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TOPOLOGICAL STRUCTURE IN THE SPACE OF (WEIGHTED) COMPOSITION OPERATORS ON WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We consider the topological structure problem for the space of composition operators as well as the space of weighted composition operators on weighted Banach spaces with sup-norm. For the first space, we prove that the set of all composition operators that differ from the given one by a compact operator is path connected; however, in general, it is not always a component. Furthermore, we show that the set of compact weighted composition operators is path connected, but it is not a component in the second space.

1. Introduction

Let $H(\mathbb{D})$ be the space of all holomorphic functions on the unit disc $\mathbb{D}$ and $\mathcal{S}(\mathbb{D})$ the set of all holomorphic self-maps of $\mathbb{D}$. For two functions $\psi \in H(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$, a *weighted composition operator* $W_{\psi,\varphi}$ is defined by $W_{\psi,\varphi}f = \psi \cdot (f \circ \varphi)$, $f \in H(\mathbb{D})$. When the function ψ is identically 1, the operator $W_{\psi,\varphi}$ reduces to the *composition operator* C_φ. These operators C_φ and $W_{\psi,\varphi}$ have been studied extensively on various function spaces during the past few decades in many directions (see [9, 21] and references therein for an overview). Among them, the study of topological structure problem for spaces of bounded (weighted) composition operators endowed with the operator norm topology has gained a special interest. We refer the reader for more information to [10, 16, 22] (for Hardy spaces), [11, 15, 19] (for the space H^∞ of all bounded holomorphic functions on $\mathbb{D}$), [13] (for Bloch spaces), and [6] (for weighted Banach spaces with sup-norm). In this paper we continue the study of topological structure problem for both spaces of composition operators and weighted composition operators on weighted Banach spaces with sup-norm generated by some radial weight and

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establish several results that essentially complement or improve the previous ones in this direction.

Recall that a *radial weight* (briefly, a *weight*) on $\mathbb{D}$ is a positive function v on $\mathbb{D}$ with $v(z) = v(|z|), z \in \mathbb{D}$, where $v(r)$ is continuous and increasing on $[0, 1)$ and $v(r) \to \infty$ as $r \to 1^-$. For a weight v on $\mathbb{D}$, we define the following weighted Banach spaces of holomorphic functions on $\mathbb{D}$:

$$H_v(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} \frac{|f(z)|}{v(z)} < \infty \right\}$$

and

$$H_v^0(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} \frac{|f(z)|}{v(z)} = 0 \right\},$$

endowed with the norm $\|\cdot\|_v$, where, as usual, for a function $g : \mathbb{D} \to [0, \infty)$, $\lim\limits_{|z|\to 1^-} g(z) := \lim\limits_{r\to 1^-} \sup\limits_{|z|>r} g(z)$. These spaces are also known as weighted Bergman spaces of infinite order. We will use the common symbol X_v to denote either of the spaces $H_v(\mathbb{D})$ or $H_v^0(\mathbb{D})$ when some statement is true for both of them.

It is well known (see, for instance, [1, 2]) that to characterize topological properties of spaces X_v or linear operators between them in term of weights, the standard way is to use the so-called associated weights. According to [3, Definition 1.1], for a given weight v on $\mathbb{D}$, its *associated weight* is defined by

$$\widetilde{v}(z) = \sup\{|f(z)| : f \in H_v(\mathbb{D}), \|f\|_v \le 1\}.$$

Note that $\widetilde{v}(z) = \widetilde{v}(|z|)$ and $0 < \widetilde{v}(z) \le v(z)$ for all $z \in \mathbb{D}$, $\widetilde{v}(r)$ is increasing and *log-convex* on $[0, 1)$ (i.e. the function $\log \widetilde{v}(e^x)$ is convex on $(-\infty, 0)$), and $H_v(\mathbb{D}) = H_{\widetilde{v}}(\mathbb{D})$ isometrically. Moreover, in [14, Lemma 2.2] it was shown that for a log-convex weight v on $\mathbb{D}$, there is some constant M such that $\widetilde{v}(z) \le v(z) \le M\widetilde{v}(z), z \in \mathbb{D}$. Thus, for our main purposes it seems reasonable to use log-convex weights.

Next, for a log-convex weight v, by [4, Theorem 2.3] the following condition from [17, p. 310] and [18, Definition 2.1]

$$\limsup_{k\to\infty} \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} < \infty \tag{1.1}$$

is equivalent to the continuity of all compositions operators $C_\varphi, \varphi \in S(\mathbb{D})$, on X_v. Consequently, for log-convex weights satisfying (1.1) and only for them, the spaces $\mathcal{C}(H_v(\mathbb{D}))$ and $\mathcal{C}(H_v^0(\mathbb{D}))$ of all bounded composition operators on $H_v(\mathbb{D})$ and, respectively, $H_v^0(\mathbb{D})$ coincide and equal to the space $\{C_\varphi : \varphi \in S(\mathbb{D})\}$ of all composition operators. Note that some conditions of various types that are equivalent to (1.1) were stated in [1, Lemma 2.6]. In particular, (1.1) $\iff v(r) = O(v(r^2))$ as $r \to 1^-$.

In view of the above reasons it is natural to consider the topological structure problem for the spaces of (weighted) composition operators on spaces X_v given by log-convex weights v satisfying (1.1). Let $\mathcal{V}$ denote the set of all such weights. The standard weights $v_\alpha(z) = (1-|z|^2)^{-\alpha}$, $\alpha > 0$, belong to $\mathcal{V}$. Throughout the present paper we will always suppose that $v \in \mathcal{V}$.

The paper is organized as follows. In Section 2 we recall some auxiliary results on spaces X_v and (weighted) composition operators on them.

The topological structure of the space $\mathcal{C}(X_v)$ of all composition operators on X_v is studied in Section 3. We prove that the set $[C_\varphi]$ of all composition operators that differ from the given one C_φ by a compact operator is path connected in $\mathcal{C}(X_v)$ (Theorem 3.1). Nevertheless, a component in $\mathcal{C}(X_v)$ is not in general the set of such type (Example 3.4). Note that MacCluer, Ohno, and Zhao [19] showed these results for the space H^∞ and we now extend them to the family of all Bergman spaces of infinite order given by weights from $\mathcal{V}$ (see Remark 3.5). Moreover, in Proposition 3.6 we prove that the condition that completely characterizes isolated composition operators C_φ in the setting of the space H^∞ (see [19, Corollary 9] and [12, Theorem 4.1]) is necessary for C_φ to be isolated in all spaces $\mathcal{C}(X_v)$ with v in $\mathcal{V}$.

Section 4 is devoted to the space $\mathcal{C}_w(X_v)$ of all bounded **nonzero** weighted composition operators on X_v. It should be noted that Bonet, Lindström and Wolf [6] investigated the topological structure problem for the space $\mathcal{C}_w^0(X_v)$ of **all** bounded weighted composition operators on X_v. However, it is easy to see that every operator $W_{\psi,\varphi}$ in $\mathcal{C}_w^0(X_v)$ and the zero operator 0 are always connected by the path $W_{t\psi,\varphi}, t \in [0,1]$. This implies that the space $\mathcal{C}_w^0(X_v)$ is path connected. In view of this, some results in [6] should be considered again in the context of the space $\mathcal{C}_w(X_v)$. It is worth to emphasize that some arguments used in [6] cannot be applied to the space $\mathcal{C}_w(X_v)$. So we develop some new ideas to prove that the set $\mathcal{C}_{w,0}(X_v)$ of all nonzero compact weighted composition operators is path connected in $\mathcal{C}_w(X_v)$; unfortunately, it is not a path component (Theorem 4.2). We also give a simple sufficient condition to ensure that two operators in $\mathcal{C}_w(X_v)$ belong to the same path component of this space (Proposition 4.3). These results clarify and improve the corresponding ones in [6, Theorems 3.2 and 4.2]. Moreover, we describe two path connected sets of the same type in $\mathcal{C}_w(X_v)$, one of which is a path component, while another is not (Examples 4.6 and 4.8).

2. Preliminaries

In this section we collect some notation and results concerning properties of functions in the spaces X_v and (weighted) composition operators and their differences on these spaces.

The pseudo-hyperbolic distance between z and ζ in $\mathbb{D}$ is defined by

$$\rho(z,\zeta)=\left|\frac{z-\zeta}{1-\overline{z}\zeta}\right|.$$

For a function $\varphi\in H(\mathbb{D})$, put

$$\|\varphi\|_\infty=\sup_{z\in\mathbb{D}}|\varphi(z)| \text{ and } M(\varphi,r)=\sup_{|z|\le r}|\varphi(z)|,\ r\in(0,1).$$

For a subset E of the unit circle $\partial\mathbb{D}$, we denote by $|E|$ the Lebesgue measure of E on $\partial\mathbb{D}$.

Lemma 2.1. *There is a constant $C>0$, dependent only on v, such that for every $f\in H_v(\mathbb{D})$ and $z,\zeta\in\mathbb{D}$,*

$$|f'(z)|\le C\frac{v(z)}{1-|z|}\|f\|_v, \tag{2.1}$$

and

$$|f(z)-f(\zeta)|\le C\|f\|_v\rho(z,\zeta)\max\{v(z),v(\zeta)\}. \tag{2.2}$$

Proof. In [1, Theorem 2.8] it was proved that for every weight $v\in\mathcal{V}$, the differentiation operator D is bounded from $H_v(\mathbb{D})$ to $H_{v_1}(\mathbb{D})$ with $v_1(r)=v(r)/(1-r)$, which implies (2.1).

The inequality (2.2) was obtained in [5, Lemma 1].

□

The next result follows from [4, Proposition 2.1 and Theorems 2.3 and 3.3].

Proposition 2.2. *Let $\varphi\in\mathcal{S}(\mathbb{D})$.*

(a) *The operator C_φ is bounded on X_v. Moreover,*

$$\|C_\varphi\|\le\sup_{z\in\mathbb{D}}\frac{v(\varphi(z))}{v(z)}<\infty.$$

(b) *The operator C_φ is compact on X_v if and only if*

$$\lim_{|z|\to1^-}\frac{v(\varphi(z))}{v(z)}=0.$$

This proposition implies that

$$\mathcal{C}(H_v(\mathbb{D}))=\mathcal{C}(H_v^0(\mathbb{D}))=\{C_\varphi:\varphi\in\mathcal{S}(\mathbb{D})\},$$

and the sets $\mathcal{C}_0(H_v(\mathbb{D}))$ and $\mathcal{C}_0(H_v^0(\mathbb{D}))$ of all compact composition operators on $H_v(\mathbb{D})$ and $H_v^0(\mathbb{D})$, respectively, coincide; more precisely

$$\mathcal{C}_0(H_v(\mathbb{D}))=\mathcal{C}_0(H_v^0(\mathbb{D}))=\{C_\varphi:\varphi\in\mathcal{S}(\mathbb{D}),v(\varphi(z))=o(v(z)),|z|\to1^-\}.$$

Thus, all results and arguments in Section 3 will be stated simultaneously for both spaces $\mathcal{C}(H_v(\mathbb{D}))$ and $\mathcal{C}(H_v^0(\mathbb{D}))$.

Compactness of differences of two composition operators between weighted Banach spaces with sup-norm was characterized in [5, Corollary 7 and Theorem 9]. To state these results for composition operators from X_v into itself with $v \in \mathcal{V}$, we need the following observation. Note that we will also use it in Section 4.

Remark 2.3. Let $\varphi \in \mathcal{S}(\mathbb{D})$ and $g : \mathbb{D} \to [0, \infty)$. As usual, we put

$$\lim_{|\varphi(z)| \to 1^-} g(z) := \begin{cases} \lim\limits_{r \to 1^-} \sup\limits_{|\varphi(z)| > r} g(z) & \text{if } \|\varphi\|_\infty = 1 \\ 0 & \text{if } \|\varphi\|_\infty < 1. \end{cases}$$

Then

$$\lim_{|z| \to 1^-} g(z) = 0 \ \text{ implies that } \lim_{|\varphi(z)| \to 1^-} g(z) = 0. \tag{2.3}$$

Indeed, it is enough to check this statement for φ with $\|\varphi\|_\infty = 1$. Given $r \in (0, 1)$, letting $\widetilde{r} := M(\varphi, r)$, we get that

$$\sup_{|\varphi(z)| > \widetilde{r}} g(z) \leq \sup_{|z| > r} g(z) \ \text{ and } \ \widetilde{r} \to 1^- \ \text{ as } r \to 1^-,$$

which implies (2.3).

Proposition 2.4. *Let $\varphi, \phi \in \mathcal{S}(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The difference $C_\varphi - C_\phi$ is compact on $H_v(\mathbb{D})$.*
(ii) *The difference $C_\varphi - C_\phi$ is compact on $H_v^0(\mathbb{D})$.*
(iii)

$$\lim_{|z| \to 1^-} \frac{v(\varphi(z))}{v(z)} \rho(\varphi(z), \phi(z)) = \lim_{|z| \to 1^-} \frac{v(\phi(z))}{v(z)} \rho(\varphi(z), \phi(z)) = 0.$$

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii). Suppose that $C_\varphi - C_\phi$ is compact on $H_v^0(\mathbb{D})$. If $\|\varphi\|_\infty = \|\phi\|_\infty = 1$, then the assertion follows from [5, Theorem 9]. If $\|\varphi\|_\infty < 1$ (similarly to the case $\|\phi\|_\infty < 1$), then C_φ is compact on $H_v^0(\mathbb{D})$. Hence, C_ϕ is also compact on $H_v^0(\mathbb{D})$, which and Proposition 2.2(b) imply that

$$\lim_{|z| \to 1^-} \frac{v(\varphi(z))}{v(z)} = \lim_{|z| \to 1^-} \frac{v(\phi(z))}{v(z)} = 0.$$

Using this and the fact that $\rho(\varphi(z), \phi(z)) \leq 1$, $z \in \mathbb{D}$, we get (iii).

(iii) $\Longrightarrow$ (i). By Remark 2.3, (iii) implies that

$$\lim_{|\varphi(z)| \to 1^-} \frac{v(\varphi(z))}{v(z)} \rho(\varphi(z), \phi(z)) = \lim_{|\phi(z)| \to 1^-} \frac{v(\phi(z))}{v(z)} \rho(\varphi(z), \phi(z)) = 0.$$

It remains to use [5, Corollary 7] to obtain (i). □

Boundedness and compactness of the operator $W_{\psi,\varphi}$ between weighted Banach spaces with sup-norm were characterized in [8, Propositions 3.1 and 3.2, Corollaries 4.3 and 4.5], from which we get the following result.

Proposition 2.5. *Let $\varphi \in S(\mathbb{D})$ and $\psi \in H(\mathbb{D})$. Then the next two assertions hold:*

(a) *The operator* $W_{\psi,\varphi} : X_v \to X_v$ *is bounded if and only if* $\psi \in X_v$ *and*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(\varphi(z))}{v(z)} < \infty.$$

(b) *The operator* $W_{\psi,\varphi} : X_v \to X_v$ *is compact if and only if* $\psi \in X_v$ *and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{|\psi(z)|v(\varphi(z))}{v(z)} = 0 \text{ for } X_v = H_v(\mathbb{D})$$

or

$$\lim_{|z|\to 1^-} \frac{|\psi(z)|v(\varphi(z))}{v(z)} = 0 \text{ for } X_v = H_v^0(\mathbb{D}).$$

From this proposition it follows that $\mathcal{C}_w(H_v^0(\mathbb{D}))$ and $\mathcal{C}_{w,0}(H_v^0(\mathbb{D}))$ are proper subsets of $\mathcal{C}_w(H_v(\mathbb{D}))$ and, respectively, $\mathcal{C}_{w,0}(H_v(\mathbb{D}))$. In view of this, some arguments in Section 4 will be presented separately for spaces $\mathcal{C}_w(H_v(\mathbb{D}))$ and $\mathcal{C}_w(H_v^0(\mathbb{D}))$.

We also need the following lemma.

Lemma 2.6. *Let* $[z,\zeta]$ *denote the closed interval connecting points* z *and* ζ *in* $\mathbb{D}$. *Then* $\rho(\xi,\eta) \le \rho(z,\zeta)$ *for all* $\xi,\eta \in [z,\zeta]$.

Proof. Without loss of generality we may assume that the points lie in the interval in the following order: $z \to \xi \to \eta \to \zeta$. We have the next obvious relations:

$$|\xi - \eta| = |z - \zeta| - (|z - \xi| + |\zeta - \eta|),$$

$$\begin{aligned} |1 - \overline{\xi}\eta| &\ge |1 - \overline{z}\zeta| - |\overline{z}\zeta - \overline{\xi}\eta| \\ &\ge |1 - \overline{z}\zeta| - (|\zeta||\overline{z} - \overline{\xi}| + |\overline{\xi}||\zeta - \eta|) \\ &\ge |1 - \overline{z}\zeta| - (|z - \xi| + |\zeta - \eta|), \end{aligned}$$

and

$$|z - \zeta| < |1 - \overline{z}\zeta|.$$

Then

$$\begin{aligned} \rho(\xi,\eta) = \frac{|\xi - \eta|}{|1 - \overline{\xi}\eta|} &\le \frac{|z - \zeta| - (|z - \xi| + |\zeta - \eta|)}{|1 - \overline{z}\zeta| - (|z - \xi| + |\zeta - \eta|)} \\ &\le \frac{|z - \zeta|}{|1 - \overline{z}\zeta|} = \rho(z,\zeta). \end{aligned}$$

□

3. The space of composition operators

In this section we consider the topological structure problem for the space $\mathcal{C}(X_v)$ of all composition operators on X_v under the operator norm topology. To simplify our considerations, we will write $C_\varphi \sim C_\phi$ in $\mathcal{C}(X_v)$ if these operators are in the same path component of $\mathcal{C}(X_v)$. Two composition operators C_φ and C_ϕ are said to be *compactly* *equivalent* (or, briefly, *equivalent*) in $\mathcal{C}(X_v)$ if their difference $C_\varphi - C_\phi$ is compact on X_v. Obviously, this relation is an equivalence one. Denote by $[C_\varphi]$ the equivalence class of all composition operators that are equivalent to the given operator C_φ. Note that the set $\mathcal{C}_0(X_v)$ of all compact composition operators on X_v coincide with the class $[C_0]$ of all operators from $\mathcal{C}(X_v)$ that are equivalent to the operator $C_0 : f \mapsto f(0)$.

Theorem 3.1. *Each equivalence class $[C_\varphi]$ is path connected in the space $\mathcal{C}(X_v)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{D})$ and C_ϕ be an arbitrary operator from $[C_\varphi]$. Then $C_\varphi - C_\phi$ is compact on X_v and, by Proposition 2.4, we have

$$\lim_{|z|\to 1^-} \frac{v(\varphi(z))}{v(z)}\rho(\varphi(z),\phi(z)) = \lim_{|z|\to 1^-} \frac{v(\phi(z))}{v(z)}\rho(\varphi(z),\phi(z)) = 0. \tag{3.1}$$

For each $t \in [0,1]$, put $\varphi_t(z) = (1-t)\varphi(z) + t\phi(z), z \in \mathbb{D}$. Clearly, $\varphi_t \in \mathcal{S}(\mathbb{D})$ for all $t \in [0,1]$ and, by Proposition 2.2(a), the corresponding operators $C_{\varphi_t}, t \in [0,1]$, are bounded on X_v. Moreover, all differences $C_\varphi - C_{\varphi_t}$ are compact on X_v. Indeed, $|\varphi_t(z)| \le \max\{|\varphi(z)|, |\phi(z)|\}$ and, hence, $v(\varphi_t(z)) \le \max\{v(\varphi(z)), v(\phi(z))\}$ for all $z \in \mathbb{D}$ and $t \in [0,1]$. Next, by Lemma 2.6, $\rho(\varphi(z),\varphi_t(z)) \le \rho(\varphi(z),\phi(z))$. From the above inequalities and (3.1) it follows that

$$\lim_{|z|\to 1^-} \frac{v(\varphi(z))}{v(z)}\rho(\varphi(z),\varphi_t(z)) \le \lim_{|z|\to 1^-} \frac{v(\varphi(z))}{v(z)}\rho(\varphi(z),\phi(z)) = 0,$$

and

$$\begin{aligned} &\lim_{|z|\to 1^-} \frac{v(\varphi_t(z))}{v(z)}\rho(\varphi(z),\varphi_t(z)) \\ &\le \lim_{|z|\to 1^-} \max\left\{\frac{v(\varphi(z))}{v(z)}\rho(\varphi(z),\phi(z)), \frac{v(\phi(z))}{v(z)}\rho(\varphi(z),\phi(z))\right\} = 0. \end{aligned}$$

Using Proposition 2.4 once again, we conclude that $C_\varphi - C_{\varphi_t}$ is compact on X_v for every $t \in [0,1]$.

Thus, $C_{\varphi_t} \in [C_\varphi]$ for all $t \in [0,1]$ and, to finish the proof, it remains to show that the map

$$[0,1] \to \mathcal{C}(X_v), t \mapsto C_{\varphi_t},$$

is continuous on $[0,1]$. That is, $\|C_{\varphi_s} - C_{\varphi_t}\| \to 0$ as $s \to t$ for all $t \in [0,1]$.

Fix a number $t \in [0,1]$. For every $r \in (0,1)$, $s \in [0,1]$, and $f \in X_v$, by (2.2) we get

$$\begin{aligned}\|C_{\varphi_s} f - C_{\varphi_t} f\|_v &= \sup_{z\in\mathbb{D}} \frac{|f(\varphi_s(z)) - f(\varphi_t(z))|}{v(z)} \\ &\le C\|f\|_v \sup_{z\in\mathbb{D}} \rho(\varphi_s(z),\varphi_t(z)) \frac{\max\{v(\varphi_s(z)), v(\varphi_t(z))\}}{v(z)} \\ &\le C\|f\|_v \sup_{z\in\mathbb{D}} \rho(\varphi_s(z),\varphi_t(z)) \frac{\max\{v(\varphi(z)), v(\phi(z))\}}{v(z)}.\end{aligned}$$

Consequently, for every $r \in (0,1)$ and $s \in [0,1]$,

$$\|C_{\varphi_s} - C_{\varphi_t}\| \le C \max\{\mathcal{I}(r,s), \mathcal{J}(r,s)\},$$

where

$$\begin{aligned}\mathcal{I}(r,s) &:= \sup_{|z|\le r} \rho(\varphi_s(z),\varphi_t(z)) \frac{\max\{v(\varphi(z)), v(\phi(z))\}}{v(z)} \\ &\le |s-t| \, \frac{v(M_r)}{v(0)} \, \sup_{|z|\le r} \frac{|\varphi(z)-\phi(z)|}{(1-|\varphi_s(z)||\varphi_t(z)|)} \\ &\le \frac{2v(M_r)}{v(0)\big(1-M_r^2\big)}|s-t| \quad \text{with} \quad M_r := \max\{M(\varphi,r), M(\phi,r)\},\end{aligned}$$

and, by Lemma 2.6,

$$\begin{aligned}\mathcal{J}(r,s) &:= \sup_{|z|>r} \rho(\varphi_s(z),\varphi_t(z)) \frac{\max\{v(\varphi(z)), v(\phi(z))\}}{v(z)} \\ &\le \sup_{|z|>r} \rho(\varphi(z),\phi(z)) \frac{\max\{v(\varphi(z)), v(\phi(z))\}}{v(z)} =: J(r).\end{aligned}$$

Therefore, for every $r \in (0,1)$ and $s \in [0,1]$,

$$\|C_{\varphi_s} - C_{\varphi_t}\| \le C \max\left\{ \frac{2v(M_r)}{v(0)\big(1-M_r^2\big)}|s-t|, J(r) \right\}.$$

By letting $s \to t$, this implies that

$$\limsup_{s\to t} \|C_{\varphi_s} - C_{\varphi_t}\| \le CJ(r) \quad \text{for all } r \in (0,1).$$

Next, letting $r \to 1^-$ and using that $J(r) \to 0$ as $r \to 1^-$ by (3.1), we then get

$$\lim_{s\to t} \|C_{\varphi_s} - C_{\varphi_t}\| = 0,$$

which completes the proof. □

Corollary 3.2. *The set $\mathcal{C}_0(X_v)$ of all compact composition operators on X_v is path connected in $\mathcal{C}(X_v)$.*

Remark 3.3. Corollary 3.2 follows also from [6, Theorem 3.2]. Indeed, putting $\psi^{(1)} \equiv 1$ and $\psi^{(2)} \equiv 1$ in the proof of this theorem we get that two compact composition operators $C_{\phi^{(1)}}$ and $C_{\phi^{(2)}}$ on X_v belong to the same path component of the space $\mathcal{C}(X_v)$.

Now we use the main idea from [19, Example 1] to show that in general $[C_\varphi]$ might not be a whole path component of $\mathcal{C}(X_v)$.

Example 3.4. Let $\varphi_0(z) = 1 + a(z-1)$ with $0 < a < 1$. Then the class $[C_{\varphi_0}]$ is not a path component in $\mathcal{C}(X_v)$.

Proof. Let $\delta = \frac{a(1-a)}{4}$. For each $t \in [-\delta, \delta]$, we put

$$\varphi_t(z) = \varphi_0(z) + t(z-1)^2$$

and show that, for every $0 < |t| \le \delta$, $C_{\varphi_t} \sim C_{\varphi_0}$ in $\mathcal{C}(X_v)$ but $C_{\varphi_t} \notin [C_{\varphi_0}]$. Evidently, this gives the result.

Since $1 - |\varphi_0(z)|^2 \ge a(1-a)|z-1|^2$ for all $z \in \mathbb{D}$,

$$\begin{aligned} |\varphi_t(z)| &\le |\varphi_0(z)| + |t||z-1|^2 \le \sqrt{1 - a(1-a)|z-1|^2} + |t||z-1|^2 \\ &\le 1 - \frac{a(1-a)}{2}|z-1|^2 + |t||z-1|^2 < 1, \end{aligned}$$

for all $z \in \mathbb{D}$ and $t \in [-\delta, \delta]$. Hence, $\varphi_t \in \mathcal{S}(\mathbb{D})$, and, by Proposition 2.2(a), $C_{\varphi_t} \in \mathcal{C}(X_v)$ for all $t \in [-\delta, \delta]$.

Next, similarly to [19, Example 1], consider a sequence $(z_n) \subset \mathbb{D}$ such that $z_n \to 1$ along the arc $|1-z|^2 = 1 - |z|^2$. For each $n \in \mathbb{N}$, we have

$$1 - |\varphi_0(z_n)|^2 = a(2-a)|z_n - 1|^2 = a(2-a)(1 - |z_n|^2).$$

Hence,

$$|\varphi_0(z_n)| = \sqrt{(1-a)^2 + a(2-a)|z_n|^2} \ge |z_n|,$$

and

$$\begin{aligned} \rho(\varphi_0(z_n), \varphi_t(z_n)) &\ge \frac{|t||z_n-1|^2}{1 - |\varphi_0(z_n)|^2 + |\varphi_0(z_n)||t||z_n-1|^2} \\ &\ge \frac{|t|}{a(2-a) + |t|}. \end{aligned}$$

Thus, for all $n \ge 1$ and $t \in [-\delta, \delta]$,

$$\frac{v(\varphi_0(z_n))}{v(z_n)} \rho(\varphi_0(z_n), \varphi_t(z_n)) \ge \frac{|t|}{a(2-a) + |t|}.$$

From this and Proposition 2.4 it follows that $C_{\varphi_t} - C_{\varphi_0}$ is not compact on X_v and, consequently, $C_{\varphi_t} \notin [C_{\varphi_0}]$ for all $0 < |t| \le \delta$.

To complete the proof, it is enough to check that the path $C_{\varphi_t}, t \in [-\delta, \delta]$, is continuous in $\mathcal{C}(X_v)$. For every $t, s \in [-\delta, \delta]$ and $f \in X_v$, using (2.2), we have

$$\|C_{\varphi_s}f - C_{\varphi_t}f\|_v = \sup_{z\in\mathbb{D}} \frac{|f(\varphi_s(z)) - f(\varphi_t(z))|}{v(z)}$$
$$\leq C\|f\|_v \sup_{z\in\mathbb{D}} \rho(\varphi_s(z),\varphi_t(z)) \frac{\max\{v(\varphi_s(z)), v(\varphi_t(z))\}}{v(z)}.$$

Therefore,

$$\|C_{\varphi_s} - C_{\varphi_t}\| \leq C \sup_{z\in\mathbb{D}} \rho(\varphi_s(z),\varphi_t(z)) \frac{\max\{v(\varphi_s(z)), v(\varphi_t(z))\}}{v(z)}. \tag{3.2}$$

Moreover, for every $t, s \in [-\delta,\delta]$ and $z \in \mathbb{D}$,

$$\begin{aligned}
\rho(\varphi_s(z),\varphi_t(z)) &= \frac{|t-s||z-1|^2}{|1-\overline{\varphi_s(z)}\varphi_t(z)|} \\
&\leq \frac{|t-s||z-1|^2}{1-|\varphi_0(z)|^2 - (|t|+|s|)|\varphi_0(z)||z-1|^2 - |ts||z-1|^4} \\
&\leq \frac{|t-s||z-1|^2}{a(1-a)|z-1|^2 - (|t|+|s|)|z-1|^2 - |ts||z-1|^4} \\
&\leq \frac{|t-s|}{a(1-a)-2\delta-4\delta^2} \leq \frac{4|t-s|}{a(1-a)}. \qquad (3.3)
\end{aligned}$$

Next, for each $s \in [-\delta,\delta]$, we put

$$a_s = \varphi_s(0) = 1-a+s, \beta_s(z) = \frac{z-a_s}{1-a_s z}, \text{ and } \phi_s = \beta_s \circ \varphi_s.$$

Then $\phi_s(0) = \beta_s \circ \varphi_s(0) = 0$. Hence, by the Schwarz lemma, $|\phi_s(z)| \leq |z|$ for every $z \in \mathbb{D}$. From this it follows that, for every $r \in (0,1)$,

$$\sup_{|z|\leq r} |\varphi_s(z)| = \sup_{|z|\leq r} |(\beta_s^{-1}\circ\phi_s)(z)| \leq \sup_{|z|\leq r} |\beta_s^{-1}(z)| = \frac{r+|a_s|}{1+r|a_s|} \leq \frac{r+r_0}{1+rr_0},$$

where $r_0 = 1-a+\delta \in (0,1)$. This, (3.2), and (3.3) imply that

$$\|C_{\varphi_s} - C_{\varphi_t}\| \leq \frac{4C}{a(1-a)}|t-s| \sup_{r\in[0,1)} \frac{v\left(\frac{r+r_0}{1+rr_0}\right)}{v(r)}$$

and it remains to check that the last supremum is finite.

By [1, Lemma 2.6](i), there is some constant $M > 0$, dependent only on v, such that

$$(\log v)'(r) = \frac{v'(r)}{v(r)} \leq \frac{M}{1-r} \quad \text{for all } r \in (0,1).$$

Then, using the arguments in the proof of [1, Theorem 2.8, (i)$\Longrightarrow$(vii)], we get

$$\begin{aligned}\log v\left(\frac{r+r_0}{1+rr_0}\right)-\log v(r) &\le (\log v)'\left(\frac{r+r_0}{1+rr_0}\right)\frac{r+r_0}{1+rr_0}\log\frac{r+r_0}{(1+rr_0)r}\\ &\le \frac{M(r+r_0)}{(1-r)(1-r_0)}\log\left(1+\frac{r_0(1-r^2)}{(1+rr_0)r}\right)\\ &\le \frac{M(r+r_0)r_0(1+r)}{(1-r_0)(1+rr_0)r}\le\frac{8M}{1-r_0},\end{aligned}$$

for every $r\in[\frac{1}{2},1)$. Thus, there is some number $M_0>1$, dependent only on v and r_0, such that

$$v\left(\frac{r+r_0}{1+rr_0}\right)\le M_0 v(r)\quad\text{for all } r\in[0,1).$$

Consequently,

$$\sup_{r\in[0,1)}\frac{v\left(\frac{r+r_0}{1+rr_0}\right)}{v(r)}\le M_0,$$

which completes the proof. $\square$

Remark 3.5. Note that to characterize components in the space of composition operators on Hardy space H^2, Shapiro and Sundberg [22] conjectured that the set of all composition operators that differ from the given one by a compact operator forms a component. Later, Bourdon [7], Moorhouse and Toews [20] independently showed that this conjecture is false. In [19] MacCluer, Ohno and Zhao also gave a negative answer to this conjecture for the setting of space H^∞. In fact, in Theorem 3.1 we proved that the sets of such a type are path connected in the space $\mathcal{C}(X_v)$. Nevertheless, Example 3.4 shows that, in general, they are not components of $\mathcal{C}(X_v)$. Therefore, Shapiro and Sundberg's conjecture is also not true for all spaces X_v given by weights v from the class $\mathcal{V}$.

We end this section with a result concerning isolated points in the spaces $\mathcal{C}(X_v)$. Recall that the result in [6, Theorem 5.7] can be reformulated as follows: If the set

$$E(v,\varphi)=\left\{\omega\in\partial\mathbb{D}\mid\ \exists(z_n)\subset\mathbb{D}:\lim_{n\to\infty}z_n=\omega\text{ and }\lim_{n\to\infty}\frac{v(\varphi(z_n))}{v(z_n)}>0\right\}$$

has Lebesgue measure strictly positive, then C_φ is isolated in $\mathcal{C}(X_v)$. This is an analog of [19, Corollary 8]. On the other hand, in [19, Corollary 9] it was established that if

$$\int_0^{2\pi}\log(1-|\varphi(e^{i\theta})|)d\theta>-\infty,\tag{3.4}$$

then C_φ is not isolated in $\mathcal{C}(H^\infty)$. Equivalently, the condition

$$\int_0^{2\pi} \log(1-|\varphi(e^{i\theta})|)d\theta = -\infty \tag{3.5}$$

is necessary for the operator C_φ to be isolated in $\mathcal{C}(H^\infty)$. In the next proposition we extend this result to all weighted spaces X_v with $v \in \mathcal{V}$. Note that Hosokawa, Izuchi, and Zheng [12, Theorem 4.1] proved that (3.5) gives the complete description of isolated operators C_φ in $\mathcal{C}(H^\infty)$.

Proposition 3.6. *If $\varphi \in \mathcal{S}(\mathbb{D})$ satisfies* (3.4), *then the operator C_φ is not isolated in $\mathcal{C}(X_v)$.*

Proof. Following [19, Corollary 9], consider the next bounded outer function in $\mathbb{D}$:

$$\phi(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1-|\varphi(e^{i\theta})|)d\theta\right), \ z \in \mathbb{D}.$$

As is known, $|\phi| \le 1-|\varphi|$ in $\mathbb{D}$ and $|\phi| = 1-|\varphi|$ almost everywhere on $\partial\mathbb{D}$. This implies, in particular, that the functions $\varphi_t(z) = \varphi(z)+t\phi(z)$ are in $\mathcal{S}(\mathbb{D})$ for every $|t|<1$. Hence, by Proposition 2.2(a), all operators C_{φ_t}, $|t|<1$, belong to $\mathcal{C}(X_v)$.

We will show that the path C_{φ_t}, $|t| \le \frac{1}{4}$, is continuous in $\mathcal{C}(X_v)$ and, consequently, C_φ is not isolated.

By the proof of (i) $\Longrightarrow$ (vii) in [1, Theorem 2.8], there exists a constant $M>0$ such that

$$v\left(\frac{1+r}{2}\right) \le Mv(r) \ \text{ for all } r \in [0,1).$$

From this it follows that, for each $|t| \le \frac{1}{2}$ and all $z \in \mathbb{D}$,

$$\begin{aligned} v(\varphi_t(z)) &= v(|\varphi(z)+t\phi(z)|) \le v(|\varphi(z)|+|t||\phi(z)|) \\ &\le v(|\varphi(z)|+|t|(1-|\varphi(z)|)) \le v\left(\frac{1+|\varphi(z)|}{2}\right) \le Mv(\varphi(z)). \end{aligned}$$

Using this and (2.2), we get that, for each $f \in X_v$ and $s,t \in [-\frac{1}{2},\frac{1}{2}]$,

$$\begin{aligned} \|C_{\varphi_t}f - C_{\varphi_s}f\|_v &= \sup_{z\in\mathbb{D}} \frac{|f(\varphi_t(z)) - f(\varphi_s(z))|}{v(z)} \\ &\le C\|f\|_v \sup_{z\in\mathbb{D}} \rho(\varphi_t(z),\varphi_s(z))\frac{\max\{v(\varphi_t(z)),v(\varphi_s(z))\}}{v(z)} \\ &\le CM\|f\|_v \sup_{z\in\mathbb{D}} \rho(\varphi_t(z),\varphi_s(z))\frac{v(\varphi(z))}{v(z)} \\ &\le M_0\|f\|_v \sup_{z\in\mathbb{D}} \rho(\varphi_t(z),\varphi_s(z)), \end{aligned}$$

where $M_0 = CM \sup_{z\in\mathbb{D}} \dfrac{v(\varphi(z))}{v(z)} < \infty$ by Proposition 2.2(a). Thus, for every $s, t \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\|C_{\varphi_t} - C_{\varphi_s}\| \leq M_0 \sup_{z\in\mathbb{D}} \rho(\varphi_t(z), \varphi_s(z)).$$

Next, for each $s, t \in [-\frac{1}{4}, \frac{1}{4}]$ and $z \in \mathbb{D}$,

$$\begin{aligned}
\rho(\varphi_t(z), \varphi_s(z)) &= \left| \frac{\varphi_t(z) - \varphi_s(z)}{1 - \overline{\varphi_t(z)}\varphi_s(z)} \right| \\
&\leq |t-s| \frac{|\phi(z)|}{1 - |\varphi(z)|^2 - (|t|+|s|)|\varphi(z)||\phi(z)| - |ts||\phi(z)|^2} \\
&= |t-s| \frac{1}{\frac{1-|\varphi(z)|^2}{|\phi(z)|} - (|t|+|s|)|\varphi(z)| - |ts||\phi(z)|} \\
&\leq |t-s| \frac{1}{\frac{1-|\varphi(z)|}{|\phi(z)|} - (|t|+|s|) - |ts|} \\
&\leq |t-s| \frac{1}{1 - (|t|+|s|) - |ts|} \leq \frac{16}{7}|t-s|.
\end{aligned}$$

Consequently,

$$\|C_{\varphi_t} - C_{\varphi_s}\| \leq \frac{16}{7} M_0 |t-s| \quad \text{for all } s, t \in \Big[-\frac{1}{4}, \frac{1}{4}\Big],$$

which implies that $C_{\varphi_t}, t \in [-\frac{1}{4}, \frac{1}{4}]$, is a continuous path in $\mathcal{C}(X_v)$. $\square$

4. The space of weighted composition operators

In this section we study the topological structure of the space $\mathcal{C}_w(X_v)$ of all nonzero bounded weighted composition operators on X_v under the operator norm topology. For simplicity, we write $W_{\psi,\varphi} \sim W_{\chi,\phi}$ in $\mathcal{C}_w(X_v)$ if the operators $W_{\psi,\varphi}$ and $W_{\chi,\phi}$ are in the same path component of $\mathcal{C}_w(X_v)$.

As it was pointed out in Introduction, the space $C_w^0(X_v)$ of all bounded weighted composition operators is always path connected. In view of this, Theorem 3.2 in [6] should be revised for the setting of nonzero weighted composition operators. But on this way we cannot use some arguments from the proof of this theorem in [6]. More precisely, to prove that two compact operators $W_{\psi^{(1)},\phi^{(1)}}$ and $W_{\psi^{(2)},\phi^{(2)}}$ in $C_w^0(H_v^0(\mathbb{D}))$ are path connected, the authors in [6] showed that

$$W_{\psi^{(1)},\phi^{(1)}} \sim W_{\psi^{(1)}(0),\phi^{(1)}(0)} \sim W_{\psi^{(2)}(0),\phi^{(2)}(0)} \sim W_{\psi^{(2)},\phi^{(2)}} \text{ in } C_w^0(H_v^0(\mathbb{D})),$$

which cannot be applied to the space $\mathcal{C}_w(H_v^0(\mathbb{D}))$ when $\psi^{(1)}(0) = 0$ or $\psi^{(2)}(0) = 0$. From this reason, we develop some new ideas to establish this result for the spaces $\mathcal{C}_w(X_v)$. Moreover, we prove a bit more by showing that the set of all nonzero compact weighted composition operators on X_v is not a path component of $\mathcal{C}_w(X_v)$ for all $v \in \mathcal{V}$.

In our further considerations we will use the next simple result, which is proved similarly to [23, Lemma 4.8].

Lemma 4.1. *Every operator $W_{\psi,\varphi} \in \mathcal{C}_w(X_v)$ is path connected with the operator C_φ in $\mathcal{C}_w(X_v)$.*

Theorem 4.2. *The set $\mathcal{C}_{w,0}(X_v)$ of all nonzero compact weighted composition operators on X_v is path connected in the space $\mathcal{C}_w(X_v)$; but it is not a path component in this space.*

Proof. (a) To prove that the set $\mathcal{C}_{w,0}(X_v)$ is path connected in the space $\mathcal{C}_w(X_v)$, it suffices to show that every operator $W_{\psi,\varphi}$ in $\mathcal{C}_{w,0}(X_v)$ and the operator C_0 belong to the same path component of $\mathcal{C}_w(X_v)$ via a path in $\mathcal{C}_{w,0}(X_v)$.

If $\psi(z) \equiv const$, then the assertion follows from Lemma 4.1 and Corollary 3.2.

Now suppose that $\psi \in X_v$ is non-constant. We put

$$\psi_t(z) = 1 - t + t\psi(z) \text{ and } \varphi_t(z) = t\varphi(z), z \in \mathbb{D}, t \in [0,1].$$

Then, for every $t \in [0,1)$, ψ_t is a nonzero function in X_v and $\overline{\varphi_t(\mathbb{D})} \subset \overline{t\varphi(\mathbb{D})} \subset \mathbb{D}$. From this and Proposition 2.5(b) it follows that all operators $W_{\psi_t,\varphi_t}, t \in [0,1)$, are compact on X_v. Hence, $W_{\psi_t,\varphi_t} \in \mathcal{C}_{w,0}(X_v)$ for all $t \in [0,1]$; moreover, $W_{\psi_0,\varphi_0} = C_0$ and $W_{\psi_1,\varphi_1} = W_{\psi,\varphi}$. We claim that the map

$$[0,1] \to \mathcal{C}_w(X_v), t \mapsto W_{\psi_t,\varphi_t},$$

is continuous on $[0,1]$. Then $W_{\psi,\varphi} \sim C_0$ in $\mathcal{C}_w(X_v)$ via a path W_{ψ_t,φ_t} in $\mathcal{C}_{w,0}(X_v)$.

It remains to prove the claim. Obviously, $W_{\psi_t,\varphi_t} = (1-t)C_{t\varphi} + W_{t\psi,t\varphi}$, and hence,

$$\|W_{\psi_s,\varphi_s} - W_{\psi_t,\varphi_t}\| \le \|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\| + \|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\|,$$

for every $t, s \in [0,1]$. Consequently, to prove the claim, it is enough to show that for every $t \in [0,1]$,

(i) $\lim\limits_{s\to t} \|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\| = 0$ and **(ii)** $\lim\limits_{s\to t} \|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\| = 0$.

In our further demonstration we will use the next obvious inequality for functions $f \in H(\mathbb{D})$:

$$|f(sz) - f(tz)| \le |t-s||z| \max_{\tau\in[s,t]} |f'(\tau z)|, \ z \in \mathbb{D}, \ t, s \in [0,1], \tag{4.1}$$

where we briefly write $[s,t]$ for the interval between s and t.

First, we prove **(i)**. If $t = 1$, then by Proposition 2.2(a),

$$\|(1-s)C_{s\varphi}\| \le (1-s) \sup_{z\in\mathbb{D}} \frac{v(s\varphi(z))}{v(z)} \le (1-s) \sup_{z\in\mathbb{D}} \frac{v(\varphi(z))}{v(z)} \to 0, s \to 1.$$

Let now $t \in [0,1)$ and $t_0 \in (t,1)$. For every $s \in [0,t_0)$ and $f \in X_v$, using (2.1) and (4.1), we get

$$\begin{aligned}
&\|(1-s)C_{s\varphi}f - (1-t)C_{t\varphi}f\|_v \\
&= \sup_{z\in\mathbb{D}} \frac{|(1-s)f(s\varphi(z)) - (1-t)f(t\varphi(z))|}{v(z)} \\
&\le (1-s)\sup_{z\in\mathbb{D}} \frac{|f(s\varphi(z)) - f(t\varphi(z))|}{v(z)} + |s-t|\sup_{z\in\mathbb{D}} \frac{|f(t\varphi(z))|}{v(z)} \\
&\le |s-t|\sup_{z\in\mathbb{D}} \frac{|\varphi(z)|}{v(z)} \max_{\tau\in[s,t]} |f'(\tau\varphi(z))| + |s-t|\sup_{z\in\mathbb{D}} \frac{v(t\varphi(z))}{v(z)}\|f\|_v \\
&\le C|s-t|\|f\|_v \sup_{z\in\mathbb{D}} \frac{|\varphi(z)|}{v(z)} \max_{\tau\in[s,t]} \frac{v(\tau\varphi(z))}{1-|\tau\varphi(z)|} + |s-t|\sup_{z\in\mathbb{D}} \frac{v(t\varphi(z))}{v(z)}\|f\|_v \\
&\le \frac{Cv(t_0)}{(1-t_0)v(0)}|s-t|\|f\|_v + \frac{v(t_0)}{v(0)}|s-t|\|f\|_v \\
&= \left(\frac{C}{1-t_0}+1\right)\frac{v(t_0)}{v(0)}|s-t|\|f\|_v.
\end{aligned}$$

Therefore,

$$\|(1-s)C_{s\varphi} - (1-t)C_{t\varphi}\| \le \left(\frac{C}{1-t_0}+1\right)\frac{v(t_0)}{v(0)}|s-t| \to 0 \text{ as } s \to t,$$

which completes the proof of **(i)**.

Next, we prove **(ii)**. Fix a number $t \in [0,1]$. For every $s \in [0,1]$ and $f \in X_v$, we have

$$\begin{aligned}
&\|W_{s\psi,s\varphi}f - W_{t\psi,t\varphi}f\|_v = \sup_{z\in\mathbb{D}} \frac{|s\psi(z)f(s\varphi(z)) - t\psi(z)f(t\varphi(z))|}{v(z)} \\
&\le |s|\sup_{z\in\mathbb{D}} \frac{|\psi(z)(f(s\varphi(z)) - f(t\varphi(z)))|}{v(z)} + |s-t|\sup_{z\in\mathbb{D}} \frac{|\psi(z)f(t\varphi(z))|}{v(z)}.
\end{aligned}$$

To continue, we need several auxiliary estimates.

Estimate 1: We have

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \frac{|\psi(z)f(t\varphi(z))|}{v(z)} &\le \|f\|_v \sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(t\varphi(z))}{v(z)} \\
&\le \|f\|_v \sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(\varphi(z))}{v(z)} = M\|f\|_v,
\end{aligned}$$

where $M := \sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(\varphi(z))}{v(z)}$ is finite by Proposition 2.5(a).

Estimate 2: Obviously, for every $r \in (0,1)$ and $s \in [0,1]$,

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)(f(s\varphi(z)) - f(t\varphi(z)))|}{v(z)} = \max\{\mathcal{I}(r,s,f), \mathcal{J}(r,s,f)\},$$

where, by using (2.1) and (4.1),

$$\begin{aligned}\mathcal{I}(r,s,f) &:= \sup_{|\varphi(z)|\le r} \frac{|\psi(z)(f(s\varphi(z)) - f(t\varphi(z)))|}{v(z)}\\ &= |s-t| \sup_{|\varphi(z)|\le r} \frac{|\psi(z)\varphi(z)|}{v(z)} \max_{\tau\in[s,t]} |f'(\tau\varphi(z))|\\ &\le C|s-t|\|f\|_v \sup_{|\varphi(z)|\le r} \frac{|\psi(z)|}{v(z)} \max_{\tau\in[s,t]} \frac{v(\tau\varphi(z))}{1-|\tau\varphi(z)|}\\ &\le \frac{Cv(r)}{1-r}\|\psi\|_v\|f\|_v|s-t|,\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}(r,s,f) &:= \sup_{|\varphi(z)|> r} \frac{|\psi(z)(f(s\varphi(z)) - f(t\varphi(z)))|}{v(z)}\\ &\le \sup_{|\varphi(z)|> r} \frac{|\psi(z)|(|f(s\varphi(z))| + |f(t\varphi(z))|)}{v(z)}\\ &\le \|f\|_v \sup_{|\varphi(z)|> r} \frac{|\psi(z)|(v(s\varphi(z)) + v(t\varphi(z)))}{v(z)}\\ &\le 2\|f\|_v \sup_{|\varphi(z)|> r} \frac{|\psi(z)|v(\varphi(z))}{v(z)}.\end{aligned}$$

Using the above estimates, we obtain

$$\begin{aligned}&\|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\|\\ &\le \max\left\{\frac{Cv(r)}{1-r}\|\psi\|_v|s-t|, 2\sup_{|\varphi(z)|>r}\frac{|\psi(z)|v(\varphi(z))}{v(z)}\right\} + M|s-t|,\end{aligned}$$

for every $r \in (0,1)$ and $s \in [0,1]$. By letting $s \to t$, and then $r \to 1^-$ in the last inequality, we get

$$\limsup_{s\to t} \|W_{s\psi,s\varphi} - W_{t\psi,t\varphi}\| \le 2\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{|\psi(z)|v(\varphi(z))}{v(z)}.$$

Moreover, applying Proposition 2.5(b) to the compact operator $W_{\psi,\varphi}$ on X_v, we obtain

$$\lim_{|\varphi(z)|\to 1^-} \frac{|\psi(z)|v(\varphi(z))}{v(z)} = 0 \text{ if } X_v = H_v(\mathbb{D})$$

or

$$\lim_{|z|\to 1^-} \frac{|\psi(z)|v(\varphi(z))}{v(z)} = 0 \text{ if } X_v = H_v^0(\mathbb{D}),$$

which both imply, by Remark 2.3, that

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{|\psi(z)|v(\varphi(z))}{v(z)} = 0.$$

Consequently, $\lim_{s\to t}\|W_{s\psi,s\varphi}-W_{t\psi,t\varphi}\|=0$. This establishes the result claimed.

(b) Now we consider the operators W_{ψ_0,φ_0} and C_{φ_0}, where $\psi_0(z)=1-z$ and $\varphi_0(z)=1+a(z-1)$ with $0<a<1$. Obviously, W_{ψ_0,φ_0} and C_{φ_0} belong to $\mathcal{C}_w(X_v)$. However, it is easy to check that W_{ψ_0,φ_0} is compact, while C_{φ_0} is not compact on X_v.

Indeed, for all $r\in(0,1)$,

$$\frac{v(\varphi_0(r))}{v(r)}=\frac{v(1+a(r-1))}{v(r)}\ge 1.$$

Hence, by Proposition 2.2(b), C_{φ_0} is not compact on X_v.

Next, for any sequence $(z_n)_n$ in $\mathbb{D}$ with $|z_n|\to 1$ as $n\to\infty$, without loss of generality we suppose that $z_n\to\eta\in\partial\mathbb{D}$. If $\eta\ne 1$, then $1+a(\eta-1)\in\mathbb{D}$, hence,

$$\frac{|\psi_0(z_n)|v(\varphi_0(z_n))}{v(z_n)}\le 2\frac{v(\varphi_0(z_n))}{v(z_n)}\to 0 \text{ as } n\to\infty.$$

If $\eta=1$, then $\psi_0(z_n)\to 0$ as $n\to\infty$, hence,

$$\frac{|\psi_0(z_n)|v(\varphi_0(z_n))}{v(z_n)}\le |\psi_0(z_n)|\sup_{z\in\mathbb{D}}\frac{v(\varphi_0(z))}{v(z)}\to 0 \text{ as } n\to\infty,$$

since the last supremum is finite by Proposition 2.2(a).

Consequently,

$$\lim_{|z|\to 1^-}\frac{|\psi_0(z)|v(\varphi_0(z))}{v(z)}=0,$$

which implies, by Remark 2.3 and Proposition 2.5(b), that W_{ψ_0,φ_0} is compact on X_v.

It remains to note that, by Lemma 4.1, $W_{\psi_0,\varphi_0}\sim C_{\varphi_0}$ in $\mathcal{C}_w(X_v)$. From this it follows that the set $\mathcal{C}_{w,0}(X_v)$ is not a path component of $\mathcal{C}_w(X_v)$. □

Now we establish some analogs of the results of Section 3 for weighted composition operators.

Proposition 4.3. *Let φ and ϕ be two functions in $\mathcal{S}(\mathbb{D})$. If the difference $C_\varphi-C_\phi$ is compact on X_v, then all the operators $W_{\psi,\varphi}$ and $W_{\chi,\phi}$ from the space $\mathcal{C}_w(X_v)$ belong to the same path component of this space.*

Proof. By Lemma 4.1, $W_{\psi,\varphi}\sim C_\varphi$ and $W_{\chi,\phi}\sim C_\phi$ in $\mathcal{C}_w(X_v)$. On the other hand, by Theorem 3.1, $C_\phi\sim C_\varphi$ in $\mathcal{C}(X_v)$ and, hence, in $\mathcal{C}_w(X_v)$. Consequently, $W_{\chi,\phi}\sim W_{\psi,\varphi}$ in $\mathcal{C}_w(X_v)$. □

Remark 4.4. In [6, Theorem 4.2] a similar result to Proposition 4.3 was stated in the setting of the space $C_w^0(H_v^0(\mathbb{D}))$ under some additional restrictions on functions φ,ϕ,ψ, and χ that are strictly stronger than the ones in Proposition 4.3. In particular, in this theorem the authors required that $\lim_{|z|\to 1^-} \rho(\varphi(z), \phi(z)) = 0$, which implies, by Proposition 2.4, that the difference $C_\varphi - C_\phi$ is a compact operator on X_v.

For $\varphi \in \mathcal{S}(\mathbb{D})$, denote by $\mathcal{W}([C_\varphi])$ the set of all weighted composition operators $W_{\psi,\phi} \in \mathcal{C}_w(X_v)$ with $C_\phi \in [C_\varphi]$. The following result follows immediately from Proposition 4.3.

Corollary 4.5. *Each set $\mathcal{W}([C_\varphi])$, $\varphi \in \mathcal{S}(\mathbb{D})$, is path connected in $\mathcal{C}_w(X_v)$.*

Now we show that the sets $\mathcal{W}([C_\varphi])$ may be path components of the space $\mathcal{C}_w(X_v)$ and may be not. To see this, we consider the next examples.

Example 4.6. For $\varphi_0(z) = 1 + a(z-1)$ with $0 < a < 1$, the set $\mathcal{W}([C_{\varphi_0}])$ is not a path component of $\mathcal{C}_w(X_v)$. More precisely, $\mathcal{W}([C_{\varphi_0}])$ is a proper subset of the path component of $\mathcal{C}_w(X_v)$ containing $\mathcal{C}_{w,0}(X_v)$.

Proof. By part (b) in the proof of Theorem 4.2, the operator W_{ψ_0,φ_0} with $\psi_0(z) = 1 - z$ and $\varphi_0(z) = 1 + a(z-1)$ is compact, while C_{φ_0} is not compact on X_v. Then, by Theorem 4.2, $W_{\psi_0,\varphi_0} \sim C_0$ in $\mathcal{C}_w(X_v)$. But $C_{\varphi_0} - C_0$ is not compact on X_v, which implies that the operator C_0 does not belong to $\mathcal{W}([C_{\varphi_0}])$ and completes the proof. □

Remark 4.7. The arguments in Example 4.6 work as well for those sets $\mathcal{W}([C_\varphi])$ that generated by $\varphi \in \mathcal{S}(\mathbb{D})$ with the finite set $E(v, \varphi)$. Thus, all these sets being path connected in the space $\mathcal{C}_w(X_v)$ are proper subsets of the corresponding path components of $\mathcal{C}_w(X_v)$ containing $\mathcal{C}_{w,0}(X_v)$.

Example 4.8. For $\varphi_1(z) = z$, the set $\mathcal{W}([C_{\varphi_1}])$ is a path component of $\mathcal{C}_w(X_v)$.

Proof. Obviously, $E(v, \varphi_1) = \partial\mathbb{D}$. Hence, by [6, Theorem 5.7], C_{φ_1} is isolated in $\mathcal{C}(X_v)$, which implies that $[C_{\varphi_1}] = \{C_{\varphi_1}\}$. From this and Proposition 2.5(a) it follows that

$$\mathcal{W}([C_{\varphi_1}]) = \{W_{\psi,\varphi_1} : 0 < \|\psi\|_\infty < \infty\}.$$

We will prove that $\mathcal{W}([C_{\varphi_1}])$ is open and, simultaneously, closed in $\mathcal{C}_w(X_v)$, from which the assertion follows.

Let (W_{ψ_n,φ_1}) be a sequence in $\mathcal{W}([C_{\varphi_1}])$ converging to some operator $W_{\chi,\phi}$ in $\mathcal{C}_w(X_v)$. Then $W_{\psi_n,\varphi_1}(f) \to W_{\chi,\phi}(f)$ in X_v for all $f \in X_v$. Taking here $f(z) \equiv 1$ and $f(z) \equiv z$, we obtain that $\psi_n \to \chi$ and $\psi_n\varphi_1 \to \chi\phi$ in X_v as $n \to \infty$. Therefore,

$$\chi(\varphi_1 - \phi) = (\chi - \psi_n)\varphi_1 + (\psi_n\varphi_1 - \chi\phi) \to 0 \text{ in } X_v.$$

Since $\chi \not\equiv 0$, this implies that $\phi = \varphi_1$. Thus, the set $\mathcal{W}([C_{\varphi_1}])$ is closed in $\mathcal{C}_w(X_v)$. The fact that it is open in $\mathcal{C}_w(X_v)$ follows immediately from the following auxiliary lemma, in which we will use the next notation: $F(\psi,\varepsilon) := \{\omega \in \partial\mathbb{D} : |\psi(\omega)| \geq \varepsilon\}$ and $\|\psi\|_e := \inf\{\varepsilon > 0 : |F(\psi,\varepsilon)| = 0\}$.

□

Lemma 4.9. *Let W_{ψ,φ_1} be an operator in $\mathcal{W}([C_{\varphi_1}])$. Then, for every operator $W_{\chi,\phi}$ in $\mathcal{C}_w(X_v)$ with $\phi \neq \varphi_1$,*

$$\|W_{\psi,\varphi_1} - W_{\chi,\phi}\| \geq \|\psi\|_e.$$

Proof. Since ψ is a nonzero function, $\|\psi\|_e > 0$. Take an arbitrary number $r \in (0, \|\psi\|_e)$. Then $|F(\psi,r)| > 0$.

Since $\phi \neq \varphi_1$, $|\{\omega \in \partial\mathbb{D} : \phi(\omega) = \omega\}| = 0$. So there exist a point $\omega \in F(\psi,r)$ and a sequence $(z_n) \subset \mathbb{D}$ such that $z_n \to \omega$, $|\psi(z_n)| \to |\psi(\omega)| \geq r$, and $\phi(z_n) \to \eta \neq \omega$. Then $\rho(z_n,\phi(z_n)) \to 1$ as $n \to \infty$.

Next, by [3, Subsection 1.2(iv), Theorem 1.13 and comments after it], for each $n \in \mathbb{N}$, there is a function f_n in the unit ball of X_v such that $f_n(z_n) = \widetilde{v}(z_n)$ (recall that by $\widetilde{v}$ it is denoted the weight associated with v). We put

$$h_n(z) = f_n(z)\frac{z - \phi(z_n)}{1 - z\overline{\phi(z_n)}}, \ z \in \mathbb{D}.$$

Then $h_n \in X_v$ with $\|h_n\|_v \leq 1$ for all n. Taking into account that $W_{\psi,\varphi_1}h_n(z_n) = \psi(z_n)\rho(z_n,\phi(z_n))\widetilde{v}(z_n)$ and $W_{\chi,\phi}h_n(z_n) = 0$, we get

$$\begin{aligned}\|W_{\psi,\varphi_1} - W_{\chi,\phi}\| &\geq \|W_{\psi,\varphi_1}h_n - W_{\chi,\phi}h_n\|_{\widetilde{v}} \\ &\geq \frac{|W_{\psi,\varphi_1}h_n(z_n) - W_{\chi,\phi}h_n(z_n)|}{\widetilde{v}(z_n)} \quad = |\psi(z_n)|\rho(z_n,\phi(z_n))\end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$\|W_{\psi,\varphi_1} - W_{\chi,\phi}\| \geq \limsup_{n\to\infty} |\psi(z_n)|\rho(z_n,\phi(z_n)) \geq r$$

and, consequently, $\|W_{\psi,\varphi_1} - W_{\chi,\phi}\| \geq \|\psi\|_e$. □

Remark 4.10. Some of the arguments used in the proof of Example 4.8 work as well for any isolated operator C_φ in $\mathcal{C}(X_v)$. More precisely, by the same reasons as in this example, one can easily check that the corresponding sets $\mathcal{W}([C_\varphi]) = \mathcal{W}(\{C_\varphi\})$ are all closed in the space $\mathcal{C}_w(X_v)$. Moreover, by Corollary 4.5 they are path connected in this space. We think but could not prove that they are also open in $\mathcal{C}_w(X_v)$.

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