.6)] and [28, Proposition 3.2.2 and Theorem 3.3.2]. Motivated by this fact, we prove many results for the symmetric and exterior powers of Young permutation modules that are similar to this fact. One can refer to Theorem 6.1.6 for the details.

To describe our final result, recall that for each λ there is a unique Young module Y^μ such that $Y^\mu \mid M^\lambda$ and Y^μ is a Scott module. Moreover, in Section 6.4, we show that μ can be explicitly determined by λ . As our final result, we prove a similar result for the symmetric and exterior squares of Young permutation modules (see Theorem 6.1.7).

We end this introduction by stating the organization of this thesis as follows.

Chapter 2 includes eleven sections. In these sections, we set up the basic notation used in this thesis and give a brief summary of the required knowledge.

Chapter 3 consists of two sections. In Section 3.1, the definition of a Littlewood-Richardson sequence is introduced. By this definition, we can recall a known result presented in [43]. By this result, we obtain some results about the short exact sequences of modules of cyclic groups. These results are presented in Section 3.2. Some results in Section 3.2 shall be used in Chapter 4.

Chapter 4 has three sections. In Section 4.1, we first recall the motivation of computing the complexities of simple $\mathbb{F}\mathfrak{S}_n$ -modules and state Theorems 4.1.2 and 4.1.3. They are the main results of Chapter 4. For further reference, we also provide some known results. In Section 4.2, we show Theorem 4.1.2. In Section 4.3, we provide the proof of Theorem 4.1.3.

Chapter 5 is a collection of five sections. In Section 5.1, we first discuss Hemmer's conjecture. For our viewpoint, this is also another motivation for the classification of trivial source Specht modules. We then state the main results of Chapter 5. They are Theorems 5.1.4, 5.1.5 and 5.1.6. We also present some known results so that we can use them in the following sections. In Section 5.2, we introduce the fixed-point functors of symmetric groups and give many properties of them. Using these properties, in Sections 5.3 and 5.4, we show Theorems 5.1.4 and 5.1.5 respectively. In Section 5.5, we use Broué correspondence to prove Theorem 5.1.6. This theorem obviously implies a conjecture of Hudson.

Chapter 6 contains eight sections. In Section 6.1, we gather some known results and state the main results of Chapter 6. They are Theorems 6.1.3, 6.1.6 and 6.1.7. In Section 6.2, we prove Theorem 6.1.3. Moreover, we also determine the symmetric and exterior powers of a Young permutation module that are projective. In Section 6.3, we classify all the indecomposable exterior powers of Young permutation modules. We use Sections 6.4 and 6.5 to present the proof of Theorem 6.1.6. To strengthen Theorem 6.1.6, in Section 6.6, we consider some special cases of Theorem 6.1.6. To prove Theorem 6.1.7, in Section 6.7, we prove a result of some p -subgroups of $\mathfrak{S}_n$. Using this result, in Section 6.8, we prove Theorem 6.1.7.

Chapter 2

Basic Notation and Preliminaries

The main aim of this chapter is to introduce the basic notation of this thesis. Moreover, we also collect many known results that shall be repeatedly used in the following chapters. To make our presentation of this chapter as concise as possible, we assume that the reader is familiar with the modular representation theory of finite groups. For a general background of this topic, one may refer to [2] or [55].

2.1 General Conventions

Throughout the whole thesis, we fix an algebraically closed field $\mathbb{F}$ of positive characteristic p . For a given finite group G , we write $H \leq G$ (resp. $H < G$) to indicate that H is a subgroup (resp. proper subgroup) of G . If $H \leq G$, let $N_G(H)$ be the normalizer of H in G . As usual, we also use 1 to denote the trivial group if there is no confusion. For a non-empty set S of an $\mathbb{F}$ -linear space, let $\langle S \rangle_{\mathbb{F}}$ be the $\mathbb{F}$ -linear space generated by S . Let $\langle \emptyset \rangle_{\mathbb{F}}$ denote the zero space.

All the $\mathbb{F}G$ -modules occurring in this thesis are finitely generated left $\mathbb{F}G$ -modules. Induction and restriction of $\mathbb{F}G$ -modules are presented by $\uparrow$ and $\downarrow$ respectively. Use $\otimes$ and $\boxtimes$ to denote inner tensor product and outer tensor product of modules respectively. All the inner tensor products and outer tensor products of modules are over $\mathbb{F}$. The usual dual of an $\mathbb{F}G$ -module M is denoted by M^* . Use $\mathbb{F}_G$ to denote the trivial $\mathbb{F}G$ -module throughout the whole thesis and omit the subscript if there is no confusion.

Let M and N be $\mathbb{F}G$ -modules. Write $N \mid M$ if N is isomorphic to a direct summand of M , i.e., $M \cong L \oplus N$ for some $\mathbb{F}G$ -module L . If N is indecomposable, for a decomposition of M into a direct sum of indecomposable modules, the number of indecomposable direct summands that are isomorphic to N is well-defined by the Krull–Schmidt Theorem and is denoted by $[M : N]$. Furthermore, if N is simple, the number of composition factors of M that are isomorphic to N is well-defined by the Jordan–Hölder Theorem and is denoted by $(M : N)$. Let m be a positive integer and N_i be an $\mathbb{F}G$ -module for all $1 \leq i \leq m$. Write $M \sim \sum_{i=1}^m N_i$ to indicate that there exists an $\mathbb{F}G$ -filtration of M such that all successive subquotients are exactly isomorphic to these N_i 's. The Jacobson radical of M is denoted by $\text{Rad}(M)$.

2.2 Vertices and Sources of Indecomposable Modules

Let G be a finite group and $P \leq G$. Let M be an indecomposable $\mathbb{F}G$ -module. Following [27], say P a vertex of M if P is a minimal (with respect to inclusion of subgroups) subgroup of G subject to the condition that $M \mid (M \downarrow_P) \uparrow^G$. All the vertices of M are known to form a G -conjugacy class of p -subgroups of G (see [27, Theorem 4 (2)]). It is clear that M and the indecomposable $\mathbb{F}G$ -modules isomorphic to M have same vertices. Moreover, note that M and M^* have same vertices by the definition.

Let P be a vertex of M . There exists some indecomposable $\mathbb{F}P$ -module S such that $M \mid S \uparrow^G$. The module S is called a P -source of M . All the P -sources of M are $N_G(P)$ -conjugate to each other (see [27, Theorem 5]). Call M a trivial source module if all the P -sources of M are trivial modules. Note that M is a trivial source module if and only if M^* is a trivial source module.

2.3 Decompositions of Finite-Dimensional Algebras

Throughout this section, we fix A to be a finite-dimensional associative $\mathbb{F}$ -algebra with identity 1_A . As usual, A can be viewed as an (A, A) -bimodule naturally. We use ${}_A A_A$ to denote this (A, A) -bimodule.

Note that ${}_A A_A$ can be decomposed into a direct sum of indecomposable (A, A) -bimodules. More precisely, we have

$${}_A A_A = B_1 \oplus \cdots \oplus B_\ell, \quad (2.3.1)$$

where ℓ is a positive integer and B_i is an indecomposable (A, A) -bimodule for any $1 \leq i \leq \ell$. In particular, for any $1 \leq i \leq \ell$, B_i is a two-sided ideal of A . We call each B_i a p -block of A for any $1 \leq i \leq \ell$. By (2.3.1), we get

$$1_A = e_{B_1} + \cdots + e_{B_\ell}, \quad (2.3.2)$$

where $e_{B_i} \in B_i$ for any $1 \leq i \leq \ell$. It is not difficult to see that e_{B_i} is a central primitive idempotent of A for any $1 \leq i \leq \ell$. We call e_{B_i} the block idempotent of B_i for any $1 \leq i \leq \ell$.

Let M be an indecomposable A -module. Then there is a unique $i \in \{1, \dots, \ell\}$ such that $e_{B_i} v = v$ for any $v \in M$. Moreover, we also have $e_{B_j} v = 0$ for any $j \in \{1, \dots, \ell\} \setminus \{i\}$ and $v \in M$. We thus say that M belongs to B_i or B_i contains M .

We now specialize our discussion to the group algebras. Let G be a finite group. The natural $(\mathbb{F}G, \mathbb{F}G)$ -bimodule structure of $\mathbb{F}G$ makes $\mathbb{F}G$ become an $\mathbb{F}[G \times G]$ -module. More precisely, the group $G \times G$ acts on $\mathbb{F}G$ with the action $(h, k)v = hvk^{-1}$ for any $h, k \in G$ and $v \in \mathbb{F}G$. So $\mathbb{F}G$ is an $\mathbb{F}[G \times G]$ -module by scalar extension. So every p -block of the $(\mathbb{F}G, \mathbb{F}G)$ -bimodule $\mathbb{F}G$ is an indecomposable $\mathbb{F}[G \times G]$ -module. For such a p -block B , by [2, Section 13, Theorem 4], B has a vertex

$$\Delta D = \{(d, d) : d \in D\},$$

where D is some p -subgroup of G . The p -subgroup D is called a defect group of B . If M is an indecomposable $\mathbb{F}G$ -module that belongs to B , then it is known that D contains a vertex of M (see [2, Section 13, Theorem 5]).

2.4 p -Permutation Modules and Scott Modules

Let G be a finite group and M be an $\mathbb{F}G$ -module. Say M a p -permutation module if, for every p -subgroup P of G , there exists an $\mathbb{F}$ -basis $\mathcal{B}_P$ of M depending on P such that $\mathcal{B}_P$ is permuted by P . We observe that the class of p -permutation modules is closed under taking direct sums and non-zero direct summands. Moreover, the class of p -permutation modules is also closed under the restriction, induction, inner tensor products and outer tensor products of modules. Therefore, the non-zero projective $\mathbb{F}G$ -modules are p -permutation $\mathbb{F}G$ -modules.

It is known that indecomposable p -permutation $\mathbb{F}G$ -modules are exactly trivial source $\mathbb{F}G$ -modules (see [8, (0.4)]). Therefore, a p -permutation $\mathbb{F}G$ -module is always equivalently defined to be a non-zero direct summand of a permutation $\mathbb{F}G$ -module, where a permutation $\mathbb{F}G$ -module is an $\mathbb{F}G$ -module with an $\mathbb{F}$ -basis permuted by G .

Let $H \leq G$. For the transitive permutation $\mathbb{F}G$ -module $(\mathbb{F}_H)^\uparrow^G$, it is known that $(\mathbb{F}_H)^\uparrow^G$ has a unique trivial $\mathbb{F}G$ -submodule and a unique trivial $\mathbb{F}G$ -quotient module. Moreover, there exists an indecomposable $\mathbb{F}G$ -direct summand S of $(\mathbb{F}_H)^\uparrow^G$ with a trivial $\mathbb{F}G$ -submodule and a trivial $\mathbb{F}G$ -quotient module (see [9, Theorem 1]). The module S , unique up to isomorphism, is called the Scott module of $(\mathbb{F}_H)^\uparrow^G$ and is denoted by $Sc_G(H)$. By the definition of $Sc_G(H)$, $Sc_G(H)$ is a self-dual trivial source $\mathbb{F}G$ -module. It is known that the vertices of $Sc_G(H)$ are G -conjugate to a Sylow p -subgroup of H (see [9, Theorem 1]).

Assume further that $K \leq G$ and Q is a Sylow p -subgroup of H . According to [9, Page 1859], we remark that

$$Sc_G(H) \cong Sc_G(K)$$

if and only if a Sylow p -subgroup of H is G -conjugate to a Sylow p -subgroup of K . Therefore, we usually use $Sc_G(Q)$ to denote $Sc_G(H)$. For more properties of $Sc_G(Q)$, one may refer to [9].

2.5 Green Correspondences of Indecomposable Modules

Let G be a finite group and P be a p -subgroup of G . Assume further that $N_G(P) \leq H \leq G$. We define

$$\mathfrak{X} = \{Q \leq G : Q \leq gPg^{-1} \cap P \text{ for some } g \in G \setminus H\}$$

and

$$\mathfrak{Y} = \{Q \leq G : Q \leq gPg^{-1} \cap H \text{ for some } g \in G \setminus H\}.$$

It is obvious that $\mathfrak{X} \subseteq \mathfrak{Y}$ and $P \notin \mathfrak{Y}$. We are now ready to describe the Green correspondences of indecomposable modules.

Theorem 2.5.1. [55, §4.4] *Let G be a finite group and P be a p -subgroup of G . Let $N_G(P) \leq H \leq G$. Then there is a one-to-one correspondence between the indecomposable $\mathbb{F}G$ -modules with a vertex P and the indecomposable $\mathbb{F}H$ -modules with a vertex P . The correspondence is given as follows.*

- (i) *Assume that M is an indecomposable $\mathbb{F}G$ -module with a vertex P . Then $M \downarrow_H$ has a unique indecomposable direct summand $\mathcal{G}_{G,H}(M)$ with a vertex P . The vertices of all the other indecomposable direct summands of $M \downarrow_H$ are in $\mathfrak{Y}$.*
- (ii) *Assume that N is an indecomposable $\mathbb{F}H$ -module with a vertex P . Then $N \uparrow^G$ has a unique indecomposable direct summand $\mathcal{F}_{G,H}(N)$ with a vertex P . The vertices of all the other indecomposable direct summands of $N \uparrow^G$ are in $\mathfrak{X}$.*
- (iii) *We have $\mathcal{F}_{G,H}(\mathcal{G}_{G,H}(M)) \cong M$ and $\mathcal{G}_{G,H}(\mathcal{F}_{G,H}(N)) \cong N$.*

The one-to-one correspondence in Theorem 2.5.1 is usually called the Green correspondence with respect to G and H . For convenience, we shall omit the subscript G if there is no confusion. Therefore, if M is an indecomposable $\mathbb{F}G$ -module with a vertex P , then the Green correspondent of M with respect to H is denoted by $\mathcal{G}_H(M)$. Recall that Green correspondences send trivial source modules to trivial source modules (see [5, 2.12.2 (iv)]). In Lemma 5.5.3, we shall see that Green correspondences send self-dual indecomposable modules to self-dual indecomposable modules. For more properties of $\mathcal{G}_H(M)$, one can refer to [55, § 4.4].

2.6 Brauer Quotients of Modules

We now describe the Brauer quotients of p -permutation modules developed by Broué in [8]. Let G be a finite group and $P \leq G$. Let M be an $\mathbb{F}G$ -module. For a given non-empty set $S \subseteq M$, we set $S^P = \{v \in S : \forall q \in P, qv = v\}$. If S is an $\mathbb{F}G$ -submodule of M , then S^P is an $\mathbb{F}[N_G(P)/P]$ -module. For any p -subgroup Q of P , the relative trace map from M^Q to M^P , denoted by Tr_Q^P , is defined to be

$$\text{Tr}_Q^P(v) = \sum_{g \in \{P/Q\}} gv,$$

where $\{P/Q\}$ is a complete set of representatives of left cosets of Q in P . Note that the $\mathbb{F}$ -linear map Tr_Q^P is independent of the choices of $\{P/Q\}$. We also define

$$\text{Tr}^P(M) = \sum \text{Tr}_Q^P(M^Q),$$

where the sum runs over all the members of $\{Q < P : Q \text{ is a } p\text{-group}\}$. It is clear to see that $\text{Tr}^P(M)$ is an $\mathbb{F}[N_G(P)/P]$ -submodule of M^P . The Brauer quotient of M with respect to P , written as $M(P)$, is the $\mathbb{F}[N_G(P)/P]$ -module

$$M^P / \text{Tr}^P(M).$$

Notice that $M(P) = 0$ unless P is a p -subgroup of G . If P is a p -subgroup of G and M is indecomposable, by [8, (1.3)], we know that $M(P) \neq 0$ only if P is G -conjugate to a p -subgroup of a vertex of M . For our discussion, we list some known results as follows.

Lemma 2.6.1. *Let P be a p -subgroup of a finite group G . Let M be a trivial source $\mathbb{F}G$ -module with a vertex V . Let N be a p -permutation $\mathbb{F}G$ -module and $\mathcal{B}_P(N)$ be an $\mathbb{F}$ -basis of N that is permuted by P .*

- (i) [8, (3.2) (1)] *We have $M(P) \neq 0$ if and only if P is G -conjugate to a p -subgroup of V .*
- (ii) [59, Proposition 2.1] *We have $N(P) \cong U(P) \oplus V(P)$ as $\mathbb{F}[N_G(P)/P]$ -modules if we have $N \cong U \oplus V$ for some $\mathbb{F}G$ -modules U and V .*
- (iii) [23, Lemma 2.7] *Let G_1, G_2 be finite groups and P_1, P_2 be p -groups such that $P_1 \leq G_1$ and $P_2 \leq G_2$. Let M_i be a p -permutation $\mathbb{F}G_i$ -module for any $1 \leq i \leq 2$. Then*

$$(M_1 \boxtimes M_2)(P_1 \times P_2) \cong M_1(P_1) \boxtimes M_2(P_2)$$

as modules under the action of $N_{G_1 \times G_2}(P_1 \times P_2)/(P_1 \times P_2) \cong (N_{G_1}(P_1)/P_1) \times (N_{G_2}(P_2)/P_2)$.

- (iv) [59, Proposition 2.2] *We have $N(P) \cong \langle \mathcal{B}_P(N)^P \rangle_{\mathbb{F}}$ as $\mathbb{F}[N_G(P)/P]$ -modules.*

We end this section with the famous Broué correspondence summarized as follows. We will repeatedly use this tool in the following chapters.

Theorem 2.6.2. [8, (3.2) and (3.4)] *Let P be a p -subgroup of a finite group G and M be a trivial source $\mathbb{F}G$ -module with a vertex P . Let L be a p -permutation $\mathbb{F}G$ -module.*

- (i) *The correspondence $M \mapsto M(P)$ is in fact a bijection between the isomorphism classes of trivial source $\mathbb{F}G$ -modules with a vertex P and the isomorphism classes of indecomposable projective $\mathbb{F}[N_G(P)/P]$ -modules.*
- (ii) *The inflation of $M(P)$ from $N_G(P)/P$ to $N_G(P)$ is isomorphic to $\mathcal{G}_{N_G(P)}(M)$.*
- (iii) *Let N be a trivial source $\mathbb{F}G$ -module with a vertex Q . Then we have $N \mid L$ if and only if $N(Q) \mid L(Q)$. We also have $[L : N] = [L(Q) : N(Q)]$.*

2.7 Rank Varieties and Complexities of Modules

The p -rank of a finite group G is defined to be the largest non-negative integer r subject to the condition that G has an elementary abelian p -subgroup of order p^r . We will use $\text{rank}(G)$ to denote the p -rank of G .

Let E be an elementary abelian p -group of order p^n with a generator set $\{g_1, \dots, g_n\}$ and M be an $\mathbb{F}E$ -module. For each non-zero point $\alpha = (\alpha_1, \dots, \alpha_n)$ in the affine space $\mathbb{F}^n$, one defines

$$u(\alpha) = 1 + \sum_{i=1}^n \alpha_i (g_i - 1).$$

Note that $\langle u(\alpha) \rangle$ is a cyclic group of order p , which means that $M \downarrow_{\langle u(\alpha) \rangle}$ is projective if and only if $M \downarrow_{\langle u(\alpha) \rangle}$ is free. The rank variety of M over E , denoted by $V_E^\#(M)$, is defined to be the set

$$\{0 \neq \alpha \in \mathbb{F}^n : M \text{ is not free as an } \mathbb{F}\langle u(\alpha) \rangle\text{-module}\} \cup \{0\}.$$

It is independent of the choices and orders of generators of E up to isomorphism. For a fixed ordered generator set of E , the $\mathbb{F}E$ -modules that are mutually isomorphic have the same rank variety. As an algebraic variety, the dimension of $V_E^\#(M)$ is denoted by $\dim V_E^\#(M)$. Notice that $\dim V_E^\#(M) \leq n$ and $\dim V_E^\#(\mathbb{F}) = n$ by the definition.

Given a finite group G and an $\mathbb{F}G$ -module M , if $E \leq G$, then we note that $\dim V_E^\#(M \downarrow_E)$ is an invariant of M up to G -conjugation of E . The complexity of M , denoted by $c_G(M)$, is defined to be the non-negative integer

$$\max_{E \in \mathcal{E}^{\max}} \{\dim V_E^\#(M \downarrow_E)\},$$

where $\mathcal{E}^{\max}$ is a set of representatives of all maximal elementary abelian p -subgroups of G up to G -conjugation. Observe that $c_G(\mathbb{F})$ is the p -rank of G . The definition provided here is quite convenient for us in the following discussion. It is also of course equivalent to the classical definition of the complexity of M described in [3] by [3, §1 Theorem], [4, Theorem 1.1] and [10, Theorem 7.6]. If M is an indecomposable $\mathbb{F}G$ -module, it is known that $c_G(M)$ is no more than the p -rank of the defect groups of the p -block containing M (see [3, Corollaries 4 and 5]). Moreover, it attains this upper bound if $p \nmid \dim_{\mathbb{F}} M$ (see [46, Proposition 2.1 (iii)]).

Lemma 2.7.1. *Let G be a finite group and M be an $\mathbb{F}G$ -module.*

- (i) *Let N and L be $\mathbb{F}G$ -modules. If $M \cong N \oplus L$, then $c_G(M) = \max\{c_G(N), c_G(L)\}$.*
- (ii) *Let $H \leq G$ and N be an $\mathbb{F}H$ -module. If $M \mid N \uparrow^G$ and $N \mid M \downarrow_H$, then $c_G(M) = c_H(N)$.*
- (iii) *Assume that M is an indecomposable $\mathbb{F}G$ -module with a vertex P and a P -source S . Then $c_G(M) = c_P(S)$.*
- (iv) *Let H be a finite group and N be an $\mathbb{F}H$ -module. Then $c_{G \times H}(M \boxtimes N) = c_G(M) + c_H(N)$.*
- (v) *Let $H \leq G$ and N be an $\mathbb{F}H$ -module. Then $c_G(N \uparrow^G) = c_H(N)$.*
- (vi) *Let ε^G be the set of all elementary abelian p -subgroups of G . If M is a p -permutation $\mathbb{F}G$ -module, then $c_G(M) = \max\{\text{rank}(E) : E \in \varepsilon^G, M(E) \neq 0\}$.*

Proof. All results except (vi) are known (see [5, 2.24.4], [50, Theorem 2.3 (vi)]). We prove (vi).

Set $m = \max\{\text{rank}(E) : E \in \varepsilon^G, M(E) \neq 0\}$. As M is a p -permutation $\mathbb{F}G$ -module, M is a direct sum of some trivial source $\mathbb{F}G$ -modules. By (i) and (iii), note that $c_G(M)$ is the largest p -rank of the vertices of all the indecomposable direct summands of M . Let V be a vertex of some indecomposable direct summand of M that has p -rank $c_G(M)$. There exists an elementary abelian p -group E such that $E \leq V$ and $\text{rank}(E) = c_G(M)$. By Lemma 2.6.1 (i) and (ii), we have $M(E) \neq 0$, which implies that $c_G(M) \leq m$.

For any elementary abelian p -subgroup F of G , if $M(F) \neq 0$, by Lemma 2.6.1 (ii), there must exist some indecomposable $\mathbb{F}G$ -module L such that $L \mid M$ and $L(F) \neq 0$. Using Lemma 2.6.1 (i) again, we get $\text{rank}(F) \leq \text{rank}(Q) \leq c_G(M)$, where Q is a vertex of L . Therefore, we deduce that $c_G(M) = m$. The proof is now complete. $\square$

2.8 Generic Jordan Types of Modules

Let $\langle g \rangle$ be a cyclic group of order p and M be an $\mathbb{F}\langle g \rangle$ -module. By the representation theory of cyclic groups, a non-zero $\mathbb{F}\langle g \rangle$ -module is indecomposable only if its $\mathbb{F}$ -dimension is no more than p . Moreover, for any $1 \leq i \leq p$, there is a unique indecomposable $\mathbb{F}\langle g \rangle$ -module of $\mathbb{F}$ -dimension i up to isomorphism. The matrix representation of g on this $\mathbb{F}\langle g \rangle$ -module is a Jordan block with all eigenvalues 1 and size i . So M can be viewed as a direct sum of the Jordan blocks. We call this direct sum of the Jordan blocks the Jordan type of M and denote it by $[1]^{n_1} \dots [p]^{n_p}$, where $[i]$ denotes a Jordan block of size i and n_i is the number of Jordan blocks of size i in this direct sum. For convenience, when we display the Jordan type $[1]^{m_1} \dots [p]^{m_p}$ of a non-zero $\mathbb{F}\langle g \rangle$ -module, we usually omit the component $[i]$ if $m_i = 0$. For example, if $p = 5$ and M has Jordan type $[1]^2[2]^0[3]^5[4]^1[5]^0$, write $[1]^2[3]^5[4]^1$ for the Jordan type of M .

Now take E to be an elementary abelian p -group of order p^n with a generator set $\{g_1, \dots, g_n\}$ and M to be an $\mathbb{F}E$ -module. Let $\mathbb{K}$ denote the extension field $\mathbb{F}(\alpha_1, \dots, \alpha_n)$ over $\mathbb{F}$ where $\alpha = \{\alpha_1, \dots, \alpha_n\}$ is a set of indeterminates. Denote by u_α the element $1 + \sum_{i=1}^n \alpha_i(g_i - 1)$ of $\mathbb{K}E$. Note that $\langle u_\alpha \rangle$ is a cyclic group of order p and M can be viewed as a $\mathbb{K}\langle u_\alpha \rangle$ -module by extending scalars and restriction. Therefore, the Jordan type of $M \downarrow_{\langle u_\alpha \rangle}$ is defined and is called the generic Jordan type of M . Wheeler in [61] showed that the generic Jordan type of M is independent of the choices of generators of E . The stable generic Jordan type of M is the generic Jordan type of M modulo free direct summands. Let M have generic Jordan type $[1]^{n_1} \dots [p]^{n_p}$. The module M is said to be generically free if $n_1 = \dots = n_{p-1} = 0$. Finally, for an $\mathbb{F}E$ -module N , say N and M have mutually complementary stable generic Jordan types if N has stable generic Jordan type $[1]^{n_{p-1}} \dots [p-1]^{n_1}$.

Notice that mutually isomorphic $\mathbb{F}E$ -modules have the same generic Jordan type. Let G be a finite group containing E and N be an $\mathbb{F}G$ -module. For any $g \in G$, let $E^g = g^{-1}Eg$ and note that $N \downarrow_{E^g}$ and $N \downarrow_E$ have the same generic Jordan type. For more properties of generic Jordan types of modules, one may refer to [20] and [61]. Some needed results of generic Jordan types of modules are collected as follows. For obtaining their proofs, the reader may refer to [22, Proposition 7.1] and [47, Proposition 2.2].

Proposition 2.8.1. [22, 47] *Let E be an elementary abelian p -subgroup of a finite group G . Let M be an $\mathbb{F}E$ -module and N be an indecomposable $\mathbb{F}G$ -module.*

- (i) *The module M is generically free only if $p \mid \dim_{\mathbb{F}} M$.*
- (ii) *The modules M and M^* have the same stable generic Jordan type.*
- (iii) *The stable generic Jordan type of a direct sum of $\mathbb{F}E$ -modules is the direct sum of the stable generic Jordan types of these modules.*
- (iv) *Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be a short exact sequence of $\mathbb{F}E$ -modules.*
 - (a) *If M_1 is generically free, then we get that M_2 and M_3 have the same stable generic Jordan type.*
 - (b) *If M_2 is generically free, then we get that M_1 and M_3 have mutually complementary stable generic Jordan types.*
 - (c) *If M_3 is generically free, then we get that M_1 and M_2 have the same stable generic Jordan type.*
- (v) *Let F be a proper subgroup of E and L be an $\mathbb{F}F$ -module. Then $L \uparrow^E$ is generically free.*
- (vi) *The module M is not generically free if and only if $V_E^{\#}(M) = V_E^{\#}(\mathbb{F})$.*
- (vii) *Let E have p -rank n . If $N \downarrow_E$ is not generically free, then $n \leq c_G(N)$. When $N \downarrow_E$ is not generically free, we have $c_G(N) = n$ if n is also the p -rank of the defect groups of the p -block containing N .*

Let E be an elementary abelian p -subgroup of G . If M is a p -permutation $\mathbb{F}G$ -module, as a corollary of Proposition 2.8.1, the stable generic Jordan type of $M \downarrow_E$ is described as follows.

Lemma 2.8.2. *Let G be a finite group and E be an elementary abelian p -subgroup of G . Let M be a p -permutation $\mathbb{F}G$ -module and $\mathcal{B}_E(M)$ be an $\mathbb{F}$ -basis of M permuted by E . Then $M \downarrow_E$ has stable generic Jordan type $[1]^n$, where n is the number of fixed points of $\mathcal{B}_E(M)$ under the action of E .*

Proof. We may assume that $M \downarrow_E$ is not a direct sum of trivial $\mathbb{F}E$ -modules. Let $\mathcal{O}_1, \dots, \mathcal{O}_s$ be all the pairwise distinct E -orbits of $\mathcal{B}_E(M)$ whose sizes are strictly larger than one. According to the hypotheses and the definitions of $\mathcal{O}_1, \dots, \mathcal{O}_s$, we have

$$M \downarrow_E \cong \underbrace{\mathbb{F} \oplus \dots \oplus \mathbb{F}}_{n \text{ copies}} \oplus \bigoplus_{i=1}^s (\mathbb{F}_{E_i}) \uparrow^E, \quad (2.8.1)$$

where $(\mathbb{F}_{E_i}) \uparrow^E \cong \langle \mathcal{O}_i \rangle_{\mathbb{F}}$ as $\mathbb{F}E$ -modules for all $1 \leq i \leq s$. For any $1 \leq i \leq s$, we have $E_i < E$ as $|\mathcal{O}_i| > 1$. The lemma thus follows by (2.8.1), Proposition 2.8.1 (iii) and (v). $\square$

2.9 Symmetric and Exterior Powers of Modules

We now describe the constructions of symmetric and exterior powers of modules. Let M be a finite-dimensional $\mathbb{F}$ -linear space and write $d = \dim_{\mathbb{F}} M$. If $d = 0$, then all the symmetric and exterior powers of M are defined to be the zero space. We thus assume that $d > 0$ and M has an $\mathbb{F}$ -basis $\{v_1, \dots, v_d\}$. Let x be a positive integer and $x > 1$. Use $\bigotimes^x M$ to denote the inner tensor product of x copies of M . So $\bigotimes^x M$ has an $\mathbb{F}$ -basis

$$\{v_{i_1} \otimes \dots \otimes v_{i_x} : \forall 1 \leq j \leq x, i_j \in \{1, \dots, d\}\}.$$

Let $\mathfrak{S}_S$ be the symmetric group acting on a finite set S . By convention, we set $\mathfrak{S}_0 = \mathfrak{S}_\emptyset = 1$ and put $\mathfrak{S}_z = \mathfrak{S}_{\{1, \dots, z\}}$ for any positive integer z . Let $S_x(M)$ be the $\mathbb{F}$ -linear subspace of $\bigotimes^x M$ spanned by the set

$$\{(w_1 \otimes \cdots \otimes w_x) - (w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(x)}) : \sigma \in \mathfrak{S}_x, \forall 1 \leq i \leq x, w_i \in M\}.$$

Let $S^x M$ be the $\mathbb{F}$ -linear quotient space $(\bigotimes^x M)/S_x(M)$. It is called the x th symmetric power of M . For any $w_1, \dots, w_x \in M$, we write

$$w_1 \dots w_x$$

to denote $w_1 \otimes \cdots \otimes w_x + S_x(M)$. It is clear that $w_1 \dots w_x = w_{\sigma(1)} \dots w_{\sigma(x)}$ for any $\sigma \in \mathfrak{S}_x$. Therefore, it is obvious to see that $S^x M$ has an $\mathbb{F}$ -basis

$$\{v_{i_1} \dots v_{i_x} : 1 \leq i_1 \leq \cdots \leq i_x \leq d\}. \quad (2.9.1)$$

To continue our discussion, it is necessary to introduce a group homomorphism. If $p > 2$, let sign_x denote the well-known non-trivial group homomorphism from $\mathfrak{S}_x$ to the subgroup $\{1, -1\}$ of the group $\mathbb{F} \setminus \{0\}$. If $p = 2$, we view sign_x as the trivial group homomorphism from $\mathfrak{S}_x$ to the group $\mathbb{F} \setminus \{0\}$. For convenience, we omit the subscript x of sign_x if there is no confusion. We write $\Lambda_x(M)$ for the $\mathbb{F}$ -linear subspace of $\bigotimes^x M$ spanned by the set

$$\{w_1 \otimes \cdots \otimes w_x : \forall 1 \leq i \leq x, w_i \in M, \exists 1 \leq j < k \leq x, w_j = w_k\}. \quad (2.9.2)$$

Let $\Lambda^x M$ be the $\mathbb{F}$ -linear quotient space $(\bigotimes^x M)/\Lambda_x(M)$. It is called the x th exterior power of M . For any $w_1, \dots, w_x \in M$, we write

$$w_1 \wedge \cdots \wedge w_x$$

to denote $w_1 \otimes \cdots \otimes w_x + \Lambda_x(M)$. We have $w_1 \wedge \cdots \wedge w_x = 0$ if there exist some $1 \leq i < j \leq x$ such that $w_i = w_j$. Assume further that we have $w_1 \wedge \cdots \wedge w_x \neq 0$. We thus have $d \geq x$. For any $\sigma \in \mathfrak{S}_x$, by (2.9.2), it is not difficult to note that $w_{\sigma(1)} \wedge \cdots \wedge w_{\sigma(x)} = \text{sign}(\sigma)(w_1 \wedge \cdots \wedge w_x)$. We thus deduce that $\Lambda^x M$ has an $\mathbb{F}$ -basis

$$\{v_{i_1} \wedge \cdots \wedge v_{i_x} : 1 \leq i_1 < \cdots < i_x \leq d\}. \quad (2.9.3)$$

Moreover, $\Lambda^x M$ is the zero space if $d < x$. By convention, we set $S^1 M = \Lambda^1 M = M$.

Let G be a finite group. Assume further that M is an $\mathbb{F}G$ -module. Notice that $\bigotimes^x M$ is an $\mathbb{F}G$ -module under the diagonal action of G . By the definitions of $S_x(M)$ and $\Lambda_x(M)$, both $S_x(M)$ and $\Lambda_x(M)$ are $\mathbb{F}G$ -submodules of $\bigotimes^x M$. Therefore, it is obvious to see that both $S^x M$ and $\Lambda^x M$ are $\mathbb{F}G$ -modules. We put $S^0 M = \Lambda^0 M = \mathbb{F}$ and notice that both $S^y M$ and $\Lambda^y M$ are $\mathbb{F}G$ -modules for any non-negative integer y .

2.10 Combinatorics

Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout the remaining part of this thesis, we let $n \in \mathbb{N}$ and $[1, n] = \{1, \dots, n\}$. In this thesis, we multiply the permutations of symmetric groups from right to left. So $(1, 2)(2, 3) = (1, 2, 3) \in \mathfrak{S}_3$.

A composition of n is a finite sequence $(\lambda_1, \dots, \lambda_\ell)$ of non-negative integers such that $\sum_{i=1}^\ell \lambda_i = n$. If this sequence is non-increasing and $\lambda_\ell \neq 0$, we call this composition a partition of n . Throughout this thesis, we do not distinguish the compositions of 0. Therefore, there is a unique composition of 0. As the empty set, the unique composition of 0 is denoted by $\emptyset$. It is also the unique partition of 0. Write $\lambda \models n$ (resp. $\lambda \vdash n$) to indicate that λ is a composition (resp. partition) of n . Write $\lambda \models 0$ (resp. $\lambda \vdash 0$) if $\lambda = \emptyset$. In this thesis, we use the exponential expression of sequences of integers. For an example, we have $(1, 2, 2, 1, 0, 0, 0) = (1, 2^2, 1, 0^3)$. We now define a partial order $\trianglelefteq$ as follows. Let $\alpha = (\alpha_1, \dots, \alpha_r) \vdash n$ and $\beta = (\beta_1, \dots, \beta_s) \vdash n$. If $r \leq s$, then write $\beta \trianglelefteq \alpha$ if $\sum_{i=1}^t \beta_i \leq \sum_{i=1}^t \alpha_i$ for any $1 \leq t \leq r$. Also write $\beta \triangleleft \alpha$ if $\beta \trianglelefteq \alpha$ and $\beta \neq \alpha$. Recall that $\trianglelefteq$ is the usual dominance order of partitions.

Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \models n$ and call each entry of λ a part of λ . Let $|\lambda| = \sum_{i=1}^\ell \lambda_i = n$ and $\bar{\lambda}$ be the partition obtained from λ by omitting the zero parts of λ and rearranging the non-zero parts of λ . Let $\ell(\lambda)$ be the number of parts of $\bar{\lambda}$. By convention, $\ell(\emptyset) = |\emptyset| = 0$ and $\bar{\emptyset} = \emptyset$. Let $a, m \in \mathbb{N}_0$ and $\mu = (\mu_1, \dots, \mu_{\ell'}) \models m$. We define the following compositions and partitions:

$$\begin{aligned} a\mu &= (a\mu_1, \dots, a\mu_{\ell'}) \models am, \\ \mu \cup \lambda &= (\mu_1, \dots, \mu_{\ell'}, \lambda_1, \dots, \lambda_\ell) \vdash m + n, \\ \mu \cup \emptyset &= \emptyset \cup \mu = \bar{\mu}. \end{aligned}$$

Moreover, if $\ell' \leq \ell$, we define $\emptyset + \emptyset = \emptyset$ and

$$\lambda + \mu = \mu + \lambda = (\lambda_1 + \mu_1, \dots, \lambda_{\ell'} + \mu_{\ell'}, \lambda_{\ell'+1}, \dots, \lambda_\ell) \models m + n.$$

Assume further that $m > 0$ and $\mu \vdash m$. Then μ is a p -restricted partition of m if $\mu_i - \mu_{i+1} < p$ for all $1 \leq i < \ell'$ and $\mu_{\ell'} < p$. The unique p -restricted partition of 0 is $\emptyset$. By [26, Lemma 7.5], the p -adic expansion of μ is the unique sum $\sum_{i=0}^x p^i \mu(i)$ for some $x \in \mathbb{N}_0$, where $\mu(i)$ is a p -restricted partition of $|\mu(i)|$ for all $0 \leq i \leq x$, $\mu(x) \neq \emptyset$ and $\mu = \sum_{i=0}^x p^i \mu(i)$. We say μ is p -regular if there is no $b \in \mathbb{N}_0$ such that $b + p \leq \ell'$ and $\mu_{b+1} = \dots = \mu_{b+p}$. The set of all the p -regular partitions of n is denoted by $\mathfrak{Reg}(n)$. If μ has exactly two parts, call μ a two-part partition. Say μ a hook partition if $\mu = (n - r, 1^r)$ for some integer r such that $0 \leq r < n$.

The Young diagram of λ , denoted by $[\lambda]$, is a left aligned array of boxes, where the i th row of this array has exactly λ_i boxes for all $1 \leq i \leq \ell$. In particular, the i th row of $[\lambda]$ is empty if $\lambda_i = 0$. We regard the 1st row (resp. ℓ th row) of $[\lambda]$ as the top row (resp. bottom row) of $[\lambda]$. The columns of $[\lambda]$ are counted from left to right. So the 1st column (resp. last column) is the leftmost (rightmost) column of $[\lambda]$. We will not distinguish between λ and $[\lambda]$.

Recall that $m > 0$ and $\mu \vdash m$. For any box of μ , we can identify it with the pair (i, j) , where i is the row number of this box and j is the column number of this box. This identification gives us a bijection between μ and the following set

$$\mathbb{Y}_\mu = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq \ell', 1 \leq j \leq \mu_i\}.$$

Note that there exists $\mu' \vdash m$ such that $\mathbb{Y}_{\mu'} = \{(i, j) : (j, i) \in \mathbb{Y}_\mu\}$. The partition μ' is called the conjugate of μ . By convention, set $\emptyset' = \emptyset$. So the conjugate of $\emptyset$ is still $\emptyset$. Each member of $\mathbb{Y}_\mu$ is called a node of μ . For any node (i, j) of μ , the hook of the node (i, j) is the set $H_{i,j}^\mu = \{(i, k) \in \mathbb{Y}_\mu : k \geq j\} \cup \{(k, j) \in \mathbb{Y}_\mu : k > i\}$. For convenience, set $h_{i,j}^\mu = |H_{i,j}^\mu|$.

The rim of μ is its south-east border. A rim p -hook of μ is p connected boxes of the rim of μ whose removal makes the remaining boxes of μ form the Young diagram of a partition. The p -core $\kappa(\mu)$ of μ is the partition whose Young diagram is constructed by removing all rim p -hooks from μ successively. The number of rim p -hooks removed from μ to obtain $\kappa(\mu)$ is called the p -weight of μ . Recall that $\kappa(\mu)$ and the p -weight of μ are independent of the manner of removing p -hooks (see [37, Pages 76-78]). So $\kappa(\mu)$ and the p -weight of μ are well-defined.

Finally recall that $\lambda \models n$. Via a bijection, all the boxes of λ are replaced by the numbers $1, \dots, n$. Such a replacement is called a λ -tableau. Following this definition, we view each row of a λ -tableau as a set of numbers. Let $T(\lambda)$ be the set of all the λ -tableaux. Given $s, t \in T(\lambda)$, write $s \approx t$ if the i th rows of s and t are equal as sets for all $1 \leq i \leq \ell$. Let $\{t\}$ be the equivalence class containing t with respect to $\approx$ and call it a λ -tabloid. Regard the rows of t as the rows of $\{t\}$. Let $\mathcal{T}(\lambda)$ be the set of all the λ -tabloids. To display a λ -tabloid $\{t\}$, we draw lines between the rows of t and rewrite the numbers of each row of t in increasing order from left to right. For example, if $\lambda = (2, 0, 3) \models 5$ and

$$t = \begin{array}{ccc} 4 & 1 & \\ 3 & 5 & 2 \end{array},$$

then we have

$$\{t\} = \frac{\overline{1 \ 4}}{\overline{2 \ 3 \ 5}}.$$

2.11 Modules of Symmetric Groups

We now briefly present some material of representation theory of symmetric groups needed in this thesis. One can refer to [35] or [37] for a background on this topic. For a given subgroup H of $\mathfrak{S}_n$, denote by $[1, n]/H$ the set of orbits of $[1, n]$ under the natural action of H . For any subset S of $[1, n]$, we view $\mathfrak{S}_S$ as a subgroup of $\mathfrak{S}_n$ naturally. Given $m_1, \dots, m_\ell \in \mathbb{N}_0$, if $\sum_{i=1}^\ell m_i \leq n$, we define

$$\mathfrak{S}_{(m_1, \dots, m_\ell)} = \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_\ell} \leq \mathfrak{S}_n,$$

where $\mathfrak{S}_{m_1}$ acts on the set $\{1, \dots, m_1\}$ (if $m_1 > 0$), $\mathfrak{S}_{m_2}$ acts on the set $\{m_1 + 1, \dots, m_1 + m_2\}$ (if $m_2 > 0$) and so on. Let $\lambda \vdash n$. The Young subgroup of $\mathfrak{S}_n$ associated with λ is defined to be $\mathfrak{S}_\lambda$. Let $a, b \in \mathbb{N}$ and $a \leq b \leq n$. We set $\mathfrak{S}_{b-a}^{+a} = \mathfrak{S}_{\{a+1, \dots, b\}}$. Notice that $\mathfrak{S}_a$ and $\mathfrak{S}_{b-a}^{+a}$ can commute with each other. Moreover, we have $\mathfrak{S}_{b-a}^{+a} \cong \mathfrak{S}_{b-a}$ and $\mathfrak{S}_{(a, b-a)} = \mathfrak{S}_a \times \mathfrak{S}_{b-a}^{+a}$. Use $\text{sgn}(n)$ to denote the sign module of $\mathfrak{S}_n$ and omit the parameter if the related symmetric group is clear. Note that we always have $\text{sgn}(n) \cong \mathbb{F}_{\mathfrak{S}_n}$ if $p = 2$.

The Young permutation module with respect to λ , denoted by M^λ , is the $\mathbb{F}$ -linear space generated by all λ -tabloids, where $\mathfrak{S}_n$ permutes these λ -tabloids. It is also isomorphic to $(\mathbb{F}_{\mathfrak{S}_\lambda})^{\uparrow \mathfrak{S}_n}$. Since M^λ is a permutation $\mathbb{F}\mathfrak{S}_n$ -module, we notice that M^λ is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module. Let P be a p -subgroup of $\mathfrak{S}_n$. As $\mathcal{T}(\lambda)$ is an $\mathbb{F}$ -basis of M^λ that can be permuted by any p -subgroup of $\mathfrak{S}_n$, according to Lemma 2.6.1 (iv), we can easily compute $\dim_{\mathbb{F}} M^\lambda(P)$. In particular, the proof of the following lemma is clear.

Lemma 2.11.1. *Let $\lambda \vdash n$ and P be a p -subgroup of $\mathfrak{S}_n$. Then $\{t\} \in \mathcal{T}(\lambda)^P$ if and only if the set of numbers in each row of $\{t\}$ is a union set of some members of $[1, n]/P$. In particular, both $|\mathcal{T}(\lambda)^P|$ and $\dim_{\mathbb{F}} M^\lambda(P)$ are the number of unordered ways to insert the orbits in $[1, n]/P$ into the rows of λ .*

To describe the Specht modules of symmetric groups, we should introduce the polytabloids of group algebras of symmetric groups. Let $t \in \mathcal{T}(\lambda)$. Call t a row standard λ -tableau if the entries of each row of t are increasing from left to right. Call t a column standard λ -tableau if the entries of each column of t are increasing from top to the bottom. If t is both row standard and column standard, then t is called a standard λ -tableau. Recall that we view each row of t as a set of numbers. Similarly, we also view each column of t as a set of numbers. The column stabilizer of t , denoted by C_t , is the subset of $\mathfrak{S}_n$ that contains precisely the elements stabilizing every column of t . It is obvious to see that $C_t \leq \mathfrak{S}_n$. The λ -polytabloid with respect to t , denoted by e_t , is the element of M^λ that is defined by

$$\sum_{g \in C_t} \text{sign}(g) g\{t\}.$$

The Specht module S^λ is an $\mathbb{F}\mathfrak{S}_n$ -submodule of M^λ generated by all λ -polytabloids. The dual of S^λ is denoted by $S_{\lambda'}$. It is known that $\{e_u : u \text{ is a standard } \lambda\text{-tableau}\}$ is an $\mathbb{F}$ -basis of S^λ (see [35, 8.4]). This $\mathbb{F}$ -basis is called the standard basis of S^λ . By [63, Theorem 3.2], we also know that S^λ is indecomposable if $p > 2$ or $p = 2$ and λ is 2-regular (or 2-restricted). By [35, 8.15], the relationship between S^λ and $S_{\lambda'}$ is given by the formula

$$S^\lambda \otimes \text{sgn} \cong S_{\lambda'}.$$

In particular, if $p = 2$, we have

$$S^\lambda \cong S_{\lambda'}.$$

If we have $p = 2$ and S^λ is self-dual, the formula $S^\lambda \cong S_{\lambda'}$ implies that $S^\lambda \cong S^{\lambda'}$. Unlike the characteristic zero case, $\{S^\lambda : \lambda \vdash n\}$ is usually not the set of all simple $\mathbb{F}\mathfrak{S}_n$ -modules up to isomorphism. However, James in [35, 11.5] showed that the module

$$D^\lambda = S^\lambda / \text{Rad}(S^\lambda)$$

is either zero or simple. It is simple if and only if $\lambda \in \mathfrak{Reg}(n)$. Moreover, the set

$$\{D^\mu : \mu \in \mathfrak{Reg}(n)\}$$

forms a complete set of representatives of isomorphism classes of simple $\mathbb{F}\mathfrak{S}_n$ -modules. It is known that all the simple $\mathbb{F}\mathfrak{S}_n$ -modules are self-dual (see [35, Page 42]). For any $\alpha \vdash n$ and $\beta \in \mathfrak{Reg}(n)$, we have $(S^\beta : D^\beta) = 1$ and $(S^\alpha : D^\beta) \neq 0$ only if $\alpha \leq \beta$ (see [35, 12.2]). Following the notation used in [15, Section 4], we write

$$JM(n) = \{\lambda \vdash n : S^\lambda \text{ is simple}\}.$$

The set $JM(n)$ will be repeatedly utilized in Chapter 5. A Specht filtration of an $\mathbb{F}\mathfrak{S}_n$ -module is a filtration of the module with all the successive quotient factors isomorphic to some Specht modules. Recall that every Young permutation module has a Specht filtration (see [35, 17.14]). The following lemma shall be repeatedly used in Chapter 5.

Lemma 2.11.2. *Let $\lambda \vdash n$ and M be an $\mathbb{F}\mathfrak{S}_n$ -module. Then*

- (i) *M is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module if and only if M^* is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module.*
- (ii) *M is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module if and only if we have $M \otimes \text{sgn}$ is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module.*
- (iii) *S^λ is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module if and only if $S^{\lambda'}$ is a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module.*

Proof. Note that $*$ preserves direct sum of $\mathbb{F}\mathfrak{S}_n$ -modules and takes a permutation $\mathbb{F}\mathfrak{S}_n$ -module to a permutation $\mathbb{F}\mathfrak{S}_n$ -module. (i) thus follows by noticing that $M \cong (M^*)^*$. For (ii), let P be a p -subgroup of $\mathfrak{S}_n$ and $\mathcal{B}_P$ be an $\mathbb{F}$ -basis of M which is permuted by P . Let v be a generator of sgn . Then $\mathcal{B}_{P,v} = \{u \otimes v : u \in \mathcal{B}_P\}$ is an $\mathbb{F}$ -basis of $M \otimes \text{sgn}$ which is permuted by P . Therefore, the assertion of (ii) is shown by the definition. As $S^{\lambda'} \cong (S^\lambda \otimes \text{sgn})^*$, (iii) thus follows by combining (i) and (ii). The lemma follows. $\square$

The representatives of all isomorphism classes of indecomposable direct summands of Young permutation modules are usually called Young modules and are already parameterized by James in [36, Theorem 3.1]. Let $\lambda \vdash n$ and $\mu \vdash n$. For a given decomposition of M^λ into a direct sum of indecomposable modules, there exists a unique indecomposable direct summand of M^λ that contains S^λ . The module, unique up to isomorphism, is denoted by Y^λ . It is known that Y^λ is a self-dual trivial source $\mathbb{F}\mathfrak{S}_n$ -module (see [24, Corollary 1.53.1]). In [36, Theorem 3.1], James proved that $[M^\lambda : Y^\lambda] = 1$ and $[M^\lambda : Y^\mu] \neq 0$ only if $\lambda \leq \mu$. We thus get

$$M^\lambda \cong Y^\lambda \oplus \bigoplus_{\lambda \triangleleft \mu} [M^\lambda : Y^\mu] Y^\mu. \quad (2.11.1)$$

In [18], Donkin introduced the signed Young modules for the case $p > 2$ and parameterized them by some pairs of partitions. It is known that they are also self-dual trivial source modules (see [15, Theorem 3.8 (b)]). If $p > 2$, we use $Y(\alpha|p\beta)$ to denote the $\mathbb{F}\mathfrak{S}_n$ -signed Young module labelled by the partitions α, β , where $|\alpha| + p|\beta| = n$. For the details of the definition of a signed Young module, one may refer to [15, 3.2] or [18].

We also need to recall another known result. It is known that every Young module has a Specht filtration. Namely, according to [16, 2.6], for some $m \in \mathbb{N}_0$, Y^λ has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{m+1} = Y^\lambda, \quad (2.11.2)$$

where M_{i+1}/M_i is isomorphic to some Specht module S^{λ_i} and $\lambda \leq \lambda_i$ for all $0 \leq i \leq m$. Moreover, $M_1 \cong S^\lambda$ and the subquotient S^λ occurs exactly once in this filtration of Y^λ .

The Brauer quotients of Young modules with respect to their vertices are studied in [19] and [26]. To describe them, let λ have p -adic expansion $\sum_{i=0}^m p^i \lambda(i)$ for some non-negative integer m , where $|\lambda(i)| = a_i$ for all $0 \leq i \leq m$. Denote by $\mathcal{O}_\lambda$ the partition

$$\underbrace{(p^m, \dots, p^m)}_{a_m \text{ times}}, \underbrace{(p^{m-1}, \dots, p^{m-1})}_{a_{m-1} \text{ times}}, \dots, \underbrace{(1, \dots, 1)}_{a_0 \text{ times}}.$$

Let $a, b \in \mathbb{N}$ and $a \leq b \leq n$. As $\mathfrak{S}_{b-a}^{+a} \cong \mathfrak{S}_{b-a}$, by the definitions of Young permutation modules, Specht modules and Young modules, we can similarly define Young permutation modules, Specht modules and Young modules for the group algebra $\mathbb{F}\mathfrak{S}_{b-a}^{+a}$. Let $\eta \vdash b-a$. If $a < b$, let $M_{a,b}^\eta$ be the $\mathbb{F}\mathfrak{S}_{b-a}^{+a}$ -Young permutation module labelled by η . Let $S_{a,b}^\eta$ denote the $\mathbb{F}\mathfrak{S}_{b-a}^{+a}$ -Specht module labelled by η . Let $Y_{a,b}^\eta$ denote the $\mathbb{F}\mathfrak{S}_{b-a}^{+a}$ -Young module labelled by η . If $a = b$, set $M_{a,a}^\emptyset = S_{a,a}^\emptyset = Y_{a,a}^\emptyset = \mathbb{F}_{\mathfrak{S}_\emptyset}$. The $\mathbb{F}\mathfrak{S}_{b-a}^{+a}$ -modules $M_{a,b}^\eta$, $S_{a,b}^\eta$, $Y_{a,b}^\eta$ shall be frequently used in Chapter 5.

We now describe the Brauer quotients of Young modules with respect to their vertices.

Theorem 2.11.3. [19, 26] *Let $\lambda \vdash n$. Then all the vertices of Y^λ are $\mathfrak{S}_n$ -conjugate to the Sylow p -subgroups of $\mathfrak{S}_{\mathcal{O}_\lambda}$. Moreover, Y^λ is projective if and only if λ is p -restricted.*

Theorem 2.11.4. [19, 26] *Let $\lambda \vdash n$ and λ have p -adic expansion $\sum_{i=0}^m p^i \lambda(i)$ for some $m \in \mathbb{N}_0$, where $|\lambda(i)| = a_i$ for all $0 \leq i \leq m$. Let $\alpha = (a_0, a_1, \dots, a_m)$ and P_λ be a Sylow p -subgroup of $\mathfrak{S}_{\mathcal{O}_\lambda}$.*

- (i) *We have $\mathfrak{S}_\alpha \cong N_{\mathfrak{S}_n}(P_\lambda)/N_{\mathfrak{S}_{\mathcal{O}_\lambda}}(P_\lambda) \cong (N_{\mathfrak{S}_n}(P_\lambda)/P_\lambda)/(N_{\mathfrak{S}_{\mathcal{O}_\lambda}}(P_\lambda)/P_\lambda)$.*
- (ii) *The group $N_{\mathfrak{S}_{\mathcal{O}_\lambda}}(P_\lambda)/P_\lambda$ acts trivially on $Y^\lambda(P_\lambda)$. Therefore, $Y^\lambda(P_\lambda)$ can be viewed as an $\mathbb{F}\mathfrak{S}_\alpha$ -module.*
- (iii) *We have $Y^\lambda(P_\lambda) \cong Y^{\lambda(0)} \boxtimes Y_{a_0, a_0+a_1}^{\lambda(1)} \boxtimes \dots \boxtimes Y_{\sum_{i=0}^{m-1} a_i, \sum_{i=0}^m a_i}^{\lambda(m)}$ as $\mathbb{F}\mathfrak{S}_\alpha$ -modules.*

Recall that $\lambda \vdash n$ and $\kappa(\lambda)$ is the p -core of λ . We close the whole chapter by describing the p -blocks of $\mathbb{F}\mathfrak{S}_n$. To fulfill this goal, let us state a proven conjecture of Nakayama as follows.

Theorem 2.11.5. [37, 6.1.21] *Every p -block of $\mathbb{F}\mathfrak{S}_n$ is labelled by the p -core of a partition of n . Let c be the p -core of a partition of n and B be the p -block of $\mathbb{F}\mathfrak{S}_n$ labelled by c . Let $\lambda \vdash n$. Then S^λ belongs to B if and only if $\kappa(\lambda) = c$. Similarly, Y^λ belongs to B if and only if $\kappa(\lambda) = c$. Moreover, let $\mu \in \mathfrak{Rcg}(n)$. Then D^μ belongs to B if and only if $\kappa(\mu) = c$.*

By Theorem 2.11.5, let $B_{\kappa(\lambda)}$ denote the p -block of $\mathbb{F}\mathfrak{S}_n$ labelled by $\kappa(\lambda)$. By the definition of p -weight of λ , observe that the p -weight w of λ satisfies the equation $|\kappa(\lambda)| + pw = n$. Call w the p -weight of $B_{\kappa(\lambda)}$. Finally, recall that a defect group of $B_{\kappa(\lambda)}$ can be chosen to be a Sylow p -subgroup of $\mathfrak{S}_{pw}$ (see [37, 6.2.45]).

Chapter 3

Green-Klein Theorem and Some Computational Results

This chapter essentially provides preliminary knowledge for the next chapter. We shall utilize the Green-Klein Theorem to study some short exact sequences of modules of a cyclic group of order p . The obtained results will be used to compute the complexities of some simple modules of symmetric groups in the next chapter.

3.1 Green-Klein Theorem

This section contains a short introduction to the Green-Klein Theorem. Though the full version of the Green-Klein Theorem holds for a more general situation (see [43, Theorems 4.1 and 4.2]), for our purpose, we only state a special case of this theorem and call it the Green-Klein Theorem. Moreover, we also include two lemmas that will be used in this chapter.

Let A be a finite-dimensional $\mathbb{F}$ -algebra. It is known that understanding the extensions of A -modules is important for studying the cohomological theory and the representation theory of A . However, even if A is the group algebra of a finite group, it is still extremely difficult to completely understand the extensions of A -modules. For example, if the representation type of A is tame or wild, then it seems to be impossible to completely understand the extensions of A -modules. Assume further that A is of finite representation type. If A is the group algebra of a cyclic group of order p , then the Green-Klein Theorem tells us that we may understand the extensions of A -modules from the viewpoint of combinatorics. To describe the Green-Klein Theorem, we need to introduce the following two definitions.

Definition 3.1.1. Let $t \in \mathbb{N}$. For any $0 \leq i \leq t$, let $s^{(i)} = (s_1^{(i)}, s_2^{(i)}, \dots)$ be a composition whose parts are non-increasing. By adding zeros after the last part of every composition, we can require that all compositions in $\{s^{(0)}, \dots, s^{(t)}\}$ have the same number of parts. Then the sequence $(s^{(0)}, \dots, s^{(t)})$ is called a Littlewood-Richardson sequence if $(s^{(0)}, \dots, s^{(t)})$ satisfies the following conditions:

(a) For any $1 \leq i \leq t$ and each subscript j , we have

$$0 \leq s_j^{(i)} - s_j^{(i-1)} \leq 1;$$

(b) For any $2 \leq i \leq t$ and each subscript k , we have

$$\sum_{j \geq k} (s_j^{(i)} - s_j^{(i-1)}) \leq \sum_{j \geq k} (s_j^{(i-1)} - s_j^{(i-2)}).$$

Assume that $(s^{(0)}, \dots, s^{(t)})$ is a Littlewood-Richardson sequence. Let $\alpha = \overline{s^{(0)}}$ and $\gamma = \overline{s^{(t)}}$. For any $1 \leq i \leq t$, set $\beta_i = |s^{(i)}| - |s^{(i-1)}|$ and $\beta = (\beta_1, \dots, \beta_t)$. Then $[\alpha, \beta; \gamma]$ is called the type of $(s^{(0)}, \dots, s^{(t)})$. For convenience, we also call a Littlewood-Richardson sequence a L-R sequence.

Definition 3.1.2. Let G be a cyclic group of order p and M be an $\mathbb{F}G$ -module. The Jordan type of M can be expressed by a partition. More precisely, if the Jordan type of M is $[1]^{n_1} \dots [p]^{n_p}$, then the Jordan type of M can be expressed by the partition

$$(\underbrace{p, \dots, p}_{n_p \text{ times}}, \underbrace{p-1, \dots, p-1}_{n_{p-1} \text{ times}}, \dots, \underbrace{1, \dots, 1}_{n_1 \text{ times}}).$$

The displayed partition is called the type of M . We denote the type of M by J_M . For any $\mathbb{F}G$ -modules U and V , notice that $J_U = J_V$ if and only if $U \cong V$ as $\mathbb{F}G$ -modules.

We now state the Green-Klein Theorem. It is a special case of [43, Theorems 4.1 and 4.2].

Theorem 3.1.3. Let G be a cyclic group of order p .

- (i) If there exists a short exact sequence of $\mathbb{F}G$ -modules

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0,$$

then there exists a L-R sequence of type $[J_W, J'_U; J_V]$.

- (ii) If there exists a L-R sequence of type $[\alpha, \beta; \gamma]$, then there exists a short exact sequence of $\mathbb{F}G$ -modules

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

such that $\alpha = J_W$, $\beta = J'_U$ and $\gamma = J_V$.

We close this section by two lemmas. These lemmas will be used in the next section.

Lemma 3.1.4. Let $t \in \mathbb{N}$ and $(s^{(0)}, \dots, s^{(t)})$ be a L-R sequence, where $s^{(i)} = (s_1^{(i)}, s_2^{(i)}, \dots)$ for any $0 \leq i \leq t$. Assume that $(s^{(0)}, \dots, s^{(t)})$ has type $[\alpha, (1^t); \gamma]$. If there exist $j > 1$ and $2 \leq k \leq t$ such that $s_j^{(k)} = s_j^{(k-1)} + 1$, then $s_i^{(k-1)} = s_i^{(k-2)}$ for any $1 \leq i < j$.

Proof. Write S for $(s^{(0)}, \dots, s^{(t)})$. Suppose that there exists $1 \leq \ell < j$ such that $s_\ell^{(k-1)} \neq s_\ell^{(k-2)}$. Since S is a L-R sequence, by Definition 3.1.1 (a), we get $s_\ell^{(k-1)} = s_\ell^{(k-2)} + 1$. Since S has type $[\alpha, (1^t); \gamma]$, by the definition of the type of a L-R sequence, we have $|s^{(m)}| - |s^{(m-1)}| = 1$ for every $1 \leq m \leq t$. As $s_\ell^{(k-1)} = s_\ell^{(k-2)} + 1$ and $|s^{(k-1)}| - |s^{(k-2)}| = 1$, notice that $s_u^{(k-1)} = s_u^{(k-2)}$ for every subscript $u > \ell$. As $s_j^{(k)} = s_j^{(k-1)} + 1$ and $j > \ell$, we can deduce that

$$\sum_{v \geq j} (s_v^{(k-1)} - s_v^{(k-2)}) = 0 < 1 = |s^{(k)}| - |s^{(k-1)}| = \sum_{v \geq j} (s_v^{(k)} - s_v^{(k-1)}),$$

which implies that S does not satisfy Definition 3.1.1 (b). In particular, S is not a L-R sequence, which contradicts the assumption that S is a L-R sequence. So we get the desired result. $\square$

Lemma 3.1.5. Let $t \in \mathbb{N}$ and $(s^{(0)}, \dots, s^{(t)})$ be a L-R sequence, where $s^{(i)} = (s_1^{(i)}, \dots, s_u^{(i)})$ for any $0 \leq i \leq t$. Assume that $(s^{(0)}, \dots, s^{(t)})$ has type $[\alpha, \beta; \gamma]$ and $\beta = (\beta_1, \dots, \beta_t)$.

- (i) Given $0 \leq i \leq t$, there exist $x_{i1}, \dots, x_{iu} \in \mathbb{N}_0$ such that $s^{(i)} = (s_1^{(i)} - x_{i1}, \dots, s_u^{(i)} - x_{iu})$ and $\sum_{v=1}^u x_{iv} = \sum_{w=i+1}^t \beta_w$.
- (ii) There is a L-R sequence $(\tilde{s}^{(0)}, \dots, \tilde{s}^{(t-1)}, \gamma)$ of type $[\alpha, \beta; \gamma]$.

Proof. Write S for $(s^{(0)}, \dots, s^{(t)})$. For (i), as S is a L-R sequence, Definition 3.1.1 (a) tells us that there exist $x_{i1}, \dots, x_{iu} \in \mathbb{N}_0$ such that $s^{(i)} = (s_1^{(i)} - x_{i1}, \dots, s_u^{(i)} - x_{iu})$. Since S has type $[\alpha, \beta; \gamma]$, by the definition of the type of a L-R sequence, $|s^{(j)}| - |s^{(j-1)}| = \beta_j$ for any $1 \leq j \leq t$. We thus have $|s^{(t)}| - \sum_{v=1}^u x_{iv} = |s^{(i)}| = |s^{(t)}| - \sum_{w=i+1}^t \beta_w$. So $\sum_{v=1}^u x_{iv} = \sum_{w=i+1}^t \beta_w$.

For (ii), the case $s^{(t)} = \emptyset$ is trivial. Assume that $s^{(t)} \neq \emptyset$. Let x be the largest subscript such that $s_x^{(t)} \neq 0$. Since S is a L-R sequence of type $[\alpha, \beta; \gamma]$, notice that $\gamma = (s_1^{(t)}, \dots, s_x^{(t)})$. By (i), note that $s_y^{(z)} = 0$ for every $x < y \leq u$ and $0 \leq z \leq t$. Set $\tilde{s}^{(k)} = (s_1^{(k)}, \dots, s_x^{(k)})$ for every $0 \leq k \leq t-1$. As S is a L-R sequence of type $[\alpha, \beta; \gamma]$, note that Definition 3.1.1 (a) and (b) hold for $(\tilde{s}^{(0)}, \dots, \tilde{s}^{(t-1)}, \gamma)$ and the L-R sequence $(\tilde{s}^{(0)}, \dots, \tilde{s}^{(t-1)}, \gamma)$ has type $[\alpha, \beta; \gamma]$. (ii) thus follows. $\square$

3.2 Some Computational Results

In this section, we use the Green-Klein Theorem to study some short exact sequences of modules of a cyclic group of order p . The obtained computational results will be used in the next chapter. For convenience, throughout this section, G is a cyclic group of order p . We use a sequence of propositions to present our main results.

Proposition 3.2.1. *Let U, V, W be $\mathbb{F}G$ -modules. Assume that U has Jordan type $[1]^1$ and V has Jordan type $[1]^{n_1} \dots [p]^{n_p}$. If there exists a short exact sequence*

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0,$$

then the Jordan type of W is contained in the set

$$\{[1]^{n_1} \dots [i-1]^{n_{i-1}+1} [i]^{n_i-1} \dots [p]^{n_p} : 1 < i \leq p, n_i > 0\} \cup \{[1]^{n_1-1} \dots [p]^{n_p} : n_1 > 0\}.$$

Proof. By the assumptions and Theorem 3.1.3 (i), there is a L-R sequence of type $[J_W, (1); J_V]$. By the definition of the type of a L-R sequence and Lemma 3.1.5 (ii), there is a L-R sequence $(s^{(0)}, J_V)$ such that $\overline{s^{(0)}} = J_W$ and $|J_V| - |s^{(0)}| = 1$. Let $J_V = (\lambda_1, \dots, \lambda_b)$. By Lemma 3.1.5 (i), observe that $J_W = \overline{s^{(0)}} = (\lambda_1, \dots, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_b)$ for some $1 \leq i \leq b$. As V has Jordan type $[1]^{n_1} \dots [p]^{n_p}$, we get the desired result by Definition 3.1.2. $\square$

Proposition 3.2.2. *Let U, V, W be $\mathbb{F}G$ -modules. Assume that U has Jordan type $[a]^1$ and V has Jordan type $[1]^x [p]^y$, where we have $0 < a < p$ and $x, y > 0$. If there exists a short exact sequence*

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0,$$

then the Jordan type of W is either $[1]^x [p-a]^1 [p]^{y-1}$ or $[1]^{x-1} [p-a+1]^1 [p]^{y-1}$.

Proof. The case $a = 1$ is clear by Proposition 3.2.1. We thus can assume further that $1 < a < p$. By the assumptions and Theorem 3.1.3 (i), there is a L-R sequence of type $[J_W, (1^a); J_V]$. By the definition of the type of a L-R sequence and Lemma 3.1.5 (ii), there exists a L-R sequence $(s^{(0)}, \dots, s^{(a)})$ such that $\overline{s^{(0)}} = J_W$ and $s^{(a)} = J_V$. In particular, $s^{(a)} = (p^y, 1^x)$ by Definition 3.1.2. Moreover, we also have $|s^{(i)}| - |s^{(i-1)}| = 1$ for every $1 \leq i \leq a$.

We claim that $s^{(a-1)} = (p^{y-1}, p-1, 1^x)$. As $|s^{(a)}| - |s^{(a-1)}| = 1$ and the parts of $s^{(a-1)}$ are non-increasing, by Lemma 3.1.5 (i), it is enough to prove $s^{(a-1)} \neq (p^y, 1^{x-1}, 0)$. Suppose that $s^{(a-1)} = (p^y, 1^{x-1}, 0)$. Since $s^{(a)} = (p^y, 1^x)$ and $a > 1$, by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a)}, s^{(a-1)}, s^{(a-2)}$, we have $s^{(a-2)} = s^{(a-1)}$, which contradicts the equality $|s^{(a-1)}| - |s^{(a-2)}| = 1$. The desired claim thus follows.

By the proven claim and the fact $s^{(a)} = (p^y, 1^x)$, we set

$$b = \max\{c \in \mathbb{N} : c \leq a, s^{(a-d)} = (p^{y-1}, p-d, 1^x) \text{ for every } 0 \leq d \leq c\}.$$

If $b = a$, then $J_W = \overline{s^{(0)}} = s^{(0)} = s^{(a-b)} = (p^{y-1}, p-a, 1^x)$, which implies that W has Jordan type $[1]^x [p-a]^1 [p]^{y-1}$ by Definition 3.1.2. We thus assume further that $b < a$. By the definition of b , note that $s^{(a-b+1)} = (p^{y-1}, p-b+1, 1^x)$ and $s^{(a-b)} = (p^{y-1}, p-b, 1^x)$. Moreover, we also have $s^{(a-b-1)} \neq (p^{y-1}, p-b-1, 1^x)$. As $|s^{(a-b)}| - |s^{(a-b-1)}| = 1$ and the parts of $s^{(a-b-1)}$ are non-increasing, we have $s^{(a-b-1)} = (p^{y-1}, p-b, 1^{x-1}, 0)$ by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-b+1)}, s^{(a-b)}, s^{(a-b-1)}$.

Observe that $b = a-1$. Otherwise, $b < a-1$ as $b < a$. By Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-b)}, s^{(a-b-1)}, s^{(a-b-2)}$, we have $s^{(a-b-2)} = s^{(a-b-1)} = (p^{y-1}, p-b, 1^{x-1}, 0)$, which contradicts the equality $|s^{(a-b-1)}| - |s^{(a-b-2)}| = 1$. So $b = a-1$, which implies that $s^{(0)} = s^{(a-b-1)} = (p^{y-1}, p-a+1, 1^{x-1}, 0)$. So $J_W = (p^{y-1}, p-a+1, 1^{x-1})$ as $J_W = \overline{s^{(0)}}$. So W has Jordan type $[1]^{x-1} [p-a+1]^1 [p]^{y-1}$ by Definition 3.1.2. The proof is now complete. $\square$

Example 3.2.3. Let us use some examples to illustrate Proposition 3.2.2. Let U, V, W be $\mathbb{F}G$ -modules and V have Jordan type $[1]^x [p]^y$, where $x, y > 0$. Assume that there is a short exact sequence of $\mathbb{F}G$ -modules $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$. If U has Jordan type $[p-1]^1$, then the Jordan type of W is either $[1]^{x+1} [p]^{y-1}$ or $[1]^{x-1} [2]^1 [p]^{y-1}$. If $p > 2$ and U has Jordan type $[p-2]^1$, then the Jordan type of W is either $[1]^x [2]^1 [p]^{y-1}$ or $[1]^{x-1} [3]^1 [p]^{y-1}$.

Proposition 3.2.4. *Let U, V, W be $\mathbb{F}G$ -modules. Assume that U has Jordan type $[a]^1$ and V has Jordan type $[1]^x[b]^y[p]^z$, where we have $1 < b < a < p$ and $x, y, z > 0$. If there exists a short exact sequence*

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0,$$

then there exist $c, d \in \mathbb{N}_0$ such that $c + d \leq a \leq c + d + 1 \leq b + c \leq p$ and the Jordan type of W is either $[1]^x[b - d]^1[b]^y[p - c]^1[p]^z$ or $[1]^{x-1}[b - d]^1[b]^y[p - c]^1[p]^z$.

Proof. By the assumptions and Theorem 3.1.3 (i), there is a L-R sequence of type $[J_W, (1^a); J_V]$. By the definition of the type of a L-R sequence and Lemma 3.1.5 (ii), there exists a L-R sequence $(s^{(0)}, \dots, s^{(a)})$ such that $\overline{s^{(0)}} = J_W$ and $s^{(a)} = J_V$. Moreover, we also have $|s^{(i)}| - |s^{(i-1)}| = 1$ for every $1 \leq i \leq a$. By Definition 3.1.2, note that $s^{(a)} = (p^z, b^y, 1^x)$.

We claim that $s^{(a-1)} = (p^{z-1}, p-1, b^y, 1^x)$. As $|s^{(a)}| - |s^{(a-1)}| = 1$ and the parts of $s^{(a-1)}$ are non-increasing, by Lemma 3.1.5 (i), it suffices to check that $s^{(a-1)}$ is neither $(p^z, b^{y-1}, b-1, 1^x)$ nor $(p^z, b^y, 1^{x-1}, 0)$. Suppose that $s^{(a-1)} = (p^z, b^{y-1}, b-1, 1^x)$. As $s^{(a)} = (p^z, b^y, 1^x)$, we set

$$f = \max\{g \in \mathbb{N} : g \leq a, s^{(a-h)} = (p^z, b^{y-1}, b-h, 1^x) \text{ for every } 0 \leq h \leq g\}.$$

We thus have $s^{(a-f+1)} = (p^z, b^{y-1}, b-f+1, 1^x)$ and $s^{(a-f)} = (p^z, b^{y-1}, b-f, 1^x)$. So we have $1 \leq b-f < a-f$ as $b < a$ and the parts of $s^{(a-f)}$ are non-increasing. In particular, $a-f-1 > 0$. We thus can deduce that $s^{(a-f-1)} \neq (p^z, b^{y-1}, b-f-1, 1^x)$. As $|s^{(a-f)}| - |s^{(a-f-1)}| = 1$ and the parts of $s^{(a-f-1)}$ are non-increasing, $s^{(a-f-1)} = (p^z, b^{y-1}, b-f, 1^{x-1}, 0)$ by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-f+1)}, s^{(a-f)}, s^{(a-f-1)}$. As $a-f-1 > 0$, by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-f)}, s^{(a-f-1)}, s^{(a-f-2)}$, we have $s^{(a-f-2)} = s^{(a-f-1)}$, which contradicts the equality $|s^{(a-f-1)}| - |s^{(a-f-2)}| = 1$. Suppose that $s^{(a-1)} = (p^z, b^y, 1^{x-1}, 0)$. Since $s^{(a)} = (p^z, b^y, 1^x)$ and $a > 1$, by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a)}, s^{(a-1)}, s^{(a-2)}$, notice that $s^{(a-1)} = s^{(a-2)}$, which is absurd as $|s^{(a-1)}| - |s^{(a-2)}| = 1$. The desired claim thus follows.

By the proven claim and the fact $s^{(a)} = (p^z, b^y, 1^x)$, we put

$$c = \max\{j \in \mathbb{N} : j \leq a, s^{(a-k)} = (p^{z-1}, p-k, b^y, 1^x) \text{ for every } 0 \leq k \leq j\}.$$

We thus have $c \leq a$, $s^{(a-c+1)} = (p^{z-1}, p-c+1, b^y, 1^x)$ and $s^{(a-c)} = (p^{z-1}, p-c, b^y, 1^x)$. So we have $b+c \leq p$ as the parts of $s^{(a-c)}$ are non-increasing. If $c < a$, by the definition of c , observe that $s^{(a-c-1)} \neq (p^{z-1}, p-c-1, b^y, 1^x)$. As $|s^{(a-c)}| - |s^{(a-c-1)}| = 1$ and the parts of $s^{(a-c-1)}$ are non-increasing, by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-c+1)}, s^{(a-c)}, s^{(a-c-1)}$, note that $s^{(a-c-1)}$ is $(p^{z-1}, p-c, b^y, 1^{x-1}, 0)$ or $(p^{z-1}, p-c, b^{y-1}, b-1, 1^x)$. Therefore, we can distinguish three cases to finish our proof.

Case 1: $c = a$.

Set $d = 0$. As $c = a > b > 1$ and $b+c \leq p$, observe that $c+d = a < c+d+1 < b+c \leq p$. Moreover, we have $J_W = s^{(0)} = s^{(a-c)} = (p^{z-1}, p-a, b^y, 1^x)$ as $J_W = \overline{s^{(0)}}$. So W has Jordan type $[1]^x[b]^y[p-a]^1[p]^z$ by Definition 3.1.2.

Case 2: $c < a$ and $s^{(a-c-1)} = (p^{z-1}, p-c, b^y, 1^{x-1}, 0)$.

Since $s^{(a-c)} = (p^{z-1}, p-c, b^y, 1^x)$ and $s^{(a-c-1)} = (p^{z-1}, p-c, b^y, 1^{x-1}, 0)$, if $a > c+1$, we can deduce that $s^{(a-c-1)} = s^{(a-c-2)}$ by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-c)}, s^{(a-c-1)}, s^{(a-c-2)}$. This is a contradiction as $|s^{(a-c-1)}| - |s^{(a-c-2)}| = 1$. We thus have $c = a-1$ and put $d = 0$. Since $b+c \leq p$ and $b > 1$, $c+d < a = c+d+1 < b+c \leq p$. Moreover, $s^{(0)} = s^{(a-c-1)} = (p^{z-1}, p-a+1, b^y, 1^{x-1}, 0)$, which implies that $J_W = (p^{z-1}, p-a+1, b^y, 1^{x-1})$ as $J_W = \overline{s^{(0)}}$. So W has Jordan type $[1]^{x-1}[b]^y[p-a+1]^1[p]^z$ by Definition 3.1.2.

Case 3: $c < a$ and $s^{(a-c-1)} = (p^{z-1}, p-c, b^{y-1}, b-1, 1^x)$.

As $s^{(a-c)} = (p^{z-1}, p-c, b^y, 1^x)$ and $s^{(a-c-1)} = (p^{z-1}, p-c, b^{y-1}, b-1, 1^x)$, we set

$$d = \max\{\ell \in \mathbb{N} : \ell \leq a-c, s^{(a-c-m)} = (p^{z-1}, p-c, b^{y-1}, b-m, 1^x) \text{ for every } 0 \leq m \leq \ell\}.$$

We thus have $c+d \leq a$ and $s^{(a-c-d)} = (p^{z-1}, p-c, b^{y-1}, b-d, 1^x)$. As the parts of $s^{(a-c-d)}$ are non-increasing, we also have $d+1 \leq b$.

Assume that $c + d = a$. As $c + d + 1 \leq b + c \leq p$, note that $c + d = a < c + d + 1 \leq \overline{b + c} \leq p$. Moreover, we have $J_W = s^{(0)} = s^{(a-c-d)} = (p^{z-1}, p-c, b^{y-1}, b-a+c, 1^x)$ as $J_W = \overline{s^{(0)}}$. So W has Jordan type $[1]^x [b-a+c]^1 [b]^{y-1} [p-c]^1 [p]^{z-1}$ by Definition 3.1.2.

Assume that $c + d < a$. We recall that $s^{(a-c-d)} = (p^{z-1}, p-c, b^{y-1}, b-d, 1^x)$. Observe that $s^{(a-c-d+1)} = (p^{z-1}, p-c, b^{y-1}, b-d+1, 1^x)$ and $s^{(a-c-d-1)} \neq (p^{z-1}, p-c, b^{y-1}, b-d-1, 1^x)$ by the definition of d . As the parts of $s^{(a-c-d-1)}$ are non-increasing and $|s^{(a-c-d)}| - |s^{(a-c-d-1)}| = 1$, notice that $s^{(a-c-d-1)} = (p^{z-1}, p-c, b^{y-1}, b-d, 1^{x-1}, 0)$ by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-c-d+1)}, s^{(a-c-d)}, s^{(a-c-d-1)}$. Notice that $a = c + d + 1$. Otherwise, by Definition 3.1.1 (a) and applying Lemma 3.1.4 to $s^{(a-c-d)}, s^{(a-c-d-1)}, s^{(a-c-d-2)}$, we deduce that $s^{(a-c-d-1)} = s^{(a-c-d-2)}$, which contradicts the equality $|s^{(a-c-d-1)}| - |s^{(a-c-d-2)}| = 1$. As $c + d + 1 \leq b + c \leq p$, we thus have $c + d < a = c + d + 1 \leq b + c \leq p$. Moreover, we also have $s^{(0)} = s^{(a-c-d-1)} = (p^{z-1}, p-c, b^{y-1}, b-a+c+1, 1^{x-1}, 0)$, which implies that $J_W = (p^{z-1}, p-c, b^{y-1}, b-a+c+1, 1^{x-1})$ as $J_W = \overline{s^{(0)}}$. Note that the Jordan type of W is $[1]^{x-1} [b-a+c+1]^1 [b]^{y-1} [p-c]^1 [p]^{z-1}$ by Definition 3.1.2.

Note that the desired proposition holds for each one of the three listed cases. The proof is now complete by combining the three listed cases. $\square$

Example 3.2.5. We use some examples to illustrate Proposition 3.2.4. Let $p > 3$ and U, V, W be $\mathbb{F}G$ -modules. Assume that there exists a short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$.

- (i) Assume that U has Jordan type $[p-1]^1$ and V has Jordan type $[1]^x [2]^1 [p]^z$, where $x, z > 0$. All pairs (c, d) that satisfy the inequality in Proposition 3.2.4 are exactly $(p-2, 1)$, $(p-2, 0)$, $(p-3, 1)$. By Proposition 3.2.4 and comparing the dimensions of modules, the Jordan type of W is contained in the set $\{[1]^{x+1} [2]^1 [p]^{z-1}, [1]^x [3]^1 [p]^{z-1}, [1]^{x-1} [2]^2 [p]^{z-1}\}$.
- (ii) Assume that U has Jordan type $[p-1]^1$ and V has Jordan type $[1]^x [3]^1 [p]^z$, where $x, z > 0$. All pairs (c, d) that satisfy the inequality in Proposition 3.2.4 are exactly $(p-3, 2)$, $(p-3, 1)$, $(p-4, 2)$. By Proposition 3.2.4 and comparing the dimensions of modules, the Jordan type of W is contained in the set $\{[1]^{x+1} [3]^1 [p]^{z-1}, [1]^x [4]^1 [p]^{z-1}, [1]^{x-1} [2]^1 [3]^1 [p]^{z-1}\}$.
- (iii) Assume that U has Jordan type $[p-1]^1$ and V has Jordan type $[1]^x [2]^2 [p]^z$, where $x, z > 0$. All pairs (c, d) that satisfy the inequality in Proposition 3.2.4 are exactly $(p-2, 1)$, $(p-2, 0)$, $(p-3, 1)$. By Proposition 3.2.4 and comparing the dimensions of modules, the Jordan type of W is contained in the set $\{[1]^{x+1} [2]^2 [p]^{z-1}, [1]^x [2]^1 [3]^1 [p]^{z-1}, [1]^{x-1} [2]^3 [p]^{z-1}\}$.
- (iv) Assume that U has Jordan type $[p-2]^1$ and V has Jordan type $[1]^x [2]^1 [p]^z$, where $x, z > 0$. All pairs (c, d) that satisfy the inequality in Proposition 3.2.4 are exactly $(p-2, 0)$, $(p-3, 1)$, $(p-3, 0)$, $(p-4, 1)$. By Proposition 3.2.4 and comparing the dimensions of modules, the Jordan type of W is contained in the set

$$\{[1]^{x+1} [3]^1 [p]^{z-1}, [1]^x [4]^1 [p]^{z-1}, [1]^x [2]^2 [p]^{z-1}, [1]^{x-1} [2]^1 [3]^1 [p]^{z-1}\}.$$

We end this chapter with two additional examples.

Example 3.2.6. Let $p > 3$ and U, V, W be $\mathbb{F}G$ -modules, where U has Jordan type $[p-1]^2$. Assume that there exists a short exact sequence

$$0 \rightarrow U \xrightarrow{\phi} V \rightarrow W \rightarrow 0.$$

So $\phi(U) = M_1 \oplus M_2$, where modules M_1 and M_2 have the same Jordan type $[p-1]^1$. Observe that $W \cong V/\phi(U) \cong (V/M_1)/(\phi(U)/M_1)$, where $\phi(U)/M_1$ has Jordan type $[p-1]^1$. So we have

$$0 \rightarrow M_1 \rightarrow V \rightarrow V/M_1 \rightarrow 0, \quad (3.2.1)$$

$$0 \rightarrow \phi(U)/M_1 \rightarrow V/M_1 \rightarrow W \rightarrow 0. \quad (3.2.2)$$

- (i) Assume that V has Jordan type $[1]^x [p]^z$, where $x, z > 1$. By (3.2.1) and Example 3.2.3, the Jordan types of V/M_1 is either $[1]^{x+1} [p]^{z-1}$ or $[1]^{x-1} [2]^1 [p]^{z-1}$. If V/M_1 has Jordan type $[1]^{x+1} [p]^{z-1}$, by (3.2.2) and Example 3.2.3, the Jordan type of W is either $[1]^{x+2} [p]^{z-2}$ or $[1]^x [2]^1 [p]^{z-2}$. If V/M_1 has Jordan type $[1]^{x-1} [2]^1 [p]^{z-1}$, by (3.2.2) and Example 3.2.5 (i), the Jordan type of W lies in the set $\{[1]^x [2]^1 [p]^{z-2}, [1]^{x-1} [3]^1 [p]^{z-2}, [1]^{x-2} [2]^2 [p]^{z-2}\}$.

- (ii) Assume that V has Jordan type $[1]^x[2]^1[p]^z$, where $x, z > 1$. By (3.2.1) and Example 3.2.5 (i), the Jordan type of V/M_1 lies in the set $\{[1]^{x+1}[2]^1[p]^{z-1}, [1]^x[3]^1[p]^{z-1}, [1]^{x-1}[2]^2[p]^{z-1}\}$. If V/M_1 has Jordan type $[1]^{x+1}[2]^1[p]^{z-1}$, according to (3.2.2) and Example 3.2.5 (i), the Jordan type of W lies in the set $\{[1]^{x+2}[2]^1[p]^{z-2}, [1]^{x+1}[3]^1[p]^{z-2}, [1]^x[2]^2[p]^{z-2}\}$. If V/M_1 has Jordan type $[1]^x[3]^1[p]^{z-1}$, by (3.2.2) and Example 3.2.5 (ii), the Jordan type of W lies in the set $\{[1]^{x+1}[3]^1[p]^{z-2}, [1]^x[4]^1[p]^{z-2}, [1]^{x-1}[2]^1[3]^1[p]^{z-2}\}$. If V/M_1 has Jordan type $[1]^{x-1}[2]^2[p]^{z-1}$, by (3.2.2) and Example 3.2.5 (iii), the Jordan type of W lies in the set $\{[1]^x[2]^2[p]^{z-2}, [1]^{x-1}[2]^1[3]^1[p]^{z-2}, [1]^{x-2}[2]^3[p]^{z-2}\}$.

Example 3.2.7. Let $p > 3$ and U, V, W be $\mathbb{F}G$ -modules, where U has Jordan type $[p-1]^2$. Assume that there is a short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$.

- (i) Assume that V has Jordan type $[1]^x[p]^z$, where $x, z > 1$. By Example 3.2.6 (i), the Jordan type of W lies in the set

$$\{[1]^{x+2}[p]^{z-2}, [1]^x[2]^1[p]^{z-2}, [1]^{x-1}[3]^1[p]^{z-2}, [1]^{x-2}[2]^2[p]^{z-2}\}.$$

- (ii) Assume that V has Jordan type $[1]^x[2]^1[p]^z$, where $x, z > 1$. By Example 3.2.6 (ii), the Jordan type of W lies in the set that contains precisely

$$\begin{aligned} &[1]^{x+2}[2]^1[p]^{z-2}, [1]^{x+1}[3]^1[p]^{z-2}, [1]^x[2]^2[p]^{z-2}, \\ &[1]^x[4]^1[p]^{z-2}, [1]^{x-1}[2]^1[3]^1[p]^{z-2}, [1]^{x-2}[2]^3[p]^{z-2}. \end{aligned}$$

Chapter 4

The Complexities of Some Simple Modules of Symmetric Groups

In this chapter, we explicitly compute the complexities of simple modules of symmetric groups labelled by some two-part partitions. Most of the simple modules considered in this chapter are contained in p -blocks with non-abelian defect groups.

4.1 Main Results and Some Known Results

The aim of this section is to state the motivation and main results of this chapter. Moreover, for convenience, we also include some known results for further reference.

Since the complexities of modules can relate the cohomological theory of finite groups to the representation theory of finite groups, this definition gains more and more attention from the researchers. However, given a finite group G and an $\mathbb{F}G$ -module M , it is usually very difficult to compute $c_G(M)$ explicitly.

Though the general case is quite complicated, many advancements have been obtained for the computation of complexities of $\mathbb{F}\mathfrak{S}_n$ -modules. In particular, recall that the complexities of Young permutation modules, Young modules, signed Young modules and Lie modules have already been explicitly computed in [28], [15] and [12]. Moreover, there are also closed formulas for some Specht modules and simple modules of $\mathfrak{S}_n$ (see [28], [32], [46], [47], [48] and [50]). However, according to our knowledge, little knowledge is known about the complexities of simple $\mathbb{F}\mathfrak{S}_n$ -modules that may not be contained in p -blocks with abelian defect groups. In particular, the complexities of simple $\mathbb{F}\mathfrak{S}_n$ -modules labelled by two-part partitions are not completely known. Therefore, we are interested in the following project.

Project A. Compute $c_{\mathfrak{S}_n}(D^\lambda)$ for $\lambda = (n - k, k) \in \mathfrak{Reg}(n)$.

The following known result motivates us to use the p -weights of partitions in Project A.

Lemma 4.1.1. [15, Corollary 6.5 and Remark 6.8] *Let $\lambda \in \mathfrak{Reg}(n)$. If λ has p -weight w and $D^\lambda \cong S^\mu$ for some partition μ of n , then $c_{\mathfrak{S}_n}(D^\lambda) = c_{\mathfrak{S}_n}(S^\mu) = w$.*

The main results of this chapter contribute to Project A. We summarize them as follows.

Theorem 4.1.2. *The following assertions hold.*

- (i) *Let $\lambda = (n - 1, 1)$ be a p -regular partition with p -weight w . Then*

$$c_{\mathfrak{S}_n}(D^\lambda) = \begin{cases} w - 1, & \text{if } p = 2 \text{ and } n \equiv 2 \pmod{4}, \\ w, & \text{otherwise.} \end{cases}$$

- (ii) *Let $\lambda = (n - 2, 2)$ be a p -regular partition with p -weight w . Then*

$$c_{\mathfrak{S}_n}(D^\lambda) = \begin{cases} w - 1, & \text{if } p = 2 \text{ and } n = 5 \text{ or } 6, \\ w, & \text{otherwise.} \end{cases}$$

Theorem 4.1.3. *Let $k \in \mathbb{N}_0$ and $\lambda = (n-k, k) \vdash n$, where $k \leq 2p$ if $p > 3$, and $k < 6$ if $p = 3$. If $p > 2$ and λ has p -weight w , then $c_{\mathfrak{S}_n}(D^\lambda) = w$.*

We introduce the required notation and list some known results. Let non-negative integers a and b have the p -adic sums $\sum_{i \geq 0} p^i a_i$ and $\sum_{i \geq 0} p^i b_i$ respectively. Write $a \subseteq b$ if $a_i \leq b_i$ for all $i \geq 0$ and write $a \sqsubseteq b$ if $a_i = b_i$ or $a_i = 0$ for all $i \geq 0$. For any $\lambda \vdash n$ and $\mu \in \mathfrak{Reg}(n)$, recall that $(S^\lambda : D^\mu)$ is the number of the composition factors of S^λ that are isomorphic to D^μ .

Theorem 4.1.4. [34, Theorem 3.3] *Let s and k be integers such that $0 \leq 2s \leq 2k \leq n$. Then $[M^{(n-k, k)} : Y^{(n-s, s)}] = \varphi(n, k, s)$, where*

$$\varphi(n, k, s) = \begin{cases} 1, & \text{if } k - s \subseteq n - 2s, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.1.5. [35, 24.15] *Let s and k be integers such that $0 \leq 2s \leq 2k \leq n$ and $(n-s, s) \in \mathfrak{Reg}(n)$. Then $(S^{(n-k, k)} : D^{(n-s, s)}) = \psi(n, k, s)$, where*

$$\psi(n, k, s) = \begin{cases} 1, & \text{if } k - s \sqsubseteq n - 2s + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.1.6. *Let $(n-k, k) \vdash n$. Then $M^{(n-k, k)}$ has a filtration of $\mathbb{F}\mathfrak{S}_n$ -modules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k+1} = M^{(n-k, k)}, \quad (4.1.1)$$

where $M_{i+1}/M_i \cong S^{(n-k+i, k-i)}$ for all $0 \leq i \leq k$. In particular, we have

$$\dim_{\mathbb{F}} S^{(n-k, k)} = \binom{n}{k} - \binom{n}{k-1}. \quad (4.1.2)$$

Proof. The first assertion is from [35, 17.13 and 17.17]. The second assertion is from the first one. $\square$

The following lemma will be used in the proof of Theorem 4.1.3. To show it, recall the definition of the block idempotent of a p -block of $\mathbb{F}\mathfrak{S}_n$ given in (2.3.2).

Lemma 4.1.7. *Let λ and μ be p -regular two-part partitions of n . Suppose they satisfy the following conditions:*

- (a) $\lambda \triangleleft \mu \triangleleft (n)$, $\kappa(\lambda) = \kappa(\mu) = \kappa$;
- (b) *there does not exist a partition ν such that $\lambda \triangleleft \nu \triangleleft \mu$ and $\kappa(\nu) = \kappa$;*
- (c) *there does not exist a partition η such that $\mu \triangleleft \eta \triangleleft (n)$ and $\kappa(\eta) = \kappa$.*

If $\kappa((n)) = \kappa$, $Y^{(n)} \mid M^\lambda$ and $Y^\mu \nmid M^\lambda$, then there exists a short exact sequence

$$0 \rightarrow S^\lambda \rightarrow Y^\lambda \rightarrow S^\mu \rightarrow 0.$$

Proof. Let e_κ be the block idempotent of the p -block of $\mathbb{F}\mathfrak{S}_n$ labelled by κ and $M = e_\kappa M^\lambda$. As $\kappa((n)) = \kappa$, note that $M \sim S^\lambda + S^\mu + S^{(n)}$ by (4.1.1) and the conditions. We also get $M \cong Y^\lambda \oplus Y^{(n)}$ from (2.11.1), Theorem 4.1.4, the Krull-Schmidt Theorem and the hypotheses. Now consider the filtration of Y^λ in (2.11.2) and suppose that S^μ is not a subquotient of this filtration. By the conditions again, note that all the subquotients of this filtration are the trivial modules except one S^λ . This contradicts the equation $(M : D^\mu) = (Y^\lambda : D^\mu)$ by the condition (a). So S^μ is a subquotient of the filtration. The lemma now follows by (2.11.2), the condition (a) and counting dimensions of M and Y^λ . $\square$

We end this section by discussing induction and restriction of the simple $\mathbb{F}\mathfrak{S}_n$ -modules labelled by two-part partitions. The restriction of these modules has been studied by Sheth in [57]. For general results of induction and restriction of simple $\mathbb{F}\mathfrak{S}_n$ -modules, which are referred as modular branching rules, one can refer to [44, Theorems 11.2.7 and 11.2.8].

The following theorems will be used in the proof of Theorem 4.1.3.

Theorem 4.1.8. Let $p > 2$ and a, b be integers such that $0 < b < a$. Let $c = b - 2$. We have

$$D^{(a,b)} \downarrow_{\mathfrak{S}_{a+b-1}} \cong \begin{cases} D^{(a-1,b)} \oplus D^{(a,b-1)}, & \text{if } c \not\equiv a \pmod{p} \text{ and } c \not\equiv a-1 \pmod{p}, \\ D^{(a-1,b)}, & \text{if } c \equiv a \pmod{p}. \end{cases}$$

Proof. The theorem is from [57, Theorem (ii) and (iii)]. $\square$

Theorem 4.1.9. Let $p > 2$ and $m \in \mathbb{N}_0$. Let $s \in \mathbb{N}$ and $s < p$. Let $\alpha = (n - mp - s, mp + s)$. Let $\beta = (n - mp - s, mp + s - 1)$. Let $\gamma = (n - mp - s - 1, mp + s - 1)$. Assume that α, β, γ are partitions.

- (i) If $n \not\equiv 2s - 1 \pmod{p}$ and $n \not\equiv 2s - 2 \pmod{p}$, we have $D^\beta \mid D^\alpha \downarrow_{\mathfrak{S}_{n-1}}$, $D^\alpha \mid D^\beta \uparrow^{\mathfrak{S}_n}$ and $c_{\mathfrak{S}_n}(D^\alpha) = c_{\mathfrak{S}_{n-1}}(D^\beta)$.
- (ii) If $n \equiv 2s - 1 \pmod{p}$, we have $D^\gamma \mid D^\alpha \downarrow_{\mathfrak{S}_{n-2}}$, $D^\alpha \mid D^\gamma \uparrow^{\mathfrak{S}_n}$ and $c_{\mathfrak{S}_n}(D^\alpha) = c_{\mathfrak{S}_{n-2}}(D^\gamma)$.
- (iii) If $n \equiv 2s - 2 \pmod{p}$ and $s > 1$, we also have $D^\gamma \mid D^\alpha \downarrow_{\mathfrak{S}_{n-2}}$, $D^\alpha \mid D^\gamma \uparrow^{\mathfrak{S}_n}$ and $c_{\mathfrak{S}_n}(D^\alpha) = c_{\mathfrak{S}_{n-2}}(D^\gamma)$.

Proof. The theorem follows from the proof of [14, Proposition 4.4] and Lemma 2.7.1 (ii). $\square$

4.2 Proof of Theorem 4.1.2

The aim of this section is to prove Theorem 4.1.2. Throughout the whole section, for the partitions $(n - 1, 1)$ and $(n - 2, 2)$, let $S_1 = S^{(n-1,1)}$ and $S_2 = S^{(n-2,2)}$. Moreover, we also let $D_1 = D^{(n-1,1)}$ and $D_2 = D^{(n-2,2)}$ if these partitions are p -regular. Following Proposition 2.8.1 (vii), our main strategy is to show that D_1 and D_2 , when restricted to some appropriate elementary abelian p -subgroups of $\mathfrak{S}_n$, are not generically free. Moreover, when $p = 2$, by the definition of the complexity of a module, we also need to compute some rank varieties of modules and their dimensions. For any elementary abelian p -group E , notice that the trivial $\mathbb{F}E$ -module always has stable generic Jordan type $[1]^1$.

Notation 4.2.1. We fix the required notation as follows.

- (i) Let P be a p -subgroup of $\mathfrak{S}_n$. When P acts on $[1, n]$ naturally, write $a_0(P)$ and $a_1(P)$ to denote the number of fixed points and the number of orbits of size p respectively.
- (ii) Let $n \geq 2$ and $\{t_i\}$ be the $(n - 1, 1)$ -tabloid with i lying in the second row. When $p = 2$, $2 \mid n$ and $n > 2$, D_1 has an $\mathbb{F}$ -basis $\{\{t_1\} + \{t_i\} + \text{Rad}(S_1) : 2 \leq i \leq n - 1\}$, where $\text{Rad}(S_1)$ is generated by $\sum_{i=1}^n \{t_i\}$. When $p = 2$, $2 \nmid n$ and $n > 2$, set

$$v_i = \{t_1\} + \{t_i\} + \text{Rad}(S_1)$$

for all $2 \leq i \leq n - 1$. The ordered $\mathbb{F}$ -basis $\{v_2, \dots, v_{n-1}\}$ of D_1 is denoted by $\mathcal{B}_2^n$.

- (iii) Let $n \geq 4$ and $\{t_{i,j}\}$ be the $(n - 2, 2)$ -tabloid with i, j lying in the second row. Denote by $M_0^2(n)$ the set

$$\left\{ \sum_{1 \leq i < j \leq n} k_{i,j} \{t_{i,j}\} \in M^{(n-2,2)} : \sum_{1 \leq i < j \leq n} k_{i,j} = 0 \right\}.$$

Note that $M_0^2(n)$ is an $\mathbb{F}\mathfrak{S}_n$ -submodule of $M^{(n-2,2)}$. Also note that $M_0^2(n)$ has an $\mathbb{F}$ -basis $\{\{t_{i,j}\} - \{t_{1,2}\} : 1 \leq i < j \leq n, \{i, j\} \neq \{1, 2\}\}$ which is denoted by $\mathcal{B}_{2,0}^n$. Moreover, there exists a short exact sequence

$$0 \rightarrow M_0^2(n) \rightarrow M^{(n-2,2)} \xrightarrow{\delta} \mathbb{F} \rightarrow 0, \quad (4.2.1)$$

where δ is the augmentation map from $M^{(n-2,2)}$ to $\mathbb{F}$.

- (iv) Let $m \in \mathbb{N}$ and R_m be an elementary abelian p -subgroup of $\mathfrak{S}_{p^m}$ acting regularly on $\{1, \dots, p^m\}$. Set $R_0 = 1$. For example, if $p = 2$, then

$$R_3 = \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 3)(2, 4)(5, 7)(6, 8), (1, 5)(2, 6)(3, 7)(4, 8) \rangle.$$

- (v) Let $m \in \mathbb{N}_0$ and $0 \leq pm \leq n$. If $m > 0$, let E_m be the elementary abelian p -subgroup $\langle \bigcup_{i=1}^m \{s_i\} \rangle$ of $\mathfrak{S}_n$, where

$$s_i = ((i-1)p + 1, \dots, ip).$$

Set $E_0 = 1$ and note that $\text{rank}(E_m) = m$. Let $s \in \mathbb{N}_0$ and $0 \leq 4s \leq n$. If $s > 0$, let K_s be the elementary abelian 2-subgroup $\langle \bigcup_{i=1}^s \{k_{i,1}, k_{i,2}\} \rangle$ of $\mathfrak{S}_n$, where

$$k_{i,1} = (4i-3, 4i-2)(4i-1, 4i) \text{ and } k_{i,2} = (4i-3, 4i-1)(4i-2, 4i).$$

Set $K_0 = 1$ and note that $\text{rank}(K_s) = 2s$.

- (vi) Let $2 \mid n$ and $x, y_x \in \mathbb{N}_0$, where $0 \leq 4x \leq n$ and $y_x = \frac{n-4x}{2}$. If $x < \frac{n}{4}$, let H_x be the elementary abelian 2-subgroup $\langle \bigcup_{i=1}^{y_x} \{s_{x,i}\} \rangle$ of $\mathfrak{S}_n$, where

$$s_{x,i} = (4x+2i-1, 4x+2i).$$

Set $H_{n/4} = 1$ if $4 \mid n$. Also note that $\text{rank}(H_x) = y_x$. If $p = 2$, observe that $H_0 = E_{n/2}$.

Recall the definition of the standard basis of a Specht module (see Page 21). We need the following lemmas as preparation.

Lemma 4.2.2. *Let $n \geq 2$ and E be an elementary abelian p -subgroup of $\mathfrak{S}_n$. Let $a_0 = a_0(E)$. Then $S_1 \downarrow_E$ has stable generic Jordan type*

$$\begin{cases} [p-1]^1, & \text{if } a_0 = 0, \\ [1]^{a_0-1}, & \text{if } a_0 > 0. \end{cases}$$

Proof. When $a_0 = 0$, observe that $p \mid n$. In this case, we have $M^{(n-1,1)} \cong Y^{(n-1,1)}$ by Theorem 4.1.4 and $Y^{(n-1,1)} \downarrow_E$ is generically free by Lemmas 2.8.2 and 2.11.1. We also have a short exact sequence

$$0 \rightarrow S_1 \downarrow_E \rightarrow Y^{(n-1,1)} \downarrow_E \rightarrow S^{(n)} \downarrow_E \rightarrow 0.$$

The stable generic Jordan type of $S_1 \downarrow_E$ is $[p-1]^1$ by applying Proposition 2.8.1 (iv) (b) to the above short exact sequence.

For the remaining case $a_0 \neq 0$, we may assume that 1 is fixed by E . Then we note that $S_1 \downarrow_E$ is itself a p -permutation $\mathbb{F}E$ -module since the standard basis of S_1 can be permuted by E and its subgroups. Note that a standard basis vector of S_1 is fixed by E if and only if it is labelled by a standard $(n-1, 1)$ -tableau whose second row contains an E -fixed point of $[1, n]$. The result in this case now follows from Lemma 2.8.2. $\square$

Lemma 4.2.3. *Let $n \geq 4$ and E be an elementary abelian p -subgroup of $\mathfrak{S}_n$. Let $a_0 = a_0(E)$ and $a_1 = a_1(E)$. Then $M_0^2(n) \downarrow_E$ has stable generic Jordan type*

$$\begin{cases} [p-1]^1, & \text{if } p > 2, a_0 < 2 \text{ or } p = 2, a_0 < 2, a_1 = 0, \\ [1]^{\binom{a_0}{2}-1}, & \text{if } p > 2 \text{ and } a_0 \geq 2, \\ [1]^{\binom{a_0}{2}+a_1-1}, & \text{otherwise.} \end{cases}$$

Proof. When $p > 2$, $a_0 < 2$ or $p = 2$, $a_0 < 2$, $a_1 = 0$, observe that $M^{(n-2,2)} \downarrow_E$ is generically free by Lemmas 2.8.2 and 2.11.1, which implies that $M_0^2(n) \downarrow_E$ has stable generic Jordan type $[p-1]^1$ by applying Proposition 2.8.1 (iv) (b) to the short exact sequence in (4.2.1).

When $p > 2$ and $a_0 \geq 2$, we may assume that $\{1, 2\}$ is fixed by E pointwise. Note that $M_0^2(n) \downarrow_E$ is a p -permutation $\mathbb{F}E$ -module since $\mathcal{B}_{2,0}^n$ can be permuted by E and its subgroups. We get the desired stable generic Jordan type of $M_0^2(n) \downarrow_E$ by Lemma 2.8.2.

In all other cases, namely $p = 2$, $a_0 \geq 2$ or $p = 2$, $a_0 < 2$, $a_1 > 0$, we may also assume that $\{t_{1,2}\}$ is fixed by E . So $M_0^2(n) \downarrow_E$ is still a p -permutation $\mathbb{F}E$ -module as $\mathcal{B}_{2,0}^n$ is also permuted by E and its subgroups. We thus get the desired stable generic Jordan type of $M_0^2(n) \downarrow_E$ by using Lemma 2.8.2 again. $\square$

Lemma 4.2.4. [46, Lemma 5.9] *Let $p = 2$, $2 \mid n$ and $n \geq 4$. Then $S_2 \downarrow_{E_{n/2}}$ has stable generic Jordan type $[1]^{\frac{n}{2}-2}$.*

Lemma 4.2.5. *Let $p \nmid n$ and $n \geq 4$. Let E be an elementary abelian p -subgroup of $\mathfrak{S}_n$. Let $a_0 = a_0(E)$ and $a_1 = a_1(E)$. Then $S_2 \downarrow_E$ has stable generic Jordan type*

$$\begin{cases} [p-1]^1, & \text{if } p > 2, a_0 < 3 \text{ or } p = 2, a_0 < 3, a_1 = 0, \\ [1]^{(a_0-1)-1}, & \text{if } p > 2 \text{ and } a_0 \geq 3, \\ [1]^{(a_0-1)+a_1-1}, & \text{otherwise.} \end{cases}$$

Proof. When $p \nmid n$, note that E may be viewed as an elementary abelian p -subgroup of $\mathfrak{S}_{n-1}$. In the case $n > 4$, we have $S_2 \downarrow_E = (S_2 \downarrow_{\mathfrak{S}_{n-1}}) \downarrow_E \cong M_0^2(n-1) \downarrow_E$ according to [56, Lemma 3]. When $n = 4$, we have $S^{(2,2)} \downarrow_E \cong S^{(2,1)} \downarrow_E$ by the Branching Rule. The proof is now complete by Lemmas 4.2.3 and 4.2.2. $\square$

We state a result to figure out all maximal elementary abelian p -subgroups of $\mathfrak{S}_n$. For our purpose, we need another notation. Let $b, c \in \mathbb{N}$ and assume that $B \leq \mathfrak{S}_b$ and $C \leq \mathfrak{S}_c$. If $b + c \leq n$, we write $B \times C \hookrightarrow \mathfrak{S}_b \times \mathfrak{S}_c \hookrightarrow \mathfrak{S}_n$ for the obvious inclusions of groups.

Theorem 4.2.6. [1, § VI Theorem 1.3] *Representatives of the conjugacy classes of all maximal elementary abelian p -subgroups of $\mathfrak{S}_n$ are*

$$\underbrace{R_1 \times \cdots \times R_1}_{t_1 \text{ times}} \times \cdots \times \underbrace{R_\ell \times \cdots \times R_\ell}_{t_\ell \text{ times}} \hookrightarrow \underbrace{\mathfrak{S}_p \times \cdots \times \mathfrak{S}_p}_{t_1 \text{ times}} \times \cdots \times \underbrace{\mathfrak{S}_{p^\ell} \times \cdots \times \mathfrak{S}_{p^\ell}}_{t_\ell \text{ times}} \hookrightarrow \mathfrak{S}_n,$$

where we have $t_1, \dots, t_\ell \in \mathbb{N}_0$ and $0 \leq n - \sum_{i=1}^\ell p^i t_i < p$.

Notice that the maximal elementary abelian p -subgroup displayed in Theorem 4.2.6 has p -rank $\sum_{i=1}^\ell i t_i$. In particular, any elementary abelian p -subgroup of $\mathfrak{S}_n$ has p -rank no more than $\lfloor \frac{n}{p} \rfloor$. We thus deduce that $\text{rank}(\mathfrak{S}_n) = \lfloor \frac{n}{p} \rfloor$. We can also use Theorem 4.2.6 to deduce the following result.

Lemma 4.2.7. *Let $p \mid n$ and E be an elementary abelian p -subgroup of $\mathfrak{S}_n$ with p -rank $\frac{n}{p}$.*

- (i) *If $p > 2$, then E is $\mathfrak{S}_n$ -conjugate to $E_{n/p}$.*
- (ii) *If $p = 2$, then E is $\mathfrak{S}_n$ -conjugate to some $K_\ell \times H_\ell$, where $0 \leq \ell \leq \lfloor \frac{n}{4} \rfloor$.*

Proof. By Theorem 4.2.6, E is $\mathfrak{S}_n$ -conjugate to an elementary abelian p -subgroup

$$(R_1)^{t_1} \times \cdots \times (R_m)^{t_m} \hookrightarrow (\mathfrak{S}_p)^{t_1} \times \cdots \times (\mathfrak{S}_{p^m})^{t_m} \hookrightarrow \mathfrak{S}_n$$

for some $m \in \mathbb{N}$ and $t_1, \dots, t_m \in \mathbb{N}_0$. We thus have $\sum_{i=1}^m p^i t_i = n$ and $\sum_{i=1}^m i t_i = \frac{n}{p}$ by the hypotheses. These equations imply that $\sum_{i=1}^m p^{i-1} t_i = \sum_{i=1}^m i t_i$.

For (i), as $p > 2$, for any $a \in \mathbb{N}$, note that $a < p^{a-1}$ if $a > 1$. Therefore, t_1 is the unique non-zero number among $t_1, \dots, t_m$. (i) is shown.

For (ii), as $p = 2$, for any $a \in \mathbb{N}$, note that $a < 2^{a-1}$ if $a > 2$. We thus deduce that $t_i = 0$ for all $2 < i \leq m$. The lemma thus follows. $\square$

When $p = 2$, $2 \mid n$ and $n > 2$, recall that $v_i = \{t_1\} + \{t_i\} + \text{Rad}(S_1) \in D_1$ for all $2 \leq i \leq n-1$. The following two lemmas are from direct computation.

Lemma 4.2.8. *Let $p = 2$, $2 \mid n$ and $n > 2$. Let $i, j \in \mathbb{N}$ such that $1 \leq i \leq \frac{n}{2}$ and $2 \leq j \leq n-1$. Then*

$$s_i v_j = \begin{cases} v_j, & \text{if } i \neq 1 \text{ and } j \notin \{2i-1, 2i\}, \\ v_{2i}, & \text{if } i \neq 1, i \neq \frac{n}{2} \text{ and } j = 2i-1, \\ v_{2i-1}, & \text{if } i \neq 1, i \neq \frac{n}{2} \text{ and } j = 2i, \\ v_2, & \text{if } i = 1 \text{ and } j = 2, \\ v_2 + v_j, & \text{if } i = 1 \text{ and } 2 < j \leq n-1, \\ \sum_{i=2}^{n-1} v_i, & \text{if } i = \frac{n}{2} \text{ and } j = n-1. \end{cases}$$

Proof. We only present the computation of the case $i = \frac{n}{2}$ and $j = n - 1$. We have

$$\begin{aligned} s_i v_j &= (n - 1, n)(\{t_1\} + \{t_{n-1}\} + \text{Rad}(S_1)) = \{t_1\} + \{t_n\} + \text{Rad}(S_1) \\ &= \sum_{i=2}^{n-1} \{t_i\} + \text{Rad}(S_1) = \sum_{i=2}^{n-1} v_i. \end{aligned}$$

Similarly, one can finish the computation of all the other cases. $\square$

Lemma 4.2.9. *Let $p = 2$, $2 \mid n$ and $n \geq 4$. Let $i, j \in \mathbb{N}$ such that $1 \leq i \leq \lfloor \frac{n}{4} \rfloor$ and $2 \leq j \leq n - 1$. If $n = 4$, then $k_{1,1}v_j = k_{1,2}v_j = v_j$. Moreover, if $n > 4$, then*

$$k_{i,1}v_j = \begin{cases} v_j, & \text{if } i \neq 1 \text{ and } j \notin \{4i - 3, 4i - 2, 4i - 1, 4i\}, \\ v_{4i-2}, & \text{if } i \neq 1 \text{ and } j = 4i - 3, \\ v_{4i-3}, & \text{if } i \neq 1 \text{ and } j = 4i - 2, \\ v_{4i}, & \text{if } i \neq 1, i \neq \frac{n}{4} \text{ and } j = 4i - 1, \\ v_{4i-1}, & \text{if } i \neq 1, i \neq \frac{n}{4} \text{ and } j = 4i, \\ v_2, & \text{if } i = 1 \text{ and } j = 2, \\ v_2 + v_4, & \text{if } i = 1 \text{ and } j = 3, \\ v_2 + v_3, & \text{if } i = 1 \text{ and } j = 4, \\ v_2 + v_j, & \text{if } i = 1 \text{ and } 4 < j \leq n - 1, \\ \sum_{i=2}^{n-1} v_i, & \text{if } 4 \mid n, i = \frac{n}{4} \text{ and } j = n - 1, \end{cases}$$

and

$$k_{i,2}v_j = \begin{cases} v_j, & \text{if } i \neq 1 \text{ and } j \notin \{4i - 3, 4i - 2, 4i - 1, 4i\}, \\ v_{4i-1}, & \text{if } i \neq 1 \text{ and } j = 4i - 3, \\ v_{4i-3}, & \text{if } i \neq 1 \text{ and } j = 4i - 1, \\ v_{4i}, & \text{if } i \neq 1, i \neq \frac{n}{4} \text{ and } j = 4i - 2, \\ v_{4i-2}, & \text{if } i \neq 1, i \neq \frac{n}{4} \text{ and } j = 4i, \\ v_3 + v_4, & \text{if } i = 1 \text{ and } j = 2, \\ v_3, & \text{if } i = 1 \text{ and } j = 3, \\ v_3 + v_2, & \text{if } i = 1 \text{ and } j = 4, \\ v_3 + v_j, & \text{if } i = 1 \text{ and } 4 < j \leq n - 1, \\ \sum_{i=2}^{n-1} v_i. & \text{if } 4 \mid n, i = \frac{n}{4} \text{ and } j = n - 2. \end{cases}$$

Proof. The case $n = 4$ is clear. We thus consider the case $n > 4$. For the first collection, we only present the computation of the case $4 \mid n$, $i = \frac{n}{4}$ and $j = n - 1$. We have

$$\begin{aligned} k_{i,1}v_j &= (n - 3, n - 2)(n - 1, n)(\{t_1\} + \{t_{n-1}\} + \text{Rad}(S_1)) = \{t_1\} + \{t_n\} + \text{Rad}(S_1) \\ &= \sum_{i=2}^{n-1} \{t_i\} + \text{Rad}(S_1) = \sum_{i=2}^{n-1} v_i. \end{aligned}$$

Similarly, one can finish the computation of all the other cases of the first collection.

For the second collection, we only present the computation of the case $4 \mid n$, $i = \frac{n}{4}$ and $j = n - 2$. We have

$$\begin{aligned} k_{i,2}v_j &= (n - 3, n - 1)(n - 2, n)(\{t_1\} + \{t_{n-2}\} + \text{Rad}(S_1)) = \{t_1\} + \{t_n\} + \text{Rad}(S_1) \\ &= \sum_{i=2}^{n-1} \{t_i\} + \text{Rad}(S_1) = \sum_{i=2}^{n-1} v_i. \end{aligned}$$

Similarly, one can finish the computation of all the other cases of the second collection. The desired lemma thus follows. $\square$

The following lemma is important in our discussion.

Lemma 4.2.10. *Let $p = 2$, $2 \mid n$ and $n > 2$.*

- (i) *If $n \equiv 0 \pmod{4}$, then $D_1 \downarrow_{K_{n/4}}$ has stable generic Jordan type $[1]^2$.*
- (ii) *The module $D_1 \downarrow_{E_{n/2}}$ is generically free.*

Proof. Let $K = K_{n/4}$ if $n \equiv 0 \pmod{4}$. Set $E = E_{n/2}$. For (i), let $m = \frac{n}{4}$ and $\alpha = \bigcup_{i=1}^{2m} \{\alpha_i\}$ be a set of indeterminates. Let $\mathbb{K} = \mathbb{F}(\alpha_1, \dots, \alpha_{n/2})$ and view $D_1 \downarrow_K$ as a $\mathbb{K}K$ -module by scalar extension. Set

$$u_\alpha = 1 + \sum_{i=1}^m (\alpha_{2i-1}(k_{i,1} + 1) + \alpha_{2i}(k_{i,2} + 1)).$$

Recall that $\mathcal{B}_2^n = \{v_2, \dots, v_{n-1}\}$. Following the order of the basis vectors in $\mathcal{B}_2^n$, let N_n be the matrix representing $u_\alpha + 1$ on $D_1 \downarrow_{\langle u_\alpha \rangle}$ with respect to the ordered basis $\mathcal{B}_2^n$. Note that N_n can be obtained from the definition of u_α and Lemma 4.2.9. When $n \geq 12$, N_n can be viewed as the column block matrix $(B_1, B(4, \alpha_3, \alpha_4), B(8, \alpha_5, \alpha_6), \dots, B(n-8, \alpha_{(n/2)-3}, \alpha_{(n/2)-2}), B_2)$. Here $B(a, \beta, \gamma)$ denotes the $((n-2) \times 4)$ -matrix with entries in $\mathbb{K}$

$$\begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \beta + \gamma & \beta & \gamma & 0 \\ \beta & \beta + \gamma & 0 & \gamma \\ \gamma & 0 & \beta + \gamma & \beta \\ 0 & \gamma & \beta & \beta + \gamma \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the integer a satisfies $4 \mid a$, $4 \leq a \leq n-8$ and the symbols β, γ occur from row a to row $a+3$. The remaining $((n-2) \times 3)$ -matrices with entries in $\mathbb{K}$ have definitions

$$B_1 = \begin{pmatrix} \alpha_2 & \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_1 + \alpha_2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \alpha_1 & \alpha_1 + \alpha_{n/2} & \alpha_1 + \alpha_{(n/2)-1} \\ \alpha_2 & \alpha_2 + \alpha_{n/2} & \alpha_2 + \alpha_{(n/2)-1} \\ 0 & \alpha_{n/2} & \alpha_{(n/2)-1} \\ \vdots & \vdots & \vdots \\ 0 & \alpha_{n/2} & \alpha_{(n/2)-1} \\ \alpha_{(n/2)-1} + \alpha_{n/2} & \alpha_{(n/2)-1} + \alpha_{n/2} & \alpha_{(n/2)-1} + \alpha_{n/2} \\ \alpha_{(n/2)-1} & \alpha_{(n/2)-1} & \alpha_{(n/2)-1} \\ \alpha_{n/2} & \alpha_{n/2} & \alpha_{n/2} \end{pmatrix}.$$

The column block matrix $N_8 = (B_1, B_2)$ and N_4 is the zero matrix as $D^{(3,1)} \downarrow_{\langle u_\alpha \rangle} \cong \mathbb{F} \oplus \mathbb{F}$. As $p = 2$, if $n > 4$, the two rightmost column vectors of B_2 can be written as $\mathbb{K}$ -linear combinations of the remaining column vectors of N_n . Moreover, the largest $\mathbb{K}$ -linearly independent set of the remaining column vectors of N_n has size $\frac{n}{2} - 2$. So $\text{rank}(N_n) = \frac{n}{2} - 2$ for any $n \geq 4$. (i) follows as the stable generic Jordan type of $D_1 \downarrow_K$ is independent of the choice of generators of K .

For (ii), suppose that $D_1 \downarrow_E$ is not generically free. We thus have $V_E^\#(D_1 \downarrow_E) = V_E^\#(\mathbb{F})$ by Proposition 2.8.1 (vi). Also note that $D_1 \downarrow_{\mathfrak{S}_{n-1}} \cong D^{(n-2,1)}$ by [57, Theorem 3.1]. Take $\tilde{E}$ to be

$E_{(n/2)-1}$ and notice that $\tilde{E} = \mathfrak{S}_{n-1} \cap E$. We have $V_E^\#(D^{(n-2,1)} \downarrow_{\tilde{E}}) = V_E^\#(D_1 \downarrow_{\tilde{E}}) = V_E^\#(\mathbb{F})$ by the definition of the rank variety of a module. This implies the following calculation:

$$\frac{n}{2} - 1 = \dim V_E^\#(\mathbb{F}) = \dim V_E^\#(D^{(n-2,1)} \downarrow_{\tilde{E}}) \leq c_{\mathfrak{S}_{n-1}}(D^{(n-2,1)}) \leq \frac{n}{2} - 2.$$

The first inequality of the calculation comes from the definition of the complexity of a module. The second one is clear since the partition $(n-2, 1)$ has 2-weight $\frac{n}{2} - 2$. This calculation shows an obvious contradiction. So $D_1 \downarrow_E$ is generically free. $\square$

Lemma 4.2.11. *Let $p = 2$, $n \equiv 2 \pmod{4}$ and $n > 2$. Then $c_{\mathfrak{S}_n}(D_1) = \frac{n}{2} - 1$.*

Proof. Set $F_\ell = K_\ell \times H_\ell$ for any $\ell \in \mathbb{N}_0$ and $\ell \leq \frac{n-2}{4}$. By Lemma 4.2.7 (ii) and the definition of the complexity of a module, it suffices to show that $\max_{0 \leq \ell \leq \frac{n-2}{4}} \dim V_{F_\ell}^\#(D_1 \downarrow_{F_\ell}) = \frac{n}{2} - 1$. Let $\ell \in \mathbb{N}_0$ and $\ell \leq \frac{n-2}{4}$. For any non-zero point $\alpha = (\alpha_1, \dots, \alpha_{n/2}) \in \mathbb{F}^{\frac{n}{2}}$, put

$$u_{\alpha, \ell} = 1 + \sum_{i=1}^{\ell} (\alpha_{2i-1}(k_{i,1} + 1) + \alpha_{2i}(k_{i,2} + 1)) + \sum_{i=1}^{m_\ell} \alpha_{2\ell+i}(s_{\ell,i} + 1),$$

where $m_\ell = \frac{n-4\ell}{2}$. Recall that $\mathcal{B}_2^n = \{v_2, \dots, v_{n-1}\}$. Following the order of the members in $\mathcal{B}_2^n$, let $N(\alpha, \ell)$ be the matrix representing $u_{\alpha, \ell} + 1$ on $D_1 \downarrow_{\langle u_{\alpha, \ell} \rangle}$ with respect to the ordered $\mathbb{F}$ -basis $\mathcal{B}_2^n$. Note that $N(\alpha, \ell)$ can be obtained from the definition of $u_{\alpha, \ell}$ and Lemmas 4.2.8, 4.2.9. Since $p = 2$ and $2 \mid n$, also notice that $D_1 \downarrow_{\langle u_{\alpha, \ell} \rangle}$ is not free if and only if $\text{rank}(N(\alpha, \ell)) < \frac{n}{2} - 1$. In the following cases, we determine $V_{F_\ell}^\#(D_1 \downarrow_{F_\ell})$ by computing $\text{rank}(N(\alpha, \ell))$. In particular, we get $\dim V_{F_\ell}^\#(D_1 \downarrow_{F_\ell})$.

Case 1: $\ell = 0$.

By Notation 4.2.1 (v) and (vi), note that $F_0 = H_0 = E_{n/2}$. When $n \geq 10$, view $N(\alpha, 0)$ as the column block matrix $(C_1, C(4, \alpha_3, \alpha_4), C(8, \alpha_5, \alpha_6), \dots, C(n-6, \alpha_{(n/2)-2}, \alpha_{(n/2)-1}), C_2)$. Here $C(a, \beta, \gamma)$ denotes the $((n-2) \times 4)$ -matrix with entries in $\mathbb{F}$

$$\begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 \\ \beta & \beta & 0 & 0 \\ 0 & 0 & \gamma & \gamma \\ 0 & 0 & \gamma & \gamma \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the integer a satisfies $4 \mid a$, $4 \leq a \leq n-6$ and the symbols β, γ occur from row a to row $a+3$. The $\mathbb{F}$ -matrices C_1 and C_2 are the $((n-2) \times 3)$ -matrix and $((n-2) \times 1)$ -matrix respectively, where

$$C_1 = \begin{pmatrix} 0 & \alpha_1 & \alpha_1 \\ 0 & \alpha_2 & \alpha_2 \\ 0 & \alpha_2 & \alpha_2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} \alpha_1 + \alpha_{n/2} \\ \alpha_{n/2} \\ \vdots \\ \alpha_{n/2} \\ 0 \end{pmatrix}.$$

When $n = 6$, the column block matrix $N(\alpha, 0) = (C_1, C_2)$. In general case, if α has only one zero entry, then $\text{rank}(N(\alpha, 0)) = \frac{n}{2} - 1$. If α has no zero entry, then $\text{rank}(N(\alpha, 0)) < \frac{n}{2} - 1$ if and only if C_2 is an $\mathbb{F}$ -linear combination of the remaining column vectors of $N(\alpha, 0)$. Note

that the latter condition holds if and only if $\sum_{i=1}^{\frac{n}{2}} \bar{\alpha}_i = 0$, where $\bar{\alpha}_i = \prod_{j=1, j \neq i}^{\frac{n}{2}} \alpha_j$. If α has at least two zero entries, note that $\text{rank}(N(\alpha, 0)) < \frac{n}{2} - 1$. We thus have

$$V_{F_0}^\#(D_1 \downarrow_{F_0}) = \{(\gamma_1, \dots, \gamma_{n/2}) \in \mathbb{F}^{\frac{n}{2}} : \sum_{i=1}^{\frac{n}{2}} \bar{\gamma}_i = 0, \bar{\gamma}_i = \prod_{\substack{j=1 \\ j \neq i}}^{\frac{n}{2}} \gamma_j\}$$

and $\dim V_{F_0}^\#(D_1 \downarrow_{F_0}) = \frac{n}{2} - 1$.

Case 2: $\ell = \frac{n-2}{4}$.

When $n \geq 10$, view $N(\alpha, \frac{n-2}{4})$ as the column block matrix

$$(B_1, B(4, \alpha_3, \alpha_4), B(8, \alpha_5, \alpha_6), \dots, B(n-6, \alpha_{(n/2)-2}, \alpha_{(n/2)-1}), \bar{B}_2).$$

The $B_1, B(a, \beta, \gamma)$, defined in the proof of Lemma 4.2.10 (i), are viewed as $\mathbb{F}$ -matrices by regarding their entries α_1, α_2 as corresponding components of α . Moreover, the integer a satisfies $4 \mid a$ and $4 \leq a \leq n-6$. The matrix $\bar{B}_2$ is the $((n-2) \times 1)$ -matrix

$$\begin{pmatrix} \alpha_1 + \alpha_{n/2} \\ \alpha_2 + \alpha_{n/2} \\ \alpha_{n/2} \\ \vdots \\ \alpha_{n/2} \\ 0 \end{pmatrix}.$$

When $n = 6$, the column block matrix $N(\alpha, 1) = (B_1, \bar{B}_2)$. As $p = 2$, if α has an entry α_s such that $1 \leq s \leq \frac{n}{2} - 2$, $2 \nmid s$ and $\alpha_s = \alpha_{s+1} = 0$, note that $\text{rank}(N(\alpha, \frac{n-2}{4})) < \frac{n}{2} - 1$. Otherwise, note that $\text{rank}(B_1) = 1$ and $\text{rank}(B(4, \alpha_3, \alpha_4)) = \dots = \text{rank}(B(n-6, \alpha_{(n/2)-2}, \alpha_{(n/2)-1})) = 2$ if $n > 6$. Moreover, also note that $\bar{B}_2$ is not an $\mathbb{F}$ -linear combination of the remaining column vectors of $N(\alpha, \frac{n-2}{4})$. So $\text{rank}(N(\alpha, \frac{n-2}{4})) = \frac{n}{2} - 1$. Therefore,

$$V_{F_\ell}^\#(D_1 \downarrow_{F_\ell}) = \bigcup_{\substack{s=1 \\ 2 \nmid s}}^x \{(\gamma_1, \dots, \gamma_{n/2}) \in \mathbb{F}^{\frac{n}{2}} : \gamma_s = \gamma_{s+1} = 0\}$$

and $\dim V_{F_\ell}^\#(D_1 \downarrow_{F_\ell}) = \frac{n}{2} - 2$, where $\ell = \frac{n-2}{4}$ and $x = \frac{n}{2} - 2$.

Case 3: $0 < \ell < \frac{n-2}{4}$.

When $\ell > 1$, $N(\alpha, \ell)$ is the column block matrix $(B_1, B, D, \bar{B}_2)$. As column block matrices,

$$\begin{aligned} B &= (B(4, \alpha_3, \alpha_4), B(8, \alpha_5, \alpha_6), \dots, B(4\ell-4, \alpha_{2\ell-1}, \alpha_{2\ell})) \text{ and} \\ D &= (D(4\ell, \alpha_{2\ell+1}, \alpha_{2\ell+2}), \dots, D(n-6, \alpha_{(n/2)-2}, \alpha_{(n/2)-1})). \end{aligned}$$

$((n-2) \times 4)$ - $\mathbb{F}$ -matrix

$$D(a, \beta, \gamma) = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 \\ \beta & \beta & 0 & 0 \\ 0 & 0 & \gamma & \gamma \\ 0 & 0 & \gamma & \gamma \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the integer a satisfies $4 \mid a$, $4\ell \leq a \leq n-6$ and the symbols β, γ occur from row a to row $a+3$. The definitions of other components of $N(\alpha, \ell)$ are from Case 2. Given a component $B(a', \beta, \gamma)$ of B , the integer a' satisfies $4 \mid a'$ and $4 \leq a' \leq 4\ell-4$. When $\ell = 1$, the column block matrix $N(\alpha, 1) = (B_1, D, \bar{B}_2)$. We now study $\text{rank}(N(\alpha, \ell))$ for the general case. Define two sets

$$V_1 = \bigcup_{\substack{s=1 \\ 2 \nmid s}}^{\frac{n}{2}-2} \{(\gamma_1, \dots, \gamma_{n/2}) \in \mathbb{F}^{\frac{n}{2}} : \gamma_s = \gamma_{s+1} = 0\},$$

$$V_2(\ell) = \{(\gamma_1, \dots, \gamma_{n/2}) \in \mathbb{F}^{\frac{n}{2}} : \sum_{i=2\ell+1}^{\frac{n}{2}} \bar{\gamma}_i = 0, \bar{\gamma}_i = \prod_{\substack{j=2\ell+1 \\ j \neq i}}^{\frac{n}{2}} \gamma_j\}.$$

Since $p = 2$, if α has an entry α_s such that $1 \leq s \leq \frac{n}{2}-2$, $2 \nmid s$ and $\alpha_s = \alpha_{s+1} = 0$, note that $\text{rank}(N(\alpha, \ell)) < \frac{n}{2}-1$. Otherwise, note that $\text{rank}(\bar{B}_1) = 1$ and all remaining column block components of $N(\alpha, \ell)$ except $\bar{B}_2$ have rank two. In this case, note that $\text{rank}(N(\alpha, \ell)) < \frac{n}{2}-1$ if and only if $\alpha \in V_2(\ell)$. Therefore, $V_{F_\ell}^\#(D_1 \downarrow_{F_\ell}) = V_1 \cup V_2(\ell)$ and $\dim V_{F_\ell}^\#(D_1 \downarrow_{F_\ell}) = \frac{n}{2}-1$. The lemma now follows by combining the three cases. $\square$

The proof of Theorem 4.1.2 relies on the following propositions.

Proposition 4.2.12. *Let $(n-1, 1) \in \mathfrak{Rcg}(n)$ having p -weight w . We have*

$$c_{\mathfrak{S}_n}(D_1) = \begin{cases} w-1, & \text{if } p=2 \text{ and } n \equiv 2 \pmod{4}, \\ w, & \text{otherwise.} \end{cases}$$

Proof. When $p \nmid n$, we have $S_1 \cong D_1$ by Theorem 4.1.5. The result follows from Lemma 4.1.1. When $p > 2$ and $p \mid n$, we have $\dim_{\mathbb{F}} D_1 \equiv p-2 \pmod{p}$, which implies that $c_{\mathfrak{S}_n}(D_1) = w$. For the case where $p=2$ and $2 \mid n$, note that $w = \frac{n}{2}$. We also get the desired result by Lemma 4.2.10 (i), Proposition 2.8.1 (vii) and Lemma 4.2.11. We are done. $\square$

Proposition 4.2.13. *Let $(n-2, 2) \in \mathfrak{Rcg}(n)$ having p -weight w . We have*

$$c_{\mathfrak{S}_n}(D_2) = \begin{cases} w-1, & \text{if } p=2, n=5 \text{ or } 6, \\ w, & \text{otherwise.} \end{cases}$$

Proof. When $p > 2$, by Theorem 4.1.5, we have

$$S_2 \sim \begin{cases} D_2, & \text{if } n \not\equiv 1 \pmod{p} \text{ and } n \not\equiv 2 \pmod{p}, \\ D_2 + D^{(n)}, & \text{if } n \equiv 1 \pmod{p}, \\ D_2 + D_1, & \text{if } n \equiv 2 \pmod{p}. \end{cases} \quad (4.2.2)$$

When $n \not\equiv 1 \pmod{p}$ and $n \not\equiv 2 \pmod{p}$, note that $S_2 \cong D_2$. Therefore, $c_{\mathfrak{S}_n}(D_2) = w$ by Lemma 4.1.1. By (4.2.2) and (4.1.2), one gets $\dim_{\mathbb{F}} D_2 \equiv p-2 \pmod{p}$ when $n \equiv 1 \pmod{p}$ or $n \equiv 2 \pmod{p}$, which forces $c_{\mathfrak{S}_n}(D_2) = w$.

We deal with the case $p=2$. When $n \equiv 1 \pmod{4}$ and $n > 5$, set $E = E_{(n-1)/2}$ and note that $w = \frac{n-1}{2}$. By Theorem 4.1.5 and Lemma 4.2.5, we have $S_2 \sim D_2 + D^{(n)}$ and $S_2 \downarrow_E$ has stable generic Jordan type $[1]^{\frac{n-3}{2}}$. As $1 < \frac{n-3}{2}$, the two facts force that $D_2 \downarrow_E$ is not generically free by Proposition 2.8.1 (iv) (c). We thus get that $c_{\mathfrak{S}_n}(D_2) = w$ by Proposition 2.8.1 (vii). For the remaining case $p=2$ and $n=5$, we have $c_{\mathfrak{S}_5}(D^{(3,2)}) = 1$ by [58, Example 1].

When $n \equiv 3 \pmod{4}$, $S_2 \cong D_2$ by Theorem 4.1.5. So $c_{\mathfrak{S}_n}(D_2) = w$ by Lemma 4.1.1.

When $2 \mid n$, assign $E = E_{n/2}$ and note that $w = \frac{n}{2}$. When $n \equiv 0 \pmod{4}$, by Theorem 4.1.5, we have $S_2 \sim D_2 + D_1$. Notice that $D_1 \downarrow_E$ is generically free by Lemma 4.2.10 (ii). This implies that both $D_2 \downarrow_E$ and $S_2 \downarrow_E$ have stable generic Jordan type $[1]^{\frac{n}{2}-2}$ by Proposition 2.8.1 (iv) (a) and Lemma 4.2.4. We conclude that $D_2 \downarrow_E$ is not generically free since $n \neq 4$. Therefore, we have $c_{\mathfrak{S}_n}(D_2) = w$ by Proposition 2.8.1 (vii).

Finally, when $n \equiv 2 \pmod{4}$ and $n > 6$, we have $S_2 \sim D_2 + D_1 + D^{(n)}$ by Theorem 4.1.5. Put $R = \text{Rad}(S_2)$. By [56, Theorem 6 (ii)], we have the short exact sequences

$$0 \rightarrow R \downarrow_E \rightarrow S_2 \downarrow_E \rightarrow D_2 \downarrow_E \rightarrow 0 \text{ and } 0 \rightarrow D_1 \downarrow_E \rightarrow R \downarrow_E \rightarrow D^{(n)} \downarrow_E \rightarrow 0.$$

Suppose $D_2 \downarrow_E$ is generically free. Note that both $R \downarrow_E$ and $S_2 \downarrow_E$ have stable generic Jordan type $[1]^{\frac{n}{2}-2}$ by the short exact sequence on the left-hand side, Proposition 2.8.1 (iv) (c) and Lemma 4.2.4. Using the other short exact sequence, Lemma 4.2.10 (ii) and Proposition 2.8.1 (iv) (a), we know $R \downarrow_E$ and $D^{(n)} \downarrow_E$ have the same stable generic Jordan type. The fact implies that $\frac{n}{2} - 2 = 1$. This is a contradiction since $n \neq 6$. Hence $D_2 \downarrow_E$ is not generically free, which implies that $c_{\mathfrak{S}_n}(D_2) = w$ by Proposition 2.8.1 (vii).

For the remaining case where $p = 2$ and $n = 6$, we set $E = \langle (1, 2), (3, 4), (5, 6) \rangle$ and put $F = \langle (1, 2)(3, 4), (1, 3)(2, 4), (5, 6) \rangle$. By Theorem 4.2.6, they are exactly all maximal elementary abelian 2-subgroups of $\mathfrak{S}_6$ up to $\mathfrak{S}_6$ -conjugation. According to the definition of the rank variety of a module, we can determine $V_E^\#(D^{(4,2)} \downarrow_E)$ (resp. $V_F^\#(D^{(4,2)} \downarrow_F)$) with respect to an ordered generator set of E (resp. F) and an $\mathbb{F}$ -basis of $D^{(4,2)}$. In particular, direct computation shows that $\dim V_E^\#(D^{(4,2)} \downarrow_E) = 1$ and $\dim V_F^\#(D^{(4,2)} \downarrow_F) = 2$. This shows that $c_{\mathfrak{S}_6}(D^{(4,2)}) = 2$ by the definition of the complexity of a module. The proof is now complete. $\square$

Theorem 4.1.2 is now proved by combining Propositions 4.2.12 and 4.2.13.

4.3 Proof of Theorem 4.1.3

This section focuses on proving Theorem 4.1.3. We first show some needed results and then finish the proof of Theorem 4.1.3. For convenience, throughout this section, we fix r to be the remainder of n when n is divided by p . The assumption of r is important as we shall repeatedly use it in this section. In particular, we shall formulate most results of this section by using r .

We begin with a theorem which is useful in the calculation of dimensions of simple modules of symmetric groups labelled by two-part partitions.

Theorem 4.3.1. [51, Lucas Theorem] *Let non-negative integers a and b have the p -adic sums $\sum_{i \geq 0} p^i a_i$ and $\sum_{i \geq 0} p^i b_i$ respectively. Then $\binom{a}{b} \equiv \prod_{i \geq 0} \binom{a_i}{b_i} \pmod{p}$.*

The following lemma is a property of p -restricted two-part partitions.

Lemma 4.3.2. [39, Lemma 5.2] *Given a p -restricted partition (a, b) , it has p -weight either zero or one. Furthermore, it has p -weight one if and only if $a - b < p - 1 \leq a$.*

For a partition $(n - m, m)$, note that it has p -core (r) if $p \mid m$. The following lemma determines its p -core if $p \nmid m$.

Lemma 4.3.3. *Let $s \in \mathbb{N}$ and $s < p$. Let $m \in \mathbb{N}_0$ and $2mp + 2s \leq n$. Let $\lambda = (n - mp - s, mp + s)$.*

$$(i) \text{ When } p = 2, \kappa(\lambda) = \begin{cases} \emptyset, & \text{if } r = 0, \\ (2, 1), & \text{if } r = 1. \end{cases}$$

$$(ii) \text{ When } p > 2 \text{ and } 0 < s \leq \frac{p-1}{2},$$

$$\kappa(\lambda) = \begin{cases} (p + r - s, s), & \text{if } 0 \leq r < s - 1, \\ (s - 1, r - s + 1), & \text{if } s - 1 \leq r < 2s - 1, \\ (p + s - 1, s), & \text{if } r = 2s - 1, \\ (r - s, s), & \text{if } 2s - 1 < r \leq p - 1. \end{cases}$$

$$(iii) \text{ When } p > 2 \text{ and } s = \frac{p+1}{2},$$

$$\kappa(\lambda) = \begin{cases} (\frac{3p-1}{2}, \frac{p+1}{2}), & \text{if } r = 0, \\ (\frac{p-1}{2} + r, \frac{p+1}{2}), & \text{if } 0 < r < \frac{p-1}{2}, \\ (\frac{p-1}{2}, r - \frac{p-1}{2}), & \text{if } \frac{p-1}{2} \leq r \leq p - 1. \end{cases}$$

(iv) When $p > 3$ and $\frac{p+1}{2} < s < p$,

$$\kappa(\lambda) = \begin{cases} (s-1, p+r-s+1), & \text{if } 0 \leq r < 2s-p-1, \\ (p+s-1, s), & \text{if } r = 2s-p-1, \\ (p+r-s, s), & \text{if } 2s-p-1 < r < s-1, \\ \overline{(s-1, r-s+1)}, & \text{if } s-1 \leq r \leq p-1. \end{cases}$$

Proof. One removes all horizontal p -hooks of λ successively to make it become a p -restricted partition and then uses Lemma 4.3.2 to compute directly. $\square$

Some corollaries of Lemma 4.3.3 are given as follows.

Corollary 4.3.4. *A partition $(n-m, m)$ has p -core (r) if and only if $m \equiv 0 \pmod{p}$ or $m \equiv r+1 \pmod{p}$.*

Corollary 4.3.5. *Let $p > 2$ and $m \in \mathbb{N}_0$. Let $s \in \mathbb{N}$ and $s < p$. Let $\alpha = (n-mp-s, mp+s)$. Let $\beta = (n-mp-s, mp+s-1)$. Let $\gamma = (n-mp-s-1, mp+s-1)$. Assume that α, β, γ are partitions. Let w_α, w_β and w_γ be the p -weights of α, β and γ respectively.*

- (i) *If $r \not\equiv 2s-1 \pmod{p}$ and $r \not\equiv 2s-2 \pmod{p}$, then $w_\alpha = w_\beta$.*
- (ii) *If $r \equiv 2s-1 \pmod{p}$, then $w_\alpha = w_\gamma$.*
- (iii) *If $r \equiv 2s-2 \pmod{p}$ and $s > 1$, then $w_\alpha = w_\gamma$.*

Proof. For (i), it suffices to check that $|\kappa(\alpha)| = |\kappa(\beta)| + 1$. We consider three cases.

Case 1: $0 < s \leq \frac{p-1}{2}$.

If $0 \leq r < s-1$, then $\kappa(\alpha) = (p+r-s, s)$ and $\kappa(\beta) = (p+r-s, s-1)$ by Lemma 4.3.3 (ii). If $s-1 \leq r < 2s-1$, note that $0 < r$ and $1 < s$. Otherwise, $r = 0$ if and only if $s = 1$. We get a contradiction as $r \not\equiv 2s-2 \pmod{p}$. Also notice that $r < 2s-2$ as $r \not\equiv 2s-2 \pmod{p}$. We thus have $s-2 \leq r-1 < 2s-3 = 2(s-1)-1$, which implies that $\kappa(\alpha) = \overline{(s-1, r-s+1)}$ and $\kappa(\beta) = \overline{(s-2, r-s+1)}$ by Lemma 4.3.3 (ii). Since $r \not\equiv 2s-1 \pmod{p}$, observe that $r \not\equiv 2s-1$. If $2s-1 < r \leq p-1$, notice that $0 < r$. We have $\kappa(\alpha) = (r-s, s)$ and $\kappa(\beta) = (r-s, s-1)$ by Lemma 4.3.3 (ii).

Case 2: $s = \frac{p+1}{2}$.

Note that $r > 0$ since $r \not\equiv 2s-1 \pmod{p}$. If $0 < r < \frac{p-1}{2}$, then $\kappa(\alpha) = (\frac{p-1}{2} + r, \frac{p+1}{2})$ and $\kappa(\beta) = (\frac{p-1}{2} + r, \frac{p-1}{2})$ by Lemma 4.3.3 (iii) and (ii). If $\frac{p-1}{2} \leq r \leq p-1$, then $r < p-1$ since $r \not\equiv 2s-2 \pmod{p}$. From Lemma 4.3.3 (iii) and (ii), we get $\kappa(\alpha) = \overline{(\frac{p-1}{2}, r - \frac{p-1}{2})}$ and $\kappa(\beta) = \overline{(\frac{p-3}{2}, r - \frac{p-1}{2})}$.

Case 3: $p > 3$ and $\frac{p+1}{2} < s < p$.

If $0 \leq r < 2s-p-1$, as $r \not\equiv 2s-2 \pmod{p}$, notice that $r = 0$ if $s = \frac{p+3}{2}$ and $r < 2s-p-2$. We get $\kappa(\alpha) = (s-1, p+r-s+1)$ and $\kappa(\beta) = (s-2, p+r-s+1)$ by Lemma 4.3.3 (iv) and (iii). We have $r \not\equiv 2s-p-1$ since $r \not\equiv 2s-1 \pmod{p}$. If $2s-p-1 < r < s-1$, we get $\kappa(\alpha) = (p+r-s, s)$ and $\kappa(\beta) = (p+r-s, s-1)$ by Lemma 4.3.3 (iv) and (iii). If $s-1 \leq r \leq p-1$, by Lemma 4.3.3 (iv) and (iii) again, we have $\kappa(\alpha) = \overline{(s-1, r-s+1)}$ and $\kappa(\beta) = \overline{(s-2, r-s+1)}$.

Therefore, we get $|\kappa(\alpha)| = |\kappa(\beta)| + 1$ by combining the three cases and (i) follows. Similarly, one can use Lemma 4.3.3 to obtain $|\kappa(\alpha)| = |\kappa(\gamma)| + 2$ and the correctness of (ii) and (iii). $\square$

For the remaining part of this section, we assume that $p > 2$ and recall the definition of E_a in Notation 4.2.1 (v). We collect some preliminary results in the following lemma.

Lemma 4.3.6. *Let $i \in \mathbb{N}$ and $i \leq p-1$. Let $n \equiv ip \pmod{p^2}$ and $\lambda = (n-p, p)$. Let $\mu = (n-p+1, p-1)$. Let $\nu = (n-1, 1)$ and $E = E_{n/p}$. Assume that λ, μ, ν are partitions.*

(i) The module $Y^\lambda \downarrow_E$ has stable generic Jordan type $[1]^{\frac{n}{p}-1}$.

(ii) There exists a short exact sequence $0 \rightarrow S^\lambda \rightarrow Y^\lambda \rightarrow S^\nu \rightarrow 0$.

Proof. We claim that $M^\lambda \cong Y^\lambda \oplus M \oplus Y^{(n)}$, where $M \mid M^\mu$. Let $m \in \mathbb{N}$ such that $2m \leq n$, $0 < m < p$ and $\eta = (n-m, m)$. We have $p-m-1 \leq n-2m$ if $p-m \leq n-2m$. By Theorem 4.1.4, the relation implies that $Y^\eta \mid M^\mu$ if $Y^\eta \mid M^\lambda$. Using Theorem 4.1.4 again, we also have $Y^{(n)} \mid M^\lambda$ as $p \leq n$. The claim at the beginning of the proof is shown. Using Lemmas 2.8.2 and 2.11.1, we obtain that $M^\lambda \downarrow_E$ has stable generic Jordan type $[1]^{\frac{n}{p}}$ and $M^\mu \downarrow_E$ is generically free. Then (i) follows by the above claim and Proposition 2.8.1 (iii).

For (ii), by Corollary 4.3.4, observe that $\kappa(\lambda) = \kappa((n))$ and ν is the unique partition of n such that $\lambda \triangleleft \nu \triangleleft (n)$ and $\kappa(\nu) = \kappa(\lambda)$. As $p-1 \not\leq n-2$, we get $Y^\nu \nmid M^\lambda$ by Theorem 4.1.4. These facts and the above claim imply the existence of the desired short exact sequence by Lemma 4.1.7. $\square$

We now begin to finish the proof of Theorem 4.1.3. Our main strategy is to apply Theorem 4.1.9 and Proposition 2.8.1 (vii) to the simple modules of Theorem 4.1.3. To apply Theorem 4.1.9, we introduce a notation as follows.

Definition 4.3.7. For any two-part partition $\lambda = (a-bp-c, bp+c)$ where integers a, b, c satisfy $a > 0$, $b \geq 0$ and $0 \leq c < p$, set $u = 2c-1$, $v = 2c-2$ and $\phi^0(\lambda) = \lambda$. If $c > 0$, define

$$\phi^1(\lambda) = \begin{cases} (a-bp-c, bp+c-1), & \text{if } a \not\equiv u \pmod{p} \text{ and } a \not\equiv v \pmod{p}, \\ (a-bp-c-1, bp+c-1), & \text{if } a \equiv u \pmod{p}, \\ (a-bp-c-1, bp+c-1), & \text{if } a \equiv v \pmod{p} \text{ and } c > 1, \\ \emptyset, & \text{if } a \equiv 0 \pmod{p} \text{ and } c = 1. \end{cases}$$

If $c = 0$, put $\phi^1(\lambda) = \emptyset$. Note that $\phi^1(\lambda)$ is a well-defined partition. Recursively, for $1 < k \in \mathbb{N}$, if $\phi^{k-1}(\lambda) \neq \emptyset$, define $\phi^k(\lambda) = \phi^1(\phi^{k-1}(\lambda))$. By this definition and the definition of $\phi^1(\lambda)$, there exists some $z \in \mathbb{N}$ such that $\phi^z(\lambda) = \emptyset$ and $\phi^{z-1}(\lambda) \neq \emptyset$. Define $\Phi(\lambda) = \phi^{z-1}(\lambda)$. Note that $\Phi(\lambda)$ is either a partition $(x-bp, bp)$ or a partition $(xp-bp-1, bp+1)$ for some $x \in \mathbb{N}$. For an example, if $p = 3$, then $\phi^1((6, 5)) = (5, 4)$ and $\phi^2((6, 5)) = \emptyset$. So $\Phi((6, 5)) = (5, 4)$.

We have the following corollary by Definition 4.3.7. This corollary shall be repeatedly used in the following discussion.

Corollary 4.3.8. Let λ be a two-part partition of n . Let $m \in \mathbb{N}$ and $\Phi(\lambda) \vdash m$.

(i) We have $c_{\mathfrak{S}_n}(D^\lambda) = c_{\mathfrak{S}_m}(D^{\Phi(\lambda)})$.

(ii) Both λ and $\Phi(\lambda)$ have the same p -weight.

Proof. By Definition 4.3.7, there exists $z \in \mathbb{N}$ such that $\Phi(\lambda) = \phi^{z-1}(\lambda) \neq \emptyset$. The case $z = 1$ is clear since $\phi^0(\lambda) = \lambda$. If $z > 1$, note that $\phi^k(\lambda) \neq \emptyset$ for all $1 \leq k \leq z-1$. By Definition 4.3.7, Theorem 4.1.9 and Corollary 4.3.5, for all $0 \leq k < z-1$, $D^{\phi^k(\lambda)}$, $D^{\phi^{k+1}(\lambda)}$ have the same complexity and $\phi^k(\lambda)$, $\phi^{k+1}(\lambda)$ have the same p -weight. In particular, as $\phi^0(\lambda) = \lambda$ and $\phi^{z-1}(\lambda) = \Phi(\lambda)$, we get (i) and (ii). $\square$

The entire proof of Theorem 4.1.3 will be divided into the following propositions.

Proposition 4.3.9. Let $s \in \mathbb{N}$ and $s < p$. If $2s \leq n$ and $\lambda = (n-s, s)$ has p -weight w , then $c_{\mathfrak{S}_n}(D^\lambda) = w$.

Proof. By Corollary 4.3.8 (i), it suffices to compute the complexity of $D^{\Phi(\lambda)}$. By Definition 4.3.7, for some $m \in \mathbb{N}$, we have either $\Phi(\lambda) = (m)$ or $\Phi(\lambda) = (mp-1, 1)$. By Proposition 4.2.12 and Corollary 4.3.8 (ii), for the two possibilities, the complexity of $D^{\Phi(\lambda)}$ is exactly w . Therefore, $c_{\mathfrak{S}_n}(D^\lambda) = w$, as desired. $\square$

Recall that r is the remainder of n when n is divided by p . Note that a partition $(n-m, m)$ has p -weight $\frac{n-r}{p}$ if $p \mid m$. The following lemma is important in the proof of Theorem 4.1.3.

Lemma 4.3.10. *Let $p > 3$ and $m \in \mathbb{N}$ where $m < \frac{p-1}{2}$ and $2mp < n$. Let $n \equiv ip \pmod{p^2}$ where $2m < i \leq p-1$. Let $E = E_{n/p}$, $\lambda = (n-mp, mp)$ and $\mu = (n-mp+p-1, mp-p+1)$. If $D^\mu \downarrow_E$ is not generically free, then $D^\lambda \downarrow_E$ is not generically free; in particular, $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.*

Proof. By Corollary 4.3.4, all the partitions of n that dominate λ and have the same p -core as λ are exactly $(n-mp+jp, mp-jp)$ and $(n-mp+kp-1, mp-kp+1)$, where the integers j, k satisfy $0 \leq j \leq m$ and $0 < k \leq m$. For all $0 < j \leq m$, since $2m < i \leq p-1$, note that

$$j < i - 2m + 2j = i - 2(m-j) \leq p-1,$$

which implies that $jp \not\sqsubseteq n - 2mp + 2jp + 1$. Moreover, $kp-1 \sqsubseteq n - 2mp + 2kp - 1$ if and only if $k = 1$. Therefore, we conclude that $S^\lambda \sim D^\lambda + D^\mu$ by Theorem 4.1.5. Let $\alpha = (n-mp-1, mp)$ and $\beta = (n-mp-1, mp+1)$. Similarly, using Corollary 4.3.4 and Theorem 4.1.5 again, we obtain that $S^\alpha \cong D^\alpha$ and $S^\beta \sim D^\beta + D^\lambda$. Let $\gamma = (n-mp-1, mp, 1)$. By the Branching Rule and the block decomposition, we have

$$D^{\alpha \uparrow \mathfrak{S}_n} \cong S^{\alpha \uparrow \mathfrak{S}_n} \cong M \oplus S^\gamma, \quad (4.3.1)$$

where $M \sim S^\lambda + S^\beta$. Using [44, Theorem 11.2.8], we also get

$$D^{\alpha \uparrow \mathfrak{S}_n} \cong N \oplus D^\gamma, \quad (4.3.2)$$

where N is a self-dual and indecomposable $\mathbb{F}\mathfrak{S}_n$ -module whose head and socle are isomorphic to D^λ . As $0 < m < \frac{p-1}{2}$, note that $S^\gamma \cong D^\gamma$ by [37, 7.3.23]. Combining (4.3.1), (4.3.2) and the Krull–Schmidt Theorem, we have $M \cong N$. Therefore, we deduce that N has three Loewy layers and its heart is isomorphic to $D^\mu \oplus D^\beta$. Set $D = D^\mu \oplus D^\beta$. As $D^\mu \downarrow_E$ is not generically free, $D \downarrow_E$ is not generically free by Proposition 2.8.1 (iii). By the Mackey Formula, $D^{\alpha \uparrow \mathfrak{S}_n} \downarrow_E \cong \bigoplus_{t \in \mathcal{T}} D_t \uparrow^E$, where $\mathcal{T}$ is a complete set of the representatives of $(\mathfrak{S}_{n-1}, E)$ -double cosets in $\mathfrak{S}_n$ and D_t is an $\mathbb{F}[t\mathfrak{S}_{n-1}t^{-1} \cap E]$ -module for any $t \in \mathcal{T}$. Note that $t\mathfrak{S}_{n-1}t^{-1} \cap E \neq E$ for any $t \in \mathcal{T}$. So $D^{\alpha \uparrow \mathfrak{S}_n} \downarrow_E$ is generically free by Proposition 2.8.1 (iii) and (v), which implies that $N \downarrow_E$ is generically free by (4.3.2) and Proposition 2.8.1 (iii).

Now put $R = \text{Rad}(N)$ and suppose that $D^\lambda \downarrow_E$ is generically free. We can apply Proposition 2.8.1 (iv) (c) to the short exact sequence $0 \rightarrow R \downarrow_E \rightarrow N \downarrow_E \rightarrow D^\lambda \downarrow_E \rightarrow 0$ and get that $R \downarrow_E$ is generically free. We thus apply Proposition 2.8.1 (iv) (a) to the short exact sequence $0 \rightarrow D^\lambda \downarrow_E \rightarrow R \downarrow_E \rightarrow D \downarrow_E \rightarrow 0$ and deduce that $D \downarrow_E$ is generically free. However, this is a contradiction since we know that $D \downarrow_E$ is not generically free. Therefore, $D^\lambda \downarrow_E$ is not generically free. The second assertion follows by the first one and Proposition 2.8.1 (vii). $\square$

We need the following two lemmas to compute $c_{\mathfrak{S}_n}(D^{(n-p,p)})$ if $n \geq 2p$.

Lemma 4.3.11. *Let $n \geq 2p$ and $r = 0$. Let $\lambda = (n-p, p)$ and $E = E_{n/p}$. Then $D^\lambda \downarrow_E$ is not generically free. Moreover, we have $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.*

Proof. Let $\mu = (n-1, 1)$. By Corollary 4.3.4 and Theorem 4.1.5, we have

$$S^\lambda \sim \begin{cases} D^\lambda + D^\mu + D^{(n)}, & \text{if } n \equiv p \pmod{p^2}, \\ D^\lambda + D^\mu, & \text{otherwise.} \end{cases} \quad (4.3.3)$$

By (4.3.3), (4.1.2) and Theorem 4.3.1, we have

$$\dim_{\mathbb{F}} D^\lambda \equiv \begin{cases} 2 & \pmod{p}, & \text{if } n \equiv p \pmod{p^2}, \\ i+2 & \pmod{p}, & \text{if } n \equiv ip \pmod{p^2} \text{ for some } i \in \mathbb{N}_0 \text{ and } 1 \neq i \leq p-1. \end{cases}$$

So we get that $p \nmid \dim_{\mathbb{F}} D^\lambda$ unless $p > 3$ and $n \equiv (p-2)p \pmod{p^2}$. When $p = 3$, or both $p > 3$ and $n \not\equiv (p-2)p \pmod{p^2}$, $D^\lambda \downarrow_E$ is not generically free by Proposition 2.8.1 (i). Moreover, $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.

When $p > 3$ and $n \equiv (p-2)p \pmod{p^2}$, as $\dim_{\mathbb{F}} D^\mu \equiv p-2 \pmod{p}$, note that $D^\mu \downarrow_E$ is not generically free by Proposition 2.8.1 (i). It is clear that λ and μ now satisfy all the assumptions of Lemma 4.3.10. By Lemma 4.3.10, $D^\lambda \downarrow_E$ is not generically free and $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$. $\square$

Lemma 4.3.12. *Let $m \in \mathbb{N}_0$ and $2mp + r \leq n$. Let $\lambda = (n - mp, mp)$ and $\mu = (n - mp - r, mp)$. If $0 < r < p - 1$, then $D^\mu \mid D^\lambda \downarrow_{\mathfrak{S}_{n-r}}$.*

Proof. The case $m = 0$ is trivial. The case where $p = 3$ and $r = 1$ is also clear by Theorem 4.1.8. For any integer a satisfying the condition $0 \leq a \leq r$, let $\lambda_a = (n - mp - a, mp)$. We deal with the remaining case where $p > 3$ and $m > 0$. Let $t \in \mathbb{N}_0$ and $t < r$. As $0 < r < p - 1$, we have $0 < r - t \leq r < p - 1$, which implies that $p - 2 \not\equiv r - t - 1 \pmod{p}$. The relation $p - 2 \not\equiv r - t - 1 \pmod{p}$ and Theorem 4.1.8 say that $D^{\lambda_{t+1}} \mid D^{\lambda_t} \downarrow_{\mathfrak{S}_{n-t-1}}$. Therefore, as $\lambda_0 = \lambda$ and $\lambda_r = \mu$, we get $D^\mu \mid D^\lambda \downarrow_{\mathfrak{S}_{n-r}}$ by induction. $\square$

Proposition 4.3.13. *Let $n \geq 2p$ and $\lambda = (n - p, p)$. Then $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n-r}{p}$.*

Proof. By Lemma 4.3.11, we may assume that $r \neq 0$. Let $\mu = (n - p - r, p)$. When $0 < r < p - 1$, set $E = E_{(n-r)/p}$ and note that $\mathfrak{S}_{n-r}$ contains E . By Lemma 4.3.12, we have $D^\mu \downarrow_E \mid D^\lambda \downarrow_E$. Notice that $D^\mu \downarrow_E$ is not generically free by Lemma 4.3.11. Therefore, we deduce that $D^\lambda \downarrow_E$ is not generically free by Proposition 2.8.1 (iii). We thus get $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n-r}{p}$ by applying Proposition 2.8.1 (vii) to $D^\lambda \downarrow_E$.

When $r = p - 1$, using Corollary 4.3.4 and Theorem 4.1.5, we have $S^\lambda \sim D^\lambda + D^{(n)}$ if $n \equiv p - 1 \pmod{p^2}$ and $S^\lambda \cong D^\lambda$ otherwise. The case $S^\lambda \cong D^\lambda$ is clear by Lemma 4.1.1. When $n \equiv p - 1 \pmod{p^2}$, by (4.1.2) and Theorem 4.3.1, we have $\dim_{\mathbb{F}} D^\lambda \equiv p - 2 \pmod{p}$ and $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n-p+1}{p}$. This completes the proof. $\square$

Lemma 4.3.14. *Let $n \geq 3p$ and $r = 0$. Let $\lambda = (n - p - 1, p + 1)$ and $\mu = (n - p, p)$. Let $E = E_{n/p}$. Then $S^\lambda \downarrow_E$ and $S^\mu \downarrow_E$ have mutually complementary stable generic Jordan types.*

Proof. The lemma is deduced by [46, Lemma 5.8]. $\square$

To prove the following lemma, we need to use Examples 3.2.3, 3.2.5, 3.2.7.

Lemma 4.3.15. *Let $n \geq 3p$ and $r = 0$. Let $\lambda = (n - p - 1, p + 1)$ and $E = E_{n/p}$. Then $D^\lambda \downarrow_E$ is not generically free. In particular, $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.*

Proof. Let $\mu = (n - p, p)$ and $\nu = (n - 1, 1)$. Note that λ has p -weight $\frac{n}{p}$. By Corollary 4.3.4 and Theorem 4.1.5, we have

$$S^\lambda \sim \begin{cases} D^\lambda + D^\mu + D^{(n)}, & \text{if } n \equiv p \pmod{p^2}, \\ D^\lambda + D^\mu + D^\nu, & \text{if } n \equiv 2p \pmod{p^2}, \\ D^\lambda + D^\mu, & \text{otherwise.} \end{cases} \quad (4.3.4)$$

By (4.3.3), (4.3.4), (4.1.2) and Theorem 4.3.1, we have

$$\dim_{\mathbb{F}} D^\lambda \equiv \begin{cases} p - 4 \pmod{p}, & \text{if } n \equiv 2p \pmod{p^2}, \\ -2i - 2 \pmod{p}, & \text{if } n \equiv ip \pmod{p^2} \text{ for some } i \in \mathbb{N}_0 \text{ and } 2 \neq i \leq p - 1. \end{cases}$$

So we have $p \nmid \dim_{\mathbb{F}} D^\lambda$ unless $p > 3$ and $n \equiv (p - 1)p \pmod{p^2}$. When $p = 3$, or both $p > 3$ and $n \not\equiv (p - 1)p \pmod{p^2}$, note that $D^\lambda \downarrow_E$ is not generically free by Proposition 2.8.1 (i). Moreover, $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.

When $p > 3$ and $n \equiv (p - 1)p \pmod{p^2}$, we set $m = \frac{n}{p}$. Let $\alpha = \bigcup_{i=1}^m \{\alpha_i\}$ be a set of indeterminates and $\mathbb{K} = \mathbb{F}(\alpha_1, \dots, \alpha_m)$. Let

$$u_\alpha = 1 + \sum_{i=1}^m \alpha_i (s_i - 1) \text{ and } A = \mathbb{K}\langle u_\alpha \rangle.$$

Suppose that $D^\lambda \downarrow_E$ is generically free. Then $S^\lambda \downarrow_E$ and $D^\mu \downarrow_E$ have the same stable generic Jordan type by (4.3.4) and Proposition 2.8.1 (iv) (c). By Lemmas 4.3.6 (i), 4.2.2 and Proposition 2.8.1 (ii), notice that the stable generic Jordan types of $Y^\mu \downarrow_E$ and $S_\nu \downarrow_E$ are $[1]^{\frac{n}{p}-1}$ and $[p - 1]^1$ respectively. We use Example 3.2.3 to study the short exact sequence

$$0 \rightarrow S_\nu \downarrow_A \rightarrow Y^\mu \downarrow_A \rightarrow S_\mu \downarrow_A \rightarrow 0$$

given by Lemma 4.3.6 (ii) and get that the stable generic Jordan type of $S_\mu \downarrow_E$ is contained in the set $\{[1]^{\frac{n}{p}}, [1]^{\frac{n}{p}-2}[2]^1\}$. This obtained set and Proposition 2.8.1 (ii) imply that the stable generic Jordan type of $S^\mu \downarrow_E$ is also contained in $\{[1]^{\frac{n}{p}}, [1]^{\frac{n}{p}-2}[2]^1\}$. Therefore, by Lemma 4.3.14, the stable generic Jordan type of $S^\lambda \downarrow_E$ lies in the set

$$C = \{[p-1]^{\frac{n}{p}}, [p-2]^1[p-1]^{\frac{n}{p}-2}\}.$$

We also observe that the stable generic Jordan type of $D^\nu \downarrow_E$ lies in $\{[p-2]^1, [p-1]^2\}$ by Lemma 4.2.2 and applying Proposition 3.2.1 to the short exact sequence

$$0 \rightarrow D^{(n)} \downarrow_A \rightarrow S^\nu \downarrow_A \rightarrow D^\nu \downarrow_A \rightarrow 0.$$

Now we can use Examples 3.2.3, 3.2.5 (iv), 3.2.7 to study the short exact sequence

$$0 \rightarrow D^\nu \downarrow_A \rightarrow S^\mu \downarrow_A \rightarrow D^\mu \downarrow_A \rightarrow 0$$

provided by (4.3.3) and get that the stable generic Jordan type of $D^\mu \downarrow_E$ is contained in the set

$$\{[1]^{\frac{n}{p}+2}, [1]^{\frac{n}{p}}[2]^1, [1]^{\frac{n}{p}-1}[3]^1, [1]^{\frac{n}{p}-2}[4]^1, [1]^{\frac{n}{p}-2}[2]^2, [1]^{\frac{n}{p}-3}[2]^1[3]^1, [1]^{\frac{n}{p}-4}[2]^3\}.$$

Note that all members of this set are well-defined since $n \equiv (p-1)p \pmod{p^2}$ and $p \geq 5$. Name this set by S and notice that $C \cap S = \emptyset$. This is a contradiction since we know that $S^\lambda \downarrow_E$ and $D^\mu \downarrow_E$ have the same stable generic Jordan type. So $D^\lambda \downarrow_E$ is not generically free. Moreover, we have $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$ by Proposition 2.8.1 (vii). This completes the proof. $\square$

Proposition 4.3.16. *Let $s \in \mathbb{N}$ and $s < p$. Let $n \geq 2p + 2s$ and $\lambda = (n - p - s, p + s)$ have p -weight w . Then $c_{\mathfrak{S}_n}(D^\lambda) = w$.*

Proof. By Corollary 4.3.8 (i), we just need to compute the complexity of $D^{\Phi(\lambda)}$. By Definition 4.3.7, for some positive integer m , we have either $\Phi(\lambda) = (m - p, p)$ or $\Phi(\lambda) = (mp - p - 1, p + 1)$. By Proposition 4.3.13, Lemma 4.3.15 and Corollary 4.3.8 (ii), for the two possibilities, the complexity of $D^{\Phi(\lambda)}$ is exactly w . Therefore, we have $c_{\mathfrak{S}_n}(D^\lambda) = w$. We are done. $\square$

We need the following lemma to deduce the final proposition.

Lemma 4.3.17. *Let $p > 3$ and $r = 0$. Let $n \geq 4p$ and $\lambda = (n - 2p, 2p)$. Let $E = E_{n/p}$. Then $D^\lambda \downarrow_E$ is not generically free. Moreover, we have $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.*

Proof. Let $\mu = (n - p - 1, p + 1)$, $\nu = (n - p, p)$ and $\eta = (n - 1, 1)$. Following Corollary 4.3.4 and Theorem 4.1.5, we get

$$S^\lambda \sim \begin{cases} D^\lambda + D^\mu + D^\eta + D^{(n)}, & \text{if } n \equiv 2p \pmod{p^2}, \\ D^\lambda + D^\mu + D^\nu, & \text{if } n \equiv 3p \pmod{p^2}, \\ D^\lambda + D^\mu, & \text{otherwise.} \end{cases} \quad (4.3.5)$$

According to (4.3.3), (4.3.4), (4.3.5), (4.1.2) and Theorem 4.3.1, we have

$$\dim_{\mathbb{F}} D^\lambda \equiv \begin{cases} 2 & \pmod{p}, & \text{if } n \equiv 0 & \pmod{p^2}, \\ 4 & \pmod{p}, & \text{if } n \equiv p & \pmod{p^2}, \\ 6 & \pmod{p}, & \text{if } n \equiv 2p \text{ or } 3p & \pmod{p^2}, \\ 16 & \pmod{p}, & \text{if } n \equiv 4p & \pmod{p^2}. \end{cases}$$

Let i be an integer. When $n \equiv ip \pmod{p^2}$ and $0 \leq i \leq 4$, we conclude that $p \nmid \dim_{\mathbb{F}} D^\lambda$, which implies that $D^\lambda \downarrow_E$ is not generically free by Proposition 2.8.1 (i). Moreover, $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$.

For the remaining case where $p > 5$, $n \equiv ip \pmod{p^2}$ and $4 < i \leq p - 1$, by Lemma 4.3.15, $D^\mu \downarrow_E$ is not generically free. Also notice that λ and μ now satisfy all the assumptions of Lemma 4.3.10. Using Lemma 4.3.10 again, $D^\lambda \downarrow_E$ is not generically free and $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n}{p}$. $\square$

Proposition 4.3.18. *Let $p > 3$, $n \geq 4p$ and $\lambda = (n - 2p, 2p)$. Then $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n-r}{p}$.*

Proof. We may assume that $r \neq 0$ by Lemma 4.3.17. Let $\mu = (n - 2p - r, 2p)$ and $\nu = (n - p, p)$. When $0 < r < p - 1$, let $E = E_{(n-r)/p}$ and note that $\mathfrak{S}_{n-r}$ contains E . By Lemma 4.3.12, we have $D^\mu \downarrow_E \mid D^\lambda \downarrow_E$. Observe that $D^\mu \downarrow_E$ is not generically free by Lemma 4.3.17. Therefore, $D^\lambda \downarrow_E$ is not generically free by Proposition 2.8.1 (iii). We thus get $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n-r}{p}$ by applying Proposition 2.8.1 (vii) to $D^\lambda \downarrow_E$.

When $r = p - 1$, using Corollary 4.3.4 and Theorem 4.1.5, we have

$$S^\lambda \sim \begin{cases} D^\lambda + D^{(n)}, & \text{if } n \equiv 2p - 1 \pmod{p^2}, \\ D^\lambda + D^\nu, & \text{if } n \equiv 3p - 1 \pmod{p^2}, \\ D^\lambda, & \text{otherwise.} \end{cases} \quad (4.3.6)$$

The case $S^\lambda \cong D^\lambda$ is clear by Lemma 4.1.1. For the remaining situations, we deduce that $\dim_{\mathbb{F}} D^\lambda \equiv p - 2 \pmod{p}$ by (4.3.6), (4.1.2) and Theorem 4.3.1, which implies $c_{\mathfrak{S}_n}(D^\lambda) = \frac{n-r}{p}$. The proof is now complete. $\square$

We are now ready to present the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. As the complexity of the trivial $\mathbb{F}\mathfrak{S}_n$ -module is clear, we now finish the proof of Theorem 4.1.3 by collecting Propositions 4.3.9, 4.3.13, 4.3.16 and 4.3.18. $\square$

We close this chapter by mentioning the following remark.

Remark 4.3.19. The work presented in this chapter focuses on computing the complexities of simple $\mathbb{F}\mathfrak{S}_n$ -modules labelled by two-part partitions. By Proposition 2.8.1 (vii), our main strategy is to prove that the modules that we are interested in are not generically free. As it is not easy to compute the stable generic Jordan types of modules directly, we want to find some short exact sequences of modules and apply Proposition 2.8.1 (iv). By Theorem 4.1.5, we can obtain some short exact sequences of modules, which let us get the main results of this chapter.

Let $(n - a, a) \in \mathfrak{Reg}(n)$. If $a > p = 2$, the short exact sequences obtained from Theorem 4.1.5 become more complicated. Furthermore, we cannot completely understand the stable generic Jordan type of $S^{(n-a,a)} \downarrow_E$ if the elementary abelian p -subgroup E of $\mathfrak{S}_n$ has large p -rank. So the approaches used in this chapter are not sufficient to compute $c_{\mathfrak{S}_n}(D^{(n-a,a)})$ if $a > p = 2$. If $p > 2$, by Corollary 4.3.8, we get $c_{\mathfrak{S}_n}(D^{(n-a,a)})$ by computing $c_{\mathfrak{S}_n}(D^{\Phi((n-a,a))})$. If $(p, a) = (3, 6)$ or $a > 2p > 4$, we cannot completely determine $c_{\mathfrak{S}_n}(D^{\Phi((n-a,a))})$. Therefore, the approaches used in this chapter are not enough to compute $c_{\mathfrak{S}_n}(D^{(n-a,a)})$ if $(p, a) = (3, 6)$ or $a > 2p > 4$.

Chapter 5

Some Trivial Source Specht Modules

This chapter focuses on the classification of trivial source Specht modules. We completely classify the trivial source Specht modules labelled by hook partitions. We also classify the trivial source Specht modules labelled by two-part partitions in the odd characteristic case. Moreover, in the even characteristic case, we prove a result for the classification of the trivial source Specht modules labelled by partitions with 2-weight 2, which justifies a conjecture given in [33].

5.1 Main Results and Some Known Results

In this section, we first state the motivation of classifying trivial source Specht modules. We then summarize the main results of this chapter. For further reference, we also collect some known results.

Recall that a Specht filtration of an $\mathbb{F}\mathfrak{S}_n$ -module M is a filtration of M with all successive quotient factors isomorphic to some Specht modules. Recall the definition of a trivial source $\mathbb{F}\mathfrak{S}_n$ -module introduced in Section 2.2. Motivated by some properties of the tilting modules of Schur algebras, Hemmer proposed the following:

Conjecture 5.1.1. [31, Conjecture 7.1] *Let M be an $\mathbb{F}\mathfrak{S}_n$ -module that is indecomposable and self-dual. If M has a Specht filtration, then M is a trivial source $\mathbb{F}\mathfrak{S}_n$ -module.*

As simple modules of symmetric groups are self-dual, we have the following evidence for Conjecture 5.1.1.

Theorem 5.1.2. [13, Proposition 4.6] *The trivial source simple $\mathbb{F}\mathfrak{S}_n$ -modules are exactly the simple $\mathbb{F}\mathfrak{S}_n$ -Specht modules.*

The self-dual Specht modules are simple in the odd characteristic case (see [35, 13.17]). To check Conjecture 5.1.1 for all indecomposable self-dual Specht modules, it is interesting to consider the following project.

Project B. Classify all the trivial source $\mathbb{F}\mathfrak{S}_n$ -Specht modules.

According to our knowledge, the project of classifying all the trivial source $\mathbb{F}\mathfrak{S}_n$ -Specht modules first begins in the thesis of Hudson (see [33]). In her thesis, Hudson classified all the trivial source Specht modules labelled by partitions with p -weight 1 when $p > 3$ and identified the trivial source Specht modules of $\mathbb{F}\mathfrak{S}_{2p}$ and $\mathbb{F}\mathfrak{S}_{2p+1}$ when p is odd. Moreover, for the case $p = 2$, she provided the following conjecture.

Conjecture 5.1.3. [33, Conjecture 5.0.1] *Let $p = 2$. Then the trivial source Specht modules labelled by partitions with 2-weight 2 are either simple or isomorphic to Young modules.*

The main results of this chapter continue contributing to Project B. The main approaches used in this chapter depend on the fixed-point functors of symmetric groups defined in [29] and the Broué correspondences of trivial source $\mathbb{F}\mathfrak{S}_n$ -modules. We completely classify the trivial source Specht modules labelled by hook partitions. When $p > 2$, we also classify the trivial source Specht modules labelled by two-part partitions. Moreover, we justify Conjecture 5.1.3. To state our results more precisely, recall that $JM(n)$ is the set of partitions labelling simple $\mathbb{F}\mathfrak{S}_n$ -Specht modules. We will show the following:

Theorem 5.1.4. *Let n and r be integers such that $0 \leq r < n$. Then $S^{(n-r, 1^r)}$ is a trivial source module if and only if the partition $(n-r, 1^r)$ falls into precisely one of the following cases:*

- (i) $(n-r, 1^r) \in JM(n)$;
- (ii) $p > 3$, $n = p$, $0 < r < p-1$ and $2 \mid r$;
- (iii) $p = 2$, $2 \nmid n$, $n \geq 2r > 2$, $r \neq n-2$, $n-1$ and $n \equiv 2r+1 \pmod{2^L}$ where the integer L satisfies $2^{L-1} \leq r < 2^L$;
- (iv) $p = 2$, $2 \nmid n$, $2r > n$, $r \neq n-2$, $n-1$ and $n \equiv 2r+1 \pmod{2^{L'}}$ where the integer L' satisfies $2^{L'-1} \leq n-r-1 < 2^{L'}$.

Theorem 5.1.5. *Let $p > 2$ and n, r be integers such that $n > 0$ and $0 \leq 2r \leq n$. Then $S^{(n-r, r)}$ is a trivial source module if and only if the partition $(n-r, r) \in JM(n)$.*

Theorem 5.1.6. *Let $p = 2$ and n be an integer such that $n \geq 4$. Let λ be a partition of n with 2-weight 2 and $\kappa(\lambda)$ be its 2-core. Then S^λ is a trivial source module if and only if λ falls into exactly one of the following cases:*

- (i) $\lambda \in JM(n)$;
- (ii) $\lambda \notin JM(n)$, $S^\lambda \cong Y^\mu$ and $\mu = \kappa(\lambda) + (2, 2)$.

The case (ii) of Theorem 5.1.6 indeed happens. For example, let $p = 2$. Note that $S^{(3,1,1)}$ is an indecomposable, non-simple, trivial source module. Theorem 5.1.6 says that $S^{(3,1,1)} \cong Y^{(3,2)}$.

5.2 Fixed-Point Functors of Symmetric Groups

The aim of this section is to study fixed-point functors of symmetric groups. After a short summary of these objects, we present some properties of these functors. Some results will be used in the following two sections while others may be of independent interest.

Let $m \in \mathbb{N}$ and $m \leq n$. Throughout this chapter, recall that $\mathfrak{S}_{n-m}^{+m} = \mathfrak{S}_{\{m+1, \dots, n\}} \cong \mathfrak{S}_{n-m}$. So we have $\mathfrak{S}_{(m, n-m)} = \mathfrak{S}_m \times \mathfrak{S}_{n-m}^{+m}$. Let V be an $\mathbb{F}\mathfrak{S}_n$ -module. As $\mathfrak{S}_{n-m}^{+m}$ centralizes $\mathfrak{S}_m$, the $\mathbb{F}$ -linear space $V^{\mathfrak{S}_m}$ can be regarded as an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module naturally. The fixed-point functor with the parameter m , introduced by Hemmer in [29], is defined to be

$$\mathcal{F}_m : \mathbb{F}\mathfrak{S}_n\text{-mod} \mapsto \mathbb{F}\mathfrak{S}_{n-m}^{+m}\text{-mod}$$

by setting

$$\begin{aligned} \mathcal{F}_m(V) &= V^{\mathfrak{S}_m} \cong \text{Hom}_{\mathbb{F}\mathfrak{S}_m}(\mathbb{F}, V \downarrow_{\mathfrak{S}_m}) \text{ (}\mathbb{F}\mathfrak{S}_{n-m}^{+m}\text{-isomorphism)} \\ &\cong \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(M^{(m, 1^{n-m})}, V) \text{ (}\mathbb{F}\text{-linear isomorphism).} \end{aligned}$$

Moreover, we also have

$$\mathcal{F}_m(V) = V^{\mathfrak{S}_m} = (V \downarrow_{\mathfrak{S}_{(m, n-m)}})^{\mathfrak{S}_m}.$$

Therefore, it is convenient to regard $\mathcal{F}_m(V)$ as the largest $\mathbb{F}$ -linear subspace of $V \downarrow_{\mathfrak{S}_{(m, n-m)}}$ where $\mathfrak{S}_m$ acts trivially. We know that $\mathcal{F}_m$ is exact if $m < p$ and is not exact in general if $m \geq p$ (see [29, Page 300]). In [30], Hemmer obtained some results of extensions of simple modules of symmetric groups by using fixed-point functors.

We begin to prove some properties of fixed-point functors. Throughout the whole section, fix an integer m satisfying $1 \leq m \leq n$.

Lemma 5.2.1. *Let M and N be $\mathbb{F}\mathfrak{S}_n$ -modules. Then $\mathcal{F}_m(M \oplus N) = \mathcal{F}_m(M) \oplus \mathcal{F}_m(N)$.*

Proof. Note that $(M \oplus N)^{\mathfrak{S}_m} = M^{\mathfrak{S}_m} \oplus N^{\mathfrak{S}_m}$. The lemma follows by restricting both sides of the equality to $\mathfrak{S}_{n-m}^{+m}$ and the definition of fixed-point functors. $\square$

Proposition 5.2.2. *Let P be a projective $\mathbb{F}\mathfrak{S}_n$ -module. Then $\mathcal{F}_m(P)$ is also a projective $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module.*

Proof. We may assume that P is non-zero. As P is projective, for some positive integer ℓ , we have

$$(P \downarrow_{\mathfrak{S}_{(m,n-m)}})^{\mathfrak{S}_m} \cong \bigoplus_{i=1}^{\ell} (P_i \boxtimes Q_i)^{\mathfrak{S}_m} = \bigoplus_{i=1}^{\ell} (P_i^{\mathfrak{S}_m} \boxtimes Q_i), \quad (5.2.1)$$

where P_i and Q_i are indecomposable projective $\mathbb{F}\mathfrak{S}_m$ -module and $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module respectively for each $1 \leq i \leq \ell$. Regarding both sides of (5.2.1) as $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules, we have

$$\mathcal{F}_m(P) \cong \bigoplus_{i=1}^{\ell} (\dim_{\mathbb{F}} P_i^{\mathfrak{S}_m}) Q_i,$$

which implies the desired result. $\square$

Let G be a finite group. We need some elementary knowledge from the cohomological theory of $\mathbb{F}G$ -modules. For any $\mathbb{F}G$ -module M , a projective (resp. injective) resolution of M is an exact sequence of $\mathbb{F}G$ -modules

$$\mathbb{P} : \cdots \rightarrow P_i \xrightarrow{\alpha_i} \cdots \rightarrow P_0 \xrightarrow{\alpha_0} M \rightarrow 0 \text{ (resp. } \mathbb{I} : 0 \rightarrow M \xrightarrow{\beta_0} I_0 \rightarrow \cdots \xrightarrow{\beta_i} I_i \rightarrow \cdots), \quad (5.2.2)$$

where P_k (resp. I_k) is projective for each $k \geq 0$. It is already known that M has a projective (resp. injective) resolution (see [5, Page 8]). Assume that $\mathbb{P}$ (resp. $\mathbb{I}$) in (5.2.2) is a projective (resp. injective) resolution of M . Then $\mathbb{P}$ (resp. $\mathbb{I}$) is called a minimal projective (resp. injective) resolution of M if no indecomposable $\mathbb{F}G$ -direct summand of P_k (resp. I_k) is contained in the kernel (resp. cokernel) of α_k (resp. β_k) for each $k \geq 0$. It is known that M has a unique minimal projective (resp. injective) resolution up to $\mathbb{F}G$ -isomorphism (see [5, 1.4.3]).

Assume further that $\mathbb{P}$ (resp. $\mathbb{I}$) is the minimal projective (resp. injective) resolution of M . If M is not projective, for each $\ell > 0$, let $\Omega^\ell(M)$ (resp. $\Omega^{-\ell}(M)$) be the kernel (resp. cokernel) of $\alpha_{\ell-1}$ (resp. $\beta_{\ell-1}$). Write $M = N \oplus P$, where each indecomposable $\mathbb{F}G$ -direct summand of N is not projective and P is projective. Set $\Omega^0(M) = N$. If M is projective, set $\Omega^\ell(M) = 0$ for all integer ℓ . We are now ready to present the following corollary.

Corollary 5.2.3. *Let M be an $\mathbb{F}\mathfrak{S}_n$ -module. If $m < p$, for any integer ℓ , we have*

$$\mathcal{F}_m(\Omega^\ell(M)) \cong \Omega^\ell(\mathcal{F}_m(M)) \oplus P$$

for some projective $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module P .

Proof. By Proposition 5.2.2, we may assume that M is not projective. The case $\ell = 0$ is clear by Lemma 5.2.1 and Proposition 5.2.2. When $\ell \neq 0$, $\mathcal{F}_m$ is exact. By Proposition 5.2.2, $\mathcal{F}_m$ takes the minimal projective (resp. injective) resolution of M to a projective (resp. injective) resolution of $\mathcal{F}_m(M)$. The corollary thus follows from Schanuel's lemma (see [5, Page 6]). $\square$

Let G be a finite group and $H \leq G$. Recall that the set of all pairwise distinct left cosets of H in G forms an $\mathbb{F}$ -basis of $(\mathbb{F}_H)^\uparrow^G$. We write $|G : H|$ to denote the index of H in G .

Lemma 5.2.4. *Let $H \leq \mathfrak{S}_n$. Then $\mathcal{F}_m((\mathbb{F}_H)^\uparrow^{\mathfrak{S}_n})$ is a permutation $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module.*

Proof. It is sufficient to check that $\mathcal{F}_m((\mathbb{F}_H)^\uparrow^{\mathfrak{S}_n})$ has a permutation basis under the action of $\mathfrak{S}_{n-m}^{+m}$. Let $s = |\mathfrak{S}_n : H|$ and $\mathcal{B} = \bigcup_{i=1}^s \{g_i H\}$ be the set of all pairwise distinct left cosets of H in $\mathfrak{S}_n$. For any $\sigma \in \mathfrak{S}_n$ and $g_i H \in \mathcal{B}$, there exists a unique $g_j H \in \mathcal{B}$ such that $\sigma g_i H = g_j H$. We denote g_j by $g_{\sigma,i}$ and view $\mathcal{B}$ as an $\mathbb{F}$ -basis of $(\mathbb{F}_H)^\uparrow^{\mathfrak{S}_n}$.

For any $1 \leq i \leq s$, let $\mathcal{O}(g_i)$ be the orbit of $\mathcal{B}$ containing $g_i H$ under the action of $\mathfrak{S}_m$ and $\mathcal{O}(g_i) = \{x_{i,1}g_i H, \dots, x_{i,t_i}g_i H\}$. So $\mathcal{B} = \biguplus_{j=1}^t \mathcal{O}(g_{i_j})$, where, for all $1 \leq j \leq t \leq s$, $t_{i_j} = |\mathcal{O}(g_{i_j})|$ and $x_{i_j,1}, \dots, x_{i_j,t_{i_j}} \in \mathfrak{S}_m$. For any $W \subseteq \mathfrak{S}_n$, let $W\mathcal{O}(g_i) = \{\eta x H : \eta \in W, x H \in \mathcal{O}(g_i)\}$. For any $\sigma \in \mathfrak{S}_n$, set $\sigma\mathcal{O}(g_i) = \{\sigma\}\mathcal{O}(g_i)$. We write

$$\overline{\mathcal{O}(g_i)} = \sum_{xH \in \mathcal{O}(g_i)} xH$$

and notice that $\overline{\mathcal{B}} = \bigcup_{j=1}^t \{\overline{\mathcal{O}(g_{i_j})}\}$ is an $\mathbb{F}$ -basis of $((\mathbb{F}_H)^\uparrow^{\mathfrak{S}_n})^{\mathfrak{S}_m}$. For any $\tau \in \mathfrak{S}_{n-m}^{+m}$ and $\mathcal{O}(g_{i_j})$, note that $\tau\mathcal{O}(g_{i_j}) = \tau\mathfrak{S}_m g_{i_j} H = \mathfrak{S}_m \tau g_{i_j} H = \mathfrak{S}_m g_{\tau, i_j} H = \mathcal{O}(g_{\tau, i_j}) = \mathcal{O}(g_{i_a})$ for some $a \in \mathbb{N}$ and $a \leq t$. We thus have

$$\tau\overline{\mathcal{O}(g_{i_j})} = \tau \sum_{k=1}^{t_{i_j}} x_{i_j,k} g_{i_j} H = \sum_{k=1}^{t_{i_j}} x_{i_j,k} \tau g_{i_j} H = \sum_{k=1}^{t_{i_j}} x_{i_j,k} g_{\tau, i_j} H = \overline{\mathcal{O}(g_{\tau, i_j})} = \overline{\mathcal{O}(g_{i_a})}.$$

So $\overline{\mathcal{B}}$ is an $\mathbb{F}$ -basis of $\mathcal{F}_m((\mathbb{F}_H)^\uparrow^{\mathfrak{S}_n})$ permuted by $\mathfrak{S}_{n-m}^{+m}$. The proof is now complete. $\square$

We close this section by a corollary. It will be used in the following discussion.

Corollary 5.2.5. *Let M be a p -permutation $\mathbb{F}\mathfrak{S}_n$ -module. If $\mathcal{F}_m(M) \neq 0$, then $\mathcal{F}_m(M)$ is a p -permutation $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module.*

Proof. We may view M as a direct summand of a direct sum of some transitive permutation $\mathbb{F}\mathfrak{S}_n$ -modules. By Lemmas 5.2.1 and 5.2.4, $\mathcal{F}_m(M)$ is a non-zero direct summand of a direct sum of some permutation $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules. So $\mathcal{F}_m(M)$ is indeed a p -permutation $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module by the definition. $\square$

5.3 The Specht Modules Labelled by Hook Partitions

In this section, we focus on proving Theorem 5.1.4. Throughout the whole section, fix $n > 1$ and integers m, r satisfying $1 \leq m, r < n$. Let $\lambda \vdash n - m$. We recall the definitions of $M_{m,n}^\lambda$ and $S_{m,n}^\lambda$ (see Page 23). We first present some explicit calculation results of $\mathcal{F}_m(S^{(n-r, 1^r)})$ and then finish the proof of Theorem 5.1.4.

Notation 5.3.1. For convenience, we introduce the required notation.

- (i) Let V_n be a n -dimensional $\mathbb{F}$ -linear space with an $\mathbb{F}$ -basis $\{e_1, \dots, e_n\}$. For any $1 \leq b \leq n$, set $V_b = \langle \{e_1, \dots, e_b\} \rangle_{\mathbb{F}}$. We write $f_i = e_i - e_n$ for all $1 \leq i \leq n$. Also observe that V_b is an $\mathbb{F}\mathfrak{S}_b$ -module as $\mathfrak{S}_b$ permutes the subscripts of members of $\{e_1, \dots, e_b\}$. So $V_n \cong M^{(n-1, 1)}$ and $S^{(n-1, 1)} \cong \langle \{f_1, \dots, f_{n-1}\} \rangle_{\mathbb{F}}$. Identify an $\mathbb{F}$ -basis of $S^{(n-1, 1)}$ with $\{f_1, \dots, f_{n-1}\}$. If $n - m \geq 2$, also identify an $\mathbb{F}$ -basis of $S_{m,n}^{(n-m-1, 1)}$ with $\{f_{m+1}, \dots, f_{n-1}\}$.
- (ii) Let $a, b, c \in \mathbb{N}$. Set $J(a, b, c) = \{\mathbf{i} = (i_1, \dots, i_b) \in \mathbb{N}^b : a \leq i_1 < \dots < i_b \leq c\}$ and $J^+(a, b, c) = \{\mathbf{i} = (i_1, \dots, i_b) \in \mathbb{N}^b : a \leq i_1 < \dots < i_b < c\}$. Write $J(a, b)$ and $J^+(a, b)$ to denote $J(a, b, n)$ and $J^+(a, b, n)$ respectively. Let $b \leq n$ and $\Lambda^c V_b$, $\Lambda^c S^{(n-1, 1)}$ be the c th exterior powers of V_b , $S^{(n-1, 1)}$ respectively. Recall that we put $\Lambda^0 S^{(n-1, 1)} = \mathbb{F}$ and $\Lambda^1 S^{(n-1, 1)} = S^{(n-1, 1)}$. By (2.9.3), if $c \leq b$, then $\Lambda^c V_b$ has an $\mathbb{F}$ -basis

$$\mathcal{B}_{V_b}^c = \{e_{\mathbf{i}} = e_{i_1} \wedge \dots \wedge e_{i_c} : \mathbf{i} = (i_1, \dots, i_c) \in J(1, c, b)\}.$$

By (2.9.3) again, if $c < n$, then $\Lambda^c S^{(n-1, 1)}$ has an $\mathbb{F}$ -basis

$$\mathcal{B}_S^c = \{f_{\mathbf{i}} = f_{i_1} \wedge \dots \wedge f_{i_c} : \mathbf{i} = (i_1, \dots, i_c) \in J^+(1, c)\}.$$

If $r + m - n < c < r$ and $n - m \geq 2$, then the $(r - c)$ th exterior power $\Lambda^{r-c} S_{m,n}^{(n-m-1, 1)}$ has an $\mathbb{F}$ -basis $\{f_{\mathbf{i}} : \mathbf{i} \in J^+(m+1, r-c)\}$. For any $v \in \Lambda^r S^{(n-1, 1)}$ and $f_{\mathbf{i}} \in \mathcal{B}_S^r$, let $d_{\mathbf{i}}^v$ be the coefficient of $f_{\mathbf{i}}$ in the $\mathbb{F}$ -linear combination of members of $\mathcal{B}_S^r$ representing v .

(iii) Let $\ell \in \mathbb{N}_0$ and $\ell \leq r$. We set

$$B_\ell = \{f_{\underline{i}} = f_{i_1} \wedge \cdots \wedge f_{i_r} \in \mathcal{B}_S^r : |\{i_1, \dots, i_r\} \cap \{1, \dots, m\}| = \ell\}.$$

Note that $B_u \cap B_v = \emptyset$ if $u \neq v$ and observe that $B_\ell \neq \emptyset$ if and only if

$$r + m - n + 1 \leq \ell \leq m. \quad (5.3.1)$$

For any $v \in (\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$ and $B_\ell \neq \emptyset$, put $\overline{B}_\ell^v = \sum_{w \in B_\ell} d_w^v w$, where $d_w^v = d_{\underline{i}}^v$ if $w = f_{\underline{i}}$. For completeness, let $\overline{B}_\ell^v = 0$ if $B_\ell = \emptyset$. Then $v = \sum_{i=0}^r \overline{B}_i^v$. Moreover, for any $g \in \mathfrak{S}_m$, $g\overline{B}_i^v = \overline{B}_i^{gv}$ for all $0 \leq i \leq r$ as $gv = v$.

(iv) Let $b, c \in \mathbb{N}$ and $b, c < n$. Set

$$v_{b,c} = \begin{cases} \sum_{\underline{i} \in J(1,c,b)} f_{\underline{i}}, & \text{if } c \leq b, \\ 0, & \text{if } b < c. \end{cases}$$

So $v_{b,c} \in \Lambda^c S^{(n-1,1)}$. If $c \leq b$, then $\langle \{gv_{b,c} : g \in \mathfrak{S}_b\} \rangle_{\mathbb{F}} \cong \langle \{g(\sum_{\underline{i} \in J(1,c,b)} e_{\underline{i}}) : g \in \mathfrak{S}_b\} \rangle_{\mathbb{F}}$ as $\mathbb{F}\mathfrak{S}_b$ -modules. Therefore, $\langle \{gv_{b,c} : g \in \mathfrak{S}_b\} \rangle_{\mathbb{F}}$ is contained in $\Lambda^c V_b$ up to $\mathbb{F}\mathfrak{S}_b$ -isomorphism.

We now present our calculation results.

Lemma 5.3.2. *We have*

$$\mathcal{F}_m(S^{(n-1,1)}) \cong \begin{cases} S_{m,n}^{(n-m-1,1)} \oplus \mathbb{F}, & \text{if } p \mid m \text{ and } 1 < n-m, \\ M_{m,n}^{(n-m-1,1)}, & \text{if } p \nmid m \text{ and } 1 < n-m, \\ \mathbb{F}, & \text{if } n-m=1. \end{cases}$$

Proof. When $n-m \geq 2$, note that

$$(S^{(n-1,1)})^{\mathfrak{S}_m} = \langle \{f_{m+1}, \dots, f_{n-1}\} \rangle_{\mathbb{F}} \oplus \langle \{f_1 + \cdots + f_m\} \rangle_{\mathbb{F}}.$$

If $p \mid m$, we get $\sum_{i=1}^m f_i = \sum_{i=1}^m e_i$ and

$$\begin{aligned} \mathcal{F}_m(S^{(n-1,1)}) &= \langle \{f_{m+1}, \dots, f_{n-1}\} \rangle_{\mathbb{F}} \oplus \langle \{e_1 + \cdots + e_m\} \rangle_{\mathbb{F}} \\ &\cong S_{m,n}^{(n-m-1,1)} \oplus \mathbb{F}. \end{aligned}$$

If $p \nmid m$, denote $\frac{1}{m} \sum_{i=1}^m e_i$ by x and we have

$$\mathcal{F}_m(S^{(n-1,1)}) = \langle \{x - e_{m+1}, \dots, x - e_n\} \rangle_{\mathbb{F}} \cong M_{m,n}^{(n-m-1,1)}.$$

When $n-m=1$, we get $\mathcal{F}_m(S^{(n-1,1)}) = \langle \{\sum_{i=1}^{n-1} f_i\} \rangle_{\mathbb{F}}$, which is a trivial $\mathbb{F}\mathfrak{S}_1^{+(n-1)}$ -module. The lemma now follows. $\square$

Lemma 5.3.3. *Let $p > 2$, $b, c \in \mathbb{N}$ and $c \leq b \leq n$. We have*

$$(\Lambda^c V_b)^{\mathfrak{S}_b} = \begin{cases} 0, & \text{if } c > 1, \\ \langle \{e_1 + \cdots + e_b\} \rangle_{\mathbb{F}}, & \text{if } c = 1. \end{cases}$$

Proof. For any $u \in (\Lambda^c V_b)^{\mathfrak{S}_b}$ and $u = \sum_{\underline{i} \in J(1,c,b)} k_{\underline{i}} e_{\underline{i}}$, we have $k_{\underline{i}} = k_{\underline{j}}$ for any $\underline{i}, \underline{j} \in J(1,c,b)$ by using suitable permutations to act on u and comparing the coefficients. Therefore, we have $(\Lambda^c V_b)^{\mathfrak{S}_b} \subseteq \langle \{\sum_{\underline{i} \in J(1,c,b)} e_{\underline{i}}\} \rangle_{\mathbb{F}}$. When $c > 1$, set $v = \sum_{\underline{i} \in J(1,c,b)} e_{\underline{i}}$ and get

$$v = w + \sum_{(1,2,i_3,\dots,i_c) \in J(1,c,b)} e_1 \wedge e_2 \wedge e_{i_3} \wedge \cdots \wedge e_{i_c}. \quad (5.3.2)$$

Note that no basis element of the form $e_1 \wedge e_2 \wedge e_{j_3} \wedge \cdots \wedge e_{j_c}$ of $\mathcal{B}_{V_b}^c$ is involved in w . Now use

the transposition $(1, 2)$ to act on both sides of (5.3.2) and obtain that

$$(1, 2)v = (1, 2)w - \sum_{(1, 2, i_3, \dots, i_c) \in J(1, c, b)} e_1 \wedge e_2 \wedge e_{i_3} \wedge \dots \wedge e_{i_c}.$$

So $(1, 2)v \neq v$ and $(\Lambda^c V_b)^{\mathfrak{S}_b} = 0$. When $c = 1$, it is clear that $V_b^{\mathfrak{S}_b} = \langle \{e_1 + \dots + e_b\} \rangle_{\mathbb{F}}$. The lemma follows. $\square$

Lemma 5.3.4. *Let $p > 2$. If $r > 1$, for any $v \in (\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$, we have*

$$v = \begin{cases} \overline{B_0^v} + \overline{B_1^v}, & \text{if } r < n - m, \\ \overline{B_1^v}, & \text{if } r = n - m, \\ 0, & \text{if } r > n - m, \end{cases}$$

where $\overline{B_1^v} = \sum_{\mathbf{i} \in J^+(m+1, r-1)} k_{\mathbf{i}} v_{m,1} \wedge f_{\mathbf{i}}$ and $k_{\mathbf{i}} \in \mathbb{F}$ for all $\mathbf{i} \in J^+(m+1, r-1)$.

Proof. We recall that $v = \sum_{i=0}^r \overline{B_i^v}$. Moreover, $\overline{B_i^v} \in (\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$ for all $0 \leq i \leq r$ as $v \in (\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$. We claim that $\overline{B_i^v} = 0$ if $1 < i \leq r$. If $B_i \neq \emptyset$, by the definition and an easy calculation, we get that

$$\overline{B_i^v} \in \begin{cases} \langle \{v_{m,i} \wedge w_i\} \rangle_{\mathbb{F}}, & \text{if } 0 < i < r, \\ \langle \{v_{m,r}\} \rangle_{\mathbb{F}}, & \text{if } i = r, \end{cases} \quad (5.3.3)$$

where $w_i \in \langle \{f_{\mathbf{i}} : \mathbf{i} \in J^+(m+1, r-i)\} \rangle_{\mathbb{F}}$.

Suppose that $\overline{B_u^v} \neq 0$ for some $1 < u \leq r$. We thus have $u \leq m$, $v_{m,u} \neq 0$, $w_u \neq 0$ if $u < r$. We also recall that $\langle \{gv_{m,u} : g \in \mathfrak{S}_m\} \rangle_{\mathbb{F}} \subseteq \Lambda^u V_m$ up to $\mathbb{F}\mathfrak{S}_m$ -isomorphism. By Lemma 5.3.3, we get that $v_{m,u}$ is not fixed by $\mathfrak{S}_m$ as $(\Lambda^u V_m)^{\mathfrak{S}_m} = 0$. In particular, $v_{m,u} \wedge w_u \notin (\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$ if $1 < u < r$ and $v_{m,r} \notin (\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$ if $u = r$. By (5.3.3), these facts imply that $\overline{B_u^v} = 0$, which is a contradiction. The claim is shown.

When $r < n - m$, by (5.3.1), note that $B_0 \neq \emptyset$ and $B_1 \neq \emptyset$. We therefore get $v = \overline{B_0^v} + \overline{B_1^v}$ by the claim. For the remaining cases, we can determine whether B_0 or B_1 is empty by the given conditions and (5.3.1). Therefore, we obtain the corresponding results by the claim. The proof is now complete. $\square$

Let $s \in \mathbb{N}_0$ and $s < n$. To continue our discussion, we recall that $S^{(n-s, 1^s)} \cong \Lambda^s S^{(n-1, 1)}$ (see [52, Proposition 2.3]).

Lemma 5.3.5. *Let $p > 2$. If $r > 1$, then we have*

$$\mathcal{F}_m(S^{(n-r, 1^r)}) \cong \begin{cases} M, & \text{if } p \mid m \text{ and } r < n - m, \\ N, & \text{if } p \nmid m \text{ and } r < n - m, \\ \text{sgn} \downarrow_{\mathfrak{S}_r^+}, & \text{if } r = n - m, \\ 0, & \text{if } r > n - m, \end{cases}$$

where $M \cong S_{m,n}^{(n-m-r, 1^r)} \oplus S_{m,n}^{(n-m-r+1, 1^{r-1})}$ and $N \sim S_{m,n}^{(n-m-r, 1^r)} + S_{m,n}^{(n-m-r+1, 1^{r-1})}$.

Proof. It is sufficient to determine $\mathcal{F}_m(\Lambda^r S^{(n-1, 1)})$. When $r < n - m$, define

$$\mathcal{C}_0 = \{f_{\mathbf{i}} : \mathbf{i} \in J^+(m+1, r)\} \text{ and } \mathcal{C}_1 = \{v_{m,1} \wedge f_{\mathbf{j}} : \mathbf{j} \in J^+(m+1, r-1)\}.$$

Observe that the members of $\mathcal{C}_0 \cup \mathcal{C}_1$ are all fixed by $\mathfrak{S}_m$. Moreover, $\mathcal{C}_0 \cup \mathcal{C}_1$ is linearly independent over $\mathbb{F}$. By Lemma 5.3.4, for any $v \in (\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$, we have $v = \overline{B_0^v} + \overline{B_1^v}$. Moreover, $\overline{B_0^v} \in \langle \mathcal{C}_0 \rangle_{\mathbb{F}}$ and $\overline{B_1^v} = \sum_{\mathbf{i} \in J^+(m+1, r-1)} k_{\mathbf{i}} v_{m,1} \wedge f_{\mathbf{i}}$, where $k_{\mathbf{i}} \in \mathbb{F}$ for all $\mathbf{i} \in J^+(m+1, r-1)$. We thus conclude that $\mathcal{C}_0 \cup \mathcal{C}_1$ is an $\mathbb{F}$ -basis of $(\Lambda^r S^{(n-1, 1)})^{\mathfrak{S}_m}$ by these facts. If $p \mid m$, note that $\mathcal{C}_1$ equals

$$\{(\sum_{i=1}^m e_i) \wedge f_{\mathbf{j}} : \mathbf{j} \in J^+(m+1, r-1)\}.$$

Therefore, we obtain that $\mathcal{F}_m(\Lambda^r S^{(n-1,1)}) = \langle \mathcal{C}_0 \rangle_{\mathbb{F}} \oplus \langle \mathcal{C}_1 \rangle_{\mathbb{F}}$, where $\langle \mathcal{C}_0 \rangle_{\mathbb{F}} \cong S_{m,n}^{(n-m-r,1^r)}$ and $\langle \mathcal{C}_1 \rangle_{\mathbb{F}} \cong S_{m,n}^{(n-m-r+1,1^{r-1})}$ as $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules. If $p \nmid m$, we set $S = \mathcal{F}_m(\Lambda^r S^{(n-1,1)})$ and $P = \langle \mathcal{C}_0 \rangle_{\mathbb{F}}$. Notice that $P \cong S_{m,n}^{(n-m-r,1^r)}$ as $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules. To obtain the desired result, we need to show that $S/P \cong S_{m,n}^{(n-m-r+1,1^{r-1})}$ as $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules. For any $v \in S$, let $\bar{v}$ be the image of v under the natural map from S to S/P . Define an $\mathbb{F}$ -linear map ϕ from S/P to $\Lambda^{r-1} S_{m,n}^{(n-m-1,1)}$ by sending each $\overline{v_{m,1} \wedge f_{\underline{j}}}$ to $f_{\underline{j}}$. It is obvious to see that ϕ is an $\mathbb{F}$ -linear isomorphism. To show that ϕ is an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -isomorphism, it is enough to check that the action of transposition $(n-1, n)$ is preserved by ϕ . Set $\sigma = (n-1, n)$ and $s = \sum_{i=1}^m e_i$. Note that $v_{m,1} = s - me_n$ by the definition. For any $v_{m,1} \wedge f_{\underline{j}} \in \mathcal{C}_1$ where $\underline{j} = (j_1, \dots, j_{r-1})$, we distinguish two cases.

Case 1: $j_{r-1} < n-1$.

We have

$$\begin{aligned} & \phi(\sigma(\overline{v_{m,1} \wedge f_{j_1} \wedge \dots \wedge f_{j_{r-1}}})) \\ &= \phi(\overline{(s - me_{n-1}) \wedge (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-1}} - e_{n-1})}) \\ &= \phi(\overline{(s - me_n + me_n - me_{n-1}) \wedge (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-1}} - e_{n-1})}) \\ &= \phi(\overline{(s - me_n) \wedge (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-1}} - e_{n-1})}) \\ &= (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-1}} - e_{n-1}) \\ &= \sigma(f_{j_1} \wedge \dots \wedge f_{j_{r-1}}) \\ &= \sigma\phi(\overline{v_{m,1} \wedge f_{j_1} \wedge \dots \wedge f_{j_{r-1}}}). \end{aligned}$$

Case 2: $j_{r-1} = n-1$.

We have

$$\begin{aligned} & \phi(\sigma(\overline{v_{m,1} \wedge f_{j_1} \wedge \dots \wedge f_{j_{r-2}} \wedge f_{n-1}})) \\ &= \phi(\overline{(s - me_{n-1}) \wedge (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-2}} - e_{n-1}) \wedge (e_n - e_{n-1})}) \\ &= \phi(\overline{(s - me_n + me_n - me_{n-1}) \wedge (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-2}} - e_{n-1}) \wedge (e_n - e_{n-1})}) \\ &= \phi(\overline{(s - me_n) \wedge (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-2}} - e_{n-1}) \wedge (e_n - e_{n-1})}) \\ &= (e_{j_1} - e_{n-1}) \wedge \dots \wedge (e_{j_{r-2}} - e_{n-1}) \wedge (e_n - e_{n-1}) \\ &= -f_{j_1} \wedge \dots \wedge f_{j_{r-2}} \wedge f_{n-1} \\ &= \sigma(f_{j_1} \wedge \dots \wedge f_{j_{r-2}} \wedge f_{n-1}) \\ &= \sigma\phi(\overline{v_{m,1} \wedge f_{j_1} \wedge \dots \wedge f_{j_{r-2}} \wedge f_{n-1}}). \end{aligned}$$

So ϕ is an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -isomorphism. We thus get that $S \sim S_{m,n}^{(n-m-r,1^r)} + S_{m,n}^{(n-m-r+1,1^{r-1})}$ as $\Lambda^{r-1} S_{m,n}^{(n-m-1,1)} \cong S_{m,n}^{(n-m-r+1,1^{r-1})}$.

When $r = n - m$, by Lemma 5.3.4 and a similar discussion as above, we obtain

$$(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m} = \langle \{v_{m,1} \wedge f_{m+1} \wedge \dots \wedge f_{n-1}\} \rangle_{\mathbb{F}} \text{ and } \mathcal{F}_m(\Lambda^r S^{(n-1,1)}) \cong \text{sgn} \downarrow_{\mathfrak{S}_r^{+m}}.$$

When $r > n - m$, by Lemma 5.3.4, $\mathcal{F}_m(\Lambda^r S^{(n-1,1)}) = 0$ as $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m} = 0$. The proof is now complete. $\square$

We now present some results for the case $p = 2$.

Lemma 5.3.6. *Let $p = 2$, $a, b \in \mathbb{N}$ and $a \leq b < n$. For any $s \in \{b+1, \dots, n\}$, we have*

$$\sum_{(i_1, \dots, i_a) \in J(1, a, b)} (e_{i_1} + e_s) \wedge \dots \wedge (e_{i_a} + e_s) = \begin{cases} v_{b,a} + bf_s, & \text{if } a = 1, \\ v_{b,a} + (b-a+1)v_{b,a-1} \wedge f_s, & \text{if } a > 1. \end{cases}$$

Proof. The case $a = 1$ is trivial by a direct calculation. We now consider the case $a > 1$. For any $(i_1, \dots, i_\ell, \dots, i_a) \in J(1, a, b)$, we shall write $f_{i_1} \wedge \dots \wedge \widehat{f_{i_\ell}} \wedge \dots \wedge f_{i_a}$ to denote the corresponding vector of $\Lambda^{a-1} S^{(n-1,1)}$ without the component f_{i_ℓ} . Since $p = 2$, we have

$$\begin{aligned}
& \sum_{(i_1, \dots, i_a) \in J(1, a, b)} (e_{i_1} + e_s) \wedge \dots \wedge (e_{i_a} + e_s) \\
&= \sum_{(i_1, \dots, i_a) \in J(1, a, b)} (f_{i_1} + f_s) \wedge \dots \wedge (f_{i_a} + f_s) \\
&= v_{b,a} + f_s \wedge \left(\sum_{(i_1, \dots, i_a) \in J(1, a, b)} \sum_{c=1}^a f_{i_1} \wedge \dots \wedge \widehat{f_{i_c}} \wedge \dots \wedge f_{i_a} \right) \\
&= v_{b,a} + (b-a+1)f_s \wedge \left(\sum_{\mathbf{j} \in J(1, a-1, b)} f_{\mathbf{j}} \right) \\
&= v_{b,a} + (b-a+1)v_{b,a-1} \wedge f_s,
\end{aligned}$$

as desired. $\square$

Lemma 5.3.7. *Let $p = 2$ and $s = r + m - n + 1$. If $r > 1$, for any $v \in (\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$, we have*

$$v = \begin{cases} \sum_{i=0}^r \overline{B_i^v}, & \text{if } r < n-m \text{ and } r \leq m, \\ \sum_{i=0}^m \overline{B_i^v}, & \text{if } m < r < n-m, \\ \sum_{i=s}^r \overline{B_i^v}, & \text{if } n-m \leq r \leq m, \\ \sum_{i=s}^m \overline{B_i^v}, & \text{if } n-m \leq r \text{ and } r > m, \end{cases}$$

where, for all $0 < i < r$, $\overline{B_i^v} = \sum_{\mathbf{i} \in J^+(m+1, r-i)} k_{\mathbf{i}} v_{m,i} \wedge f_{\mathbf{i}}$ and $k_{\mathbf{i}} \in \mathbb{F}$ for all $\mathbf{i} \in J^+(m+1, r-i)$. Furthermore, $\overline{B_r^v} \in \langle \{v_{m,r}\} \rangle_{\mathbb{F}}$.

Proof. Recall that $v = \sum_{i=0}^r \overline{B_i^v}$. As $v \in (\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$, we have $\overline{B_i^v} \in (\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$ for all $0 \leq i \leq r$. Therefore, if $B_i \neq \emptyset$ and $0 < i \leq r$, direct computation shows that (5.3.3) holds for $\overline{B_i^v}$. In particular, if $B_i \neq \emptyset$ and $0 < i \leq r$, then $\overline{B_i^v}$ has the desired form. Therefore, it suffices to show that v is the desired $\mathbb{F}$ -linear combination of $\{\overline{B_i^v} : 0 \leq i \leq r\}$ for every given condition.

When $m < r$, by (5.3.1), note that $B_i = \emptyset$ for all $m < i \leq r$. When $n-m \leq r$, notice that $0 < s$. Therefore, by (5.3.1) again, $B_i = \emptyset$ for all $0 \leq i < s$. We thus can observe that v is the desired $\mathbb{F}$ -linear combination of $\{\overline{B_i^v} : 0 \leq i \leq r\}$ for every given condition. The lemma follows. $\square$

Lemma 5.3.8. *Let $p = 2$ and $s = r + m - n + 1$. If $r > 1$, then we have*

$$\mathcal{F}_m(S^{(n-r, 1^r)}) \sim \begin{cases} \sum_{i=0}^r S_{m,n}^{(n-m-r+i, 1^{r-i})}, & \text{if } r < n-m \text{ and } r \leq m, \\ \sum_{i=0}^m S_{m,n}^{(n-m-r+i, 1^{r-i})}, & \text{if } m < r < n-m, \\ \sum_{i=s}^r S_{m,n}^{(n-m-r+i, 1^{r-i})}, & \text{if } n-m \leq r \leq m, \\ \sum_{i=s}^m S_{m,n}^{(n-m-r+i, 1^{r-i})}, & \text{if } n-m \leq r \text{ and } r > m. \end{cases}$$

Proof. It is sufficient to determine $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. Let $\ell \in \mathbb{N}$ and $\ell < r$. When $J^+(m+1, r)$, $J^+(m+1, r-\ell) \neq \emptyset$ and $\ell \leq m$, define

$$\mathcal{C}_0 = \{f_{\mathbf{i}} : \mathbf{i} \in J^+(m+1, r)\} \text{ and } \mathcal{C}_\ell = \{v_{m,\ell} \wedge f_{\mathbf{j}} : \mathbf{j} \in J^+(m+1, r-\ell)\}.$$

Set $\mathcal{C}_\ell = \emptyset$ if $J^+(m+1, r-\ell) = \emptyset$ or $m < \ell$ and put $\mathcal{C}_{i,j} = \biguplus_{k=i}^j \mathcal{C}_k$ for all $1 \leq i \leq j < r$.

When $r < n-m$ and $r \leq m$, write $\mathcal{C}$ for $\{v_{m,r}\} \cup \mathcal{C}_0 \cup \mathcal{C}_{1,r-1}$. By the definition of $\mathcal{C}$, notice that $\mathcal{C}$ is an $\mathbb{F}$ -linearly independent subset of $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$. For any $v \in (\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$, by Lemma 5.3.7, $v = \sum_{a=0}^r \overline{B_a^v}$, $\overline{B_0^v} \in \langle \mathcal{C}_0 \rangle_{\mathbb{F}}$, $\overline{B_r^v} \in \langle \{v_{m,r}\} \rangle_{\mathbb{F}}$, $\overline{B_a^v} = \sum_{\mathbf{i} \in J^+(m+1, r-a)} k_{\mathbf{i}} v_{m,a} \wedge f_{\mathbf{i}}$, where $0 < a < r$ and $k_{\mathbf{i}} \in \mathbb{F}$ for all $\mathbf{i} \in J^+(m+1, r-a)$. By these facts, we deduce that $\mathcal{C}$ is an $\mathbb{F}$ -basis of $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$.

We shall construct the desired Specht filtration for $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. Let $U_0 = \langle \mathcal{C}_0 \rangle_{\mathbb{F}}$ and $U_i = \langle \mathcal{C}_0 \cup \mathcal{C}_{1,i} \rangle_{\mathbb{F}}$ for all $1 \leq i < r$. Set $U_r = \mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. We have a chain of subspaces of $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$ given by

$$0 \subset U_0 \subset U_1 \subset \cdots \subset U_r.$$

By the definition of U_0 , note that $U_0 \cong S_{m,n}^{(n-m-r,1^r)}$ as $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules. We claim that U_i is an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -module for all $1 \leq i < r$. According to the definition of U_i , it is enough to check that every member of $\mathcal{C}_{1,i}$ is still in U_i under the action of the transposition $(n-1, n)$. Write $\sigma = (n-1, n)$. For any $v_{m,\ell} \wedge f_{\underline{j}} \in \mathcal{C}_{1,i}$, note that $1 \leq \ell \leq i < r \leq m$. If $\ell > 1$, we have

$$\sigma(v_{m,\ell} \wedge f_{\underline{j}}) = \sigma v_{m,\ell} \wedge \sigma f_{\underline{j}} = (v_{m,\ell} + (m-\ell+1)v_{m,\ell-1} \wedge f_{n-1}) \wedge \sigma f_{\underline{j}} \in U_i, \quad (5.3.4)$$

where the last equality is from Lemma 5.3.6. Similarly, if $\ell = 1$, by Lemma 5.3.6 again, we also have

$$\sigma(v_{m,1} \wedge f_{\underline{j}}) = \sigma v_{m,1} \wedge \sigma f_{\underline{j}} = (v_{m,1} + m f_{n-1}) \wedge \sigma f_{\underline{j}} \in U_i. \quad (5.3.5)$$

The claim is shown.

Finally, we will prove that $U_i/U_{i-1} \cong \Lambda^{r-i} S_{m,n}^{(n-m-1,1)}$ as $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules for all $1 \leq i \leq r$. For any $v \in U_i$, write $\bar{v}$ to denote the image of v under the natural map from U_i to U_i/U_{i-1} . When $1 \leq i < r$, define an $\mathbb{F}$ -linear map ϕ_i from U_i/U_{i-1} to $\Lambda^{r-i} S_{m,n}^{(n-m-1,1)}$ by sending each $\overline{v_{m,i} \wedge f_{\underline{j}}}$ to $\bar{f}_{\underline{j}}$. It is easy to observe that ϕ_i is an $\mathbb{F}$ -linear isomorphism. To show that ϕ_i is an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -isomorphism, we only need to check that the action of σ is preserved by ϕ_i . If $i = 1$, for any $v_{m,1} \wedge f_{\underline{j}} \in \mathcal{C}_1$, we have

$$\begin{aligned} \phi_1(\sigma(\overline{v_{m,1} \wedge f_{\underline{j}}})) &= \phi_1(\overline{(v_{m,1} + m f_{n-1}) \wedge \sigma f_{\underline{j}}}) \quad (\text{by (5.3.5)}) \\ &= \phi_1(\overline{v_{m,1} \wedge \sigma f_{\underline{j}}}) \\ &= \sigma f_{\underline{j}} = \sigma \phi_1(\overline{v_{m,1} \wedge f_{\underline{j}}}). \end{aligned}$$

If $1 < i < r$, for any $v_{m,i} \wedge f_{\underline{j}} \in \mathcal{C}_i$, we also have

$$\begin{aligned} \phi_i(\sigma(\overline{v_{m,i} \wedge f_{\underline{j}}})) &= \phi_i(\overline{(v_{m,i} + (m-i+1)v_{m,i-1} \wedge f_{n-1}) \wedge \sigma f_{\underline{j}}}) \quad (\text{by (5.3.4)}) \\ &= \phi_i(\overline{v_{m,i} \wedge \sigma f_{\underline{j}} + (m-i+1)v_{m,i-1} \wedge \sigma f_{\underline{j}} \wedge f_{n-1}}) \quad (\text{since } p = 2) \\ &= \phi_i(\overline{v_{m,i} \wedge \sigma f_{\underline{j}}}) \\ &= \sigma f_{\underline{j}} = \sigma \phi_i(\overline{v_{m,i} \wedge f_{\underline{j}}}). \end{aligned}$$

Therefore, ϕ_i is an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -isomorphism for all $1 \leq i < r$. We define an $\mathbb{F}$ -linear map ϕ_r from U_r/U_{r-1} to $\mathbb{F}$ by sending $\overline{v_{m,r}}$ to a generator of $\mathbb{F}$. By Lemma 5.3.6, it is obvious to see that ϕ_r is an $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -isomorphism. We now have shown that $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$ has the desired Specht filtration as $\Lambda^{r-i} S_{m,n}^{(n-m-1,1)} \cong S_{m,n}^{(n-m-r+i,1^{r-i})}$ for all $1 \leq i \leq r$.

The proofs of all the other cases are similar to the one for $r < n-m$ and $r \leq m$. We only list corresponding assertions for each case. One may follow the above proof to justify them.

When $m < r < n-m$, $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$ has an $\mathbb{F}$ -basis $\mathcal{C}_0 \cup \mathcal{C}_{1,m}$. We set $U_0 = \langle \mathcal{C}_0 \rangle_{\mathbb{F}}$, $U_i = \langle \mathcal{C}_0 \cup \mathcal{C}_{1,i} \rangle_{\mathbb{F}}$ for all $1 \leq i < m$ and $U_m = \mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. Then the chain of these $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules

$$0 \subset U_0 \subset U_1 \subset \cdots \subset U_m$$

gives the desired Specht filtration to $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$.

When $n-m \leq r \leq m < n-1$, $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$ has an $\mathbb{F}$ -basis $\{v_{m,r}\} \cup \mathcal{C}_{s,r-1}$. Let $U_i = \langle \mathcal{C}_{s,i} \rangle_{\mathbb{F}}$ for all $s \leq i < r$ and $U_r = \mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. Then the chain of these $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules

$$0 \subset U_s \subset U_{s+1} \subset \cdots \subset U_r$$

provides the desired Specht filtration for $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. When $m = n-1$, $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$ has an $\mathbb{F}$ -basis $\{v_{m,r}\}$. So $\mathcal{F}_m(\Lambda^r S^{(n-1,1)}) \cong \mathbb{F}$.

When $n - m \leq r$ and $r > m$, $(\Lambda^r S^{(n-1,1)})^{\mathfrak{S}_m}$ has an $\mathbb{F}$ -basis $\mathcal{C}_{s,m}$. Set $U_i = \langle \mathcal{C}_{s,i} \rangle_{\mathbb{F}}$ for all $s \leq i < m$ and $U_m = \mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. Then the chain of these $\mathbb{F}\mathfrak{S}_{n-m}^{+m}$ -modules

$$0 \subset U_s \subset U_{s+1} \subset \cdots \subset U_m$$

induces the desired Specht filtration of $\mathcal{F}_m(\Lambda^r S^{(n-1,1)})$. The proof is now complete. $\square$

A direct corollary of Lemmas 5.3.2, 5.3.5 and 5.3.8 is given as follows.

Corollary 5.3.9. *Let a and b be integers such that $0 \leq a < n$ and $1 \leq b \leq n$. Then the following assertions hold.*

- (i) *If $\mathcal{F}_b(S^{(n-a,1^a)}) \neq 0$, then there exists a Specht filtration of $\mathcal{F}_b(S^{(n-a,1^a)})$ such that all the successive quotient factors of the filtration are labelled by the hook partitions of $n - b$.*
- (ii) *Let M be an $\mathbb{F}\mathfrak{S}_n$ -module with a Specht filtration and $\mathcal{F}_b(M) \neq 0$. If $b < p$ and all the successive quotient factors of the filtration are labelled by hook partitions of n , then $\mathcal{F}_b(M)$ has a Specht filtration with all successive quotient factors labelled by hook partitions of $n - b$.*

Proof. Note that $\mathcal{F}_n$ takes an $\mathbb{F}\mathfrak{S}_n$ -module to zero or a direct sum of trivial modules. Also notice that $\mathcal{F}_b(S^{(n)})$ is a trivial module. (i) is clear by Lemmas 5.3.2, 5.3.5 and 5.3.8.

For (ii), as $\mathcal{F}_b$ is now exact, we get the desired result by (i) and induction on the number of quotient factors of the filtration. $\square$

One more result is needed to prove Theorem 5.1.4. It is a part of [47, Theorem 4.5]. To state it and finish the proof of Theorem 5.1.4, we need another notation. Let $a \in \mathbb{N}$ and $ap \leq n$. Recall that $s_i = ((i-1)p+1, \dots, ip)$ (see Notation 4.2.1 (v)). Let $E_{\geq a}$ denote the elementary abelian p -subgroup $\langle \bigcup_{i=a}^b \{s_i\} \rangle$ of $\mathfrak{S}_n$, where $b = \lfloor \frac{n}{p} \rfloor$.

Lemma 5.3.10. *Let $p \mid n$ and a be the remainder of r when r is divided by p . If $2 \nmid a$, then the stable generic Jordan type of $S^{(n-r,1^r)} \downarrow_{E_{\geq 1}}$ is $[p-1]^\beta$, where $\beta \in \mathbb{N}$.*

We now begin to finish the proof of Theorem 5.1.4.

Proposition 5.3.11. *Let $n > p > 2$. Then $S^{(n-r,1^r)}$ is a trivial source module if and only if the partition $(n-r, 1^r) \in JM(n)$.*

Proof. One direction is clear by Theorem 5.1.2. For the other direction, since it is well-known that $S^{(n-r,1^r)}$ is simple if $p \nmid n$ (see [35, 24.1]), we may assume that $p \mid n$. Let a be the remainder of r when r is divided by p . By Lemmas 5.3.10 and 2.8.2, we may assume further that $r > 1$ and $2 \mid a$. We consider the following cases.

Case 1: $a = 0$.

When $n - r \geq 2p$, by Lemma 5.3.5, we have

$$\begin{aligned} \mathcal{F}_p(S^{(n-r,1^r)}) &\cong S_{p,n}^{(n-p-r,1^r)} \oplus S_{p,n}^{(n-p-r+1,1^{r-1})}, \\ \mathcal{F}_p(S_{p,n}^{(n-p-r+1,1^{r-1})}) &\cong S_{2p,n}^{(n-2p-r+1,1^{r-1})} \oplus S_{2p,n}^{(n-2p-r+2,1^{r-2})}. \end{aligned}$$

Using Lemma 5.3.10, we get that the stable generic Jordan type of $S_{2p,n}^{(n-2p-r+2,1^{r-2})} \downarrow_{E_{\geq 3}}$ is $[p-1]^e$, where $e \in \mathbb{N}$. Therefore, $S_{2p,n}^{(n-2p-r+2,1^{r-2})}$ is not a trivial source module by Lemma 2.8.2. By Corollary 5.2.5, the fact implies that $S^{(n-r,1^r)}$ is not a trivial source module.

When $n - r < 2p$, note that $n - r = p$ since $p \mid n - r$ and $n > r$. By Lemma 2.11.2 (iii), we know that $S^{(r+1,1^{p-1})}$ is a trivial source module if and only if $S^{(p,1^r)}$ is a trivial source module. Notice that $r > p - 1$ as $r > 1$ and $a = 0$. We now apply $\mathcal{F}_p$ to $S^{(r+1,1^{p-1})}$ and obtain that

$$\mathcal{F}_p(S^{(r+1,1^{p-1})}) \cong S_{p,p+r}^{(r-p+1,1^{p-1})} \oplus S_{p,p+r}^{(r-p+2,1^{p-2})}$$

by Lemma 5.3.5. By Lemma 5.3.10, the stable generic Jordan type of $S_{p,p+r}^{(r-p+2,1^{p-2})} \downarrow_{E_{\geq 2}}$ is $[p-1]^c$, where $c \in \mathbb{N}$. Therefore, by the same reason mentioned in the case $n - r \geq 2p$ and Lemma 2.11.2 (iii), we deduce that $S^{(p,1^r)}$ is not a trivial source module.

Case 2: $0 < a < p$.

When $n - r > p$, by Lemma 5.3.5, we have

$$\mathcal{F}_p(S^{(n-r, 1^r)}) \cong S_{p,n}^{(n-p-r, 1^r)} \oplus S_{p,n}^{(n-p-r+1, 1^{r-1})}.$$

By Lemma 5.3.10 again, the stable generic Jordan type of $S_{p,n}^{(n-p-r+1, 1^{r-1})} \downarrow_{E_{\geq 2}}$ is $[p-1]^d$, where $d \in \mathbb{N}$. So $S_{p,n}^{(n-p-r+1, 1^{r-1})}$ is not a trivial source module by Lemma 2.8.2. We thus deduce that $S^{(n-r, 1^r)}$ is not a trivial source module by Corollary 5.2.5. For the remaining case, note that $n - r \neq p$ as $p \mid n$ and $0 < a < p$.

When $n - r < p$, if $a = p - 1$, we have $n - r = 1$ as $p \mid n$ and $n - r < p$. So $S^{(n-r, 1^r)}$ is in fact the simple module $S^{(1^n)}$. We thus assume that $a < p - 1$ in the following discussion. By Lemma 2.11.2 (iii), we know that $S^{(r+1, 1^{n-r-1})}$ is a trivial source module if and only if $S^{(n-r, 1^r)}$ is a trivial source module. Note that $r > p$. Otherwise, we get $2p \leq n - r + r < p + p = 2p$, which is absurd. Moreover, observe that $n - r - 1 \equiv p - 1 - a \pmod{p}$, $2 \mid p - 1 - a$ and $p - 1 - a > 0$. One may write a similar proof as the one given in the case $n - r > p$ to deduce that $S^{(r+1, 1^{n-r-1})}$ is not a trivial source module. Using Lemma 2.11.2 (iii) again, we get that $S^{(n-r, 1^r)}$ is not a trivial source module.

By the analysis of above two cases, we have shown that $S^{(n-r, 1^r)}$ is not a trivial source module if $n > p > 2$, $p \mid n$ and $n - r > 1$. Therefore, if $n > p > 2$, $S^{(n-r, 1^r)}$ is a trivial source module only if it is simple. The proof is now complete. $\square$

Remark 5.3.12. Let $p > 2$. If $n < p$, all the Specht modules are trivial source modules. Moreover, they are all simple. If $n = p$, let $a \in \mathbb{N}_0$ and $a < p$. It is known that $S^{(p-a, 1^a)}$ is a trivial source module if and only if $2 \mid a$ (see [19]). Note that these latter trivial source Specht modules are non-simple except for $S^{(p)}$ and $S^{(1^p)}$.

We now deal with the case $p = 2$. In this case, all the indecomposable Specht modules labelled by hook partitions of n were classified by Murphy in [53]. Let us state her result as follows. If $2 \nmid n$ and $n \geq 2r$, $S^{(n-r, 1^r)}$ is indecomposable if and only if $n \equiv 2r + 1 \pmod{2^L}$, where $L \in \mathbb{N}$ and $2^{L-1} \leq r < 2^L$. If $2 \nmid n$ and $2r > n$, $S^{(n-r, 1^r)}$ is indecomposable if and only if $n \equiv 2r + 1 \pmod{2^{L'}}$, where $L' \in \mathbb{N}$ and $2^{L'-1} \leq n - r - 1 < 2^{L'}$. If $2 \mid n$, all Specht modules labelled by hook partitions of n are indecomposable. We need a lemma to conclude this case.

Lemma 5.3.13. *Let $p = 2$ and $2 \nmid n$. If $S^{(n-r, 1^r)}$ is indecomposable, then we have*

- (i) $S^{(n-r, 1^r)}$ is a trivial source module.
- (ii) $S^{(n-r, 1^r)}$ is non-simple if and only if $r > 1$ and $r \neq n - 2$, $n - 1$.

Proof. Recall that $r \geq 1$. We now need to work on $\Lambda^r S^{(n-1, 1)}$. As $2 \nmid n$, it is known that $\Lambda^r S^{(n-1, 1)} \mid \Lambda^r M^{(n-1, 1)}$. (i) thus follows as $\Lambda^r M^{(n-1, 1)}$ is a permutation module. (ii) is a part of [35, 23.7]. $\square$

Remark 5.3.14. Let $p = 2$. If $2 \nmid n$, all the non-simple trivial source Specht modules labelled by hook partitions of n are clear by Lemma 5.3.13 and the Murphy's result. If $2 \mid n$, let P be a Sylow 2-subgroup of $\mathfrak{S}_n$. By [54, Corollary 4.4 and Theorem 4.5], note that P is a vertex of $S^{(n-r, 1^r)}$ and $S^{(n-r, 1^r)} \downarrow_P$ is a P -source of $S^{(n-r, 1^r)}$. In particular, only the trivial module is a trivial source Specht module in this case.

It is trivial to see that $S^{(1)}$ is a trivial source module. Theorem 5.1.4 is now proved by combining Theorem 5.1.2, Proposition 5.3.11 and Remarks 5.3.12, 5.3.14.

5.4 The Specht Modules Labelled by Two-Part Partitions

The proof of Theorem 5.1.5 will be presented in this section. To finish the proof, we first work on the Specht modules labelled by partitions with at most two columns and then translate the obtained results to the Specht modules labelled by two-part partitions by Lemma 2.11.2 (iii).

For any $m \in \mathbb{N}_0$, write $\ell_p(m)$ to denote the smallest non-negative integer a such that $m < p^a$. We need some preliminary results.

Lemma 5.4.1. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. We have*

- (i) $\text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, S^\lambda) \neq 0$ *if and only if* $\lambda_i \equiv -1 \pmod{p^{\ell_p(\lambda_{i+1})}}$ *for all* $1 \leq i < \ell$.
- (ii) $\text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, S_\lambda) \neq 0$ *implies that* $\lambda = (n)$ *if* $p > 2$.

Proof. (i) is from [35, 24.4]. (ii) is a direct deduction of [35, 13.17]. $\square$

Let $\lambda \vdash n$. We will call λ a James partition if $\text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, S^\lambda) \neq 0$.

Lemma 5.4.2. *Let $p > 2$ and λ be a James partition of n . Then S^λ is a trivial source module if and only if $\lambda = (n)$.*

Proof. One direction is clear. For the other direction, let P be a vertex of S^λ and suppose that $\lambda \neq (n)$. As S^λ is a trivial source module, we have $S^\lambda \mid (\mathbb{F}_P)^\uparrow \mathfrak{S}_n$. Note that Lemma 5.4.1 (ii) implies that $S^\lambda \not\cong \text{Sc}_{\mathfrak{S}_n}(P)$. However, since λ is a James partition, we come to a contradiction as we get two trivial submodules for the transitive permutation module $(\mathbb{F}_P)^\uparrow \mathfrak{S}_n$. So λ has to be (n) if S^λ is a trivial source module. The lemma is shown. $\square$

Let a and b be non-negative integers such that $b < p - 1$ and $n = a(p - 1) + b$. If $p > 2$, let $\epsilon = ((a + 1)^b, a^{p-1-b})$. It is known that S^ϵ has the simple head sgn (see [35, 24.5 (iii)]). We now prove a result which may have its own interest.

Proposition 5.4.3. *Let $p > 2$ and $n = a(p - 1) + b$, where $a, b \in \mathbb{N}_0$ and $b < p - 1$. Then $S^{\overline{((a+1)^b, a^{p-1-b})}}$ is a trivial source module if and only if $n < p$.*

Proof. One direction is trivial. To prove the other direction, write ϵ to denote $\overline{((a + 1)^b, a^{p-1-b})}$. If S^ϵ is a trivial source module, then we have the following deductions:

- (a) Note that $S_{\epsilon'}$ is also a trivial source module by Lemma 2.11.2 (ii);
- (b) Note that $S_{\epsilon'}$ has a trivial simple head as S^ϵ has the simple head sgn .

Let P be a vertex of $S_{\epsilon'}$. (a) and (b) imply that $S_{\epsilon'} \mid (\mathbb{F}_P)^\uparrow \mathfrak{S}_n$ and $S_{\epsilon'} \cong \text{Sc}_{\mathfrak{S}_n}(P)$. Therefore, $S_{\epsilon'}$ has a trivial submodule by the definition of $\text{Sc}_{\mathfrak{S}_n}(P)$. Using Lemma 5.4.1 (ii), we deduce that $\epsilon = (1^n)$. So we have $ab = 0$ by the definition of ϵ . If $a = 0$, then $n = b < p - 1 < p$. If $b = 0$, we have $a = 1$ and $n = p - 1 < p$ by the definition of ϵ . The proof is now complete. $\square$

For any $m \in \mathbb{N}$, define $\nu_p(m)$ to be the largest non-negative integer a such that $p^a \mid m$. In particular, $\nu_p(m) \geq 0$. By convention, put $\nu_p(0) = \infty$. The following theorem is known as the Carter criterion.

Theorem 5.4.4. [37, 7.3.23] *Let λ be a p -restricted partition of n . Then S^λ is simple if and only if $\nu_p(h_{a,b}^\lambda) = \nu_p(h_{a,c}^\lambda)$ for any two nodes (a, b) and (a, c) of λ .*

We now introduce another notation. Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, $s \in \mathbb{N}$ and $s \leq \ell$. Define $\bar{\lambda}^s$ to be the partition obtained from λ by deleting the first s rows of λ . Namely, we have

$$\bar{\lambda}^s = (\lambda_{s+1}, \dots, \lambda_\ell) \vdash n - \sum_{i=1}^s \lambda_i$$

if $s < \ell$ and get $\emptyset$ if $s = \ell$. To finish the preparation, we need to recall the definition of $S_{\lambda_1, n}^{\bar{\lambda}^1}$ (see Page 23) and the following result discovered by Hemmer.

Theorem 5.4.5. [30, Theorem 4.5] *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $\lambda_1 < p$. Then we have $\mathcal{F}_{\lambda_1}(S^\lambda) \cong S_{\lambda_1, n}^{\bar{\lambda}^1}$.*

We now begin to finish the proof of Theorem 5.1.5. Until the end of the section, fix further that $p > 2$ and an integer r satisfying $0 \leq 2r \leq n$. Note that the partition $(2^r, 1^{n-2r})$ is a p -restricted partition of n . The proof of Theorem 5.1.5 will be based on a sequence of lemmas. We begin with some combinatorial results.

Lemma 5.4.6. *Let $\lambda = (2^r, 1^{n-2r}) \vdash n$ and $r > 0$. Assume that a is the largest positive integer such that $\nu_p(h_{a,1}^\lambda) \neq \nu_p(h_{a,2}^\lambda)$ for the nodes $(a, 1)$ and $(a, 2)$ of λ . Then there are non-negative integers u, v and w such that $0 \leq u, v < p$ and $h_{a,2}^\lambda = u + vp^w$.*

Proof. Note that $h_{a,2}^\lambda > 0$ and let $h_{a,2}^\lambda$ have p -adic sum $\sum_{i \geq 0} b_i p^i$ where $0 \leq b_i < p$ for all $i \geq 0$. If $h_{a,2}^\lambda = b_0$, we may assign $u = b_0$ and $v = w = 0$. If $b_0 < h_{a,2}^\lambda$, let s be the smallest subscript such that $s > 0$ and $b_s > 0$. We claim that $\sum_{i > s} b_i p^i = 0$. Suppose that $\sum_{i > s} b_i p^i > 0$. Set $t = a + \sum_{i > s} b_i p^i$ and note that λ has nodes $(t, 1)$ and $(t, 2)$. Moreover, we have $h_{t,1}^\lambda = h_{a,1}^\lambda - \sum_{i > s} b_i p^i$ and $h_{t,2}^\lambda = b_0 + b_s p^s$. If $b_0 \neq 0$, $\nu_p(h_{a,2}^\lambda) = \nu_p(h_{t,2}^\lambda) = 0$. However, as $\nu_p(h_{a,1}^\lambda) \neq \nu_p(h_{a,2}^\lambda)$, observe that both $\nu_p(h_{a,1}^\lambda)$ and $\nu_p(h_{t,1}^\lambda)$ are non-zero. So $\nu_p(h_{t,1}^\lambda) \neq \nu_p(h_{t,2}^\lambda)$. It contradicts the choice of a as $t > a$. If $b_0 = 0$, $\nu_p(h_{a,2}^\lambda) = \nu_p(h_{t,2}^\lambda) = s$. By the choice of a , we conclude that $\nu_p(h_{t,1}^\lambda) = s$. It implies that $\nu_p(h_{a,1}^\lambda) = s$. We thus get $\nu_p(h_{a,1}^\lambda) = \nu_p(h_{a,2}^\lambda) = s$, which also contradicts the choice of a . Therefore, the claim is shown. We can set $u = b_0$, $v = b_s$ and $w = s$ by this claim. The lemma follows. $\square$

Lemma 5.4.7. *Let $\lambda = (2^r, 1^{n-2r}) \vdash n$ and $\lambda \notin JM(n)$. If $h_{1,2}^\lambda \geq p$, $\nu_p(h_{1,1}^\lambda) \neq \nu_p(h_{1,2}^\lambda)$ and $\nu_p(h_{a,1}^\lambda) = \nu_p(h_{a,2}^\lambda)$ for all nodes $(a, 1), (a, 2)$ of λ such that $a > 1$, then $\nu_p(h_{1,1}^\lambda), \nu_p(h_{1,2}^\lambda) > 0$.*

Proof. By the hypotheses and Lemma 5.4.6, let $h_{1,1}^\lambda$ and $h_{1,2}^\lambda$ have p -adic sums $\sum_{i \geq 0} a_i p^i$ and $b_0 + b_s p^s$ respectively, where $0 \leq a_i, b_0, b_s < p$ for all $i \geq 0$ and $b_s, s > 0$. Suppose that either $\nu_p(h_{1,1}^\lambda)$ or $\nu_p(h_{1,2}^\lambda)$ is zero. We first assume that $\nu_p(h_{1,1}^\lambda) = 0$. So we have $a_0 \neq 0$ while $b_0 = 0$. Note that $h_{1,1}^\lambda > a_0$ otherwise $\lambda \in JM(n)$ as λ is a p -core. Let $u = a_0 + 1$ and observe that λ has the nodes $(u, 1)$ and $(u, 2)$. Moreover, $h_{u,1}^\lambda = \sum_{i > 0} a_i p^i$ and $h_{u,2}^\lambda = b_s p^s - a_0$. In particular, from the hypotheses, we have $\nu_p(h_{u,1}^\lambda) = \nu_p(h_{u,2}^\lambda)$. But it is clear to see that $\nu_p(h_{u,1}^\lambda) \geq 1$ and $\nu_p(h_{u,2}^\lambda) = 0$. It shows that $\nu_p(h_{u,1}^\lambda) \neq \nu_p(h_{u,2}^\lambda)$, which is a contradiction. One may write a similar proof to get a contradiction for the case $\nu_p(h_{1,2}^\lambda) = 0$. Therefore, we conclude that $\nu_p(h_{1,1}^\lambda), \nu_p(h_{1,2}^\lambda) > 0$. The proof is now finished. $\square$

We now return to the proof of Theorem 5.1.5.

Lemma 5.4.8. *Let $\lambda = (2^r, 1^{n-2r}) \vdash n$ and $\lambda \notin JM(n)$. If we have $\nu_p(h_{1,1}^\lambda) \neq \nu_p(h_{1,2}^\lambda)$ and $\nu_p(h_{a,1}^\lambda) = \nu_p(h_{a,2}^\lambda)$ for all nodes $(a, 1), (a, 2)$ of λ such that $a > 1$, then S^λ is not a trivial source module.*

Proof. By the hypotheses and Lemmas 5.4.6, 5.4.7, if $h_{1,2}^\lambda \geq p$, let $h_{1,1}^\lambda$ and $h_{1,2}^\lambda$ have p -adic sums $\sum_{i > 0} a_i p^i$ and $b_s p^s$ respectively, where $0 \leq a_i, b_s < p$ for all $i > 0$ and $b_s, s > 0$. Let t be the smallest subscript such that $t > 0$ and $a_t > 0$. Note that $\nu_p(h_{1,1}^\lambda) = t \neq s = \nu_p(h_{1,2}^\lambda)$ by computation and the hypotheses. We claim that $s < t$. Suppose that $s > t$. We set $u = 1 + a_t p^t$ and note that the nodes $(u, 1), (u, 2)$ are in λ . Moreover, $h_{u,1}^\lambda = \sum_{i \geq t+1} a_i p^i$ and $h_{u,2}^\lambda = b_s p^s - a_t p^t$. Especially, from the hypotheses, $\nu_p(h_{u,1}^\lambda) = \nu_p(h_{u,2}^\lambda)$. However, we also get that $\nu_p(h_{u,1}^\lambda) = \nu_p(\sum_{i \geq t+1} a_i p^i) \geq t+1$ but $\nu_p(h_{u,2}^\lambda) = \nu_p(p^t(b_s p^{s-t} - a_t)) = t$. The calculation implies that $\nu_p(h_{u,1}^\lambda) \neq \nu_p(h_{u,2}^\lambda)$, which is a contradiction. The claim is now shown.

Suppose that S^λ is a trivial source module. Notice that $\lambda' = (h_{1,1}^\lambda - 1, h_{1,2}^\lambda)$. By Lemma 2.11.2 (iii), we know that $S^{\lambda'}$ is a trivial source module. If $h_{1,2}^\lambda < p$, we have $\nu_p(h_{1,1}^\lambda) > 0$ as $\nu_p(h_{1,1}^\lambda) \neq \nu_p(h_{1,2}^\lambda) = 0$. Note that λ' is a James partition by Lemma 5.4.1 (i). Due to Lemma 5.4.2, we get $\lambda = (1^n)$. It is a contradiction since $\lambda \notin JM(n)$. If $h_{1,2}^\lambda \geq p$, by the claim and Lemmas 5.4.7, 5.4.1 (i), we observe that λ' is also a James partition as $s < t$. Therefore, by Lemma 5.4.2, we get $\lambda' = (n)$. It also contradicts the condition that $\lambda \notin JM(n)$. Hence S^λ is not a trivial source module. The proof is now complete. $\square$

Lemma 5.4.9. *Let $\lambda = (2^r, 1^{n-2r}) \vdash n$ and $\lambda \notin JM(n)$. Then S^λ is not a trivial source module.*

Proof. According to the assumption that $\lambda \notin JM(n)$ and Theorem 5.4.4, there exists a pair of nodes $(a, 1), (a, 2)$ of λ such that $\nu_p(h_{a,1}^\lambda) \neq \nu_p(h_{a,2}^\lambda)$ and $\nu_p(h_{b,1}^\lambda) = \nu_p(h_{b,2}^\lambda)$ for all nodes $(b, 1), (b, 2)$ of λ such that $b > a$. When $a = 1$, the desired result follows by Lemma 5.4.8. When $a > 1$, put $u = a - 1$ and $v = 2a - 2$. By Theorem 5.4.5 and Corollary 5.2.5, S^λ is a trivial

source $\mathbb{F}\mathfrak{S}_n$ -module only if $S_{v,n}^{\bar{\lambda}^u}$ is a trivial source $\mathbb{F}\mathfrak{S}_{n-v}^{+v}$ -module. Note that $\bar{\lambda}^u$ satisfies all the hypotheses of Lemma 5.4.8. Therefore, we deduce that $S_{v,n}^{\bar{\lambda}^u}$ is not a trivial source module, which shows that S^λ is not a trivial source module. The proof is now complete. $\square$

Corollary 5.4.10. *Let $\lambda = (2^r, 1^{n-2r}) \vdash n$. Then S^λ is a trivial source module if and only if $\lambda \in JM(n)$.*

Proof. One direction is clear by Theorem 5.1.2 while Lemma 5.4.9 implies the other direction. The corollary follows. $\square$

We close this section by presenting the proof of Theorem 5.1.5.

Proof of Theorem 5.1.5. One direction is from Theorem 5.1.2. Note that S^λ is simple if and only if $S^{\lambda'}$ is simple. Therefore, the other direction can be justified by using Lemma 2.11.2 (iii) and Corollary 5.4.10. We are done. $\square$

5.5 Verification of a Conjecture of Hudson

This section contains the proof of Theorem 5.1.6. Theorem 5.1.6 will be deduced from Theorem 5.5.1 after a necessary preparation. Unlike the former sections, our main tool used here is the Broué correspondences of trivial source $\mathbb{F}\mathfrak{S}_n$ -modules.

We state a stronger result than the conjecture of Hudson as follows and prove it after a preparation. This result implies her conjecture obviously.

Theorem 5.5.1. *Let $p = 2$, $n \geq 4$ and $\lambda \vdash n$ with 2-weight 2. If S^λ is an indecomposable, non-simple, trivial source module, then $S^\lambda \cong Y^\mu$ where $\mu = \kappa_2(\lambda) + (2^2)$.*

We list some lemmas as preparation. We begin with two results of modules of finite groups.

Lemma 5.5.2. *Let G and H be finite groups. Assume that M and N are an $\mathbb{F}G$ -module and an $\mathbb{F}H$ -module respectively. If both M and N are self-dual, then $M \boxtimes N$ is self-dual as an $\mathbb{F}[G \times H]$ -module.*

Proof. We view M as an $\mathbb{F}[G \times H]$ -module by letting the elements of $\mathbb{F}H$ act trivially on M . Similarly, we also view N as an $\mathbb{F}[G \times H]$ -module by letting the elements of $\mathbb{F}G$ act trivially on N . By this viewpoint, we use M_H and N_G to denote M and N respectively. As M is self-dual as an $\mathbb{F}G$ -module and N is self-dual as an $\mathbb{F}H$ -module, we have the following two deductions:

- (a) Note that both M_H and N_G are self-dual as $\mathbb{F}[G \times H]$ -modules;
- (b) Note that $M \boxtimes N \cong M_H \otimes N_G$ as $\mathbb{F}[G \times H]$ -modules.

By (a) and (b), it is obvious that $M \boxtimes N$ is self-dual as an $\mathbb{F}[G \times H]$ -module. $\square$

For our discussion, recall the notation of Green correspondents of modules in Section 2.5.

Lemma 5.5.3. *Let G be a finite group. Let M be an indecomposable $\mathbb{F}G$ -module with a vertex P and $N_G(P) \leq H \leq G$. Then M is self-dual if and only if $\mathcal{G}_H(M)$ is self-dual.*

Proof. By properties of the Green correspondent $\mathcal{G}_H(M)$, we have

$$M \downarrow_H \cong \mathcal{G}_H(M) \oplus U \text{ and } \mathcal{G}_H(M) \uparrow^G \cong M \oplus V, \quad (5.5.1)$$

where, for any indecomposable direct summand W of the $\mathbb{F}H$ -module U or the $\mathbb{F}G$ -module V , P is not a vertex of W . When $M \cong M^*$, by (5.5.1), we get

$$\mathcal{G}_H(M)^* \oplus U^* \cong (M \downarrow_H)^* = (M^*) \downarrow_H \cong M \downarrow_H \cong \mathcal{G}_H(M) \oplus U. \quad (5.5.2)$$

Note that $*$ preserves vertices of indecomposable modules. We thus get $\mathcal{G}_H(M) \cong \mathcal{G}_H(M)^*$ by (5.5.2) and the Krull-Schmidt Theorem. When $\mathcal{G}_H(M) \cong \mathcal{G}_H(M)^*$, by (5.5.1) again, we obtain

$$M^* \oplus V^* \cong (\mathcal{G}_H(M) \uparrow^G)^* \cong (\mathcal{G}_H(M)^*) \uparrow^G \cong \mathcal{G}_H(M) \uparrow^G \cong M \oplus V. \quad (5.5.3)$$

Using (5.5.3) and the Krull-Schmidt Theorem, we have $M \cong M^*$. The lemma follows. $\square$

The following three lemmas are about the symmetric groups.

Lemma 5.5.4. *Let λ be a p -regular or p -restricted partition of n . If S^λ is self-dual, then S^λ is simple.*

Proof. Suppose that S^λ is not simple. When λ is p -regular, S^λ has a simple head D^λ . As S^λ is self-dual, it forces that S^λ has at least two composition factors isomorphic to D^λ . So we have $(S^\lambda : D^\lambda) \geq 2$, which contradicts the fact $(S^\lambda : D^\lambda) = 1$. When λ is p -restricted, note that $S^\lambda \otimes \text{sgn} \cong S^{\lambda'}$ as S^λ is self-dual. Also note that λ' is p -regular and $S^{\lambda'}$ is self-dual. We get a similar contradiction as we assume that S^λ is not simple. The lemma thus follows. $\square$

Lemma 5.5.5. [59, Lemma 2.5] *Let $H \leq \mathfrak{S}_n$ and r be an integer such that $0 < r \leq n$. Let H act on $[1, n]$. If H fixes $n - r + 1, \dots, n$, then*

$$N_{\mathfrak{S}_n}(H) = N_{\mathfrak{S}_{n-r}}(H) \times \mathfrak{S}_r^{+(n-r)}.$$

Moreover, $N_{\mathfrak{S}_n}(H)/H \cong (N_{\mathfrak{S}_{n-r}}(H)/H) \times \mathfrak{S}_r^{+(n-r)} \cong (N_{\mathfrak{S}_{n-r}}(H)/H) \times \mathfrak{S}_r$.

Lemma 5.5.6. [60, Corollary 2] *Let P be a Sylow 2-subgroup of $\mathfrak{S}_n$. Then $N_{\mathfrak{S}_n}(P) = P$.*

We now deal with Theorem 5.5.1. From now on, we shall use C_4 , $\mathfrak{S}_{(2^2)}$ and K_4 to denote the subgroups $\langle (1, 2, 3, 4) \rangle$, $\langle (1, 2), (3, 4) \rangle$ and $\langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$ of $\mathfrak{S}_4$ respectively. Note that they are exactly all 2-subgroups of $\mathfrak{S}_4$ of order 4 up to $\mathfrak{S}_4$ -conjugation. Recall that B_λ is the p -block of $\mathbb{F}\mathfrak{S}_n$ labelled by the p -core λ of a partition of n . Let $Y^\varnothing$ be the trivial $\mathbb{F}\mathfrak{S}_\varnothing$ -module.

Lemma 5.5.7. *Let $p = 2$ and M be an indecomposable $\mathbb{F}\mathfrak{S}_n$ -module. Let λ be the 2-core of a partition of n and w be the 2-weight of B_λ . If B_λ contains M and M is a trivial source module whose vertices are defect groups of B_λ , then $M \cong Y^\alpha \cong S^\alpha$, where $\alpha = \lambda + (2w) \in JM(n)$.*

Proof. Let P be a Sylow 2-subgroup of $\mathfrak{S}_{2w}$. Note that P is also a vertex of M . By Lemmas 5.5.5 and 5.5.6, we know that

$$N_{\mathfrak{S}_n}(P) = N_{\mathfrak{S}_{2w}}(P) \times \mathfrak{S}_{n-2w}^{+2w} = P \times \mathfrak{S}_{n-2w}^{+2w}.$$

Moreover, we also have $N_{\mathfrak{S}_n}(P)/P \cong \mathfrak{S}_{n-2w}$. Since M is an indecomposable module with a vertex P and a trivial P -source, by Theorem 2.6.2 (i), we get that $M(P)$ is an indecomposable projective $\mathbb{F}\mathfrak{S}_{n-2w}$ -module. Since the regular $\mathbb{F}\mathfrak{S}_{n-2w}$ -module is a Young permutation module, by (2.11.1) and Theorem 2.11.3, $M(P) \cong Y^\mu$ as $\mathbb{F}\mathfrak{S}_{n-2w}$ -modules, where μ is a 2-restricted partition of $n - 2w$. Let $\nu = \mu + (2w)$. By Theorem 2.11.3, note that Y^ν is an indecomposable $\mathbb{F}\mathfrak{S}_n$ -module with a vertex P and a trivial P -source. By the Sylow Theorem, we can require that P is a Sylow 2-subgroup of $\mathfrak{S}_{\mathcal{O}_\nu}$. According to Theorem 2.11.4 and Lemma 5.5.6, we have

$$\mathfrak{S}_{n-2w} \cong N_{\mathfrak{S}_n}(P)/N_{\mathfrak{S}_{\mathcal{O}_\nu}}(P) = N_{\mathfrak{S}_n}(P)/P \text{ and } Y^\nu(P) \cong Y^\mu.$$

We thus obtain that $M \cong Y^\nu$ by Theorem 2.6.2 (i). It forces that $\alpha = \nu$ as B_λ contains M . It is not difficult to see that $Y^\alpha \cong S^\alpha$. Moreover, since α is a 2-regular partition and $S^\alpha \cong Y^\alpha$, S^α is simple by Lemma 5.5.4. The proof is now complete. $\square$

Lemma 5.5.8. *Let $p = 2$, $n \geq 4$ and M be an indecomposable $\mathbb{F}\mathfrak{S}_n$ -module. Let λ be the 2-core of a partition of n and B_λ have 2-weight 2. If M is a trivial source module with a vertex $\mathfrak{S}_{(2^2)}$ and B_λ contains M , then $M \cong Y^\alpha$, where $\alpha = \lambda + (2^2)$.*

Proof. Notice that M is isomorphic to a Young module since $M \mid M^{(2^2, 1^{n-4})}$. Since $\mathfrak{S}_{(2^2)}$ is a vertex of M , by Theorem 2.11.3, there exists a 2-restricted partition of $n - 4$, say μ , such that $M \cong Y^\nu$, where $\nu = \mu + (2^2)$. Since B_λ contains M and B_λ has 2-weight 2, we get that $\alpha = \nu$. The lemma is shown. $\square$

Lemma 5.5.9. *Let $p = 2$. Then*

$$(\mathbb{F}_{K_4})^\uparrow^{\mathfrak{S}_4} \cong 2D^{(3,1)} \oplus Sc_{\mathfrak{S}_4}(K_4).$$

In particular, all the indecomposable direct summands of the decomposition of $(\mathbb{F}_{K_4})^\uparrow^{\mathfrak{S}_4}$ are self-dual.

Proof. Note that $N_{\mathfrak{S}_4}(K_4)/K_4 \cong \mathfrak{S}_3$. By an easy calculation, we have $(\mathbb{F}_{K_4})^{\uparrow \mathfrak{S}_4}(K_4) \cong M^{(1^3)}$ as $\mathbb{F}\mathfrak{S}_3$ -modules. By [24, Appendix A] and [62, Lemma 6], we have $M^{(1^3)} \cong Y^{(1^3)} \oplus 2Y^{(2,1)}$ and $D^{(3,1)}$ has the vertex K_4 . As both $D^{(3,1)}$ and $Sc_{\mathfrak{S}_4}(K_4)$ have the vertex K_4 , by Lemma 2.6.1 (ii), Theorem 2.6.2 (i), (iii) and counting dimensions, we can deduce that $D^{(3,1)}(K_4) \cong Y^{(2,1)}$ and $(Sc_{\mathfrak{S}_4}(K_4))(K_4) \cong Y^{(1^3)}$. The desired isomorphic formula follows by Theorem 2.6.2 (iii). The second assertion is clear by the isomorphic formula. $\square$

Let $a, b \in \mathbb{N}$ and $a \leq b \leq n$. Let $\eta \vdash b - a$. We recall the definition of $Y_{a,b}^\eta$ (see Page 23).

Proposition 5.5.10. *Let $p = 2$, $n \geq 4$ and $H = N_{\mathfrak{S}_n}(K_4)$. Let M be an indecomposable $\mathbb{F}\mathfrak{S}_n$ -module with a vertex K_4 . If M is a trivial source module, then, for some 2-restricted partition λ of $n - 4$, we have $\mathcal{G}_H(M) \cong D^{(3,1)} \boxtimes Y_{4,n}^\lambda$ or $Sc_{\mathfrak{S}_4}(K_4) \boxtimes Y_{4,n}^\lambda$. In particular, M is self-dual.*

Proof. According to Lemma 5.5.5, we have $H = N_{\mathfrak{S}_4}(K_4) \times \mathfrak{S}_{n-4}^{+4} = \mathfrak{S}_{(4,n-4)}$. Moreover, we have $H/K_4 \cong \mathfrak{S}_3 \times \mathfrak{S}_{n-4}^{+4}$. By Theorem 2.6.2 (i), note that $M(K_4) \cong Y^{(2,1)} \boxtimes Y_{4,n}^\lambda$ or $Y^{(1^3)} \boxtimes Y_{4,n}^\lambda$ as $\mathbb{F}[\mathfrak{S}_3 \times \mathfrak{S}_{n-4}^{+4}]$ -modules, where λ is a 2-restricted partition of $n - 4$. Furthermore, according to [45, Propositions 1.1 and 1.2], notice that $D^{(3,1)} \boxtimes Y_{4,n}^\lambda$ and $Sc_{\mathfrak{S}_4}(K_4) \boxtimes Y_{4,n}^\lambda$ are indecomposable $\mathbb{F}\mathfrak{S}_{(4,n-4)}$ -modules with a vertex K_4 and a trivial K_4 -source. By Lemma 2.6.1 (iii) and the proof of Lemma 5.5.9, we have

$$(D^{(3,1)} \boxtimes Y_{4,n}^\lambda)(K_4) \cong D^{(3,1)}(K_4) \boxtimes Y_{4,n}^\lambda \cong Y^{(2,1)} \boxtimes Y_{4,n}^\lambda, \quad (5.5.4)$$

$$(Sc_{\mathfrak{S}_4}(K_4) \boxtimes Y_{4,n}^\lambda)(K_4) \cong (Sc_{\mathfrak{S}_4}(K_4))(K_4) \boxtimes Y_{4,n}^\lambda \cong Y^{(1^3)} \boxtimes Y_{4,n}^\lambda \quad (5.5.5)$$

as $\mathbb{F}[\mathfrak{S}_3 \times \mathfrak{S}_{n-4}^{+4}]$ -modules. Notice that $N_{\mathfrak{S}_{(4,n-4)}}(K_4) = \mathfrak{S}_{(4,n-4)} = H$. Therefore, (5.5.4) and (5.5.5) are isomorphic formulae for $\mathbb{F}[H/K_4]$ -modules. From Theorem 2.6.2 (ii), we obtain that $\mathcal{G}_H(M) \cong \mathcal{G}_H(D^{(3,1)} \boxtimes Y_{4,n}^\lambda)$ or $\mathcal{G}_H(Sc_{\mathfrak{S}_4}(K_4) \boxtimes Y_{4,n}^\lambda)$. However, observe that

$$\mathcal{G}_H(D^{(3,1)} \boxtimes Y_{4,n}^\lambda) = D^{(3,1)} \boxtimes Y_{4,n}^\lambda \text{ and } \mathcal{G}_H(Sc_{\mathfrak{S}_4}(K_4) \boxtimes Y_{4,n}^\lambda) = Sc_{\mathfrak{S}_4}(K_4) \boxtimes Y_{4,n}^\lambda.$$

The first assertion thus follows. The second one is clear by Lemmas 5.5.9, 5.5.2 and 5.5.3. $\square$

We are now ready to prove Theorem 5.5.1.

Proof of Theorem 5.5.1. Let P be a vertex of S^λ . We may choose P to be a 2-subgroup of $\mathfrak{S}_4$. It is known that $|P| = 1$ if and only if λ has 2-weight 0 (see [54, Page 88]). So $|P| > 1$ as λ has 2-weight 2. As $|P| > 1$, by [62, Theorem 1], P is cyclic if and only if λ has 2-weight 1. As λ has 2-weight 2, we thus have $|P| \geq 4$ and P is not cyclic. So P is not $\mathfrak{S}_n$ -conjugate to C_4 . As S^λ is non-simple, we have $|P| \neq 8$ by Lemma 5.5.7. So $|P| = 4$. If P is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_{(2^2)}$, by Lemma 5.5.8, we get the desired result.

To prove Theorem 5.5.1, it suffices to show that $P \neq K_4$. Suppose that $P = K_4$. By Proposition 5.5.10, S^λ is self-dual. This implies that λ is neither 2-regular nor 2-restricted by Lemma 5.5.4. As λ has 2-weight 2, when $n > 4$, it forces that $\lambda = (s + 2, s - 1, s - 2, \dots, 2, 1^3)$ for some positive integer s . By this fact and [21, Theorem A], we get that $\mathfrak{S}_{(2^2)}$ is $\mathfrak{S}_n$ -conjugate to P , which contradicts the assumption that $P = K_4$. When $n = 4$, as λ is neither 2-regular nor 2-restricted, we have $\lambda = (2^2)$. It is clear that $S^{(2^2)} \cong D^{(3,1)}$, which contradicts the assumption that S^λ is non-simple. So $P \neq K_4$. The proof is now complete. $\square$

Theorem 5.1.6 is now shown by Theorems 5.1.2 and 5.5.1. We end this chapter with a corollary which may have its own interest.

Corollary 5.5.11. *Let $p = 2$, $n \geq 4$ and $\lambda \vdash n$ with 2-weight 2. Then S^λ is an indecomposable $\mathbb{F}\mathfrak{S}_n$ -module with a vertex K_4 and a trivial K_4 -source if and only if $n = 4$ and $\lambda = (2^2)$.*

Proof. One direction is clear. For the other direction, notice that K_4 cannot be $\mathfrak{S}_n$ -conjugate to the Sylow 2-subgroups of the Young subgroups of $\mathfrak{S}_n$. By Theorems 2.11.3 and 5.1.6, we thus deduce that $\lambda \in JM(n)$. By [38, Main Theorem], if λ is neither 2-regular nor 2-restricted, then $\lambda = (2^2)$. Since $\lambda \in JM(n)$, it is known that S^λ is isomorphic to a Young module if λ is either 2-regular or 2-restricted (see [15, Remark 6.8]). Therefore, as S^λ has a vertex K_4 , λ is neither 2-regular nor 2-restricted. So we have $n = 4$ and $\lambda = (2^2)$. The corollary is shown. $\square$

Chapter 6

Symmetric and Exterior Powers of Young Permutation Modules

Given $\lambda \vdash n$, in this chapter, we study symmetric and exterior powers of M^λ in positive characteristic case. We determine the symmetric and exterior powers of M^λ that are projective. All the indecomposable exterior powers of M^λ are also classified. We then prove some results for indecomposable direct summands that have the largest complexity in direct sum decompositions of some symmetric and exterior powers of M^λ . We close this chapter by parameterizing all the Scott modules that are isomorphic to direct summands of the symmetric or exterior square of M^λ and determining their corresponding multiplicities explicitly.

6.1 Main Results and Some Known Results

In this section, we state our main results of this chapter. Moreover, we also introduce the required notation and collect some known results. They will be used in the following sections.

Let G be a finite group and M be an $\mathbb{F}G$ -module. Let $1 < a \in \mathbb{N}$. Throughout this chapter, following the notation given in Section 2.9, let $S^a M$ and $\Lambda^a M$ be the a th symmetric and exterior power of M respectively. By convention, if $M \neq 0$, set $S^0 M = \Lambda^0 M = \mathbb{F}$ and $S^1 M = \Lambda^1 M = M$. As mentioned in Section 2.9, the symmetric and exterior powers of M are natural $\mathbb{F}G$ -modules. We collect two known results as follows.

Lemma 6.1.1. [6, Lemmas 1.14.1, 1.14.3, 1.14.4] *Let G be a finite group and $b \in \mathbb{N}_0$. Let M and L be $\mathbb{F}G$ -modules.*

- (i) *Let $\bigotimes^b M$ be the b -fold inner tensor product of M , where $\bigotimes^1 M = M$ and $\bigotimes^0 M = \mathbb{F}$. If $b < p$, then $S^b M \mid \bigotimes^b M$ and $\Lambda^b M \mid \bigotimes^b M$.*
- (ii) *We have $S^b(M \oplus L) \cong \bigoplus_{i=0}^b (S^i M \otimes S^{b-i} L)$ and $\Lambda^b(M \oplus L) \cong \bigoplus_{i=0}^b (\Lambda^i M \otimes \Lambda^{b-i} L)$.*

Assume further that $M \neq 0$. Let $d = \dim_{\mathbb{F}} M$ and $\{v_1, \dots, v_d\}$ be an $\mathbb{F}$ -basis of M . By (2.9.1), $S^a M$ has an $\mathbb{F}$ -basis $\{u : u = v_{i_1} \dots v_{i_a} \in S^a M, 1 \leq i_1 \leq \dots \leq i_a \leq d\}$. Moreover, if $a \leq d$, $\Lambda^a M$ has an $\mathbb{F}$ -basis $\{u : u = v_{i_1} \wedge \dots \wedge v_{i_a} \in \Lambda^a M, 1 \leq i_1 < \dots < i_a \leq d\}$ by (2.9.3). If M is a p -permutation $\mathbb{F}G$ -module, for any p -subgroup P of G , let $\mathcal{B}_P(M)$ denote a fixed $\mathbb{F}$ -basis of M permuted by P . We shall need the following lemma.

Lemma 6.1.2. *Let P be a p -subgroup of a finite group G and $1 < b \in \mathbb{N}$. Let M be a p -permutation $\mathbb{F}G$ -module and $d = \dim_{\mathbb{F}} M$. Let $\mathcal{B}_P(M) = \{v_1, \dots, v_d\}$.*

- (i) *The set $\{u : u = v_{i_1} \dots v_{i_b} \in S^b M, 1 \leq i_1 \leq \dots \leq i_b \leq d\}$ is an $\mathbb{F}$ -basis of $S^b M$ permuted by P .*
- (ii) *If $b \leq d$, the set $\{\alpha_u u : u = v_{i_1} \wedge \dots \wedge v_{i_b} \in \Lambda^b M, 1 \leq i_1 < \dots < i_b \leq d\}$ is an $\mathbb{F}$ -basis of $\Lambda^b M$ permuted by P , where the $\mathbb{F}$ -scalar α_u depends on u and is either 1 or -1 .*
- (iii) *$S^b M$ is a p -permutation $\mathbb{F}G$ -module. If $b \leq d$, then $\Lambda^b M$ is a p -permutation $\mathbb{F}G$ -module.*

Proof. (i) is clear. As $b \leq d$, let $\mathcal{B} = \{u : u = v_{i_1} \wedge \cdots \wedge v_{i_b} \in \Lambda^b M, 1 \leq i_1 < \cdots < i_b \leq d\}$ and note that $\mathcal{B}$ is an $\mathbb{F}$ -basis of $\Lambda^b M$. By the definition of $\mathcal{B}_P(M)$, for any $g \in P$ and $u \in \mathcal{B}$, there exists some $v \in \mathcal{B}$ such that $gu = v$ or $gu = -v$. We thus get (ii) by [23, Lemma 2.6]. By (i) and (ii), for any p -subgroup Q of G and the fixed $\mathcal{B}_Q(M)$, both $S^b M$ and $\Lambda^b M$ have an $\mathbb{F}$ -basis permuted by Q . Therefore, both $S^b M$ and $\Lambda^b M$ are p -permutation $\mathbb{F}G$ -modules, which is exactly (iii). $\square$

Assume further that M is a p -permutation $\mathbb{F}G$ -module. For any p -subgroup P of G and the fixed $\mathcal{B}_P(M)$, use $\mathcal{B}_{P,a}^s(M)$ to denote the $\mathbb{F}$ -basis of $S^a M$ in Lemma 6.1.2 (i). Similarly, use $\mathcal{B}_{P,a}^e(M)$ to denote the $\mathbb{F}$ -basis of $\Lambda^a M$ in Lemma 6.1.2 (ii).

In [7, Lemma 1 (i) and (ii)], Benson and Lim got an optimal upper bound for the complexities of symmetric and exterior powers of R , where R is a direct sum of finitely many copies of the regular $\mathbb{F}G$ -module. Let $b \in \mathbb{N}$ and recall that $\nu_p(b)$ is the largest non-negative integer c such that $p^c \mid b$. We set $\nu_p(0) = \infty$. Using the Brauer quotients of p -permutation $\mathbb{F}G$ -modules, we generalize their result as follows.

Theorem 6.1.3. *Let G be a finite group and P be a non-zero projective $\mathbb{F}G$ -module. Let $b \in \mathbb{N}_0$ and $r = \text{rank}(G)$. Then $c_G(S^b P) = \min\{\nu_p(b), r\}$. Moreover, if $b \leq \dim_{\mathbb{F}} P$, then $c_G(\Lambda^b P) = \min\{\nu_p(b), r\}$.*

We shall prove Theorem 6.1.3 in the next section. Theorem 6.1.3 motivates us to study the symmetric and exterior powers of $\mathbb{F}G$ -modules. In particular, we are interested in the symmetric and exterior powers of $\mathbb{F}\mathfrak{S}_n$ -modules. For the $\mathbb{F}\mathfrak{S}_n$ -modules, notice that the class of Young permutation modules contains the regular module. Therefore, it is natural to consider the following project.

Project C. Let $\lambda \vdash n$ and $b \in \mathbb{N}$. Generalize known results of M^λ to the cases of $S^b M^\lambda$ and $\Lambda^b M^\lambda$.

The aim of this chapter is to report some progress for Project C. The main results of this chapter are stated as follows. Their proofs will be presented in the following sections.

Given $\lambda \vdash n$, we first determine the symmetric and exterior powers of M^λ that are projective (see Propositions 6.2.6 and 6.2.7). Moreover, all the indecomposable exterior powers of M^λ are also classified (see Theorem 6.3.7 and Remark 6.3.8 (i)).

To state our next result, we list two known results.

Theorem 6.1.4. [17, (3.6)] *Let $\lambda \vdash n$ and $\mu \vdash n$. Let λ have the p -adic expansion $\sum_{i=0}^m p^i \lambda(i)$ for some $m \in \mathbb{N}_0$. Then $Y^\lambda \mid M^\mu$ if and only if there exists an equality $\mu = \sum_{i=0}^m p^i \nu[i]$ such that $\nu[i] \models |\lambda(i)|$ and $\overline{\nu[i]} \trianglelefteq \lambda(i)$ for all $0 \leq i \leq m$.*

Theorem 6.1.5. [28, 3.2.2 and 3.3.2] *Let $\lambda \vdash n$ and λ have the p -adic expansion $\sum_{i=0}^m p^i \lambda(i)$ for some $m \in \mathbb{N}_0$. Then $c_{\mathfrak{S}_n}(M^\lambda) = \text{rank}(\mathfrak{S}_\lambda)$ and $c_{\mathfrak{S}_n}(Y^\lambda) = \frac{n - |\lambda(0)|}{p}$.*

By (2.11.1) and Lemma 2.7.1 (i), there is $\mu \vdash n$ such that $Y^\mu \mid M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(M^\lambda)$. Moreover, by Theorems 6.1.4 and 6.1.5, all the partitions of n that satisfy the two conditions can be explicitly determined by λ . We want to generalize this fact for some symmetric and exterior powers of M^λ . In particular, we prove the following:

Theorem 6.1.6. *Let $b, n \in \mathbb{N}$ and $b > 1$. Let $\lambda \vdash n$.*

- (i) *There exists some $\mu \vdash n$ such that $Y^\mu \mid S^b M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(S^b M^\lambda)$.*
- (ii) *If $\lambda \neq (n)$, then there exists some $\mu \vdash n$ such that $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$.*

In fact, the partition μ in Theorem 6.1.6 depends on λ and b and may not be unique. However, in the proof of Theorem 6.1.6, we can explicitly determine some choices for the partition μ in Theorem 6.1.6 (see Propositions 6.4.14 and 6.5.11). For some special cases, we have a stronger theorem (see Theorem 6.6.11). Another well-known fact is as follows.

Theorem 6.1.7. *Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Then there exists a unique $\mu \vdash n$ such that $Y^\mu \mid M^\lambda$ and Y^μ is a Scott module. Moreover, we have $[M^\lambda : Y^\mu] = 1$ and the vertices of Y^μ are $\mathfrak{S}_n$ -conjugate to a Sylow p -subgroup of $\mathfrak{S}_\lambda$.*

The partition μ in Theorem 6.1.7 is uniquely determined by λ (see Lemma 6.4.13). Our final goal of this chapter is to prove a similar result for $S^2 M^\lambda$ and $\Lambda^2 M^\lambda$ (see Theorem 6.8.10).

6.2 Proof of Theorem 6.1.3

The aim of this section is to finish the proof of Theorem 6.1.3. Moreover, we also determine the symmetric and exterior powers of a Young permutation module that are projective. Some lemmas are required as a preparation.

Let H be a subgroup of a finite group G and M be an $\mathbb{F}G$ -module. Let S be a non-empty multiset whose members are some elements of M . Let S' be the underlying set of S . Write $H \Rightarrow S$ if $S' \not\subseteq M^H$ and S is closed under the action of H on M . Also write $H \Rightarrow S$ if $H \Rightarrow S$ and S is itself a set. In particular, $H \Rightarrow S$ implies that S may have repeated members.

Lemma 6.2.1. *Let G be a finite group and P be a p -subgroup of G . Let M be a p -permutation $\mathbb{F}G$ -module and $d = \dim_{\mathbb{F}} M$. Let $1 < a \in \mathbb{N}$ and $\mathcal{B}_P(M) = \{v_1, \dots, v_d\}$. Then we have $(S^a M)(P) \cong S^a(M(P)) \oplus S_a(M, P)$ as $\mathbb{F}$ -linear spaces, where $S_a(M, P)$ is the $\mathbb{F}$ -linear space generated by*

$$\{u \in \mathcal{B}_{P,a}^s(M) : u = v_{i_1} \dots v_{i_a}, 1 \leq i_1 \leq \dots \leq i_a \leq d, P \Rightarrow \{v_{i_1}, \dots, v_{i_a}\}\}.$$

Proof. The case $P = 1$ is trivial. We thus assume that $P \neq 1$. By the definitions of $\mathcal{B}_{P,a}^s(M)$ and $(S^a M)(P)$, notice that $(S^a M)(P) = \langle \{u + \text{Tr}^P(S^a M) : u \in (\mathcal{B}_{P,a}^s(M))^P\} \rangle_{\mathbb{F}}$. Moreover, $\{u + \text{Tr}^P(S^a M) : u \in (\mathcal{B}_{P,a}^s(M))^P\}$ is an $\mathbb{F}$ -basis of $(S^a M)(P)$. It is also not difficult to see that $v_{i_1} \dots v_{i_a} \in (\mathcal{B}_{P,a}^s(M))^P$ if and only if $\{v_{i_1}, \dots, v_{i_a}\}$ is closed under the action of P on M . Therefore, $(S^a M)(P) = M_1 \oplus M_2$ as $\mathbb{F}$ -linear spaces, where

$$\begin{aligned} M_1 &= \langle \{u + \text{Tr}^P(S^a M) : u = v_{i_1} \dots v_{i_a} \in \mathcal{B}_{P,a}^s(M), v_{i_1}, \dots, v_{i_a} \in (\mathcal{B}_P(M))^P\} \rangle_{\mathbb{F}}, \\ M_2 &= \langle \{u + \text{Tr}^P(S^a M) : u = v_{i_1} \dots v_{i_a} \in \mathcal{B}_{P,a}^s(M), P \Rightarrow \{v_{i_1}, \dots, v_{i_a}\}\} \rangle_{\mathbb{F}}. \end{aligned}$$

Note that $M_2 \cong S_a(M, P)$ as $\mathbb{F}$ -linear spaces. By the definition of M_1 and Lemma 2.6.1 (iv), note that $M_1 \cong S^a(M(P))$ as $\mathbb{F}$ -linear spaces. So $(S^a M)(P) = M_1 \oplus M_2 \cong S^a(M(P)) \oplus S_a(M, P)$ as $\mathbb{F}$ -linear spaces, as desired. $\square$

By mimicking the proof of Lemma 6.2.1, one deduces the following result.

Lemma 6.2.2. *Let G be a finite group and P be a p -subgroup of G . Let M be a p -permutation $\mathbb{F}G$ -module and $d = \dim_{\mathbb{F}} M$. Let $1 < a \in \mathbb{N}$ and $\mathcal{B}_P(M) = \{v_1, \dots, v_d\}$. If $a \leq d$, then $(\Lambda^a M)(P) \cong \Lambda^a(M(P)) \oplus \Lambda_a(M, P)$ as $\mathbb{F}$ -linear spaces, where $\Lambda_a(M, P)$ is the $\mathbb{F}$ -linear space generated by*

$$\{\alpha_u u \in \mathcal{B}_{P,a}^e(M) : u = v_{i_1} \wedge \dots \wedge v_{i_a}, 1 \leq i_1 < \dots < i_a \leq d, P \Rightarrow \{v_{i_1}, \dots, v_{i_a}\}\}.$$

The correctness of the following lemma is obvious.

Lemma 6.2.3. *Let G be a finite group and P be a p -subgroup of G . Let M be a p -permutation $\mathbb{F}G$ -module and $d = \dim_{\mathbb{F}} M$. Let $1 < a \in \mathbb{N}$ and $\mathcal{B}_P(M) = \{v_1, \dots, v_d\}$.*

- (i) *We have $S_a(M, P) \neq 0$ if and only if $P \Rightarrow \{v_{i_1}, \dots, v_{i_a}\}$ for some $1 \leq i_1 \leq \dots \leq i_a \leq d$.*
- (ii) *Assume further that $a \leq d$. We have $\Lambda_a(M, P) \neq 0$ if and only if $P \Rightarrow \{v_{i_1}, \dots, v_{i_a}\}$ for some $1 \leq i_1 < \dots < i_a \leq d$.*

Let G be a finite group and M be a p -permutation $\mathbb{F}G$ -module. By Lemma 2.7.1 (vi), recall that $c_G(M) = \max\{\text{rank}(E) : E \in \varepsilon^G, M(E) \neq 0\}$. To determine $c_G(M)$, our strategy is to find an elementary abelian p -subgroup E of G with the largest p -rank such that $M(E) \neq 0$. For our purpose, the following inequality is useful.

Lemma 6.2.4. *Let G be a finite group and M be a p -permutation $\mathbb{F}G$ -module. Let $1 < a \in \mathbb{N}$ and $d = \dim_{\mathbb{F}} M$. If $a \leq d$, then $c_G(\Lambda^a M) \leq c_G(S^a M)$.*

Proof. For any elementary abelian p -subgroup E of G , if $(\Lambda^a M)(E) \neq 0$, we show that $(S^a M)(E) \neq 0$, which will complete the proof by Lemma 2.7.1 (vi). For the given $\mathcal{B}_E(M)$, by Lemma 6.2.2, the condition $(\Lambda^a M)(E) \neq 0$ implies $\Lambda^a(M(E)) \neq 0$ or $\Lambda_a(M, E) \neq 0$. If

$\Lambda^a(M(E)) \neq 0$, then $M(E) \neq 0$ and $(S^a M)(E) \neq 0$ by Lemma 6.2.1. If $\Lambda_a(M, E) \neq 0$, by Lemma 6.2.3 (ii), there exists some $S \subseteq \mathcal{B}_E(M)$ such that $|S| = a$ and $E \Rightarrow S$. So $E \Rightarrow S$ and $S_a(M, E) \neq 0$ by Lemma 6.2.3 (i). We have $(S^a M)(E) \neq 0$ by Lemma 6.2.1. The lemma follows. $\square$

We are now ready to prove Theorem 6.1.3.

Proof of Theorem 6.1.3. Set $u = \min\{\nu_p(b), r\}$ and use $\Delta^b P$ to denote either $\Lambda^b P$ or $S^b P$. We require that $b \leq \dim_{\mathbb{F}} P$ if $\Delta^b P = \Lambda^b P$. The case $b \in \{0, 1\}$ is trivial. We thus assume that $1 < b$. We first verify that $(\Delta^b P)(E) \neq 0$ for any elementary abelian p -subgroup E of G with p -rank u . It then suffices to prove that $\text{rank}(F) \leq u$ for any elementary abelian p -subgroup F of G such that $(\Delta^b P)(F) \neq 0$.

Since $u \leq r$, there indeed exists an elementary abelian p -subgroup of G with p -rank u . For any elementary abelian p -subgroup E of G with p -rank u , if $u = 0$, then $(\Delta^b P)(E) = \Delta^b P \neq 0$. If $u > 0$, let $\mathcal{B}_E(P) = \{v_1, \dots, v_d\}$, where $d = \dim_{\mathbb{F}} P$. As P is projective, we claim that $\mathcal{B}_E(P)$ has no fixed points under the action of any non-trivial subgroup of E . Otherwise, by Lemma 2.6.1 (iv), $P(K) \neq 0$ for some non-trivial $K \leq E$, which contradicts Lemma 2.6.1 (i). The claim is shown. So $\mathcal{B}_E(P)$ is a union set of some orbits of size p^u under the action of E . If $\Delta^b P = \Lambda^b P$, as $p^u \mid b$ and $b \leq d$, one can choose b/p^u E -orbits from $\mathcal{B}_E(P)$ and define S to be the union set of these E -orbits. We thus have $E \Rightarrow S$, which implies that $\Lambda_b(P, E) \neq 0$ by Lemma 6.2.3 (ii). So $(\Delta^b P)(E) \neq 0$ by Lemma 6.2.2. If $\Delta^b P = S^b P$, as $p^u \mid b$, one can choose a single E -orbit $\mathcal{O}$ from $\mathcal{B}_E(P)$ and put S to be the multiset satisfying the conditions that $S' = \mathcal{O}$ and each member of S' has multiplicity b/p^u in S . So S has b members and $P \Rightarrow S$, which implies that $S_b(P, E) \neq 0$ by Lemma 6.2.3 (i). We also have $(\Delta^b P)(E) \neq 0$ by Lemma 6.2.1.

Let F be an elementary abelian p -subgroup of G such that $(\Delta^b P)(F) \neq 0$. So $|F| \mid p^r$ as $\text{rank}(G) = r$. If $|F| = 1$, we are done. If $|F| > 1$, for the given $\mathcal{B}_F(P)$, as P is projective, $(\Delta^b P)(F) \neq 0$ implies that $\Lambda_b(P, F) \neq 0$ and $S_b(P, F) \neq 0$ by Lemmas 2.6.1 (i), 6.2.1 and 6.2.2. If $\Delta^b P = \Lambda^b P$, since $\Lambda_b(P, F) \neq 0$, by Lemma 6.2.3 (ii), there exists some $S \subseteq \mathcal{B}_F(P)$ such that $|S| = b$ and $F \Rightarrow S$. By letting F and S play the roles of E and $\mathcal{B}_E(P)$ respectively in the claim, note that S is a union set of some orbits of size $|F|$ under the action of F . So $|F| \mid b$. We thus have $\text{rank}(F) \leq \nu_p(b)$ and $\text{rank}(F) \leq \min\{\nu_p(b), r\}$. The case $\Delta^b P = S^b P$ is similar. The proof is now complete. $\square$

Our next goal is to determine the symmetric and exterior powers of a Young permutation module that are projective. Given $\lambda \vdash n$, throughout the remaining part of this chapter, we fix $\mathcal{B}_P(M^\lambda) = \mathcal{T}(\lambda)$ for any p -subgroup P of $\mathfrak{S}_n$. Also recall that $\text{rank}(\mathfrak{S}_n) = \lfloor \frac{n}{p} \rfloor$.

Lemma 6.2.5. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. Let $r = \text{rank}(\mathfrak{S}_\lambda)$. Then*

$$c_{\mathfrak{S}_n}(S^a M^\lambda) = \begin{cases} r, & \text{if } \nu_p(a) = 0, \\ r + \nu_p(a), & \text{if } \nu_p(a) > 0, \nu_p(a) + r \leq \lfloor \frac{n}{p} \rfloor, \\ \lfloor \frac{n}{p} \rfloor, & \text{if } \nu_p(a) > 0, \nu_p(a) + r > \lfloor \frac{n}{p} \rfloor. \end{cases}$$

Proof. It suffices to find an elementary abelian p -subgroup E of $\mathfrak{S}_n$ with the largest p -rank such that $(S^a M^\lambda)(E) \neq 0$. Let $u = \nu_p(a)$ and $b = \lfloor \frac{n}{p} \rfloor$. For any $1 \leq i \leq b$, recall that the p -cycle $s_i = ((i-1)p+1, \dots, ip)$. We distinguish three cases.

Case 1: $u = 0$.

Let E be an elementary abelian p -subgroup of $\mathfrak{S}_n$ such that $\text{rank}(E) = r$ and $M^\lambda(E) \neq 0$. We thus have $S^a(M^\lambda(E)) \neq 0$ and $(S^a M^\lambda)(E) \neq 0$ by Lemma 6.2.1. For any elementary abelian p -subgroup F of $\mathfrak{S}_n$ satisfying $\text{rank}(F) > r$, note that $M^\lambda(F) = 0$. Otherwise, by Lemma 2.7.1 (vi) and Theorem 6.1.5, we have $r < \text{rank}(F) \leq c_{\mathfrak{S}_n}(M^\lambda) = r$, which is a contradiction. So $S^a(M^\lambda(F)) = 0$ and $|\mathcal{T}(\lambda)^F| = 0$ by Lemma 2.6.1 (iv). Therefore, if there exists some multiset S such that S has a members, $S' \subseteq \mathcal{T}(\lambda)$ and $F \Rightarrow S$, we get $p \mid a$, which contradicts the assumption $u = 0$. By Lemma 6.2.3 (i), we deduce that $S_a(M^\lambda, F) = 0$. So $(S^a M^\lambda)(F) = 0$ by Lemma 6.2.1. Therefore, $c_{\mathfrak{S}_n}(S^a M^\lambda) = r$.

Case 2: $u > 0$ and $u + r \leq b$.

As $u + r \leq b$, let $E = \langle \bigcup_{i=1}^{u+r} \{s_i\} \rangle$ and $F = \langle \bigcup_{i=1}^r \{s_i\} \rangle$. By Lemma 2.11.1, there exists some $\{t\} \in \mathcal{T}(\lambda)^F$. Note that the orbit $\mathcal{O}$ of $\mathcal{T}(\lambda)$ containing $\{t\}$ under the action of E has size p^u . Otherwise, there exists some $F < H \leq E$ such that $\{t\} \in \mathcal{T}(\lambda)^H$. By Lemma 2.6.1 (iv), $M^\lambda(H) \neq 0$. This implies that $r < \text{rank}(H) \leq c_{\mathfrak{S}_n}(M^\lambda) = r$ by Lemma 2.7.1 (vi) and Theorem 6.1.5. This is a contradiction. As $p^u \mid a$, let S be the multiset satisfying the conditions $S' = \mathcal{O}$ and each member of S' has multiplicity a/p^u in S . So S has a members and $E \Rightarrow S$, which implies that $S_a(M^\lambda, E) \neq 0$ by Lemma 6.2.3 (i). So $(S^a M^\lambda)(E) \neq 0$ by Lemma 6.2.1.

For any elementary abelian p -subgroup L of $\mathfrak{S}_n$ satisfying $\text{rank}(L) > u + r$, note that $S^a(M^\lambda(L)) = 0$ as $M^\lambda(L) = 0$. Also notice that each orbit of $\mathcal{T}(\lambda)$ under the action of L has size at least p^{u+1} . If $S_a(M^\lambda, L) \neq 0$, by Lemma 6.2.3 (i), there exists some multiset S such that S has a members, $S' \subseteq \mathcal{T}(\lambda)$ and $L \Rightarrow S$. We thus have $p^{u+1} \mid a$. This contradicts the fact $u = \nu_p(a)$. So $(S^a M^\lambda)(L) = 0$ by Lemma 6.2.1. Therefore, $c_{\mathfrak{S}_n}(S^a M^\lambda) = u + r$.

Case 3: $u > 0$ and $u + r > b$.

Let $\tilde{E} = \langle \bigcup_{i=1}^b \{s_i\} \rangle$ and $\tilde{F} = \langle \bigcup_{i=1}^r \{s_i\} \rangle$. Let $\tilde{E}$ and $\tilde{F}$ play the roles of E and F in Case 2 respectively. As $b - r < u$, the proof of Case 2 shows that we have $S^a(M^\lambda(\tilde{E})) \neq 0$ or $S_a(M^\lambda, \tilde{E}) \neq 0$. Therefore, $(S^a M^\lambda)(\tilde{E}) \neq 0$ by Lemma 6.2.1. The fact $\text{rank}(\mathfrak{S}_n) = b$ thus forces that $c_{\mathfrak{S}_n}(S^a M^\lambda) = b$. The proof is now complete. $\square$

According to Theorem 6.1.5 and Lemma 6.2.5, we deduce the following result.

Proposition 6.2.6. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $b \in \mathbb{N}_0$. Then $S^b M^\lambda$ is projective if and only if one of the following cases occurs: (i) $n < p$; (ii) $p \nmid b$ and $\lambda_i < p$ for all $1 \leq i \leq \ell$.*

We need another notation to study the exterior powers of a Young permutation module that are projective. Let $b = \lfloor \frac{n}{p} \rfloor$. Set $t_i = \prod_{j=1}^i s_j \in \mathfrak{S}_n$ and put $F_i = \langle t_i \rangle$ for all $1 \leq i \leq b$ (see Notation 4.2.1 (v)). Given $\lambda \vdash n$, for all $1 \leq i \leq b$, put $m_{\lambda,i} = \dim_{\mathbb{F}} M^\lambda(F_i)$ and $\overline{m}_{\lambda,i} = \dim_{\mathbb{F}} M^\lambda - m_{\lambda,i}$. By Lemma 2.11.1, it is easy to compute $m_{\lambda,i}$ for all $1 \leq i \leq b$. For our purpose, note that $\overline{m}_{\lambda,i} \geq 0$ for all $1 \leq i \leq b$.

Proposition 6.2.7. *Let $\lambda \vdash n$, $1 < a \in \mathbb{N}$ and $b = \lfloor \frac{n}{p} \rfloor$. If $a \leq \dim_{\mathbb{F}} M^\lambda$, then $\Lambda^a M^\lambda$ is projective if and only if, for any $1 \leq i \leq b$ and $a = x + py$ where $x, y \in \mathbb{N}_0$, we have either $m_{\lambda,i} < x$ or $\overline{m}_{\lambda,i} < py$.*

Proof. If $\Lambda^a M^\lambda$ is projective, by Lemma 2.6.1 (i), $(\Lambda^a M^\lambda)(F_i) = 0$ for all $1 \leq i \leq b$. For any $1 \leq i \leq b$, by Lemma 6.2.2, $0 = (\Lambda^a M^\lambda)(F_i) \cong \Lambda^a(M^\lambda(F_i)) \oplus \Lambda_a(M^\lambda, F_i)$. In particular, $\Lambda^a(M^\lambda(F_i)) = 0$ and $\Lambda_a(M^\lambda, F_i) = 0$. Note that $\Lambda^a(M^\lambda(F_i)) = 0$ gives us $m_{\lambda,i} < a$. Therefore, it suffices to check the decompositions $a = x + py$, where $x, y \in \mathbb{N}_0$ and $x < a$. For some $1 \leq i \leq b$, suppose that $m_{\lambda,i} \geq x$ and $\overline{m}_{\lambda,i} \geq py$ for such a decomposition $a = x + py$. By Lemma 2.6.1 (iv), $|\mathcal{T}(\lambda)^{F_i}| \geq x$. Moreover, as F_i is a cyclic group of order p and $\overline{m}_{\lambda,i} \geq py$, when acted by F_i , the number of orbits of $\mathcal{T}(\lambda)$ having size p is at least y . Let $\mathcal{X} \subseteq \mathcal{T}(\lambda)^{F_i}$ and $\mathcal{Y} \subseteq \mathcal{T}(\lambda)$, where $|\mathcal{X}| = x$ and $\mathcal{Y}$ is a union set of exactly y orbits of $\mathcal{T}(\lambda)$ having size p under the action of F_i . Since $x < a$ and $y > 0$, we have $|\mathcal{X} \cup \mathcal{Y}| = a$ and $F_i \Rightarrow \mathcal{X} \cup \mathcal{Y}$, which implies that $\Lambda_a(M^\lambda, F_i) \neq 0$ by Lemma 6.2.3 (ii). So $(\Lambda^a M^\lambda)(F_i) \neq 0$ by Lemma 6.2.2. This is a contradiction. We get the desired assertion.

Conversely, if the assertion holds, suppose that $\Lambda^a M^\lambda$ is not projective. By Lemma 2.7.1 (vi), there exists a non-trivial p -subgroup P of $\mathfrak{S}_n$ such that $(\Lambda^a M^\lambda)(P) \neq 0$. We may assume further that $P = F_i$ for some $1 \leq i \leq b$. By Lemma 6.2.2, we have $\Lambda^a(M^\lambda(F_i)) \neq 0$ or $\Lambda_a(M^\lambda, F_i) \neq 0$. For the decomposition $a = a + 0$, we deduce that $\dim_{\mathbb{F}} M^\lambda(F_i) = m_{\lambda,i} < a$ by the assertion. So $\Lambda^a(M^\lambda(F_i)) = 0$ and $\Lambda_a(M^\lambda, F_i) \neq 0$. By Lemma 6.2.3 (ii), we deduce that there exists some $S \subseteq \mathcal{T}(\lambda)$ such that $|S| = a$ and $F_i \Rightarrow S$. Let $x = |S^{F_i}|$ and y be the number of orbits of S having size p under the action of F_i . We have $a = x + py$, $m_{\lambda,i} \geq x$ and $\overline{m}_{\lambda,i} \geq py$. This contradicts the assertion. So $\Lambda^a M^\lambda$ is projective and we are done. $\square$

Example 6.2.8. We end this section with an example. Let $p = 3$, $n = 5$, $a = 8$, $\lambda = (3, 2) \vdash 5$. All allowable decompositions of 8 in Proposition 6.2.7 are exactly $8 + 0$, $5 + 3$, $2 + 6$. By Lemma 2.11.1, $m_{(3,2),1} = 1 < 2$, 5, 8. By Proposition 6.2.7, $\Lambda^8 M^{(3,2)}$ is projective. By this example and Lemma 6.2.5, the inequality in Lemma 6.2.4 can be strict.

6.3 Classification of Indecomposable Exterior Powers of Young Permutation Modules

In this section, we classify all the indecomposable exterior powers of $\mathbb{F}\mathfrak{S}_n$ -Young permutation modules. To formulate the main result of this section, recall that $Y(\alpha|p\beta)$ is the $\mathbb{F}\mathfrak{S}_n$ -signed Young module labelled by partitions α and β , where $|\alpha| + p|\beta| = n$. We need a list of lemmas as a preparation.

Lemma 6.3.1. *Let $a \in \mathbb{N}$ and $1 < a < n$. Then $\Lambda^a M^{(n-1,1)}$ is indecomposable if and only if precisely one of the following cases occurs:*

- (i) $p > 2$, $p \mid n$, $a = pb + r$, where $b, r \in \mathbb{N}_0$ and $r < p$;
- (ii) $p = 2$, $2 \mid n$, $2^m \leq n < 2^{m+1}$ for some $m \in \mathbb{N}$, $2^{i-1} \leq \min\{n-a, a\} < 2^i$ and $\min\{n-a, a\} \equiv \frac{n-2^m}{2} \pmod{2^{i-1}}$ for some $1 \leq i \leq m$.

In (i), we have $\Lambda^a M^{(n-1,1)} \cong Y((n-a, 1^r)|(pb))$. In (ii), we have $\Lambda^a M^{(n-1,1)} \cong Y(\overline{(n-a, a)})$.

Proof. When $p > 2$, according to [23, Theorem 1.4 (i) and Proposition 7.1], $\Lambda^a M^{(n-1,1)}$ is indecomposable if and only if (i) holds. When $p = 2$, note that $\Lambda^a M^{(n-1,1)} \cong M^{(n-a, a)}$. By [25, Theorem 2 (b)], $\Lambda^a M^{(n-1,1)}$ is indecomposable if and only if (ii) holds. We are done. $\square$

One more notation is required for the next lemma. Given $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, let $c(\lambda)$ be the largest subscript c of parts of λ such that $\lambda_c \geq 2$. Set $c(\lambda) = 0$ if $\lambda = (1^n)$. Write

$$f^\lambda = \begin{cases} \dim_{\mathbb{F}} M^\lambda - \sum_{i=1}^{c(\lambda)} \frac{(n-2)^!}{(\lambda_i-2)! \prod_{j \neq i} (\lambda_j)!}, & \text{if } c(\lambda) > 0, \\ \dim_{\mathbb{F}} M^{(1^n)}, & \text{if } c(\lambda) = 0. \end{cases}$$

Lemma 6.3.2. *Let $\lambda \vdash n$ and $d = \dim_{\mathbb{F}} M^\lambda > 1$. Then*

$$\Lambda^d M^\lambda \cong \begin{cases} \mathbb{F}, & \text{if } p = 2 \text{ or } p > 2 \text{ and } 4 \mid f^\lambda, \\ \text{sgn}, & \text{otherwise.} \end{cases}$$

Proof. As $\dim_{\mathbb{F}} \Lambda^d M^\lambda = 1$ and $\mathbb{F}, \text{sgn}$ are the unique one-dimensional $\mathbb{F}\mathfrak{S}_n$ -modules. It suffices to consider $(\Lambda^d M^\lambda)^{\downarrow \langle (1,2) \rangle}$. Let $S = \mathcal{T}(\lambda)^{\langle (1,2) \rangle}$ and note that $\langle (1,2) \rangle$ acts on $\mathcal{T}(\lambda) \setminus S$. Set $x = |S|$ and observe that each orbit of $\mathcal{T}(\lambda) \setminus S$ under the action of $\langle (1,2) \rangle$ has size 2. Let $2y = f^\lambda$. By Lemma 2.11.1, $x = d - f^\lambda$ and $\mathcal{T}(\lambda) \setminus S$ has y orbits under the action of $\langle (1,2) \rangle$. Let $\mathcal{T}(\lambda) = \{\{t_1\}, \dots, \{t_d\}\}$ and define

$$t = \{t_1\} \wedge \{t_2\} \wedge \dots \wedge \{t_{d-x-1}\} \wedge \{t_{d-x}\} \wedge \{t_{d-x+1}\} \wedge \dots \wedge \{t_d\} \in \Lambda^d M^\lambda,$$

where $\{t_i\} \in S$ for all $d-x+1 \leq i \leq d$. Moreover, for all $1 \leq j \leq d-x-1$ and $2 \nmid j$, $\{\{t_j\}, \{t_{j+1}\}\}$ is an orbit of $\mathcal{T}(\lambda) \setminus S$ under the action of $\langle (1,2) \rangle$. We thus have $(1,2)t = (-1)^y t$. The lemma thus follows. $\square$

Let $a \in \mathbb{N}$ and M, N be $\mathbb{F}\mathfrak{S}_n$ -modules. Recall that $\Lambda^a(M \oplus N) \cong \bigoplus_{i=0}^a (\Lambda^i M \otimes \Lambda^{a-i} N)$. This fact will be repeatedly used in the following two lemmas.

Lemma 6.3.3. *Let $\lambda \vdash n$ and $d = \dim_{\mathbb{F}} M^\lambda > 2$. If $\lambda \neq (n-1, 1)$, then we have $\Lambda^{d-1} M^\lambda$ is indecomposable if and only if $p = 2$, $2 \mid n$, λ is one of the m partitions $(n-k_i, k_i)$ except for $(n-1, 1)$, where $m \in \mathbb{N}$, $2^m \leq n < 2^{m+1}$, $2^{i-1} \leq k_i < 2^i$ and $k_i \equiv \frac{n-2^m}{2} \pmod{2^{i-1}}$ for any $1 \leq i \leq m$. Moreover, if $\Lambda^{d-1} M^\lambda$ is indecomposable, then $\Lambda^{d-1} M^\lambda \cong Y^\lambda$.*

Proof. When $p > 2$, by [25, Theorem 2 (a)], M^λ is decomposable since $\lambda \neq (n-1, 1)$ and (n) . So $M^\lambda \cong M \oplus N$ for some non-zero $\mathbb{F}\mathfrak{S}_n$ -modules M and N . We set $x = \dim_{\mathbb{F}} M$, $y = \dim_{\mathbb{F}} N$ and note that $d = x + y$. We have $\Lambda^x M \otimes \Lambda^{y-1} N \mid \Lambda^{d-1} M^\lambda$ and $\Lambda^{x-1} M \otimes \Lambda^y N \mid \Lambda^{d-1} M^\lambda$. So $\Lambda^{d-1} M^\lambda$ is decomposable. When $p = 2$, notice that $\Lambda^{d-1} M^\lambda \cong M^\lambda$. By [25, Theorem 2 (b)], $\Lambda^{d-1} M^\lambda$ is indecomposable if and only if the case stated in the lemma occurs. Moreover, if the case stated in the lemma occurs, we have $\Lambda^{d-1} M^\lambda \cong Y^\lambda$. $\square$

Lemma 6.3.4. *Let $\lambda \vdash n$, $1 < a \in \mathbb{N}$ and $d = \dim_{\mathbb{F}} M^\lambda$. If $\ell(\lambda) \geq 3$ and $a \leq d-2$, then $\Lambda^a M^\lambda$ is decomposable.*

Proof. Since $\ell(\lambda) \geq 3$, M^λ is decomposable by [25, Theorem 2]. So we have $M^\lambda \cong M \oplus N$ for some non-zero $\mathbb{F}\mathfrak{S}_n$ -modules M and N . Let $x = \dim_{\mathbb{F}} M$, $y = \dim_{\mathbb{F}} N$ and note that $d = x + y$. If $x \leq a$, as $1 < a \leq d-2$, note that $0 \leq a-x \leq d-2-x = y-2$ and $1 \leq a-x+1 \leq d-2-(x-1) = y-1$. So $\Lambda^x M \otimes \Lambda^{a-x} N \mid \Lambda^a M^\lambda$ and $\Lambda^{x-1} M \otimes \Lambda^{a-x+1} N \mid \Lambda^a M^\lambda$. If $x > a$, $\Lambda^a M \mid \Lambda^a M^\lambda$ and $\Lambda^{a-1} M \otimes N \mid \Lambda^a M^\lambda$. We thus get that $\Lambda^a M^\lambda$ is decomposable. $\square$

We are left with the case $\lambda = (n-r, r) \vdash n$ and $r > 1$. In this case, we view M^λ naturally as the $\mathbb{F}\mathfrak{S}_n$ -module generated by the subsets of $[1, n]$ having size r . Let $P(r) = \{s \subseteq [1, n] : |s| = r\}$ and define a total order $\leq_\ell$ for $P(r)$ as follows. Let $s = \{a_1, \dots, a_r\}$, $\tilde{s} = \{b_1, \dots, b_r\} \in P(r)$, where $a_1 < \dots < a_r$ and $b_1 < \dots < b_r$. Write $s <_\ell \tilde{s}$ if $a_1 < b_1$ or there exists some $i \in \mathbb{N}$ such that $1 \leq i < r$, $a_j = b_j$ for all $1 \leq j \leq i$ and $a_{i+1} < b_{i+1}$. We have $s =_\ell \tilde{s}$ if and only if $s = \tilde{s}$. We write $P(r) = \{s_1 <_\ell \dots <_\ell s_d\}$, where $d = \dim_{\mathbb{F}} M^{(n-r, r)}$. If $1 < a \in \mathbb{N}$ and $a \leq d$, $P(r, a) = \{s_{i_1} \wedge \dots \wedge s_{i_a} \in \Lambda^a M^{(n-r, r)} : 1 \leq i_1 < \dots < i_a \leq d\}$ is an $\mathbb{F}$ -basis of $\Lambda^a M^{(n-r, r)}$. Given $v = s_{i_1} \wedge \dots \wedge s_{i_a} \in P(r, a)$, for all $1 \leq j \leq n$, define n_j^v to be the number of j in the sets $s_{i_1}, \dots, s_{i_a}$. Put $m_k^v = |\{1 \leq i \leq n : n_i^v = k\}|$ for all $k \geq 0$. Note that $ra = \sum_{i=1}^n n_i^v$ and $n = \sum_{i \geq 0} m_i^v$. For any $u, v \in P(r, a)$, if there exists $g \in \mathfrak{S}_n$ such that $gu = v$ or $gu = -v$, then note that $m_k^u = m_k^v$ for all $k \geq 0$.

Example 6.3.5. We use an example to illustrate the above definitions. Let $a = 3$, $\lambda = (3, 2) \vdash 5$ and $v = \{1, 2\} \wedge \{1, 3\} \wedge \{1, 4\} \in P(2, 3)$. So $\{1, 2\} <_\ell \{1, 3\}$. We have $n_1^v = 3$, $n_2^v = n_3^v = n_4^v = 1$, $n_5^v = 0$, $m_0^v = m_3^v = 1$, $m_1^v = 3$ and $m_i^v = 0$ for all $i \in \mathbb{N} \setminus \{1, 3\}$.

Lemma 6.3.6. *Let $a, r \in \mathbb{N}$, $2 < 2r \leq n$ and $1 < a \leq d = \dim_{\mathbb{F}} M^{(n-r, r)}$. If $a \leq d-2$, then $\Lambda^a M^{(n-r, r)}$ is decomposable.*

Proof. We shall pick $x, y \in P(r, a)$ and show that $m_i^x \neq m_i^y$ for some $i \geq 0$. This fact implies the correctness of the lemma. Let $u = s_1 \wedge \dots \wedge s_a \in P(r, a)$ and $v = s_2 \wedge \dots \wedge s_a \wedge s_d$. As $a \leq d-2$, we have $s_a <_\ell s_{d-1}$ and $v \in P(r, a)$. Following the total order $\leq_\ell$, $s_1 = \{1, \dots, r\}$, $s_{d-1} = \{n-r, n-r+2, \dots, n\}$ and $s_d = \{n-r+1, \dots, n\}$. Given $b \in [1, n]$, let $t = |S_b|$, where

$$S_b = \{s \in P(r) : b \in s\}.$$

Note that $t = \binom{n-1}{r-1} = \frac{rd}{n}$, $n_1^u = \min\{a, t\}$ and $n_c^u \leq t$ for all $1 \leq c \leq n$. We have two cases.

Case 1: $a \geq t$.

In this case, $n_1^u = t$. As $a \leq d-2$, neither s_{d-1} nor s_d occurs in u as a component. So, for all $n-r+2 \leq i \leq n$, we have $n_i^u \leq t-2$. If $n_{n-r+1}^u \leq t-2$, then we have $m_t^u > m_t^v$, which implies that $\Lambda^a M^{(n-r, r)}$ is decomposable. If $n_{n-r+1}^u = t-1$ and $n-r > 2$, then $\{n-r, n-r+1, \dots, n-1\}$ has to occur in u as a component. By the definitions of u and $\leq_\ell$, as $r > 1$, notice that $n_2^u = t$. So we still have $m_t^u > m_t^v$ and $\Lambda^a M^{(n-r, r)}$ is decomposable. If $n-r = 2$ and $n_3^u = t-1$, as $2 < 2r \leq n$, $n_3^u = \frac{rd}{n} - 1 = 2$, $a = 4$ and $u = \{1, 2\} \wedge \{1, 3\} \wedge \{1, 4\} \wedge \{2, 3\}$. Set $z = \{1, 2\} \wedge \{1, 3\} \wedge \{2, 4\} \wedge \{3, 4\}$ and note that $m_2^u = 2 < 4 = m_2^z$. This means that $\Lambda^4 M^{(2, 2)}$ is decomposable.

Case 2: $a < t$.

In this case, $n_1^u = a$. As $1 < a$, both $\{1, \dots, r\}$ and $\{1, \dots, r-1, r+1\}$ occur in u as components. So, for all $n-r+2 \leq i \leq n$, $n_i^u \leq a-2$. If $2r < n$, then $r+1 < n-r+1$ and $n_{n-r+1}^u \leq a-2$ as well. So $m_a^u > m_a^v$ and $\Lambda^a M^{(n-r, r)}$ is decomposable. If $2r = n$ and $a > 2$, note that $\{1, \dots, r\}$, $\{1, \dots, r-1, r+1\}$ and $\{1, \dots, r-1, r+2\}$ occur in u as components. In particular, we have $n_{r+1}^u \leq a-2$. Therefore, following the case $2r < n$, we have $m_a^u > m_a^v$ and $\Lambda^a M^{(n-r, r)}$ is decomposable. If $2r = n$ and $a = 2$, then $u = \{1, \dots, r\} \wedge \{1, \dots, r-1, r+1\}$. Set $z = \{1, \dots, r\} \wedge \{r+1, \dots, 2r\}$. We have $m_2^u = r-1 > 0 = m_2^z$, which implies that $\Lambda^2 M^{(n-r, r)}$ is decomposable. The lemma follows by combining the two listed cases. $\square$

We now summarize what we have got by the following theorem. The following theorem is also the main result of this section. Note that the complete list of possible cases of this theorem is covered by Lemmas 6.3.1, 6.3.2, 6.3.3, 6.3.4 and 6.3.6.

Theorem 6.3.7. *Let $\lambda \vdash n$, $1 < a \in \mathbb{N}$ and $a \leq d = \dim_{\mathbb{F}} M^\lambda$. Then $\Lambda^a M^\lambda$ is indecomposable if and only if exactly one of the following cases occurs:*

(i) $a = d$. In this case,

$$\Lambda^d M^\lambda \cong \begin{cases} \mathbb{F}, & \text{if } p = 2 \text{ or } p > 2 \text{ and } 4 \mid f^\lambda, \\ \text{sgn}, & \text{otherwise;} \end{cases}$$

(ii) $p = 2$, $2 \mid n$, $a = d - 1$, λ is one of the m partitions $(n - k_i, k_i)$ except for $(n - 1, 1)$, where $m \in \mathbb{N}$, $2^m \leq n < 2^{m+1}$, $2^{i-1} \leq k_i < 2^i$ and $k_i \equiv \frac{n-2^m}{2} \pmod{2^{i-1}}$ for any $1 \leq i \leq m$. In this case, $\Lambda^{d-1} M^\lambda \cong Y^\lambda$;

(iii) $p > 2$, $p \mid n$, $a < n$, $\lambda = (n - 1, 1)$, $a = pb + r$, where $b, r \in \mathbb{N}_0$ and $0 \leq r < p$. In this case, $\Lambda^a M^{(n-1,1)} \cong Y((n-a, 1^r)|(pb))$;

(iv) $p = 2$, $2 \mid n$, $a < n$, $\lambda = (n - 1, 1)$, $2^m \leq n < 2^{m+1}$ for some $m \in \mathbb{N}$, $2^{i-1} \leq \min\{n-a, a\} < 2^i$ and $\min\{n-a, a\} \equiv \frac{n-2^m}{2} \pmod{2^{i-1}}$ for some $1 \leq i \leq m$. In this case, $\Lambda^a M^{(n-1,1)} \cong Y^{\overline{(n-a, a)}}$.

Remark 6.3.8. We have the following remarks.

- (i) In [25], Gill classified all the indecomposable $\mathbb{F}\mathfrak{S}_n$ -Young permutation modules. As the trivial $\mathbb{F}\mathfrak{S}_n$ -module is indecomposable, according to [25, Theorem 2] and Theorem 6.3.7, note that all the indecomposable exterior powers of $\mathbb{F}\mathfrak{S}_n$ -Young permutation modules are now classified.
- (ii) Let $1 < a \in \mathbb{N}$ and $\lambda \vdash n$. Note that $S^a M^\lambda$ is indecomposable if and only if $\lambda = (n)$. Therefore, all the indecomposable symmetric powers of $\mathbb{F}\mathfrak{S}_n$ -Young permutation modules are also clear.

6.4 Proof of Theorem 6.1.6: Symmetric Powers of Young Permutation Modules

In this section, we prove Theorem 6.1.6 (i). More specifically, let $1 < a \in \mathbb{N}$ and $\lambda \vdash n$. We find a Young module Y^μ such that $Y^\mu \mid S^a M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(S^a M^\lambda)$. We begin by fixing the required notation.

Notation 6.4.1. Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $1 < a \in \mathbb{N}$.

- (i) Let $m \in \mathbb{N}$, $\mu = (\mu_1, \dots, \mu_{\ell'}) \models m$ and $\{t\} \in \mathcal{T}(\mu)$. For all $1 \leq i \leq \ell'$, if $\mu_i \neq 0$, let $R_i(\{t\})$ be the set of entries in the i th row of $\{t\}$. Otherwise, $R_i(\{t\}) = \emptyset$. Let the numbers $1, \dots, m$ replace the boxes of μ from left to right and from the top to the bottom successively and use t^μ to denote the obtained μ -tableau. This μ -tableau t^μ is called the initial μ -tableau. Put $d_{\lambda,a} = \min\{\nu_p(a), \lfloor \frac{n}{p} \rfloor - \text{rank}(\mathfrak{S}_\lambda)\}$ and $e_{\lambda,a} = n - pd_{\lambda,a}$. Note that $e_{\lambda,a} \geq 0$.
- (ii) For all $1 \leq i \leq \ell$, we write $\lambda_i = pq_i + r_i$, where $q_i, r_i \in \mathbb{N}_0$ and $r_i < p$. We set $q_\lambda = (pq_1, \dots, pq_\ell)$ and $r_\lambda = (r_1, \dots, r_\ell)$. It is obvious to see that $|q_\lambda| = \text{rank}(\mathfrak{S}_\lambda)p$ and $|r_\lambda| = n - \text{rank}(\mathfrak{S}_\lambda)p \geq pd_{\lambda,a}$. Moreover, q_λ is a non-increasing sequence. Let $r_{\lambda,a}$ be the composition obtained from r_λ by removing $pd_{\lambda,a}$ boxes from right to left and from the bottom to the top successively. Let $\lambda^a = q_\lambda + r_{\lambda,a} \models e_{\lambda,a}$.
- (iii) Define a λ -tabloid $\mathfrak{s}_\lambda$ as follows. If $0 < \text{rank}(\mathfrak{S}_\lambda) < \frac{n}{p}$, for all $1 \leq i \leq \ell$, put

$$R_i(\mathfrak{s}_\lambda) = \begin{cases} R_i(\{t^{q_\lambda}\}) \cup \{\text{rank}(\mathfrak{S}_\lambda)p + x : x \in R_i(\{t^{r_\lambda}\})\}, & \text{if } r_i > 0, \\ R_i(\{t^{q_\lambda}\}), & \text{if } r_i = 0. \end{cases}$$

Otherwise, set $\mathfrak{s}_\lambda = \{t^\lambda\}$. For the case $e_{\lambda,a} > 0$, define a λ^a -tabloid $\mathfrak{s}_{\lambda,a}$ as follows. If $0 < e_{\lambda,a} < n$, for all $1 \leq i \leq \ell$, set $R_i(\mathfrak{s}_{\lambda,a}) = R_i(\mathfrak{s}_\lambda) \setminus \{e_{\lambda,a} + 1, \dots, n\}$. If $e_{\lambda,a} = n$, we have $d_{\lambda,a} = 0$, $r_{\lambda,a} = r_\lambda$ and $\lambda^a = \lambda$. Set $\mathfrak{s}_{\lambda,a} = \mathfrak{s}_\lambda$.

- (iv) Let $H_{\lambda,a} = \{g \in \mathfrak{S}_{e_{\lambda,a}} : g\mathfrak{s}_{\lambda,a} = \mathfrak{s}_{\lambda,a}\} \leq \mathfrak{S}_{e_{\lambda,a}}$ for the case $e_{\lambda,a} > 0$. If $e_{\lambda,a} = 0$, put $H_{\lambda,a} = 1$. If $d_{\lambda,a} > 0$, define a p -cycle $s_{\lambda,a,i} = (e_{\lambda,a} + (i-1)p + 1, \dots, e_{\lambda,a} + ip)$ for all $1 \leq i \leq d_{\lambda,a}$. Let P be a Sylow p -subgroup of $H_{\lambda,a}$ and put

$$P_{\lambda,a} = \begin{cases} P \times \langle \bigcup_{i=1}^{d_{\lambda,a}} \{s_{\lambda,a,i}\} \rangle, & \text{if } d_{\lambda,a} > 0, \\ P, & \text{if } d_{\lambda,a} = 0. \end{cases}$$

So $P_{\lambda,a}$ is unique up to $H_{\lambda,a}$ -conjugation. Let $\mathfrak{S}_{\lambda,a} = H_{\lambda,a} \times (\mathfrak{S}_p)^{d_{\lambda,a}}$, where, if $d_{\lambda,a} > 0$, the first factor $\mathfrak{S}_p$ acts on the set $\{e_{\lambda,a} + 1, \dots, e_{\lambda,a} + p\}$, the second factor $\mathfrak{S}_p$ acts on the set $\{e_{\lambda,a} + p + 1, \dots, e_{\lambda,a} + 2p\}$ and so on. Notice that $P_{\lambda,a}$ is a Sylow p -subgroup of $\mathfrak{S}_{\lambda,a}$ and $\mathfrak{S}_{\lambda,a}$ is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_\eta$, where $\eta = \lambda^a \cup (p^{d_{\lambda,a}})$.

- (v) Let $d = \dim_{\mathbb{F}} M^\lambda$ and label $\mathcal{T}(\lambda) = \{\{t_1\}, \dots, \{t_d\}\}$, where $\{t_1\} = \{t^\lambda\}$. Write

$$\begin{aligned} \mathcal{T}(\lambda)_a^s &= \{\{t_{i_1}\} \dots \{t_{i_a}\} \in S^a M^\lambda : 1 \leq i_1 \leq \dots \leq i_a \leq d\}, \\ \mathcal{T}(\lambda)_a^e &= \{\{t_{i_1}\} \wedge \dots \wedge \{t_{i_a}\} \in \Lambda^a M^\lambda : 1 \leq i_1 < \dots < i_a \leq d\}. \end{aligned}$$

Note that $\mathcal{T}(\lambda)_a^s$ is an $\mathbb{F}$ -basis of $S^a M^\lambda$. Moreover, if $a \leq d$, also note that $\mathcal{T}(\lambda)_a^e$ is an $\mathbb{F}$ -basis of $\Lambda^a M^\lambda$. Given $t \in \mathcal{T}(\lambda)_a^s$, let $\mathcal{O}^s(t)$ be the orbit of $\mathcal{T}(\lambda)_a^s$ containing t under the action of $\mathfrak{S}_n$ and set $V(t) = \langle \mathcal{O}^s(t) \rangle_{\mathbb{F}}$. Therefore, $V(t) \cong (\mathbb{F}_{K(t)})^{\uparrow \mathfrak{S}_n}$, where $K(t) = \{g \in \mathfrak{S}_n : gt = t\} \leq \mathfrak{S}_n$. For some $x \in \mathbb{N}$,

$$S^a M^\lambda = \bigoplus_{i=1}^x V(t_i) \cong \bigoplus_{i=1}^x (\mathbb{F}_{K(t_i)})^{\uparrow \mathfrak{S}_n}, \quad (6.4.1)$$

where $x \leq d$ and $t_1, \dots, t_x \in \mathcal{T}(\lambda)_a^s$. We now assume further that $a \leq d$. Let $t \in \mathcal{T}(\lambda)_a^e$ and $\mathcal{O}^e(t) = \{s \in \mathcal{T}(\lambda)_a^e : \exists g \in \mathfrak{S}_n, gt = s \text{ or } gt = -s\}$. Let $W(t) = \langle \mathcal{O}^e(t) \rangle_{\mathbb{F}}$ and $L(t) = \{g \in \mathfrak{S}_n : gt = t \text{ or } gt = -t\} \leq \mathfrak{S}_n$. Note that $W(t) \cong (\mathbb{F}^t)^{\uparrow \mathfrak{S}_n}$, where $\mathbb{F}^t$ is a one-dimensional $\mathbb{F}L(t)$ -module. If $t' \in \mathcal{O}^e(t)$, also note that $L(t')$ is $\mathfrak{S}_n$ -conjugate to $L(t)$ and $(\mathbb{F}^{t'})^{\uparrow \mathfrak{S}_n} \cong (\mathbb{F}^t)^{\uparrow \mathfrak{S}_n}$. Moreover, $\mathbb{F}^t = \mathbb{F}$ if $p = 2$. For some $y \in \mathbb{N}$, $y \leq d$ and $u_1, \dots, u_y \in \mathcal{T}(\lambda)_a^e$,

$$\Lambda^a M^\lambda = \bigoplus_{i=1}^y W(u_i) \cong \bigoplus_{i=1}^y (\mathbb{F}^{u_i})^{\uparrow \mathfrak{S}_n}. \quad (6.4.2)$$

Example 6.4.2. We use an example to illustrate most of the definitions in Notation 6.4.1. Let $p = 3$, $n = 11$, $a = 6$ and $\lambda = (5, 4, 2) \vdash 11$. We have $d_{\lambda,a} = 1$, $e_{\lambda,a} = 8$, $q_\lambda = (3^2, 0) \models 6$, $r_\lambda = (2, 1, 2) \models 5$, $r_{\lambda,a} = (2) \vdash 2$ and $\lambda^a = (5, 3, 0) \models 8$. Moreover,

$$\{t^\lambda\} = \frac{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & \\ \hline 10 & 11 & & & \end{array}}{\quad}, \quad \mathfrak{s}_\lambda = \frac{\begin{array}{ccccc} 1 & 2 & 3 & 7 & 8 \\ \hline 4 & 5 & 6 & 9 & \\ \hline 10 & 11 & & & \end{array}}{\quad}, \quad \mathfrak{s}_{\lambda,a} = \frac{\begin{array}{ccccc} 1 & 2 & 3 & 7 & 8 \\ \hline 4 & 5 & 6 & & \\ \hline & & & & \end{array}}{\quad},$$

$H_{\lambda,a} = \mathfrak{S}_{\{1,2,3,7,8\}} \times \mathfrak{S}_{\{4,5,6\}}$ and $\mathfrak{S}_{\lambda,a} = H_{\lambda,a} \times \mathfrak{S}_{\{9,10,11\}}$. Also note that $P_{\lambda,a}$ is $H_{\lambda,a}$ -conjugate to $\langle (1, 2, 3), (4, 5, 6), (9, 10, 11) \rangle$.

The following lemma contains some properties of the definitions given in Notation 6.4.1.

Lemma 6.4.3. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. Let $\mathcal{O}_{\lambda,a}$ be the orbit of $\mathcal{T}(\lambda)$ containing $\mathfrak{s}_\lambda$ under the action of $P_{\lambda,a}$.*

- (i) *For any $\{t\} \in \mathcal{O}_{\lambda,a}$, if $e_{\lambda,a} > 0$, then $R_i(\mathfrak{s}_{\lambda,a}) \subseteq R_i(\{t\})$ for all $1 \leq i \leq \ell(\lambda)$.*
- (ii) *We have $|\mathcal{O}_{\lambda,a}| = p^{d_{\lambda,a}}$. In particular, $|\mathcal{O}_{\lambda,a}| \mid a$.*

Proof. By the definition of $P_{\lambda,a}$, $P_{\lambda,a} = P \times E$, where $P \leq H_{\lambda,a}$, $E \leq \mathfrak{S}_n$ and $|E| = p^{d_{\lambda,a}}$. Moreover, if $e_{\lambda,a} > 0$, for any $e \in E$ and $x \in \{1, \dots, e_{\lambda,a}\}$, $e(x) = x$.

For (i), as $e_{\lambda,a} > 0$, by the definition of $\mathfrak{s}_{\lambda,a}$, $R_i(\mathfrak{s}_{\lambda,a}) \subseteq R_i(\mathfrak{s}_\lambda)$ for all $1 \leq i \leq \ell(\lambda)$. As $H_{\lambda,a}$ fixes $\mathfrak{s}_{\lambda,a}$ and all the entries of $\mathfrak{s}_{\lambda,a}$ are exactly the numbers $1, \dots, e_{\lambda,a}$, if $g\mathfrak{s}_\lambda = \{t\}$ for some

$g \in P_{\lambda,a}$, we have $g\mathfrak{s}_{\lambda,a} = \mathfrak{s}_{\lambda,a}$ and $R_i(g\mathfrak{s}_{\lambda,a}) \subseteq R_i(g\mathfrak{s}_\lambda) = R_i(\{t\})$ for all $1 \leq i \leq \ell(\lambda)$. The two facts imply the correctness of (i).

For (ii), if $e_{\lambda,a} = n$, then $d_{\lambda,a} = 0$, $\mathfrak{s}_\lambda = \mathfrak{s}_{\lambda,a}$ and $P_{\lambda,a} = P \leq H_{\lambda,a}$. Therefore, $P_{\lambda,a}$ fixes $\mathfrak{s}_\lambda$ and $|\mathcal{O}_{\lambda,a}| = 1$. If $0 < e_{\lambda,a} < n$, by (i), P fixes $\mathfrak{s}_\lambda$. By the definition of $\mathfrak{s}_\lambda$, for all $1 \leq i \leq \ell(\lambda)$, $|R_i(\mathfrak{s}_\lambda) \cap \{e_{\lambda,a} + 1, \dots, n\}| < p$. As $e(x) = x$ for any $e \in E$ and $x \in \{1, \dots, e_{\lambda,a}\}$, by Lemma 2.11.1, this implies that $e\mathfrak{s}_\lambda \neq \mathfrak{s}_\lambda$ for all $1 \neq e \in E$. So $|\mathcal{O}_{\lambda,a}| = p^{d_{\lambda,a}}$. If $e_{\lambda,a} = 0$, by the definitions of $H_{\lambda,a}$ and $d_{\lambda,a}$, $P_{\lambda,a} = E$ and every part of λ is strictly less than p . By Lemma 2.11.1, $e\mathfrak{s}_\lambda \neq \mathfrak{s}_\lambda$ for all $1 \neq e \in E$. So $|\mathcal{O}_{\lambda,a}| = p^{d_{\lambda,a}}$. The first assertion is shown. As $d_{\lambda,a} \leq \nu_p(a)$ and $|\mathcal{O}_{\lambda,a}| = p^{d_{\lambda,a}}$, $|\mathcal{O}_{\lambda,a}| \mid a$, as desired. $\square$

Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. Let $\mathcal{O}_{\lambda,a}$ be the orbit of $\mathcal{T}(\lambda)$ containing $\mathfrak{s}_\lambda$ under the action of $P_{\lambda,a}$. Let $o_{\lambda,a} = |\mathcal{O}_{\lambda,a}|$ and $\mathcal{O}_{\lambda,a} = \{\{s_1\}, \dots, \{s_{o_{\lambda,a}}\}\}$. By Lemma 6.4.3 (ii), $a = o_{\lambda,a}x_{\lambda,a}$ for some $x_{\lambda,a} \in \mathbb{N}$. For all $1 \leq i \leq o_{\lambda,a}$, formally write

$$\{s_i\}^{\otimes x_{\lambda,a}} = \underbrace{\{s_i\} \dots \{s_i\}}_{x_{\lambda,a} \text{ times}}$$

and let $t_{\lambda,a} = \{s_1\}^{\otimes x_{\lambda,a}} \dots \{s_{o_{\lambda,a}}\}^{\otimes x_{\lambda,a}} \in \mathcal{T}(\lambda)_a^s$. Notice that $P_{\lambda,a} \leq K(t_{\lambda,a})$. Moreover, $g\{s_i\} \in \mathcal{O}_{\lambda,a}$ for any $g \in K(t_{\lambda,a})$ and $1 \leq i \leq o_{\lambda,a}$.

We use the following three lemmas to discuss the relationship between $K(t_{\lambda,a})$ and $\mathfrak{S}_{\lambda,a}$.

Lemma 6.4.4. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. If $d_{\lambda,a} = 0$, then $K(t_{\lambda,a}) = \mathfrak{S}_{\lambda,a}$.*

Proof. As $d_{\lambda,a} = 0$, by the definitions of $\mathcal{O}_{\lambda,a}$, $t_{\lambda,a}$, $\mathfrak{s}_{\lambda,a}$, $\mathfrak{S}_{\lambda,a}$ and Lemma 6.4.3 (ii), $e_{\lambda,a} = n$, $\mathcal{O}_{\lambda,a} = \{\mathfrak{s}_\lambda\}$, $t_{\lambda,a} = \mathfrak{s}_\lambda^{\otimes a}$, $\mathfrak{s}_\lambda = \mathfrak{s}_{\lambda,a}$ and $K(t_{\lambda,a}) = H_{\lambda,a} = \mathfrak{S}_{\lambda,a}$. This completes the proof. $\square$

Lemma 6.4.5. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. If $d_{\lambda,a} > 0$, then $K(t_{\lambda,a}) = H_{\lambda,a} \times K$, where $K \leq \mathfrak{S}_{\{e_{\lambda,a}+1, \dots, n\}}$.*

Proof. As $d_{\lambda,a} > 0$, we have $0 \leq e_{\lambda,a} < n$. If $e_{\lambda,a} = 0$, by the definition of $H_{\lambda,a}$, $H_{\lambda,a} = 1$. The desired equality holds trivially. We thus assume that $0 < e_{\lambda,a} < n$.

For any given $g \in K(t_{\lambda,a})$, we claim that there does not exist a pair $i, j \in [1, n]$ such that $i \in \{1, \dots, e_{\lambda,a}\}$, $j \in \{e_{\lambda,a}+1, \dots, n\}$ and $g(j) = i$. Suppose that such g, i and j exist. Then, for any $\{t\} \in \mathcal{O}_{\lambda,a}$, i and j must lie in the same row of $\{t\}$. Otherwise, if $i \in R_u(\{t\})$, $j \in R_v(\{t\})$ and $u \neq v$, by Lemma 6.4.3 (i), $i \in R_u(\mathfrak{s}_{\lambda,a}) \cap R_v(g\{t\})$. As $g \in K(t_{\lambda,a})$, $g\{t\} \in \mathcal{O}_{\lambda,a}$. By Lemma 6.4.3 (i) again, $i \in R_u(g\{t\})$. This is a contradiction. We thus assume that $i, j \in R_u(\{t\})$. As $d_{\lambda,a} > 0$, by the definition of $d_{\lambda,a}$, $\lambda \neq (n)$. By the definition of $P_{\lambda,a}$, note that there exist some $m, w \in \mathbb{N}$ and $k \in \{e_{\lambda,a}+1, \dots, n\} \setminus \{j\}$ such that $1 \leq m < p$, $1 \leq w \leq d_{\lambda,a}$ and $s_{\lambda,a,w}^m(k) = j$. As $\{t\} \in \mathcal{O}_{\lambda,a}$, by the definition of $\mathcal{O}_{\lambda,a}$, $|R_u(\{t\}) \cap \{e_{\lambda,a}+1, \dots, n\}| < p$. Therefore, we may require that $k \in R_v(\{t\})$ and $u \neq v$. We have $i \in R_u(s_{\lambda,a,w}^m\{t\})$ and $j \in R_v(s_{\lambda,a,w}^m\{t\})$. Since $s_{\lambda,a,w} \in P_{\lambda,a}$, $s_{\lambda,a,w}^m\{t\} \in \mathcal{O}_{\lambda,a}$. By above discussion, we get a contradiction as i, j are not in the same row of $s_{\lambda,a,w}^m\{t\}$. The claim is shown.

By this claim, $K(t_{\lambda,a}) \leq H \times K$, where H is the projection of $K(t_{\lambda,a})$ with respect to $\mathfrak{S}_{e_{\lambda,a}}$ and K is the projection of $K(t_{\lambda,a})$ with respect to $\mathfrak{S}_{\{e_{\lambda,a}+1, \dots, n\}}$. Note that H fixes $\mathfrak{s}_{\lambda,a}$. Otherwise, if there exist some $h \in H$, $i \in R_c(\mathfrak{s}_{\lambda,a})$ and $j \in R_d(\mathfrak{s}_{\lambda,a})$ such that $c \neq d$ and $h(i) = j$, as there exists some $x \in K(t_{\lambda,a})$ such that $x = hk$ and $k \in K$, we get that $x(i) = j$ and $j \in R_c(x\mathfrak{s}_{\lambda,a}) \cap R_d(\mathfrak{s}_{\lambda,a})$. By Lemma 6.4.3 (i), $j \in R_c(x\mathfrak{s}_\lambda) \cap R_d(x\mathfrak{s}_\lambda)$, which is a contradiction. So $H \leq H_{\lambda,a}$. Since $H_{\lambda,a} \leq \mathfrak{S}_{e_{\lambda,a}}$, by Lemma 6.4.3 (i) again, $H_{\lambda,a} \leq K(t_{\lambda,a})$. We have $H = H_{\lambda,a}$ as $H \leq H_{\lambda,a} \leq H \cap K(t_{\lambda,a})$. This equality implies that $K \leq K(t_{\lambda,a})$ and $K(t_{\lambda,a}) = H_{\lambda,a} \times K$. The proof is now complete. $\square$

Lemma 6.4.6. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. If $d_{\lambda,a} > 0$, then $K(t_{\lambda,a}) \leq \mathfrak{S}_{\lambda,a}$.*

Proof. By Lemma 6.4.5 and the definitions of $d_{\lambda,a}$ and $\mathfrak{S}_{\lambda,a}$, as $d_{\lambda,a} > 0$, we have $0 \leq e_{\lambda,a} < n$, $\lambda \neq (n)$, $K(t_{\lambda,a}) = H_{\lambda,a} \times K$ and $K \leq \mathfrak{S}_{\{e_{\lambda,a}+1, \dots, n\}}$. It suffices to show that K is contained in

$$\underbrace{\mathfrak{S}_p \times \dots \times \mathfrak{S}_p}_{d_{\lambda,a} \text{ times}}$$

where the first factor $\mathfrak{S}_p$ acts on the set $\{e_{\lambda,a} + 1, \dots, e_{\lambda,a} + p\}$, the second factor $\mathfrak{S}_p$ acts on the set $\{e_{\lambda,a} + p + 1, \dots, e_{\lambda,a} + 2p\}$ and so on. For all $1 \leq u < v \leq d_{\lambda,a}$, we claim that there do not exist $k \in K$, $i, j \in [1, n]$ such that $i \in \{e_{\lambda,a} + (u-1)p + 1, \dots, e_{\lambda,a} + up\}$, $j \in \{e_{\lambda,a} + (v-1)p + 1, \dots, e_{\lambda,a} + vp\}$ and $k(i) = j$.

Suppose that such u, v, i, j and k exist. For any $\{t\} \in \mathcal{O}_{\lambda,a}$, by the definitions of $\mathfrak{s}_\lambda$ and $\mathcal{O}_{\lambda,a}$, there exist some $b, c, d, e \in \mathbb{N}$ (independent of the choices of $\{t\}$) such that $b < c \leq d < e$, the numbers $e_{\lambda,a} + (u-1)p + 1, \dots, e_{\lambda,a} + up$ lie in the b th row, $\dots$, c th row of $\{t\}$, the numbers $e_{\lambda,a} + (v-1)p + 1, \dots, e_{\lambda,a} + vp$ lie in the d th row, $\dots$, e th row of $\{t\}$. Moreover, we also have $|R_h(\{t\}) \cap \{e_{\lambda,a} + 1, \dots, n\}| < p$ for all $b \leq h \leq e$. As $k \in K(t_{\lambda,a})$ by Lemma 6.4.5, we have $k\{t\} \in \mathcal{O}_{\lambda,a}$. Therefore, $j \in R_x(k\{t\})$, where $b \leq x \leq c$ and $d \leq x \leq e$. This forces that $c = d$ and $i \in R_c(\{t\})$. Recall that $s_{\lambda,a,u} = (e_{\lambda,a} + (u-1)p + 1, \dots, e_{\lambda,a} + up)$ and $s_{\lambda,a,u} \in P_{\lambda,a} \leq K(t_{\lambda,a})$. Note that there exists some $m \in \mathbb{N}$ such that $1 \leq m < p$, $i \in R_y(s_{\lambda,a,u}^m \{t\})$ and $b \leq y < c$. Note that $s_{\lambda,a,u}^m \{t\} \in \mathcal{O}_{\lambda,a}$ while $i \notin R_c(s_{\lambda,a,u}^m \{t\})$. This is a contradiction. The claim is shown and the desired containment follows. $\square$

The following corollary may have its own interest.

Corollary 6.4.7. *Let $p = 2$, $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. Then $K(t_{\lambda,a}) = \mathfrak{S}_{\lambda,a}$.*

Proof. By Lemma 6.4.4, we may assume that $d_{\lambda,a} > 0$. As $p = 2$, by the definitions of $P_{\lambda,a}$ and $\mathfrak{S}_{\lambda,a}$, Lemmas 6.4.5 and 6.4.6 imply that $K(t_{\lambda,a}) \leq \mathfrak{S}_{\lambda,a} = H_{\lambda,a} \times \langle \bigcup_{i=1}^{d_{\lambda,a}} \{s_{\lambda,a,i}\} \rangle \leq K(t_{\lambda,a})$, which completes the proof. $\square$

Example 6.4.8. Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. We may not always have $K(t_{\lambda,a}) = \mathfrak{S}_{\lambda,a}$. For a counterexample, let $p = a = 3$, $n = 6$ and $\lambda = (3, 1^3) \vdash 6$. Note that $d_{\lambda,a} = 1$, $e_{\lambda,a} = 3$, $\mathfrak{s}_\lambda = \{t^\lambda\}$ and $P_{\lambda,a} = \langle (1, 2, 3), (4, 5, 6) \rangle$. Therefore, $\mathcal{O}_{\lambda,a}$ contains exactly

$$\begin{array}{ccc|ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ \hline 4 & & & 5 & & & 6 & & \\ \hline 5 & & & 6 & & & 4 & & \\ \hline 6 & & & 4 & & & 5 & & \end{array}.$$

Also notice that $\mathfrak{S}_{\lambda,a} = \mathfrak{S}_{(3^2)}$. Observe that $K(t_{\lambda,a}) < \mathfrak{S}_{\lambda,a}$.

To present the main result of this section, we list two results as follows.

Theorem 6.4.9. [11, Propositions 12.1.1 and 13.1.2] *Let $\lambda \vdash n$ and λ have the p -adic expansion $\sum_{i=0}^m p^i \lambda(i)$ for some $m \in \mathbb{N}_0$. Then Y^λ is a Scott module if and only if $\lambda(i) = ((p-1)^{m_i}, n_i)$, $m_i \geq 0$ and $0 \leq n_i < p-1$ for all $0 \leq i \leq m$.*

Lemma 6.4.10. *Let $\lambda \vdash n$. Then there exists a unique sum $\sum_{i=0}^m p^i \lambda[i]$ such that $m \in \mathbb{N}_0$, $\lambda[i]$ is a composition and each part of $\lambda[i]$ is no more than $p-1$ for all $0 \leq i \leq m$, $\lambda[m] \neq \emptyset$ and $\lambda = \sum_{i=0}^m p^i \lambda[i]$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$. For all $1 \leq b \leq \ell$, let $\lambda_b = \sum_{c \geq 0} p^c a_{b,c}$, where $a_{b,c} \in \mathbb{N}_0$ and $a_{b,c} < p$ for all $c \geq 0$. For all $c \geq 0$, set $\lambda[c] = (a_{1,c}, \dots, a_{\ell,c})$. Note that $\lambda = \sum_{i=0}^m p^i \lambda[i]$ and $\lambda[m] \neq \emptyset$ for some $m \in \mathbb{N}_0$. Moreover, for all $0 \leq i \leq m$ and $\lambda[i] \neq \emptyset$, each part of $\lambda[i]$ is no more than $p-1$. We get the existence of the desired sum. For all $1 \leq b \leq \ell$ and $c \geq 0$, if $a_{b,c} \neq 0$, then $a_{b,c}$ is uniquely determined by λ_b . This fact gives us the uniqueness of the sum. The proof is now complete. $\square$

Definition 6.4.11. Let $\lambda \vdash n$ and $\lambda = \sum_{i=0}^m p^i \lambda[i]$ satisfy all the conditions given in Lemma 6.4.10. For all $0 \leq i \leq m$, let $|\lambda[i]| = a_i(p-1) + b_i$, where $a_i, b_i \in \mathbb{N}_0$ and $b_i < p-1$. Put $\lambda(i) = ((p-1)^{a_i}, b_i)$ for all $0 \leq i \leq m$ and set $s(\lambda) = \sum_{i=0}^m p^i \lambda(i)$. By Lemma 6.4.10, $s(\lambda)$ is a well-defined partition.

Example 6.4.12. Let us illustrate the notation $s(\lambda)$. If $p = 3$, we have $s((5, 3, 3)) = (8, 3) \vdash 11$.

According to Theorems 6.1.4 and 6.4.9, we can prove the following lemma.

Lemma 6.4.13. *Let $\lambda \vdash n$. Then we have $Y^{s(\lambda)} \mid M^\lambda$ and $Y^{s(\lambda)}$ is a Scott module.*

The following proposition is the main result of this section.

Proposition 6.4.14. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. Let $\mu = s(\lambda^a \cup (p^{d_{\lambda,a}}))$. Then $Y^\mu \mid S^a M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(S^a M^\lambda)$.*

Proof. Let $\eta = \lambda^a \cup (p^{d_{\lambda,a}})$. By (6.4.1), note that $(\mathbb{F}_{K(t_{\lambda,a})})^{\uparrow \mathfrak{S}_n} \mid S^a M^\lambda$. As $P_{\lambda,a}$ is a Sylow p -subgroup of $\mathfrak{S}_{\lambda,a}$ and $P_{\lambda,a} \leq K(t_{\lambda,a})$, by Lemmas 6.4.4 and 6.4.6, it is clear that $P_{\lambda,a}$ is also a Sylow p -subgroup of $K(t_{\lambda,a})$. Therefore, we have $Sc_{\mathfrak{S}_n}(P_{\lambda,a}) \mid (\mathbb{F}_{K(t_{\lambda,a})})^{\uparrow \mathfrak{S}_n}$. In particular, $Sc_{\mathfrak{S}_n}(P_{\lambda,a}) \mid S^a M^\lambda$. By the definition of $\mathfrak{S}_{\lambda,a}$, also notice that $\mathfrak{S}_{\lambda,a}$ is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_\eta$ and $Sc_{\mathfrak{S}_n}(P_{\lambda,a}) \mid M^\eta$. Therefore, by Lemma 6.4.13, note that $Sc_{\mathfrak{S}_n}(P_{\lambda,a}) \cong Y^\mu$ and $Y^\mu \mid S^a M^\lambda$.

It suffices to show that both Y^μ and $S^a M^\lambda$ have the same complexity. Set $r = \text{rank}(P_{\lambda,a})$. If $d_{\lambda,a} = 0$, by the definitions of $P_{\lambda,a}$ and $H_{\lambda,a}$, $P_{\lambda,a}$ is $\mathfrak{S}_n$ -conjugate to a Sylow p -subgroup of $\mathfrak{S}_\lambda$. In particular, $r = \text{rank}(\mathfrak{S}_\lambda)$. If $d_{\lambda,a} > 0$, by the definitions of $P_{\lambda,a}$ and λ^a , $r = \text{rank}(\mathfrak{S}_\lambda) + d_{\lambda,a}$. Due to Lemmas 2.7.1 (iii) and 6.2.5, both Y^μ and $S^a M^\lambda$ have the same complexity. $\square$

Theorem 6.1.6 (i) follows by Proposition 6.4.14. We conclude this section by providing two other examples.

Example 6.4.15. We have the following examples.

- (i) Let $p = 2$. We have $D^{(3,1)} \mid S^2 M^{(2^2)}$ and $D^{(3,1)} \mid \Lambda^2 M^{(2^2)}$. Moreover, by Lemmas 2.7.1 (i), 6.2.4 and 6.2.5, note that $c_{\mathfrak{S}_4}(D^{(3,1)}) = c_{\mathfrak{S}_4}(S^2 M^{(2^2)}) = c_{\mathfrak{S}_4}(\Lambda^2 M^{(2^2)}) = 2$. However, $D^{(3,1)}$ is not a Young module.
- (ii) Let $p = 2$. Note that $Y^{(4,2)} \mid S^4 M^{(5,1)}$ and $Y^{(4,2)} \mid \Lambda^4 M^{(5,1)}$. Moreover, by Lemmas 2.7.1 (i), 6.1.5, 6.2.4 and 6.2.5, $c_{\mathfrak{S}_6}(Y^{(4,2)}) = c_{\mathfrak{S}_6}(S^4 M^{(5,1)}) = c_{\mathfrak{S}_6}(\Lambda^4 M^{(5,1)}) = 3$. However, although the Young module that we have found in Proposition 6.4.14 is a Scott module, note that $Y^{(4,2)}$ is not a Scott module by Theorem 6.4.9.

6.5 Proof of Theorem 6.1.6: Exterior Squares of Young Permutation Modules

In this section, we prove Theorem 6.1.6. For our purpose, recall the definitions in Notation 6.4.1. Given $\lambda = (\lambda_1, \dots, \lambda_\ell) \models n$ and $\{t\} \in \mathcal{T}(\lambda)$, we define $R(\{t\}) = \{g \in \mathfrak{S}_n : g\{t\} = \{t\}\}$. Note that $R(\{t\}) = \prod_{i=1}^\ell \mathfrak{S}_{R_i(\{t\})}$. So $R(\{t\})$ is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_{\bar{\lambda}}$, where we recall that $\bar{\lambda}$ is the partition whose parts are exactly the non-zero parts of λ . If $\lambda \vdash n$, also notice that $\ell(r_\lambda) = |\{1 \leq i \leq \ell : p \nmid \lambda_i\}|$. We present some lemmas as our preliminary results.

Lemma 6.5.1. *Let $\lambda \vdash n$ and $\lambda \neq (n)$. Let $\{u\}, \{v\} \in \mathcal{T}(\lambda)$ and $\{u\} \neq \{v\}$. Let t denote $\{u\}\{v\}$ or $\{u\} \wedge \{v\}$, where $\{u\}\{v\} \in \mathcal{T}(\lambda)_2^s$ and $\{u\} \wedge \{v\} \in \mathcal{T}(\lambda)_2^e$. Correspondingly, let $H(t)$ be $K(t)$ or $L(t)$. If there does not exist some $g \in H(t)$ such that $g\{u\} = \{v\}$ and $g\{v\} = \{u\}$, then $H(t) = R(\{u\}) \cap R(\{v\})$.*

Proof. By the hypotheses, for any $h \in H(t)$, $h\{u\} = \{u\}$ and $h\{v\} = \{v\}$. Therefore, we have $H(t) \leq R(\{u\}) \cap R(\{v\}) \leq H(t)$. The lemma follows. $\square$

Throughout the remaining part of this chapter, use $G \rtimes H$ to denote the semidirect product of finite groups G and H . Therefore, note that G is normal in $G \rtimes H$ and $G \cap H = 1$.

Lemma 6.5.2. *Let $\lambda \vdash n$ and $\lambda \neq (n)$. Let $\{u\}, \{v\} \in \mathcal{T}(\lambda)$ and $\{u\} \neq \{v\}$. Let t denote $\{u\}\{v\}$ or $\{u\} \wedge \{v\}$, where $\{u\}\{v\} \in \mathcal{T}(\lambda)_2^s$ and $\{u\} \wedge \{v\} \in \mathcal{T}(\lambda)_2^e$. Correspondingly, let $H(t)$ be $K(t)$ or $L(t)$. If there exists some $g \in H(t)$ such that $g\{u\} = \{v\}$ and $g\{v\} = \{u\}$, then there is some involution e of $\mathfrak{S}_n$ such that $e\{u\} = \{v\}$, $e\{v\} = \{u\}$ and $H(t) = (R(\{u\}) \cap R(\{v\})) \rtimes \langle e \rangle$. Moreover, if $g^2 = 1$, e can be chosen to be g .*

Proof. Let $R = R(\{u\}) \cap R(\{v\})$. For any $h \in H(t)$, if $h\{u\} = \{v\}$ and $h\{v\} = \{u\}$, we claim that $H(t) = R \cup hR$. Note that $R \leq H(t)$. For any $k \in H(t)$, if $k\{u\} = \{u\}$ and $k\{v\} = \{v\}$, then $k \in R$. If $k\{u\} = \{v\}$ and $k\{v\} = \{u\}$, then $h^{-1}k\{u\} = \{u\}$ and $h^{-1}k\{v\} = \{v\}$. Therefore, $k \in hR$. The claim is shown.

By this claim and the hypotheses, $g \in N_{\mathfrak{S}_n}(R)$. If $g^2 = 1$, as $g \notin R$, we are done. If $g^2 \neq 1$, let S be the set of orbits of $[1, n]$ under the action of R . Notice that each row of $\{u\}$ or $\{v\}$ is a union of some members of S . As $g \in N_{\mathfrak{S}_n}(R)$, g permutes the members of S with the same size. For any $\mathcal{O}_1, \mathcal{O}_2 \in S$, if $|\mathcal{O}_1| = |\mathcal{O}_2|$, $\mathcal{O}_1 \subseteq R_i(\{u\})$, $\mathcal{O}_2 \subseteq R_j(\{u\})$ and g sends $\mathcal{O}_1$ to $\mathcal{O}_2$, $\mathcal{O}_2 \subseteq R_i(\{v\})$ as $g\{u\} = \{v\}$. Moreover, note that $\mathcal{O}_1 \subseteq R_j(\{v\})$. Otherwise, as $g\{v\} = \{u\}$, if $\mathcal{O}_1 \not\subseteq R_j(\{v\})$, $\mathcal{O}_2 \not\subseteq R_j(\{u\})$. This is a contradiction. Therefore, for all $\mathcal{O}_1, \mathcal{O}_2 \in S$ satisfying that $\mathcal{O}_1 \neq \mathcal{O}_2$ and g sends $\mathcal{O}_1$ to $\mathcal{O}_2$, let e be an involution of $\mathfrak{S}_n$ swapping all these $\mathcal{O}_1, \mathcal{O}_2$. By the above discussion, $e\{u\} = \{v\}$, $e\{v\} = \{u\}$, $e \in N_{\mathfrak{S}_n}(R)$, $e \notin R$ and $e^2 = 1$. The lemma thus follows. $\square$

Lemma 6.5.3. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, $\lambda \neq (n)$ and $r = \text{rank}(\mathfrak{S}_\lambda)$. Let $a \in R_i(\{t^\lambda\})$ and $b \in R_j(\{t^\lambda\})$, where $1 \leq i \neq j \leq \ell$. Let $t = (a, b)t^\lambda$ and $s = \{t^\lambda\} \wedge \{t\}$. Then we have $L(s) = (R(\{t^\lambda\}) \cap R(\{t\})) \times \langle (a, b) \rangle$, where $R(\{t^\lambda\}) \cap R(\{t\})$ is $\mathfrak{S}_n$ -conjugate to a Young subgroup of $\mathfrak{S}_n$ with p -rank*

$$\begin{cases} r, & \text{if } p \nmid \lambda_i \text{ and } p \nmid \lambda_j, \\ r-1, & \text{if } p \nmid \lambda_i \text{ and } p \mid \lambda_j \text{ or } p \mid \lambda_i \text{ and } p \nmid \lambda_j, \\ r-2, & \text{if } p \mid \lambda_i \text{ and } p \mid \lambda_j. \end{cases}$$

Proof. Observe that $(a, b)\{t^\lambda\} = \{t\}$ and $(a, b)\{t\} = \{t^\lambda\}$. By Notation 6.4.1 (v), note that $s = \{t^\lambda\} \wedge \{t\} \in \mathcal{T}(\lambda)_2^e$. Also notice that a and b are fixed under the action of $R(\{t^\lambda\}) \cap R(\{t\})$ on $[1, n]$. Therefore, by Lemma 6.5.2, $L(s) = (R(\{t^\lambda\}) \cap R(\{t\})) \times \langle (a, b) \rangle$. Let $S_i = R_i(\{t^\lambda\}) \setminus \{a\}$ and $S_j = R_j(\{t^\lambda\}) \setminus \{b\}$. By the definition of $\{t\}$,

$$R(\{t^\lambda\}) \cap R(\{t\}) = \mathfrak{S}_{S_i} \times \mathfrak{S}_{S_j} \times \prod_{\substack{k=1 \\ k \neq i, j}}^{\ell} \mathfrak{S}_{R_k(\{t^\lambda\})} \leq \mathfrak{S}_\lambda. \quad (6.5.1)$$

So $R(\{t^\lambda\}) \cap R(\{t\})$ is $\mathfrak{S}_n$ -conjugate to a Young subgroup of $\mathfrak{S}_n$ with the desired p -rank. The proof is now complete. $\square$

Lemma 6.5.4. *Let $p > 2$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. If $p \nmid \lambda_i$ and $p \nmid \lambda_j$ for some $1 \leq i < j \leq \ell$, let $\nu = (\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_j - 1, \dots, \lambda_\ell)$ and $\mu = \overline{r_{\nu \cup (1^2)}} + \overline{q_{\nu \cup (1^2)}}$. Then $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \text{rank}(\mathfrak{S}_\lambda)$.*

Proof. Let $r = \text{rank}(\mathfrak{S}_\lambda)$ and $a_k \in R_k(\{t^\lambda\})$ for any $k \in \{i, j\}$. Let $t = (a_i, a_j)t^\lambda$ and note that $\{t\} \neq \{t^\lambda\}$. By Notation 6.4.1 (v), also note that $s = \{t^\lambda\} \wedge \{t\} \in \mathcal{T}(\lambda)_2^e$. As $i \neq j$, by Lemma 6.5.3, $L(s) = (R(\{t^\lambda\}) \cap R(\{t\})) \times \langle (a_i, a_j) \rangle$. Let $\eta = \nu \cup (1^2)$ and $\tilde{\eta} = \nu \cup (2)$. As $p > 2$, note that $M^\eta \cong M^{\tilde{\eta}} \oplus (\mathbb{F}^s)^{\uparrow \mathfrak{S}_n}$ by [49, Corollary 4.6 (i)]. By Theorem 6.1.4, observe that $Y^\mu \mid M^\eta$. Let μ have the p -adic expansion $\sum_{i=0}^m p^i \mu(i)$ for some $m \in \mathbb{N}_0$. For any a p -adic decomposition $\tilde{\eta} = \sum_{i=0}^m p^i \sigma[i]$, if $\sigma[i] \models |\mu(i)|$ for all $0 \leq i \leq m$, then $\mu(0) = \overline{r_{\tilde{\eta}}}$ and $\sigma[0] = r_{\tilde{\eta}}$. Moreover, $\ell(r_\eta) = \ell(r_{\tilde{\nu}}) + 2 > \ell(r_{\tilde{\nu}}) + 1 = \ell(r_{\tilde{\eta}})$. Therefore, $\overline{r_{\tilde{\eta}}} \not\leq \overline{r_{\tilde{\eta}}}$ and $Y^\mu \nmid M^{\tilde{\eta}}$ by Theorem 6.1.4, which implies that $Y^\mu \mid (\mathbb{F}^s)^{\uparrow \mathfrak{S}_n}$ by the Krull-Schmidt Theorem. As $(\mathbb{F}^s)^{\uparrow \mathfrak{S}_n} \mid \Lambda^2 M^\lambda$ by (6.4.2), we have $Y^\mu \mid \Lambda^2 M^\lambda$. As $p \nmid \lambda_i$ and $p \nmid \lambda_j$, by Lemma 2.7.1 (i) and Theorem 6.1.5, we also have $c_{\mathfrak{S}_n}(Y^\mu) = r \leq c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$. Since $p > 2$, by Lemmas 6.2.4 and 6.2.5, this inequality implies that $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. The proof is now complete. $\square$

Lemma 6.5.5. *Let $p > 2$ and $\lambda = (\lambda_1, \lambda_2) \vdash n$, where $p \mid \lambda_1$ and $\lambda_2 < p$. Define $\mu = (\lambda_1 - 1, \lambda_2 - 1) \cup (1^2)$. Then $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \text{rank}(\mathfrak{S}_\lambda) - 1$.*

Proof. Let $r = \text{rank}(\mathfrak{S}_\lambda)$ and $t = (1, n)t^\lambda$. As $\{t\} \neq \{t^\lambda\}$, by Notation 6.4.1 (v), notice that $s = \{t^\lambda\} \wedge \{t\} \in \mathcal{T}(\lambda)_2^e$. By Lemma 6.5.3, $L(s) = (R(\{t^\lambda\}) \cap R(\{t\})) \times \langle (1, n) \rangle$. As $p > 2$, by [49, Corollary 4.8 (i)] and (6.4.2), we have $Y^\mu \mid (\mathbb{F}^s)^{\uparrow \mathfrak{S}_n}$ and $(\mathbb{F}^s)^{\uparrow \mathfrak{S}_n} \mid \Lambda^2 M^\lambda$, which implies that $Y^\mu \mid \Lambda^2 M^\lambda$.

As $p \mid \lambda_1$ and $\lambda_2 < p$, by Lemmas 2.7.1 (i), 6.2.4, 6.2.5 and Theorem 6.1.5, notice that $r - 1 = c_{\mathfrak{S}_n}(Y^\mu) \leq c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \leq r$. For any elementary abelian p -subgroup E of $\mathfrak{S}_n$ with p -rank r , we claim that $\dim_{\mathbb{F}} M^\lambda(E) \leq 1$. If $M^\lambda(E) \neq 0$, by Lemma 2.11.1, note that $[1, n]$ has exactly λ_2 fixed points under the action of E . Otherwise, as $p \mid \lambda_1$ and $\lambda_2 < p$, by Lemma

2.11.1, note that $[1, n]$ has at least $p + \lambda_2$ fixed points under the action of E . We thus can choose an elementary abelian p -subgroup F of $\mathfrak{S}_n$ such that $\text{rank}(F) > r$ and $M^\lambda(F) \neq 0$, which contradicts Lemma 2.7.1 (vi) and Theorem 6.1.5. So $\dim_{\mathbb{F}} M^\lambda(E) = 1$ by Lemma 2.11.1. The claim is shown. As $p > 2$, by this claim and Lemma 6.2.2, $(\Lambda^2 M^\lambda)(E) = 0$. Therefore, $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r - 1$ by Lemma 2.7.1 (vi). The proof is now complete. $\square$

Lemma 6.5.6. *Let $p > 2$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. Assume that $\lambda_i \geq p$ and $\lambda_j \geq p$ for some $1 \leq i < j \leq \ell$. Let $\nu = (\lambda_1, \dots, \lambda_i - p, \dots, \lambda_j - p, \dots, \lambda_\ell)$ and $\mu = \overline{r_{\nu \cup (p, p)}} + \overline{q_{\nu \cup (p, p)}}$. Then $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \text{rank}(\mathfrak{S}_\lambda)$.*

Proof. Let $r = \text{rank}(\mathfrak{S}_\lambda)$, $\mathcal{O}_1 = \{1, \dots, p\}$ and $\mathcal{O}_2 = \{p+1, \dots, 2p\}$. Let $\{u\} \in \mathcal{T}(\lambda)$, where we have $\mathcal{O}_1 \subseteq R_i(\{u\})$ and $\mathcal{O}_2 \subseteq R_j(\{u\})$. Let $e = \prod_{k=1}^p (k, p+k) \in \mathfrak{S}_n$ and $\{v\} = e\{u\} \in \mathcal{T}(\lambda)$. As $\{u\} \neq \{v\}$, we may assume that $t = \{u\} \wedge \{v\} \in \mathcal{T}(\lambda)_2^e$. By Lemma 6.5.2, it is clear that $L(t) = (R(\{u\}) \cap R(\{v\})) \rtimes \langle e \rangle$. Also notice that $L(t) = H \times K$, where the subgroup H is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_{\bar{\nu}}$ and $K = \mathfrak{S}_{(p, p)} \rtimes \langle e \rangle$. Recall that $W(t) \cong (\mathbb{F}^t)^{\uparrow \mathfrak{S}_n}$, where $\mathbb{F}^t$ denotes a one-dimensional $\mathbb{F}L(t)$ -module. Therefore, we have $\mathbb{F}^t \cong \mathbb{F} \boxtimes \mathbb{F}'$ as $\mathbb{F}[H \times K]$ -modules, where $\mathbb{F}' = \langle \{w\} \rangle_{\mathbb{F}}$. Note that $ew = -w$ and $gw = w$ for all $g \in \mathfrak{S}_{(p, p)}$. Let $M = (\mathbb{F}')^{\uparrow \mathfrak{S}_{2p}}$. By Lemma 6.5.2 and (6.4.2), observe that $M \mid \Lambda^2 M^{(p, p)}$. As $p > 2$, it is clear that both $\Lambda^2 M^{(p, p)}$ and M are direct sums of Young modules. As $\mathbb{F} \nmid M$, by Lemmas 2.7.1 (i), (v) and Theorem 6.1.5, we have $c_{\mathfrak{S}_{2p}}(M) = 2$ and $Y^{(p, p)} \mid M$. So $(\mathbb{F}_H \boxtimes Y^{(p, p)})^{\uparrow \mathfrak{S}_n} \mid (\mathbb{F}_H \boxtimes M)^{\uparrow \mathfrak{S}_n} \cong (\mathbb{F}^t)^{\uparrow \mathfrak{S}_n}$. By Lemmas 2.7.1 (iv), (v) and Theorem 6.1.5, $(\mathbb{F}_H \boxtimes Y^{(p, p)})^{\uparrow \mathfrak{S}_n}$ has complexity r . Write η for $\nu \cup (p, p)$ and note that

$$M^\eta \cong (\mathbb{F}_H \boxtimes M^{(p, p)})^{\uparrow \mathfrak{S}_n} \cong (\mathbb{F}_H \boxtimes Y^{(p, p)})^{\uparrow \mathfrak{S}_n} \oplus \bigoplus_{(p, p) \triangleleft \gamma} [M^{(p, p)} : Y^\gamma] (\mathbb{F}_H \boxtimes Y^\gamma)^{\uparrow \mathfrak{S}_n}. \quad (6.5.2)$$

By Theorem 6.1.5, we have $c_{\mathfrak{S}_n}(M^\eta) = r$. Let $\tilde{\eta} = \nu \cup (2p)$. By (6.5.2), the Krull-Schmidt Theorem, Lemma 2.7.1 (i), (iv), (v) and Theorem 6.1.5, for any $\alpha \vdash n$ satisfying that $Y^\alpha \mid M^\eta$ and $c_{\mathfrak{S}_n}(Y^\alpha) = r$, we have either $Y^\alpha \mid (\mathbb{F}_H \boxtimes Y^{(p, p)})^{\uparrow \mathfrak{S}_n}$ or $Y^\alpha \mid M^{\tilde{\eta}}$. By Theorems 6.1.4 and 6.1.5, note that $Y^\mu \mid M^\eta$ and $c_{\mathfrak{S}_n}(Y^\mu) = r$. Let μ have the p -adic expansion $\sum_{i=0}^m p^i \mu(i)$ for some $m \in \mathbb{N}$. For any a p -adic decomposition $\tilde{\eta} = \sum_{i=0}^m p^i \sigma[i]$, if $\sigma[i] \models |\mu(i)|$ for all $0 \leq i \leq m$, then we have $\mu(0) = \bar{r}_{\tilde{\eta}}$, $\sigma[0] = r_{\tilde{\eta}}$ and $\bar{r}_{\tilde{\eta}} = \bar{r}_{\tilde{\eta}}$. Moreover, we have

$$\ell(\mu(1)) = \ell(q_\eta) = \ell(q_{\bar{\nu}}) + 2 > \ell(q_{\bar{\nu}}) + 1 = \ell(q_{\tilde{\eta}}) \geq \ell(\sigma[1]).$$

Therefore, $Y^\mu \nmid M^{\tilde{\eta}}$ by Theorem 6.1.4, which implies that $Y^\mu \mid (\mathbb{F}_H \boxtimes Y^{(p, p)})^{\uparrow \mathfrak{S}_n}$. By (6.4.2) and the above discussion, note that $Y^\mu \mid (\mathbb{F}^t)^{\uparrow \mathfrak{S}_n}$ and $(\mathbb{F}^t)^{\uparrow \mathfrak{S}_n} \mid \Lambda^2 M^\lambda$. Since $p > 2$, by Lemmas 2.7.1 (i), 6.2.4 and 6.2.5, we have $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. The proof is now complete. $\square$

We now present some lemmas for the case $p = 2$. For convenience, we define some partitions of n . Let $p = 2$, $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $\lambda \neq (n)$. Set

$$\lambda_{i,j} = \begin{cases} (\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_j - 1, \dots, \lambda_\ell) \cup (2), & \text{if } \ell(r_\lambda) > 0 \text{ and } 1 \leq i < j \leq \ell, \\ (\lambda_1, \dots, \lambda_i - 2, \dots, \lambda_j - 2, \dots, \lambda_\ell) \cup (4), & \text{if } \ell(r_\lambda) = 0 \text{ and } 1 \leq i < j \leq \ell. \end{cases}$$

Lemma 6.5.7. *Let $p = 2$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. If $2 \nmid \lambda_i$ and $2 \nmid \lambda_j$ for some $1 \leq i < j \leq \ell$, then $M^{\lambda_{i,j}} \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(M^{\lambda_{i,j}}) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \text{rank}(\mathfrak{S}_\lambda) + 1$.*

Proof. Let $r = \text{rank}(\mathfrak{S}_\lambda)$, $a \in R_i(\{t^\lambda\})$ and $b \in R_j(\{t^\lambda\})$. Let $t = (a, b)t^\lambda$ and note that $\{t\} \neq \{t^\lambda\}$. By Notation 6.4.1 (v), also notice that $u = \{t^\lambda\} \wedge \{t\} \in \mathcal{T}(\lambda)_2^e$. As $i \neq j$, $2 \nmid \lambda_i$ and $2 \nmid \lambda_j$, by Lemma 6.5.3 and (6.5.1), $L(u) = (R(\{t^\lambda\}) \cap R(\{t\})) \times \langle (a, b) \rangle$, where $L(u)$ has 2-rank $r + 1$ and $L(u)$ is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_{\lambda_{i,j}}$. As $p = 2$ and (6.4.2) holds, we have $M^{\lambda_{i,j}} \cong (\mathbb{F}_{L(u)})^{\uparrow \mathfrak{S}_n}$ and $(\mathbb{F}_{L(u)})^{\uparrow \mathfrak{S}_n} \mid \Lambda^2 M^\lambda$. Therefore, $M^{\lambda_{i,j}} \mid \Lambda^2 M^\lambda$. By Lemma 2.7.1 (v), we have $c_{\mathfrak{S}_n}(M^{\lambda_{i,j}}) = r + 1$. This implies that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \geq r + 1$ by Lemma 2.7.1 (i). As $p = 2$ and $r + 1 = \text{rank}(L(u)) \leq \text{rank}(\mathfrak{S}_n) \leq \frac{n}{2}$, by Lemmas 6.2.4 and 6.2.5, notice that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \leq r + 1$. We have $c_{\mathfrak{S}_n}(M^{\lambda_{i,j}}) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r + 1$. The lemma follows. $\square$

Lemma 6.5.8. *Let $p = 2$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. If $\ell(r_\lambda) = 1$ and $2 \nmid \lambda_i + \lambda_j$ for some $1 \leq i < j \leq \ell$, then $M^{\lambda_{i,j}} \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(M^{\lambda_{i,j}}) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \text{rank}(\mathfrak{S}_\lambda)$.*

Proof. Let $r = \text{rank}(\mathfrak{S}_\lambda)$, $a \in R_i(\{t^\lambda\})$ and $b \in R_j(\{t^\lambda\})$. We set $t = (a, b)t^\lambda$ and note that $\{t\} \neq \{t^\lambda\}$. By Notation 6.4.1 (v), also note that $u = \{t^\lambda\} \wedge \{t\} \in \mathcal{T}(\lambda)_2^\circ$. As $i \neq j$ and $2 \nmid \lambda_i + \lambda_j$, by Lemma 6.5.3 and (6.5.1), notice that $L(u) = (R(\{t^\lambda\}) \cap R(\{t\})) \times \langle (a, b) \rangle$, where $L(u)$ has 2-rank r and $L(u)$ is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_{\lambda_{i,j}}$. As $p = 2$ and (6.4.2) holds, $M^{\lambda_{i,j}} \cong (\mathbb{F}_{L(u)})^{\uparrow \mathfrak{S}_n}$ and $(\mathbb{F}_{L(u)})^{\uparrow \mathfrak{S}_n} \mid \Lambda^2 M^\lambda$. Therefore, $M^{\lambda_{i,j}} \mid \Lambda^2 M^\lambda$. By Lemma 2.7.1 (v), note that $c_{\mathfrak{S}_n}(M^{\lambda_{i,j}}) = r$, which implies that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \geq r$ by Lemma 2.7.1 (i). As $p = 2$ and $\ell(r_\lambda) = 1$, we have $r = \lfloor \frac{n}{2} \rfloor$. Therefore, by Lemma 2.7.1 (vi), note that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \leq \text{rank}(\mathfrak{S}_n) = r$. So we obtain that $c_{\mathfrak{S}_n}(M^{\lambda_{i,j}}) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. The lemma follows. $\square$

To continue the discussion, given $\lambda \vdash n$, by Lemma 6.4.13, recall that $Y^{s(\lambda)} \mid M^\lambda$ and $Y^{s(\lambda)}$ is a Scott module. Moreover, $s(\lambda)$ can be explicitly determined by λ (see Definition 6.4.11).

Lemma 6.5.9. *Let $p = 2$ and $\lambda \vdash n$. If $\lambda \neq (n)$ and $\ell(r_\lambda) = 0$, then $Y^{s(\lambda_{i,j})} \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^{s(\lambda_{i,j})}) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \text{rank}(\mathfrak{S}_\lambda)$ for all $1 \leq i < j \leq \ell(\lambda)$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $r = \text{rank}(\mathfrak{S}_\lambda)$. For any $1 \leq i < j \leq \ell$, since $\ell(r_\lambda) = 0$, note that $2 \mid \lambda_i$ and $2 \mid \lambda_j$. Let $\{a, c\} \subseteq R_i(\{t^\lambda\})$ and $\{b, d\} \subseteq R_j(\{t^\lambda\})$. Let $e = (a, b)(c, d) \in \mathfrak{S}_n$. Let $t = et^\lambda$ and note that $\{t\} \neq \{t^\lambda\}$. By Notation 6.4.1 (v), $u = \{t^\lambda\} \wedge \{t\} \in \mathcal{T}(\lambda)_2^\circ$. Moreover, we have $e\{t^\lambda\} = \{t\}$ and $e\{t\} = \{t^\lambda\}$. Let $S_i = R_i(\{t^\lambda\}) \setminus \{a, c\}$ and $S_j = R_j(\{t^\lambda\}) \setminus \{b, d\}$. According to Lemma 6.5.2, observe that $L(u) = (R(\{t^\lambda\}) \cap R(\{t\})) \rtimes \langle e \rangle$, where

$$R(\{t^\lambda\}) \cap R(\{t\}) = \mathfrak{S}_{S_i} \times \mathfrak{S}_{S_j} \times \left(\prod_{\substack{k=1 \\ k \neq i, j}}^{\ell} \mathfrak{S}_{R_k(\{t^\lambda\})} \right) \times \mathfrak{S}_{\{a, c\}} \times \mathfrak{S}_{\{b, d\}} \leq \mathfrak{S}_\lambda. \quad (6.5.3)$$

Note that a Sylow 2-subgroup P of $L(u)$ is $\mathfrak{S}_n$ -conjugate to a Sylow 2-subgroup of $\mathfrak{S}_{\lambda_{i,j}}$. Therefore, $Sc_{\mathfrak{S}_n}(P) \cong Y^{s(\lambda_{i,j})}$. Since $p = 2$ and (6.4.2) holds, we have $Y^{s(\lambda_{i,j})} \mid (\mathbb{F}_{L(u)})^{\uparrow \mathfrak{S}_n}$ and $(\mathbb{F}_{L(u)})^{\uparrow \mathfrak{S}_n} \mid \Lambda^2 M^\lambda$. So $Y^{s(\lambda_{i,j})} \mid \Lambda^2 M^\lambda$. As $2 \mid \lambda_i$ and $2 \mid \lambda_j$, by (6.5.3), note that $\text{rank}(P) = r$. By Lemma 2.7.1 (iii), we deduce that $c_{\mathfrak{S}_n}(Y^{s(\lambda_{i,j})}) = r$. Also note that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \geq r$ by Lemma 2.7.1 (i). As $\ell(r_\lambda) = 0$, we have $r = \text{rank}(\mathfrak{S}_n) = \frac{n}{2}$. Therefore, by Lemma 2.7.1 (vi), notice that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) \leq \text{rank}(\mathfrak{S}_n) = r$. We thus have $c_{\mathfrak{S}_n}(Y^{s(\lambda_{i,j})}) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. The lemma follows. $\square$

We deduce the following corollary.

Corollary 6.5.10. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, $\lambda \neq (n)$ and $r = \text{rank}(\mathfrak{S}_\lambda)$.*

(i) *If $p > 2$, then*

$$c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \begin{cases} r - 1, & \text{if } \ell = 2, p \mid \lambda_1 \text{ and } \lambda_2 < p, \\ r, & \text{otherwise.} \end{cases}$$

(ii) *If $p = 2$, then*

$$c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \begin{cases} r + 1, & \text{if } \ell(r_\lambda) \geq 2, \\ r, & \text{otherwise.} \end{cases}$$

Proof. For (i), if $\ell \geq 3$ and $\lambda_2 < p$, by Lemma 6.5.4, note that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. If $\lambda_2 \geq p$, by Lemma 6.5.6, also notice that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. For the remaining case, as $\lambda \neq (n)$, we have $\ell = 2$ and $\lambda_2 < p$. This case is solved by Lemmas 6.5.4 and 6.5.5. The proof of (i) is now complete.

For (ii), if $\ell(r_\lambda) \geq 2$, for some $1 \leq i < j \leq \ell$, we have $2 \nmid \lambda_i$ and $2 \nmid \lambda_j$. By Lemma 6.5.7, we get the desired complexity for $\Lambda^2 M^\lambda$. For the case $\ell(r_\lambda) < 2$, according to Lemmas 6.5.8 and 6.5.9, note that $c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = r$. This completes the proof. $\square$

The following proposition is the main result of this section.

Proposition 6.5.11. *Let $\lambda \vdash n$ and $\lambda \neq (n)$. Then there exists some $\mu \vdash n$ such that $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $R = \{\nu \vdash n : c_{\mathfrak{S}_n}(Y^\nu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)\}$. By Corollary 6.5.10 and Theorem 6.1.5, note that R can be explicitly determined. We have two cases.

Case 1: $p > 2$.

Let $S = \{(i, j) : 1 \leq i < j \leq \ell, \lambda_i < p \text{ or } \lambda_j \geq p\}$ and assume that $(u, v) \in S$. If $\lambda_u < p$, let $\alpha = (\lambda_1, \dots, \lambda_u - 1, \dots, \lambda_v - 1, \dots, \lambda_\ell) \cup (1^2)$ and $\mu = \bar{r}_\alpha + \bar{q}_\alpha$. If $\lambda_v \geq p$, then we set $\beta = (\lambda_1, \dots, \lambda_u - p, \dots, \lambda_v - p, \dots, \lambda_\ell) \cup (p, p)$ and $\mu = \bar{r}_\beta + \bar{q}_\beta$. For the two possibilities, according to Lemmas 6.5.4 and 6.5.6, note that $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$. We thus assume that $S = \emptyset$. As $\lambda \neq (n)$, note that we have $\ell = 2$, $\lambda_1 \geq p$ and $\lambda_2 < p$. In this case, if $p \nmid \lambda_1$, let $\gamma = (\lambda_1 - 1, \lambda_2 - 1) \cup (1^2)$ and $\mu = \bar{r}_\gamma + \bar{q}_\gamma$. If $p \mid \lambda_1$, set $\mu = \gamma$. For the two possibilities, by Lemmas 6.5.4 and 6.5.5, we also have $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$.

Case 2: $p = 2$.

If $\ell(r_\lambda) > 0$, as $\lambda \neq (n)$, let $T = \{\nu : 1 \leq i < j \leq \ell, Y^\nu \mid M^{\lambda_{i,j}}, 2 \nmid \lambda_i \lambda_j \text{ or } 2 \nmid \lambda_i + \lambda_j\}$ and note that $T \neq \emptyset$. Moreover, T can be explicitly determined by Theorem 6.1.4. Let $\mu \in R \cap T$. If $\ell(r_\lambda) = 0$, as $\lambda \neq (n)$, for any given $1 \leq i < j \leq \ell$, let $\mu = s(\lambda_{i,j})$. The case $p = 2$ is solved by Lemmas 6.5.7, 6.5.8 and 6.5.9. The proposition follows. $\square$

Example 6.5.12. We illustrate Proposition 6.5.11 with an example. Let $\lambda = (3^2, 2) \vdash 8$. By the proof of Proposition 6.5.11, the partition μ of Proposition 6.5.11 has the following possibilities:

$$\begin{cases} (2^4), (4, 2^2), & \text{if } p = 2, \\ (5, 3), & \text{if } p = 3, \\ (2^3, 1^2), (3, 2, 1^3), & \text{if } p > 3. \end{cases}$$

Theorem 6.1.6 is now proved by combining Propositions 6.4.14 and 6.5.11.

6.6 Some Special Cases of Theorem 6.1.6

In this section, we would like to consider some special cases of Theorem 6.1.6. For our purpose, recall the definitions in Notation 6.4.1. Given $\lambda \models n$, also recall that $\bar{\lambda}$ is the partition whose parts are exactly the non-zero parts of λ .

Let $\lambda \vdash n$ such that $\bar{r}_\lambda = (p - 1, 1)$. Therefore, we have $p \mid n$ and $d_{\lambda,p} = 1$. Moreover, also notice that $\mathfrak{S}_{\lambda,p} = H_{\lambda,p} \times \mathfrak{S}_{\{n-p+1, \dots, n\}}$. Recall that $\mathcal{O}_{\lambda,p} = \{g\mathfrak{s}_\lambda : g \in P_{\lambda,p}\}$. By Lemma 6.4.3 (ii), label $\mathcal{O}_{\lambda,p} = \{\{s_1\}, \dots, \{s_p\}\}$ and recall that $t_{\lambda,p} = \{s_1\} \dots \{s_p\} \in \mathcal{T}(\lambda)_p^s$. As $H_{\lambda,p} \leq K(t_{\lambda,p})$ by Lemma 6.4.5 and $\bar{r}_\lambda = (p - 1, 1)$, note that $\mathfrak{S}_{\lambda,p} \leq K(t_{\lambda,p})$. The following lemma is obvious.

Lemma 6.6.1. *Let $\lambda \vdash n$ such that $\bar{r}_\lambda = (p - 1, 1)$. Then $K(t_{\lambda,p}) = \mathfrak{S}_{\lambda,p}$.*

Proof. It suffices to show that $K(t_{\lambda,p}) \leq \mathfrak{S}_{\lambda,p}$. This inequality is from Lemma 6.4.6. $\square$

We shall need the following definition.

Definition 6.6.2. Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$. For any $t = \{u_1\} \dots \{u_a\} \in \mathcal{T}(\lambda)_a^s$, we now define $n_{t,i} = |\bigcap_{j=1}^a R_i(\{u_j\})|$ for all $1 \leq i \leq \ell(\lambda)$ and put $\phi(t) = (n_{t,1}, \dots, n_{t,\ell(\lambda)})$. For instance, let $p = 3$ and $\lambda = (5, 4) \vdash 9$. As $r_\lambda = (2, 1)$, $\phi(t_{\lambda,p}) = q_\lambda = (3^2)$. Let $t_1, t_2 \in \mathcal{T}(\lambda)_a^s$. If there exists some $g \in \mathfrak{S}_n$ such that $gt_1 = t_2$, note that $\phi(t_1) = \phi(t_2)$.

By Definition 6.6.2, we can present the following lemma.

Lemma 6.6.3. *Let $\lambda \vdash n$ such that $\bar{r}_\lambda = (p - 1, 1)$. Then $\mathcal{O}^s(t_{\lambda,p}) = \{t \in \mathcal{T}(\lambda)_p^s : \phi(t) = q_\lambda\}$.*

Proof. Let $S = \{t \in \mathcal{T}(\lambda)_p^s : \phi(t) = q_\lambda\}$. By the definition of $t_{\lambda,p}$, observe that $\phi(t_{\lambda,p}) = q_\lambda$. So we have $\mathcal{O}^s(t_{\lambda,p}) \subseteq S$. It suffices to show that $|\mathcal{O}^s(t_{\lambda,p})| = |S|$. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$, where $\lambda_u \equiv p - 1 \pmod{p}$ and $\lambda_v \equiv 1 \pmod{p}$ for some $1 \leq u \neq v \leq \ell$. As $\bar{r}_\lambda = (p - 1, 1)$, by the definition of S , we have

$$|S| = \frac{n!}{p!(\lambda_u - p + 1)!(\lambda_v - 1)! \prod_{i \neq u, v} (\lambda_i!)}. \quad (6.6.1)$$

Recall that $\mathcal{O}^s(t_{\lambda,p})$ is an $\mathbb{F}$ -basis of $V(t_{\lambda,p})$ and $V(t_{\lambda,p}) \cong (\mathbb{F}_{K(t_{\lambda,p})})^{\uparrow \mathfrak{S}_n}$. By Lemma 6.6.1 and the definition of $\mathfrak{S}_{\lambda,p}$, note that $|\mathcal{O}^s(t_{\lambda,p})||K(t_{\lambda,p})| = |\mathcal{O}^s(t_{\lambda,p})||\mathfrak{S}_{\lambda,p}| = n!$ and $\mathfrak{S}_{\lambda,p}$ is $\mathfrak{S}_n$ -conjugate to $\mathfrak{S}_\eta$, where $\eta = q_\lambda \cup (p)$. By (6.6.1), we have $|\mathcal{O}^s(t_{\lambda,p})| = |S|$, as desired. $\square$

We recall the definitions in Notation 4.2.1 (v), (vi) and prove some lemmas. For our purpose, we shall need to use Lemma 4.2.7.

Lemma 6.6.4. *Let $\lambda \vdash n$ such that $\overline{r_\lambda} = (p-1, 1)$. Let $b = \frac{n}{p}$ and $t = \{u_1\} \dots \{u_p\} \in \mathcal{T}(\lambda)_p^s$, where $\{u_i\} \neq \{u_j\}$ for all $1 \leq i \neq j \leq p$. Let $S = \{\{u_1\}, \dots, \{u_p\}\}$ and assume that we have $c_{\mathfrak{S}_n}(V(t)) = c_{\mathfrak{S}_n}(S^p M^\lambda)$.*

- (i) *There exists an elementary abelian p -subgroup E of $K(t)$ with p -rank b .*
- (ii) *If $E_b \leq K(t)$, then there exists a unique $s_k \in \{s_1, \dots, s_b\}$ such that $\langle s_k \rangle$ acts on S transitively. Moreover, $s_i\{u_j\} = \{u_j\}$ for all $1 \leq i \neq k \leq b$ and $1 \leq j \leq p$.*
- (iii) *If $p = 2$ and $K_\ell \times H_\ell \leq K(t)$ for some $0 \leq \ell \leq \lfloor \frac{n}{4} \rfloor$, then $\ell \neq \frac{n}{4}$ and there exists a unique $s_{\ell,q} \in \{s_{\ell,1}, \dots, s_{\ell,y_\ell}\}$ such that $\langle s_{\ell,q} \rangle$ acts on S transitively. Moreover, let $U = \bigcup_{i=1}^\ell \{k_{i,1}, k_{i,2}\} \cup \{s_{\ell,1}, \dots, s_{\ell,y_\ell}\}$. Then $g\{u_j\} = \{u_j\}$ for all $g \in U \setminus \{s_{\ell,q}\}$ and $1 \leq j \leq 2$.*

Proof. As $\overline{r_\lambda} = (p-1, 1)$, note that $\text{rank}(\mathfrak{S}_\lambda) = b-1$. By Lemma 6.2.5, we notice that $c_{\mathfrak{S}_n}(V(t)) = c_{\mathfrak{S}_n}(S^p M^\lambda) = b$. As $V(t) \cong (\mathbb{F}_{K(t)})^{\uparrow \mathfrak{S}_n}$, by Lemma 2.7.1 (v), there exists an elementary abelian p -subgroup E of $K(t)$ with p -rank b . (i) is shown. By the definition of $K(t)$, observe that $K(t)$ acts on S . For any element g of $K(t)$ with p -power order, if $\langle g \rangle$ does not act transitively on S , then g fixes $\{u_i\}$ for all $1 \leq i \leq p$.

For (ii), as $E_b \leq K(t)$, note that there exists some $s_k \in \{s_1, \dots, s_b\} \subseteq E_b$ such that $\langle s_k \rangle$ acts transitively on S . Otherwise, we have $g\{u_j\} = \{u_j\}$ for all $g \in E_b$ and $1 \leq j \leq p$. So E_b is $\mathfrak{S}_n$ -conjugate to a subgroup of $\mathfrak{S}_\lambda$. This fact implies that $b = \text{rank}(E_b) \leq \text{rank}(\mathfrak{S}_\lambda) = b-1$, which is absurd. Also note that there does not exist $s_\ell \in \{s_1, \dots, s_b\}$ such that $\ell \neq k$ and $\langle s_\ell \rangle$ acts transitively on S . Otherwise, there exists some $m \in \mathbb{N}$ such that $1 \leq m < p$ and $s_\ell^m s_k \{u_i\} = \{u_i\}$ for all $1 \leq i \leq p$. This implies that the numbers $(k-1)p+1, \dots, kp$ are totally contained in some row of $\{u_i\}$ for all $1 \leq i \leq p$. Therefore, by Lemma 2.11.1, s_k fixes $\{u_i\}$ for all $1 \leq i \leq p$, which is a contradiction. (ii) thus follows.

For (iii), as $K_\ell \times H_\ell \leq K(t)$, there exists some $g \in U$ such that $\langle g \rangle$ acts on S transitively. Otherwise, $z\{u_j\} = \{u_j\}$ for all $z \in K_\ell \times H_\ell$ and $1 \leq j \leq 2$. So $K_\ell \times H_\ell$ is $\mathfrak{S}_n$ -conjugate to a subgroup of $\mathfrak{S}_\lambda$. We have $b = \text{rank}(K_\ell \times H_\ell) \leq \text{rank}(\mathfrak{S}_\lambda) = b-1$, which is absurd. For such an element g , we claim that $g \in \{s_{\ell,1}, \dots, s_{\ell,y_\ell}\}$. If $g = k_{i,1}$ or $k_{i,2}$ for some $1 \leq i \leq \ell$, notice that $S = \{\{u_1\}, \{u_2\}\}$ and $\langle k_{i,1}, k_{i,2} \rangle$ acts on S . Therefore, there is some $e = (a, b)(c, d) \in \langle k_{i,1}, k_{i,2} \rangle$ such that $e\{u_1\} = \{u_1\}$. By Lemma 2.11.1, this fact implies that we have $\{a, b\} \subseteq R_x(\{u_1\})$, $\{c, d\} \subseteq R_y(\{u_1\})$ and $x \neq y$. Otherwise, if a, b, c, d lie in a single row of $\{u_1\}$, then note that $g\{u_1\} = \{u_1\}$ by Lemma 2.11.1. This is a contradiction. As $g\{u_1\} = \{u_2\}$ and $e \in \langle k_{i,1}, k_{i,2} \rangle$, we get that $\{a, b\} \subseteq R_y(\{u_2\})$ and $\{c, d\} \subseteq R_x(\{u_2\})$. Therefore, for all $h \in U \setminus \{k_{i,1}, k_{i,2}\}$, as $\{a, b\} \subseteq R_x(h\{u_1\})$ and $K(t)$ acts on S , $h\{u_1\} = \{u_1\}$. By Lemma 2.11.1, this forces that all the parts of λ are even, which contradicts the fact that $\overline{r_\lambda} = (1^2)$. The claim is shown. By this claim, $\ell \neq \frac{n}{4}$ and there exists some $s_{\ell,q} \in U$ such that $\langle s_{\ell,q} \rangle$ acts on S transitively. One can write a proof as the one given in (ii) to show that $w\{u_j\} = \{u_j\}$ for all $w \in U \setminus \{s_{\ell,q}\}$ and $1 \leq j \leq 2$. The proof is now complete. $\square$

Lemma 6.6.5. *Let $p = 2$ and $\lambda \vdash n$ such that $\overline{r_\lambda} = (1^2)$. Let $t = \{u_1\} \wedge \{u_2\} \in \mathcal{T}(\lambda)_2^e$. Assume that $c_{\mathfrak{S}_n}(W(t)) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$.*

- (i) *There exists an elementary abelian 2-subgroup E of $L(t)$ with 2-rank $\frac{n}{2}$.*
- (ii) *If we have $K_\ell \times H_\ell \leq L(t)$ for some $0 \leq \ell \leq \lfloor \frac{n}{4} \rfloor$, then $\ell \neq \frac{n}{4}$ and there exists a unique $s_{\ell,q} \in \{s_{\ell,1}, \dots, s_{\ell,y_\ell}\}$ such that $\langle s_{\ell,q} \rangle$ acts on $\{\{u_1\}, \{u_2\}\}$ transitively. Furthermore, define $U = \bigcup_{i=1}^\ell \{k_{i,1}, k_{i,2}\} \cup \{s_{\ell,1}, \dots, s_{\ell,y_\ell}\}$. Then $g\{u_j\} = \{u_j\}$ for all $g \in U \setminus \{s_{\ell,q}\}$ and $1 \leq j \leq 2$.*

Proof. As $p = 2$ and $\overline{r_\lambda} = (1^2)$, note that $W(t) \cong (\mathbb{F}_{L(t)})^{\uparrow \mathfrak{S}_n}$ and $\text{rank}(\mathfrak{S}_\lambda) = \frac{n}{2} - 1$. By Corollary 6.5.10 (ii), we have $c_{\mathfrak{S}_n}(W(t)) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda) = \frac{n}{2}$. Therefore, (i) and (ii) are shown by mimicking the proofs given in (i) and (iii) of Lemma 6.6.4 respectively. $\square$

Lemma 6.6.6. *Let $\lambda \vdash n$ such that $\overline{r_\lambda} = (p-1, 1)$. Let $t = \{u_1\} \dots \{u_p\} \in \mathcal{T}(\lambda)_p^s$ and $c_{\mathfrak{S}_n}(V(t)) = c_{\mathfrak{S}_n}(S^p M^\lambda)$. Then $\{u_i\} \neq \{u_j\}$ for all $1 \leq i \neq j \leq p$.*

Proof. Let $r = \text{rank}(\mathfrak{S}_\lambda)$. Note that $K(t)$ acts on the set of the λ -tabloids among $\{u_1\}, \dots, \{u_p\}$. If the desired assertion is false, a Sylow p -subgroup P of $K(t)$ fixes $\{u_i\}$ for all $1 \leq i \leq p$. So P is $\mathfrak{S}_n$ -conjugate to a subgroup of $\mathfrak{S}_\lambda$. As $\overline{r_\lambda} = (p-1, 1)$ and $V(t) \cong (\mathbb{F}_{K(t)})^{\uparrow \mathfrak{S}_n}$, by Lemmas 2.7.1 (v) and 6.2.5, $c_{\mathfrak{S}_n}(V(t)) \leq r$ and $c_{\mathfrak{S}_n}(S^p M^\lambda) = r+1$, which contradicts the fact $c_{\mathfrak{S}_n}(V(t)) = c_{\mathfrak{S}_n}(S^p M^\lambda)$. The lemma follows. $\square$

Lemma 6.6.7. *Let $p > 2$ and $\lambda \vdash n$ such that $\overline{r_\lambda} = (p-1, 1)$. If $t = \{u_1\} \dots \{u_p\} \in \mathcal{T}(\lambda)_p^s$ and $c_{\mathfrak{S}_n}(V(t)) = c_{\mathfrak{S}_n}(S^p M^\lambda)$, then $\phi(t) = q_\lambda$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $b = \frac{n}{p}$. By Lemma 6.6.6, notice that $\{u_1\}, \dots, \{u_p\}$ are pairwise distinct. Let $S = \{\{u_1\}, \dots, \{u_p\}\}$. By Lemma 6.6.4 (i), there exists an elementary abelian p -subgroup E of $K(t)$ with p -rank b . As $p > 2$, by Lemma 4.2.7 (i), E is $\mathfrak{S}_n$ -conjugate to E_b . As $\phi(t) = \phi(gt)$ for all $g \in \mathfrak{S}_n$, we may assume further that $E = E_b$. By Lemmas 6.6.4 (ii) and 2.11.1, there exists some s_k such that $1 \leq k \leq b$ and $\langle s_k \rangle$ acts transitively on S . Moreover, for all $1 \leq i \neq k \leq b$, s_i fixes $\{u_1\}$ and the numbers $(i-1)p+1, \dots, ip$ are totally contained in some row of $\{u_1\}$. As $\overline{r_\lambda} = (p-1, 1)$, also note that $p-1$ members of $\{(k-1)p+1, \dots, kp\}$ are in $R_x(\{u_1\})$ and the remaining member of $\{(k-1)p+1, \dots, kp\}$ is in $R_y(\{u_1\})$, where $\lambda_x \equiv p-1 \pmod{p}$ and $\lambda_y \equiv 1 \pmod{p}$. Moreover, for any $a \in \{(k-1)p+1, \dots, kp\}$, we have $a \notin \bigcap_{i=1}^p R_x(\{u_i\})$ and $a \notin \bigcap_{i=1}^p R_y(\{u_i\})$. As $\langle s_k \rangle$ acts transitively on S , by the above discussion and the definition of $\phi(t)$, we have $\phi(t) = q_\lambda$. The lemma follows. $\square$

Lemma 6.6.8. *Let $p = 2$ and $\lambda \vdash n$ such that $\overline{r_\lambda} = (1^2)$. Assume that $t = \{u_1\}\{u_2\} \in \mathcal{T}(\lambda)_2^s$ and $c_{\mathfrak{S}_n}(V(t)) = c_{\mathfrak{S}_n}(S^2 M^\lambda)$. Then $\phi(t) = q_\lambda$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $b = \frac{n}{2}$ and $S = \{\{u_1\}, \{u_2\}\}$. By Lemma 6.6.6, observe that $\{u_1\} \neq \{u_2\}$. By Lemma 6.6.4 (i), there exists an elementary abelian 2-subgroup E of $K(t)$ with 2-rank b . By Lemma 4.2.7 (ii), E is $\mathfrak{S}_n$ -conjugate to $K_c \times H_c$ for some $0 \leq c \leq \lfloor \frac{b}{2} \rfloor$. As $\phi(t) = \phi(gt)$ for all $g \in \mathfrak{S}_n$, we may assume that $E = K_c \times H_c$. By Lemmas 6.6.4 (iii) and 2.11.1, there exists some $s_{c,q}$ such that $1 \leq q \leq y_c$ and $\langle s_{c,q} \rangle$ acts transitively on S . Moreover, let $U = \bigcup_{i=1}^c \{k_{i,1}, k_{i,2}\} \cup \{s_{c,1}, \dots, s_{c,y_c}\}$. For all $g \in U \setminus \{s_{c,q}\}$, g fixes $\{u_1\}$ and each orbit of $\text{Supp}(g)$ under the action of $\langle g \rangle$ is totally contained in some row of $\{u_1\}$, where $\text{Supp}(g) = \{i \in [1, n] : g(i) \neq i\}$. Since $\overline{r_\lambda} = (1^2)$, note that $R_x(\{u_1\})$ contains a member of $\{4c+2q-1, 4c+2q\}$ and $R_y(\{u_1\})$ contains the other member of $\{4c+2q-1, 4c+2q\}$, where $2 \nmid \lambda_x$ and $2 \nmid \lambda_y$. For any $a \in \{4c+2q-1, 4c+2q\}$, note that $a \notin (\bigcap_{i=1}^2 R_x(\{u_i\})) \cup (\bigcap_{i=1}^2 R_y(\{u_i\}))$. As $s_{c,q}\{u_1\} = \{u_2\}$, by the above discussion and the definition of $\phi(t)$, we have $\phi(t) = q_\lambda$. The lemma follows. $\square$

We have the following two corollaries.

Corollary 6.6.9. *Let $\lambda \vdash n$ such that $\overline{r_\lambda} = (p-1, 1)$. For some $z \in \mathbb{N}$, there exists a decomposition*

$$S^p M^\lambda = V(t_{\lambda,p}) \oplus \bigoplus_{i=1}^z V(t_i),$$

where $t_1, \dots, t_z \in \mathcal{T}(\lambda)_p^s$, $V(t_{\lambda,p}) \cong M^\mu$, $\mu = q_\lambda \cup (p)$ and $c_{\mathfrak{S}_n}(V(t_i)) < c_{\mathfrak{S}_n}(S^p M^\lambda)$ for all $1 \leq i \leq z$.

Proof. By (6.4.1), for some $z \in \mathbb{N}$, recall that $S^p M^\lambda = V(t_{\lambda,p}) \oplus \bigoplus_{i=1}^z V(t_i)$, where we have $t_1, \dots, t_z \in \mathcal{T}(\lambda)_p^s$. By Lemma 6.6.1 and the definition of $\mathfrak{S}_{\lambda,p}$, note that $V(t_{\lambda,p}) \cong M^\mu$, where $\mu = q_\lambda \cup (p)$. For all $1 \leq i \leq z$, as $V(t_i) = \langle \mathcal{O}^s(t_i) \rangle_{\mathbb{F}}$ and $\mathcal{O}^s(t_{\lambda,p}) \cap \mathcal{O}^s(t_i) = \emptyset$, by Lemmas 2.7.1 (i), 6.6.3, 6.6.7 and 6.6.8, $c_{\mathfrak{S}_n}(V(t_i)) < c_{\mathfrak{S}_n}(S^p M^\lambda)$ for all $1 \leq i \leq z$. The corollary follows. $\square$

Corollary 6.6.10. *Let $p = 2$ and $\lambda \vdash n$ such that $\overline{r_\lambda} = (1^2)$. For some $z \in \mathbb{N}$, there is a decomposition*

$$\Lambda^2 M^\lambda = \bigoplus_{i=1}^z W(t_i),$$

where $t_1, \dots, t_z \in \mathcal{T}(\lambda)_2^e$, $W(t_1) \cong M^\mu$, $\mu = q_\lambda \cup (2)$ and $c_{\mathfrak{S}_n}(W(t_i)) < c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$ for all $1 < i \leq z$.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$. By (6.4.2), for some $z \in \mathbb{N}$, we have $\Lambda^2 M^\lambda = \bigoplus_{i=1}^z W(t_i)$, where $t_1, \dots, t_z \in \mathcal{T}(\lambda)_2^e$. By Lemma 2.7.1 (i), we may assume further that $t_1 = \{u\} \wedge \{v\}$ and $c_{\mathfrak{S}_n}(W(t_1)) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$. By Lemmas 6.6.5 (i), (ii) and 4.2.7 (ii), there exists some $(a, b) \in L(t_1)$ such that $(a, b)\{u\} = \{v\}$. Moreover, as $\overline{r_\lambda} = (1^2)$, a member of $\{a, b\}$ is in $R_c(\{u\})$ and the other member of $\{a, b\}$ is in $R_d(\{u\})$, where $1 \leq c \neq d \leq \ell$, $2 \nmid \lambda_c$ and $2 \nmid \lambda_d$. As $p = 2$, by Lemma 6.5.2, $W(t_1) \cong (\mathbb{F}_{L(t_1)})^{\uparrow \mathfrak{S}_n} \cong M^\mu$, where $\mu = q_\lambda \cup (2)$. If we have $c_{\mathfrak{S}_n}(W(t_x)) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$ for some $1 < x \leq z$, note that $\mathcal{O}^e(t_x) \cap \mathcal{O}^e(t_1) = \emptyset$. Let $t_x = \{\tilde{u}\} \wedge \{\tilde{v}\}$. By Lemmas 6.6.5 (i), (ii) and 4.2.7 (ii) again, there exists some $(j, k) \in L(t_x)$ such that $(j, k)\{\tilde{u}\} = \{\tilde{v}\}$. Moreover, as $\overline{r_\lambda} = (1^2)$, a member of $\{j, k\}$ is in $R_c(\{\tilde{u}\})$ and the other member of $\{j, k\}$ is in $R_d(\{\tilde{u}\})$. It is easy to note that there exists some $g \in \mathfrak{S}_n$ such that $gt_1 = t_x$, which contradicts the fact $\mathcal{O}^e(t_x) \cap \mathcal{O}^e(t_1) = \emptyset$. Therefore, by Lemma 2.7.1 (i), $c_{\mathfrak{S}_n}(W(t_i)) < c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$ for all $1 < i \leq z$. The proof is now complete. $\square$

The main result of this section is from Corollaries 6.6.9 and 6.6.10. We state it as follows.

Theorem 6.6.11. *Let $\lambda \vdash n$ such that $\overline{r_\lambda} = (p-1, 1)$. Let $\mu = q_\lambda \cup (p)$ and M be an indecomposable $\mathbb{F}\mathfrak{S}_n$ -module.*

- (i) *If $M \mid S^p M^\lambda$ and $c_{\mathfrak{S}_n}(M) = c_{\mathfrak{S}_n}(S^p M^\lambda)$, then $M \cong Y^\nu$, where $Y^\nu \mid M^\mu$. Moreover, all the partitions labelling these Young modules can be explicitly determined by μ .*
- (ii) *If $p = 2$, $M \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(M) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$, then $M \cong Y^\nu$, where $Y^\nu \mid M^\mu$. Moreover, all the partitions labelling these Young modules can be explicitly determined by μ .*

Proof. Note that (i) follows from Lemmas 2.7.1 (i), 6.2.5, Corollary 6.6.9 and Theorems 6.1.4, 6.1.5. Also note that (ii) follows from Lemma 2.7.1 (i), Corollaries 6.6.10, 6.5.10 and Theorems 6.1.4, 6.1.5. $\square$

Following Theorems 6.1.6 and 6.6.11, we propose the following question.

Question 6.6.12. *Let $\lambda \vdash n$ and $1 < a \in \mathbb{N}$.*

- (i) *For any $\mu \vdash n$ satisfying $Y^\mu \mid S^a M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(S^a M^\lambda)$, can one determine the partition μ explicitly?*
- (ii) *If $\lambda \neq (n)$, for any $\mu \vdash n$ satisfying $Y^\mu \mid \Lambda^2 M^\lambda$ and $c_{\mathfrak{S}_n}(Y^\mu) = c_{\mathfrak{S}_n}(\Lambda^2 M^\lambda)$, can one determine the partition μ explicitly?*

Note that Theorem 6.6.11 nicely answers Question 6.6.12 for some special cases. Therefore, Theorem 6.6.11 is a stronger result than Theorem 6.1.6 for some special cases.

6.7 A Result of Some p -Subgroups of $\mathfrak{S}_n$

The main result of this section is Theorem 6.7.8. This result is important for the generalization of Theorem 6.1.7. Given $m \in \mathbb{N}$, let $\mathcal{M}_m$ be the set of all the $(m \times m)$ -matrices over $\mathbb{N}_0$. We fix the required notation as follows.

Notation 6.7.1. Let $1 < m \in \mathbb{N}$ and $\mathcal{D}_m = \{M \in \mathcal{M}_m : M \text{ is a diagonal matrix}\}$.

- (i) Let $S \subseteq [1, n]$ and P_S be a fixed Sylow p -subgroup of $\mathfrak{S}_S$. For any $g \in \mathfrak{S}_n$, we define $\text{Supp}(g) = \{i \in [1, n] : g(i) \neq i\}$. Therefore, $g = 1$ if and only if $\text{Supp}(g) = \emptyset$. Set $g\emptyset = \emptyset$ and put $gT = \{g(i) : i \in T\}$ for any non-empty subset T of $[1, n]$.

- (ii) Let $M = (a_{ij}) \in \mathcal{M}_m$. For all $1 \leq i, j \leq m$, write $a_{ij} = \sum_{k \geq 0} p^k a_{i,j,k}$, where $a_{i,j,k} \in \mathbb{N}_0$ and $a_{i,j,k} < p$ for all $k \geq 0$. Define

$$\begin{aligned} D(M) &= (\sum_{1 \leq i \leq m} a_{i,i,k})_{k \geq 0}, \\ U(M) &= (\sum_{1 \leq i < j \leq m} a_{i,j,k})_{k \geq 0}, \\ T(M) &= (\sum_{1 \leq i, j \leq m} a_{i,j,k})_{k \geq 0}. \end{aligned}$$

Let M^T be the transpose of M . If $M = M^T$, then $T(M) = D(M) + 2U(M)$, where the addition and the scalar multiplication are componentwise. To write the defined $D(M)$, $U(M)$ and $T(M)$ down explicitly, we shall end these sequences with their last non-zero entries and use (0) to denote an infinite sequence of zeros.

- (iii) Define $\mathcal{M}_{m,n} = \{(a_{ij}) \in \mathcal{M}_m : \sum_{1 \leq i, j \leq m} a_{ij} = n\} \setminus \mathcal{D}_m$. For any $M = (b_{ij}) \in \mathcal{M}_{m,n}$, we assign a subset $s_{i,j,M}$ of $[1, n]$ to b_{ij} for any $1 \leq i, j \leq m$. In this assignment, for all $1 \leq i, j, u, v \leq m$, we require that $|s_{i,j,M}| = b_{ij}$ and $s_{i,j,M} \cap s_{u,v,M} = \emptyset$ if $(i, j) \neq (u, v)$. By this assignment, we define

$$P_M = \begin{cases} \prod_{1 \leq i, j \leq m} P_{s_{i,j,M}}, & \text{if } M \neq M^T \text{ or } p > 2, \\ (\prod_{1 \leq k \leq m} P_{s_{k,k,M}} \times \prod_{1 \leq i < j \leq m} (P_{s_{i,j,M}} \times P_{s_{i,j,M}}^{e_M})) \rtimes \langle e_M \rangle, & \text{otherwise,} \end{cases} \quad (6.7.1)$$

where e_M is a fixed involution of $\mathfrak{S}_n$, $\text{Supp}(e_M) = [1, n] \setminus \bigcup_{k=1}^m s_{k,k,M}$, $P_{s_{i,j,M}}^{e_M} = e_M P_{s_{i,j,M}} e_M$ and $e_M s_{i,j,M} = s_{j,i,M}$ for all $1 \leq i, j \leq m$. For any $1 \leq i < j \leq m$, note that $P_{s_{i,j,M}}^{e_M}$ is $\mathfrak{S}_{s_{j,i,M}}$ -conjugate to $P_{s_{j,i,M}}$. In particular, for any $1 \leq i < j \leq m$, $|P_{s_{i,j,M}}^{e_M}| = |P_{s_{j,i,M}}|$. A p -subgroup P of $\mathfrak{S}_n$ is called a Young p -subgroup of $\mathfrak{S}_n$ if P is $\mathfrak{S}_n$ -conjugate to a Sylow p -subgroup of some Young subgroup of $\mathfrak{S}_n$. Let $\mathcal{Y}_n$ be the set of all the Young p -subgroups of $\mathfrak{S}_n$.

Example 6.7.2. An example illustrates most of the definitions in Notation 6.7.1. Let $p = 2$ and $M_1, M_2 \in \mathcal{M}_{3,10}$, where

$$M_1 = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

It is not difficult to get that $D(M_1) = (0, 2) \neq (0) = D(M_2)$, $U(M_1) = (1^2) \neq (1, 2) = U(M_2)$ and $T(M_1) = (2, 4) = T(M_2)$. Moreover, by (6.7.1), observe that P_{M_1} is $\mathfrak{S}_{10}$ -conjugate to $\langle (1, 2), (3, 4), (5, 6), (7, 8), (1, 5)(2, 6)(9, 10) \rangle$. By (6.7.1) again, we can also observe that P_{M_2} is $\mathfrak{S}_{10}$ -conjugate to $\langle (1, 2), (3, 4), (5, 6), (7, 8), (1, 3)(2, 4)(5, 7)(6, 8)(9, 10) \rangle$.

Let $\emptyset \neq S \subseteq [1, n]$. We recall some facts of the construction of a Sylow p -subgroup of $\mathfrak{S}_S$. Let $i \in \mathbb{N}_0$ and P_i be a Sylow p -subgroup of $\mathfrak{S}_{p^i}$. If $|S| = \sum_{j \geq 0} p^j n_j$, where $n_j \in \mathbb{N}_0$ and $n_j < p$ for all $j \geq 0$, then $P_S \cong \prod_{j \geq 0} (P_j)^{n_j}$. If $p = 2$ and $i > 0$, recall that $P_i = (P_{i-1,1} \times P_{i-1,2}) \rtimes \langle w \rangle$, where the groups $P_{i-1,1}$ and $P_{i-1,2}$ are $\mathfrak{S}_{2^{i-1}}$ -conjugate to P_{i-1} and $w P_{i-1,1} w = P_{i-1,2}$ for the involution w . Also recall that $[1, n]/P$ is the set of orbits of $[1, n]$ under the action of a subgroup P of $\mathfrak{S}_n$. To prove the main result of this section, we need some lemmas as preparation. The first lemma describes a characterization of the Young p -subgroups of $\mathfrak{S}_n$.

Lemma 6.7.3. *Let P be a p -subgroup of $\mathfrak{S}_n$ and $[1, n]/P = \{\mathcal{O}_1, \dots, \mathcal{O}_s\}$. Then $P \in \mathcal{Y}_n$ if and only if P is a Sylow p -subgroup of $\mathfrak{S}_{\mathcal{O}_1} \times \dots \times \mathfrak{S}_{\mathcal{O}_s}$.*

Proof. According to the definition of $\mathcal{Y}_n$, it suffices to show that $P \in \mathcal{Y}_n$ only if P is a Sylow p -subgroup of $\mathfrak{S}_{\mathcal{O}_1} \times \dots \times \mathfrak{S}_{\mathcal{O}_s}$. Without loss of generality, as $P \in \mathcal{Y}_n$, we may assume that P is a Sylow p -subgroup of $\mathfrak{S}_\lambda$, where $\lambda \vdash n$. It is not difficult to see that $P \leq \mathfrak{S}_{\mathcal{O}_1} \times \dots \times \mathfrak{S}_{\mathcal{O}_s} \leq \mathfrak{S}_\lambda$, which implies that P is a Sylow p -subgroup of $\mathfrak{S}_{\mathcal{O}_1} \times \dots \times \mathfrak{S}_{\mathcal{O}_s}$. The proof is now complete. $\square$

For some special cases, we can use the following lemma to describe some members of $\mathcal{Y}_n$.

Lemma 6.7.4. *Let $p = 2$ and $1 < m \in \mathbb{N}$. If $M \in \mathcal{M}_{m,n}$ and $M = M^T$, then $P_M \in \mathcal{Y}_n$ if and only if $U(M) = (0^a, 1)$ for some $a \in \mathbb{N}_0$.*

Proof. Recall the assignment of M in Notation 6.7.1 (iii) and let $D_M = \prod_{i=1}^m P_{s_{i,i},M}$. As $M = M^T$, notice that $|s_{i,j},M| = |s_{j,i},M|$ for any $1 \leq i, j \leq m$. If $U(M) = (0^a, 1)$ for some $a \in \mathbb{N}_0$, by (6.7.1) and the definition of $U(M)$, it is obvious to see that $P_M \in \mathcal{Y}_n$.

Conversely, for any $1 \leq i < j \leq m$, let $S_{i,j,M} = s_{i,j,M} \cup s_{j,i,M}$ and $Q_{i,j,M}$ be a fixed Sylow 2-subgroup of $\mathfrak{S}_{S_{i,j,M}}$. We define $Q_M = D_M \times \prod_{1 \leq i < j \leq m} Q_{i,j,M}$ and note that $Q_M \in \mathcal{Y}_n$. Moreover, by (6.7.1), for all $i \geq 0$, the number of P_M -orbits of $[1, n]$ of size 2^i is equal to the number of Q_M -orbits of $[1, n]$ of size 2^i . As we have assumed that $P_M \in \mathcal{Y}_n$, by Lemma 6.7.3, note that P_M is $\mathfrak{S}_n$ -conjugate to Q_M . In particular, $|P_M| = |Q_M|$. For any $1 \leq i < j \leq m$, notice that $|P_{s_{i,j},M} \times P_{s_{j,i},M}| \leq |Q_{i,j,M}|$. For all $w \in \mathbb{N}_0$, if we have $1 \leq u < v \leq m$, $s_{u,v,M} \neq \emptyset$ and $P_{s_{u,v},M}$ is not $\mathfrak{S}_n$ -conjugate to a Sylow 2-subgroup of $\mathfrak{S}_{2^w}$, by the construction of a Sylow 2-subgroup of $\mathfrak{S}_{s_{u,v},M}$, note that $|P_{s_{u,v},M} \times P_{s_{v,u},M}| < 4|P_{s_{u,v},M} \times P_{s_{v,u},M}| \leq |Q_{u,v,M}|$. By (6.7.1), we thus have

$$\begin{aligned} |P_M| &= 2|D_M| |P_{s_{u,v},M} \times P_{s_{v,u},M}^{e_M}| \prod_{\substack{1 \leq i < j \leq m \\ (i,j) \neq (u,v)}} |P_{s_{i,j},M} \times P_{s_{j,i},M}^{e_M}| \\ &= 2|D_M| |P_{s_{u,v},M} \times P_{s_{v,u},M}| \prod_{\substack{1 \leq i < j \leq m \\ (i,j) \neq (u,v)}} |P_{s_{i,j},M} \times P_{s_{j,i},M}| \\ &< 4|D_M| |P_{s_{u,v},M} \times P_{s_{v,u},M}| \prod_{\substack{1 \leq i < j \leq m \\ (i,j) \neq (u,v)}} |P_{s_{i,j},M} \times P_{s_{j,i},M}| \\ &\leq |D_M| \prod_{1 \leq i < j \leq m} |Q_{i,j,M}| = |Q_M|, \end{aligned}$$

which contradicts the fact that $|P_M| = |Q_M|$. Therefore, for any $1 \leq i < j \leq m$, $P_{s_{i,j},M}$ is $\mathfrak{S}_n$ -conjugate to a Sylow 2-subgroup of $\mathfrak{S}_{2^k}$ for some $k \in \mathbb{N}_0$. So $2|P_{s_{i,j},M} \times P_{s_{j,i},M}| = |Q_{i,j,M}|$ for all $1 \leq i < j \leq m$ and $s_{i,j,M} \neq \emptyset$. By (6.7.1), we have

$$|P_M| = 2|D_M| \prod_{1 \leq i < j \leq m} |P_{s_{i,j},M} \times P_{s_{j,i},M}| = |D_M| \prod_{1 \leq i < j \leq m} |Q_{i,j,M}| = |Q_M|, \quad (6.7.2)$$

which forces that $|s_{u,v},M| = |s_{v,u},M| = 2^k$ for some $1 \leq u < v \leq m$ and $k \in \mathbb{N}_0$. For any $1 \leq i < j \leq m$, (6.7.2) also implies that $s_{i,j,M} \neq \emptyset$ only if $i = u$ and $j = v$. By the definition of $U(M)$, we have $U(M) = (0^a, 1)$ for some $a \in \mathbb{N}_0$. The lemma follows. $\square$

Let $1 < m \in \mathbb{N}$ and $M \in \mathcal{M}_{m,n}$. If $p = 2$, $M = M^T$ and $U(M) = (0^a, 1)$ for some $a \in \mathbb{N}_0$, set $T'(M) = D(M) + (0^{a+1}, 1, 0, 0, \dots)$, where the sum is componentwise. Otherwise, put $T'(M) = T(M)$. Recall the assignment of M in (6.7.1) and let $p = 2$. For any $1 \leq i, j \leq m$, if $s_{i,j,M} \neq \emptyset$, there exist unique $x \in \mathbb{N}$ and $m_1, \dots, m_x \in \mathbb{N}_0$ such that $m_1 < \dots < m_x$, $|s_{i,j},M| = \sum_{u=1}^x 2^{m_u}$ and

$$P_{s_{i,j},M} = Q_{i,j,M,m_1} \times \dots \times Q_{i,j,M,m_x}, \quad (6.7.3)$$

where, for any $k \in \{m_1, \dots, m_x\}$, $Q_{i,j,M,k}$ is $\mathfrak{S}_n$ -conjugate to a Sylow 2-subgroup of $\mathfrak{S}_{2^k}$.

Lemma 6.7.5. *Let $1 < m \in \mathbb{N}$ and $M_1, M_2 \in \mathcal{M}_{m,n}$. If we have $P_{M_1}, P_{M_2} \in \mathcal{Y}_n$, then P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} if and only if $T'(M_1) = T'(M_2)$.*

Proof. Let $M \in \mathcal{M}_{m,n}$ and $P_M \in \mathcal{Y}_n$. By (6.7.1), Lemma 6.7.4 and the definition of $T'(M)$, for any $i > 0$, note that the i th entry of $T'(M)$ is the number of orbits of $[1, n]$ of size p^{i-1} under the action of P_M . If $T'(M_1) = T'(M_2)$, as $P_{M_1}, P_{M_2} \in \mathcal{Y}_n$, by Lemma 6.7.3, P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} . Conversely, if P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} , then there is a size preserving one-to-one correspondence between the members of $[1, n]/P_{M_1}$ and $[1, n]/P_{M_2}$. By the explanation of the entries of $T'(M)$, we have $T'(M_1) = T'(M_2)$. The proof is now complete. $\square$

Throughout the remaining part of this chapter, recall that the permutations are multiplied from right to left (see Section 2.10). We need the next two lemmas to deduce our main result.

Lemma 6.7.6. *Let $p = 2$, $1 < m \in \mathbb{N}$ and $M_1, M_2 \in \mathcal{M}_{m,n}$. If $M_1 = M_1^T$, $M_2 = M_2^T$, $D(M_1) = D(M_2)$ and $U(M_1) = U(M_2)$, then P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} .*

Proof. Recall the assignments of M_1 and M_2 in Notation 6.7.1 (iii). By (6.7.1), we have

$$\begin{aligned} P_{M_1} &= \left(\prod_{1 \leq k \leq m} P_{s_{k,k}, M_1} \times \prod_{1 \leq i < j \leq m} (P_{s_{i,j}, M_1} \times P_{s_{i,j}, M_1}^{e_{M_1}}) \right) \rtimes \langle e_{M_1} \rangle, \\ P_{M_2} &= \left(\prod_{1 \leq k \leq m} P_{s_{k,k}, M_2} \times \prod_{1 \leq i < j \leq m} (P_{s_{i,j}, M_2} \times P_{s_{i,j}, M_2}^{e_{M_2}}) \right) \rtimes \langle e_{M_2} \rangle. \end{aligned}$$

For any given $s_{a,b,M_1} \neq \emptyset$, if $a \leq b$, by (6.7.3), there exist unique $c \in \mathbb{N}$ and $d_1, \dots, d_c \in \mathbb{N}_0$ such that $d_1 < \dots < d_c$ and

$$P_{s_{a,b,M_1}} = Q_{a,b,M_1,d_1} \times \dots \times Q_{a,b,M_1,d_c}. \quad (6.7.4)$$

If $a < b$, as $U(M_1) = U(M_2)$, by the definitions of $U(M_1)$ and $U(M_2)$, there exist

$$Q_{a_1,b_1,M_2,d_1}, \dots, Q_{a_c,b_c,M_2,d_c},$$

where $1 \leq a_i < b_i \leq m$ for all $1 \leq i \leq c$. Moreover, we also have

$$e_{M_2} Q_{a_1,b_1,M_2,d_1} e_{M_2}, \dots, e_{M_2} Q_{a_c,b_c,M_2,d_c} e_{M_2}.$$

According to (6.7.4) and the Sylow Theorem, it is not difficult to see that there exists some $g_{a,b} \in \mathfrak{S}_n$ such that

$$g_{a,b} Q_{a,b,M_1,d_i} g_{a,b}^{-1} = Q_{a_i,b_i,M_2,d_i} \text{ and } g_{a,b} e_{M_1} Q_{a,b,M_1,d_i} e_{M_1} g_{a,b}^{-1} = e_{M_2} Q_{a_i,b_i,M_2,d_i} e_{M_2}$$

for all $1 \leq i \leq c$. If $a = b$, as $D(M_1) = D(M_2)$, by the definitions of $D(M_1)$ and $D(M_2)$, there exist

$$Q_{a_1,a_1,M_2,d_1}, \dots, Q_{a_c,a_c,M_2,d_c},$$

where $1 \leq a_i \leq m$ for all $1 \leq i \leq c$. By (6.7.4) and the Sylow Theorem, notice that there exists some $g_{a,a} \in \mathfrak{S}_n$ such that

$$g_{a,a} Q_{a,a,M_1,d_i} g_{a,a}^{-1} = Q_{a_i,a_i,M_2,d_i}$$

for all $1 \leq i \leq c$. Also set $g_{i,j} = 1$ for all $s_{i,j,M_1} = \emptyset$ or $i > j$. Since $D(M_1) = D(M_2)$ and $U(M_1) = U(M_2)$, for any $1 \leq i, j, u, v \leq m$, if $(i, j) \neq (u, v)$, by the definitions of $g_{i,j}$ and $g_{u,v}$, we can require that $\text{Supp}(g_{i,j}) \cap \text{Supp}(g_{u,v}) = \emptyset$. Set $g = \prod_{1 \leq i, j \leq m} g_{i,j}$. By the definition of g , note that

$$g \left(\prod_{1 \leq k \leq m} P_{s_{k,k}, M_1} \times \prod_{1 \leq i < j \leq m} (P_{s_{i,j}, M_1} \times P_{s_{i,j}, M_1}^{e_{M_1}}) \right) g^{-1} = \prod_{1 \leq k \leq m} P_{s_{k,k}, M_2} \times \prod_{1 \leq i < j \leq m} (P_{s_{i,j}, M_2} \times P_{s_{i,j}, M_2}^{e_{M_2}}).$$

As $U(M_1) = U(M_2)$, by the definitions of e_{M_1} and e_{M_2} , if $g e_{M_1} g^{-1}$ is a product of h pairwise disjoint 2-cycles, so is e_{M_2} . For all $1 \leq i < j \leq m$, we claim that $g e_{M_1} g^{-1}$ swaps s_{i,j,M_2} and s_{j,i,M_2} . Let $e_{M_1} = (i_1, j_1) \dots (i_h, j_h)$ for some $h \in \mathbb{N}$, where $\text{Supp}((i_u, j_u)) \cap \text{Supp}((i_v, j_v)) = \emptyset$ for any $1 \leq u \neq v \leq h$. By the definition of e_{M_1} , for any $1 \leq k \leq h$, note that $i_k \in s_{x,y,M_1}$ and $j_k \in s_{y,x,M_1}$ for some $x \neq y$. We have $g e_{M_1} g^{-1} = (g(i_1), g(j_1)) \dots (g(i_h), g(j_h))$. The claim follows by the definition of g . By this claim, both $g P_{M_1} g^{-1}$ and P_{M_2} are $\mathfrak{S}_n$ -conjugate to the Sylow 2-subgroups of

$$\left(\prod_{k=1}^m \mathfrak{S}_{s_{k,k}, M_2} \times \prod_{1 \leq i < j \leq m} (\mathfrak{S}_{s_{i,j}, M_2} \times \mathfrak{S}_{s_{j,i}, M_2}) \right) \rtimes \langle e_{M_2} \rangle.$$

The lemma thus follows. $\square$

Lemma 6.7.7. *Let $p = 2$, $1 < m \in \mathbb{N}$ and $M_1, M_2 \in \mathcal{M}_{m,n}$. Let $M_1 = M_1^T$, $M_2 = M_2^T$ and $P_{M_1}, P_{M_2} \notin \mathcal{Y}_n$. If P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} , then $D(M_1) = D(M_2)$ and $U(M_1) = U(M_2)$.*

Proof. Recall the assignments of M_1 and M_2 in Notation 6.7.1 (iii). Since P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} , let $gP_{M_1}g^{-1} = P_{M_2}$ for some $g \in \mathfrak{S}_n$. For any $x \in \{1, 2\}$, set

$$\mathfrak{S}_{M_x} = \prod_{1 \leq i, j \leq m} \mathfrak{S}_{s_{i,j}, M_x} \text{ and } R_{M_x} = \prod_{1 \leq k \leq m} P_{s_{k,k}, M_x} \times \prod_{1 \leq i < j \leq m} (P_{s_{i,j}, M_x} \times P_{s_{i,j}, M_x}^{e_{M_x}}).$$

By (6.7.1), for any $x \in \{1, 2\}$, we have $R_{M_x} \leq \mathfrak{S}_{M_x}$, $P_{M_x} = R_{M_x} \rtimes \langle e_{M_x} \rangle$ and $|s_{i,j}, M_x| = |s_{j,i}, M_x|$ for any $1 \leq i, j \leq m$. We split the whole proof into four steps.

Step 1: Prove that $gR_{M_1}g^{-1} = R_{M_2}$.

For any $u \leq v$ and $s_{u,v}, M_2 \neq \emptyset$, by (6.7.3), we have $P_{s_{u,v}, M_2} = Q_{u,v, M_2, k_1} \times \cdots \times Q_{u,v, M_2, k_s}$ for unique $s \in \mathbb{N}$ and pairwise distinct non-negative integers $k_1, \dots, k_s$. We may assume that $Q_{u,v, M_2, k_i} \neq 1$ for all $1 \leq i \leq s$. Therefore, for any given $1 \leq i \leq s$, note that

$$Q_{u,v, M_2, k_i} = (Q_{k_i, 1} \times Q_{k_i, 2}) \rtimes \langle w \rangle,$$

where $Q_{k_i, j}$ is $\mathfrak{S}_n$ -conjugate to a Sylow 2-subgroup of $\mathfrak{S}_{2^{k_i-1}}$ for any $j \in \{1, 2\}$ and we have $wQ_{k_i, 1}w = Q_{k_i, 2}$ for the involution w . Let $\sigma_{k_i, 1}$ be a 2^{k_i-1} -cycle of $Q_{k_i, 1}$. We can also write $\sigma_{k_i, 2} = w\sigma_{k_i, 1}w$. We may assume that $\text{Supp}(\sigma_{k_i, 1}) \cap \text{Supp}(\sigma_{k_i, 2}) = \emptyset$. Define $\rho_{k_i} = \sigma_{k_i, 1}w$ and note that ρ_{k_i} is a 2^{k_i} -cycle of Q_{u,v, M_2, k_i} . Moreover, $\rho_{k_i}^2 = \sigma_{k_i, 1}w\sigma_{k_i, 1}w = \sigma_{k_i, 1}\sigma_{k_i, 2}$. As $gP_{M_1}g^{-1} = P_{M_2}$, we have $\rho_{k_i} = gqg^{-1}$ or $\rho_{k_i} = gqe_{M_1}g^{-1}$ for some $q \in R_{M_1}$. As we have $e_{M_1} \in N_{\mathfrak{S}_n}(R_{M_1})$, note that $\rho_{k_i}^2 = grg^{-1}$ for some $r \in R_{M_1}$. As $\text{Supp}(\sigma_{k_i, 1}) \cap \text{Supp}(\sigma_{k_i, 2}) = \emptyset$, there are $r_1, r_2 \in \mathfrak{S}_n$ such that $r = r_1r_2$, $\text{Supp}(r_1) \cap \text{Supp}(r_2) = \emptyset$ and $\sigma_{k_i, j} = gr_jg^{-1}$ for any $j \in \{1, 2\}$. As $r \in R_{M_1}$ and both $\sigma_{k_i, 1}$ and $\sigma_{k_i, 2}$ are 2^{k_i-1} -cycles, note that

$$\text{Supp}(\sigma_{k_i, 1}) = \text{Supp}(gr_1g^{-1}) \subseteq gs_{a,b}, M_1 \text{ and } \text{Supp}(\sigma_{k_i, 2}) = \text{Supp}(gr_2g^{-1}) \subseteq gs_{c,d}, M_1$$

for some $1 \leq a, b, c, d \leq m$. We claim that $\rho_{k_i} = ghg^{-1}$ for some $h \in R_{M_1}$. Otherwise, if $\rho_{k_i} = ghe_{M_1}g^{-1}$ for some $h \in R_{M_1}$, we shall show the following results.

- (i) We have $a \neq b$ and $(c, d) = (b, a)$.
- (ii) We have $|gs_{a,b}, M_1| = |gs_{b,a}, M_1| = |\text{Supp}(\sigma_{k_i, 1})| = 2^{k_i-1}$.
- (iii) We have $|gs_{j,k}, M_1| = |gs_{k,j}, M_1| = 0$ for any $1 \leq j \neq k \leq m$ and $\{j, k\} \neq \{a, b\}$.

For (i), assume that $a = b$. Recall that $\rho_{k_i} = \sigma_{k_i, 1}w = ghe_{M_1}g^{-1}$ and $\sigma_{k_i, j} = gr_jg^{-1}$ for any $j \in \{1, 2\}$. Let $w = gxg^{-1}$ for some $x \in P_{M_1}$. We note that $r_2 \neq 1$. Otherwise, since $|\text{Supp}(\sigma_{k_i, 1})| = |\text{Supp}(\sigma_{k_i, 2})|$, we have $\sigma_{k_i, 1} = \sigma_{k_i, 2} = gr_2g^{-1} = 1$ and ρ_{k_i} is a transposition. As $\rho_{k_i} = ghe_{M_1}g^{-1}$, it is clear to see that $he_{M_1} = (\alpha, \beta) \in P_{M_1}$ and $h = (\alpha, \beta)e_{M_1} \in R_{M_1}$. As $P_{M_1} \notin \mathcal{Y}_n$, by Lemma 6.7.4 and the definition of e_{M_1} , note that $|\text{Supp}(e_{M_1})| > 3$. Therefore, by the definition of e_{M_1} , there exist γ, η such that $e_{M_1}(\gamma) = \eta$, $\eta \notin \{\alpha, \beta\}$, $\gamma \in s_{\lambda, \mu}, M_1$ and $\eta \in s_{\mu, \lambda}, M_1$ for some $1 \leq \lambda \neq \mu \leq m$. So $h(\gamma) = \eta$, which contradicts the fact that $h \in R_{M_1}$. As $w^2 = x^2 = 1$, we have

$$r_2 = g^{-1}gr_2g^{-1}g = g^{-1}\sigma_{k_i, 2}g = g^{-1}w\sigma_{k_i, 1}wg = g^{-1}gxg^{-1}gr_1g^{-1}gxg^{-1}g = xr_1x. \quad (6.7.5)$$

As $\text{Supp}(\sigma_{k_i, 1}) \subseteq gs_{a,a}, M_1$, $\text{Supp}(\sigma_{k_i, 2}) \subseteq gs_{c,d}, M_1$, $\sigma_{k_i, 1} = gr_1g^{-1}$ and $\sigma_{k_i, 2} = gr_2g^{-1}$, we have $\text{Supp}(r_1) \subseteq s_{a,a}, M_1$ and $\text{Supp}(r_2) \subseteq s_{c,d}, M_1$. As $x \in P_{M_1}$, by (6.7.1) and (6.7.5), note that

$$xs_{a,a}, M_1 = s_{a,a}, M_1 \text{ and } \text{Supp}(r_2) = \text{Supp}(xr_1x) \subseteq s_{a,a}, M_1 \cap s_{c,d}, M_1,$$

which forces that $c = d = a$. Otherwise, as $s_{a,a}, M_1 \cap s_{c,d}, M_1 = \emptyset$, we get $r_2 = 1$, which is a contradiction. So we deduce that

$$\text{Supp}(he_{M_1}) = \text{Supp}(g^{-1}\rho_{k_i}g) = \text{Supp}(r_1) \cup \text{Supp}(r_2) \subseteq s_{a,a}, M_1.$$

Therefore, we naturally view he_{M_1} as an element of $\mathfrak{S}_{s_{a,a},M_1}$. So $he_{M_1} \in \mathfrak{S}_{s_{a,a},M_1} \leq \mathfrak{S}_{M_1}$. As $h \in R_{M_1} \leq \mathfrak{S}_{M_1}$, we get $e_{M_1} \in \mathfrak{S}_{M_1}$, which contradicts the definition of e_{M_1} . So we have $a \neq b$.

As $r_2 = xr_1x$, $\text{Supp}(r_1) \subseteq s_{a,b,M_1}$ and $x \in P_{M_1}$, by (6.7.1), we have $\text{Supp}(r_2) \subseteq s_{a,b,M_1}$ or $\text{Supp}(r_2) \subseteq s_{b,a,M_1}$. If $\{c, d\} \neq \{a, b\}$, then $s_{a,b,M_1} \cap s_{c,d,M_1} = s_{b,a,M_1} \cap s_{c,d,M_1} = \emptyset$. Recall that $\text{Supp}(r_2) \subseteq s_{c,d,M_1}$. Therefore, we have $\text{Supp}(r_2) = \emptyset$ and $r_2 = 1$, which is a contradiction. So $\{c, d\} = \{a, b\}$. If $(a, b) = (c, d)$, then we have

$$\text{Supp}(he_{M_1}) = \text{Supp}(g^{-1}\rho_{k_i}g) = \text{Supp}(r_1) \cup \text{Supp}(r_2) \subseteq s_{a,b,M_1}.$$

Similarly, we naturally view he_{M_1} as an element of $\mathfrak{S}_{s_{a,b},M_1}$. So we have $he_{M_1} \in \mathfrak{S}_{s_{a,b},M_1} \leq \mathfrak{S}_{M_1}$. As $h \in R_{M_1} \leq \mathfrak{S}_{M_1}$, we also get $e_{M_1} \in \mathfrak{S}_{M_1}$, which also contradicts the definition of e_{M_1} . So we have $(c, d) = (b, a)$.

To prove (ii), let $S = \text{Supp}(\sigma_{k_i,1})$ and $T = \text{Supp}(\sigma_{k_i,2})$. Recall that $S \subseteq gs_{a,b,M_1}$. Therefore, $|S| \leq |gs_{a,b,M_1}| = |gs_{b,a,M_1}|$. Assume that this inequality is strict. We now pick $y \in (gs_{a,b,M_1}) \setminus S$ and note that $y \notin S \cup T = \text{Supp}(\rho_{k_i})$. Since $h \in R_{M_1}$ and $e_{M_1}s_{a,b,M_1} = s_{b,a,M_1}$, we have

$$hs_{b,a,M_1} = s_{b,a,M_1} \text{ and } z = ge_{M_1}g^{-1}(y) \in gs_{b,a,M_1}.$$

We thus have

$$y = \rho_{k_i}(y) = ghg^{-1}ge_{M_1}g^{-1}(y) = ghg^{-1}(z) \in gs_{a,b,M_1} \cap gs_{b,a,M_1},$$

which forces that $a = b$. By (i), this is impossible. So $|S| = |gs_{a,b,M_1}| = |gs_{b,a,M_1}| = 2^{k_i-1}$. The last equality holds as we have shown that $\sigma_{k_i,2} = gr_2g^{-1} \neq 1$ in (i).

For (iii), suppose that $|gs_{j,k,M_1}| = |gs_{k,j,M_1}| \neq 0$ for some $1 \leq j \neq k \leq m$ and $\{j, k\} \neq \{a, b\}$. Then $g(s_{j,k,M_1} \cup s_{k,j,M_1}) \cap g(s_{a,b,M_1} \cup s_{b,a,M_1}) = \emptyset$. By the proof of (i), recall that

$$\text{Supp}(\rho_{k_i}) = \text{Supp}(\sigma_{k_i,1}) \cup \text{Supp}(\sigma_{k_i,2}) \subseteq gs_{a,b,M_1} \cup gs_{b,a,M_1} = g(s_{a,b,M_1} \cup s_{b,a,M_1}).$$

So there exists some $y \in gs_{j,k,M_1}$ such that $y \notin \text{Supp}(\rho_{k_i})$. Since we have $h \in R_{M_1}$ and $e_{M_1}s_{j,k,M_1} = s_{k,j,M_1}$, we have $hs_{k,j,M_1} = s_{k,j,M_1}$ and $z = ge_{M_1}g^{-1}(y) \in gs_{k,j,M_1}$. As $j \neq k$, by mimicking the proof of (ii), we deduce that $y \in gs_{j,k,M_1} \cap gs_{k,j,M_1} = \emptyset$, which is absurd. (iii) is shown.

By (i), (ii), (iii), the definition of $U(M_1)$ and Lemma 6.7.4, we have $P_{M_1} \in \mathcal{Y}_n$, which is a contradiction. Therefore, $\rho_{k_i} = ghg^{-1}$ for some $h \in R_{M_1}$. The claim is shown. Since ρ_{k_i} is a 2^{k_i} -cycle and $h \in R_{M_1}$, there exist some $1 \leq e, f \leq m$ such that $\text{Supp}(h) \subseteq s_{e,f,M_1}$. For any $\varrho \in Q_{u,v,M_2,k_i}$, if $\varrho = g\kappa e_{M_1}g^{-1}$ for some $\kappa \in R_{M_1}$, note that

$$\text{Supp}(\kappa e_{M_1}) = \text{Supp}(g^{-1}\varrho g) \subseteq \text{Supp}(g^{-1}\rho_{k_i}g) = \text{Supp}(h) \subseteq s_{e,f,M_1}.$$

Therefore, we naturally view κe_{M_1} as an element of $\mathfrak{S}_{s_{e,f},M_1}$. So $\kappa e_{M_1} \in \mathfrak{S}_{s_{e,f},M_1} \leq \mathfrak{S}_{M_1}$. Since $\kappa \in R_{M_1} \leq \mathfrak{S}_{M_1}$, we have $e_{M_1} \in \mathfrak{S}_{M_1}$, which contradicts the definition of e_{M_1} . So $\varrho \in gR_{M_1}g^{-1}$. Recall that we choose Q_{u,v,M_2,k_i} arbitrarily. So we in fact have shown that $P_{s_{u,v},M_2} \leq gR_{M_1}g^{-1}$. As $gP_{M_1}g^{-1} = P_{M_2}$ and $P_{M_1} \leq N_{\mathfrak{S}_n}(R_{M_1})$, if $u < v$, then

$$g^{-1}e_{M_2}g \in P_{M_1} \text{ and } P_{s_{u,v},M_2}^{e_{M_2}} \leq e_{M_2}gR_{M_1}g^{-1}e_{M_2} = gg^{-1}e_{M_2}gR_{M_1}g^{-1}e_{M_2}gg^{-1} = gR_{M_1}g^{-1}.$$

As $P_{s_{u,v},M_2}$ is chosen arbitrarily, we have shown that $R_{M_2} \leq gR_{M_1}g^{-1}$. As $|R_{M_1}| = |R_{M_2}|$, we have $gR_{M_1}g^{-1} = R_{M_2}$.

Step 2: Prove that $e_{M_2} = g\rho e_{M_1}g^{-1}$ for some $\rho \in R_{M_1}$.

If $e_{M_2} = g\rho g^{-1}$ for some $\rho \in R_{M_1}$, by (6.7.1) and Step 1, we have $P_{M_2} \leq gR_{M_1}g^{-1}$, which contradicts the equality $|P_{M_2}| = |gP_{M_1}g^{-1}| = 2|R_{M_1}|$. We are done.

Step 3: Prove that $g(\prod_{k=1}^m P_{s_{k,k},M_1})g^{-1} = \prod_{k=1}^m P_{s_{k,k},M_2}$.

It suffices to show that $\prod_{k=1}^m P_{s_{k,k},M_2} \leq g(\prod_{k=1}^m P_{s_{k,k},M_1})g^{-1}$. For any $s_{u,u,M_2} \neq \emptyset$, by (6.7.3), recall that $P_{s_{u,u},M_2} = Q_{u,u,M_2,k_1} \times \cdots \times Q_{u,u,M_2,k_s}$ for unique $s \in \mathbb{N}$ and pairwise distinct non-negative integers $k_1, \dots, k_s$. We may assume that $Q_{u,u,M_2,k_i} \neq 1$ for all $1 \leq i \leq s$. For any

given Q_{u,u,M_2,k_i} , let ρ_{k_i} be a 2^{k_i} -cycle of Q_{u,u,M_2,k_i} . By Steps 1 and 2, we have $\rho_{k_i} = gzg^{-1}$ and $e_{M_2} = g\rho e_{M_1}g^{-1}$ for some $z, \rho \in R_{M_1}$. As ρ_{k_i} is a 2^{k_i} -cycle, there exist some $1 \leq a, b \leq m$ such that $\text{Supp}(z) \subseteq s_{a,b,M_1}$. As $\rho \in R_{M_1}$ and $e_{M_1}s_{a,b,M_1} = s_{b,a,M_1}$, notice that $\rho s_{b,a,M_1} = s_{b,a,M_1}$ and $\rho e_{M_1}s_{a,b,M_1} = s_{b,a,M_1}$. If $a \neq b$, as

$$\rho_{k_i} = e_{M_2}\rho_{k_i}e_{M_2} = g\rho e_{M_1}g^{-1}gzg^{-1}ge_{M_1}\rho^{-1}g^{-1} = g\rho e_{M_1}ze_{M_1}\rho^{-1}g^{-1},$$

we thus have

$$\text{Supp}(\rho_{k_i}) = \text{Supp}(gzg^{-1}) = \text{Supp}(g\rho e_{M_1}ze_{M_1}\rho^{-1}g^{-1}) \subseteq gs_{a,b,M_1} \cap gs_{b,a,M_1} = \emptyset.$$

Therefore, we have $\rho_{k_i} = 1$, which is a contradiction as $Q_{u,u,M_2,k_i} \neq 1$. So we obtain that $\text{Supp}(\rho_{k_i}) = \text{Supp}(gzg^{-1}) \subseteq gs_{a,a,M_1}$. For any $\tau \in Q_{u,u,M_2,k_i}$, we have

$$\text{Supp}(\tau) \subseteq \text{Supp}(\rho_{k_i}) \subseteq gs_{a,a,M_1}. \quad (6.7.6)$$

By Step 1 and (6.7.6), we naturally view τ as an element of $gP_{s_{a,a,M_1}}g^{-1}$. We thus have

$$\tau \in gP_{s_{a,a,M_1}}g^{-1} \leq g\left(\prod_{k=1}^m P_{s_{k,k,M_1}}\right)g^{-1}.$$

As we choose $P_{s_{u,u,M_2}}$ and Q_{u,u,M_2,k_i} arbitrarily, the desired inequality is shown.

Step 4: Obtain the conclusion.

By Step 1, (6.7.1), (6.7.3), the definitions of $T(M_1)$ and $T(M_2)$, it is not difficult to see that $T(M_1) = T(M_2)$. By Step 3, (6.7.1), (6.7.3), the definitions of $D(M_1)$ and $D(M_2)$, also note that $D(M_1) = D(M_2)$. As $M_1 = M_1^T$ and $M_2 = M_2^T$, recall that we have $T(M_i) = D(M_i) + 2U(M_i)$ for any $i \in \{1, 2\}$, where the addition and the scalar multiplication are componentwise. So we also obtain that $U(M_1) = U(M_2)$. The proof is now complete. $\square$

We are ready to deduce the main result of this section.

Theorem 6.7.8. *Let $1 < m \in \mathbb{N}$ and $M_1, M_2 \in \mathcal{M}_{m,n}$. Then P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} if and only if precisely one of the following cases occurs:*

- (i) $P_{M_1}, P_{M_2} \in \mathcal{Y}_n$ and $T'(M_1) = T'(M_2)$;
- (ii) $P_{M_1}, P_{M_2} \notin \mathcal{Y}_n$, $D(M_1) = D(M_2)$ and $U(M_1) = U(M_2)$.

Proof. If P_{M_1} is $\mathfrak{S}_n$ -conjugate to P_{M_2} , it is clear that we have either $P_{M_1}, P_{M_2} \in \mathcal{Y}_n$ or $P_{M_1}, P_{M_2} \notin \mathcal{Y}_n$. When $P_{M_1}, P_{M_2} \notin \mathcal{Y}_n$, by (6.7.1), we deduce that $p = 2$, $M_1 = M_1^T$ and $M_2 = M_2^T$. Therefore, the theorem follows by Lemmas 6.7.5, 6.7.6 and 6.7.7. $\square$

We close this section by providing an example of Theorem 6.7.8.

Example 6.7.9. Let us consider Example 6.7.2 again. According to Lemma 6.7.4, note that $P_{M_1}, P_{M_2} \notin \mathcal{Y}_{10}$. Since $D(M_1) \neq D(M_2)$ and $U(M_1) \neq U(M_2)$, Theorem 6.7.8 tells us that P_{M_1} is not $\mathfrak{S}_{10}$ -conjugate to P_{M_2} .

6.8 A Similar Result of Theorem 6.1.7

Let $\lambda \vdash n$. In this section, we prove a similar result of Theorem 6.1.7 for S^2M^λ and Λ^2M^λ . For convenience, we shall mainly work on Λ^2M^λ and mimic the presented proofs to deduce the corresponding results of S^2M^λ . For our purpose, recall the definitions in Notations 6.4.1 (v) and 6.7.1. Let $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda$ be a fixed complete set of representatives of $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ -double cosets in $\mathfrak{S}_n$. We also require that $1 \in \mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda$.

We first introduce the required notation. Let $m \in \mathbb{N}$ and $M = (a_{ij}) \in \mathcal{M}_{m,n}$. We set $r(M) = (\sum_{i=1}^m a_{1i}, \dots, \sum_{i=1}^m a_{mi})$. Similarly, we put $c(M) = (\sum_{i=1}^m a_{i1}, \dots, \sum_{i=1}^m a_{im})$. Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $\lambda \neq (n)$. Define $\mathcal{M}_\lambda = \{N \in \mathcal{M}_{\ell,n} : \lambda = r(N) = c(N)\}$ and $\mathbf{n}_i^\lambda = \{(\sum_{j=1}^{i-1} \lambda_j) + 1, \dots, \sum_{j=1}^i \lambda_j\}$ for all $1 \leq i \leq \ell$. We shall repeatedly use the following theorem. It is in fact a special case of [37, 1.3.8 and 1.3.10].

Theorem 6.8.1. [37, 1.3.8 and 1.3.10] Assume that $\lambda \vdash n$ and $\lambda \neq (n)$. If $g \in \mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda$ and $h \in \mathfrak{S}_\lambda g \mathfrak{S}_\lambda$, then we have $|\mathbf{n}_i^\lambda \cap h \mathbf{n}_j^\lambda| = |\mathbf{n}_i^\lambda \cap g \mathbf{n}_j^\lambda|$ for all $1 \leq i, j \leq \ell(\lambda)$. Moreover, the map sending g to $(\mathbf{n}_i^\lambda \cap g \mathbf{n}_j^\lambda)$ is a bijection between $(\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ and $\mathcal{M}_\lambda$.

Let $\lambda \vdash n$, $\lambda \neq (n)$ and $M = (a_{ij}) \in \mathcal{M}_\lambda$. By Theorem 6.8.1, there exists a unique $g \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ such that $|\mathbf{n}_i^\lambda \cap g \mathbf{n}_j^\lambda| = a_{ij}$ for all $1 \leq i, j \leq \ell(\lambda)$. In particular, we have $\{t^\lambda\} \neq g\{t^\lambda\}$. Set $t_{M,s} = \{t^\lambda\}(g\{t^\lambda\})$ and $t_{M,e} = \{t^\lambda\} \wedge g\{t^\lambda\}$. According to Notation 6.4.1 (v), note that $t_{M,s} \in \mathcal{T}(\lambda)_2^s$ and $t_{M,e} \in \mathcal{T}(\lambda)_2^e$. We present some lemmas as preliminary results.

Lemma 6.8.2. Let $\lambda \vdash n$, $\lambda \neq (n)$ and $M \in \mathcal{M}_\lambda$. Then $\mathcal{O}^e(t_{M,e}) = \mathcal{O}^e(t_{M^T,e})$. In particular, $W(t_{M,e}) = W(t_{M^T,e})$.

Proof. We set $M = (a_{ij})$ and $M^T = (b_{ij})$. According to Theorem 6.8.1, there exists a unique $g \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ such that $t_{M,e} = \{t^\lambda\} \wedge g\{t^\lambda\}$ and $|\mathbf{n}_i^\lambda \cap g \mathbf{n}_j^\lambda| = a_{ij}$ for all $1 \leq i, j \leq \ell(\lambda)$. Note that $b_{ij} = |\mathbf{n}_j^\lambda \cap g \mathbf{n}_i^\lambda| = |g^{-1} \mathbf{n}_j^\lambda \cap \mathbf{n}_i^\lambda| = |\mathbf{n}_i^\lambda \cap g^{-1} \mathbf{n}_j^\lambda|$ for all $1 \leq i, j \leq \ell(\lambda)$. By Theorem 6.8.1, there exists a unique $h \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ such that $g^{-1} \in \mathfrak{S}_\lambda h \mathfrak{S}_\lambda$ and $t_{M^T,e} = \{t^\lambda\} \wedge h\{t^\lambda\}$. Therefore, $g^{-1} = u h v$ for some $u, v \in \mathfrak{S}_\lambda$. Let $x = (gu)^{-1} \in \mathfrak{S}_n$ and notice that $u\{t^\lambda\} = v\{t^\lambda\} = \{t^\lambda\}$. We thus have

$$\begin{aligned} x t_{M,e} &= (u^{-1} g^{-1})(\{t^\lambda\} \wedge g\{t^\lambda\}) = u^{-1}(g^{-1}\{t^\lambda\} \wedge \{t^\lambda\}) \\ &= -u^{-1}(\{t^\lambda\} \wedge g^{-1}\{t^\lambda\}) \\ &= -u^{-1}(\{t^\lambda\} \wedge u h v \{t^\lambda\}) \\ &= -(\{t^\lambda\} \wedge h\{t^\lambda\}) = -t_{M^T,e}, \end{aligned}$$

which shows that $\mathcal{O}^e(t_{M,e}) = \mathcal{O}^e(t_{M^T,e})$ by the definitions of $\mathcal{O}^e(t_{M,e})$ and $\mathcal{O}^e(t_{M^T,e})$. The second equality also follows by the definitions of $W(t_{M,e})$ and $W(t_{M^T,e})$. $\square$

Let $\lambda \vdash n$ and $\{t\} \in \mathcal{T}(\lambda)$. Recall that $R(\{t\}) = \{g \in \mathfrak{S}_n : g\{t\} = \{t\}\}$.

Lemma 6.8.3. Let $\lambda \vdash n$, $\lambda \neq (n)$ and $M_1, M_2 \in \mathcal{M}_\lambda$. If we have $M_1 \neq M_2$, then we have $\mathcal{O}^e(t_{M_1,e}) = \mathcal{O}^e(t_{M_2,e})$ if and only if $M_1^T = M_2$. In particular, if we have $M_1 \neq M_2$, then we have $W(t_{M_1,e}) = W(t_{M_2,e})$ if and only if $M_1^T = M_2$.

Proof. By Lemma 6.8.2, it suffices to show that the equality $\mathcal{O}^e(t_{M_1,e}) = \mathcal{O}^e(t_{M_2,e})$ implies that $M_1^T = M_2$. Let $t_{M_1,e} = \{t^\lambda\} \wedge g_1\{t^\lambda\}$ and $t_{M_2,e} = \{t^\lambda\} \wedge g_2\{t^\lambda\}$, where $g_1 \neq g_2$ and $g_1, g_2 \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$. If $\mathcal{O}^e(t_{M_1,e}) = \mathcal{O}^e(t_{M_2,e})$, we claim that $g_1^{-1} \in \mathfrak{S}_\lambda g_2 \mathfrak{S}_\lambda$. As $g_1 \neq g_2$ and $\mathcal{O}^e(t_{M_1,e}) = \mathcal{O}^e(t_{M_2,e})$, by the definitions of $\mathcal{O}^e(t_{M_1,e})$ and $\mathcal{O}^e(t_{M_2,e})$, we have $g t_{M_2,e} = -t_{M_1,e}$ for some $g \in \mathfrak{S}_n$. So $g\{t^\lambda\} = g_1\{t^\lambda\}$ and $g g_2\{t^\lambda\} = \{t^\lambda\}$. Therefore, $g_1^{-1} g = u$ and $g g_2 = v$, where $u, v \in R(\{t^\lambda\}) = \mathfrak{S}_\lambda$. We thus have

$$g_1^{-1} = u g^{-1} v v^{-1} = u g^{-1} g g_2 v^{-1} = u g_2 v^{-1} \in \mathfrak{S}_\lambda g_2 \mathfrak{S}_\lambda.$$

The claim is shown. Let $M_1 = (a_{ij})$ and $M_2 = (b_{ij})$. By this claim and Theorem 6.8.1, for any $1 \leq i, j \leq \ell(\lambda)$, $b_{ij} = |\mathbf{n}_i^\lambda \cap g_1^{-1} \mathbf{n}_j^\lambda| = |g_1 \mathbf{n}_i^\lambda \cap \mathbf{n}_j^\lambda| = |\mathbf{n}_j^\lambda \cap g_1 \mathbf{n}_i^\lambda| = a_{ji}$. So $M_1^T = M_2$ and the first assertion is shown. By Notation 6.4.1 (v), note that $W(t_{M_1,e}) = W(t_{M_2,e})$ if and only if $\mathcal{O}^e(t_{M_1,e}) = \mathcal{O}^e(t_{M_2,e})$. The second assertion follows by the first one. $\square$

Lemma 6.8.4. Let $\lambda \vdash n$, $\lambda \neq (n)$ and $M \in \mathcal{M}_\lambda$. Assume that $g \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ and $t_{M,e} = \{t^\lambda\} \wedge g\{t^\lambda\}$. Then $L(t_{M,e}) = (R(\{t^\lambda\}) \cap R(g\{t^\lambda\})) \rtimes \langle z \rangle$ for some involution $z \in \mathfrak{S}_n$ if and only if $M = M^T$.

Proof. Let $M = (a_{ij})$ and $R = R(\{t^\lambda\}) \cap R(g\{t^\lambda\})$. Note that

$$R = \mathfrak{S}_\lambda \cap g \mathfrak{S}_\lambda g^{-1} = \prod_{1 \leq i, j \leq \ell(\lambda)} \mathfrak{S}_{\mathbf{n}_i^\lambda \cap g \mathbf{n}_j^\lambda}. \quad (6.8.1)$$

Moreover, for any $1 \leq i \leq \ell(\lambda)$, the i th row of $\{t^\lambda\}$ contains exactly all the members of $\mathbf{n}_i^\lambda$. Similarly, the i th row of $g\{t^\lambda\}$ contains exactly all the members of $g \mathbf{n}_i^\lambda$. If $L(t_{M,e}) = R \rtimes \langle z \rangle$ and $z^2 = 1$, as z normalizes R , by (6.8.1), given $1 \leq i, j \leq \ell(\lambda)$, note that $z(\mathbf{n}_i^\lambda \cap g \mathbf{n}_j^\lambda) = \mathbf{n}_x^\lambda \cap g \mathbf{n}_y^\lambda$

for some $1 \leq x, y \leq \ell(\lambda)$. Moreover, by Lemmas 6.5.1 and 6.5.2, also note that z swaps $\{t^\lambda\}$ and $g\{t^\lambda\}$. For any $1 \leq i, j \leq \ell(\lambda)$, $\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda$ lies in the i th row of $\{t^\lambda\}$. As $z\{t^\lambda\} = g\{t^\lambda\}$, there exists some $1 \leq k \leq \ell(\lambda)$ such that $z(\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda) = \mathbf{n}_k^\lambda \cap g\mathbf{n}_i^\lambda$. Since $z^2 = 1$, we also have $z(\mathbf{n}_k^\lambda \cap g\mathbf{n}_i^\lambda) = \mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda$. This forces that $j = k$ as $\mathbf{n}_k^\lambda \cap g\mathbf{n}_i^\lambda$ is in the k th row of $\{t^\lambda\}$. Therefore, by the definition of $t_{M,e}$ and Theorem 6.8.1, we have $a_{ij} = |\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda| = |\mathbf{n}_j^\lambda \cap g\mathbf{n}_i^\lambda| = a_{ji}$ for any $1 \leq i, j \leq \ell(\lambda)$. We thus have $M = M^T$.

Conversely, we claim that $h\{t^\lambda\} = g\{t^\lambda\}$ and $hg\{t^\lambda\} = \{t^\lambda\}$ for some $h \in \mathfrak{S}_n$. As $M = M^T$, by the definition of $t_{M,e}$ and Theorem 6.8.1, for any $1 \leq i, j \leq \ell(\lambda)$, we have

$$|\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda| = a_{ij} = a_{ji} = |\mathbf{n}_j^\lambda \cap g\mathbf{n}_i^\lambda| = |g^{-1}\mathbf{n}_j^\lambda \cap \mathbf{n}_i^\lambda| = |\mathbf{n}_i^\lambda \cap g^{-1}\mathbf{n}_j^\lambda|. \quad (6.8.2)$$

By (6.8.2) and Theorem 6.8.1, note that $g^{-1} \in \mathfrak{S}_\lambda g \mathfrak{S}_\lambda$. So $g^{-1} = ugv$ for some $u, v \in \mathfrak{S}_\lambda$. Set $h = (gu)^{-1}$. As $R(\{t^\lambda\}) = \mathfrak{S}_\lambda$, we have

$$h\{t^\lambda\} = u^{-1}g^{-1}\{t^\lambda\} = u^{-1}ugv\{t^\lambda\} = g\{t^\lambda\} \text{ and } hg\{t^\lambda\} = u^{-1}g^{-1}g\{t^\lambda\} = \{t^\lambda\}.$$

Therefore, h swaps $\{t^\lambda\}$ and $g\{t^\lambda\}$ and the claim is shown. By this claim and Lemma 6.5.2, we deduce that $L(t_{M,e}) = R \rtimes \langle z \rangle$ for some involution $z \in \mathfrak{S}_n$. $\square$

Lemma 6.8.5. *Let $\lambda \vdash n$, $\lambda \neq (n)$ and $M \in \mathcal{M}_\lambda$.*

- (i) *Let $N \in \mathcal{M}_\lambda$. Then $V(t_{M,s}) = V(t_{N,s})$ if and only if $N = M$ or $N = M^T$.*
- (ii) *Let $g \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ and $t_{M,s} = \{t^\lambda\}(g\{t^\lambda\})$. Then we have $M = M^T$ if and only if $K(t_{M,s}) = (R(\{t^\lambda\}) \cap R(g\{t^\lambda\})) \rtimes \langle z \rangle$ for some involution $z \in \mathfrak{S}_n$.*

Proof. By mimicking the proof of Lemma 6.8.2, we can prove that $\mathcal{O}^s(t_{M,s}) = \mathcal{O}^s(t_{M^T,s})$ and $V(t_{M,s}) = V(t_{M^T,s})$. We now can show (i) and (ii) by mimicking the proofs given in Lemmas 6.8.3 and 6.8.4 respectively. $\square$

Let $\lambda \vdash n$ and $\lambda \neq (n)$. For our following results, let $\mathcal{S}_\lambda = \{M \in \mathcal{M}_\lambda : M = M^T\}$. If $\mathcal{S}_\lambda \neq \mathcal{M}_\lambda$, note that $2 \mid |\mathcal{M}_\lambda| - |\mathcal{S}_\lambda|$ and let $2x_\lambda + |\mathcal{S}_\lambda| = |\mathcal{M}_\lambda|$. Pick $M_i \in \mathcal{M}_\lambda \setminus \mathcal{S}_\lambda$ for each $1 \leq i \leq x_\lambda$ and require that $\{M_i, M_i^T\} \cap \{M_j, M_j^T\} = \emptyset$ for all $1 \leq i \neq j \leq x_\lambda$. Set $\mathcal{N}_\lambda = \{M_i : 1 \leq i \leq x_\lambda\}$. If $\mathcal{S}_\lambda = \mathcal{M}_\lambda$, put $\mathcal{N}_\lambda = \emptyset$. We also set $\mathcal{S}_{(n)} = \mathcal{N}_{(n)} = \emptyset$.

We are now ready to deduce two propositions.

Proposition 6.8.6. *Let $\lambda \vdash n$ and $\lambda \neq (n)$.*

- (i) *We have $\Lambda^2 M^\lambda = \bigoplus_{M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda} W(t_{M,e})$.*
- (ii) *If $M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$, then P_M is $\mathfrak{S}_n$ -conjugate to a Sylow p -subgroup of $L(t_{M,e})$.*

Proof. As $\lambda \neq (n)$, by the definitions of $\mathcal{S}_\lambda$ and $\mathcal{N}_\lambda$, note that $\mathcal{S}_\lambda \cup \mathcal{N}_\lambda \neq \emptyset$. Recall that $\mathcal{T}(\lambda)_2^e$ is an $\mathbb{F}$ -basis of $\Lambda^2 M^\lambda$. Set $U = \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$. For any distinct $M_1, M_2 \in U$, by Lemma 6.8.3 and the definitions of $\mathcal{S}_\lambda, \mathcal{N}_\lambda, \mathcal{O}^e(t_{M_1,e})$ and $\mathcal{O}^e(t_{M_2,e})$, note that $\mathcal{O}^e(t_{M_1,e}) \cap \mathcal{O}^e(t_{M_2,e}) = \emptyset$. Therefore, as $W(t_{M,e}) = \langle \mathcal{O}^e(t_{M,e}) \rangle_{\mathbb{F}}$ for all $M \in U$, to get the desired equality, it suffices to show that $\mathcal{T}(\lambda)_2^e = \bigcup_{M \in U} \mathcal{O}^e(t_{M,e})$. It is clear that $\bigcup_{M \in U} \mathcal{O}^e(t_{M,e}) \subseteq \mathcal{T}(\lambda)_2^e$. For any $t \in \mathcal{T}(\lambda)_2^e$, there exists some $g \in \mathfrak{S}_n$ such that $x = \{t^\lambda\} \wedge g\{t^\lambda\} \in \mathcal{T}(\lambda)_2^e$ and $t \in \mathcal{O}^e(x)$. Note that $g = uhv$, where $h \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ and $u, v \in \mathfrak{S}_\lambda$. By Theorem 6.8.1, there exists $N \in \mathcal{M}_\lambda$ such that $t_{N,e} = \{t^\lambda\} \wedge h\{t^\lambda\}$. As $u, v \in \mathfrak{S}_\lambda = R(\{t^\lambda\})$, notice that $ut_{N,e} = x$, which implies that $\mathcal{O}^e(x) = \mathcal{O}^e(t_{N,e})$ by the definitions of $\mathcal{O}^e(x)$ and $\mathcal{O}^e(t_{N,e})$. Therefore, we have

$$t \in \mathcal{O}^e(x) = \mathcal{O}^e(t_{N,e}) \subseteq \bigcup_{M \in U} \mathcal{O}^e(t_{M,e})$$

by Lemma 6.8.3 and the definitions of $\mathcal{S}_\lambda$ and $\mathcal{N}_\lambda$. So $\mathcal{T}(\lambda)_2^e \subseteq \bigcup_{M \in U} \mathcal{O}^e(t_{M,e})$ and the desired equality thus follows. (i) is shown.

Let $M = (a_{ij}) \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$. By Theorem 6.8.1, there exists $g \in (\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ such that $a_{ij} = |\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda|$ for any $1 \leq i, j \leq \ell(\lambda)$. So $t_{M,e} = \{t^\lambda\} \wedge g\{t^\lambda\}$. If $p > 2$ or $M \neq M^T$, by Lemmas 6.8.4, 6.5.1 and 6.5.2, note that a Sylow p -subgroup P of $L(t_{M,e})$ is also a Sylow p -subgroup of $R(\{t^\lambda\}) \cap R(g\{t^\lambda\})$. By (6.8.1) and (6.7.1), P_M is $\mathfrak{S}_n$ -conjugate to P .

If $p = 2$ and $M = M^T$, let z be an involution of $\mathfrak{S}_n$ that swaps $\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda$ and $\mathbf{n}_j^\lambda \cap g\mathbf{n}_i^\lambda$ for all $1 \leq i < j \leq \ell(\lambda)$ and fixes the remaining numbers of $[1, n]$. Note that

$$z\mathbf{n}_i^\lambda = z(\mathbf{n}_i^\lambda \cap g[1, n]) = z\left(\bigcup_{j=1}^{\ell(\lambda)} (\mathbf{n}_i^\lambda \cap g\mathbf{n}_j^\lambda)\right) = \bigcup_{j=1}^{\ell(\lambda)} (\mathbf{n}_j^\lambda \cap g\mathbf{n}_i^\lambda) = [1, n] \cap g\mathbf{n}_i^\lambda = g\mathbf{n}_i^\lambda. \quad (6.8.3)$$

As $z^2 = 1$, by (6.8.3), we have $z\{t^\lambda\} = g\{t^\lambda\}$ and $zg\{t^\lambda\} = \{t^\lambda\}$. By Lemma 6.5.2, we thus have $L(t_{M,e}) = (R(\{t^\lambda\}) \cap R(g\{t^\lambda\})) \rtimes \langle z \rangle$. Therefore, by (6.8.1), (6.7.1) and the definition of z , note that P_M is $\mathfrak{S}_n$ -conjugate to a Sylow 2-subgroup of $L(t_{M,e})$. (ii) is also proved. $\square$

Proposition 6.8.7. *Let $\lambda \vdash n$.*

- (i) *We have $S^2M^\lambda = V(b_\lambda) \oplus \bigoplus_{M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda} V(t_{M,s})$, where $b_\lambda = \{t^\lambda\}\{t^\lambda\}$ and $K(b_\lambda) = \mathfrak{S}_\lambda$.*
- (ii) *If $M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$, then P_M is $\mathfrak{S}_n$ -conjugate to a Sylow p -subgroup of $K(t_{M,s})$.*

Proof. By the definitions of $\mathcal{S}_\lambda$ and $\mathcal{N}_\lambda$, note that $\mathcal{S}_\lambda \cup \mathcal{N}_\lambda = \emptyset$ if and only if $\lambda = (n)$. As $S^2M^{(n)} \cong \mathbb{F}$, the proposition holds for $S^2M^{(n)}$. We thus assume that $\lambda \neq (n)$. Therefore, $\mathcal{S}_\lambda \cup \mathcal{N}_\lambda \neq \emptyset$. Observe that $\mathcal{O}^s(b_\lambda) \cap \mathcal{O}^s(t_{M,s}) = \emptyset$ for any $M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$ and $K(b_\lambda) = \mathfrak{S}_\lambda$. By Lemma 6.8.5 (i) and mimicking the proof of Proposition 6.8.6 (i), (i) is shown. By Lemma 6.8.5 (ii) and mimicking the proof of Proposition 6.8.6 (ii), (ii) is also proved. We are done. $\square$

Lemma 6.8.8. *Let $\lambda \vdash n$ and $\lambda \neq (n)$. If $M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$, then*

$$\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, W(t_{M,e})) = \begin{cases} 0, & \text{if } p > 2 \text{ and } M \in \mathcal{S}_\lambda, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. As $M \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda \subseteq \mathcal{M}_\lambda$, by Theorem 6.8.1 and the definition of $t_{M,e}$, there exists $g \in (\mathfrak{S}_\lambda \setminus \mathfrak{S}_n / \mathfrak{S}_\lambda) \setminus \{1\}$ such that $t_{M,e} = \{t^\lambda\} \wedge g\{t^\lambda\}$. Let $N = \langle \{t_{M,e}\} \rangle_{\mathbb{F}}$ and note that N is a one-dimensional $\mathbb{F}L(t_{M,e})$ -module. As $W(t_{M,e}) = \langle \mathcal{O}^e(t_{M,e}) \rangle_{\mathbb{F}}$, we also notice that $W(t_{M,e}) \cong N^{\uparrow \mathfrak{S}_n}$.

If $p > 2$ and $M \in \mathcal{S}_\lambda$, by Lemma 6.8.4, there exists some $z \in \mathfrak{S}_n$ such that $z \in L(t_{M,e})$, $z^2 = 1$, $z\{t^\lambda\} = g\{t^\lambda\}$ and $zg\{t^\lambda\} = \{t^\lambda\}$. We thus have $zt_{M,e} = -t_{M,e}$ and $N \not\cong \mathbb{F}$ as $\mathbb{F}L(t_{M,e})$ -modules. By the Frobenius reciprocity, we get $\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, W(t_{M,e})) = 0$.

If $M \in \mathcal{N}_\lambda$, note that $M \neq M^T$. By Lemmas 6.8.4, 6.5.1 and 6.5.2, we also note that $L(t_{M,e}) = R(\{t^\lambda\}) \cap R(g\{t^\lambda\})$. Therefore, we have $N \cong \mathbb{F}$ as $\mathbb{F}L(t_{M,e})$ -modules. By the Frobenius reciprocity, notice that $\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, W(t_{M,e})) = 1$. If $p = 2$, then it is clear that $W(t_{M,e}) \cong (\mathbb{F}_{L(t_{M,e})})^{\uparrow \mathfrak{S}_n}$. We thus have $\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, W(t_{M,e})) = 1$ by the Frobenius reciprocity. The lemma follows by the three cases. $\square$

Corollary 6.8.9. *Let $\lambda \vdash n$.*

- (i) *We have $\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, S^2M^\lambda) = 1 + |\mathcal{S}_\lambda| + |\mathcal{N}_\lambda|$.*
- (ii) *If $\lambda \neq (n)$, then we have*

$$\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, \Lambda^2 M^\lambda) = \begin{cases} |\mathcal{N}_\lambda|, & \text{if } p > 2, \\ |\mathcal{S}_\lambda| + |\mathcal{N}_\lambda|, & \text{if } p = 2. \end{cases}$$

Proof. By the definitions of $\mathcal{S}_\lambda$ and $\mathcal{N}_\lambda$, we can observe that $|\mathcal{S}_\lambda \cup \mathcal{N}_\lambda| = |\mathcal{S}_\lambda| + |\mathcal{N}_\lambda|$. For (i), for any $t \in \mathcal{T}(\lambda)_2^s$, recall that $V(t) \cong (\mathbb{F}_{K(t)})^{\uparrow \mathfrak{S}_n}$. Therefore, (i) follows by Proposition 6.8.7 (i) and the Frobenius reciprocity. (ii) is also shown by Proposition 6.8.6 (i) and Lemma 6.8.8. $\square$

Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $\mathcal{R}_\lambda = \mathcal{S}_\lambda \cup \mathcal{N}_\lambda \cup \{D_\lambda\}$, where $D_\lambda = \text{diag}(\lambda_1, \dots, \lambda_\ell)$. Let P_{D_λ} be a Sylow p -subgroup of $\mathfrak{S}_\lambda$ and $T'(D_\lambda) = T(D_\lambda)$. We shall define an equivalence relation $\sim$ for $\mathcal{R}_\lambda$. For any $M, N \in \mathcal{R}_\lambda$, write $M \sim N$ if and only if P_M is $\mathfrak{S}_n$ -conjugate to P_N . Let $[M]_{\mathcal{R}_\lambda}$ be the equivalence class of M with respect to $\sim$. Set $\mathcal{X}_\lambda = \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$ if $p = 2$. Otherwise, put $\mathcal{X}_\lambda = \mathcal{N}_\lambda$. If $\mathcal{X}_\lambda \neq \emptyset$, as $\mathcal{X}_\lambda \subseteq \mathcal{R}_\lambda$, the restriction of $\sim$ is also an equivalence relation for $\mathcal{X}_\lambda$. For any $R \in \mathcal{X}_\lambda$, let $[R]_{\mathcal{X}_\lambda} = \{X \in \mathcal{X}_\lambda : X \sim R\}$. Note that $[R]_{\mathcal{X}_\lambda}$ is the equivalence class

of R with respect to the restriction of $\sim$. By (6.7.1), Lemma 6.7.4 and Theorem 6.7.8, it is easy to compute both $|[M]_{\mathcal{R}_\lambda}|$ and $|[R]_{\mathcal{X}_\lambda}|$. The main result of this section is as follows. It is obvious that this result is similar to Theorem 6.1.7.

Theorem 6.8.10. *Let $\lambda \vdash n$ and P be a p -subgroup of $\mathfrak{S}_n$.*

- (i) *We have $Sc_{\mathfrak{S}_n}(P) \mid S^2M^\lambda$ if and only if P is $\mathfrak{S}_n$ -conjugate to P_M for some $M \in \mathcal{R}_\lambda$. Moreover, for any $M \in \mathcal{R}_\lambda$, $[S^2M^\lambda : Sc_{\mathfrak{S}_n}(P_M)] = |[M]_{\mathcal{R}_\lambda}|$.*
- (ii) *If $\lambda \neq (n)$ and $\mathcal{X}_\lambda \neq \emptyset$, $Sc_{\mathfrak{S}_n}(P) \mid \Lambda^2M^\lambda$ if and only if P is $\mathfrak{S}_n$ -conjugate to P_M for some $M \in \mathcal{X}_\lambda$. Moreover, for any $M \in \mathcal{X}_\lambda$, $[\Lambda^2M^\lambda : Sc_{\mathfrak{S}_n}(P_M)] = |[M]_{\mathcal{X}_\lambda}|$.*
- (iii) *If $\lambda \neq (n)$, then $\mathcal{X}_\lambda = \emptyset$ if and only if $p > 2$ and $\ell(\lambda) = 2$. If $p > 2$ and $\ell(\lambda) = 2$, then $[\Lambda^2M^\lambda : Sc_{\mathfrak{S}_n}(Q)] = 0$ for any p -subgroup Q of $\mathfrak{S}_n$.*

Proof. By Proposition 6.8.7 (i) and the Krull-Schmidt Theorem, we have $Sc_{\mathfrak{S}_n}(P) \mid S^2M^\lambda$ if and only if $Sc_{\mathfrak{S}_n}(P) \mid V(b_\lambda)$ or $Sc_{\mathfrak{S}_n}(P) \mid V(t_{N,s})$ for some $N \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$. As $K(b_\lambda) = \mathfrak{S}_\lambda$ by Proposition 6.8.7 (i) and $V(b_\lambda) \cong (\mathbb{F}_{K(b_\lambda)})^{\uparrow \mathfrak{S}_n}$, note that $Sc_{\mathfrak{S}_n}(P) \mid V(b_\lambda)$ if and only if P is $\mathfrak{S}_n$ -conjugate to P_{D_λ} . For any $U \in \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$, as $V(t_{U,s}) \cong (\mathbb{F}_{K(t_{U,s})})^{\uparrow \mathfrak{S}_n}$, by Proposition 6.8.7 (ii), also note that $Sc_{\mathfrak{S}_n}(P) \mid V(t_{U,s})$ if and only if P is $\mathfrak{S}_n$ -conjugate to P_U . As we have $\mathcal{R}_\lambda = \mathcal{S}_\lambda \cup \mathcal{N}_\lambda \cup \{D_\lambda\}$, the first statement of (i) is shown. The second statement of (i) follows by Proposition 6.8.7 (i), (ii) and the definition of $[M]_{\mathcal{R}_\lambda}$.

For (ii), we first list the following three facts: (a) If $p > 2$ and $U \in \mathcal{S}_\lambda$, by Lemma 6.8.8, observe that $W(t_{U,e})$ does not have a trivial $\mathbb{F}\mathfrak{S}_n$ -submodule. Since $\text{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, Sc_{\mathfrak{S}_n}(P)) \neq 0$, we thus have $Sc_{\mathfrak{S}_n}(P) \nmid W(t_{U,e})$; (b) If $U \in \mathcal{N}_\lambda$, by Lemma 6.8.8 again, note that $W(t_{U,e}) \cong (\mathbb{F}_{L(t_{U,e})})^{\uparrow \mathfrak{S}_n}$; (c) If $p = 2$ and $U \in \mathcal{X}_\lambda$, note that $W(t_{U,e}) \cong (\mathbb{F}_{L(t_{U,e})})^{\uparrow \mathfrak{S}_n}$.

By Proposition 6.8.6 (i), the Krull-Schmidt Theorem and (a), (b), (c), we have $Sc_{\mathfrak{S}_n}(P) \mid \Lambda^2M^\lambda$ if and only if $Sc_{\mathfrak{S}_n}(P) \mid W(t_{N,e})$ for some $N \in \mathcal{X}_\lambda$. For any $U \in \mathcal{X}_\lambda$, (b) and (c) also tell us that $W(t_{U,e}) \cong (\mathbb{F}_{L(t_{U,e})})^{\uparrow \mathfrak{S}_n}$. Therefore, by Proposition 6.8.6 (ii), note that $Sc_{\mathfrak{S}_n}(P) \mid W(t_{U,e})$ if and only if P is $\mathfrak{S}_n$ -conjugate to P_U . The first statement of (ii) is shown. The second statement of (ii) follows by Proposition 6.8.6 (i), (ii), the first statement of (ii) and the definition of $[M]_{\mathcal{X}_\lambda}$.

For (iii), if $p > 2$ and $\ell(\lambda) = 2$, by the definitions of $\mathcal{N}_\lambda$, $\mathcal{M}_\lambda$ and $\mathcal{S}_\lambda$, we note that $\mathcal{X}_\lambda = \mathcal{N}_\lambda \subseteq \mathcal{M}_\lambda = \mathcal{S}_\lambda$ and $\mathcal{S}_\lambda \cap \mathcal{N}_\lambda = \emptyset$. So $\mathcal{X}_\lambda = \emptyset$. Conversely, for the case $p = 2$, we have $\mathcal{X}_\lambda = \mathcal{S}_\lambda \cup \mathcal{N}_\lambda$. As $\lambda \neq (n)$, Proposition 6.8.6 (i) forces that $\mathcal{X}_\lambda \neq \emptyset$. For the case $p > 2$, we have $\mathcal{X}_\lambda = \mathcal{N}_\lambda$. By the definitions of $\mathcal{M}_\lambda$, $\mathcal{N}_\lambda$ and $\mathcal{S}_\lambda$, the condition $\mathcal{X}_\lambda = \emptyset$ forces that $\mathcal{M}_\lambda = \mathcal{S}_\lambda$. It is not too difficult to see that $\mathcal{M}_\lambda = \mathcal{S}_\lambda$ if and only if $\ell(\lambda) = 2$. The first statement of (iii) follows. If $p > 2$ and $\ell(\lambda) = 2$, by the first statement of (iii), we have $\mathcal{X}_\lambda = \mathcal{N}_\lambda = \emptyset$. By Corollary 6.8.9 (ii), this fact implies that Λ^2M^λ has no trivial $\mathbb{F}\mathfrak{S}_n$ -submodule. As any $\mathbb{F}\mathfrak{S}_n$ -Scott module contains a trivial $\mathbb{F}\mathfrak{S}_n$ -submodule, the second statement of (iii) follows. $\square$

We conclude this thesis by presenting an example that illustrates Theorem 6.8.10.

Example 6.8.11. Let $p = 2$ and $\lambda = (2, 1^2) \vdash 4$. Note that $\mathcal{M}_\lambda \cup \{D_\lambda\}$ contains precisely

$$\begin{aligned} M_1 &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ M_5 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Also note that $\mathcal{S}_\lambda = \{M_1, M_2, M_5, M_6\}$ and let $\mathcal{N}_\lambda = \{M_3\}$. By (6.8.1), Lemma 6.7.4 and Theorem 6.7.8, $\{M_1\}$, $\{M_2, M_5, M_7\}$, $\{M_3\}$, $\{M_6\}$ are all the equivalence classes of $\mathcal{R}_\lambda$ with respect to $\sim$. Similarly, $\{M_1\}$, $\{M_2, M_5\}$, $\{M_3\}$, $\{M_6\}$ are all the equivalence classes of $\mathcal{X}_\lambda$ with respect to $\sim$. Let $Sc_i = Sc_{\mathfrak{S}_4}(P_{M_i})$ for all $1 \leq i \leq 7$ and note that $Sc_2 \cong Sc_5 \cong Sc_7$. Let $m_{i,s} = [S^2M^{(2,1^2)} : Sc_i]$ and $m_{i,e} = [\Lambda^2M^{(2,1^2)} : Sc_i]$. By Theorem 6.8.10, we have

$$m_{2,s} = 3, \quad m_{2,e} = 2 \quad \text{and} \quad m_{1,s} = m_{3,s} = m_{6,s} = m_{1,e} = m_{3,e} = m_{6,e} = 1.$$

Summary

This thesis contributes to studying the representation theory of symmetric groups. In particular, the author of this thesis mainly focuses on some classical modules of symmetric groups in the positive characteristic case. The research theme of this thesis is to understand these classical modules and some related isomorphic invariants. The considered modules mainly include the Young permutation modules, the Young modules, the Specht modules, the simple modules of symmetric groups. The considered isomorphic invariants mainly include the complexities of modules, the vertices of indecomposable modules, the sources of indecomposable modules.

The purpose of Chapter 4 is to understand the complexities of simple modules of symmetric groups. In this chapter, the complexities of some simple modules labelled by two-part partitions are explicitly computed. The applied approach is a combination of the modular branching rules, the generic Jordan types of modules, some computational results developed in Chapter 3.

The purpose of Chapter 5 is to classify the trivial source Specht modules. In this chapter, the trivial source Specht modules labelled by hook partitions are classified. The trivial source Specht modules labelled by two-part partitions are classified in the odd characteristic case. In the even characteristic case, a conjecture of Hudson is verified. The employed approaches mainly include the fixed-point functors of symmetric groups, the generic Jordan types of modules, the Brauer quotients of modules.

The purpose of Chapter 6 is to generalize the known results of Young permutation modules to their symmetric and exterior powers. In this chapter, many known results of Young permutation modules are generalized to their symmetric and exterior powers. The used approach is a mixture of the Brauer quotients of modules and some combinatorial properties of Young permutation modules.

Bibliography

- [1] A. Adem, R. J. Milgram, Cohomology of Finite Groups, **309**, Springer-Verlag, 2004.
- [2] J. L. Alperin, Local Representation Theory, Cambridge Studies in Advanced Mathematics, **11**, Cambridge University Press, 1986.
- [3] J. L. Alperin, L. Evens, Representations, resolutions and Quillen's dimension theorem, J. Pure Appl. Algebra **22** (1981), 1-9.
- [4] G. S. Avrunin, L. L. Scott, Quillen stratification for modules, Invent. Math. **66** (1982), 277-286.
- [5] D. J. Benson, Modular representation theory: New trends and methods, Lecture Notes in Mathematics, **1081**, Springer-Verlag, Berlin, 2006.
- [6] D. J. Benson, Representations of elementary abelian p -groups and vector bundles, Cambridge Tracts in Mathematics, **208**, Cambridge University Press, 2017.
- [7] D. J. Benson, K. J. Lim, Generic Jordan type of the symmetric and exterior powers, J. Algebra Appl. **13** (2014).
- [8] M. Broué, On Scott modules and p -permutation modules: an approach through the Brauer morphism, Proc. Amer. Math. Soc. **93** (1985), 401-408.
- [9] D. W. Burry, Scott modules and lower defect groups, Comm. Algebra **10** (1982), 1855-1872.
- [10] J. F. Carlson, The varieties and the cohomology ring of a module, J. Algebra **85** (1983), 104-143.
- [11] F. R. Cohen, D. J. Hemmer, D. K. Nakano, On the cohomology of Young modules for the symmetric group, Adv. Math. **224** (2010), 1419-1461.
- [12] F. R. Cohen, D. J. Hemmer, D. K. Nakano, The Lie module and its complexity, Bull. Lond. Math. Soc. **48** (2016), 109-114.
- [13] S. Danz, B. Külshammer, R. Zimmermann, On vertices of simple modules for symmetric groups of small degrees, J. Algebra **320** (2008), 680-707.
- [14] S. Danz, On vertices of completely splittable modules for symmetric groups and simple modules labelled by two part partitions, J. Group Theory **12** (2009), 351-385.
- [15] S. Danz, K. J. Lim, Signed Young modules and simple Specht modules, Adv. Math. **307** (2017), 369-416.
- [16] S. Donkin, On Schur algebras and related algebras, II. J. Algebra **111** (1987), 354-364.
- [17] S. Donkin, On tilting modules for algebraic groups, Math. Z. **212** (1993), 39-60.
- [18] S. Donkin, Symmetric and exterior powers, linear source modules and representations of Schur superalgebras, Proc. London Math. Soc. **83** (2001), 647-680.
- [19] K. Erdmann, Young modules for symmetric groups, J. Aust. Math. Soc. **71** (2001), 201-210.

- [20] E. M. Friedlander, J. Pevtsova, A. Suslin, Generic and maximal Jordan types, *Invent. Math.* **168** (2007), 485-522.
- [21] E. Giannelli, A lower bound on the vertices of Specht modules for symmetric groups, *Arch. Math. (Basel)* **103** (2014), 1-9.
- [22] E. Giannelli, K. J. Lim, M. Wildon, Sylow subgroups of symmetric and alternating groups and the vertex of $S^{(kp-p, 1^p)}$ in characteristic p , *J. Algebra* **455** (2016), 358-385.
- [23] E. Giannelli, K. J. Lim, W. O'Donovan, M. Wildon, On signed Young permutation modules and signed p -Kostka numbers, *J. Group Theory* **20** (2017), 637-679.
- [24] C. C. Gill, Tensor products, trivial source modules and related algebras, PhD thesis, University of Oxford, 2010.
- [25] C. C. Gill, Young module multiplicities, decomposition numbers and the indecomposable Young permutation modules, *J. Algebra Appl.* **13** (2014).
- [26] J. Grabmeier, Unzerlegbare Moduln mit trivialer Youngquelle und Darstellungstheorie der Schuralgebra, *Bayreuth. Math. Schr.* **20** (1985), 9-152.
- [27] J. A. Green, On the indecomposable representations of a finite group, *Math. Z.* **70** (1959), 430-445.
- [28] D. J. Hemmer, D. K. Nakano, Support varieties for modules over symmetric groups, *J. Algebra* **254** (2002), 422-440.
- [29] D. J. Hemmer, Fixed-point functors for symmetric groups and Schur algebras, *J. Algebra* **280** (2004), 295-312.
- [30] D. J. Hemmer, A row removal theorem for the Ext^1 quiver of symmetric groups and Schur algebras, *Proc. Amer. Math. Soc.* **133** (2005), 403-414.
- [31] D. J. Hemmer, Symmetric group modules with Specht and dual Specht filtrations. *Comm. Algebra* **35** (2007), 3292-3306.
- [32] D. J. Hemmer, The complexity of certain Specht modules for the symmetric group. *J. Algebraic Combin.* **30** (2009), 421-427.
- [33] T. A. Hudson, Specht Modules of Trivial Source and the Endomorphism Ring of the Lie Module, Thesis (Ph.D.)-State University of New York at Buffalo, 2018.
- [34] A. Henke, On p -Kostka numbers and Young modules, *European J. Combin.* **26** (2005), 923-942.
- [35] G. D. James, The Representation Theory of the Symmetric Groups, *Lecture Notes in Mathematics*, **682**, Springer, Berlin, 1978.
- [36] G. D. James, Trivial source modules for symmetric groups, *Arch. Math. (Basel)* **41** (1983), 294-300.
- [37] G. D. James, A. Kerber, The Representation Theory of the Symmetric Group, *Encyclopedia of Mathematics and its Applications*, **16**, Addison-Wesley Publishing Co., 1981.
- [38] G. D. James, A. Mathas, The irreducible Specht modules in characteristic 2, *Bull. London Math. Soc.* **31** (1999), 457-462.
- [39] Y. Jiang, K. J. Lim, J. L. Wang, On the Brauer constructions and generic Jordan types of Young modules, *arXiv: 1707.04075*.
- [40] Y. Jiang, On the complexities of some simple modules of symmetric groups, *Beitr. Algebra Geom.* **60** (2019), 599-625.
- [41] Y. Jiang, On some trivial source Specht modules, *J. Algebra* **556** (2020), 1073-1100.

- [42] Y. Jiang, On the symmetric and exterior powers of Young permutation modules, *J. Algebra* **567** (2021), 660-704.
- [43] T. Klein, The multiplication of Schur-functions and extensions of p -modules, *J. London Math. Soc.* **43** (1968), 280-284.
- [44] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge Tracts in Mathematics, **163**, Cambridge University Press, 2005.
- [45] B. Külshammer, Some indecomposable modules and their vertices, *J. Pure Appl. Algebra* **86** (1993), 65-73.
- [46] K. J. Lim, The varieties for some Specht modules, *J. Algebra* **321** (2009), 2287-2301.
- [47] K. J. Lim, The complexity of the Specht modules corresponding to hook partitions, *Arch. Math. (Basel)* **93** (2009), 11-22.
- [48] K. J. Lim, Specht modules with abelian vertices, *J. Algebraic Combin.* **35** (2012), 157-171.
- [49] K. J. Lim, Straightening rule for an m' -truncated polynomial ring, *J. Algebra* **522** (2019), 11-30.
- [50] K. J. Lim, K. M. Tan, The complexities of some simple modules of the symmetric groups, *Bull. Lond. Math. Soc.* **45** (2013), 497-510.
- [51] E. Lucas, Theorie des Fonctions Numeriques Simplement Periodiques, *Amer. J. Math.* **1** (1878), 289-321.
- [52] J. Müller, R. Zimmermann, Green vertices and sources of simple modules of the symmetric group labelled by hook partitions, *Arch. Math. (Basel)* **89** (2007), 97-108.
- [53] G. M. Murphy, On decomposability of some Specht modules for symmetric groups, *J. Algebra* **66** (1980), 156-168.
- [54] G. M. Murphy, M. H. Peel, Vertices of Specht modules, *J. Algebra* **86** (1984), 85-97.
- [55] H. Nagao, Y. Tsushima, *Representations of Finite Groups*, Academic Press, San Diego, 1989.
- [56] M. H. Peel, On the second natural representation of the symmetric groups, *Glasgow Math. J.* **10** (1969), 25-37.
- [57] J. Sheth, Branching rules for two row partitions and applications to the inductive systems for symmetric groups, *Comm. Algebra* **27** (1999), 3303-3316.
- [58] K. Uno, The rank variety of irreducible modules for symmetric groups, *Sūrikaiseikikenkyūsho Kōkyūroku* (2002), 8-15.
- [59] L. Valero-Elizondo, Some simple projective Brauer quotients of simple modules for the symmetric groups in characteristic two, *J. Algebra* **236** (2001), 796-818.
- [60] L. Weisner, On the Sylow Subgroups of the Symmetric and Alternating Groups, *Amer. J. Math.* **47** (1925), 121-124.
- [61] W. W. Wheeler, Generic module theory, *J. Algebra* **185** (1996), 205-228.
- [62] M. Wildon, Two theorems on the vertices of Specht modules, *Arch. Math. (Basel)* **81** (2003), 505-511.
- [63] M. Wildon, Vertices of Specht modules and blocks of the symmetric group, *J. Algebra* **323** (2010), 2243-2256.