Efficient computation of the Nagaoka–Hayashi bound for multiparameter estimation with separable measurements

Conlon, Lorcán O.; Suzuki, Jun; Lam, Ping Koy; Assad, Syed Muhamad

2021

https://hdl.handle.net/10356/153821

https://doi.org/10.1038/s41534-021-00414-1

2

computation considerably more efficient. We demonstrate the usefulness of our bound by applying it to several examples involving collective measurements on a finite number of copies of the probe. Our bound has implications for the conditions necessary to saturate the Holevo bound.

RESULTS

In this section we first present the background theory on which our results are based. We introduce the concept of locally unbiased estimators before defining the mean squared error matrix. We then introduce the SLD bound before reviewing the Holevo bound. We then review the Nagaoka bound for two parameter estimation before introducing our generalisation of this bound, the Nagaoka–Hayashi bound. Finally, we show how our bound can be cast as a semidefinite programme and present several examples where our bound is relevant. Throughout this manuscript we use the hat notation to indicate a matrix, e.g. $\hat{X}$.

Background theory

The aim of any quantum estimation problem is to determine the value of some unknown parameters, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)^T$, which are encoded in a quantum state $\rho_{\boldsymbol{\theta}}$. To do this, a measurement, described by a positive operator-valued measure (POVM), is performed on $\rho_{\boldsymbol{\theta}}$. This POVM is defined by a collection of positive semidefinite operators, $\{\Pi_x\}$, which satisfy $\sum_x \Pi_x = \mathbb{1}$. The probability of observing a particular measurement outcome, labelled by x, is given by the Born rule, $p(x|\boldsymbol{\theta}) = \mathrm{Tr}[\rho_{\boldsymbol{\theta}} \Pi_x]$. From the outcome of this measurement we construct an estimator, $\check{\boldsymbol{\theta}} = (\check{\theta}_1, \check{\theta}_2, \ldots, \check{\theta}_n)^T$, for the unknown parameters. We restrict ourselves to the class of locally unbiased estimators. This means that at the true value of the parameters, $\boldsymbol{\theta}_0$, the following conditions are satisfied

$$\int p(x|\boldsymbol{\theta}) \check{\theta}_i \, dx = \theta_{0i}$$

and

$$\int \frac{\partial p(x|\boldsymbol{\theta})}{\partial \theta_j} \check{\theta}_i \, dx = \delta_{ij}, \quad \forall i, j.$$

In the asymptotic limit the maximum likelihood estimator saturates this condition and as such this is not a very restrictive assumption[15]. We can now introduce the mean squared error (MSE) matrix, $V(\check{\boldsymbol{\theta}})$, with matrix elements

$$V_{ij}(\check{\boldsymbol{\theta}}) = \int (\check{\theta}_i - \theta_{0i})(\check{\theta}_j - \theta_{0j}) p(x|\boldsymbol{\theta}) \, dx.$$

For any locally unbiased estimator, the MSE matrix is equal to the covariance matrix. The aim of any estimation problem is to find the minimum of this covariance matrix. However, as this is a matrix quantity, there is in general no unique minimum. Thus to define a bound we introduce a positive weight matrix, W, and try to minimise $\mathrm{Tr}[WV(\check{\boldsymbol{\theta}})]$.

We first consider the case where we only wish to estimate a single parameter. This problem can be tackled using the quantum Cramér–Rao bound, which provides a lower bound on the variance of any locally unbiased estimator. In turn the quantum Fisher information provides an upper bound on this Fisher information. The quantum Fisher information can be defined in terms of different logarithmic derivatives. We first introduce the symmetric logarithmic derivative (SLD), L_i, which is a Hermitian operator implicitly defined by

$$\frac{\partial \rho_{\boldsymbol{\theta}}}{\partial \theta_i} = \frac{1}{2}(\rho_{\boldsymbol{\theta}} L_i + L_i \rho_{\boldsymbol{\theta}}).$$

Using the SLD we can construct the SLD Fisher information matrix, $\mathcal{F}$, which has matrix elements

$$\mathcal{F}_{ij} = \frac{1}{2} \mathrm{Tr}[\rho_{\boldsymbol{\theta}}(L_i L_j + L_j L_i)].$$

For a single parameter the variance of any unbiased estimator is bounded by the inverse of the Fisher information. This is known as the Cramér–Rao bound. When estimating multiple parameters simultaneously, we have the matrix inequality

$$V(\check{\boldsymbol{\theta}}) \geq \mathcal{F}^{-1},$$

which is a matrix generalisation of the Cramér–Rao bound. The weak commutativity condition, which is defined as

$$U_{ij} = \mathrm{Tr}[\rho_{\boldsymbol{\theta}} L_i L_j],$$

encapsulates the compatibility of the measurements. When estimating multiple parameters simultaneously the SLD Cramér–Rao bound is in general not saturable. This is because the optimal measurements for different parameters may not be compatible. When $U_{ij} = 0$, the SLD bound can always be saturated. Using the weight matrix and the SLD Fisher information matrix we can define the SLD bound as

$$C^S = \mathrm{Tr}[W\mathcal{F}^{-1}].$$

The SLD bound is saturable if and only if $\mathrm{Tr}[W\mathcal{F}^{-1} U \mathcal{F}^{-1}] = 0$, which is a weaker condition than $U = 0$[40]. As mentioned above there also exists a bound based on the right logarithmic derivative (RLD)[50], but we do not consider this bound in this manuscript.

A tighter bound for the weighted trace of the MSE, which can be achieved in the asymptotic limit, is given by the Holevo Cramér–Rao bound[52]. The Holevo bound can be written as[40,61]

$$C^H = \min_{\vec{X}, V} \{ \mathrm{Tr}[WV] \mid V \geq Z(\vec{X}), Z(\vec{X}) \in \mathcal{L}(\vec{X}) \}$$

where $\mathcal{L}(\vec{X})$ means that the matrix $Z(\vec{X})$ is Hermitian, $\vec{X} = (X_1, \ldots, X_n)$ is a collection of n Hermitian matrices satisfying the locally unbiased conditions

$$\mathrm{Tr}\left[\frac{\partial \rho_{\boldsymbol{\theta}}}{\partial \theta_j} X_i\right] = \delta_{ij},$$

and $Z(\vec{X})$ is a matrix with elements

$$Z_{ij}(\vec{X}) = \mathrm{Tr}[\rho_{\boldsymbol{\theta}} X_i X_j].$$

It has been shown that the Holevo bound is a tighter bound than the SLD bound. The Holevo bound is in general only achievable asymptotically, i.e. in the limit of infinitely many collective measurements.

computation accessible. We illustrate our results with two examples which highlight some of the interesting features of finite copy metrology which are inaccessible with conventional techniques. In both of these examples, we are able to find the positive operator valued measure (POVM) which saturates the bound, however whether this is always possible remains an open question.

RESULTS

Consider an n-parameter family of states $\{S_\theta | \theta \in \Theta \subseteq \mathbb{R}^n\}$ in a finite d dimensional Hilbert space $\mathcal{H}_q$ with $\theta = (\theta_1, \ldots, \theta_n)^\top$ denoting the n independent true values that we wish to estimate. Let $\Pi = (\Pi_1, \ldots, \Pi_M)^\top$ be a column vector of M POVM elements, where $(\cdot)^\top$ denotes partial transpose with respect to the classical subsystem. We require $\Pi_m \geq 0$ and $\sum_m \Pi_m = 1$. Each outcome m assigns an estimated value for θ_j through the classical estimator function $\hat{\theta}_{jm}$. The standard measure of estimation error when restricted to separable measurements is through the n-by-n MSE matrix $V_\theta(\Pi, \hat{\theta})$ with entries

$$\left[V_\theta(\Pi, \hat{\theta})\right]_{jk} = \sum_m \left(\hat{\theta}_{jm} - \theta_j\right)\left(\hat{\theta}_{km} - \theta_k\right)\mathrm{Tr}[S_\theta \Pi_m], \quad \text{for} \quad j, k = 1, \ldots, n. \tag{1}$$

The notation $\mathrm{Tr}[\cdot]$ in serif font is used to represent the trace of an operator in $\mathcal{H}_q$. For brevity of notation, hereafter we drop the argument and write the MSE matrix as V_θ. We aim to minimise the trace of the MSE matrix under the condition that our estimates are locally unbiased

$$\sum_m \mathrm{Tr}[S_\theta \Pi_m]\hat{\theta}_{jm} = \theta_j \quad \text{and} \quad \sum_m \frac{\partial}{\partial \theta_k}\mathrm{Tr}[S_\theta \Pi_m]\hat{\theta}_{jm} = \delta_{jk}. \tag{2}$$

The Nagaoka bound for two-parameter estimation gives a lower bound on the trace of the MSE matrix as[59]

$$Tr[V_\theta] \geq \min_X \{\mathrm{Tr}[S_\theta X_1 X_1 + S_\theta X_2 X_2] + \mathrm{TrAbs}\ S_\theta[X_1, X_2]\} =: c_N, \tag{3}$$

where $Tr[\cdot]$ denotes the trace of a classical matrix in $\mathcal{H}_c$, an n dimensional vector space, TrAbs A is the sum of the absolute values of the eigenvalues of the operator A, and $X = (X_1, X_2, \ldots, X_n)^\top$ is a vector of Hermitian estimator observables X_j that satisfy the locally unbiased condition at θ

$$\mathrm{Tr}[S_\theta X_j] = \theta_j \quad \text{and} \quad \frac{\partial}{\partial \theta_j}\mathrm{Tr}[S_\theta X_k] = \delta_{jk}. \tag{4}$$

The Nagaoka bound was conjectured to be a tight bound for $Tr[V_\theta]$[60].

Computable multiparameter bound

As we shall shortly prove, the Nagaoka bound can be generalised to more than two parameters. This result is stated as the following theorem.

Theorem 1 (Nagaoka–Hayashi bound). Let V_θ be the MSE matrix of an unbiased estimate of θ for a separable measurement on a model S_θ. Then, the trace of V_θ is bounded by

$$Tr[V_\theta] \geq \min_{\mathbb{L}, X}\{\mathbb{T}r[\mathbb{S}_\theta \mathbb{L}] \mid \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian},$$
$$\mathbb{L} \geq XX^\top, X_j \text{ Hermitian satisfying } (4)\} =: c_{NH}, \tag{5}$$

where $\mathbb{S}_\theta = 1_n \otimes S_\theta$ and $\mathbb{L}$ is an n-by-n matrix of Hermitian operators $\mathbb{L}_{jk}$.

We use the symbol $\mathbb{T}r[\cdot]$ to denote trace over both classical and quantum systems, i.e., over both $\mathcal{H}_c$ and $\mathcal{H}_q$. We call this bound

the Nagaoka–Hayashi bound. However, the Nagaoka–Hayashi bound is not an explicit bound. Our second main result is that this bound, c_{NH} can be computed as a semidefinite programme.

Related bounds

Before proceeding on the proof and computation of the Nagaoka–Hayashi bound, we digress briefly to mention two related bounds. The first is the Holevo bound which can be written as[52]

$$Tr[V_\theta] \geq \min_{\mathbb{L}, X}\{\mathbb{T}r[\mathbb{S}_\theta \mathbb{L}] \mid \mathrm{Tr}[\mathbb{S}_\theta \mathbb{L}] \text{ real symmetric},$$
$$\mathrm{Tr}[\mathbb{S}_\theta \mathbb{L}] \geq \mathrm{Tr}[\mathbb{S}_\theta XX^\top], X_j \text{ Hermitian satisfying } (4)\} =: c_H. \tag{6}$$

As mentioned before, the Holevo bound is a tight bound for collective measurements in the asymptotic limit. Since the conditions in the Nagaoka–Hayashi bound $\mathbb{L}_{jk} = \mathbb{L}_{kj}$ Hermitian implies $\mathrm{Tr}[\mathbb{S}_\theta \mathbb{L}]$ real symmetric and $\mathbb{L} \geq XX^\top$ implies $\mathrm{Tr}[\mathbb{S}_\theta \mathbb{L}] \geq \mathrm{Tr}[\mathbb{S}_\theta XX^\top]$, it is clear that the Nagaoka–Hayashi bound is more restrictive and hence is more informative compared to the Holevo bound. In other words $c_{NH} \geq c_H$.

The second related bound concerns estimation of physical observables. In this setting, the operators X_j are given to us as Hermitian operators and the task is to estimate the expectation values $\mathrm{Tr}[S_\theta X_j] = x_j$. This situation is common, for example, in state tomography. Here, in place of the parameter-MSE matrix Eq. (1), we have the operator-MSE matrix

$$\left[\tilde{U}_\theta(\Pi, \hat{x})\right]_{jk} = \sum_m (\hat{x}_{jm} - x_j)(\hat{x}_{km} - x_k)\mathrm{Tr}[S_\theta \Pi_m], \quad \text{for} \quad j, k = 1, 2, \ldots, n \tag{7}$$

where we require the classical estimator $\hat{x}$ and POVM Π to satisfy

$$\sum_m \hat{x}_{jm}\Pi_m = X_j. \tag{8}$$

The derivatives of the state S with respect to θ do not play any role here. A bound on the trace of $\tilde{U}_\theta$ is given by Hayashi's bound[61]

$$Tr[\tilde{U}_\theta] \geq \min_{\mathbb{L}}\left\{\mathbb{T}r[\mathbb{S}_\theta \mathbb{L}] - \sum_j x_j^2 \mid \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian}, \mathbb{L} \geq XX^\top\right\} =: c_{NH-U}. \tag{9}$$

As Hayashi's work is only available in Japanese, we summarise its main results in Supplementary Note 1. If the given matrices X happen to satisfy the locally unbiasedness condition Eq. (4) for θ, then $\tilde{U}_\theta$ also forms a valid parameter-MSE matrix for those θ. In this case, because of the additional restriction Eq. (8), it is clear that $c_{NH-U} \geq c_{NH}$. Also in this setting, Watanabe et al.[62] derived bounds for estimating two observables when restricted to certain classes of random and noisy measurements. In the case when both the observables and state S are two-dimensional, these bounds are achievable. In fact, when the number of observables $n = 2$, the minimisation over $\mathbb{L}$ can be performed analytically and c_{NH-U} takes the explicit form[61]

$$c_{NH-U} = \mathrm{Tr}[S_\theta X_1 X_1 + S_\theta X_2 X_2] + \mathrm{TrAbs}\ S_\theta[X_1, X_2] - x_1^2 - x_2^2. \tag{10}$$

Proof of main results

In this section, we shall prove Theorem 1. To that end, we need to introduce some definitions. We rewrite the elements of the MSE matrix as

$$[V_\theta]_{jk} = \mathrm{Tr}\left[S_\theta \underbrace{\sum_m \left(\hat{\theta}_{jm} - \theta_j\right)\Pi_m\left(\hat{\theta}_{km} - \theta_k\right)}_{[\mathbb{L}_\theta]_{jk}}\right], \tag{11}$$

where the MSE-matrix operator $\mathbb{L}_\theta(\Pi, \hat{\theta})$ is an n-by-n matrix with operator elements. We introduce a classical matrix ξ with elements

$\xi_{jm} := \hat{\theta}_{jm} - \theta_j$ so that

$$\mathbb{L}_\theta = \sum_m \begin{pmatrix} \xi_{1m}\Pi_m\xi_{1m} & \xi_{1m}\Pi_m\xi_{2m} & \xi_{1m}\Pi_m\xi_{3m} \\ \xi_{2m}\Pi_m\xi_{1m} & \xi_{2m}\Pi_m\xi_{2m} & \xi_{2m}\Pi_m\xi_{3m} \\ \xi_{3m}\Pi_m\xi_{1m} & \xi_{3m}\Pi_m\xi_{2m} & \xi_{3m}\Pi_m\xi_{3m} \end{pmatrix} \tag{12}$$

$$= \sum_m \begin{pmatrix} \xi_{1m} \\ \xi_{2m} \\ \xi_{3m} \end{pmatrix} (\xi_{1m}\ \xi_{2m}\ \xi_{3m}) \otimes \Pi_m, \tag{13}$$

where we have set $n = 3$ to simplify the presentation. The generalisation to arbitrary n is straightforward. With this notation, it is clear that $\mathbb{L}_\theta$ is an operator on the extended Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_q$. To anticipate the proof, it is useful to write $\mathbb{L}_\theta$ in the following form:

$$\mathbb{L}_\theta = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \dots & \Xi_{1M} \\ \Xi_{21} & \Xi_{22} & \dots & \Xi_{2M} \\ \Xi_{31} & \Xi_{32} & \dots & \Xi_{3M} \end{pmatrix} \begin{pmatrix} \Pi_1 & 0 & \dots & 0 \\ 0 & \Pi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Pi_M \end{pmatrix} \begin{pmatrix} \Xi_{11} & \Xi_{21} & \Xi_{31} \\ \Xi_{12} & \Xi_{22} & \Xi_{32} \\ \vdots & \vdots & \vdots \\ \Xi_{1M} & \Xi_{2M} & \Xi_{3M} \end{pmatrix}, \tag{14}$$

where M is the number of POVM outcomes and $\Xi_{ij} = \xi_{ij}\mathbb{1}$. We can also introduce the following extension to S_θ, $\mathbb{S}_\theta = \mathbb{1} \otimes S_\theta$ so that the expression for the MSE matrix can be written as $V_\theta = \text{Tr}[\mathbb{S}_\theta\mathbb{L}_\theta]$. We are now ready to prove Theorem 1.

Proof. Suppose the optimal POVM and unbiased estimator have been found and are given by Π and $X_j^* = \sum_m \xi_{jm}\Pi_m$, for $j = 1, \dots, n$ which leads to the optimal MSE

$$v^* = \sum_{jm} \xi_{jm}^2 \text{Tr}[S_\theta \Pi_m] = \mathbb{Tr}[\mathbb{S}_\theta \mathbb{L}_\theta^*]. \tag{15}$$

We use asterisk to denote the optimal values and optimal operators. From Π and $\hat{\theta}$, we can construct the estimator matrices $X_j^* = \sum_m \xi_{jm}\Pi_m$, for $j = 1, \dots, n$ so that

$$\begin{pmatrix} X_1^* \\ X_2^* \\ X_3^* \end{pmatrix} (X_1^*\ X_2^*\ X_3^*) = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \dots & \Xi_{1M} \\ \Xi_{21} & \Xi_{22} & \dots & \Xi_{2M} \\ \Xi_{31} & \Xi_{32} & \dots & \Xi_{3M} \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_M \end{pmatrix} (\Pi_1\ \Pi_2\ \dots\ \Pi_M) \begin{pmatrix} \Xi_{11} & \Xi_{21} & \Xi_{31} \\ \Xi_{12} & \Xi_{22} & \Xi_{32} \\ \vdots & \vdots & \vdots \\ \Xi_{1M} & \Xi_{2M} & \Xi_{3M} \end{pmatrix}. \tag{16}$$

Comparing the above with Eq. (14) and using the result

$$\begin{pmatrix} \Pi_1 & 0 & \dots & 0 \\ 0 & \Pi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Pi_M \end{pmatrix} \geq \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_M \end{pmatrix} (\Pi_1\ \Pi_2\ \dots\ \Pi_M) \tag{17}$$

which holds because Π_j are positive operators that sums up to 1 (see Proposition II.9.1 of Holevo[52]), we arrive at $\mathbb{L}_\theta^* \geq X^*X^{*\top}$. With this, we can bound v^* as

$$v^* = \mathbb{Tr}[\mathbb{S}_\theta\mathbb{L}_\theta^*] \tag{18}$$

$$\geq \min_{\mathbb{L}} \{\mathbb{Tr}[\mathbb{S}_\theta\mathbb{L}] \mid \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian}, \mathbb{L} \geq X^*X^{*\top}\} \tag{19}$$

$$\geq \min_{\mathbb{L}, X} \{\mathbb{Tr}[\mathbb{S}_\theta\mathbb{L}] \mid \mathbb{L}_{jk} = \mathbb{L}_{kj} \text{ Hermitian}, \mathbb{L} \geq XX^\top,$$
$$X_j \text{ Hermitian satisfying } (4)\} \tag{20}$$

$$= c_{\text{NH}}. \quad \square \tag{21}$$

In the two-parameter case, we show in Supplementary Note 2 that c_{NH} reduces to the original Nagaoka bound c_{N} in Eq. (3). More generally, we are interested in minimising the weighted sum of the covariances which can be formalised with a positive weight

matrix $W \geq 0$ and minimising $Tr[W V_\theta]$. This problem can be handled by a suitable reparametrisation of the model which is presented in Supplementary Note 3.

The Nagaoka–Hayashi bound is not an explicit bound as it still requires a minimisation over $\mathbb{L}$ and X. Our next result concerns with the computation of this minimisation. Since $\mathbb{L} - XX^\top$ is the Schur's complement of the identity operator in $\begin{pmatrix} \mathbb{L} & X \\ X^\top & 1 \end{pmatrix}$, the condition $\mathbb{L} \geq XX^\top$ is equivalent to $\begin{pmatrix} \mathbb{L} & X \\ X^\top & 1 \end{pmatrix} \geq 0$. With this, c_{NH} can be written as the semidefinite programme

$$c_{\text{NH}} = \min_{\mathbb{L}, X} \mathbb{Tr}[\mathbb{S}_\theta\mathbb{L}],$$
$$\text{subject to } \begin{pmatrix} \mathbb{L} & X \\ X^\top & 1 \end{pmatrix} \geq 0 \tag{22}$$

where $\mathbb{L}_{jk} = \mathbb{L}_{kj}$ Hermitian and X_j Hermitian satisfying the conditions Eq. (4) for local unbiasedness. The conversion to a standard semidefinite programme is performed in Supplementary Note 4. We also show in the same Supplementary Note that the worst case computational complexity for solving the SDP to an accuracy ϵ is $O\left((nd)^{3/2}\log(1/\epsilon)\right)$.

The computation of the Holevo bound c_{H} was shown to be a semidefinite programme by Albarelli et al.[53]. The difference between the Holevo bound and the Nagaoka–Hayashi bound is that in the former, the optimisation is performed directly on the covariance matrix $V = \text{Tr}[\mathbb{S}_\theta\mathbb{L}]$, while in the latter the optimisation is performed on the operator $\mathbb{L}$. We note that both programmes can also be applied to compute the bound on the operator-MSE $c_{\text{NH}-\text{U}}$ Eq. (9) with little modification—the only changes needed are to replace the minimisation variables X with the given observables and ignore the conditions Eq. (4).

In the following, we demonstrate our results by computing the Holevo and Nagaoka–Hayashi bounds for two illustrative examples—the estimation of orthogonal qubit rotations on the Bloch sphere in a phase damping channel and the simultaneous estimation of phase and loss in an interferometer. In the former, we find that the Holevo bound is always smaller than the Nagaoka–Hayashi bound, and in the latter we find that the two bounds are always equal. The minimisation problem was solved with the Yalmip toolbox[63] for Matlab using the Mosek solver[64].

Even though the semidefinite programme only returns numerical values for X and $\mathbb{L}$, in some of these examples, the analytical forms for them can be inferred from the numerical solutions. Furthermore, every semidefinite programme Eq. (22) has a dual programme that involves performing a maximisation over the Lagrange multipliers associated with the primal programme[65]. That the inferred solutions are indeed optimal can then be verified by checking that the values for the primal and dual programmes coincide. For both of the examples considered, we present the dual solutions in Supplementary Note 7.

Example 1: estimation of qubit rotations with a two-qubit probe

Our first example concerns estimating the rotation experienced by qubit probes subject to the phase damping channel. This channel has particular relevance for modelling decoherence in trapped ions[66–68]. We consider the maximally entangled two-qubit state $(|01\rangle + |10\rangle)/\sqrt{2}$ as a probe. The first qubit acts as a signal-probe which passes through a channel imparting three small rotations: θ_x, θ_y, and θ_z about the x, y, and z axes of the Bloch sphere. The rotated probe is then subject to the phase damping channel $\mathcal{E}$ with a known damping strength ϵ

$$\mathcal{E}[S] = \left(1 - \frac{\epsilon}{2}\right)S + \frac{\epsilon}{2}(\sigma_z \otimes 1)S(\sigma_z \otimes 1). \tag{23}$$

The second idler qubit is stored in a perfect quantum memory and remains unaffected by the rotation or phase damping. The resulting two-qubit state then has an approximate matrix representation in the computational basis as

$$S_\theta = \frac{1}{4} \begin{pmatrix} 0 & -i\theta_x - \theta_y & (1-\epsilon)(-i\theta_x - \theta_y) & 0 \\ i\theta_x - \theta_y & 2 & 2(1-\epsilon)(1-i\theta_z) & (1-\epsilon)(i\theta_x + \theta_y) \\ (1-\epsilon)(i\theta_x - \theta_y) & 2(1-\epsilon)(1+i\theta_z) & 2 & i\theta_x + \theta_y \\ 0 & (1-\epsilon)(-i\theta_x + \theta_y) & -i\theta_x + \theta_y & 0 \end{pmatrix},$$

(24)

which is valid to the first order in θ. The partial derivatives of S_θ with respect to θ evaluated at $\theta = 0$ are

$$\frac{\partial S_\theta}{\partial \theta_x} = \frac{1}{4} \begin{pmatrix} 0 & -i & -i(1-\epsilon) & 0 \\ i & 0 & 0 & i(1-\epsilon) \\ i(1-\epsilon) & 0 & 0 & i \\ 0 & -i(1-\epsilon) & -i & 0 \end{pmatrix},$$

$$\frac{\partial S_\theta}{\partial \theta_y} = \frac{1}{4} \begin{pmatrix} 0 & -1 & -(1-\epsilon) & 0 \\ -1 & 0 & 0 & (1-\epsilon) \\ -(1-\epsilon) & 0 & 0 & 1 \\ 0 & (1-\epsilon) & 1 & 0 \end{pmatrix} \quad \text{and}$$

$$\frac{\partial S_\theta}{\partial \theta_z} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i(1-\epsilon) & 0 \\ 0 & i(1-\epsilon) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(25)

Single-parameter estimation. Let us start with the simple case when $\theta_y = \theta_z = 0$ and we are only estimating the single parameter θ_x. In a single-parameter estimation problem, the Holevo bound coincides with the Nagaoka–Hayashi bound and can always be saturated by a separable measurement. In this case, the two bounds can be achieved by the estimator operator

$$X_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

(26)

which gives $c_{H,1} = c_{NH,1} = 1$, independent of ϵ. The optimal measurement that saturates this bound is a projective measurement on the four orthogonal eigenvectors of X_x

$$\begin{matrix} \Pi_1 \\ \Pi_2 \end{matrix} \Big\} = \frac{1}{2} \begin{pmatrix} 1 & \mp i & 0 & 0 \\ \pm i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} \Pi_3 \\ \Pi_4 \end{matrix} \Big\} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mp i \\ 0 & 0 & \pm i & 1 \end{pmatrix}.$$

(27)

This together with the estimation coefficients $\xi = (1, -1, -1, 1)$ gives an estimation variance of $v_x = 1$. The phase damping channel has no effect on the estimation precision.

Two-parameter estimation. Next, for estimating the two parameters θ_x and θ_y when $\theta_z = 0$, the Holevo and Nagaoka bounds no longer coincide. The optimal matrices that achieve the minimum in the Holevo bound are found to be

$$X_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad X_y = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(28)

which gives $c_{H,2} = 2$. This means that there exists a sequence of collective measurements that can saturate a variance of $v_x = v_y = 1$ in the asymptotic limit.

Unlike the single-parameter case, the optimal X_x and X_y operators for the Nagaoka bound are different from those which optimise the Holevo bound. For the Nagaoka bound the optimal matrices are

$$X_x = \frac{1}{2-\epsilon} \begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & i \\ i & 0 & 0 & i \\ 0 & -i & -i & 0 \end{pmatrix}, \quad X_y = \frac{1}{2-\epsilon} \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(29)

which gives $c_{NH,2} = 4/(2-\epsilon)$. Since there is a gap between the Holevo and Nagaoka bounds, a separable measurement cannot saturate the Holevo bound—a collective measurement is required. We show in Supplementary Note 5 that the Nagaoka bound is saturated by a family of five-outcome POVMs which gives $v_x = v_y = 2/(2-\epsilon)$. This means that when restricted to separable measurements, this is the smallest pair of variances possible.

Three parameter estimation. Finally, for estimating all three angles θ_x, θ_y, and θ_z simultaneously, we find the Holevo and Nagaoka–Hayashi bounds are

$$c_{H,3} = 2 + \frac{1}{(1-\epsilon)^2} \quad \text{and} \quad c_{NH,3} = \frac{4}{2-\epsilon} + \frac{1}{(1-\epsilon)^2}.$$

(30)

Just like the two-parameter case, the gap between the two bounds implies that a collective measurement is required to saturate the Holevo bound. These bounds are achieved by the same estimator operators Eq. (28) for the Holevo bound and Eq. (29) for the Nagaoka–Hayashi bound with the additional

$$X_z = \frac{1}{1-\epsilon} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(31)

We write down an explicit POVM that can approach $c_{NH,3}$ with $v_x = v_y = 2/(2-\epsilon)$ and $v_z \to 1/(1-\epsilon)^2$ in Supplementary Note 5 showing that this bound is tight.

In order to quantify the estimation accuracy, we define the average preciseness for simultaneous estimation of n parameters with $n/(v_1 + \cdots + v_n)$ as a figure of merit on how good the estimators perform. By construction, a large average preciseness implies that all n parameters can be determined accurately. We plot this quantity in Fig. 1 for all three estimation cases. We also note that in the two and three parameter examples, it is easy to check that the SLD Fisher information matrix is diagonal. Furthermore, the model is asymptotically classical and the Holevo bound coincides with the SLD bound[21,69].

Estimation on multiple copies. We now demonstrate the usefulness of the Nagaoka–Hayashi bound and the associated SDP by examining the precision limits when we perform collective measurements on finite copies of the probe state. We denote the Nagaoka–Hayashi bound for N copies of the same probe as $c_{NH}(S_\theta^{\otimes N})$. For a large number of copies of the probe state, we expect the Nagaoka–Hayashi bound to tend to the Holevo bound, $\lim_{N \to \infty} N c_{NH}(S_\theta^{\otimes N}) = c_H$. For any finite N, we know that $N c_{NH}(S_\theta^{\otimes N}) \geq c_H$ which follows from $N c_H(S_\theta^{\otimes N}) = c_H$. Figure 2 shows how the gap between the two bounds shrinks for an increasing number of copies of the probe state. We consider up to three copies of the probe state. Without the Nagaoka–Hayashi bound, a brute force search for the optimal measurement strategy for three copies would require optimising an M outcome POVM, where

each outcome is a 64-by-64 matrix. Thus, the Nagaoka–Hayashi bound and the associated SDP offer an efficient way to investigate the asymptotic attainability of the Holevo bound. It provides a tool to address how fast optimal estimators on finite copies can converge to the asymptotic bound.

This example demonstrates several interesting features of finite copy metrology. First, we are able to definitively show that there exists a gap between the attainable precision with collective and separable measurements. Without a separable measurement bound, such a claim is not possible as any gap between a numerically optimal POVM and the Holevo bound may be a result of a deficiency in the numerical search as opposed to a physically meaningful gap. Second, as we are able to find a POVM which coincides with the Nagaoka–Hayashi bound, we are able to say with certainty that this POVM is optimal. Finally, we are able to investigate the attainability of the Holevo bound. While it is known that the Holevo bound is asymptotically attainable, it is not known how many copies of the probe state are required to get close to the Holevo bound. As mentioned above, to investigate this numerically with a POVM search is computationally very expensive. The SDP presented circumvents this and allows us to
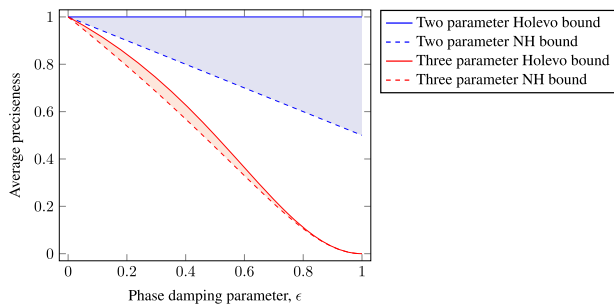
investigate the attainability of the Holevo bound in a numerically efficient manner.

Example 2: Phase and transmissivity estimation in interferometry

In our next example, we consider the problem of estimation of phase change ϕ and transmissivity η in one arm of an interferometer as shown in Fig. 3. Following Crowley et al.[33], we consider initial pure states with a definite photon-number N across the two modes $|\psi_{in}\rangle = \sum_{k=0}^{N}|k, N-k\rangle a_k$, where $|N_1, N_2\rangle$ represents a state with N_1 photons in the first mode and N_2 photons in the second mode. One family of states with a fixed photon number is the Holland–Burnett states which are obtained by interfering two Fock states with an equal number of photons on a balanced beam splitter. These states lead to a phase estimation precision better than an interferometer driven by a coherent light source with the same number of photons[70]. The Holevo bound for the Holland–Burnett state was computed by Albarelli et al.[53] for up to $N = 14$. In general, the Holevo bound requires a collective measurement on several probes to be saturated. But for some values of N and η, the Holevo bound can be saturated by a separable measurement, $\Pi^{(\phi)}$ that optimally measures the phase[53].

We compute the Nagaoka bound for these states for different values of η with $\phi = 0$ for N up to 14 using our SDP. We find that the Nagaoka and Holevo bounds always coincide (up to numerical noise). This is to be expected when $\Pi^{(\phi)}$ saturates the Holevo bound, but is not so obvious when it does not. The fact that there is no gap between the Holevo and Nagaoka bound implies one of the two possibilities: either (i) the Nagaoka bound is not tight or (ii) separable measurements are always optimal for simultaneous estimation of ϕ and η, in other words, collective measurements cannot do better. In the following, we show that the second statement is true.

Measurement saturating the Nagaoka bound. The initial pure state $|\psi_{in}\rangle = \sum_{k=0}^{N}|k, N-k\rangle a_k$ transforms in the lossy interferometer channel to the following state:

$$S_{\phi,\eta} = \bigoplus_{l=0}^{N}|\psi_l\rangle p_l\langle\psi_l|, \tag{32}$$

where each term in the direct sum

$$|\psi_l\rangle = \sum_{k=l}^{N}|k-l, N-k\rangle a_k e^{ik\phi}\sqrt{\frac{b_{kl}}{p_l}} \tag{33}$$

Fig. 1 Holevo and Nagaoka–Hayashi bounds for estimating multiple orthogonal rotation parameters simultaneously. For a maximally entangled two-qubit probe under the action of the phase damping channel, we plot the Holevo bounds (solid lines) and Nagaoka–Hayashi bounds (dashed lines) in terms of average preciseness for estimating two (blue) and three (red) orthogonal rotation parameters simultaneously. The Nagaoka–Hayashi bounds can be achieved by a separable measurement on a single probe, while the Holevo bound requires a collective measurement on a possibly infinite number of copies. The shaded area shows the gap between the two bounds. For estimating a single parameter, the Holevo and Nagaoka–Hayashi bounds coincide and are equal to the two-parameter Holevo bound.
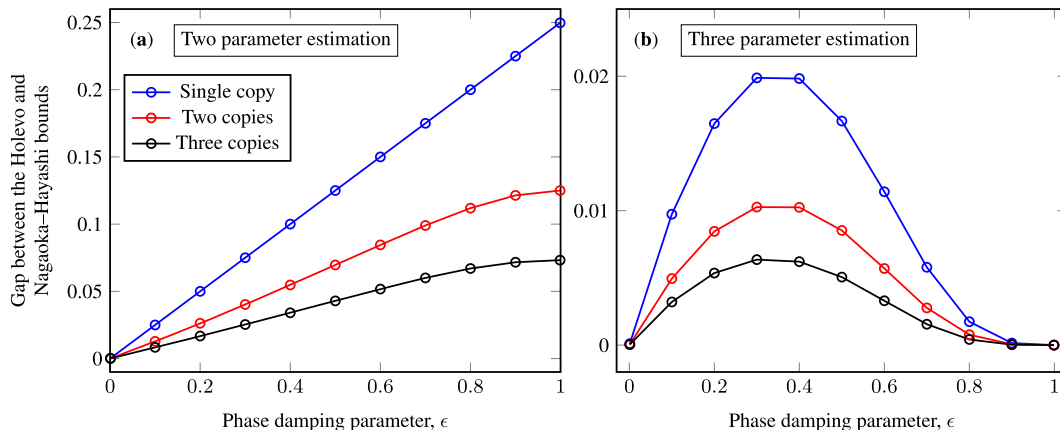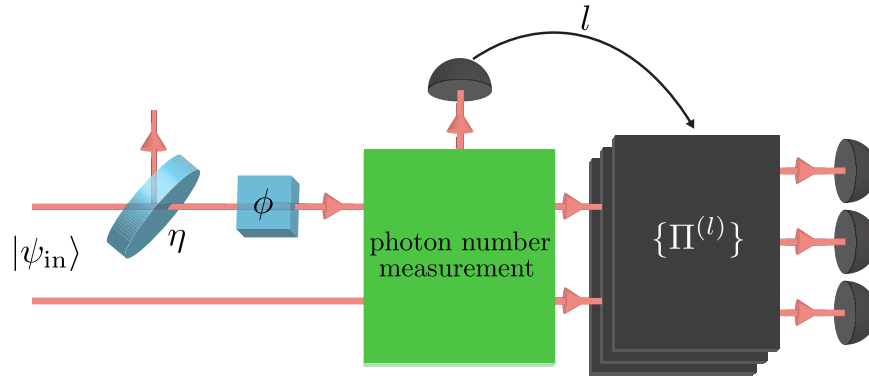

Fig. 2 Estimating multiple parameters with collective measurements on finite copies of the probe state. Both figures show how the gap between the Holevo and Nagaoka–Hayashi bounds shrinks as the number of copies of the probe state increases for estimating two (**a**) and three (**b**) parameters. The Nagaoka–Hayashi bounds are rescaled by the number of copies of the probe state to account for the resources used.

Fig. 3 Schematic for optimal estimation of the phase shift ϕ and interferometer transmissivity η using a two mode state $|\psi_{in}\rangle$ having definite photon-number N. The measurement can be performed in two stages. The first stage (green block) involves performing a projective measurement over the photon-number subspace to determine the number of photons lost, l. The outcome of this measurement is then used to select a three-outcome POVM $\{\Pi^{(l)}\}$ for the second stage (black box). This measurement strategy saturates not only the Nagaoka bound, but also the Holevo bound.

represents a state with l lost photons. The state $S_{\phi,\eta}$ is a mixed state with rank $N+1$. Here, $b_{kl} = \binom{k}{l} \eta^{k-l}(1-\eta)^l$ are the beam-splitter coefficients and p_l represents the probability of losing l photons. The partial derivatives of $S_{\phi,\eta}$ share the same direct sum structure

$$\frac{\partial S_{\phi,\eta}}{\partial \phi} = \bigoplus_{l=0}^{N} \left(|\partial_\phi \psi_l\rangle p_l \langle \psi_l| + |\psi_l\rangle p_l \langle \partial_\phi \psi_l| \right),$$

$$\frac{\partial S_{\phi,\eta}}{\partial \eta} = \bigoplus_{l=0}^{N} \left(|\psi_l\rangle \frac{\partial p_l}{\partial \eta} \langle \psi_l| + |\partial_\eta \psi_l\rangle p_l \langle \psi_l| + |\psi_l\rangle p_l \langle \partial_\eta \psi_l| \right),$$

(34)

with each block having at most rank 2. Thus, what we have is a direct sum of pure state models, and for such a model, we have a separable measurement with a direct sum structure that can achieve the Holevo bound[55]. Each block can be measured separately but we cannot minimise $v_\eta + v_\phi$ separately in each block. This is because how much weight we attach to η or ϕ in one block will depend on how much information about them that we can get from the other blocks. But regardless of the weights, each $l \neq N$ block requires at most a three outcome POVM to saturate the Holevo bound, so the total number of POVM outcomes needed is at most $3N + 1$. The extra 1 comes from the $l = N$ block where all photons are lost. An analytic POVM that saturates the Holevo bound for the $N = 1$ case is given in Supplementary Note 6. The dual solution to the Nagaoka–Hayashi bound is presented in Supplementary Note 7.

This problem demonstrates a very different but equally insightful feature of finite copy metrology compared to the qubit rotation problem. The simultaneous estimation of phase and loss has been very well studied in the literature[6,8,33,53], however until now the fact that separable measurements are sufficient to reach the ultimate attainable precision had remained unknown. This insight was only possible with our SDP, which allowed the Nagaoka–Hayashi and Holevo bounds to be compared for large N. We plot the numerically calculated Nagaoka–Hayashi and Holevo bounds for different N and η in Fig. 4. The fact that collective measurements are not required to reach the Holevo bound in this example may be important from a fundamental viewpoint.

DISCUSSION
We have presented the Nagaoka–Hayashi bound for the simultaneous estimation of multiple parameters when restricted to separable measurements. This ensures it is always tighter or just as tight as the Holevo bound. A gap between the two bounds would imply that the Holevo bound cannot be
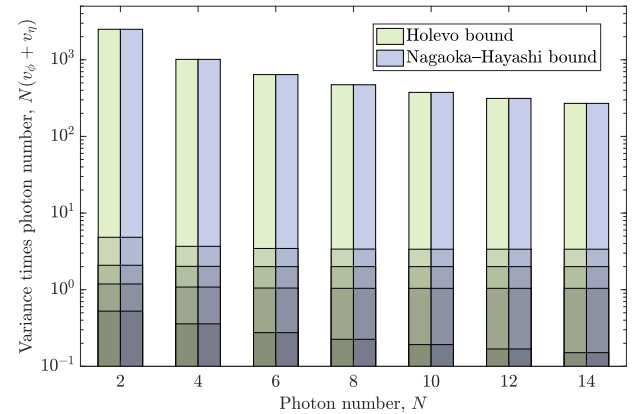

Fig. 4 The Holevo and Nagaoka–Hayashi bounds multiplied by photon number for estimating phase change ϕ and transmissivity η using the Holland–Burnett states. The numerical results show that the two bounds coincide for N up to and including 14 for any value of η. We show in the main text that there exists a separable measurement that reaches the ultimate attainable precisions in this example. In this case, a collective POVM can perform better than a separable POVM. Different shades correspond to different η values, with darker colours corresponding to larger η. Results are shown for $\eta = 0.01, 0.25, 0.5, 0.75,$ and 0.99.

achieved with a separable measurement and a collective measurement is needed to saturate it. In addition, we have shown that the Nagaoka–Hayashi bound can be formulated as a semidefinite programme, allowing it to be solved efficiently. We have demonstrated our results with two examples. These examples illustrate how our results can be used to recognise when a collective measurement is essential and when it is not. Our results can be applied to many other problems in multiparameter quantum metrology and will help quantify the maximal advantage collective measurements have to offer. In some cases, a separable measurement is already optimal, simplifying any experimental realisation.

In the first example, we have assumed that the damping strength ϵ is known. However, in a practical setting, it would be more realistic to consider ϵ as a nuisance parameter, an unknown parameter that we are not interested in which nevertheless may hinder our measurement precision[25,42,71]. The quantum Cramér–Rao bound in the presence of nuisance parameters can be computed utilising a low-rank weight matrix[25,71]. As we show in Supplementary Note 3, our SDP formalism can be immediately

applied to such cases. An interesting extension to this work would be to investigate examples which incorporate nuisance parameters.

DATA AVAILABILITY

CODE AVAILABILITY

REFERENCES

1. Giovannetti, V., Lloyd, S. & Maccone, L. Quantum-enhanced measurements: beating the standard quantum limit. *Science* **306**, 1330–1336 (2004).
2. Giovannetti, V., Lloyd, S. & Maccone, L. Advances in quantum metrology. *Nat. Photonics* **5**, 222–229 (2011).
3. Aasi, J. et al. Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light. *Nat. Photonics* **7**, 613–619 (2013).
4. Caves, C. M. Quantum-mechanical noise in an interferometer. *Phys. Rev. D* **23**, 1693–1708 (1981).
5. Barnett, S. M., Fabre, C., & Maıtre, A. Ultimate quantum limits for resolution of beam displacements. *Eur. Phys. J. D* **22**, 513–519 (2003).
6. Dorner, U. et al. Optimal quantum phase estimation. *Phys. Rev. Lett.* **102**, 040403 (2009).
7. Kacprowicz, M., Demkowicz-Dobrzański, R., Wasilewski, W., Banaszek, K. & Walmsley, I. A. Experimental quantum-enhanced estimation of a lossy phase shift. *Nat. Photonics* **4**, 357–360 (2010).
8. Demkowicz-Dobrzański, R. et al. Quantum phase estimation with lossy interferometers. *Phys. Rev. A* **80**, 013825 (2009).
9. Tsang, M., Nair, R. & Lu, X. M. Quantum theory of superresolution for two incoherent optical point sources. *Phys. Rev. X* **6**, 031033 (2016).
10. Tsang, M. Resolving starlight: a quantum perspective. *Contemp. Phys.* **60**, 279–298 (2019).
11. Yonezawa, H. et al. Quantum-enhanced optical-phase tracking. *Science* **337**, 1514–1517 (2012).
12. Zhang, L. et al. Quantum-limited fiber-optic phase tracking beyond π range. *Opt. Express* **27**, 2327–2334 (2019).
13. Giovannetti, V., Lloyd, S. & Maccone, L. Quantum-enhanced positioning and clock synchronization. *Nature* **412**, 417–419 (2001).
14. Lamine, B., Fabre, C. & Treps, N. Quantum improvement of time transfer between remote clocks. *Phys. Rev. Lett.* **101**, 123601 (2008).
15. Helstrom, C. W. The minimum variance of estimates in quantum signal detection. *IEEE Trans. Inf. Theory* **14**, 234–242 (1968).
16. Helstrom, C. W. Minimum mean-squared error of estimates in quantum statistics. *Phys. Lett. A* **25**, 101–102 (1967).
17. Belavkin, V. P. Generalized uncertainty relations and efficient measurements in quantum systems. *Theor. Math. Phys.* **26**, 213–222 (1976).
18. Robertson, H. P. The uncertainty principle. *Phys. Rev.* **34**, 163–164 (1929).
19. Cimini, V. et al. Quantum sensing for dynamical tracking of chemical processes. *Phys. Rev. A* **99**, 053817 (2019).
20. Rehacek, J. et al. Multiparameter quantum metrology of incoherent point sources: towards realistic superresolution. *Phys. Rev. A* **96**, 062107 (2017).
21. Ragy, S., Jarzyna, M. & Demkowicz-Dobrzański, R. Compatibility in multiparameter quantum metrology. *Phys. Rev. A* **94**, 052108 (2016).
22. Szczykulska, M., Baumgratz, T. & Datta, A. Multi-parameter quantum metrology. *Adv. Phys.-X* **1**, 621–639 (2016).
23. Kull, I., Guérin, P. A. & Verstraete, F. Uncertainty and trade-offs in quantum multiparameter estimation. *J. Phys. A* **53**, 244001 (2020).
24. Demkowicz-Dobrzański, R., Górecki, W. & Guţă, M. Multi-parameter estimation beyond quantum Fisher information. *J. Phys. A Math. Theor.* **53**, 363001 (2020).
25. Suzuki, J., Yang, Y. & Hayashi, M. Quantum state estimation with nuisance parameters. *J. Phys. A Math. Theor.* **53**, 453001 (2020).
26. Steinlechner, S. et al. Quantum-dense metrology. *Nat. Photonics* **7**, 626–630 (2013).
27. Hou, Z., Zhu, H., Xiang, G. Y., Li, C. F. & Guo, G. C. Achieving quantum precision limit in adaptive qubit state tomography. *npj Quantum Inf.* **2**, 16001 (2016).
28. Roccia, E. et al. Entangling measurements for multiparameter estimation with two qubits. *Quantum Sci. Technol.* **3**, 01LT01 (2018).
29. Liu, Y. et al. Loss-tolerant quantum dense metrology with SU(1, 1) interferometer. *Opt. Express* **26**, 27705–27715 (2018).
30. Hou, Z. et al. Deterministic realization of collective measurements via photonic quantum walks. *Nat. Commun.* **9**, 1414 (2018).
31. Humphreys, P. C., Barbieri, M., Datta, A. & Walmsley, I. A. Quantum enhanced multiple phase estimation. *Phys. Rev. Lett.* **111**, 070403 (2013).
32. Genoni, M. G. et al. Optimal estimation of joint parameters in phase space. *Phys. Rev. A* **87**, 012107 (2013).
33. Crowley, P., Datta, A., Barbieri, M. & Walmsley, I. A. Tradeoff in simultaneous quantum-limited phase and loss estimation in interferometry. *Phys. Rev. A* **89**, 023845 (2014).
34. Gagatsos, C. N., Branford, D. & Datta, A. Gaussian systems for quantum-enhanced multiple phase estimation. *Phys. Rev. A* **94**, 042342 (2016).
35. Baumgratz, T. & Datta, A. Quantum enhanced estimation of a multidimensional field. *Phys. Rev. Lett.* **116**, 030801 (2016).
36. Chrostowski, A., Demkowicz-Dobrzański, R., Jarzyna, M. & Banaszek, K. On super-resolution imaging as a multiparameter estimation problem. *Int. J Quantum Inf.* **15**, 1740005 (2017).
37. Pezzè, L. et al. Optimal measurements for simultaneous quantum estimation of multiple phases. *Phys. Rev. Lett.* **119**, 130504 (2017).
38. Suzuki, J. Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem. *J. Math. Phys.* **57**, 042201 (2016).
39. Szczykulska, M., Baumgratz, T. & Datta, A. Reaching for the quantum limits in the simultaneous estimation of phase and phase diffusion. *Quantum Sci. Technol.* **2**, 044004 (2017).
40. Albarelli, F., Barbieri, M., Genoni, M. G. & Gianani, I. A perspective on multi-parameter quantum metrology: from theoretical tools to applications in quantum imaging. *Phys. Lett. A* **384**, 126311 (2020).
41. Assad, S. M. et al. Accessible precisions for estimating two conjugate parameters using Gaussian probes. *Phys. Rev. Res.* **2**, 023182 (2020).
42. Tsang, M., Albarelli, F. & Datta, A. Quantum semiparametric estimation. *Phys. Rev. X* **10**, 031023 (2020).
43. Carollo, A., Spagnolo, B., Dubkov, A. A. & Valenti, D. On quantumness in multi-parameter quantum estimation. *J. Stat. Mech.: Theory Exp.* **2019**, 094010 (2019).
44. Sidhu, J. S. & Kok, P. Geometric perspective on quantum parameter estimation. *AVS Quantum Sci.* **2**, 014701 (2020).
45. Polino, E., Valeri, M., Spagnolo, N. & Sciarrino, F. Photonic quantum metrology. *AVS Quantum Sci.* **2**, 024703 (2020).
46. Suzuki, J. Parameter estimation of qubit states with unknown phase parameter. *Int. J. Quantum Inf.* **13**, 1450044 (2015).
47. Bradshaw, M., Lam, P. K. & Assad, S. M. Ultimate precision of joint quadrature parameter estimation with a Gaussian probe. *Phys. Rev. A* **97**, 012106 (2018).
48. Bradshaw, M., Assad, S. M. & Lam, P. K. A tight Cramér–Rao bound for joint parameter estimation with a pure two-mode squeezed probe. *Phys. Lett. A* **381**, 2598–2607 (2017).
49. Hayashi, M. *Asymptotic Theory of Quantum Statistical Inference: Selected Papers* (World Sci., 2005).
50. Yuen, H. & Lax, M. Multiple-parameter quantum estimation and measurement of nonselfadjoint observables. *IEEE Trans. Inf. Theory* **19**, 740–750 (1973).
51. Gill, R. D. & Massar, S. State estimation for large ensembles. *Phys. Rev. A* **61**, 042312 (2000).
52. Holevo, A. S. *Probabilistic and Statistical Aspects of Quantum Theory*, Vol. 1 (Springer Sci. Bus. Media, 2011).
53. Albarelli, F., Friel, J. F. & Datta, A. Evaluating the Holevo Cramér-Rao bound for multiparameter quantum metrology. *Phys. Rev. Lett.* **123**, 200503 (2019).
54. Sidhu, J. S., Ouyang, Y., Campbell, E. T. & Kok, P. Tight bounds on the simultaneous estimation of incompatible parameters. *Phys. Rev. X* **11**, 011028 (2021).
55. Matsumoto, K. A new approach to the Cramér-Rao-type bound of the pure-state model. *J. Phys. A: Math. Gen.* **35**, 3111–3123 (2002).
56. Kahn, J. & Guţă, M. Local asymptotic normality for finite dimensional quantum systems. *Commun. Math. Phys.* **289**, 597–652 (2009).
57. Yamagata, K., Fujiwara, A. & Gill, R. D. Quantum local asymptotic normality based on a new quantum likelihood ratio. *Ann. Stat.* **41**, 2197–2217 (2013).
58. Yang, Y., Chiribella, G. & Hayashi, M. Attaining the ultimate precision limit in quantum state estimation. *Commun. Math. Phys.* **368**, 223–293 (2019).
59. Nagaoka, H. A new approach to Cramer–Rao bounds for quantum state estimation. In *Asymptotic Theory of Quantum Statistical Inference: Selected Papers*, 100–112 (World Sci. 2005). Originally published as IEICE Technical Report, **89**, 228, IT 89–42, 9–14 (1989).
60. Nagaoka, H. A generalization of the simultaneous diagonalization of Hermitian matrices and its relation to quantum estimation theory. In *Asymptotic Theory of Quantum Statistical Inference: Selected Papers*, 133–149 (World Sci., 2005). Originally published as Trans. Jap. Soc. Indust. Appl. Math., **1**, 43–56, (1991) in Japanese. Translated to English by Y. Tsuda.

61. Hayashi, M. On simultaneous measurement of noncommutative observables. In *Development of Infinite-Dimensional Non-Commutative Anaysis*, 96–188 (Kyoto Univ., 1999).

62. Watanabe, Y., Sagawa, T. & Ueda, M. Uncertainty relation revisited from quantum estimation theory. *Phys. Rev. A* **84**, 042121 (2011).

63. Lofberg, J. Yalmip: a toolbox for modeling and optimization in matlab. in *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No. 04CH37508)*, 284–289 (IEEE, 2004).

64. ApS, M. *The MOSEK Optimization Toolbox for MATLAB Manual. Version 9.0* http://docs.mosek.com/9.0/toolbox/index.html (MOSEK ApS, 2019).

65. Boyd, S. & Vandenberghe, L. *Convex Optimization* (Cambridge Univ. Press, 2004).

66. Huelga, S. F. et al. Improvement of frequency standards with quantum entanglement. *Phys. Rev. Lett.* **79**, 3865–3868 (1997).

67. Myatt, C. J. et al. Decoherence of quantum superpositions through coupling to engineered reservoirs. *Nature* **403**, 269–273 (2000).

68. Ma, J., Huang, Y.-x., Wang, X. & Sun, C. P. Quantum Fisher information of the Greenberger-Horne-Zeilinger state in decoherence channels. *Phys. Rev. A* **84**, 022302 (2011).

69. Suzuki, J. Information geometrical characterization of quantum statistical models in quantum estimation theory. *Entropy* **21**, 703 (2019).

70. Holland, M. J. & Burnett, K. Interferometric detection of optical phase shifts at the Heisenberg limit. *Phys. Rev. Lett.* **71**, 1355–1358 (1993).

71. Suzuki, J. Nuisance parameter problem in quantum estimation theory: Tradeoff relation and qubit examples. *J. Phys. A Math. Theor.* **53**, 264001 (2020).

AUTHOR CONTRIBUTIONS

S.M.A. and J.S. conceived the project. S.M.A., J.S. and L.O.C. developed the theory, constructed the proof of Theorem 1, and worked out the optimal POVMs in the examples. L.O.C. and S.M.A. performed the numerical SDP simulations. L.O.C. and S.M.A. wrote the manuscript. All authors contributed to discussions regarding the results in this paper. P.K.L. supervised the project.

COMPETING INTERESTS

The authors declare no competing interests.

ADDITIONAL INFORMATION

Supplementary information The online version contains supplementary material available at https://doi.org/10.1038/s41534-021-00414-1.

Correspondence and requests for materials should be addressed to L.O.C., J.S. or S.M.A.

Reprints and permission information is available at http://www.nature.com/reprints

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.