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Mean-variance Hedging with Basis Risk

^aXIAOLE XUE, ^bJINGONG ZHANG, ^bCHENGGUO WENG*

^aZhongtai Securities Institute for Financial Studies
Shandong University

^bDepartment of Statistics and Actuarial Science
University of Waterloo

Abstract

Basis risk arises in a number of financial and insurance risk management problems when the hedging assets do not perfectly match the underlying asset in a hedging program. Notable examples in insurance include the hedging for longevity risks, weather index based insurance products, variable annuities, etc. In the presence of basis risk, a perfect hedging is impossible, and in this paper, we adopt a mean-variance criterion to strike a balance between the expected hedging error and its variability. Under a time-dependent diffusion model setup, explicit optimal solutions are derived for hedging target being either a European option or a forward contract. The solutions are obtained by a delicate application of linear quadratic control theory, method of backward stochastic differential equation, and Malliavin calculus. A numerical example is presented to illustrate our theoretical results and their interesting implications.

Keywords: Optimal Hedging; Basis Risk; Mean Variance Analysis, BSDE, Malliavin Derivative, Stochastic LQ Control.

*Corresponding author. Tel: +001(519)888-4567 ext. 31132. Email: c2weng@uwaterloo.ca.
Postal address: M3-200 University Avenue West, Waterloo, Ontario, Canada. N2L 3G1.

1 Introduction

Basis risk, known as the non-hedgable portion of risk as attributed to the imperfect correlation between the asset to be hedged (i.e., hedging target) and the assets used for hedging (i.e., hedging assets), widely exists in various financial and actuarial problems and brings additional complexity in risk management. A typical example in the equity market is to use an index future to hedge a basket stocks or their derivatives, which saves transaction cost compared with hedging directly with underlying stocks. The basis risk often arises in insurance problems when an index-based security is used for hedging. For example, a pension plan sponsor may choose to hedge the plan's longevity risk by resorting to standard longevity instruments which are available in the capital market. These standard longevity instruments are often based on certain mortality indices computed from one or more populations which differ from the population underlying the pension plan. The basis risk, or more specifically, the population basis risk, arises in this case (Li and Hardy, 2011; Coughlan et al., 2011). Another example from insurance is the hedging of agricultural risk using weather derivatives, where it could give rise to variable basis risk and spatial basis risk (e.g., Brockett et al., 2005; Woodard and Garcia, 2008) due to the imperfect match of the weather index and the actual crop yields. In this paper, we do not specialize in any particular area for the optimal hedging with basis risk, and instead we consider the optimal hedging solution under a general model setup.

In the literature, there are two broad streams of research dedicated into the derivation of implementable optimal hedging strategies in the presence of basis risk. In the first stream, hedging strategies are explored under an exponential utility maximization framework. The pioneering closed-form optimal hedging strategies were obtained by Davis (2006)¹, and the basic model of Davis (2006) was subsequently extended by Monoyios (2004) and Musiala and Zariphopoulou (2004) in a few interesting directions including indifference pricing, perturbation expansions, etc. The other stream of literature focus on the hedging of a stock index by future contracts written on an underlying asset which may differ from the index. The main contribution is attributed to Duffie and Richardson (1991) who obtained the optimal hedging policy under geometric Brownian motion assumptions. They demonstrated that the optimal hedging strategy can be derived from the normal equations for orthogonal projection in a Hilbert space. Their method, however, is not readily applicable to more general hedging target. This method was further exploited by Schweizer (1992) for hedging more general contingent claims.

¹The work of Davis (2006) was done in 2000 but it was not formally published until 2006.

In this paper, we consider a hedging target which is either a European option or a forward contract under a diffusion model setup where the coefficients are deterministic but allowed to be time-dependent. In the presence of basis risk, a perfect hedging is impossible, and we adopt a mean-variance criterion to strike a balance between the expected hedging error and the variability of the hedging error. A mean-variance analysis can produce an efficient frontier where one can identify the minimum achievable variance at a tolerable level of expected hedging error, or the minimum expected hedging error one can achieve at a bearable variance of the hedging error. In a dynamic setting, two approaches are mainly used to solve a mean-variance problem in the literature: the martingale method (e.g. Musiela and Rutkowski, 1997. Bielecki et al., 2005; Ji, 2010), and the indefinite stochastic Linear-Quadratic (LQ) theory (see, e.g. Zhou and Li, 2000; Lim and Zhou, 2002; Lim, 2004). Because of basis risk, the market is incomplete and thus, the martingale method fails in general. So, we follow Zhou and Li (2000), Lim and Zhou (2002) and Lim (2004) and use the stochastic LQ theory to study the optimal hedging strategies for our problem. The LQ theory allows us to consider hedging plan with multiple hedging assets while almost all the literature on optimal hedging with basis risk only consider a single hedging asset.

The standard application of the LQ theory to our problem yields a characterization of the optimal solution via one ordinary differential equation (ODE) and one backward stochastic differential equation (BSDE). For a hedging problem, we need to establish an implementable solution, and simply applying the LQ theory is not sufficient for achieving such an objective. One has to solve the ODE and the BSDE, and further explore the computability of certain critical quantities which intricately depend on the solutions of the ODE and the BSDE. In this paper, we managed to solve the ODE and BSDE, and eventually obtained an implementable optimal solution by resorting to Malliavin calculus. Certain connection between BSDE and Malliavin derivatives is utilized in the procedure, and moreover, the solution involves the derivation of the Malliavin derivative of the payoff of the hedging target. For common derivatives including forward contracts, European call options, European put options, and Asian options, we have demonstrate that the Malliavin derivatives of their payoffs have an explicit form, and this fact is essential to make the derived optimal hedging strategies implementable. A numerical example is presented to illustrate the established results and some interesting implementations.

Schweizer (1992) and Lim (2004) are the most relevant papers because they also study the quadratic hedging problem under an incomplete market as the present paper does. Below we comment on the difference of our paper from those two. As we previously mentioned, Schweizer (1992) exploited the Hilbert space projection argument as in Duffie and Richardson (1991) and

characterized the optimal hedging strategy by a stochastic differential equation (SDE). In contrast we resort to the BSDE and LQ theory and manage to derive the optimal solution in an analytical form. For common derivatives (such as futures, vanilla European options and Asian options), the optimal hedging strategy from our results is implementable with a minimum amount of computation efforts.

The major differences between Lim (2004) and our paper are as follows. First, the reason for the incompleteness of the market differs between Lim (2004) and our paper. In our paper the incompleteness arises due to the mismatch between the hedging assets and the underlying asset of the target derivative which has an additional random source. In other words, in our setting the number of hedging assets is smaller than the number of randomness sources in the market. The incompleteness of the market in Lim (2004), however, stems from the randomness of drift and volatility parameters in asset price dynamics. The covariance matrix for all the involved asset price processes in Lim (2004) is a square matrix whereas it is no longer the case for our model. Second, while it is very interesting in Lim (2004) that a general quadratic hedging problem is studied for asset prices with random drift and volatility parameters, the generality of their model also hampers the implementation of their results because one has to solve certain complex BSDEs (which is often hard) in order to derive the optimal hedging strategy. In our paper we consider deterministic drift and volatility parameters for asset prices and this enables us to explore further for more explicit solutions by utilizing Malliavin calculus.

The rest of the paper is organized as follows. Section 2 presents the mathematical formulation of our optimal hedging problem. Section 3 studies the solution of a stochastic LQ control problem which plays a crucial role in the derivation of the solution for our optimal hedging problem. Section 4 includes some background about Malliavin calculus and establishes some results regarding the Malliavin derivatives of the hedging target and the solutions to the aforementioned BSDE. Section 5 combines results from the previous sections and gives the solution to our optimal hedging problem. A formula for the efficient frontier of our mean-variance problem is also presented in this section. Section 6 presents a numerical example and section 7 concludes the paper.

2 Problem Formulation

Throughout the paper we describe all uncertainty by a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $W(t) = (W^0(t), W^1(t), \dots, W^m(t))^T$ be an $(m+1)$ -dimensional standard Brown-

nian motion on this probability space, where $M^\top$ denotes the transpose of a vector or a matrix M . Let $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ denote the augmented filtration generated by W . Further, we denote the set of all $\mathbb{R}^k$ -valued $\mathcal{F}$ -adapted processes f on $[0, T]$ with $\mathbb{E} \left[\int_0^T |f(t)|^2 dt \right] < \infty$ by $L^2_{\mathcal{F}}([0, T]; \mathbb{R}^k)$, the set of all $\mathbb{R}^k$ -valued continuous functions on $[0, T]$ by $C([0, T]; \mathbb{R}^k)$, the set of $\mathbb{R}^k$ -valued continuous $\mathcal{F}$ -adapted processes f with $\mathbb{E} \left[\sup_{0 \leq t \leq T} |f(t)|^2 \right] < \infty$ by $M^2_{\mathcal{F},c}(\Omega; \mathbb{R}^k)$, and the set of $\mathbb{R}^k$ -valued $\mathcal{F}_T$ -measurable random variables by $L^2_{\mathcal{F}}(\Omega; \mathbb{R}^k)$.

Consider an arbitrage free financial market with $m + 1$ risky assets $\{S_i(t), t \geq 0\}$, $i = 0, 1, \dots, m$, and a risk-free asset earning at a constant rate of $r > 0$. Assume that $\{S_0(t), t \geq 0\}$ is the underlying asset of our hedging objective and it has dynamics given by the following SDE:

$$\begin{cases} dS_0(t) = S_0(t) [\mu_0(t)dt + \sigma_{00}(t)dW^0(t)], \\ S_0(0) = s_0 > 0. \end{cases} \quad (1)$$

We use $\mathcal{F}^0 := \{\mathcal{F}^0_t\}_{t \geq 0}$ to denote the augmented filtration generated by W^0 , and let $G = G(S_0; T)$ be the payoff function at maturity $T > 0$ of a contingent claim that is written on asset S_0 . Our goal is to construct a hedging strategy for this financial derivative G . For technical reason, throughout the paper we assume that G satisfies the same conditions as those satisfied by the random variable ξ as stated in Proposition 2 in the sequel. The most commonly used pay-offs including European, American and Asian (call and put) options satisfy this assumption; see Nualart (2009) for details.

We assume that underlying asset S_0 is either non-tradable on the market or not liquid enough to be used for hedging the payoff G . So, we hedge the payoff using the risky assets $\{S_i(t), t \geq 0\}$, $i = 1, 2, \dots, m$, which are liquid enough and available on the market, along with the risk free asset. We assume that the price processes of risky assets $\{S_i(t), t \geq 0\}$, $i = 1, 2, \dots, m$, follow general diffusion processes under the physical measure $\mathbb{P}$ given as below:

$$\begin{cases} dS_i(t) = S_i(t) \left(\mu_i(t)dt + \sum_{j=0}^m \sigma_{ij}(t)dW^j(t) \right), \\ S_i(0) = s_i > 0, \end{cases} \quad (2)$$

where $\mu_i(t)$ and $(\sigma_{i0}(t), \sigma_{i1}(t), \dots, \sigma_{im}(t))$ are the expected return rate and volatility vector of the i -th stock, respectively, $i = 1, 2, \dots, m$. We assume that $\mu_i(t)$ and $\sigma_{ij}(t)$ are bounded deterministic functions. It is worth noting that, for each $i = 1, \dots, m$, the Brownian motion W^0 appears in the dynamics of every S_i and the strength of the correlation between assets S_i and S_0 is captured

by the coefficients σ_{00} and σ_{i0} . Since S_i , $i = 1, \dots, m$, are the assets used to hedge a claim written on S_0 , we expect a stronger correlation between S_i and S_0 would imply a better hedging effect. For ease of presentation we denote

$$\sigma_i(t) = (\sigma_{i0}(t), \sigma_{i1}(t), \dots, \sigma_{im}(t))^T, \quad i = 1, \dots, m,$$

and write

$$\sigma(t) = \begin{pmatrix} \sigma_1(t)^T \\ \sigma_2(t)^T \\ \vdots \\ \sigma_m(t)^T \end{pmatrix} = (\sigma_{ij}(t))_{m \times (m+1)} \in \mathbb{R}^{m \times (m+1)}.$$

Further, for technical reason, we assume that $\sigma(t)\sigma^T(t)$ is uniformly positive-definite, i.e., there exists a constant $\delta > 0$ such that:

$$\sigma(t)\sigma^T(t) \geq \delta I, \quad \forall t \in [0, T] \quad (3)$$

where I denotes the $m \times m$ identity matrix.

At any time $t \in [0, T]$, a hedging portfolio for the payoff G can be fully specified by the pair $\{X^\theta(t), \theta(t)\}$, where $X^\theta(t)$ denotes the time- t hedging portfolio value resulting from the hedging strategy θ , and $\theta(t) := (\theta_1(t), \theta_2(t), \dots, \theta_m(t))^T$ with $\theta_i(t)$ denoting the time- t dollar amount invested in risky asset $S_i(t)$, $i = 1, \dots, m$. This implies that the time- t dollar amount invested in risk-free asset is given by $X^\theta(t) - \sum_{i=1}^m \theta_i(t)$. Therefore, in view of the SDE for S_i in equation (2), the value process of the hedging portfolio is governed by the following SDE:

$$\begin{cases} dX^\theta(t) = [rX^\theta(t) + b(t)^T \theta(t)] dt + \theta(t)^T \sigma(t) dW(t), \\ X^\theta(0) = x_0, \end{cases} \quad (4)$$

where

$$b(t) = (\mu_1(t) - r, \mu_2(t) - r, \dots, \mu_m(t) - r)^T \quad (5)$$

is the expected excess return vector and x_0 is the initial value of the hedging portfolio, i.e., the time-0 hedging cost.

With a hedging strategy $\{X^\theta(t), \theta(t)\}$, the hedging error for the hedger at the terminal time T is given by $G(S_0; T) - X^\theta(T)$. Because of the liquidity issue with S_0 , the option G is not

replicable in general, and thus, there is no strategy θ which can ensure a zero hedging error in every state no matter what a hedging cost x_0 is given. In this paper, we resort to a mean-variance analysis for an optimal hedging strategy to strike a balance between the resulting risk exposure (measured by the variance of the hedging error) and the expected hedging error (measured by the mean of the hedging error). For ease of presentation, instead of the hedging error, we consider the profit-and-loss random variable for the hedger at terminal time T :

$$V^\theta(T) = X^\theta(T) - G(S_0; T),$$

which is simply the negative of hedging error. A larger expected value and a smaller variance of $V^\theta(T)$ implies a better hedging strategy.

By the mean-variance criterion, an optimal hedging strategy θ^* is defined as the one that solves the following optimization problem:

$$\max_{\theta \in \Theta} \left\{ J_1(\theta) := \mathbb{E} [V^\theta(T)] - \frac{\gamma}{2} \text{Var} [V^\theta(T)] \right\}, \quad (6)$$

where the admissible set is given by

$$\Theta = \{\theta \in L^2_{\mathcal{F}}(0, T; R^m) | \theta \text{ is } \mathcal{F}_t - \text{adapted}\}.$$

In order to solve problem (6), we follow Zhou and Li (2000) and construct the following auxiliary problem:

$$\max_{\theta \in \Theta} \left\{ J_2(\theta; \gamma, \lambda) := \mathbb{E} \left[\lambda V^\theta(T) - \frac{\gamma}{2} V^\theta(T)^2 \right] \right\}. \quad (7)$$

According to Zhou and Li (2000), once we have solved the above auxiliary problem, the solution of the our hedging problem (6) can be obtained based on the solution of the auxiliary problem. The connection between problems (6) and (7) is described by the following lemma.

Lemma 1. *If $\theta^*(\cdot)$ is the solution to problem (6), then $\theta^*(\cdot)$ is the optimal control for the auxiliary problem (7) with $\lambda = 1 + \gamma \mathbb{E}[V^{\theta^*}(T)]$.*

Proof. See Theorem 3.1, Zhou and Li (2000). □

The auxiliary problem (7) can be equivalently recast as follows:

$$\begin{aligned}
\max_{\theta \in \Theta} \mathbb{E} \left[\lambda V^\theta(T) - \frac{\gamma}{2} V^\theta(T)^2 \right] &= \max_{\theta \in \Theta} \mathbb{E} \left[-\frac{\gamma}{2} \left(V^\theta(T) - \frac{\lambda}{\gamma} \right)^2 \right] + \frac{\lambda^2}{2\gamma} \\
&= -\min_{\theta \in \Theta} \mathbb{E} \left[\frac{\gamma}{2} \left(V^\theta(T) - \frac{\lambda}{\gamma} \right)^2 \right] + \frac{\lambda^2}{2\gamma} \\
&= -\min_{\theta \in \Theta} \mathbb{E} \left[\frac{\gamma}{2} \left(X^\theta(T) - G(S_0; T) - \frac{\lambda}{\gamma} \right)^2 \right] + \frac{\lambda^2}{2\gamma} \quad (8)
\end{aligned}$$

where the dynamics of the state process $\{X^\theta(t), t \geq 0\}$ is given by equation (4). From equation (8), we note that the auxiliary problem is a stochastic linear quadratic (LQ) optimal control problem.

3 Stochastic LQ problem

Based on Lemma 1 and equation (8) from the last section, the solution to our optimal hedging problem (6) can be obtained by solving the following stochastic LQ problem:

$$\min_{\theta \in \Theta} J_3(\theta; \lambda) = \mathbb{E} \left[(X^\theta(T) - \xi)^2 \right], \quad (9)$$

where $\xi = G(S_0; T) + \frac{\lambda}{\gamma}$, and the state equation is given by equation (4).

We will apply the stochastic LQ technique from Lim and Zhou (2002) to solve problem (9) in the above. For ease of presentation, we define two deterministic functions as follows:

$$\begin{cases} \kappa(t) = b(t)^\top \Sigma(t)^{-1} b(t), \\ \zeta(t) = b(t)^\top \Sigma(t)^{-1} \sigma(t) \mathbf{e}_1, \end{cases} \quad (10)$$

where $\mathbf{e}_1 := (1, 0, \dots, 0)^\top$, and $\Sigma(t)^{-1}$ is the inverse of the following matrix:

$$\Sigma(t) := \sigma(t) \sigma^\top(t).$$

Solving problem (9) entails dealing with the following two equations:

$$\begin{cases} dP(t) = [\kappa(t) - 2r] P(t) dt, \\ P(T) = 1, \end{cases} \quad (11)$$

and

$$\begin{cases} d\varphi(t) = \{[\kappa(t) - r]\varphi(t) + \zeta(t)\psi_0(t)\} dt + \psi_0(t)dW^0(t), \\ \varphi(T) = -\xi. \end{cases} \quad (12)$$

Remark 1. Equation (11) is an ODE and can be easily solved. Equation (12) is a linear BSDE, and its solution is a pair of adapted processes (φ, ψ_0) . Note that all the coefficients are deterministic and bounded in the BSDE (12), and moreover, the terminal random variable $\xi \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}) \subseteq L^2_{\mathcal{F}}(\Omega; \mathbb{R})$. Thus, a unique solution pair (φ, ψ_0) exists and $(\varphi, \psi_0) \in M^2_{\mathcal{F},c}(\Omega; \mathbb{R}) \times L^2_{\mathcal{F}}([0, T]; \mathbb{R})$; see Yong and Zhou (1999, p349) for details.

ψ_0 is the Malliavin derivative of φ in pair (φ, ψ_0) as the solution to the BSDE (12) and its expression will be given in section 4. We summarize the expressions for $P(t)$ and $\varphi(t)$ in the following proposition.

Proposition 1. The solution of equation (11) is given by

$$P(t) = e^{\int_t^T [2r - \kappa(s)] ds}, \quad (13)$$

and the solution of equation (12) is given by

$$\varphi(t) = e^{-\int_t^T (\kappa(s) - r + \frac{1}{2}[\zeta(s)]^2) ds} \mathbb{E} \left[-\xi e^{-\int_t^T \zeta(s) dW^0(s)} \middle| \mathcal{F}_t^0 \right]. \quad (14)$$

Proof. Equation (11) is a simple ODE and the solution can be easily obtained as given in equation (13). The existence and uniqueness of solution to equation (12) can be found in El Karoui et al. (1997) or Yong and Zhou (1999). $\square$

Remark 2. For vanilla European option G , the random variable ξ only depends on $S_0(T)$ and we can write the random variable ξ into $G(S_0(T)) + \lambda/\gamma$, i.e., $\xi = G(S_0(T)) + \lambda/\gamma$. In this case, the calculation of $\varphi(t)$ defined in (14) can be converted into deriving the expectation of a function of a log-normal random variable. To show this, we define $\tilde{W} := \{\tilde{W}(t), 0 \leq t \leq T\}$ by $\tilde{W}(t) = W^0(t) + \int_0^t \zeta(s) ds$, $0 \leq t \leq T$. Then, by Girsanov's Theorem with $\{-\zeta(t), 0 \leq t \leq T\}$ as the kernel, $\tilde{W}$ is a standard Brownian motion under the probability measure $\tilde{\mathbb{P}}$ which is defined by the following Radon-Nikodym derivative:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ -\int_0^T \zeta(s) dW^0(s) - \frac{1}{2} \int_0^T [\zeta(s)]^2 ds \right\},$$

and we can write the dynamics of S_0 as follows

$$dS_0(t) = S_0(t) \left[\tilde{\mu}(t)dt + \sigma_{00}(t)d\tilde{W}(t) \right],$$

where $\tilde{\mu}(t) := \mu_0(t) - \zeta(t)\sigma_{00}(t)$, $0 \leq t \leq T$. Consequently, $\varphi(t)$ can be computed by

$$\begin{aligned} \varphi(t) &= -e^{-\int_t^T (\kappa(s)-r)ds} \tilde{\mathbb{E}}[\xi | \mathcal{F}_t^0] \\ &= -e^{-\int_t^T (\kappa(s)-r)ds} \left(\tilde{\mathbb{E}}[G(S_0(t)e^{Z_t}) | \mathcal{F}_t^0] + \frac{\lambda}{\gamma} \right), \end{aligned}$$

where $\tilde{\mathbb{E}}$ means taking expectation under the probability measure $\tilde{\mathbb{P}}$, and under this probability measure, Z_t is a random variable independent of the σ -field $\mathcal{F}_t^0$ and follows a normal distribution with mean $\int_t^T \left[\tilde{\mu}(s) - \frac{\sigma_{00}^2(s)}{2} \right] ds$ and variance $\int_t^T \sigma_{00}^2(s)ds$. Therefore, when G is a vanilla call or put option, the well-known Black-Scholes formula can be applied for the computation of $\varphi(t)$.

To proceed, we define the following vector:

$$\Gamma(t) := b(t)\varphi(t) + \sigma(t)\mathbf{e}_1\psi_0(t). \quad (15)$$

The following theorem shows that the solvability of (11) and (12) provides a sufficient condition for the solvability of the LQ problem (9).

Theorem 1. *Let $P \in C([0, T]; \mathbb{R})$ and $(\varphi, \psi_0) \in M_{\mathcal{F},c}^2(\Omega; \mathbb{R}) \times L_{\mathcal{F}}^2([0, T]; \mathbb{R})$ be the solutions of (11) and (12), respectively. Then the stochastic LQ problem (9) is solvable with the unique optimal feedback control given by*

$$\theta_\lambda(t) = -\frac{1}{P(t)}\Sigma(t)^{-1} [b(t)P(t)X^{\theta_\lambda}(t) + \Gamma(t)] \quad (16)$$

and the associated optimal value is given by

$$J_3(\theta_\lambda; \lambda) = P(0)x_0^2 + 2\varphi(0)x_0 + \mathbb{E}[\xi^2] - \mathbb{E} \left[\int_0^T \frac{1}{P(t)} \Gamma(t)^\top \Sigma(t)^{-1} \Gamma(t) dt \right]. \quad (17)$$

Proof. Let $(X^\theta(\cdot), \theta(\cdot))$ be an admissible pair for problem (9). Then, the objective function in problem (9) can be rewritten into the following form:

$$J_3(\theta; \lambda) = \mathbb{E}[(X^\theta(T) - \xi)^2] = \mathbb{E}[(X^\theta(T))^2] - 2\mathbb{E}[\xi X^\theta(T)] + \mathbb{E}[\xi^2]. \quad (18)$$

We apply Itô's formula to $(X^\theta(t))^2$ to obtain

$$d(X^\theta(t))^2 = \{2X^\theta(t)[rX^\theta(t) + b(t)^\top \theta(t)] + \theta(t)^\top \Sigma(t)\theta(t)\}dt + 2X^\theta(t)\theta(t)^\top \sigma(t)dW(t).$$

We further use the ODE (11) and Itô's formula to obtain

$$\begin{aligned} dP(t)(X^\theta(t))^2 &= \{P(t)[2X^\theta(t)b(t)^\top \theta(t) + \theta(t)^\top \Sigma(t)\theta(t)] \\ &\quad + X^2(t)\kappa(t)P(t)\}dt \\ &\quad + 2P(t)X^\theta(t)\theta(t)^\top \sigma(t)dW(t). \end{aligned} \quad (19)$$

Using the Burkholder-Davis-Gundy inequality and Hölder's inequality, it is easy to show

$$\int_0^s 2P(t)X^\theta(t)\theta(t)^\top \sigma(t)dW(t)$$

is a martingale. Thus, integrating equation (19) from 0 to T and taking expectation yield

$$\begin{aligned} \mathbb{E}[(X^\theta(T))^2] &= P(0)x_0^2 + \mathbb{E}\left\{\int_0^T [P(t)[2X^\theta(t)b(t)^\top \theta(t) + \theta(t)^\top \Sigma(t)\theta(t)] \right. \\ &\quad \left. + (X^\theta(t))^2 \kappa(t)P(t)\right] dt\bigg\}. \end{aligned} \quad (20)$$

Similarly, we apply Itô's formula to $\varphi(\cdot)X^\theta(\cdot)$, integrate the resulting dynamics from 0 to T and take expectation to obtain

$$-\mathbb{E}[\xi X^\theta(T)] = \varphi(0)x_0 + \mathbb{E}\left\{\int_0^T [b(t)^\top \Sigma(t)^{-1}\Gamma(t)X^\theta(t) + \theta(t)^\top \Gamma(t)] dt\right\}. \quad (21)$$

By virtue of equations (20) and (21), we plug θ_λ defined in equation (16) into equation (18) to obtain $J_3(\theta_\lambda; \lambda)$ as given by equation (17). Moreover, for any control $\theta \in \Theta$, we use equations

(18), (20) and (21) to obtain

$$\begin{aligned}
J_3(\theta; \lambda) &= P(0)x_0^2 + 2\varphi(0)x_0 + E[\xi^2] \\
&\quad + E\left[\int_0^T P(t) \left[2X^\theta(t)b(t)\theta(t) + \theta(t)^\top \Sigma(t)\theta(t) + (X^\theta(t))^2 \kappa(t)\right] dt\right] \\
&\quad + 2E\left[\int_0^T \left[b(t)^\top \Sigma(t)^{-1}\Gamma(t)X^\theta(t) + \theta(t)^\top \Gamma(t)\right] dt\right] \\
&= P(0)x_0^2 + 2\varphi(0)x_0 + E[\xi^2] - E\left[\int_0^T \frac{1}{P(t)}\Gamma(t)^\top \Sigma(t)^{-1}\Gamma(t)dt\right] \\
&\quad + E\left[\int_0^T \left(X^\theta(t)b(t) + \Sigma(t)\theta(t) + \frac{1}{P(t)}\Gamma(t)\right)^\top \right. \\
&\quad \quad \left. \cdot \Sigma(t) \left(X^\theta(t)b(t) + \Sigma(t)\theta(t) + \frac{1}{P(t)}\Gamma(t)\right) dt\right] \\
&\geq P(0)x_0^2 + 2\varphi(0)x_0 + E[\xi^2] - E\left[\int_0^T \frac{1}{P(t)}\Gamma(t)^\top \Sigma(t)^{-1}\Gamma(t)dt\right] \\
&= J_3(\theta_\lambda; \lambda),
\end{aligned} \tag{22}$$

which means that θ_λ given in (16) is an optimal control to the stochastic LQ problem (9) with an optimal value given by $J_3(\theta_\lambda; \lambda)$. Furthermore, the inequality in the last display becomes equality if and only if $\theta = \theta_\lambda$. Thus, θ_λ is the unique optimal control for the problem. $\square$

4 Malliavin Derivative and Backward Stochastic Differential Equation

In Proposition 1, we have obtained closed-form solution for φ but not ψ_0 . In this section, we resort to Malliavin derivative for the derivation of ψ_0 , which will be useful to solve the equation $\lambda = 1 + \gamma E[V^{\theta_\lambda}(T)]$ and to eventually obtain the solution to our optimal hedging problem (6).

Let F be an $\mathcal{F}_T^0$ -measurable and square integrable random variable under the physical measure $\mathbb{P}$. Then, its Malliavin derivative can be defined via chaos expansion (see Di Nunno et al. (2009) for more details):

$$F = \sum_{n=0}^{\infty} I_n(f_n(\cdot, \cdot)),$$

where $I_n(f_n)$ is the iterated stochastic integral with respect to the Brownian motion W^0 , and f_n are some deterministic symmetric functions in $L^2(\mu^n)$ which includes all functions f satisfying $\|f\|_{L^2(\mu^n)}^2 := \int_{[0,T]^n} f^2 d\mu^n < \infty$, where μ denotes the Lebesgue measure on $[0, T]$. We write $F \in \mathbb{D}_{1,2}$, if $\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=1}^{\infty} nn! \|f_n\|_{L^2(\mu^n)}^2 < \infty$, and in this case, we say that F is Malliavin

differentiable with respect to W^0 , and the Malliavin derivative of F for time t is given by

$$D_t F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0, T].$$

A random variable from $\mathbb{D}_{1,2}$ is said to be Malliavin differentiable with respect to W^0 .

The chain rule for Malliavin derivatives in Lemmas 2 and 3 below will be useful later in this section; see Nualart (2006, p28), El Karoui et al. (1997) for their proofs.

Lemma 2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded derivative f_x . Then, for any random variable $F \in \mathbb{D}_{1,2}$,*

$$f(F) \in \mathbb{D}_{1,2} \text{ and } D_t[f(F)] = f_x(F) \cdot D_t F.$$

Lemma 3. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function s.t. $|f(x) - f(y)| \leq K|x - y|$ for some constant $K > 0$ and $\forall x, y \in \mathbb{R}$. Then, for any random variable $F \in \mathbb{D}_{1,2}$, $f(F) \in \mathbb{D}_{1,2}$ and there exists a random variable G bounded by a constant almost surely such that*

$$D_t[f(F)] = G \cdot D_t F. \quad (23)$$

Remark 3. *If the law of the random variable F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then $G = f_x(F)$ in equation (23).*

Notice that $D_t f(\cdot) = 0$ if f is a deterministic function, and thus we have the following proposition, which is a simplified version of Proposition 5.3 of El Karoui et al. (1997).

Proposition 2. *Suppose $\xi \in \mathbb{D}_{1,2}$ with $E[\xi^4] < \infty$. Let (φ, ψ_0) be the solution of BSDE (12). Then, the Malliavin derivatives of φ and ψ_0 are given as follows:*

(a) *For $0 \leq t < u \leq T$,*

$$D_u \varphi(t) = D_u \psi_0(t) = 0;$$

(b) *For $u \leq t \leq T$,*

$$D_u \varphi(t) = -D_u \xi - \int_t^T \{[\kappa(s) - r] D_u \varphi(s) + \zeta(s) D_u \psi_0(s)\} ds - \int_t^T D_u \psi_0(s) dW^0(s). \quad (24)$$

Moreover, $\psi_0(t) = D_t\varphi(t)$, $0 \leq t \leq T$.

The following proposition presents an explicit expression for ψ_0 and some characterizations about ψ_0 and φ .

Proposition 3. *The solution pair to BSDE (12), (φ, ψ_0) , and their expectations admit the following expressions:*

$$\psi_0(t) = D_t\varphi(t) = \nu(t) \cdot \mathbb{E} \left[-e^{-\int_t^T \zeta(s) dW^0(s)} D_t\xi \middle| \mathcal{F}_t^0 \right], \quad (25)$$

$$\mathbb{E}[\varphi(t)] = \nu(t) \cdot \mathbb{E} \left[-\xi e^{-\int_t^T \zeta(s) dW^0(s)} \right], \quad (26)$$

$$\mathbb{E}[\psi_0(t)] = \mathbb{E}[D_t\varphi(t)] = \nu(t) \cdot \mathbb{E} \left[-e^{-\int_t^T \zeta(s) dW^0(s)} D_t\xi \right], \quad (27)$$

where

$$\nu(t) = \exp \left\{ - \int_t^T \left(\kappa(s) - r + \frac{1}{2} [\zeta(s)]^2 \right) ds \right\}.$$

Proof. Equation (24) can be viewed as a BSDE of $\{Y^\theta(t) \equiv D_\theta\varphi(t), t \geq 0\}$ and its solution can be obtained as given in equation (25) by following the same derivation procedure in the proof of Proposition 1. Thus, we omit the details for the proof. Equations (26) and (27) are obtained by taking expectation upon equations (14) and (25), respectively. $\square$

As we will see shortly in the next section, $\varphi(t)$ and $\psi_0(t)$ appear in the optimal hedging strategy, and thus, we need to find the Malliavin derivatives $D_t\xi$. Since $\xi = G(S_0; T) + \lambda/\gamma$, $D_t\xi = D_tG(S_0; T)$. Because $D_tS_0(T) = S_0(T)\sigma_{00}(t)$, we can apply Lemma 3 and Remark 3 to easily get the Malliavin derivatives for the payoffs of the following common financial instruments which only depend on the terminal price of asset S_0 :

- For a forward contract with a payoff $G(S_0; T) = S_0(T)$,

$$D_tS_0(T) = S_0(T) \cdot \sigma_{00}(t).$$

- For a European call option with a payoff $G(S_0; T) = (S_0(T) - K)_+$,

$$D_t(S_0(T) - K)_+ = \mathbf{1}_{\{S_0(T) \geq K\}} \cdot S_0(T) \cdot \sigma_{00}(t),$$

where $\mathbf{1}_{\{A\}} = 1$ if the event A holds and 0 otherwise.

- For a European put option with a payoff $G(S_0; T) = (K - S_0(T))_+$,

$$D_t(K - S_0(T))_+ = \mathbf{1}_{\{S_0(T) \leq K\}} \cdot S_0(T) \cdot \sigma_{00}(t).$$

The discussion for the Malliavin derivative of path-dependent options is challenging in general, but a closed-form expression exists for an Asian option:

- For an Asian option with a payoff $G(S_0; T) = \left(\frac{1}{T} \int_0^T S_0(r) dr - K \right)_+$, its derivative is given by

$$D_t \left(\frac{1}{T} \int_0^T S_0(r) dr - K \right)_+ = \mathbf{1}_{\{\int_0^T S_0(r) dr \geq TK\}} \cdot \frac{\sigma_{00}(t)}{T} \cdot \int_t^T S_0(r) dr.$$

See Nualart (p.334, 2009) for details.

5 Solutions to optimal hedging problem

According to Lemma 1, the optimal solution to our optimal hedging problem (6) can be obtained by solving problem (7) with $\lambda = 1 + \gamma \mathbb{E}[V^{\theta_\lambda}(T)]$, and the solution θ_λ has been obtained as given in equation (16), which however depends on (φ, ψ_0) , the solution of BSDE (12). Further note that the BSDE (12) depends on the parameter λ via $\xi \equiv G(S_0; T) + \frac{\lambda}{\gamma}$, and the profit-and-loss random variable $V^{\theta_\lambda}(T)$ essentially depends on λ . Therefore, we need to solve the equation $\lambda = 1 + \gamma \mathbb{E}[V^{\theta_\lambda}(T)]$ to obtain a value for λ and then plug it into equation (16) for an optimal hedging strategy.

We plug θ_λ given in (16) into the state equation (4) to obtain

$$\begin{cases} dX^{\theta_\lambda}(t) = \left([r - \kappa(t)] X^{\theta_\lambda}(t) - \frac{1}{P(t)} b(t)^\top \Sigma(t)^{-1} \Gamma(t) \right) dt \\ \quad - \left[X^{\theta_\lambda}(t) b(t)^\top + \frac{1}{P(t)} \right] \Sigma(t)^{-1} \sigma(t) dW(t) \\ X^{\theta_\lambda}(0) = x_0. \end{cases}$$

It follows that

$$\mathbb{E}[X^{\theta_\lambda}(T)] = x_0 + \int_0^T \left([r - \kappa(t)] \mathbb{E}[X^{\theta_\lambda}(t)] - \frac{1}{P(t)} b(t)^\top \Sigma(t)^{-1} \mathbb{E}[\Gamma(t)] \right) dt. \quad (28)$$

We then solve (28) to obtain

$$\mathbb{E} [X^{\theta_\lambda}(T)] = \alpha x_0 + \beta \quad (29)$$

where

$$\begin{cases} \alpha = e^{\int_0^T [r - \kappa(t)] dt}, \\ \beta = - \int_0^T \frac{1}{P(t)} b(t)^\top \Sigma(t)^{-1} \mathbb{E}[\Gamma(t)] e^{\int_t^T [r - \kappa(s)] ds} dt. \end{cases} \quad (30)$$

We note that β depends on λ via $\Gamma(t)$, which is defined in equation (15) and depends on λ via φ .

By equations (10), (13) and (15), we can simplify $\mathbb{E} [X^{\theta_\lambda}(T)]$ into

$$\begin{aligned} \mathbb{E} [X^{\theta_\lambda}(T)] &= \alpha x_0 - \int_0^T b(t)^\top \Sigma(t)^{-1} \mathbb{E}[\Gamma(t)] e^{-r(T-t)} dt \\ &= \alpha x_0 - \int_0^T \kappa(t) \Sigma(t)^{-1} \mathbb{E}[\varphi(t)] e^{-r(T-t)} dt - \int_0^T \zeta(t) \mathbb{E}[\psi_0(t)] dt. \end{aligned} \quad (31)$$

By lemma 1, the optimal solution of the problem (6), if exists, can be found by choosing λ as

$$\begin{aligned} \lambda &= 1 + \gamma \mathbb{E} [V^{\theta_\lambda}(T)] \\ &= 1 + \gamma \mathbb{E} [X^{\theta_\lambda}(T) - G(S_0; T)] \\ &= 1 - \gamma \mathbb{E} [G(S_0; T)] + \gamma \alpha x_0 \\ &\quad - \gamma \int_0^T \kappa(t) \Sigma(t)^{-1} \mathbb{E}[\varphi(t)] e^{-r(T-t)} dt \\ &\quad - \gamma \int_0^T \zeta(t) \mathbb{E}[\psi_0(t)] dt. \end{aligned} \quad (32)$$

Recall that $\xi = G(S_0; T) + \lambda/\gamma$, and thus, its Malliavin derivative $D_t \xi = D_t G(S_0; T)$ which is independent of γ . Consequently, using those expressions in Proposition 3 for $\mathbb{E}[\varphi(t)]$ and $\mathbb{E}[\psi_0(t)]$, we find that $\mathbb{E}[\varphi(t)]$ is the only item in (32) which depends on λ and it is given by

$$\mathbb{E}[\varphi(t)] = -\nu(t) \mathbb{E} \left[G(S_0; T) e^{-\int_t^T \zeta(s) dW^0(s)} \right] - \frac{\lambda}{\gamma} \cdot \nu(t) \mathbb{E} \left[e^{-\int_t^T \zeta(s) dW^0(s)} \right].$$

Plugging the above equation into equation (32) yields

$$\begin{aligned} \lambda = \bar{\lambda} &:= \frac{1}{N} \{ 1 - \gamma \mathbb{E} [G(S_0; T)] + \gamma \alpha x_0 \} \\ &\quad - \frac{1}{N} \int_0^T \zeta(t) \mathbb{E} [\psi_0(t)] \cdot e^{-r(T-t)} dt \\ &\quad + \frac{\gamma}{N} \int_0^T \kappa(t) e^{-\int_t^T (\kappa(s) + \frac{1}{2} [\zeta(s)]^2) ds} \cdot \mathbb{E} \left[G(S_0; T) e^{-\int_t^T \zeta(s) dW^0(s)} \right] dt \end{aligned} \quad (33)$$

where $N = 1 - \int_0^T \kappa(t) e^{\int_t^T -\kappa(s) ds} dt$.

Remark 4. Combining the analysis in the above, we have following conclusions with respect to

our hedging problem (6):

- (a) According to Lemma 1, $\theta^*(\cdot) := \theta_{\bar{\lambda}}(\cdot)$ solves our optimal hedging problem (6), where $\theta_{\lambda}(\cdot)$ is defined in equation (16).
- (b) Recall that $V^{\theta^*}(T) = X^{\theta^*}(T) - G(S_0; T)$ and $\xi = G(S_0; T) + \bar{\lambda}/\gamma$. Thus, the corresponding variance of $V^{\theta^*}(T)$ can be computed by

$$\begin{aligned} \text{Var} [V^{\theta^*}(T)] &= \mathbb{E} [V^{\theta^*}(T)^2] - (\mathbb{E} [V^{\theta^*}(T)])^2 \\ &= J_3(\theta^*; \bar{\lambda}) - \left(\mathbb{E} [X^{\theta^*}(T)] - \mathbb{E} [G(S_0; T)] - \frac{\bar{\lambda}}{\gamma} \right)^2 \end{aligned} \quad (34)$$

where $\bar{\lambda}$ and $J_3(\theta^*; \bar{\lambda})$ are respectively given by equations (33) and (17), and $\mathbb{E} [X^{\theta^*}(T)]$ is equal to $\mathbb{E}[X^{\theta_{\bar{\lambda}}}(T)]$ for which a formula is given in equation (31). In principle, the efficient frontier of the optimal hedging problem (6) can be obtained from equation (34) by varying the value of γ .

6 A numerical example

In this section we provide a numerical example to illustrate the practicability of our previously established results. We implement the optimal solution to demonstrate how the hedging effectiveness is affected by the correlation among those hedging assets. We are also interested in the benefits to include multiple hedging assets compared to a hedging strategy based on a single hedging asset.

In order to express correlation between hedging assets in a more explicit manner, we rewrite their dynamics as follows:

$$\begin{aligned} dS_i(t) &= S_i(t) \left[\mu_i(t)dt + \sum_{j=0}^m \sigma_{ij}(t)dW^j(t) \right] \\ &= S_i(t) [\mu_i(t)dt + \bar{\sigma}_i(t)dW^{*i}(t)], \end{aligned}$$

where $W^{*0}(t) := W^0(t)$, $\bar{\sigma}_i(t) = \sqrt{\sum_{j=0}^m \sigma_{ij}^2(t)}$, and $W^{*i}(t) = \sum_{j=0}^m \frac{\sigma_{ij}(t)}{\bar{\sigma}_i(t)} W^j(t)$, $i = 1, \dots, m$, for $t \in [0, T]$. Obviously, $W^{*i}(t)$, $i = 0, 1, 2, \dots, m$, are standard Brownian motions with

$dW^{*i}dW^{*j} = \rho_{ij}dt$, where

$$\rho_{ij}(t) = \sum_{k=0}^m \frac{\sigma_{ik}(t)\sigma_{jk}(t)}{\bar{\sigma}_i(t)\bar{\sigma}_j(t)} = \frac{\sigma_i(t)^\top \sigma_j(t)}{\sqrt{\sigma_i(t)^\top \sigma_i(t) \cdot \sigma_j(t)^\top \sigma_j(t)}}, \quad 0 \leq i, j \leq m.$$

Given the correlation coefficients ρ_{ij} and volatilities $\bar{\sigma}_i$, we can recover the values of σ_{ij} by the above formula.

Suppose that our hedging objective is a short position in a European call option with a strike price K , i.e., the hedging target is a payoff $G = (S_0(T) - K)_+$. We consider the case with $m = 2$, i.e., there are two hedging assets (S_1 and S_2) available for hedging the payoff G . In our numerical example, we consider those parameter values for each asset as given in table 1.

T	$S_0(0)$	K	r	μ_0	σ_{00}	μ_1	$\bar{\sigma}_1$	μ_2	$\bar{\sigma}_2$	γ	x_0
1	100	100	0.01	0.05	0.3	0.08	0.32	0.10	0.35	1	20

Table 1: Parameter values

To demonstrate the benefits of including extra hedging asset in the hedging portfolio, we consider the following three hedging plans:

- (1) **Hedging Plan 1:** Include both $S_1(t)$ and $S_2(t)$ in the hedging portfolio;
- (2) **Hedging Plan 2:** Include $S_1(t)$ only in the hedging portfolio;
- (3) **Hedging Plan 3:** Include $S_2(t)$ only in the hedging portfolio.

For hedging plan 1, while sticking to those model parameter values as specified in the caption of table 2 and those in table 1, we consider four different values for the correlation coefficient ρ_{12} to compare the hedging effectiveness for different levels of correlation among hedging assets. Four specific values for ρ_{12} are tabulated in table 2, where the values of σ_{21} and σ_{22} are recovered from the given value of ρ_{12} . From table 2, we observe that the expected terminal profit decreases and the variance of terminal profit increases as the correlation ρ_{12} increases. This means that a weaker correlation among hedging asset prices is desirable for a better hedging effectiveness. This result is consistent to our intuition. In the extreme case, it is the same as using a single asset for hedging if the two hedging assets in the hedging program behave exactly the same, in which case, there is no difference between using both hedging assets and using only one of them in hedging program. This result is instructive for practical use. Given a hedging asset has been selected, ideally one should select an additional asset which has a weak correlation with the

selected asset while a strong correlation with the hedging target, in order to enjoy the benefits from multiple hedging assets.

Parameters			Result		
σ_{21}	σ_{22}	ρ_{12}	$E[V^{\theta^*}(T)]$	$\text{Var}[V^{\theta^*}(T)]$	$J_1(\theta^*)$
-0.1633	0.2274	0.2	9.8037	128.9357	-54.6642
-0.0467	0.2761	0.4	9.2113	192.3583	-86.9679
0.07	0.2711	0.6	8.7093	232.7463	-107.6639
0.1867	0.2087	0.8	8.1814	241.2175	-112.4274

Table 2: Hedging performance with hedging assets S_1 and S_2 : $\sigma_{00} = 0.3$, $\sigma_{10} = 0.256$, $\sigma_{11} = 0.192$, $\sigma_{20} = 0.21$, $\rho_{01} = 0.8$, and $\rho_{02} = 0.6$.

The results for hedging with a single asset (either S_1 or S_2) are reported in table 3, where the row indicated by S_1 displays results for hedging with asset S_1 only and the row indicated by S_2 gives results for hedging with S_2 only. It is interesting to make a comparison between tables 2 and 3 in order to get some insights about the benefits of including multiple hedging assets. Comparing these two tables, one can see that Hedging Plan 1 which includes two hedging assets constantly outperforms the other two hedging plans. It yields higher expected terminal profit and lower variance except for the case with $\rho_{12} = 0.8$, where Hedging Plan 2 has a slightly better expected terminal profit but it also leads to a higher variance. In terms of mean-variance objective, both Hedging Plans 1 and 2 are equivalent in the case with $\rho_{12} = 0.8$. When the correlation between the two hedging assets S_1 and S_2 are weaker (e.g., $\rho = 0.2$), both the expected terminal profit and variance of Hedging Plan 1 are much better than the other two plans. Finally, it is also interesting to note that the hedging effect from Hedging Plan 3 is worse than that from Hedging Plan 2 as a result of a weaker correlation of asset S_2 to the underlying asset S_0 of the hedging target.

Asset	Parameters						Result		
	σ_{10}	σ_{11}	ρ_{01}	σ_{20}	σ_{21}	ρ_{02}	$E[V^{\theta^*}(T)]$	$\text{Var}[V^{\theta^*}(T)]$	$J_1(\theta^*)$
S_1	0.256	0.192	0.8	N/A			8.3627	241.7791	-112.5269
S_2	N/A			0.21	0.28	0.6	7.9868	387.9492	-185.9878

Table 3: Hedging performance of portfolio using a single asset (either S_1 or S_2).

7 Conclusion

Basis risk widely arises in many risk management problems from the areas of insurance and finance. In this paper, the mean-variance optimal solution is established for a general hedging

problem in the presence of basis risk. The derived optimal solution is applicable for any European options as well as forward contracts. This paper makes important contributions into the literature in at least two aspects. First, the established optimal solution allows multiple hedging assets to be included in the corresponding hedging strategy, while most literature only consider a single asset in the hedging portfolio. Second, the literature mainly consider either an exponential utility maximization framework or a hedging target as simple as a forward contract for technical reason. The optimal solution established in the present paper is based on a mean-variance analysis, and it is implementable for any European options as long as its payoff is Malliavin differentiable and the Malliavin derivative is computable either analytically or numerically.

In the present paper, the drift and volatility coefficients in the dynamics of both the hedging assets and the underlying of the hedging target are time-dependent but deterministic. Therefore, an interesting question for future research is to extend our results into a setup with general diffusion or even jump-diffusion processes. If the stochastic LQ theory is continued to be applied for optimal solutions in the way as we did in the present paper, a more delicate method has to be developed to deal with the backward stochastic differential equation (BSDE), which imperatively arises in the procedure of applying the LQ theory, and to compute the Malliavin derivative of the solution to the BSDE, which does not have a form convenient for computation in general. Another interesting direction for future research is to apply the results in this paper for specific insurance and finance hedging problems with basis risk inherently present.

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