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**INVESTIGATING THE ENERGY COST FOR
CONTROLLING COMPLEX SOCIAL NETWORKS**

CHEN HONG

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2021

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CONTROLLING COMPLEX SOCIAL NETWORKS**

CHEN HONG

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for the
degree of Doctor of Philosophy

2021

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me except where otherwise stated in this thesis. The thesis work has not been submitted for a degree or professional qualification to any other university or institution. I declare that this thesis is written by myself and is free of plagiarism and of sufficient grammatical clarity to be examined. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

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..... 10/8/2021

..... Yong Ee Hou

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Authorship Attribution Statement

This thesis contains material from 2 papers published in the following peer-reviewed journals in which I am listed as an author.

Chapter 2 is published as H. Chen and E.H. Yong. Optimizing target nodes selection for the control energy of directed complex networks. *Scientific Reports* volume 10, Article number: 18112 (2020). DOI: 10.1038/s41598-020-75101-w

Chapter 3 is published as H. Chen and E.H. Yong. Energy cost study for controlling complex social networks with conformity behavior. *Phys. Rev. E* 104, 014301 (2021). DOI: 10.1103/PhysRevE.104.014301

Chapter 4 is on the arXiv repository as H. Chen and E.H. Yong. How zealots affect the energy cost for controlling complex social networks. arXiv identifier: arXiv:2107.11744. (Submitting to *Physical Review E* for publication).

The contributions of the co-authors in all papers are as follows:

- I conceived the initial projects directions.
- I performed all analytical calculations, and wrote the programming codes to perform the numerical computations.
- I analyzed the results.
- I and Asst Prof Yong Ee Hou wrote, reviewed, and revised the manuscripts.
- Asst Prof Yong Ee Hou supervised the research.

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Abstract

The controllability of complex networks has recently emerged as a promising field of interdisciplinary research. At its core, the idea is to study, given a dynamical system modeled by state equations with linear dynamics, whether or not the state vector of a complex system could be driven towards some predefined state. Different from traditional control engineering, which deals only with small dimensions, the controllability of complex networks borrows techniques from graph theory and statistical physics to guarantee the controllability of an arbitrary complex system with arbitrary dimensions.

This thesis advances the field by exploring the controllability of such networked systems particularly in the context of social physics (socio-physics). There are three parts to the thesis. Firstly, an energy cost optimization problem is solved, where the choice of target nodes (which individuals) to control is being optimized. It was found that using the optimization algorithm, the energy cost is minimized when the target nodes are close to the driver nodes, leading to reduction in the control energy by a few orders of magnitude.

Secondly, the scaling laws of the energy cost as a function of final control time T_f when controlling a complex social network with conformity behavior is presented. It was found that, when compared to networks without conformity, the mechanism of conformity always leads to a situation where the network is easier to control.

Thirdly, how zealots, individuals with unwavering opinions, influence the amount of effort needed to control a complex social network is studied. It was found that the presence of zealots alters the energy cost at a quadratic rate with respect to their own fixed beliefs. However, whether or not the zealots' presence increases or decreases the energy cost, relative to the situation where there were no zealots present, is affected by the interplay between different parameters such as the zealots' beliefs, number of drivers, final control time regimes, network effects, network dynamics, number and configurations of normal nodes influenced by the zealots.

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Chapter 1

Introduction

The controllability of complex networks [1, 2] has seen an increasing number of studies [3–34] in recent years. This is because the results from Ref. [1] guarantees, through structural controllability [35], that a networked system evolving with linear dynamics, with arbitrary N dimensions (N number of nodes), is controllable if a suitable set of control signals could be injected into the system. For example, a television remote control is a complex system with many interacting components, with the purpose to deliver the right signals through the push of a few appropriate buttons to change the channel or adjust the television volume. This analogy works just as well with a motor vehicle: It is but a complex system with many interacting components, of which the laymen know nothing about, but operate everyday by interacting with the break, gas pedal (accelerator), and steering wheel. Thus, it would be ideal to control a complex system by interacting with just a few components of the system to gain complete control over it. In other words, a man-made system is controllable if they are behaving exactly the way that they were engineered for. By controllable, it means that the node states could be steered from any arbitrary initial state vector to any arbitrary final state vector in finite time. Here, the word ‘state’ is an arbitrary construct that describes any arbitrary quantity, depending on the type of networked system being modelled at hand. For example, a node state may represent the amount of traffic which passes through node i in a communication network [36], the transcription factor concentration in a gene regulatory network [37], or the opinion of an agent in a consensus network [38–41] and so on.

When the dimensionality of the complex system is extended beyond small dimensions, control theory meets statistical physics [42], and various complex networks can be used to model dynamical systems with high dimensions, where it would be rewarding to understand how to control them. This gives rise to several interesting research options. For example, controlling networks such as power grids [43, 44], communication networks [45, 46], gene regulatory networks [47, 48], infection networks [47, 49], neuronal networks [50, 51], food webs [52], and opinion networks [38–41]. Naturally, many of these complex systems have nonlinear dynamics [47], where the use of linear dynamics modelling may be inapplicable. Thus, the controllability of these nonlinear systems may be studied by linearizing the dynamics around their equilibrium points [2], although it is worth mentioning that a recent work questioned whether or not the linearized versions of nonlinear systems are indeed controllable as claimed [53]. Nevertheless, linear dynamics is sufficient for capturing a significant class of dynamical systems such as opinion networks [38–41], which is what this thesis is all about.

This thesis is about controlling complex social networks, where it aims to check whether or not previously published results in the controllability of complex networks still hold, if not, how do they differ, when the complex system is modified slightly to include various types of

social physics [54] features such as conformity behavior [27], or zealotry [55–61]. The reasons for focusing on social physics are two-fold. Firstly, opinion networks [38–41] usually follow linear dynamics, making them suitable to be studied in the context of network control, which requires the dynamics of the networks to be linear [2]. Secondly, there appears to be a wealth of social physics-specific features which modify the network linear dynamics slightly, but can lead to markedly different system dynamical behavior, suggesting that an investigation into this area would be fruitful. In studying controlling complex social networks, each member represents an individual, and the node states represent support/opposition of a particular idea or product. For example, when a node has a high positive node state, it represents strong support of an idea or a product, and a negative node state represents opposition or rejection of the idea or product. In particular, this thesis explores the amount of effort needed by the control signals to control complex social networks, where there may be intricacies in the modelling details that differ from the canonical continuous-time linear dynamics widely used in networked controllability. Before outlining the main contributions of the thesis in detail, chapter by chapter, it would be useful to give a succinct literature review. For example, what does “amount of effort” or energy cost, or control energy mean? (They mean the same thing and are used interchangeably throughout the thesis).

1.1 Literature review

1.1.1 Continuous-time linear dynamics

In controllability of complex networks, consider a networked system with N number of nodes evolving in time under the influence of M number of control signals as follows [1, 2]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (1.1)$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_N(t)]^T$ is the $N \times 1$ time-dependent state vector, $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_M(t)]^T$ is the $M \times 1$ time-dependent vector of control signals, $\mathbf{A} \in \mathbb{R}^{N \times N}$ describes the complex networked interactions between each node such that a_{ij} is non-zero if there is a directed link from node j to node i (for undirected, then $a_{ij} = a_{ji}$), otherwise it is zero, and $\mathbf{B} \in \mathbb{R}^{N \times M}$ is the input matrix, comprising elementary column vectors (each control signal attaches only to a single node), which says which nodes in the network have been injected with a control signal, where $b_{ij} = 1$ if control signal j attaches to node i , otherwise it is zero. The nodes which directly receive control signals are called the driver nodes, where they have their node states directly altered. Through the complex networked interactions, other nodes also have their states altered indirectly, and given that an appropriate $\mathbf{u}(t)$ is chosen, the entire state vector $\mathbf{x}(t)$ can be controlled and steered from initial state vector $\mathbf{x}_0(t_0)$ to a predefined final state vector $\mathbf{x}_f(t_f)$ in finite time $[t_0, t_f]$, where t_0 is the initial time that the control signals are applied, and t_f is the final time that the control signals are applied. Note that because $\mathbf{A}$ and $\mathbf{B}$ are time-independent matrices, the dynamics of Eq. (1.1) is said to be linear time-invariant (LTI).

The mathematical form of $\mathbf{u}(t)$ has yet been described. Depending on the control objectives [62], different types of $\mathbf{u}(t)$ may be designed to achieve control under different specific constraints. These will be discussed at a later section, in section 1.1.4, but for now it would be illustrative to plot the dynamics of Eq. (1.1) belonging to a $N = 40$, $\langle k \rangle = 6$ undirected Erdős–Rényi (ER) random network in Fig. 1.1. In Fig. 1.1(a), the node states evolution in the absence of control $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t)$ is shown, where the initial state vector $\mathbf{x}_0$ has elements drawn

from random uniform $[0, 1]$. In Fig. 1.1(b), the node states are being driven toward the final state vector $\mathbf{x}_f = [2, 2, \dots, 2]^T$ using 4 control signals attached to 4 nodes chosen at random.

Figure 1.1: Node states evolution with and without control in a $N = 40$, $\langle k \rangle = 6$ ER network. (a) Node states evolution in the absence of control. (b) Node states evolution while all N nodes of the network is driven toward $\mathbf{x}_f$ using 4 control signals. Inset shows that all node states converge to $\mathbf{x}_f = [2, 2, \dots, 2]^T$.

1.1.2 Ensuring controllability

Directed networks

While it is true that an appropriately chosen $\mathbf{u}(t)$ would lead to the dynamics shown in Fig. 1.1(b), where the state vector is controlled toward $\mathbf{x}_f(t_f)$, the choice of $\mathbf{B}$ matrix, describing where in the network should control signals be injected, is arguably more important. For a generic directed complex network, Ref. [1] proved using the statistical physics cavity method [63–65] that when a directed graph, on its equivalent bipartite graph representation, have nodes which are maximally matched, a system is controllable by attaching control signals to the unmatched nodes. Owing to the difficulty of the cavity method, the proof is not discussed here. Nevertheless, it is useful to explain how the maximum matching of nodes in bipartite graphs can lead to controllability.

Borrowing a simple example from Ref. [1] (the example is credited to Ref. [1], but figure is generated independently), maximum matching of nodes is demonstrated in Fig. 1.2. Fig. 1.2(a) shows the complex network representation of the directed network, where the networked interactions are represented by the directed links, and x_1 , x_2 , and x_3 are the node states. Equivalently, the complex network representation can be converted into bipartite representation as shown in Fig. 1.2(b). The column on the left with the superscript ‘+’ represents nodes which give a directed link to others, while the column on the right with the superscript ‘−’ represents nodes which receive a link from others or even itself. Each of the nodes, x_1 , x_2 , or x_3 can be giving or receiving links, so they are represented on both sides by x_1^+ , x_2^+ , x_3^+ and x_1^- , x_2^- , x_3^- . On the bipartite graph, a node from the ‘+’ column is matched with another node from the ‘−’ column if a line connects them (matched nodes are colored green, and unmatched nodes are colored white). Once a node has been matched with another, duplicate matching are disallowed. For example, in Fig. 1.2(b), x_1^+ is already matched with x_2^- (denoted by red line), so x_1^+ cannot also match with x_3^- (they are not matched, so the link between them are denoted

Figure 1.2: (a) Complex network representation of a simple directed network. (b) Bipartite representation of the same simple directed network. Red lines represent that two nodes are matched (matched nodes are colored green), while unmatched nodes are not colored.

by black line). However, x_3^+ can also match with x_3^- since the prior connection did not match any of these nodes. Thus, the configuration shown in Fig. 1.2(b) has a matching cardinality of size 2, which is the maximum matching for this example. The set of all possible matchings are not unique. If suppose, instead, x_1^+ is matched with x_3^- , then no other matchings are allowed based on the rules already described, and the matching cardinality of this set is size 1, which is not the maximum matching. Ref. [1] argued that by finding the maximum matching, the unmatched nodes on the ‘-’ column need to receive control signals and controllability can be guaranteed. Note, further, that the maximum matching is not unique and there may be different combinations of maximum matching in a complex network. To implement maximum matching in directed complex networks, the computationally efficient (worst case $\mathcal{O}(\sqrt{N}L)$, where N is the number of nodes, and L is the number of links) Hopcroft-Karp algorithm [66] is used.

Once the unmatched nodes have been found, the N_D minimum drivers (the minimum $b_{ij} = 1$ matrix elements) to guarantee controllability have been found, which can be checked against the $N \times NM$ Kalman controllability matrix [1, 67, 68]

$$\mathbf{K} = (\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{N-1}\mathbf{B}), \quad (1.2)$$

where if $\mathbf{K}$ has full rank such that $\text{rank}(\mathbf{K}) = N$, the system is controllable, and so the system can be moved from arbitrary $\mathbf{x}_0$ to arbitrary $\mathbf{x}_f$ in arbitrary finite time $[t_0, t_f]$ without any restriction.

By grouping up each controlled node with its nearest minimum driver nodes, ‘cactus-like’ structures comprising directed chain, ring, and dilation simple networks can be observed. Independently, it is easy to check that these simple structures are also controllable with Eq. (1.2), and so these structures form the basic controllable skeleton within a controllable directed complex network. A nice visualization on the minimum drivers and controllable network structures can found at:

<https://www.youtube.com/watch?v=9-q2qpOJfkg>

Further, it should be emphasized that the controllable structures found from the maximum matching algorithm generically holds for all network link weights a_{ij} , so long as they are non-zero (except in rare cases where some duplicate values cancel off the full rank of $\mathbf{K}$), making

this approach appealing for biological networks, where, often, there is noise or uncertainty in the network a_{ij} data. This is what structural controllability [35] means.

Generalized networks (undirected and directed networks)

While the above method is able to guarantee controllability in directed complex networks, it is not applicable for undirected complex networks, where $a_{ij} = a_{ji}$ [29]. To tackle this issue, Ref. [29] proposed a method based on the PBH [69–72] rank condition that a system is controllable if

$$\text{rank}(c\mathbf{I}_N - \mathbf{A}, \mathbf{B}) = N, \quad (1.3)$$

given that c is any complex number, and $\mathbf{I}_N$ is the $N \times N$ identity matrix. To satisfy Eq. (1.3), Ref. [29] argued that the minimum $\mathbf{B}$ matrix to ensure full controllability should satisfy the condition

$$\text{rank}[\lambda^M \mathbf{I}_N - \mathbf{A}, \mathbf{B}] = N, \quad (1.4)$$

where λ^M replaces c in Eq. (1.3), and λ^M is the most frequently-occurring eigenvalue of $\mathbf{A}$ such that λ^M belongs to the maximum geometric multiplicity $\mu(\lambda^M)$. By doing elementary column transformation on the matrix $\lambda^M \mathbf{I}_N - \mathbf{A}$, linearly dependent rows which violate full rank condition of Eq. (1.4) can be found. Thus, the minimum driver nodes are just the nodes on the linearly dependent rows that need to receive control signals so that Eq. (1.4) reaches full rank. Thus, the controllability of generalized networks, including undirected networks, can be guaranteed in this way.

Network properties and minimum driver nodes

One may wish to examine how do different types of networks differ in terms of the N_D minimum number of drivers. It would be interesting to see which networks are more controllable (require less N_D minimum drivers to ensure controllability), and which are less controllable (require more N_D minimum drivers to ensure controllability). Ref. [1] tested their theory on various types of real networks. Surprisingly, they found that the minimum driver nodes tend to be located on nodes with low average degree $\langle k \rangle$. In addition, Ref. [73] studied how the density of low node in-degree and out-degree affects controllability, and Ref. [25] studied the controllability of giant connected components in directed complex networks.

1.1.3 Self-loops: Modelling network stability

While the results that the minimum driver nodes tend to be low degree [1] is mathematically correct, Ref. [14] argued that care should be exercised when interpreting those results. This is because, during computation, self-loops a_{ii} were not taken into account. Ref. [14] argued that self-loops are important, because they model stable dynamics, which many complex systems tend to be. If self-loops are modelled, then all nodes can actually match to itself, and the minimum number of driver nodes to ensure controllability is just $N_D = 1$.

Subsequently, many networked controllability papers have taken care to model self-loops. For example, to do so, [4]

$$a_{ii} = -\delta - \sum_{j=1}^N a_{ij}, \quad (1.5)$$

where δ may be tuned to adjust the system stability.

1.1.4 Energy cost

Performance measure

An important measure in controlling complex systems is the amount of effort needed. While there can be different types of performance measures [62], for example, the minimum-time problem (bring the system from $\mathbf{x}_0$ to $\mathbf{x}_f$ in quickest time), or the terminal-control problem (minimize the deviation of $\mathbf{x}_f$ from any specified state vector), or the tracking problem (maintain the state vector $\mathbf{x}(t)$ as close as possible to any specified state vector in the time interval $[t_0, t_f]$), the performance measure of interest in this thesis is the minimum-control-effort problem (bring the system from $\mathbf{x}_0$ to $\mathbf{x}_f$ as efficiently as possible with the lowest amount of effort), which is defined to be [62, 71]

$$J = \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt, \quad (1.6)$$

where J is the cost function or performance measure that describes the magnitude of states that each of the control signals need to consume in the interval $[t_0, t_f]$ to drive the system. This quantity is of arbitrary unit (a.u.); for example, when considering states as opinions denoting support/opposition of an idea, then the energy cost is a measure of strongness of opinion that each of the driver nodes need in order to control the network. Who supplies this energy cost? The control signals do, thus when J demanded of them is too high, they cannot effectively perform the state transfer from $\mathbf{x}_0$ to $\mathbf{x}_f$, and the network cannot in practice be controlled. In an opinion network, even though the states are opinions which are mathematically unbounded, think of the energy cost as a theoretical limitation which bounds the strongness of opinion that a person can realistically express.

Energy-optimal control signal

The performance measure (1.6), minimized using Pontryagin's minimum principle [62], leads to the design of the energy-optimal control signal [62, 71]

$$\mathbf{u}^*(t) = \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} (\mathbf{x}_f - e^{\mathbf{A}t_f} \mathbf{x}_0), \quad (1.7)$$

where

$$\mathbf{W}(\mathbf{A}, \mathbf{B}, t_0, t_f) = \int_{t_0}^{t_f} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} dt \quad (1.8)$$

is the symmetric controllability Gramian, which is invertible when the system is controllable. Thus, the generic energy cost needed for controlling a network using the energy-optimal control

signal is found by substituting Eq. (1.7) back into Eq. (1.6):

$$\begin{aligned}
\mathcal{E} &= \int_{t_0}^{t_f} (\mathbf{u}^*)^T(t) \mathbf{u}^*(t) dt \\
&= \int_{t_0}^{t_f} \left[\mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} (\mathbf{x}_f - e^{\mathbf{A}t_f} \mathbf{x}_0) \right]^T \left[\mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} (\mathbf{x}_f - e^{\mathbf{A}t_f} \mathbf{x}_0) \right] dt \\
&= \int_{t_0}^{t_f} \left\{ \left[\mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} \mathbf{x}_f - \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 \right]^T \right. \\
&\quad \times \left. \left[\mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} \mathbf{x}_f - \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 \right] \right\} dt \\
&= \int_{t_0}^{t_f} \left\{ \left[\mathbf{x}_f^T \mathbf{W}^{-1} e^{\mathbf{A}(t_f-t)} \mathbf{B} - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} e^{\mathbf{A}(t_f-t)} \mathbf{B} \right] \right. \\
&\quad \times \left. \left[\mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} \mathbf{x}_f - \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 \right] \right\} dt \\
&= \int_{t_0}^{t_f} \left\{ \mathbf{x}_f^T \mathbf{W}^{-1} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} \mathbf{x}_f - \mathbf{x}_f^T \mathbf{W}^{-1} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 \right. \\
&\quad \left. - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} \mathbf{x}_f + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 \right\} dt \\
&= \mathbf{x}_f^T \mathbf{W}^{-1} \mathbf{W} \mathbf{W}^{-1} - \mathbf{x}_f^T \mathbf{W}^{-1} \mathbf{W} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} \mathbf{W} \mathbf{W}^{-1} \mathbf{x}_f + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} \mathbf{W} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 \\
&= \mathbf{x}_f^T \mathbf{W}^{-1} \mathbf{x}_f - \mathbf{x}_f^T \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0 - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} \mathbf{x}_f + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0,
\end{aligned} \tag{1.9}$$

where the integral (1.8) was applied on the time-dependent terms, and in some studies $\mathbf{x}_0$ or $\mathbf{x}_f$ may be set to 0 for convenience, leading to the simplification that $\mathcal{E} = \mathbf{x}_f^T \mathbf{W}^{-1} \mathbf{x}_f$ or $\mathcal{E} = \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{W}^{-1} e^{\mathbf{A}t_f} \mathbf{x}_0$ respectively. Note that by checking the dimensionality of the outer product of the vectors, $\mathcal{E}$ is a 1×1 scalar quantity.

Using the energy-optimal control signal $\mathbf{u}^*(t)$ to steer the state vector, the solution to dynamics (1.1) is [12]

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-\tau)} d\tau [\mathbf{W}^{-1} (\mathbf{x}_f - e^{\mathbf{A}(t_f-t_0)} \mathbf{x}_0)], \tag{1.10}$$

which is what was used to generate the solutions to Fig. 1.1(b). The solution of the node states evolution in the absence of control is just $\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_0$, which is what was plotted in Fig. 1.1(a).

Energy cost reduction

Since the amount of energy required to achieve state transfer from $\mathbf{x}_0$ to $\mathbf{x}_f$ is important, a number of studies [8, 12, 17–24] have investigated this issue. Notice in Eq. (1.9) that the energy cost is a function of different parameters including $\mathbf{x}_0$, $\mathbf{x}_f$, $\mathbf{A}$, and $\mathbf{W}^{-1}$, which is the inverse of the controllability Gramian (Eq. (1.8)), which is itself dependent on $\mathbf{A}$, $\mathbf{B}$, t_0 , and t_f . These studies typically focus on characterizing how energy cost could be reduced when each of the parameters are changed.

Energy cost reduction by t_f

Refs. [3, 5] study how the final control time t_f affects the energy cost (assuming $t_0 = 0$ for convenience). The energy cost was characterized by their lower and upper bound scaling laws with respect to t_f . It was found that when t_f is small, then raising t_f in the small t_f regime always reduces the energy cost at a rate of

$$\mathcal{E} \sim \begin{cases} t_f^{-1} & , \text{lower bound and small } t_f \\ t_f^{-\theta} & , \text{upper bound and small } t_f, \end{cases} \quad (1.11)$$

where $\theta \gg 1$, which is to be determined numerically. It turns out that [5] $\theta \approx N_0 - N_{\min}$ when using less than N number of drivers, where N_0 and $N_{\min}$ are numerical values related to the invertibility of the controllability Gramian, and when using N number of drivers such that each node directly receives a control signal, the upper bound also scales as t_f^{-1} in the small t_f regime. What this means is that demanding the control signals the drive the network node states in a small amount of time requires the most control energy, and by relaxing the demand, energy cost is reduced.

Further relaxation of the demand by raising t_f do not necessarily lead to further reduction in energy costs. Depending on whether $\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_f = \mathbf{0}$, the large t_f scaling laws presented by Ref. [3] and Ref. [5] differ because one studies the scaling laws with assumption that $\mathbf{x}_f = \mathbf{0}$, and the other $\mathbf{x}_0 = \mathbf{0}$ (also known as controllability-to-zero and reachability-from-zero [23]). Essentially, $\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_f = \mathbf{0}$ are just the reverse direction of each other, as far as system stability/instability are concerned. From the $\mathbf{x}_0 = \mathbf{0}$ results, assuming undirected networks, $\mathbf{A}$ which are negative definite (ND) and not positive definite (PD), respectively have lower and upper bound energy costs converge to constant values in the large t_f regime. Networks which are negative semi-definite (NSD) or positive semi-definite (PSD) have respectively the lower and upper bound scale as t_f^{-1} . Networks which are not ND or are PD have respectively the lower and upper bound scale as $\sim e^{-\lambda_n t_f}$, or $\sim e^{-\lambda_1 t_f}$, where λ_n and λ_1 are the largest and smallest eigenvalue of $\mathbf{A}$ [5]. All of these scaling laws represent the different implications about how the energy cost varies with respect to t_f and the inherent system stability modes. For the convergent large t_f energy cost, further increasing t_f will lead to no more energy cost reductions, and a minimum amount of effort is needed to steer the state vector. For those with t_f^{-1} , $e^{-\lambda_1 t_f}$, or $e^{-\lambda_n t_f}$ scaling, further increasing t_f leads to further reduction in energy cost, as the system semi-stability, semi-instability, or instability assists the control signals in pushing the node states away from the origin $\mathbf{0}$.

For directed networks, where $\mathbf{A}$ is in general non-symmetric, the scaling laws remain unchanged, except that λ_1 and λ_n are respectively replaced with $\text{Re}|\lambda_1|$ and $\text{Re}|\lambda_n|$ [3].

Energy cost reduction by $\mathbf{B}$

A significant body of research have focused on optimizing the energy cost reduction through the $\mathbf{B}$ matrix, which tells where in the network should control signals be attached so that energy cost is most reduced. Notice that the $\mathbf{B}$ matrix has dimensionality $N \times M$, where N is the number of nodes within the network, and M is the number of control signals used to drive the state vector. Ref. [4] derived the scaling law of the maximum control energy with respect to increasing number of driver nodes, showing that

$$\mathcal{E}_{\max} \sim e^{N/M}. \quad (1.12)$$

As such, a linear increase in the number of control signals leads to an exponential decrease in the maximum control energy. To interpret this, it just means that as more control signals are

working together to drive the state vector, the work load is shared amongst the multiple driver nodes, thus the energy cost demanded of (each of) them in driving the state vector in the most costly eigen-direction becomes less.

Assuming that the M number of control signals is fixed, the arrangement of where the control signals are placed also account for the variability in the control energy by up to several orders of magnitude. Recall that the $\mathbf{B}$ matrix has matrix elements $b_{ij} = 1$ if control signal j attaches to node i , otherwise $b_{ij} = 0$, so the problem of arranging the M number of control signals in a network with system size of N is actually a discrete optimization problem with solution space of N choose M number of combinations. As N increases, the solution space becomes infeasibly large for a brute force search. Refs. [17–21] dealt with this problem by relaxing the $\mathbf{B}$ matrix elements to allow for real numbers, so that the energy cost is continuous with respect to change in $\mathbf{B}$. With this, they constrained the energy cost along the hyper surface $\text{tr}(\mathbf{B}^T \mathbf{B}) \approx M$, and performed projected gradient descent along the hyper surface in the direction of quickest energy cost decrease, guided by the gradient $\frac{\partial \mathcal{E}}{\partial \mathbf{B}}$. Once the solution has settled on a convergent local minima, the obtained $\mathbf{B}$ matrix is selected as a candidate solution, where a numerical rounding scheme was used to convert the $\mathbf{B}$ matrix with numerical real elements back to the closest binary $\mathbf{B}$ with $b_{ij} = \{0, 1\}$. Comparing the solutions found this way to those where the control signals are placed randomly, the authors showed that the energy cost is reduced by a few to several orders of magnitude. Later on, a conceptually similar work, but using convex optimization method instead of projected gradient descent was published by Ref. [22].

Ref. [24] showed that the longest control chain (LCC) dominates the energy cost when controlling a complex network. Assuming that there are M number of driver nodes, then each of the N number of nodes which are to be controlled can be assigned to one of the M driver nodes if it is closest to the driver node (in other words, a node is assigned to a driver node if it is reached by the geodesic [74] path distance from that particular driver node). Based on these groupings, the LCC is thus the group that has the longest (unweighted) path distance from driver node to controlled node. It was shown that the energy cost increases exponentially with a linear increase in path distance of the LCC. Therefore, a way to decrease energy cost is by breaking up the maximum path distance of the LCC. Additional control signals added can optimally reduce energy cost by injecting them at nodes lying in the middle of the LCC, maximally reducing the length of the LCC. Compared to the same number of additional control signals added randomly, the LCC breaking approach reduces the energy by several orders of magnitude.

Ref. [23] studied how the eigenvalues of the $\mathbf{A}$ matrix affect the energy cost. When the real part of the eigenvalues are large, the energy cost needed for controlling them in the most costly eigen-direction is largest. To interpret this, think of the real part of the eigenvalues as how quickly the system can stabilize to 0 or destabilize away from 0. Thus, networks with eigenvalues with smaller real parts offer less resistance when controlling the state vector along the most costly eigen-direction, and would require the least energy cost to control. In addition, Ref. [23] proposed a method for weighted networks which ranks nodes by the ratios of their weighted outdegree/indegree

$$r_w(i) = \frac{w_{\text{out}}(i)}{w_{\text{in}}(i)} = \frac{\sum_{j=1}^N |a_{ji}|}{\sum_{j=1}^N |a_{ij}|}, \quad (1.13)$$

where $r_w(i)$ is the rank of node i , and a_{ij} refers to the weighted directed link pointing from node j to node i . Selecting nodes with the highest rank $r_w(i)$ to receive the control signals would lead to reduction in the energy cost by several orders of magnitude as compared to a selection strategy of randomly placing the same number of control signals.

Energy cost reduction by $\mathbf{A}$

While the driver nodes placement ($\mathbf{B}$ matrix) strategy has been a popular scope to research reducing control energy, energy cost optimization by $\mathbf{A}$ matrix remained niche. However, Ref. [75] attempted this problem by adapting the projected gradient descent method from earlier works. The search for the optimal $\mathbf{A}$ solution, guided by the gradient $\frac{\partial \mathcal{E}}{\partial \mathbf{A}}$, leads to energy cost reduction of a few to several orders of magnitude as compared to $\mathbf{A}$ initialised with random link weights. The authors suggest that energy cost reduction could be an important mechanism when examining networks with evolving weighted links, and could explain the self-organization of adaptive networks.

Energy cost reduction by $\mathbf{x}_0$ and $\mathbf{x}_f$

Beyond energy cost reduction by matrices, the state vectors $\mathbf{x}_0$ and $\mathbf{x}_f$ can also reduce energy cost. Ref. [12] examined the energy cost for driving the state vector non-strictly by introducing a penalizing parameter $\alpha \in (0, 1]$, which can be tuned to allow for the final state vector $\mathbf{x}_f$ to not be satisfied strictly. As α increases, the network is driven to $\mathbf{x}_f$ less strictly (the error of final state vector increases), and the energy cost becomes more reduced. Thus, by picking a balanced α parameter such that the final state vector $\mathbf{x}_f$ is driven to sufficient precision, energy cost can be reduced without sacrificing the intended $\mathbf{x}_f$ too much.

Energy cost reduction by target control

While most of the discussion above focused on controlling the network fully, such that all N node states are driven to $x_i(t_f)$, it may not be necessary to control the entire network in some cases [16]. Target control [8, 11, 16] refers to controlling a subset of the network, whereby only P number of nodes (where $P \leq N$) are controlled. In the case of target control, Ref. [8] derived scaling laws to show that the energy cost scales exponentially with a linear increase in P . What this means is that when the driver nodes only have to steer a small subset of the state vector, the energy cost demanded of them is low. As the cardinality of this subset increases, the workload becomes higher and energy cost increases exponentially.

1.1.5 Target control

Target control is described by [8, 16]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}\tag{1.14}$$

where $\mathbf{x}(t)$, $\mathbf{u}(t)$, $\mathbf{A}$, and $\mathbf{B}$ have already been described in Eq. (1.1), and the new addition $\mathbf{y}(t) = [y_1(t), y_2(t), \dots, y_P(t)]^T$ is the time-varying $P \times 1$ output state vector which tells the node states of the P number of subset nodes that are being target controlled, and $\mathbf{C} \in \mathbb{R}^{P \times N}$ maps which of the N nodes are chosen to be target controlled such that $c_{ij} = 1$ if node j (where $j = 1, 2, \dots, N$) is the i -th out (where $i = 1, 2, \dots, P$) to be target controlled, otherwise $c_{ij} = 0$.

To show target control, the node states evolution are plotted in Fig. 1.3 for a $N = 40$, $\langle k \rangle = 6$ undirected ER random network. In Fig. 1.3(a), the node states evolution in the absence of control $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is shown, where the initial state vector $\mathbf{x}_0$ has elements drawn from random uniform $[0, 1]$. In Fig. 1.3(b), 40% of the network is being target controlled (chosen at random) toward the final state vector of $\mathbf{y}_f = [2, 2, \dots, 2]^T$ using 4 control signals chosen at random.

Figure 1.3: Node states evolution with and without control in a $N = 40$, $\langle k \rangle = 6$ ER network. (a) Node states evolution in the absence of control. (b) Node states evolution when 40% of the network are being target controlled using 4 control signals. Inset shows that the subset of node states reach $\mathbf{y}_f = [2, 2, \dots, 2]^T$, while those that are not controlled have other node states.

Energy-optimal target control signal

The performance measure in target control is still $J = \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt$, which upon optimization leads to the energy-optimal target control signals [8, 12]

$$\mathbf{u}^*(t) = \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} (\mathbf{y}_f - \mathbf{C} e^{\mathbf{A}(t_f-t_0)} \mathbf{x}_0), \quad (1.15)$$

where $\mathbf{y}_f = [y_1(t_f), y_2(t_f), \dots, y_P(t_f)]^T$ is the $P \times 1$ final output state vector. Consequently, the generic energy cost when using Eq. (1.15) to target control the network can be found by substituting Eq. (1.15) into $J = \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt$.

The solution to dynamics (1.14) is known to be [12]

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_0 - \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-\tau)} d\tau \mathbf{C}^T \hat{\mathbf{v}}_f, \quad (1.16)$$

where $\hat{\mathbf{v}}_f = -(\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} (\mathbf{y}_f - \mathbf{C} e^{\mathbf{A}(t_f-t_0)} \mathbf{x}_0)$, which is what was used to generate the node states evolution in Fig. 1.3(b).

1.2 Applications of networked controllability

While there have been an impressive volume of work related to the theoretical or mathematical studies of the controllability of complex networks, a simple question that a typical layman might ask might be “So what can this research do?” The short answer to this question is that this research field advances the understanding of controlling complex systems, which could eventually lead to the development of applications in controlling large complex systems.

To corroborate the claim and test whether or not the framework offered by the controllability of complex networks can lead to development in application areas, Ref. [76] studied how the identified driver nodes of the connectome or brain network (the $\mathbf{A}$ matrix) related to the motor functions of *C. Elegans* (roundworms) would affect movement when ablated. When some of the identified neuronal driver nodes have been ablated, the movement of the *C. Elegans* become

restricted, suggesting that, indeed, the driver nodes are essential to ensure that the node states are controllable, since the removal of some driver nodes restricts the node states, making the system uncontrollable.

Professor Barabási gives a nice explanation here at

<https://www.youtube.com/watch?v=j6IGxgiFWFY>

starting at 31:39, where the movement of the C Elegans can be seen at 39:45.

1.3 Research aims and major contributions of the thesis

This thesis is about the energy cost needed for controlling complex social networks. It aims to check what happens when various types of social physics features are introduced into the networked controllability framework. In particular, by focusing networked controllability on the context of social physics, there are three major contributions:

- Firstly, while there has been a lot of research done on optimizing energy cost reduction in various ways, most notably the driver nodes placement optimization (optimize $\mathbf{B}$ matrix), how target nodes selection (optimize $\mathbf{C}$ matrix) can be optimized to reduce energy cost has yet been studied. By adapting the projected gradient descent method of earlier works [17–21], Chapter 2 derives the gradient $\frac{\partial \mathcal{E}}{\partial \mathbf{C}}$ to optimize the $\mathbf{C}$ matrix, finding the most energy-optimal target nodes to control, given that $\mathbf{B}$ matrix is fixed. The results show that, indeed, $\mathbf{C}$ matrix could be optimized to reduce control energy by a few orders of magnitude. The work may be applicable to opinion networks, or marketing networks, where the driver nodes are advertisers who wish to understand which customers to target their campaigning effort upon.
- Secondly, Chapter 3 revisits the energy cost scaling laws with respect to final control time t_f when conformity behavior [27] is modelled into the network dynamics, where each node mimics the opinions of their nearest neighbors over time. Previous works [3, 5] have derived the scaling laws with respect to t_f for networks with continuous-time linear dynamics. However, continuous-time linear dynamics is unable to capture conformity behavior, which the discrete-time modelling [27] can. It was found that the scaling behavior of networks with conformity behavior is similar to those of continuous-time stable linear dynamics. However, when comparing the large t_f upper bound energy cost of the two systems, networks with conformity always require less energy cost, because information spread more efficiently due to the conformity dynamics. Since conformity behavior is common in many social systems, the results have implications about the trade-off between t_f and energy cost, and would be useful for designing control strategies to control such systems.
- Thirdly, zealots, agents with unwavering opinions, are introduced to the controllability of complex networks. Previous works have studied the energy cost needed to control networks under the assumption that all nodes are compliant, passing on information neutrally without any preferences. However, the assumption on nodal neutrality should be challenged, especially in the context of a networked social system, where some people may hold fast to their personal beliefs. Chapter 4 challenges the assumption by calculating the energy cost needed to control a network, where there are both neutral and non-neutral (zealots) nodes. It was found that the presence of zealots alters the energy cost at a

quadratic rate with respect to their own fixed beliefs. However, whether or not the zealots' presence increases or decreases the energy cost, relative to the situation where there were no zealots present, is affected by the interplay between different parameters such as the zealots' beliefs, number of drivers, final control time regimes, network effects, network dynamics, number and configurations of neutral nodes influenced by the zealots. All in all, the results suggest that when controlling networks with neutral nodes and zealots, the energy cost display rich behavior that require careful consideration as the zealots alter the system behavior non-trivially.

1.4 Outline of the thesis

This thesis is outlined as follows. In Chapter 1, an introduction, literature review, and motivations of the thesis have been given. In Chapter 2, how target nodes selection is optimized to reduce energy cost is discussed. In Chapter 3, the energy cost scaling laws with respect to final control time for controlling complex social networks with conformity behavior is presented. Chapter 4 details how zealots affect the energy cost for controlling complex social networks. Finally, Chapter 5 concludes the thesis. Appendices at the end of each chapter contain additional information, including long mathematical equations, and derivations.

Chapter 2

Optimizing target nodes selection for the control energy of directed complex networks

2.1 Introduction

As mentioned in Chapter 1, there are many ways to reduce the energy cost needed in controlling complex networks. Notably, a number of studies have focused on reducing energy cost through optimal placement of the control signals (optimizing the B matrix) [17–21]. Thus, different from all previous works, a slightly different question is asked: How can the target node selection be picked in such a way that the control energy is minimized? To that end, a cost function optimization model based on projected gradient descent, similar to earlier works [17–19, 21, 77] will be used. The main difference between the present work and the existing literature is the variable matrix being optimized: While previous works have focused on optimizing the choice of input signal (matrix B), this research optimizes the choice of target nodes (matrix C). Furthermore, previous derivations of the index notation gradient information uses the I-Chain rule, which can be difficult to understand. Here, the index notation gradient information using the standard chain rule and product rule is derived in a general way, which is simpler to follow.

In this Chapter, how the choice of target nodes could be optimized to minimize the energy cost function is examined. Using the formulated energy cost function, the matrix derivative of the energy cost function with respect to matrix C , which is the energy cost function gradient information, was derived. With the gradient information obtained, an iterative search was performed using the trace-constraint-based projected gradient method (TPGM) proposed in Ref. [21] to obtain the target nodes which are energetically favorable. The energy cost obtained from the optimization algorithm is compared with those with target nodes chosen using a heuristic selection scheme such as random selection, and node degree-based selection. The simulation results show that the solution obtained from TPGM reduces the energy cost by a few orders of magnitude.

2.2 Problem formulation

In standard complex network controllability literature, the N coupled system of equations whose dynamics are linear and time invariant (LTI) is considered:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t),\end{aligned}\tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^{N \times 1}$ is the time-varying state vector, $\mathbf{y}(t) \in \mathbb{R}^{P \times 1}$ ($P \leq N$) is the subset of time-varying output state vector that is target controlled. $\mathbf{u}(t) \in \mathbb{R}^{M \times 1}$ ($M \leq N$) is a time-varying external control signal used to drive the network. The time invariant $A \in \mathbb{R}^{N \times N}$ matrix represents the network topology, where for directed complex network, the matrix element $\{a_{ij}\}$ is non-zero when there is a directed link from node j to node i , and zero element if no link exists from node j to node i . Since many complex systems tend to enjoy passive stability, self-links are modelled, where $\{a_{ii}\}$ is a negative real number [14]. The time invariant matrix $B \in \mathbb{R}^{N \times M}$ reflects the coupling between nodes and M number of external control signals $\mathbf{u}(t)$, where matrix element $\{b_{ij}\} = 1$ if node i receives a time-varying control signal from control input j , and zero element otherwise. In addition, the columns of B are elementary vectors, that is, if $b_{ij} = 1$ for some column j , then $b_{kj} = 0$ for all $k \neq i$ (no two nodes can be attached to the same control signal). Finally, the control matrix $C \in \mathbb{R}^{P \times N}$ describes the choice of the P number of subset of nodes that are to be target controlled, where matrix element $\{c_{ij}\} = 1$ if node j is the i -th node out of all possible P nodes, and zero otherwise. Note that matrix B and matrix C are column and row linear independent, where if $\{b_{ij}\} = 1$ ($\{c_{ij}\} = 1$), then no other non-zero entries may exist for column j and row i in matrix B (C). This linear independence is imposed to reflect the modeling choice for one-to-one connection between control signals and nodes, and for one-to-one connection to individually target control each of the P number of nodes [8].

The energy cost needed to drive the states of the network using the control signal $\mathbf{u}(t)$ is defined to be the cost function [71]

$$J = \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt. \quad (2.2)$$

This is the energy cost function that is to be optimized with respect to the choice of target nodes (matrix C). Similar to previous works which considered trace constraint projected matrix gradient cost function optimization [17–19, 21, 77], here it is assumed that the initial state vector $\mathbf{x}_0 \sim \mathcal{N}(0, 1)$. The energy cost is

$$\mathcal{E}(t_f, B, C) = \mathbb{E} \left[\int_0^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt \right], \quad (2.3)$$

where t_f is the total control time and initial time is set to zero throughout the rest of the Chapter ($t_0 = 0$). The energy cost is a function of the control trajectory, i.e. the state space pathway between $\mathbf{x}_0$ and $\mathbf{y}_f$. Note that Eq. (2.3) is the expected energy cost over all realizations of initial state vector $\mathbf{x}_0$, picked under standard normal distribution. In general, the cost function $\mathcal{E}(t_f, B, C)$ is dependent on the matrix B and the final time t_f . However, in this Chapter, the focus is on optimizing the cost function with respect to matrix C while keeping input matrix B and final time t_f fixed. Thus, for clarity, these two dependencies are dropped and the cost function is written only as a function of either $\mathcal{E}(C)$ or $\mathcal{E}(C^T)$ throughout the Chapter.

Consequently, the energy-optimal control signal, derived from optimal control theory [8, 62] can be calculated as

$$\mathbf{u}^*(t) = B^T e^{A^T(t_f-t)} C^T (C W C^T)^{-1} (\mathbf{y}_f - C e^{A(t_f-t_0)} \mathbf{x}_0). \quad (2.4)$$

The $N \times N$ controllability Gramian matrix

$$W = \int_{t_0}^{t_f} e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} dt \quad (2.5)$$

is an important quantity in control theory, well-known to be real, symmetric, and semi-positive definite. In practice, the controllability Gramian matrix can be used to verify that the choice

of matrix B is ensuring network (target) controllability by checking that the (output $(CW C^T)$) controllability Gramian matrix is invertible [8, 71].

Under optimization, the cost function follows:

$$\min \mathcal{E}(C^T) \quad \text{subject to} \quad \text{tr}(CC^T) = P, \quad (2.6)$$

where the solution produced by the gradient descent iterative algorithm is constrained to stay on the manifold $\text{tr}(CC^T) = P$.

2.3 Results

The energy cost function can be solved by substituting the energy-optimal control signal $\mathbf{u}^*(t)$ into the cost function $\mathcal{E}(C^T) = \mathbb{E}[\int_0^{t_f} \mathbf{u}^T(t)\mathbf{u}(t)dt]$:

$$\mathcal{E}(C^T) = \text{tr}((CW C^T)^{-1} \mathbf{y}_f \mathbf{y}_f^T) + \text{tr}(C^T (CW C^T)^{-1} C e^{At_f} e^{A^T t_f}). \quad (2.7)$$

By varying Eq. (2.7) with respect to C^T ,

$$\begin{aligned} \frac{\partial \mathcal{E}(C_k^T)}{\partial C^T} = & -2WC^T(CWC^T)^{-1} \mathbf{y}_f \mathbf{y}_f^T (CW C^T)^{-1} \\ & -2WC^T(CWC^T)^{-1} C e^{At_f} e^{A^T t_f} C^T (CW C^T)^{-1} \\ & + 2e^{At_f} e^{A^T t_f} C^T (CW C^T)^{-1}. \end{aligned} \quad (2.8)$$

The full derivation can be found in the **Appendix A Supplementary information**. Using Eqs. (2.7) and (2.8) and applying it to the TPGM algorithm, the energy-optimal optimal solution, C^* is obtained. In the simulations that follows, set $t_f = 2$, and the desired final output state vector $\mathbf{y}_f = [1, 1, \dots, 1]^T$.

2.3.1 Algorithm

To optimize the control energy by varying the selection of target nodes, the modified trace-constraint-based projected gradient method (TPGM) formulated by Li et al [21] was used. Let $\tilde{C}^T$ be the basis of the target control matrix C^T . $\tilde{C}^T$ is obtained by performing Gram Schmidt orthogonalization on matrix C^T [78]. Define the projection operator $\mathcal{T}_{\tilde{C}^T} = (I_N - \tilde{C}^T \tilde{C})$, where I_N is the $N \times N$ identity matrix. The operator $\mathcal{T}_{\tilde{C}^T}$ projects any arbitrary matrix onto the space that is perpendicular to the manifold $\text{tr}(CC^T) = P$. For two arbitrary matrices A and B which have the same dimension, an angle between them can be defined as:

$$\theta = \arccos \left(\frac{\text{tr}(A^T B)}{\|A\|_F \|B\|_F} \right), \quad (2.9)$$

where $0 \leq \theta \leq \pi$, $\|\cdot\|_F$ denotes Frobenius norm, and the matrices are perpendicular if $\theta = \pi/2 = 90^\circ$. The TPGM optimization is given in Algorithm 1.

In TPGM step 1, random initialization of matrix C_0^T means that the $N \times P$ matrix begin with elements all zeros, and at uniform random, set matrix element $[C^T]_{ij} = 1$, where node i is chosen to be the j -th target node. Once node i is chosen, maintain row and column linear independence, and thus do not allow for the selection of the same node i to be target controlled. From the numerical experiments, while starting from a pure random dense matrix is possible when system size is small or the network topology is sparse, it is computationally inefficient.

Algorithm 1: Pseudocode for finding energy-optimal target control matrix C^* using the trace-constraint-based projected gradient method (TPGM).

```

1 Initialize  $C_{k=0}^T$  as a random  $N \times P$  matrix
2 while  $\cos(\theta_k) > \xi$  do
3   Compute  $\frac{\partial \mathcal{E}(C_k^T)}{\partial C_k^T} = \nabla \mathcal{E}(C_k^T)$ 
4   Update  $\hat{C}_{k+1}^T = C_k^T - \eta \cdot (I_N - \tilde{C}_k^T \tilde{C}_k) \nabla \mathcal{E}(C_k^T)$ 
5   Normalize  $C_{k+1}^T = \sqrt{\frac{P}{\text{tr}(\tilde{C}_{k+1} \hat{C}_{k+1}^T)}} \cdot \hat{C}_{k+1}^T$ 
6   Compute  $\cos(\theta_k) = \left( \frac{\text{tr}([\nabla \mathcal{E}(C_k^T)]^T \mathcal{T}_{\tilde{C}_k^T} \nabla \mathcal{E}(C_k^T))}{\|\nabla \mathcal{E}(C_k^T)\|_F \cdot \|\mathcal{T}_{\tilde{C}_k^T} \nabla \mathcal{E}(C_k^T)\|_F} \right)$ 
7   Update  $k = k + 1$ 
8 end

```

Furthermore, when starting from a pure random dense matrix, a sensible sparse optimal matrix solution could not be obtained at the end of the optimization algorithm, when converting from dense optimal solution C^* to sparse binary optimal solution (see section 2.3.2).

In TPGM step 4, the numerical value of the learning rate or step size η is chosen empirically. When η is too large, convergence is not guaranteed and the algorithm will fail. While convergence is guaranteed when η is sufficiently small, the time taken to complete the iterative search will suffer if η is too small. Typically, the learning rate was chosen to be between 1e-8 and 1e-3, depending on the fraction P/N to be target controlled as well as the complex network topology. In general, the learning rate η can be varied to speed up the iterative process. For example, starting at $\eta = 1\text{e-}8$, and then changing to $\eta = 1\text{e-}4$ when enough iterations have been run. As observed during experimentation, the convergence of η scales inversely proportional to the numerator of the $\cos(\theta_k)$: $\text{tr}([\nabla \mathcal{E}(C_k^T)]^T \mathcal{T}_{\tilde{C}_k^T} \nabla \mathcal{E}(C_k^T))$. In TPGM step 4, $\hat{C}_{k+1}^T$ is a non-normalized quantity, while in step 5, the updated solution is constrained onto the manifold surface $\text{tr}(CC^T) = P$, obtaining the normalized C_{k+1}^T .

In TPGM step 6, the angle θ_k between k -th step gradient matrix $\nabla \mathcal{E}(C_k^T)$, and the projected gradient matrix, $\mathcal{T}_{\tilde{C}_k^T} \nabla \mathcal{E}(C_k^T)$ is calculated to check for convergence. If $\cos(\theta_k)$ approaches zero, or equivalently, when θ_k approaches $\pi/2 = 90^\circ$, the algorithm is deemed to have converged. Numerically, the while loop terminating condition, ξ , refers to a small positive quantity, for example, $\xi = 1\text{e-}2$. For some networks where the algorithm is unable to converge towards $\xi = 1\text{e-}2$, this condition is relaxed to $\xi = 1\text{e-}1$.

TPGM will iteratively update the initial proposed solution C_0 in the direction of quickest decreasing energy cost based on cost function derivative with respect to matrix variable C . When the search has finally converged, the obtained optimal target control matrix C^* corresponds to reduced control cost of a dense matrix of real numbers, which corresponds to many-to-many connections from output nodes to complex network nodes of varying link strength. Based on the obtained optimal solution C^* , the challenge is to obtain a one-to-one connection of output nodes and nodes with unity link strength, a sparse binary optimal solution, C_{binary}^* , while maintaining the characteristics of reduced control cost.

2.3.2 Selecting binary optimal target node set from C^*

There are two methods to find C_{binary}^* from C^* . The first method is based on suppressing insignificant matrix elements in C^* , and then evaluating the normalised quantity *importance index*

vector, r :

$$r = \frac{[r_1, r_2, \dots, r_N]}{\max(r_1, r_2, \dots, r_N)} \quad (2.10)$$

where $r_i = \sum_j |[C^T]_{ij}|$ is the non-normalised importance score of node i being selected as a target node, obtained by taking the row summation of the absolute of the transpose matrix, C^T . $[C^T]_{ij}$ represents the numerical contribution towards C^* for node i , target node j . Therefore, the binary optimal solution comes from C^* , where matrix entries the highest numerical contributions are kept.

During numerical experimentation, it was found that directly passing C^* into Eq. (2.10) to find C_{binary}^* usually does not lead to the optimal target control matrix. This is because of the row summation of the absolute of the numerical contribution. For example, if row a has predominantly absolute values of around 0.2 in all its columns, while row b has an absolute value of around 0.9 in one of its columns, and mostly negligible numerical values close to zero in all of its other columns, then by Eq. (2.10), node a will be ranked higher than node b . Based on experimentation with elementary topologies, it was found that numerical contribution characteristics similar to row b usually correspond to an optimal target node. Thus, the row summation should reflect that: Suppress the numerical values in each matrix element of C^* if they fall below a certain value.

The suppressed optimal matrix is computed by setting $[C^*]_{ij} = 0$ iff $[C^*]_{ij} \leq d \sigma$, where σ is computed from the standard deviation of the absolute of all matrix elements $[C^*]_{ij}$, and $d = \{0.0, 0.1, 0.2, \dots, 1.1, 1.2, \dots, 3.0\}$ is some arbitrary parameters chosen to multiply the standard deviation for the purpose of suppressing the noise terms which are believed to hide selection of ones and zeros from the optimal solution of C^* . For each suppressed matrix, check that the rank of the matrix is P , before considering it as a viable candidate solution. If the suppressed matrix has rank P , then pass the suppressed matrix into Eq. (2.10), and pick P nodes with the highest importance index. The case when $d = 0$ is similar to the proposed importance index vector formulation in refs [17, 18, 79] for finding optimal driver nodes.

Figure 2.1: When driver nodes are fixed, the arrangement of which target nodes to be controlled can affect the control energy. From numerical experimentation, the 6 target nodes which results in minimal control energy are nodes $\{1, 2, 4, 5, 7, 8\}$. The optimal target node set has reduced path distance from driver nodes to target nodes, which is consistent with the literature [24]

To illustrate the conversion process from C^* to C_{binary}^* , an example with selecting optimal target nodes in a stem network is given, as shown in Fig. 2.1. For this example network, there are a total of 9 nodes, and the driver nodes are fixed to be located at nodes 1, 4, and 7. The goal is to use TPGM to find $P = 6$ target nodes which result in reduced control energy.

In a typical TPGM run, the algorithm brings the initial proposed random matrix C_0 towards the direction of quickest decreasing energy, based on the cost function Eq. (2.7) and cost func-

tion derivative Eq. (2.8). A converged solution matrix C^* may look something like $[C^*]^T$ of Fig. 2.2. Notice that in the fifth row, for example, although the (5, 4)-th element which has the numeric value of 0.8030 larger than the rest within the row, its adjacent elements, the (5, 3)-th and (5, 6)-th elements have numeric values of 0.3092 and 0.2280, which may affect the selection of binary C_{ij}^* elements when using Eq. (2.10). It is believed that these could be noise terms that hide the optimal binary signals, and so they are suppressed. In the suppression method, numeric values that are less than $d\sigma$ (where d is some parameter setting, and σ is the standard deviation of all C_{ij}^* terms) are set to be zero, which is what is happening on the right panel of Fig. 2.2. After suppressing the noise terms, the binary selection of optimal target nodes are found by passing the suppressed matrix into Eq. (2.10) and choosing P target nodes with the highest importance index values.

Figure 2.2: Converting C^* to C_{binary}^* using the suppressed matrix method. The standard deviation, σ , of $|C^*|$ is calculated. In the top panel, passing matrix $[C^*]^T$ directly to Eq. (2.10) in the chapter main text to evaluate the importance index may not lead to the correct optimal target nodes due to the presence of insignificant matrix elements affecting the row absolute summation. In this case, by suppressing matrix elements $[|C^*|]_{ij}^T$ whose values are $\leq 2.0\sigma$ and set those to be zero (middle panel), and then passing the suppressed matrix into Eq. (2.10) in the chapter main text, the correct optimal target nodes can be found.

The second method is based on selecting the matrix elements with the largest absolute numerical values. The process is as follows: First, begin with optimal matrix transpose, $[[C^*]^T]$. Then, search for the absolute largest matrix element in each column, and order the columns in descending order, keeping track of the associate row indices. Start with a $N \times P$ zero matrix, C_{binary}^T , and set the matrix element for position (i, j) to be one, starting from the columns with the largest absolute matrix elements $[[C^*]^T]_{ij}$. At each step, check that for assigning position (i, j) to be one, no other nonzero element exists along row i or along column j . If there exists another nonzero matrix element along the row i or column j , then do not assign position (i, j) to be one, and record down column j which did not get filled. If $\text{rank}(C_{\text{binary}}^T) = P$, then stop the process; otherwise, repeat the process described for unfilled columns j 's for the case of second largest absolute matrix elements, third largest, ..., until $\text{rank}(C_{\text{binary}}) = P$. Fig. 2.3 illustrates the second approach.

Figure 2.3: Converting C^* to C_{binary}^* using the largest column absolute value method.

The two methods shown in Figs. 2.2 and 2.3 found the optimal target nodes $\{1, 2, 4, 5, 7, 8\}$, which was confirmed numerically to be the true optimal target nodes. Note that the true optimal solution can only be confirmed when the network size is small as it requires an exhaustive brute force search to find all possible combinations to verify that the identified solution is indeed globally optimal. For the example given, selecting 6 target nodes from all possible 9 nodes in the stem network is a $(9 \text{ choose } 6) \binom{9}{6} = 84$ combinatorics. As N increases, the combinatorics quickly become infeasible. In general, in a complex network, a brute force search is infeasible.

Thus, for all experiments throughout the chapter, the binary solutions are obtained from the above-described two methods. At the end of the computations of the two methods, pass all the obtained binary target node candidate solutions into the objective cost function, Eq. (2.7), and pick the target node set which yields the lowest energy cost, denoting it $[C_{\text{binary}}^*]^t$, where $t = \{1, 2, \dots, 10\}$ represents each independent iterative search. For the simulation results showing C_{binary}^* , the best solution out of all $[C_{\text{binary}}^*]^t$ was presented.

2.3.3 Numerical experiments on elementary network topologies

To understand the arrangement of energy optimal target node set, the TPGM iterative searches on elementary network topologies [1, 79] were performed. They were: Directed stem, circle, and dilation. A stem requires just a control signal, placed at the root, to become mathematically

(but not necessarily numerically) controllable [15, 25, 28]. A circle is controllable with just one control signal attached to any of the nodes in the circle. A dilation requires a minimum of two driver nodes to become controllable. To decouple the contribution from driver nodes placement to the energy cost, the driver nodes were selected in such a way that their path distances are evenly spaced, which corresponds to the most energy efficient set up as the longest path distance from driver to non-driver nodes is minimized [24]. The number of driver nodes was fixed (B fixed), and the number of target nodes is fixed at 66.7% of system size.

Figure 2.4: Energy optimal target nodes configurations when driver nodes are fixed. **(a1)** depicts a $N = 9$ directed stem network, with $M = 3$ driver nodes depicted by the cyan (grey in grayscale) control signals, and $P = 6$ optimal target nodes depicted by the magenta (dark grey) output nodes. **(a2-a3)** represent $\{N = 9, M = 2, P = 6\}$ and $\{N = 6, M = 1, P = 4\}$ directed stem network respectively. **(b1-b3)** represent circle topology with $\{N = 9, M = 3, P = 6\}$, $\{N = 9, M = 2, P = 6\}$, and $\{N = 6, M = 1, P = 4\}$ respectively. **(c1-c2)** represent dilation with $\{N = 9, M = 2, P = 6\}$ and $\{N = 9, M = 3, P = 6\}$ respectively.

For each of the 8 elementary topology labelled **(a1)** to **(c2)** as shown in Fig. 2.4, the numerical experiment was repeated independently for 1000 times in order to find the optimal target node set C_{binary}^* , given that driver nodes position are fixed. Correspondingly, TPGM was able to find the energy optimal target nodes configurations with success rates: **(a1)** - probability 58.9%, **(a2)** - 66.0%, **(a3)** - 85.9%, **(b1)** - 41.0%, **(b2)** - 40.1%, **(b3)** - 67.4%, **(c1)** - 91.2%, **(c2)** -

69.1%. This means that out of 1000 times, TPGM was able to find the optimal target nodes at least 400 times in the worst performing set up, which was **(b2)**. For **(b2)**, the search space is $9 \text{ choose } 6 = 84$ combinations, so each time TPGM finds the true optimal target nodes, it was able to find it out of a search space of 84 possible combinations. The performance of the algorithm appears reasonable, and is in-line with similar earlier works on driver nodes placement optimization by Ref. [79], which found that their proposed gradient descent methods found the optimal solution 40–65%, depending on the constraint set in the search step (such as orthogonal constraint, trace boundary constraint, or arbitrary). Finally, evidently, as seen in Fig. 2.4, the energy optimal target nodes tend to minimize their geodesic [74] path distances from the driver nodes. The findings are consistent with literature, which states that control energy cost increases exponentially with path distance [9, 24].

To test the robustness of the algorithm, repeat the experiments for elementary topologies with fixed driver nodes placement similar to configurations **(a1)**, **(a2)**, **(b1)**, **(b2)**, **(c1)**, and **(c2)** shown in Fig. 2.4, but with total target control nodes changed from $P = 6$ to $P = 5$. In general, the conclusion that energy-optimal configurations correspond to minimized path distances from driver to target nodes still holds. For the stem network with 3 driver nodes, the optimal configurations of target nodes are $\{1, 2, 4, 5, 7\}$, $\{1, 2, 4, 7, 8\}$, and $\{1, 4, 5, 7, 8\}$, given in ascending order of energy costs, with the first node set being the true optimal. TPGM was able find the optimal configurations with probabilities 30.8%, 15.2%, and 0% respectively. For stem network with 2 driver nodes, the optimal configurations are $\{1, 2, 3, 5, 6\}$ and $\{1, 2, 5, 6, 7\}$, with the former being the true optimal. TPGM success rates are 15.1% and 55.3% respectively. Thus, while control energy is minimized when target nodes are close to their nearest driver nodes, for stem network, there is a slight reduction in control energy when the target nodes are also closer to node 1, which is the root node needed to ensure controllability. It should be noted that the energy cost difference between local optimal configurations and global optimal configurations is typically smaller than 10%. For circle topology with 3 driver nodes, the optimal configurations are $\{1, 2, 5, 7, 8\}$, $\{2, 4, 5, 7, 8\}$, $\{1, 2, 4, 5, 8\}$ and $\{1, 2, 4, 7, 8\}$, $\{1, 2, 4, 5, 7\}$, $\{1, 4, 5, 7, 8\}$, with the first three node sets being the true optimal. TPGM was able to find node sets belonging to true optimal energy cost 29.6% of the time, and optimal configurations with probability 5.7%. Circle topology with 2 driver nodes has optimal configurations $\{1, 2, 3, 5, 6\}$ and $\{1, 2, 5, 6, 7\}$, with the former being the true optimal. Correspondingly, the search success rates were 5.5% and 34.0% respectively. While a circle is controllable with just one driver node placed anywhere, the difference in energy cost can be accounted for by grouping all output nodes to their nearest driver nodes: $\{1, 2, 3, 4\}$ to node 1, and $\{5, 6, 7, 8, 9\}$ to node 5. Thus, for energy-optimal control of 5 target nodes, the first 4 have to be picked to be close to their nearest driver nodes, and for the fifth node, a slight preference is given to picking target nodes belonging to the group with the lower directed path distance. In all of these, the control energy difference between true optimal and optimal are at most 2%. For dilation networks with 2 driver nodes, the optimal configurations are $\{1, 2, 6, 7, 8\}$ and $\{1, 2, 3, 6, 7\}$, with the former being the true optimal. They were respectively found by the algorithm 32.7% and 59.6% of the time. The optimal configurations of dilation networks with 3 driver nodes are $\{1, 2, 4, 6, 7\}$, $\{1, 2, 4, 5, 6\}$, and $\{1, 4, 5, 6, 7\}$, given in ascending order of energy costs. The first node set was found 97.0% of the time and the latter two are never found. The difference in energy cost between the true optimal and optimal is about 10%, and analogous explanation for grouping output nodes to their nearest driver nodes, similar to circle topologies, can be used to explain the slight difference in control energy.

2.3.4 Simulations on a small random network

Next, repeat the numerical experiment 1000 times for target controlling 5 nodes in a $N = 10$ random network (ER network), with average degree $\langle k \rangle = 2$, as shown in Fig. 2.5. The driver nodes position are fixed, and generically ensures full controllability of the network, regardless of the choice of C . The global energy-optimal configuration of target nodes selection is $\{1, 2, 4, 8, 10\}$ and TPGM was able to find this configuration 7.1% of the time. However, TPGM was also able to find the next-best performing configuration, which yields energy cost of the same order of magnitude, $\{2, 4, 6, 8, 5\}$, with success rate of 13.9%.

For complex networks, unlike elementary topologies, path distance from driver to target nodes is not the only predictor of energy cost (although it is still a strong predictor given the exponential scaling relationship between energy cost and path distance [24]), and there are other factors such as redundant paths (the number of equivalent paths from driver to target nodes) [9] that could also affect the energy cost. Therefore, for Fig. 2.5, this may be one reason why node 6 is not an optimal target node, although it is just one path distance from driver node located at node 2. In general, the exact characterization of energy cost in controlling multiple target nodes with multiple driver nodes in complex networks still remains an open problem. However, generally and heuristically speaking, it can be seen that when path distances are reduced, control energy is minimized.

Figure 2.5: Energy-optimal configuration of driver/target nodes in a random network with $N = 10$ nodes and average degree $\langle k \rangle = 2$, with fixed driver nodes at nodes $\{2, 8\}$. Driver nodes are represented by cyan (grey in grayscale), and target nodes are magenta (dark grey).

The TPGM algorithm's performance on a small random network is showed in Fig. 2.6. Fig. 2.6(a) displays the probability mass function of TPGM obtaining all the solutions found, C_{binary}^* , and their associated energy costs, indexed in ascending order in energy, TPGM $E_{\text{index } i}$. It should be noted that all 25 TPGM energy cost indexes are found from 1000 independent iterative searches, and are not exhaustive. Fig. 2.6(b) shows the energy costs associated to each TPGM $E_{\text{index } i}$. The full range of all possible energy costs is ${}^{10}C_5 = 252$, so 25 is roughly the top 10% best solutions. For comparison, the average energy cost of selecting 5 target nodes randomly without repetition, $\langle E(C_{\text{rand}}) \rangle$, is also plotted and represented by the black dashed lined on the same graph. Fig. 2.6(c) shows all 252 energy costs found from a brute force search. Comparing the solutions found from TPGM to the true list of all possible energy costs, the top 6 solutions from TPGM belong to the true top 10 energy costs, as shown in Fig. 2.6(d). Furthermore, the top 21 solutions found by TPGM are around the top 30 true optimal, yielding $\log_{10}(\text{energy cost})$ of around 3. TPGM $E_{\text{index } 22}$, $E_{\text{index } 23}$, $E_{\text{index } 24}$, and $E_{\text{index } 25}$ yields $\log_{10}(\text{energy cost})$ of around 5,

5, 6, and 8.5, which are bad performances; but Fig. 2.6(a) shows that the probabilities of TPGM finding those solutions are low. Therefore, while the rate of TPGM finding suboptimal solutions is high, the energy costs performances of the suboptimal solutions are in general comparable to the true optimal energy cost, $E_{\text{index } 1}$. From inspection, the top 15 solutions have very similar energy, which may explain why TPGM only found the global solution (index 1) 7.1% of the time. In addition, the solutions found from the search algorithm also tend to lie within the neighborhood of the lower end of all possible energy costs. Besides, they also outperform the random selection scheme by at least a few orders of magnitude.

Figure 2.6: Performance of TPGM for small random network. (a) Plot of the probability of finding a specific target node set corresponding to each TPGM energy cost index i , $\text{TPGM } E_{\text{index } i}$, given in ascending order of energy cost. (b) Plot of associated energy costs to each TPGM energy cost index. (c) Plot of all 252 energy costs in log scale for each named true $E_{\text{index } i}$, given in ascending order of energy cost. (d) Plot of the 10 lowest true energy cost as shown by the red dots, of which 6 are found by TPGM, represented by blue circles.

2.3.5 Simulation results on complex networks

Next, the TPGM algorithm is tested on different complex networks such as random networks (ER network), scale-free networks (SF network), as well as various real networks spanning a diverse range: electronic circuit networks, food web networks, and social networks. For each network, the iterative search was performed 10 times, and obtained solutions, C^* and its associated binary optimal solution C_{binary}^* , were compared against target nodes selection found from heuristic selection schemes such as random selection (repeated over 100 independent realiza-

tions) and degree-based selection of target nodes. 40% of the nodes were driver nodes, where controllability was guaranteed through finding the minimum driver nodes in the standard way [1, 66], with the additional driver nodes chosen at random uniform without repeat. For each network, once the driver nodes were picked, they remain fixed.

For the model networks, SF ($\gamma = 2.8$) and ER, with system size $N = 100$, and average degrees $k_{av} = 2.5$, the control energy needed for controlling target node set of varying cardinality $P/N = \{0.1, 0.2, \dots, 0.9, 1.0\}$ is displayed in Fig. 2.7(a) and (b). Consistent with the previous findings [8], the control energy scales exponentially with the cardinality of the target node set, regardless of the target node selection scheme chosen. However, when comparing the control energy of various selection strategies, the results show that generally, as compared to heuristic selection schemes, the energy cost for controlling target node set as found by TPGM, C^* and C_{binary}^* , is lower by a few orders of magnitude. For the result of Fig. 2.7(b), the performance of the in-degree descending selection scheme (meaning that $P/N\%$ of nodes that were ranked within the top $P/N\%$ of largest in-degrees were chosen) in the region $P/N = \{0.1, 0.2, 0.3, 0.4\}$ is generally similar to optimal target node set C_{binary}^* . This is likely due to the fact that the driver nodes which were randomly selected happened to coincide with the nodes which have high in-degrees, which results in reduced path distances between driver nodes and target nodes, and thus reduced control energy cost.

Figure 2.7: Control energy needed for controlling target nodes of increasing cardinality. Optimal target node sets control energies, $\mathcal{E}(C^*)$ and $\mathcal{E}(C_{\text{binary}}^*)$, are generally a few orders of magnitude better than selecting target node set from heuristic selection schemes. (a) SF network and (b) ER network.

Finally, TPGM was applied to real networks and model networks of system size $N = 300$. Table 2.1 details the results, where $\mathcal{E}(C_0)$ is energy cost at initialization, and $\mathcal{E}(C_{\text{rand}})$ is the energy cost when choose target nodes at random. With the exception of Circuit-s838 and Teacher-student, where the network was driven with $M/N = 0.5$ and $M/N = 0.6$ fractional number of driver nodes to target control $P/N = 0.7$ and $P/N = 0.75$ fractional number of nodes, each of the network is driven by $M/N = 0.4$ fractional number of driver nodes to target control $P/N = 0.6$ fractional number of nodes. The energy cost of target node set selected by degree-based selection scheme is presented in Table 2.2.

The heuristic selection schemes lead to target node sets C_{rand} , $C_{\text{in.asc}}$, $C_{\text{in.dsc}}$, $C_{\text{out.asc}}$, $C_{\text{out.dsc}}$. Respectively, C_{rand} corresponds to random selection of target nodes, where repeats are disallowed; $C_{\text{in.asc}}$ refers to target nodes chosen in ascending order according to their weighted in-degrees. Thus, when choosing target nodes for $C_{\text{in.asc}}$, nodes with the lowest weighted in-degrees are selected. Likewise, $C_{\text{in.dsc}}$ is associated with target nodes chosen in descending

order of weighted in-degrees, and nodes with the largest weighted in-degrees are selected. Analogously, $C_{\text{out.asc}}$ relates to target nodes picked in ascending order of weighted out-degrees; and $C_{\text{out.desc}}$ identifies target nodes of descending order of weighted out-degrees.

Table 2.1: Control energy needed in various networks when different strategies are applied to selecting target control matrix C . $\langle \mathcal{E}(C_0) \rangle$ is the average control energy from 10 independent initializations of random matrix without optimization. $\langle \mathcal{E}(C^*) \rangle$ is the average control energy from 10 independent iterative search using TPGM. These were converted into energy cost associated with binary matrices $\mathcal{E}([C_{\text{binary}}^*]^t)$, for $t = 1, 2, \dots, 10$. The lowest of these is chosen as $\mathcal{E}(C_{\text{binary}}^*)$. $\langle \mathcal{E}(C_{\text{rand}}) \rangle$ is the average control energy of 100 independent realizations of selecting target nodes randomly. Additional information such as standard deviations and mean of $\mathcal{E}([C_{\text{binary}}^*]^t)$ is presented in **Appendix A Supplementary information**.

Network	N	edges	$\langle \mathcal{E}(C_0) \rangle$	$\langle \mathcal{E}(C^*) \rangle$	$\mathcal{E}(C_{\text{binary}}^*)$	$\langle \mathcal{E}(C_{\text{rand}}) \rangle$
Model						
SF300	300	750	8.11E07	2.88E03	2.65E05	1.48E08
ER300	300	750	1.88E06	5.34E02	1.24E05	2.27E06
Electronic circuit [80]						
Circuit-s838	512	819	6.02E05	4.48E02	1.97E04	1.45E06
Circuit-s420	252	399	6.01E04	1.33E02	6.92E03	8.97E05
Circuit-s208	122	189	2.34E04	6.90E01	1.99E03	1.74E08
Food web [81–83]						
StMarks	54	356	4.83E03	6.31E01	2.77E02	5.54E03
Maspalomas	24	82	1.76E03	3.72E01	3.65E01	2.59E04
Rhode	19	53	8.07E02	2.00E01	3.65E01	1.46E05
Social Influence [84, 85]						
Phys-discuss-rev	231	565	1.90E04	1.99E02	4.11E03	1.49E05
Teacher-student	60	94	1.41E02	3.96E01	6.60E01	1.78E02
Social [84, 86]						
Phys-friend-rev	228	506	2.61E04	2.46E02	3.59E03	2.55E04
Highschool	70	366	4.32E04	1.91E02	5.45E02	4.06E04

Examining table 2.1, observe that for any network, starting from its initial, $\mathcal{E}(C_0)$, TPGM algorithm iteratively updates the C_k matrix in the direction of quickest decreasing control energy until the convergent solution, $\mathcal{E}(C^*)$, was obtained, where the energy cost is typically reduced by a few orders of magnitude. Based on section 2.3.2, the target nodes from the obtained optimal solution, C^* , were selected, thereby obtaining C_{binary}^* . While the conversion from dense real matrix solution to sparse binary matrix will in general result in increased control energy, comparing the energy cost $\mathcal{E}(C_{\text{binary}}^*)$ to heuristic selection strategies of target nodes, such as $\mathcal{E}(C_{\text{rand}})$ in table 2.1 and degree-based selection in table 2.2, the control energy compares favourably.

Table 2.2: Control energy needed in various networks when degree-based selection strategies are applied to selecting target control matrix C . $\mathcal{E}(C_{\text{in.asc}})$ refers to choosing target nodes in ascending order according to their weighted in-degrees. $\mathcal{E}(C_{\text{in.dsc}})$, $\mathcal{E}(C_{\text{out.asc}})$, and $\mathcal{E}(C_{\text{out.dsc}})$ follows analogously, where dsc refers to order of descending and out refers to weighted out-degree.

Network	N	edges	$\mathcal{E}(C_{\text{in.asc}})$	$\mathcal{E}(C_{\text{in.dsc}})$	$\mathcal{E}(C_{\text{out.asc}})$	$\mathcal{E}(C_{\text{out.dsc}})$
Model						
SF300	300	750	2.99E08	1.35E07	2.29E08	5.23E07
ER300	300	750	4.18E06	9.18E04	6.51E05	2.39E06
Electronic circuit						
Circuit-s838	512	819	4.77E05	4.99E05	4.06E04	4.80E05
Circuit-s420	252	399	8.67E04	6.68E04	8.78E05	9.59E04
Circuit-s208	122	189	3.20E06	2.65E04	4.00E04	3.20E06
Food web						
StMarks	54	356	4.73E02	4.53E02	7.16E02	5.19E02
Maspalomas	24	82	7.52E02	1.34E03	1.04E03	9.37E02
Rhode	19	53	6.52E07	1.35E06	6.11E07	5.94E06
Social Influence						
Phys-discuss-rev	231	565	4.29E04	2.62E04	1.11E04	2.95E05
Teacher-student	60	94	8.89E01	7.21E02	4.14E02	9.41E01
Social						
Phys-friend-rev	228	506	6.57E03	1.02E05	7.49E03	6.99E04
Highschool	70	366	9.45E04	1.55E03	4.10E03	1.55E05

2.4 Discussion

When the learning rate η is chosen appropriately, TPGM is convergent. To be illustrative, the success rates of convergence as a function of η for different network topologies are shown in Fig. 2.8 Generally, when η is lower, the success rate is higher. Furthermore, when P is lower and the search space is smaller, higher η tend to be accommodated. All C^* results presented in this work are convergent, even though the chosen η parameter may not have unity success rate. This is because the number of steps of the iterative search algorithm reveals if error has occurred; if the number of steps is abnormally small, the iteration is deemed to be non-convergent and to have terminated prematurely due to computation errors (for example, if the algorithm stops and the angle or energy cost is negative, this is likely an error). Therefore, searched solutions arising from error were discarded and repeated again. In this way, convergence is always guaranteed.

The convergence process is shown in Fig. 2.9(a) and (b). In Fig. 2.9(a), the k -th step cosine angle, $\cos(\theta_k)$, between energy gradient $\nabla \mathcal{E}(C_k^T)$ and projected energy gradient $\mathcal{T}_{\hat{C}_k^T} \nabla \mathcal{E}(C_k^T)$, approaches zero as TPGM iteration increases. The algorithm is deemed to have converged when the gradients are perpendicular. In Fig. 2.9(b), each iteration of TPGM brings the matrix C_k towards a lower energy cost, starting with a steep decrease in energy cost which becomes less steep with each iteration, until convergence.

It is important to initialize C_0^T properly. In the first initialization approach, all elements

Figure 2.8: The success rates of convergence as a function of $\log_{10} \eta$ for ER10 network (of Fig. 2) for target controlling 3 and 5 nodes, and rhode food web network (of table 1) for target controlling $P/N = 0.6$ number of nodes. η varies from $1e-8$ to $1e-3$. Each data point is calculated based on 100 independent iterative searches.

Figure 2.9: An illustration of one particular iterative search's convergence process on the electronic circuit network, Circuit-s208. **(a)** $\cos(\theta_k)$ moves closer to zero with iteration, although it is not always non-increasing. **(b)** Control energy is always non-increasing with each iteration, moving in the direction of largest decreasing energy cost. Insets show the first 0.3×10^5 iterations.

of C_0^T are chosen randomly and set to be $C_{ij}^T = 1$ if node i is chosen to be the j -th target node initially, otherwise all other elements are set to be zero. Row/column linear independence was ensured, which necessitates that no nodes are chosen repeatedly. If row/ column linear independence is not adhered, computations would not proceed due to mathematical error. The second initialization approach sets C_0^T to be a random matrix with dense entries of random variables. While TPGM can accommodate the second initialization approach, it is ill-advised as the computation starting from a dense matrix is in general inefficient. Furthermore, the solutions C^* and its associated C_{binary}^* also tend to perform poorly when compared to the first approach. However, no matter the initial condition, convergence is only affected by the learning rate parameter η .

When the number of nodes is small, such as in those shown in Fig. 2.4 and 2.5, it is feasi-

ble to verify TPGM's searched solutions and compare them with the true optimal target node set, which can be found via a brute force exhaustive search approach. As N increases, the N choose P brute force search is no longer viable. However, such limitation should also be viewed as a strength, because it shows the difficulty of the search problem due to its infeasibly large search space; by relaxing matrix C into continuous search space, TPGM is able to search for C^* in the direction of quickest energy cost decrease. Correspondingly, C_{binary}^* is recovered as the 'best signal' from C^* .

The computational complexity of TPGM is $\mathcal{O}(N^3)$ owing to its iterative matrix multiplication. It should be noted that the computations of the controllability Gramian matrix, $W = \int_{t_0}^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau$ as well as the calling of the exponential function for computing e^{At_f} ($e^{A^T t_f}$) are costly, which can result in bottlenecks when calculating them in the loop each time. An efficient way to code the TPGM is to compute the controllability Gramian W and e^{At_f} ($e^{A^T t_f}$) outside the while loop and storing them as variables to be retrieved within the loop. Furthermore, instead of directly computing the integral, an efficient way to compute the controllability Gramian is to use the method of ref. [87].

The condition number of the controllability Gramian matrix plays an important role in the numerical experiments. When trying to target control a network using only driver nodes found from structural controllability [1], the condition number of the Gramian may be too high and results in an infeasibly high control energy [28]. Increasing the number of control signals can lower the condition number and render the computation of the Gramian feasible [15, 28]. Note that the condition number is dependent on matrices A , B , as well as time horizon t_f . For some networks, despite increasing the number of driver nodes, the condition number of the Gramian may still be infeasibly high. To lower the condition number, the link weights of the connection matrix, $\{a_{ij}\}$, can be normalized by dividing throughout a normalization constant. For networks whose connection strengths are not specified, $\{a_{ij}\} = 1$ if there is a directed link from node j to node i . However, if this results in an infeasible condition number, then set the link strength to be random uniform $[0.5, 1.5]$, which tends to make the condition number feasible.

Building a connection between target and driver node set optimization could be very interesting. Drawing on driver nodes optimization of previous works [17, 18, 21, 77, 79] for controlling the full node set, the conclusions formed are as follows: when driver nodes are placed in such a way such that they are evenly spaced out, the control energy is most optimal. Thus, by grouping controlled nodes to their nearest driver nodes and arranging the driver nodes in such a way that no groups have excessive geodesic paths [74] from drivers to controlled nodes, energy cost is most optimized. Thus, the same conclusion should hold even for the case of target control, because full control and target control are not fundamentally different. Further, for the reverse problem of optimizing B , given that C is fixed, the same energy-optimal configurations, as presented in Fig. 2.4 and Fig. 2.5, are expected. In other words, whether optimizing driver nodes placement, or target nodes arrangement, the energy-optimal configurations are the same. Therefore, energy cost minimization could be an important mechanism in explaining structural self-organized configurations of driver/ target nodes in a natural or man-made complex system.

While the approach used in this Chapter has been algorithmic to find the optimal target nodes to reduce control energy, one may expect that a graph theoretic approach to finding optimal target nodes, given that B matrix is fixed, should lead to good solutions (low energy cost). The underlying physics follows from Ref. [24], where the longest control chain (LCC) dominates the control energy. Thus, assigning target nodes to be close to driver nodes, the path distance of LCC would be reduced, and control energy would be optimal.

In conclusion, this Chapter demonstrate the possibility of minimizing control energy of directed complex networks by optimizing the target node set C , given that network connection

matrix A , driver node placement B , and time horizon t_f are fixed. This was achieved by adapting target control [8], as well as deriving the energy gradient $\frac{\partial \mathcal{E}(C_k^T)}{\partial C^T} = \nabla \mathcal{E}(C_k^T)$ into the TPGM algorithm [21]. By ascribing target nodes to their nearest driver nodes, a directed complex network driven by driver nodes can be decomposed into elementary topologies [1, 79], such as stem, circle, and dilation. Through extensive simulations on elementary topologies, the results reveal that control energy is most optimal when target nodes are chosen such that path distances from driver nodes to target nodes are minimized, corroborating existing literature results [9, 24]. Furthermore, the results were validated on model networks (ER and SF) and real networks and show that optimal target node set (both C^* and C_{binary}^*) has control energy of a few orders of magnitude lower compared to target nodes chosen from heuristic selection schemes, such as random or nodes degree-based selection. Compared to previous works of control energy optimization [17–19, 21, 23], which focus on driver node set optimization, the present research concentrates on target node selection optimization, and show that in the context of target control, target node selections can account for variability of the control energy of a few orders of magnitude.

The problem of optimizing target node selections in the interest of control energy could be applicable to linear opinion networks [38–40], where the state vector represents opinions of individuals, and the driver nodes are modeled as agents of influence. In such a system, one may be interested in influencing a certain fractional share of opinions to align with a pre-defined favorable opinion. Much like a voter model problem [59], where one is only interested in obtaining the majority share of opinions, the specificity of which individuals to target control is not so much important as the control energy, which is to be minimized. There are some avenues of research which appear to be promising. For example, within the framework of network controllability, recent works have incorporated conformity behavior [26, 27], where individuals' opinions adapt over time to mirror the average of their neighbors', thus making the network dynamics richer and more realistic. It would be interesting to explore how optimal target nodes relate to a linear opinion network, both with and without conformity behavior in the future.

2.5 Methods

Model networks. Similar to recent works [4, 8], the model networks considered in this Chapter are modeled with stable dynamics, i.e. $\{a_{ii}\} < 0 \forall i$. Specifically, the diagonal of the connection matrix A was set to be $\{a_{ii}\} = \delta_i + \epsilon$, where δ_i is drawn from random uniform distribution $[-1, 1]$, and ϵ chosen to stabilize the nodal dynamics such that the eigenvalues of A are all negative and the largest eigenvalue is -1 . The scale-free model network is constructed from the static model [88, 89], and the link weights $\{a_{ij}\}$ are drawn randomly from a uniform interval $[0.5, 1.5]$.

Input nodes. The driver nodes selected are chosen using the method detailed in ref. [1], which uses the Hopcroft-Karp algorithm [66] to find the driver nodes to ensure controllability. To be specific, the driver nodes found in the research work presented in this Chapter are an overestimation, and guarantee full controllability of the complex network. Thus, controllability is always ensured, regardless of the choice of target node set C , and the term (CWC^T) is always invertible.

Weighted nodes degree. When considering nodes degree in the degree-based selection strategies, compute the weighted link weights of each node to determine the in-degree (out-degree). For example, the in-degree of node i is computed as: $= \sum_{j=1, j \neq i}^N \{a_{ij}\}$. Note that

self-links, $\{a_{ii}\}$, do not count as node degree as they arise from categorically different sources [14].

Gramian computation. The controllability Gramian can be efficiently calculated using the method of ref. [87]:

$$\exp \left(\begin{bmatrix} -A & BB^T \\ 0 & A^T \end{bmatrix} t_f \right) = \begin{bmatrix} F_2(t_f) & G_2(t_f) \\ 0 & F_3(t_f) \end{bmatrix}, \quad (2.11)$$

$$W = F_3(t_f)^T G_2(t_f), \quad (2.12)$$

where each block partitioned matrix of (2.11) is a $N \times N$ matrix. The controllability Gramian is computed using (2.12).

Another way is to compute the Gramian directly using numerical integration. While (2.11) and (2.12) may be computationally efficient, its numerical stability is not the best. However, for the results presented in this chapter, the numerical stability using this computation was sufficient.

Chapter 3

Energy cost study for controlling complex social networks with conformity behavior

3.1 Introduction

As mentioned in Chapter 1, within the literature, particular attention has been paid towards the energy cost needed for control. Chapter 2 has seen energy cost investigations numerically in terms of the optimization of the energy cost with respect to matrices, for example, $\mathbf{C}$ matrix optimization [33], and also introduced previous works of $\mathbf{B}$ matrix optimization [17, 22, 23]. In terms of analytical results, scaling behaviors have been derived in terms of number of drivers [4], target nodes cardinality [8], and T_f [3, 5, 6].

While the aforementioned works have been adequate in their treatment of network control, the results presented have so far generalised the network dynamics to be continuous-time and linear, without taking into account conformity. Given that networked controllability is restricted to the study of networks with linear dynamics owing to the difficulty in controlling nonlinear dynamical systems [2], continuous-time linear dynamics modelling is unable to capture conformity, which are often nonlinear [92], the same way that discrete-time linear dynamics modelling can [27]. As shown in Ref. [27], when the dynamics of the networked system has conformity behavior, where each member adapts and mimics its nearest neighbors' node states, the results obtained can differ when comparing network dynamics with and without conformity. For example, the eigenvalues of unweighted chain, ring, and complete graphs differ when comparing the systems with and without conformity. Besides, the conclusions drawn from structural controllability using a generalised network structure were inconsistent when taking into account self-dynamics [14]. Further, conformity dynamics models a significant class of social systems in a more realistic way, which a generalised linear dynamics may fail to capture. For example, evolutionary games [93], learning behaviors [94, 95] and collective movements [96–100] of animals are all described by models with conformity feature. Given the sensitivity of model specificities and the importance of control energy, it would be worthwhile to revisit previous calculations, keeping in mind conformity dynamics.

In this Chapter, the energy cost needed to control a complex network with conformity behavior is studied. The idea behind this work is to check what are the effects of the mechanism of conformity on the energy cost. To do so, the energy cost is studied analytically through a series of scaling laws. In general, the energy cost is affected by several different parameters such as initial state vector, final state vector, final control time T_f , number of drivers, and even network eigenvalues. Therefore, a systematic study of the analytical scaling laws of the control energy is advantageous, as compared to numerical ones, because it characterizes exactly the

boundaries of the energy cost based on these different parameters and allows the energy cost to be estimated. Based on the research results, it was found that for controlling networks with discrete-time linear dynamics and conformity behavior, the lower and upper bound energy cost scales as a function of T_f similarly to those of continuous-time linear dynamics without conformity behavior. While their scaling behaviors are similar, their energy costs are not. When comparing the energy cost needed to control systems with and without conformity, it was found that the energy cost of those with conformity features are always lower, suggesting that the mechanism of conformity is beneficial to the control of networked systems. Finally, the controllability of scale-free (SF) networks with degree-distribution $P(k) \sim k^{-\gamma}$ was also studied, and it was found that with or without conformity, the scaling exponent γ plays an important role in controllability, where SF networks with lower scaling exponent γ are less controllable, requiring a higher number of minimum driver nodes.

3.2 Problem formulation

In networked systems with conformity behavior, each node i mimics the strategies (or opinions) of its nearest neighbors j by adapting its node state in the next round as the average of its n_i nearest neighbors' node states in the current round [27]:

$$x_i(\tau + 1) = \frac{1}{s_i} \sum_{j=1}^{n_i} \hat{a}_{ij} x_j(\tau), \quad (3.1)$$

where $x_i(\tau)$ is the strategy of node i in discrete time τ , $s_i = \sum_{j=1}^N \hat{A}_{ij}$ is the weighted strength

of node i , with $\hat{\mathbf{A}} = \{\hat{a}_{ij}\}$ being the $N \times N$ network matrix, where $\hat{a}_{ij}$ is the non-negative directed weighted connection pointing from node j to node i , and zero otherwise. Note that x_i is a continuous variable that describes, for example, the degree of opinion in an opinion network [2, 101], where a high x_i value indicate support for opinion A , and a low x_i value indicate support for opinion B . In evolutionary games, $x_i \in [0, 1]$ denotes the probability of node i to select a particular strategy. For example, $x_i = 1$ indicates that node i always chooses to cooperate, while with $x_i = 0$, node i always defects [27]. Eq. (3.1) belongs to a class of conformity-based memory-one strategy games [102, 103], where the strategy played by an agent is influenced only by the strategies of their immediate neighbors in the round prior. For example, a fair player has a weighted probability to cooperate with other players based their immediate neighbors' strategies played in the previous round [104].

Introducing input control signal terms to Eq. (3.1), and writing the state equation in vector notation, the complex system has the following discrete linear time-invariant (LTI) dynamics [27]:

$$\begin{aligned} \mathbf{x}(\tau + 1) &= \mathbf{S}^{-1} \hat{\mathbf{A}} \mathbf{x}(\tau) + \mathbf{B} \mathbf{u}(\tau) \\ &= \mathbf{A} \mathbf{x}(\tau) + \mathbf{B} \mathbf{u}(\tau), \end{aligned} \quad (3.2)$$

where $\mathbf{x}(\tau) = [x_1(\tau), x_2(\tau), \dots, x_N(\tau)]^T$ is the state vector which captures the node states, $\mathbf{u}(\tau) = [u_1(\tau), u_2(\tau), \dots, u_M(\tau)]^T$ is the input control signals which attach to the complex network to alter the node states, $\mathbf{B} = \{b_{ij}\}$ is the $N \times M$ input matrix, comprising elementary column vectors (each control signal attaches only to a single node), which tracks where the M number of control signals are placed, where $b_{ij} = 1$ if control signal j attaches to node i (nodes

which receive control signals are called driver nodes), and $b_{ij} = 0$ otherwise, $\mathbf{S}^{-1}$ is a $N \times N$ diagonal matrix with its non-zero entries as the inverse of weighted node strength s_i (if $s_i = 0$, then set $\mathbf{S}^{-1}(i, i) = 0$), $\mathbf{A} = \mathbf{S}^{-1}\hat{\mathbf{A}} = \{a_{ij}\}$ is the $N \times N$ network matrix with conformity behavior, and is in general non-symmetric due to weighted quantity $\mathbf{S}^{-1}$.

Input control signal $\mathbf{u}(\tau)$ is responsible for driving the node states of the network and the energy cost required is defined as [6, 30, 105, 106] $\mathcal{E}(T_f) = \sum_{\tau=0}^{T_f-1} \mathbf{u}^T(\tau)\mathbf{u}(\tau)$, where T_f is the final control time, which is the allocated time that control signal $\mathbf{u}(\tau)$ must complete its tasks. Minimizing the energy cost, the discrete-time energy-optimal control signal can be derived as [6, 105]:

$$\mathbf{u}^*(\tau) = \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{W}^{-1}(\mathbf{x}_f - \mathbf{A}^{T_f}\mathbf{x}_0), \quad (3.3)$$

where

$$\mathbf{W}(T_f) = \sum_{\tau=0}^{T_f-1} \mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1} \quad (3.4)$$

is the $N \times N$ discrete-time controllability Gramian matrix, $\mathbf{x}_0 = [x_1(0), x_2(0), \dots, x_N(0)]^T$ is the initial state vector and $\mathbf{x}_f = [x_1(T_f), x_2(T_f), \dots, x_N(T_f)]^T$ is the desired final state vector. Substituting $\mathbf{u}^*(\tau)$ into $\mathcal{E}(T_f)$, and assuming that $\mathbf{x}_0 = \mathbf{0}$, the cost function becomes $\mathcal{E}(T_f) = \mathbf{x}_f^T \mathbf{W}^{-1} \mathbf{x}_f$. In addition, to simplify the scope of the research, the final state vector $\mathbf{x}_f$ is normalized by restricting the Euclidean distance $\mathbf{x}_f^T \mathbf{x}_f$ to be one, leading to the normalized cost function $E(T_f) = \mathcal{E}(T_f)/\mathbf{x}_f^T \mathbf{x}_f$, which is bounded by the eigenvalues of the Gramian using the Rayleigh-Ritz theorem [107]:

$$\frac{1}{|\eta'_{\max}(\mathbf{W}(T_f))|} = \underline{E}' \leq E(T_f) \leq \overline{E}' = \frac{1}{|\eta'_{\min}(\mathbf{W}(T_f))|}, \quad (3.5)$$

where $\underline{E}'(\overline{E}')$ is the lower (upper) bound of the energy cost, and $|\eta'_{\max}(\mathbf{W})|(|\eta'_{\min}(\mathbf{W})|)$ is the absolute maximum (minimum) eigenvalue of Gramian $\mathbf{W}$. In other words, to calculate the lower and upper bound energy costs, it is crucial to compute the inverse of the absolute maximum and minimum eigenvalues of the controllability Gramian.

The controllability Gramian exists in simplified form, which has an analytical expression comprising the complex network's eigenvalues. This can be obtained through eigendecompositions $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where $\mathbf{D}_{ii} = \lambda_i$ is the $N \times N$ diagonal matrix containing eigenvalues of $\mathbf{A}$ ($\mathbf{A}^T$), sorted in ascending order of magnitude $|\text{Re}\lambda_1| \leq |\text{Re}\lambda_2| \leq \dots \leq |\text{Re}\lambda_N|$, and $\mathbf{P}$ ($\mathbf{V}$) is the $N \times N$ associated eigenvectors of $\mathbf{A}$ ($\mathbf{A}^T$). Note that eigendecompositions are generically true for weighted network links, where the multiplicity of the eigenvalues is 1 (no two eigenvalues are the same). Substituting the eigendecompositions into Eq. (3.4), the Gramian, expressed in Hadamard product form [3–6], becomes:

$$\mathbf{W}(T_f) = \mathbf{P} \sum_{\tau=0}^{T_f-1} \mathbf{D}^\tau \mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V} \mathbf{D}^\tau \mathbf{V}^{-1} = \mathbf{P} \mathbf{M} \mathbf{V}^{-1}, \quad (3.6)$$

where $\mathbf{M}(T_f)$ is the simplified controllability Gramian, with (i, j) elements which can be obtained as (See **Appendix C Supplementary information** for derivation)

$$\mathbf{M}_{ij} = \mathbf{Q}_{ij} \frac{1 - (\lambda_i \lambda_j)^{T_f}}{1 - \lambda_i \lambda_j}, \quad (3.7)$$

with

$$\mathbf{Q}_{ij} = [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij}. \quad (3.8)$$

Therefore, to study the T_f scaling behaviors of normalized energy cost bounds, it suffices to analyze the eigenvalues of $\mathbf{M}(T_f)$ as T_f varies:

$$\frac{1}{|\eta_{\max}(\mathbf{M}(T_f))|} = \underline{E} \leq E(T_f) \leq \overline{E} = \frac{1}{|\eta_{\min}(\mathbf{M}(T_f))|}, \quad (3.9)$$

where $\underline{E}$ ($\overline{E}$) is the lower (upper) bound of the energy cost, and $|\eta_{\max}(\mathbf{M}(T_f))|$ ($|\eta_{\min}(\mathbf{M}(T_f))|$) is the absolute maximum (minimum) eigenvalue of $\mathbf{M}(T_f)$.

While most eigenvalues are computed numerically, the eigenvalues of $\mathbf{M}(T_f)$ can be estimated analytically with the formulae [5, 108]:

$$|\eta_{\max}(\mathbf{M}(T_f))| \approx \left| \operatorname{Re} \left[f(\overline{\alpha}, \overline{\beta}) \right] \right| \quad (3.10)$$

and

$$|\eta_{\min}(\mathbf{M}(T_f))| \approx \left| \operatorname{Re} \frac{1}{f(\underline{\alpha}, \underline{\beta})} \right|, \quad (3.11)$$

where $f(\alpha, \beta) = \sqrt{\frac{\alpha}{N}} + \sqrt{\frac{N-1}{N}(\beta - \frac{\alpha^2}{N})}$, and

$$\overline{\alpha} = \operatorname{trace}(\mathbf{M}^2), \quad (3.12)$$

$$\overline{\beta} = \operatorname{trace}(\mathbf{M}^4), \quad (3.13)$$

$$\underline{\alpha} = \operatorname{trace}[(\mathbf{M}^{-1})^2], \quad (3.14)$$

and

$$\underline{\beta} = \operatorname{trace}[(\mathbf{M}^{-1})^4] \quad (3.15)$$

are pertinent estimation parameters for obtaining analytical $\underline{E}$ or $\overline{E}$.

In subsequent analyses, through the use of Eq. (3.10) and (3.11), the analytical scaling behaviors of the energy bounds with respect to T_f are given in terms of small and large T_f regime, and different number of driver nodes. Where possible, analytical estimations for $\underline{E}$ and $\overline{E}$ are made, if not, then approximate, or numerical ones, which are all validated against numerical computations of $\underline{E}$ and $\overline{E}$.

3.3 Results

In this section, for the first time, the energy cost scaling laws of a complex network with discrete-time linear dynamics and conformity behavior is derived and quantified using the estimation formulae Eqs. (3.10) and (3.11). Necessarily, because the energy cost is somehow related to the network's eigenvalues, a discussion about the properties of the eigenvalues with and without conformity, and how they relate to system dynamics is given. Next, the energy costs needed to control complex networks with and without conformity are compared. Finally, the controllability properties of a SF network is studied, where it is showed, for the first time, that the scaling exponent γ of a SF network tunes its controllability.

The energy cost bounds are estimated analytically using Eqs. (3.10) and (3.11), which are themselves dependent on Eqs. (3.12)–(3.15), whose calculations depend on the traces of $\mathbf{M}$ and $\mathbf{M}^{-1}$. Recalling Eq. (3.7), $\mathbf{M}_{ij}$ is dependent on λ_i (and λ_j), the eigenvalues of network $\mathbf{A}$, which is why it is important to quantify λ_i , as the energy cost bounds are coupled to them. As

an example, using undirected Erdős–Rényi (ER) random networks with average degree $\langle k \rangle = 4$ varying from system size $N = 5$ up to $N = 195$, it is shown in Fig. 3.1(a), that when the system has conformity behavior, $|\text{Re}\lambda_i| < 1$ for $i = 1, 2, \dots, N - 1$, with $|\text{Re}\lambda_N| = 1$. In contrast, for the same network structure, but without conformity behavior, let the system dynamics be:

$$\mathbf{x}(\tau + 1) = \hat{\mathbf{A}}\mathbf{x}(\tau) + \mathbf{B}\mathbf{u}(\tau), \quad (3.16)$$

where $\hat{\mathbf{A}}$ is the network structure without conformity, with eigenvalues $\hat{\lambda}_i$ (for $i = 1, 2, \dots, N$), then it can be seen from Fig. 3.1(a) that $\hat{\mathbf{A}}$ has maximum eigenvalue $|\text{Re}\hat{\lambda}_N| > 1$, which makes the discrete-time system unstable because the node states diverge in time. The main difference between Eq. (3.2) and Eq. (3.16) lies in the fact that Eq. (3.2) is obtained as the generalised vectorised form of conformity behavior (Eq. (3.1)) plus the control terms, while Eq. (3.16) is just the discrete-time linear dynamics of the complex network structure $\hat{\mathbf{A}}$ without any conformity. This difference leads to system stability/instability as shown in Fig. 3.1. In practice, an unstable system is undesirable and stability is the basic requirement for control [109]. Thus, when the discrete-time system has conformity, the network's eigenvalues change and become less than or equal to one, causing the system to be stable and controllable.

To demonstrate system stability and instability, the evolutions of node states in an ER network with $N = 25$, and $\langle k \rangle = 4$, with and without conformity are plotted in Fig. 3.1(b) and (d), with $\mathbf{x}_0$ drawn from random uniform $[0, 1]$. In Fig. 3.1(b), when each node i adapts its opinion as the average of its n_i nearest neighbors, over time, consensus is reached within the network. On the other hand, when there is no conformity behavior, it can be seen in Fig. 3.1(d) that opinions diverge in τ , owing to $\text{Re}|\hat{\lambda}_N| > 1$.

Regardless of system stability, both types of networks can be (numerically) controlled with one driver node. Setting $\mathbf{x}_f = [4, 4, \dots, 4]^T$, and using energy-optimal control signal $\mathbf{u}^*(\tau)$, with $\hat{\mathbf{A}}$ replacing $\mathbf{A}$ in Eq. (3.3) for controlling network $\hat{\mathbf{A}}$, it is shown in Figs. 3.1(c) and (e) that the node states could be driven toward $\mathbf{x}_f$. However, due to system instability, control signal $\mathbf{u}(\hat{\mathbf{A}}, \tau)$ has to work much harder to bring the state vector towards the desired $\mathbf{x}_f$. At $T_f = 8N$, the numerical energy cost difference between controlling the system with and without conformity is about 15 orders of magnitude. While both conformity and non-conformity systems can be numerically controlled, the node states of the unstable system reach at least several orders of magnitude, which may in practice be infeasible.

Next, keeping in mind that $\lambda_i \leq 1$, the bounded estimates for the energy cost needed in controlling networks with discrete-time conformity-based linear dynamics is presented in the sections that follow. Before proceeding, it is worth mentioning that for discrete-time dynamical systems, T_f cannot be below a certain value, $\underline{T}_f$, otherwise the controllability Gramian is not invertible, and the system is not controllable. Based on numerical experiments, it was found that when using N drivers, $\underline{T}_f = 1$, otherwise,

$$\underline{T}_f = \lceil \frac{N}{M} \rceil + 1, \quad (3.17)$$

where M is the number of control signals or driver nodes.

3.3.1 Lower bound $\underline{E}$

As indicated in Fig. 3.1(a), networks $\mathbf{A}$ which have conformity behavior all have $|\text{Re}\lambda_N| = 1$, with the rest of the eigenvalues $\lambda_i < 1$ for $i = 1, 2, \dots, N - 1$. Therefore, since $\mathbf{M}_{ij} = \mathbf{Q}_{ij} \frac{1 - (\lambda_i \lambda_j)^{T_f}}{1 - \lambda_i \lambda_j}$, all $(\lambda_i \lambda_j)^{T_f}$ terms vanish in the large T_f limit, with the exception of the last

Figure 3.1: Comparison of non-conformity and conformity discrete LTI models. The magnitudes of the largest eigenvalues of $\hat{\mathbf{A}}$ and $\mathbf{A}$ denoted by $\text{Re}|\hat{\lambda}_N|$ and $\text{Re}|\lambda_N|$ respectively for random network with fixed $\langle k \rangle = 4$ and varying N is shown in (a), where each data point is computed as the mean (error bars are standard deviation) of 20 independent network realizations. State equation evolving in τ with conformity dynamics, absent control $\mathbf{u}(\tau)$ is shown in (b), while (c) shows the node states being driven by one driver node towards the desired $\mathbf{x}_f = 4$. (d) depicts the state equation of non-conformity dynamics, which is unstable and diverges in τ . (e) demonstrates the unstable system being driven by one driver node towards $\mathbf{x}_f = 4$, with inset showing the final two time step.

row/col element (using L'hopital's rule to evaluate the limit) $\mathbf{M}_{NN} = \lim_{\text{Re}\lambda_N^2 \rightarrow 1} \mathbf{Q}_{NN} \frac{1 - \text{Re}(\lambda_N^2)^{T_f}}{1 - \text{Re}(\lambda_N^2)} = \mathbf{Q}_{NN} T_f$. In the large T_f limit, all $\mathbf{M}_{ij}$ values are small relative to $\mathbf{M}_{NN}$, which dominates, and applying Eq. (3.10),

$$\underline{E} \approx \text{Re} \left| \frac{1}{\mathbf{Q}_{NN} T_f} \right| \sim T_f^{-1}, \quad (3.18)$$

regardless of number of drivers, which is validated with model networks (ER and SF) as shown in Fig. 3.2, and real networks (office [110] and karate [111]) as shown in Fig. 3.3. Note that the scaling $\underline{E} \sim T_f^{-1}$ is true for large T_f , but not necessarily true for small T_f . For small T_f , analytical $\underline{E}$ should be estimated with Eqs. (3.10), (3.12) and (3.13) using the non-vanishing $\mathbf{M}_{ij}$. However, based on Figs. 3.2 and 3.3, $\underline{E} \sim T_f^{-1}$ approximates the small T_f lower bound energy cost with good agreement between analytical and numerical results in most cases. For Fig. 3.3(g), the deviation of the numerical computations from the analytical scaling is noticeable, and the small T_f analytical $\underline{E}$ should be estimated with the non-vanishing $\mathbf{M}_{ij}$ instead.

The driver nodes placement is encoded in $\mathbf{Q}$ (Eq. (3.8)), which affects the simplified controllability Gramian $\mathbf{M}$ (Eq. (3.7)), and is responsible for shifting the $\underline{E}$ plots vertically. When controlling the network with a single control signal attached to node h , $\mathbf{B}$ is a $N \times 1$ matrix, with entry $b_{h1} = 1$, and the rest of the elements $b_{ij} = 0$, leading to $\mathbf{B}\mathbf{B}^T = \mathbf{J}^{hh}$, where $\mathbf{J}^{hh}$ is a single-entry matrix [90], and substituting into Eq. (3.8), $\mathbf{Q}_{ij} = [\mathbf{P}^{-1}]_{jh} v_{hi}$. When

Figure 3.2: Energy cost lower and upper bound for controlling model networks. Top-half (a)–(f) and bottom-half (g)–(l) panels relate to ER and SF networks (both are $N = 50$, $\langle k \rangle = 6$) respectively. Each column represents the same number of drivers, and $d = 3$. Triangles are numerical calculations of the inverse of maximum and minimum absolute eigenvalues of $\mathbf{M}$, while solid lines, dashed lines, and dash-dotted lines are respectively the analytical scaling laws, numerical linear fit, and $\overline{E}$ estimates using Eq. (3.11), calculated from numerical $\mathbf{M}^{-1}$.

Figure 3.3: Energy cost lower and upper bound for controlling real networks. Top-half (a)–(f) and bottom-half (g)–(j) panels relate to office and karate networks ($N = 40$, $\langle k \rangle = 11.9$, and $N = 34$, $\langle k \rangle = 4.6$) respectively. Each column represents the indicated number of drivers, with $d = 3$ for office network and $d = 10$ for karate network. There are no one driver node plots for karate network, because its minimum number of drivers [29] is $N_D = 10$. Triangles are numerical calculations of the inverse of maximum and minimum absolute eigenvalues of $\mathbf{M}$, while solid lines, red dashed lines, blue dashed lines, and dash-dotted lines are respectively the analytical scaling laws, numerical linear fit, small T_f analytical $\underline{E}$, and $\bar{E}$ estimates using Eq. (3.11), calculated from numerical $\mathbf{M}^{-1}$.

using d number of drivers (where $1 < d < N$), $\mathbf{B}$ is a $N \times d$ matrix with the driver nodes placement $b_{ij} = 1$ if the j -th control signal attaches to node i and zero otherwise, leading to $\mathbf{B}\mathbf{B}^T = \mathbf{J}^{d_1 d_1} + \mathbf{J}^{d_2 d_2} + \dots + \mathbf{J}^{d_d d_d}$, where d_k represents driver node k for $k = 1, 2, \dots, d$, then $\mathbf{Q}_{ij} = \sum_{k=1}^d [\mathbf{P}^{-1}]_{id_k} v_{d_k j}$. Finally, when controlling with all N drivers such that all N nodes each receive a control signal, $\mathbf{B}$ is a $N \times N$ matrix with $b_{ii} = 1$ for $i = 1, 2, \dots, N$ and zero everywhere else, leading to $\mathbf{B} = \mathbf{I}$, and $\mathbf{Q}_{ij} = [\mathbf{P}^{-1}\mathbf{V}]_{ij}$. Consequently, substituting the resultant $\mathbf{Q}_{ij}$ terms into Eq. (3.18),

$$\underline{E} \approx \begin{cases} |\text{Re}[\mathbf{P}^{-1}]_{Nh} v_{hN} T_f]^{-1}|, & \text{one driver,} \\ |\text{Re}[\sum_{k=1}^d [\mathbf{P}^{-1}]_{Nd_k} v_{d_k N} T_f]^{-1}|, & d \text{ drivers,} \\ |\text{Re}[\mathbf{P}^{-1}\mathbf{V}]_{NN} T_f]^{-1}|, & N \text{ drivers.} \end{cases} \quad (3.19)$$

Comparing Figs. 3.2(a)–(c), (g)–(i), and Figs. 3.3(a)–(c), (g)–(h), it can be seen that an increase in number of drivers leads to a decrease in $\underline{E}$.

3.3.2 Upper bound $\overline{E}$

Based on observing the numerical calculations of $\overline{E}$, there are two distinct regimes: small T_f regime, characterized by linear scaling behavior $\overline{E} \sim T_f^{-\Theta}$, where Θ is a numerical value to be found, and large T_f regime, where upper bound $\overline{E}$ converge to a constant value. Using different number of driver nodes leads to different scaling exponents such that $\overline{E} \sim T_f^{-\Theta_1}$, $\overline{E} \sim T_f^{-\Theta_d}$, and $\overline{E} \sim T_f^{-\Theta_N}$ when using one driver, d drivers, and N drivers respectively. Note that the range of small T_f regime is relatively small and decreases with increasing number of driver nodes. Small T_f regime is characterized by the linear scaling behavior $\overline{E} \sim T_f^{-\Theta}$, and not by the range of T_f . A simple algorithm can be used to perform the linear fitting of numerical $\overline{E}$:

1. Calculate numerical $\overline{E}$ at $T_f = \{\underline{T}_f, \underline{T}_f + 1, \underline{T}_f + 2, \dots, \underline{T}_f + 9\}$, where $\underline{T}_f$ is the minimum permissible T_f
2. Linear fit the first two data points computed at $\underline{T}_f$ and $\underline{T}_f + 1$ and accept this range as the valid small T_f linear scaling regime
3. Linear fit the first three data points computed at $\underline{T}_f$, $\underline{T}_f + 1$, and $\underline{T}_f + 2$, and measure the R -squared value. If R -squared value ≥ 0.85 , accept the data points as the valid small T_f linear scaling regime and move to step 4., else terminate
4. Repeat step 3. with increasing number of data points: linear fit $\overline{E}$ computations evaluated at $T_f = [\underline{T}_f, \underline{T}_f + i]$, for $i = 3, 4, \dots, 9$.

From this simple algorithm, the small T_f regime is linearly fitted, where for one driver node calculations, small T_f regime spans several data points, which decreases as number of driver nodes increases. Because using one driver node requires the most control energy [4], it can be seen that $\Theta_1 > \Theta_d > \Theta_N$ when comparing Figs. 3.2(d)–(f), (j)–(l), and Figs. 3.3(d)–(f), (i)–(j). Θ_N is in general < 1 and essentially negligible in T_f . In addition, $\Theta_1 \sim \frac{N}{\langle k \rangle}$ regardless of network topology, as shown in Fig. 3.4.

Next, $\overline{E}$ is derived for large T_f limit, when controlling the network with one driver node, where a control signal is attached to node h , leading to input matrix $\mathbf{B}$ and simplified controllability Gramian matrix $\mathbf{M}$ the same as before in Section. 3.3.1. For the upper bound, analytical $\overline{E}$

in the large T_f regime has to be estimated using Eqs. (3.11), (3.14), and (3.15), which are dependent on the inverse of the simplified controllability Gramian, $\mathbf{M}^{-1}$. While $\mathbf{M}$ has an analytical expression as shown in Eq. (3.7), analytical $\mathbf{M}^{-1}$ has to be derived by applying $\mathbf{M}^{-1} = \frac{\mathbf{M}^*}{|\mathbf{M}|}$, where $\mathbf{M}^*$ is the adjoint matrix, and $|\mathbf{M}|$ is the determinant. Using Gaussian elimination to obtain the general expressions of $\mathbf{M}^*$ and $|\mathbf{M}|$ based on $\mathbf{M}$, analytical $\mathbf{M}^{-1}$ in the large T_f limit is obtained as shown in Appendix (B.1). Note that (B.1) differs non-trivially from Ref. [6]'s discrete-time system $\mathbf{M}^{-1}$, not just in terms of eigenvectors, but also in terms of form. Inspecting (B.1), $\mathbf{M}^{-1}$ elements lying in the last row/col are negligible when T_f is large. Ignoring those terms and applying Eq. (3.14), analytical $\underline{\alpha}$ (B.2) is a constant. Similarly, $\underline{\beta}$ can be obtained through Eq. (3.15). Substituting $\underline{\alpha}$ and $\underline{\beta}$ into Eq. (3.11), analytical $\bar{E}$ is a constant.

When using more than one driver (d and N drivers) to control the complex system, it is difficult to obtain analytical $\mathbf{M}^{-1}$ in the large T_f limit. However, large T_f analytical $\bar{E}$ can still be estimated with Eqs. (3.11), (3.14), and (3.15) using $\mathbf{M}^{-1}$ computed numerically. From Figs. 3.2 and 3.3, the $\bar{E}$ scaling behavior of the system is noted to be the same as when using one driver node, and $\bar{E}$ converge to a constant value in the large T_f limit. In addition, for N drivers, $\bar{E}$ can also be approximated with (3.11) using an approximate analytical $\mathbf{M}^{-1}$, which is a diagonal matrix such that its non-zero entries $\mathbf{M}^{-1}(i, i) = [\mathbf{M}_{ii}]^{-1} = \frac{1-\lambda_i^2}{[\mathbf{P}^{-1}\mathbf{V}]_{ii}}$. This approximation can be verified against numerical computation of $\mathbf{M}^{-1}$, where it is noted that the main diagonal elements are large in comparison to the rest of matrix elements. Applying approximate $\mathbf{M}^{-1}$ to Eqs. (3.14) and (3.15), $\underline{\alpha} \approx \sum_{i=1}^{N-1} \left[\frac{1-\lambda_i^2}{[\mathbf{P}^{-1}\mathbf{V}]_{ii}} \right]^2$ and $\underline{\beta} \approx \sum_{i=1}^{N-1} \left[\frac{1-\lambda_i^2}{[\mathbf{P}^{-1}\mathbf{V}]_{ii}} \right]^4$, which when applied to Eq. (3.11), also yields a constant in the large T_f limit. The validity of the analytical or approximate results are demonstrated in Figs. 3.2 and 3.3.

Regardless of network topology, it can be seen in Figs. 3.2 and 3.3 that the scaling behaviors of the upper bound $\bar{E}$ of the energy cost with respect to T_f are very similar.

$$\bar{E} \begin{cases} \sim T_f^{-\Theta}, & \text{small } T_f, \\ \approx \text{constant}, & \text{large } T_f, \text{ one driver,} \\ \approx \text{constant}, & \text{large } T_f, d \text{ drivers,} \\ \approx \text{constant}, & \text{large } T_f, N \text{ drivers,} \end{cases} \quad (3.20)$$

where in the small T_f regime, because control signal $\mathbf{u}(\tau)$ has to drive the node states in a short amount of time, most energy is needed. Relaxing T_f requirement on the driver nodes reduces $\bar{E}$ at a scaling rate of $\bar{E} \sim T_f^{\Theta_1}$, $\bar{E} \sim T_f^{\Theta_d}$ or $\bar{E} \sim T_f^{\Theta_N}$ depending on the number of drivers used. However, increasing T_f beyond a certain point, $\bar{E}$ converge to a constant value. As expected, because increasing the number of drivers reduces control energy [4], $\bar{E}$ is highest when using just one driver node.

3.3.3 Comparing large T_f $\bar{E}$: conformity versus non-conformity

From Figs. 3.2, 3.3, and Eq. (3.20), it is apparent that $\bar{E}$ scales $\sim T_f^{-\Theta}$ in the small T_f regime, eventually converging to a constant value in the large T_f limit. This behavior is similar to those of continuous-time dynamics [5], in particular, when the non-conformity continuous-time network $\tilde{\mathbf{A}}$ is not positive definite (PD, meaning that the eigenvalues of $\tilde{\mathbf{A}}$ are all positive) or positive semi-definite (PSD, meaning that the eigenvalues of $\tilde{\mathbf{A}}$ are all positive or zeros). A consequent natural question to ask is: How do the $\bar{E}$ values compare in the absence and presence of conformity? Answering this question could offer insights into the mechanism of

Figure 3.4: Θ_1 calculated from numerical linear fit in small T_f regime of the one driver scaling law, $\overline{E} \sim T_f^{-\Theta_1}$, for varying N and $\langle k \rangle$ with random and scale-free network topologies. Each data point is the mean of 20 independent network realizations and error bar is the standard deviation.

conformity in the context of networked controllability. Before proceeding, it is worth clarifying why only large $T_f \overline{E}$ values should be compared between the conformity-based discrete-time dynamical system and the non-conformity-based continuous-time dynamical system. Firstly, it is not sensible to compare conformity and non-conformity dynamics energy costs between the discrete-time systems (3.2) and (3.16) because while the former is stable, the latter is unstable. Secondly, it is not sensible to compare the energy costs in the small T_f regime, as the discrete-time dynamical system is limited by T_f , while the continuous-time dynamical system can set T_f to an arbitrarily small value. Thirdly, it is not sensible to compare lower bound, as conformity-based $\underline{E}$ goes to zero in the large T_f limit. It should be noted that T_f has different meaning in the discrete-time and continuous-time system; for example, $T_f = 20$ is a relatively small T_f for the discrete-time system, but $T_f = 20$ is a large enough value for the continuous-time system such that further increase in T_f would not lead to further reduction in energy cost [3, 5]. For the remainder of this section, it can be assumed that the T_f values for the continuous-time or the discrete-time dynamical system are chosen sensibly such that only the convergent large $T_f \overline{E}$ values are being compared.

The continuous-time dynamical system has the following dynamics [1]:

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (3.21)$$

where $\tilde{\mathbf{A}} = \{\tilde{a}_{ij}\} \in \mathbb{R}^{N \times N}$ describes the network connection such that when node j has a directed link to node i , $\tilde{a}_{ij}$ is non-zero (in addition, it is assumed that non-zero $\tilde{a}_{ij}$ is non-negative), otherwise, $\tilde{a}_{ij} = 0$, $\mathbf{B} = \{b_{ij}\} \in \mathbb{R}^{N \times M}$ accounts for where in the network the control signals are placed, for example, $b_{ij} = 1$ if control signal j attaches to node i , otherwise $b_{ij} = 0$, t is a continuous variable depicting instantaneous time, $\dot{\mathbf{x}}(t) = [\dot{x}_1(t), \dot{x}_2(t), \dots, \dot{x}_N(t)]^T$, $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_N(t)]^T$, and $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_N(t)]^T$ are respectively the instantaneous vector of rate of change of states, the state vector, and input control signals vector. Further, the continuous-time non-conformity network $\tilde{\mathbf{A}}$ is different from the discrete-time non-conformity network $\hat{\mathbf{A}}$ on the diagonal matrix elements. For the continuous-time system, to model system stability [4], $\tilde{a}_{ii} = -\sum_{j=1}^N \tilde{a}_{ij} - \delta$, where δ represents a small perturbation to en-

Figure 3.5: Comparing the large T_f upper bound energy cost $\bar{E}$ between the continuous-time non-conformity dynamical system and the discrete-time conformity-based dynamical system. The network sizes are all $N = 50$, with average degree of $\langle k \rangle = 8$. Both continuous-time non-conformity $\log(\bar{E})$ and discrete-time conformity $\log(\bar{E})$ can be linearly fitted against N/M , indicating that $\bar{E} \sim e^{N/M}$ for both types of systems. Overall, the conformity-based system requires less control energy. (a) shows $\bar{E}$ needed to control ER networks, while (b) shows $\bar{E}$ needed to control SF networks. Each data point is computed based on the mean of 20 independent realizations, and error bars represent standard deviation.

sure system stability (in contrast, discrete-time $\hat{a}_{ii} = 0$). For the simulations that follow, $\delta = 0$, which is suitable for modelling opinion dynamics [101].

The energy cost for the continuous-time dynamical system is defined to be [71] $\mathcal{E}(T_f) = \int_0^{T_f} \mathbf{u}^T(t)\mathbf{u}(t)dt$, whereupon optimizing [3, 5], leads to the continuous-time energy-optimal control signal

$$\mathbf{u}^*(t) = \mathbf{B}^T e^{\tilde{\mathbf{A}}^T(T_f-t)} \mathbf{W}^{-1} (\mathbf{x}_f - e^{\tilde{\mathbf{A}}T_f} \mathbf{x}_0), \quad (3.22)$$

where $\mathbf{W} = \int_0^{T_f} e^{\tilde{\mathbf{A}}(T_f-t)} \mathbf{B} \mathbf{B}^T e^{\tilde{\mathbf{A}}^T(T_f-t)} dt$ is the $N \times N$ continuous-time controllability Gramian, $\mathbf{x}_0 = [x_1(0), x_2(0), \dots, x_N(0)]^T$ is the initial state vector when $t = 0$, and $\mathbf{x}_f = [x_1(T_f), x_2(T_f), \dots, x_N(T_f)]^T$ is the final state vector at $t = T_f$ that the complex system is being driven towards. Following analogous reasoning as Section. 3.2, the bounds for the energy cost needed in controlling a continuous-time dynamical system is just the same as Eq. (3.9), except that $\eta_{\min}$ and $\eta_{\max}$ are the minimum and maximum eigenvalues of the simplified continuous-time controllability Gramian [5]:

$$\mathbf{M}(T_f) = \tilde{\mathbf{P}}^T \mathbf{W} \tilde{\mathbf{P}}, \quad (3.23)$$

where $\tilde{\mathbf{P}}$ is the eigenvectors matrix of the eigen-decomposed continuous-time non-conformity matrix $\tilde{\mathbf{A}} = \tilde{\mathbf{P}} \tilde{\mathbf{D}} \tilde{\mathbf{P}}^T$, and $\mathbf{W}$ is the continuous-time controllability Gramian matrix. Thus, numerically, the upper bound energy cost is computed as:

$$\bar{E} = \frac{1}{|\eta_{\min}(\mathbf{M}(T_f))|}. \quad (3.24)$$

Computing large T_f $\bar{E}$ for the continuous-time non-conformity and the discrete-time conformity systems, their energy costs are compared in Fig. 3.5. For both Figs. 3.5(a) and (b), corresponding to ER networks and SF networks, $\log(\bar{E})$ is plotted on the vertical axes, while N/M , the total N number of nodes of the complex network over the total M number of control signals, is plotted on their horizontal axes. For ease of reading the energy costs, the equivalent

$\log_{10}(\bar{E})$ scale is also provided on the right-end vertical axes. Unsurprisingly, as noted from Fig. 3.5, the linear relationship between $\log(\bar{E})$ and N/M affirms the continuous-time results reported by Ref. [4], that the upper bound $\bar{E} \sim e^{N/M}$, indicating that an increase in the number of drivers leads to a decrease in maximum control energy at an exponential rate. Different from what has already been reported in the literature, however, is that this scaling behavior, $\bar{E} \sim e^{N/M}$, is also true for the discrete-time dynamical system with conformity. Further, regardless of number of drivers and complex network topology, the upper bound energy cost $\bar{E}$ of the discrete-time conformity-based dynamical system always requires less control energy. When using only one control signal to drive the network such that $N/M = 50$, the difference in $\bar{E}$ can be as much as 20 orders of magnitude. The disparity in $\bar{E}$ lessens as the M number of drivers increases. The results suggest that the mechanism of conformity always leads to a complex system that is easier to control, requiring less control energy.

3.3.4 The role of scaling exponent γ in the controllability of SF networks

When computing N_D minimum number of drivers [29] needed to ensure controllability of the karate network in Fig. 3.3, it was found that $N_D = 10$. Defining the degree of controllability as the fractional number of minimum drivers over the total number of nodes, $n_D = \frac{N_D}{N}$, and $\frac{\langle k \rangle}{N}$ as the density of connections of the network, the karate network was low in controllability (requiring high N_d to be controlled), with $n_D \approx 0.29$, despite being dense in connections $\frac{\langle k \rangle}{N} \approx \frac{4.6}{34} \approx 0.14$. In contrast, as shown in Ref. [27], SF networks with $\gamma = 3$ have n_D decreasing monotonically with increasing $\langle k \rangle$ towards $n_D = \frac{1}{N}$, where starting at $\frac{\langle k \rangle}{N} \approx \frac{3}{500} = 0.006$, are sufficiently dense and controllable with one driver. Therefore, with the karate network as a counterexample, $\langle k \rangle$ cannot be the only mechanism in determining the controllability of SF networks.

The scaling exponent γ also affects the controllability of SF networks. Using the static model [89] to generate SF networks with varying γ (where $2 < \gamma < \infty$), it is shown in Fig. 3.6, for fixed $N = 500$, that SF networks with lower γ (2.05) are less controllable. On the other hand, high γ (5 and 10) does not improve the controllability of SF networks significantly when compared to $\gamma = 3$. All scaling exponent γ curves display decreasing n_D with increasing $\langle k \rangle$, and the curves are the same with or without conformity dynamics.

3.4 Discussion

In this Chapter, the energy cost needed to control a complex network with conformity behavior is studied. To do so, a series of scaling laws were derived, based on the number of driver nodes,

Table 3.1: Summary of the scaling behaviors of the lower bound $\underline{E}$ and upper bound $\bar{E}$ of the energy cost with respect to T_f in small and large T_f regime using one, d , and N number of drivers.

	T_f regime	one driver	d drivers	N drivers
$\underline{E}$	small/large	$\sim T_f^{-1}$	$\sim T_f^{-1}$	$\sim T_f^{-1}$
$\bar{E}$	small	$\sim T_f^{-\Theta_1}$	$\sim T_f^{-\Theta_d}$	$\sim T_f^{-\Theta_N}$
$\bar{E}$	large	(3.11) with (B.1)	constant	constant

Figure 3.6: The degree of controllability n_D of SF networks with varying γ and $\langle k \rangle$ values. Each data point is the mean of 20 independent network realizations, with the error bar being the standard deviation.

to characterize the control energy as a function of T_f in terms of its lower bound and upper bound, allowing for the energy cost to be estimated. A summary of the scaling laws can be found at Table 3.1. In particular, it was found that lower bound scales as $\underline{E} \sim T_f^{-1}$, suggesting that in the large T_f limit, no energy cost is needed by the driver nodes to steer the state vector towards the least costly direction. If the intended final state vector coincides with the network consensus opinion, which can be reached naturally with conformity dynamics, then no further action is required by the driver nodes. Upper bound energy cost decreases with increasing T_f at a rate of $\overline{E} \sim T_f^{-\Theta}$ in the small T_f limit, suggesting that in this regime, setting a relatively larger T_f always leads to less energy cost. Further, it was found that $\Theta_1 \sim \frac{N}{\langle k \rangle}$ regardless of network topology, and the small T_f regime scaling behavior depends only on system size and average node degree. In the large T_f limit, $\overline{E}$ converges to a constant value, indicating that in this regime, the energy cost cannot be lowered any further by increasing T_f . Overall, these scaling laws have implications about the trade-off between final control time T_f and energy cost and is useful to the design of control strategies in complex social networks with conformity behavior.

It is interesting to note the difference in controllability and energy cost of the networked systems with and without conformity. From the point of view of dynamical systems, the discrete-time non-conformity model $\hat{\mathbf{A}}$ is always practically uncontrollable owing to system instability. Yet, the same network structure with conformity in discrete-time linear dynamics, $\mathbf{A} = \mathbf{S}^{-1}\hat{\mathbf{A}}$, is always stable and controllable, suggesting that the mechanism of conformity can make an otherwise unstable and uncontrollable discrete-time linear dynamical complex system stable and controllable. In terms of controllability, it was found that the fractional minimum number of drivers that guarantees the controllability of a SF network is unaffected by the mechanism of conformity. Instead, given the scale-free degree distribution $P(k) \sim k^{-\gamma}$ of a SF network, it was found that the scaling exponent γ tunes the controllability, and heterogeneous (smaller γ) [112] SF networks are less controllable, requiring a higher minimum number of driver nodes. In terms of energy cost, networks with discrete-time linear dynamics and conformity behavior and networks with continuous-time linear dynamics but no conformity behavior both scale as a function of number of drivers as $\overline{E} \sim e^{N/M}$, indicating that regardless of conformity, adding

more driver nodes always reduces the maximum control energy at an exponential rate. Nevertheless, comparing the convergent large $T_f \bar{E}$ values of the different systems shows that the conformity-based model always requires less control energy with a difference of $\bar{E}$ values by as much as 20 orders of magnitude when using one driver node to control networks of system size $N = 50$. All in all, the results suggest that the mechanism of conformity plays an important role in making a complex social system more controllable.

Objectively speaking, the degree to which a social system is controllable has neutral connotations: It is neither better nor worse to be more/less controllable. For example, a complex social system that is more difficult to control is at the same time more resilient against malicious social manipulations as it is against positive social change. The converse is true for a complex social system that is easier to control. Thus, from the point of view of governance, conformity behavior in complex social systems should be discouraged in some situations, for example, the sharing of controversial or polarizing news, and encouraged in some others, for example, the propagation of ideas which can lead to positive social change such as recycling. Recently, a few socio-physics works have been successful in corroborating their mathematical models with real world data, such as the distributions of agree, disagree, or undecided in controversial and non-controversial topics using poll data [113], and the emergence of echo chambers in social networks surrounding politically controversial topics using Twitter data [114]. It would be interesting to corroborate or even disprove, with real world data, the claim of the present research that conformity in social networks always leads to a situation where the network is easier to control or influence. Further, a limitation of networked controllability is that only linear dynamics can be studied [2], while various socio-physics models can be nonlinear or have states which are not continuous variables [54]. In these types of models, how would conformity dynamics influence the system as a whole?

3.5 Methods

System stability, network matrix diagonal entries and numerical precision require careful attention during computation. Unlike its continuous-time counterparts, which could be stabilized by introducing negative terms along the network diagonals [4], the discrete-time system cannot be stabilized the same way. Introducing diagonal entries in $\hat{\mathbf{A}}$ would cause system instability for both non-conformity and conformity discrete-time dynamics. Therefore, in this work, for discrete-time dynamics, the diagonal entries of $\hat{\mathbf{A}}$ are zero, and conformity-based model is stable with eigenvalues $\lambda_i \leq 1$. When computing the state equation of the unstable non-conformity discrete-time dynamics, which diverges, high numerical precision is needed, which can be achieved using Advanpix [115]. Computing the unstable state equation without high numerical precision, using the standard double precision, would lead to erroneous calculations where they would wrongly stabilize in large τ . Further, one driver node control typically requires high energy cost, and $\bar{E}$ has to be calculated with high numerical precision.

The model networks used in this Chapter are constructed with the standard Erdős–Rényi random network algorithm, and the static model [89], where in Fig. 3.2, $\gamma = 2.5$, in Fig. 3.5, $\gamma = 3.0$, and in Fig. 3.6, γ is varied. All model networks $\hat{\mathbf{A}}$ and $\tilde{\mathbf{A}}$ are undirected, symmetric, and weighted, with link weights $\hat{a}_{ij} = \hat{a}_{ji}$ or $\tilde{a}_{ij} = \tilde{a}_{ji}$ drawn from random uniform $(0, 1]$. The network construction algorithms do not guarantee that the graph object does not have isolated nodes, and for one driver node calculations, checks were done to ensure that each node has at least one connection, otherwise the network was discarded and replaced. For the real networks, the karate network is unweighted with link $\hat{a}_{ij} = 1$ if nodes i and j are connected and zero otherwise, the office network is weighted but with integer $\hat{a}_{ij}$ greater or equal to 1 if nodes i and

j are connected and zero otherwise.

With sufficiently high average degree $\langle k \rangle$, the conformity-based model networks used have $N_D = 1$, and any random choice of driver nodes (for d drivers and one driver) would suffice to ensure network controllability. It should be noted that k , node degree, and s , node strength, are distinct from each other; where the former is calculated from the adjacency matrix, and the latter from $\hat{\mathbf{A}}$ whose links $\hat{a}_{ij}$ are weighted. For computing minimum driver nodes [29], the Matlab software written by Patel was used [116].

Chapter 4

How zealots affect the energy cost for controlling complex social networks

4.1 Introduction

While the energy cost needed for controlling networks have been investigated sufficiently within the literature, there still lacks an investigation on how much energy is needed to control complex social networks where there are zealots present. This is important because in statistical physics of social dynamics [54] (or socio-physics), zealots, agents with unwavering opinions, have been researched in various social dynamics models [55–61]. For example, if considering a network of opinions on operating systems (such as Microsoft Windows, Apple MAC O/S, or Linux), then a zealot is someone who is fiercely loyal to a particular opinion, who refuses to accept any other views, while advocating theirs to others [60]. Theoretically, zealots represent interesting modifications to existing models to examine altered system behavior. Empirically, partisanship [118] and confirmation-bias [114] in social networks have been reported, suggesting that zealots could also be a realistic feature of networked social systems, since not all members are truly neutral. While zealots have become well-understood in the setting of socio-physics, it is still unclear how they affect the behavior of the controllability of complex networks and its associated energy cost. For example, in a campaign to steer the opinions of individuals in a complex social network, would the presence of zealots assist or sabotage the campaigning effort?

In this Chapter, zealots are introduced to the framework of network controllability, focusing on networked controllability in the context of socio-physics. Conceptually, this research is similar to an earlier work [13], which considers two types of drivers, one effecting local influences, and the other the canonical driver nodes, which steers the state vector globally. However, Ref. [13] focuses on the context of infrastructure networks and presents numerical results when the competing driver nodes induce exponentially increasing local influences to simulate infrastructural damages, and examines the amount of energy needed to neutralize these attacks. On the other hand, the present research focuses on competing driver nodes that induce constant local influences, simulating zealots' unwavering opinions, and studies the amount of energy needed to control the social network in competition or cooperation with the zealots. Furthermore, detailed analytical and numerical results are presented, where the number of canonical driver nodes and control time regimes are varied. In addition, beyond the canonical continuous-time linear dynamics [1], the analyses extend toward discrete-time linear dynamics with conformity behavior [27], which may be of particular interest to socio-physics, since it models social networks where each agent conforms to their nearest neighbors. Taken together, this Chapter presents a nuanced characterization of how zealots affect the energy cost when trying to control a complex social

network, where it shows that the interplay between different parameters such as the zealots' beliefs, number of drivers, final control time regimes, network effects, network dynamics, number and configurations of nodes influenced by the zealots, can lead to different energy cost behaviors.

4.2 Result

4.2.1 Continuous-time linear dynamics model

To study zealots affecting the energy cost in controlling a complex social network, an example is given in Fig. 4.1. Think of the node states, $x_i(t)$, as opinions on a particular topic, where a positive $x_i(t)$ models support of a particular idea, and a negative $x_i(t)$ denotes opposition. In Fig. 4.1(a), there are $N = 2$ number of normal agents (nodes 1 and 2), neutral nodes without any preferred opinion, who are open to adopting new ideas, and communicate with one another to exchange information. In Fig. 4.1(d), a control signal $u_1(t)$ attaches to node 1, making it the driver node [1] (colored red); by directly changing the state of node 1 with $u_1(t)$, the state of node 2 becomes affected and all normal agents are controlled toward consensus c . In Fig. 4.1(g), a stubborn agent, or zealot node (node z), with fixed opinion z , is introduced into the system; the zealot influences node 2, making it a zealot-influenced node (colored blue). While the driver node is still able to control all normal agents toward consensus c , less/more effort may be needed, depending on c and z . The states evolution in time are displayed in Fig. 4.1(b), where they stabilize in long time in the absence of control, and in Figs. 4.1(e) and (h), where the network is driven in the absence and presence of zealots respectively. Correspondingly, the normal agents' state space trajectories are shown in Figs. 4.1(c), (f), and (i). In Fig. 4.1(i), it is not necessary to show the 3rd dimension state of the zealot in the trajectory as its presence simply constraints $x_1(t)$ and $x_2(t)$ to the fixed $x_3 = z$ plane. For this example, $z = -5$, contrary to the control goal of driving toward $c = 2$, and it can be seen in Fig. 4.1(i) that the state space trajectory elongates as compared to Fig. 4.1(f).

There are a few rules that this model should follow:

- The mutual exchange of information between normal agents is modelled by undirected links between them
- Zealots receive no directed links from any other nodes or control signals as they hold steadfast to their beliefs
- Zealots advocate their beliefs to other normal agents through directed links
- External control signal $u(t)$ steers the opinions of normal agents with directed links, and a single control signal can only attach to one normal agent.

Although self-dynamics links are not shown in the Fig. 4.1 models, they should be present for continuous-time network dynamics. Self-dynamics stabilize the dynamics of the system [4], which is crucial for modelling complex dynamical systems realistically [14], for example, opinion dynamics [101]. While there can be multiple zealots present in an arbitrary network, to simplify the scope of the research, only one group of zealots holding the same opinion z is considered, and so it is mathematically the same to model only one zealot node affecting multiple normal agents. Throughout the rest of the Chapter, “zealot”, “zealot node”, or “zealots” are used interchangeably. For notational conciseness, the total system size is denoted to be n , where $n = N + 1$, of which N are normal agents, and the n -th node is always the zealot node.

Figure 4.1: (a) Network in the absence of control signal and zealot. (b) Node states evolution of the network in the absence of external influences. (c) State space trajectory of $x_1(t)$ and $x_2(t)$ in the absence of external influences. (d) Network with control signal $u_1(t)$ attached to node 1. (e) Node states evolution of the network being driven toward consensus $c = 2$. (f) State space trajectory of nodes being driven toward consensus. (g) Network with control signal $u_1(t)$ attached to node 1 and zealot node influencing node 2. (h) Node states evolution of the network being driven toward consensus, while node 2 is under the influence of the zealot node with fixed opinion $z = -5$. (i) State space trajectory of nodes being driven toward consensus while under the influence of the zealot.

Generalizing to an arbitrary network with n nodes, the continuous-time model with zealots influencing normal agents within the network can be modelled using target control [8]:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}\tag{4.1}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_N(t), z]^T \in \mathbb{R}^{n \times 1}$ is the time-varying state vector, with the first N elements denoting the node states of the N normal agents, and the n -th element denoting the zealot node's fixed belief, z . $\mathbf{A}$ is the full $n \times n$ network structure such that a_{ij} is non-zero if there is a directed link from node j to node i ($a_{ij} = 0$ otherwise), and comprises the symmetric reduced N -dimensional principal submatrix $\tilde{\mathbf{A}}$ (remove n -th row/column), where non-zero (and non-negative) undirected link $\tilde{a}_{ij} = \tilde{a}_{ji}$ represents normal agent nodes i and j that are connected with each other and exchange ideas, with the final row $a_{in} = 1$ if zealot node influences node i ($a_{in} = 0$ otherwise, and a_{ni} is always zero because the zealot node cannot be influenced by any other nodes). $\mathbf{B} \in \mathbb{R}^{n \times M}$ is the control input matrix, where M is the number of control signals (such that $1 \leq M \leq N$), and $b_{ij} = 1$ if control signal j attaches to node i (for $i = 1, 2, \dots, N$, where $i = n$ is not permitted as the zealot node has unwavering opinion, and $j = 1, 2, \dots, M$). In addition, it is required that a single control signal attaches only to a single normal agent. Nodes which have a control signal attached to them are called driver nodes. $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_M(t)]^T \in \mathbb{R}^{M \times 1}$ is the input vector of external control signals. $\mathbf{y}(t) = [y_1(t), y_2(t), \dots, y_N(t)]^T \in \mathbb{R}^{N \times 1}$ is the output state vector. $\mathbf{C} \in \mathbb{R}^{N \times n}$ is the target control matrix which relates which node states are being steered by $\mathbf{u}(t)$, and $c_{ij} = 1$ if node j is the i -th node (where $i, j = 1, 2, \dots, N$) to be target controlled, and $c_{ij} = 0$ otherwise. While Eq. (4.1) borrows the language of target control [8], note that for this research, full controllability of all N normal agents are considered, thus identity $\mathbf{I}_N$ is the reduced N -dimensional principal submatrix of $\mathbf{C}$, such that only the first N nodes of the state vector is being driven (final column $c_{in} = 0$ for $i = 1, 2, \dots, N$ since the zealot node has fixed opinion, z , controlling the n -th node is not permitted).

The energy cost is defined to be [71]

$$J = \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt, \tag{4.2}$$

where t_0 is the initial time, t_f is the final control time (the amount of time allocated to the control signals to steer the state vector), and when minimized leads to the energy-optimal target control signal [8]

$$\mathbf{u}^*(t) = \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} (\mathbf{y}_f - \mathbf{C} e^{\mathbf{A}(t_f-t_0)} \mathbf{x}_0), \tag{4.3}$$

where $\mathbf{W} = \int_{t_0}^{t_f} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} dt$ is the controllability Gramian, $\mathbf{y}_f = [c, c, \dots, c]^T \in \mathbb{R}^{N \times 1}$ is the final output state vector where c is the consensus opinion that the system is being steered toward, and $\mathbf{x}_0 = [0, 0, \dots, 0, z]^T \in \mathbb{R}^{n \times 1}$ is the initial state vector of the system, where it is assumed that all node states begin initially with neutral opinions at zero. When $\mathbf{B}$ matrix is chosen appropriately such that the system is controllable, $(\mathbf{C} \mathbf{W} \mathbf{C}^T)$ is invertible [8]. Substituting Eq. (4.3) into Eq. (4.2), and setting $t_0 = 0$, the energy cost when using $\mathbf{u}^*(t)$ to steer the complex system is

$$\mathcal{E} = \mathbf{y}_f^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{y}_f - 2 \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{y}_f + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \tag{4.4}$$

(for derivation of Eq. (4.4) and subsequent derivations, see **Appendix H Supplementary information**).

The reduced network connection matrix, $\tilde{\mathbf{A}}$, which consists of only normal agents' mutual interactions (undirected links), is symmetric and can be diagonalized as $\tilde{\mathbf{A}} = \tilde{\mathbf{P}}\tilde{\mathbf{D}}\tilde{\mathbf{P}}^T$, where $\tilde{\mathbf{P}}$ is the $N \times N$ orthonormal eigenvectors matrix such that $\tilde{\mathbf{P}}\tilde{\mathbf{P}}^T = \tilde{\mathbf{P}}^T\tilde{\mathbf{P}} = \mathbf{I}$, and $\tilde{\mathbf{D}} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ is a diagonal matrix containing the eigenvalues of $\tilde{\mathbf{A}}$, where they are ordered ascendingly: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. Equivalently, $\tilde{\mathbf{A}}$ is obtained by removing the n -th row/column off the full network matrix, $\mathbf{A}$. The full network matrix, $\mathbf{A}$, includes the directed links of the zealot node to normal agents in the last column, a_{in} , is non-symmetric and thus eigen-decomposed as $n \times n$ dimensional matrices $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where $\mathbf{P}$ and $\mathbf{V}$ are the eigenvectors matrices of $\mathbf{A}$ and $\mathbf{A}^T$ respectively, and $\mathbf{D} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N, 0\}$ comprises $\tilde{\mathbf{D}}$ as its reduced N -dimensional principal submatrix, with the n -th diagonal entry being 0 (zealot node does not have self-link since it has fixed z opinion). Most complex systems tend to operate near a stable state, so for the continuous-time model, the eigenvalues λ_i (for $i = 1, 2, \dots, N$) are all negative [4, 119], and network $\tilde{\mathbf{A}}$ is negative definite (ND). Note that matrix terms containing $\sim$ symbols refer to the $N \times N$ reduced matrix without the zealot node, while those without refer to the $n \times n$ full matrix.

For controlling all N normal agents, $(\mathbf{CWC}^T) = \tilde{\mathbf{W}}$, where $\tilde{\mathbf{W}} = \int_0^{t_f} e^{\tilde{\mathbf{A}}(t_f-t)} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T e^{\tilde{\mathbf{A}}^T(t_f-t)} dt$ is the controllability Gramian of the reduced network connection matrix, $\tilde{\mathbf{A}}$, and $\tilde{\mathbf{B}} \in \mathbb{R}^{N \times M}$ is the input control matrix of the reduced network matrix, with $\tilde{b}_{ij} = 1$ if control signal j attaches to node i ($\tilde{b}_{ij} = 0$ otherwise). The controllability Gramian, $\tilde{\mathbf{W}}$, can be expressed as the Hadamard product analytical form [3–5]

$$\tilde{\mathbf{W}} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{P}}^T = \tilde{\mathbf{P}}[\tilde{\mathbf{Q}} \circ \tilde{\mathbf{F}}]\tilde{\mathbf{P}}^T, \quad (4.5)$$

where $\tilde{\mathbf{M}} = \tilde{\mathbf{Q}} \circ \tilde{\mathbf{F}}$ is the simplified controllability Gramian of the reduced matrix, and

$$\tilde{M}_{ij} = \tilde{Q}_{ij}\tilde{F}_{ij} = [\tilde{\mathbf{P}}^T\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{P}}]_{ij} \frac{[e^{(\lambda_i+\lambda_j)t_f} - 1]}{\lambda_i + \lambda_j}, \quad (4.6)$$

with $\tilde{Q}_{ij} = [\tilde{\mathbf{P}}^T\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{P}}]_{ij}$, and $\tilde{F}_{ij} = \frac{[e^{(\lambda_i+\lambda_j)t_f} - 1]}{\lambda_i + \lambda_j}$. The inverse matrices follow similarly as

$$(\mathbf{CWC}^T)^{-1} = \tilde{\mathbf{W}}^{-1} = (\tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{P}}^T)^{-1} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T. \quad (4.7)$$

Substituting the eigen-decompositions $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, $\tilde{\mathbf{A}} = \tilde{\mathbf{P}}\tilde{\mathbf{D}}\tilde{\mathbf{P}}^T$, and Eq. (4.7) into Eq. (4.4), the energy cost becomes

$$\begin{aligned} \mathcal{E} = & c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [\mathbf{V}e^{\mathbf{D}t_f}\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} \\ & + z^2 \sum_{i=1}^N \sum_{j=1}^N [\mathbf{V}e^{\mathbf{D}t_f}\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} [\mathbf{P}e^{\mathbf{D}t_f}\mathbf{P}^{-1}]_{jn}. \end{aligned} \quad (4.8)$$

By inspection, Eq. (4.8) is a quadratic function in terms of z , given that c is fixed. Solving the turning point $\frac{\partial \mathcal{E}}{\partial z} = 0$, z^* can be obtained, where z^* is the opinion of the zealot node which yields the lowest energy cost when controlling the system toward consensus c . Further, the network structure $\tilde{\mathbf{A}}$, as well as the nodes which are influenced by the zealot node can also cause a change in the control energy.

Analytical equations of energy cost

Depending on the final control time, t_f , and driver nodes placement, $\tilde{\mathbf{M}}^{-1}$ changes and Eq. (4.8) changes accordingly, leading to different behaviors. Subsequently, similar to Refs. [3] and [5], the energy cost is analyzed in terms of differing number of driver nodes through control input matrix $\tilde{\mathbf{B}}$, which is encoded in $\tilde{\mathbf{M}}$, and different final control time regimes (small t_f and large t_f).

Controlling the system with N driver nodes such that all N normal agents each receive a control signal, $\tilde{\mathbf{B}} = \mathbf{I}_N$, $\tilde{\mathbf{Q}} = \mathbf{I}_N$, and $\tilde{\mathbf{M}}$ becomes a diagonal matrix with its main diagonal entries being $\tilde{\mathbf{M}}_{ii} = \tilde{\mathbf{F}}_{ii} = \frac{1}{2\lambda_i}[e^{2\lambda_i t_f} - 1]$. $\tilde{\mathbf{M}}^{-1}$ is also a diagonal matrix, with $[\tilde{\mathbf{M}}^{-1}]_{ii} = \frac{1}{\tilde{\mathbf{M}}_{ii}} = \frac{2\lambda_i}{[e^{2\lambda_i t_f} - 1]}$, which in the small t_f limit, using Taylor expansion $e^{2\lambda_i t_f} \approx 1 + 2\lambda_i t_f$, and in the large t_f limit, as the eigenvalues are negative $[e^{2\lambda_i t_f} - 1] \approx -1$, leading to

$$\tilde{\mathbf{M}}^{-1}(i, i) \approx \begin{cases} -2\lambda_i, & \text{large } t_f, \\ t_f^{-1}, & \text{small } t_f. \end{cases} \quad (4.9)$$

When controlling the system with one driver node, where there is a single control signal $u_1(t)$ attached to arbitrary node h , then $\tilde{\mathbf{B}}$ is a $N \times 1$ matrix with $\tilde{b}_{h1} = 1$, leading to $\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T = \tilde{\mathbf{J}}^{hh}$, where $\tilde{\mathbf{J}}^{hh}$ is a $N \times N$ single-entry matrix [90] such that $[\tilde{\mathbf{J}}^{hh}]_{ij} = 1$ when $i = j = h$, and zero otherwise. Consequently, $\tilde{\mathbf{Q}}_{ij} = \tilde{p}_{hi}\tilde{p}_{hj}$, $\tilde{\mathbf{M}}_{ij} = \tilde{p}_{hi}\tilde{p}_{hj}[\frac{e^{(\lambda_i+\lambda_j)t_f}-1}{\lambda_i+\lambda_j}]$, and in the large t_f limit, the exponential terms vanish because all λ_i are negative, and $\tilde{\mathbf{M}}_{ij} = \frac{-\tilde{p}_{hi}\tilde{p}_{hj}}{\lambda_i+\lambda_j}$. Using $\tilde{\mathbf{M}}^{-1} = \frac{\tilde{\mathbf{M}}^*}{|\tilde{\mathbf{M}}|}$, where $\tilde{\mathbf{M}}^*$ and $|\tilde{\mathbf{M}}|$ are the adjoint matrix and determinant of $\tilde{\mathbf{M}}$ respectively, [5]

$$\tilde{\mathbf{M}}^{-1}(i, j) = \frac{-4\lambda_i\lambda_j}{\tilde{p}_{hi}\tilde{p}_{hj}(\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k}. \quad (4.10)$$

For small t_f , neither first-order nor second-order Taylor expansion of $\tilde{\mathbf{F}}_{ij}$ yields an invertible $\tilde{\mathbf{M}}$, and $\tilde{\mathbf{M}}^{-1}(i, j)$ has to be estimated with [5] $|\tilde{\mathbf{M}}| \sim t_f^{N_0}$ and $\tilde{\mathbf{M}}^*(i, j) \sim t_f^{N_{ij}}$ such that

$$\tilde{\mathbf{M}}^{-1}(i, j) \sim t_f^{N_{ij}-N_0}, \quad (4.11)$$

where the integer exponents N_{ij} (for $i, j = 1, 2, \dots, N$) or N_0 refer to the N_{ij} -th or N_0 -th order Taylor expansion of $\tilde{\mathbf{F}}_{ij}$ where invertibility is satisfied, and are computed numerically.

Using d number of drivers to control the network, where $1 < d < N$, $\tilde{\mathbf{B}}$ is a $N \times d$ matrix with $\tilde{b}_{ij} = 1$ if control signal j (where $j = 1, 2, \dots, d$) attaches to node i (where $i = 1, 2, \dots, N$),

leading to $\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T = \sum_{k=1}^d \mathbf{J}^{d_k d_k}$, where $d_k = \{1, 2, \dots, N\}$ refers to the arbitrary k -th driver node.

Thereafter, $\tilde{\mathbf{Q}}_{ij} = \sum_{k=1}^d \tilde{p}_{d_k i} \tilde{p}_{d_k j}$, and $\tilde{\mathbf{M}}_{ij} = \sum_{k=1}^d \tilde{p}_{d_k i} \tilde{p}_{d_k j} \frac{[e^{(\lambda_i+\lambda_j)t_f}-1]}{\lambda_i+\lambda_j}$. Owing to the summation, it is difficult factor the terms to derive $\tilde{\mathbf{M}}^{-1}$ analytically. Thus, d drivers energy cost results have to be computed with numerical $\tilde{\mathbf{M}}^{-1}$.

Substituting Eqs. (4.9) and (4.10) into Eq. (4.8), the analytical energy costs are (the superscripts denote the t_f regime, and the subscripts denote the number of drivers used to control the network)

$$\mathcal{E}_N^{\text{large } t_f} = -2c^2 \sum_{i,j,k} \tilde{p}_{ik} \tilde{p}_{jk} \lambda_k + 4cz \sum_{i,j,k} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ik} \tilde{p}_{jk} \lambda_k - 2z^2 [\mathbf{P}^{-1}]_{nn} \sum_{i,j,k} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ik} \tilde{p}_{jk} p_{jn} \lambda_k, \quad (4.12)$$

where $\sum_{i,j,k} = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N$,

$$\mathcal{E}_N^{\text{small } t_f} \approx c^2 t_f^{-1} N - 2czr + z^2 r t_f, \quad (4.13)$$

where $r = \sum_{i=1}^N a_{in}$ is the integer number of nodes which are influenced by the zealot node (the n -th node),

$$\begin{aligned} \mathcal{E}_1^{\text{large } t_f} = & c^2 \sum_{i,j,l,m} \tilde{p}_{li} \tilde{p}_{mj} \frac{-4\lambda_i \lambda_j}{\tilde{p}_{hi} \tilde{p}_{hj} (\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k} \\ & - 2cz \sum_{i,j,l,m} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{li} \tilde{p}_{mj} \frac{-4\lambda_i \lambda_j}{\tilde{p}_{hi} \tilde{p}_{hj} (\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k} \\ & + z^2 \sum_{i,j,l,m} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{li} \tilde{p}_{mj} p_{mn} [\mathbf{P}^{-1}]_{nn} \frac{-4\lambda_i \lambda_j}{\tilde{p}_{hi} \tilde{p}_{hj} (\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k}, \end{aligned} \quad (4.14)$$

where $\sum_{i,j,l,m} = \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \sum_{m=1}^N$.

For small t_f regime, one driver energy cost, although Eq. (4.11) is a valid approximation, owing to the coupling terms $[\mathbf{V} e^{\mathbf{D} t_f} \mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij}$ and $[\mathbf{V} e^{\mathbf{D} t_f} \mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij} [\mathbf{P} e^{\mathbf{D} t_f} \mathbf{P}^{-1}]_{jn}$ in Eq. (4.8), it is difficult to express small t_f $\tilde{\mathbf{M}}^{-1}(i, j)$ terms in approximate form. Therefore, numerical $\tilde{\mathbf{M}}^{-1}$ is used instead. Furthermore, since d drivers energy cost also require numerical $\tilde{\mathbf{M}}^{-1}$, all three analytical energy cost equations should thus be expressed by Eq. (4.8). Letting $\mathbf{V} e^{\mathbf{D} t_f} \mathbf{V}^{-1} = e^{\mathbf{A}^T t_f}$, and $\mathbf{P} e^{\mathbf{D} t_f} \mathbf{P}^{-1} = e^{\mathbf{A} t_f}$,

$$\left. \begin{aligned} \mathcal{E}_1^{\text{small } t_f} \\ \mathcal{E}_d^{\text{large } t_f} \\ \mathcal{E}_d^{\text{small } t_f} \end{aligned} \right\} = c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij} + z^2 \sum_{i=1}^N \sum_{j=1}^N [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij} [e^{\mathbf{A} t_f}]_{jn}, \quad (4.15)$$

where the driver nodes placement are encoded in $\tilde{\mathbf{M}}^{-1}$ through the inverse of Eq. (4.6), and the time regimes are set by t_f .

From the presented analytical energy cost equations, note that the choice of zealot-influenced nodes affects the energy cost. In the large t_f regime, such as Eqs. (4.12) and (4.14), the choice of zealot-influenced nodes enters the equations through $[\mathbf{V}^{-1}]_{ni}$ and p_{jn} , which are respectively the final row and final column of the eigenvectors matrix of the full (transposed) network, $\mathbf{A}^T$ and $\mathbf{A}$. Depending on which nodes are being influenced by the zealot node (through $a_{in} = \{0, 1\}$), $[\mathbf{V}^{-1}]_{ni}$ and p_{jn} change accordingly, and the choice of influenced nodes is consequential to the energy cost. On the other hand, in the small t_f regime, the N drivers energy cost is invariant to the choice of zealot-influenced nodes. In Eq. (4.13), ceteris paribus, only the r number (and not choice) of zealot-influenced nodes tunes the energy cost.

Taking $\frac{\partial \mathcal{E}}{\partial z} = 0$, the minima are (subscripts denote number of drivers and t_f regime)

$$z_{N, \text{large } t_f}^* = \frac{c \sum_{i,j,k} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ik} \tilde{p}_{jk} \lambda_k}{[\mathbf{P}^{-1}]_{nn} \sum_{i,j,k} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ik} \tilde{p}_{jk} p_{jn} \lambda_k}, \quad (4.16)$$

$$z_{N, \text{small } t_f}^* \approx ct_f^{-1}, \quad (4.17)$$

$$z_{1, \text{large } t_f}^* = \frac{c \sum_{i,j,l,m} [\mathbf{V}^{-1}]_{nl} \tilde{p}_{li} \tilde{p}_{mj} \frac{-4\lambda_i \lambda_j}{\tilde{p}_{hi} \tilde{p}_{hj} (\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}}{\sum_{i,j,l,m} [\mathbf{V}^{-1}]_{nl} \tilde{p}_{li} \tilde{p}_{mj} p_{mn} [\mathbf{P}^{-1}]_{nn} \frac{-4\lambda_i \lambda_j}{\tilde{p}_{hi} \tilde{p}_{hj} (\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}}, \quad (4.18)$$

and

$$\left. \begin{aligned} z_{1, \text{small } t_f}^* \\ z_{d, \text{large } t_f}^* \\ z_{d, \text{small } t_f}^* \end{aligned} \right\} = \frac{c \sum_{i=1}^N \sum_{j=1}^N [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij}}{\sum_{i=1}^N \sum_{j=1}^N [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}} \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{P}}^T]_{ij} [e^{\mathbf{A} t_f}]_{jn}}. \quad (4.19)$$

When the zealots hold unwavering optimal opinion z^* , their presence assist in driving the network opinions toward c by lowering the energy cost, as compared to the situation where there were no zealots present. In the small t_f regime, when using N drivers to control the complex network, the optimal zealot $z_{N, \text{small } t_f}^*$ opinion, is unaffected by network properties, and is proportional to consensus c and t_f^{-1} . In contrast, all other turning points z^* are affected by network properties. In those cases, it is difficult to analyze z^* by inspection owing to the many coupled terms present, and numerical experiments are needed to gain more insight.

Numerical experiments

In the results that follows, it can be assumed that $t_f = 20$ and $t_f \leq 0.1$ respectively for large and small t_f regimes, and the complex networks are being driven toward consensus $c = 5$, which is fixed throughout, while z , the zealot's unwavering opinion is tuned, and the choice of nodes influenced by the zealot node (normal agent nodes which receive directed links from the zealot node) is varied.

N drivers

The results of the numerical experiments when controlling random Erdős–Rényi [42] (ER) and scale-free (SF; static model [89]) complex networks topologies toward consensus $c = 5$, with $N = 300$ normal agents, and average degree $\langle k \rangle = 6$, using N number of drivers, such that each node directly receives a control signal $u_i(t)$ are presented in Figs. 4.2 and 4.3 for large t_f and small t_f respectively.

As expected, the analytical energy cost $\mathcal{E}(z)$, as a function of varying fixed zealot opinion z (Eqs. (4.12) and (4.13)), validated against numerical computations, show a quadratic behavior with respect to z in Figs. 4.2(a), 4.2(g), 4.3(a), 4.3(b), 4.3(g), and 4.3(h). Therefore, depending on what the zealot's fixed opinion z is, the energy cost needed to control a complex network to consensus $c = 5$ follows a quadratic curve, which has a turning point at z^* (black dotted line) that assists in lowering the energy cost, as compared to the energy cost needed for controlling the complex network in a situation where there are no zealots present, denoted by the horizontal green dashed line intersecting with the quadratic curve at $z = 0$. Away from minima z^* , the energy cost increases at a z^2 rate, which when above the green dashed line, opinion z becomes detrimental to the controlling of complex networks, and the zealots' presence increases energy cost.

Figure 4.2: Continuous-time linear dynamics: Large t_f regime, N drivers results in ER and SF networks with varying configurations of zealot-influenced nodes. (a)–(f) are results from a ER network, while (g)–(l) are results from a SF network. (a) and (g) validate Eq. (4.12). (b) and (h) show that as the turning points z^* increases, their associated minima energy costs $\mathcal{E}_N^{\text{large } t_f}(z^*)$ decreases. (c) and (i) show that the average node degree $\langle k \rangle$ of a configuration of zealot-influenced nodes correlate to its turning point z^* . Inset in (c) show the linear fits of all the data points. (d) and (j) show that length l is proportional to energy cost for different configurations of nodes influenced by a zealot of fixed $z = -5$ opinion (black dashed lines in both are fitted by $\mathcal{E}_N^{\text{large } t_f}(l) \approx l^{1.77}$). (e) and (k) are the same, except that the zealot has fixed opinion $z = 5$. Likewise, (f) and (l) are the same, except that the zealot has fixed opinion $z = z^*$. The insets in (d), (e), (j), and (k) show the data points associated with $r = 1, r = 2, r = 3, r = 4$ which are clustered closely. The different markers denote the varying r number of zealot-influenced normal agents according to legend in (b) or (i), and green dashed line indicates the energy cost in the absence of any zealot.

Figure 4.3: Continuous-time linear dynamics: Small t_f regime, N drivers results in ER and SF networks. Unlike the large t_f regime results, varying the choice of zealot-influenced normal agents is inconsequential to the energy cost, and only the r number of zealot-influenced normal agents matters. (a)—(f) are results from a ER network, while (g)—(l) are results from a SF network. (a) and (g) ((b) and (h)) validate Eq. (4.13) at $t_f = 0.1$ ($t_f = 0.01$) for different r values, where the insets show the analytical energy cost curves associated with $r = 1$, $r = 2$, $r = 3$, and $r = 4$. Note that the turning points z^* is invariant to the r number of zealot-influenced nodes. (c) and (i) show the turning points z^* and associated minima energy costs (insets show the data points associated with $r = 1$, $r = 2$, $r = 3$, $r = 4$, where there are slight deviations between analytical and numerical results due to the first-order approximation $e^{\mathbf{A}t_f} \approx \mathbf{I}_n + \mathbf{A}t_f$ used in deriving analytical results). (d) and (j) display the energy cost and associated state space trajectory length l when different r number of zealot-influenced nodes is varied (the symbol markers correspond to r value according to legends in (c) or (i)), as well as the opinion z of the zealot is varied from $[-100, -20]$ (color represent the associated z values). (e) and (k) are the same, except that $z \in [20, 100]$. Likewise, (f) and (l) are the same, except that $z \in [120, 200]$. Green dashed line indicates the energy cost in the absence of any zealot.

N drivers: Large t_f regime

For the large t_f regime results, Figs. 4.2(a) and (g) correspond to the energy costs (Eq. (4.12)) needed to control ER and SF networks when a particular set of nodes have been influenced by the zealot node, with strength of zealot opinion z varying in the range $[-10, 10]$. In any one instance of a numerical experiment, the zealot node can influence r number of normal agent nodes, which are fixed, once chosen, for the entirety of that instance. Depending on the configuration of zealot-influenced nodes, the parabolas as shown in Figs. 4.2(a) and (g) vary accordingly with different turning points z^* , and their associated minimum energy costs $\mathcal{E}_N^{\text{large } t_f}(z^*)$, and steepness. To understand how the selection of r number of zealot-influenced nodes affects the energy cost, their configurations were varied with increasing r , from $r = 1$, $r = 2$, $r = 3$, $r = 4$, up to $r/N = 60\%$ of the network nodes, and the turning points z^* , minima energy costs $\mathcal{E}_N^{\text{large } t_f}(z^*)$, and associated state space trajectory lengths were measured.

The state space trajectory was first introduced in Fig. 4.1, for example, when the zealot node (holding fixed zealot opinion $z = -5$) is introduced into the system (for controlling network node states toward consensus $c = 2$), the length of the state space trajectory elongates. Mathematically, the length l of the state space trajectory is computed as follows:

$$l = \sum_{i=1}^{99} \sqrt{\sum_{j=1}^N [x_j(t_i) - x_j(t_{i+1})]^2}, \quad (4.20)$$

where it should be noted that although the time variable $t \in [0, t_f]$ is a continuous variable, it is computationally sampled at 100 evenly spaced t_i values (for $i = 1, 2, \dots, 100$). Thus, the length l of the state space trajectory is computationally approximated as the sum of the lengths of 99 pieces of straight line Euclidean distances between $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$, $\mathbf{x}(t_2)$ and $\mathbf{x}(t_3)$, ..., $\mathbf{x}(t_{99})$ and $\mathbf{x}(t_{100})$.

One may wish to ask if a higher z^* is more beneficial or less beneficial to the energy cost needed in controlling the complex network. Plotting z^* against $\mathcal{E}_N^{\text{large } t_f}(z^*)$ in Figs. 4.2(b) and (h), it can be seen that within the same r value, an increase in turning point z^* leads to a decrease in minima energy cost $\mathcal{E}_N^{\text{large } t_f}(z^*)$. Further, increasing r values leads to reduction of minima energy cost, as the more normal agents that are influenced by assisting optimal zealot opinion z^* , the less the control signals have to work to drive the network state vector toward consensus. Note that the optimal zealot opinion z^* is not necessarily always the same as consensus $c = 5$, and as r value increases, z^* decreases owing to the fact that the network node states follow linear dynamics. In other words, because linear dynamics is always adding or subtracting node states based on the coupled complex networked interactions, having more normal agents influenced by the zealot node leads to the saturation of node states (bias toward the direction of z) in the long time. Thus, when r value is relatively large, such as when $r/N = 40\%$ or $r/N = 60\%$, a zealot opinion of $z = c = 5$ will not necessarily assist in lowering energy cost due to the saturation of node states. Instead, the optimal zealot opinion z^* need to be a milder z value below 5 in order for the zealot's presence to be energy cost-assisting. Finally, (not shown in Fig. 4.2), increasing r value also leads to the $\mathcal{E}_N^{\text{large } t_f}(z)$ quadratic curve having a steeper parabola, and the effective assisting z range where energy cost is lowered (relative to green dashed line) is shortened.

Which network properties of the zealot-influenced nodes control the turning points z^* and associated minima energy costs $\mathcal{E}(z^*)$? To investigate, various common network properties, such as the eigenvalues, degree $\langle k \rangle$, centralities (betweenness, closeness, and eigenvectors), and PageRank [120] of each node are measured and used to distinguish independent configurations of zealot-influenced nodes by selecting them in descending order based on these features. From

these measurements, it was found that PageRank and node degree $\langle k \rangle$ of the zealot-influenced nodes are correlated to the turning points z^* , where an increase in average PageRank or $\langle k \rangle$ of the zealot-influenced nodes leads to an increase in z^* , as shown in Figs. 4.2(c) and (i) (only $\langle k \rangle$ results displayed).

To ensure the sampling of zealot-influenced nodes are well-represented along a diverse z^* range, the configurations of zealot-influenced nodes were chosen according to descending order of nodes degree $\langle k \rangle$. For example, for $r/N = 10\%$, the top 10% nodes with highest degree $\langle k \rangle$ were selected, followed by the next 10%, and so on. For $r/N = 20\%$, $r/N = 40\%$, and $r/N = 60\%$, the selection is similar, except that between sets, there could be overlapping choices of nodes. Using any other selection strategy other than distinguishing $\langle k \rangle$ (or PageRank) would lead to a tight cluster of data points around similar z^* values in Figs. 4.2(b) and (h). Since node degrees $\langle k \rangle$ correlate to turning points z^* , the network topological differences between ER and SF networks in degree distributions $P(k)$ [42] explain the disparity of z^* data points in Figs. 4.2(b) and (h), where the z^* of the SF network tend to be larger, due to the presence of hubs with large average degree $\langle k \rangle$. Physically, this means that when hubs in SF networks have become influenced by the zealot node with assisting z^* opinion, the energy cost is most reduced. Conversely, when hubs are influenced by zealot nodes with z opinion far away from its turning point, the energy cost is most increased as compared to the situation where there are no zealots present.

Figs. 4.2(d)–(f) and (j)–(l) reveal how the zealots sway the state space trajectory lengths l and associated energy costs. For each configuration of zealot-influenced nodes, driving the state vector toward consensus $c = 5$, the energy cost and length l are measured respectively for the zealot's z value as $z = -5$ in Figs. 4.2(d) and (j), $z = 5$ in Figs. 4.2(e) and (k), and $z = z^*$ in Figs. 4.2(f) and (l). In all of these figures, the computations show that the energy cost is proportional to length l :

$$\mathcal{E}_N^{\text{large } t_f}(l) \propto l. \quad (4.21)$$

With the exception of $z = z^*$ results for $r = 1$, $r = 2$, $r = 3$, and $r = 4$ configurations, the data points show that a configuration that leads to a control action that takes a longer l path requires a higher energy cost. Depending on the zealot's z value relative to a specific configuration's z^* value, the zealots' presence may reduce or increase energy cost relative to the energy cost in the absence of zealots. For example, in Figs. 4.2(d) and (j), when $z = -5$, contrarian to the consensus goal, and far away from their respective r values turning point z^* , all energy cost increases as compared to the no-zealot energy cost (when $z = 0$). Further, at this z value, an increase in r number of zealot-influenced nodes leads to high saturation of zealot-bias node states owing to the networked linear dynamics and the driver nodes would thus need to consume higher energy cost to overcome the zealot's influence. In the converse situation, in Figs. 4.2(e) and (h), when $z = 5$, supportive to the consensus goal, most of the energy cost are lowered, compared to the no-zealot energy cost. When $z = 5$, for configurations near the associated turning point z^* , the zealot's presence is beneficial for control, decreasing energy cost. For the magenta triangles, corresponding to data points of $r/N = 60\%$, a z value of 5 is far away from their associated turning points z^* , and in this situation, the zealot's presence is adversarial for control, increasing energy cost. At $z = z^*$, all configurations are at their respective turning points, and the zealots' presence assist in controlling the network, reducing energy cost.

N drivers: Small t_f regime

In the small t_f regime, using N drivers, the energy costs (Eq. 4.13) quadratic curves are plotted in Figs. 4.3(a), (b), (g), and (h) respectively for ER and SF networks with small $t_f = 0.1$

and $t_f = 0.01$. In these figures, the color-coded plots correspond to the r number of zealot-influenced nodes in accordance to legends in Figs. 4.3(c) or (i). As predicted analytically (Eq. (4.17)), the turning point $z_{N, \text{small } t_f}^* \approx ct_f^{-1}$ is independent of the r number, as well as the choice, of zealot-influenced nodes. From these figures, the theoretical prediction is validated, and all turning points are the same, regardless of r value, and affected only by small t_f value, where $z_{N, \text{small } t_f}^* \approx 50$ and $z_{N, \text{small } t_f}^* \approx 500$ respectively for $t_f = 0.1$ and $t_f = 0.01$, given that $c = 5$ is fixed. Its invariant turning point z^* and associated minima energy cost $\mathcal{E}_N^{\text{small } t_f}(z^*)$ are plotted in Figs. 4.3(c) and (i) for varying r values, where an increase in r leads to a decrease in minima energy cost. Unlike its large t_f counterparts, no topological effects between ER and SF networks are observed, as in the small t_f regime, there is barely enough time for the network topological effects such as node degree $\langle k \rangle$ to take effect, and only the r number (and not choice) of influenced nodes matters to the energy cost.

The energy cost $\mathcal{E}_N^{\text{small } t_f}(l)$ as a function of state space trajectory length l (Eq. (4.20)) are computed and plotted in Figs. 4.3(d), (e), (f), (j), (k), and (l). Since the small t_f regime energy cost is invariant to the choice of zealot-influenced nodes, each r value (r denoted by different symbols) only has one data point. To generate more data points to analyze $\mathcal{E}_N^{\text{small } t_f}(l)$ and l dependency, z values were varied (z values denoted by color). Respectively, $z \in [-100, -20]$ for Figs. 4.3(d) and (j), $z \in [20, 100]$ for Figs. 4.3(e) and (k), and $z \in [120, 200]$ for Figs. 4.3(f) and (l). $z \in [20, 100]$ corresponds to the z range where the zealot's presence assists in lowering the energy cost (relative to the no-zealot energy cost), while $z \in [-100, -20]$ and $z \in [120, 200]$ are all z values where the zealot's presence would increase energy cost. In Figs. 4.3(d), (f), (j), and (l), state space trajectory l correlates to energy cost, and an increase in l leads to an increase in $\mathcal{E}_N^{\text{small } t_f}(l)$. Further, increasing r value or $|z|$ values also leads to increased l and energy cost, suggesting that when more normal agents are influenced by the zealot node with strong opinions, more effort is needed to steer the network node states toward consensus. In the assisting $z \in [20, 100]$ range (Figs. 4.3(e) and (k)), l anti-correlates to $\mathcal{E}_N^{\text{small } t_f}(l)$, where an increase in l leads to a decrease in energy cost. When r value is the highest, and z value is closest to $z^* = 50$, energy cost is most reduced.

One driver

When using only one control signal to control the complex network, the various node states are indirectly driven through various paths throughout the network from the sole control signal, and the resulting state space trajectory is highly circuitous [3]. Owing to the highly circuitous state space trajectory, it is difficult to analyze the effects the choice of zealot-influenced nodes have on the energy cost in a complex network. Therefore, in the results that follows, simple network topologies such as chain, ring, and star networks [13] will be studied instead.

One driver: Large t_f regime

The results from the numerical experiments with chain, star, and ring networks are presented in Figs. 4.4, and D.1–D.3 (Figs. D.1–D.3 in **Appendix D**). In these figures, the energy costs needed in controlling these networks when there is a zealot node present and influencing $r = 1$, $r = 2$, $r = 3$, and $r = 4$ number of normal agents are measured. For each r value, the measurement is repeated for an exhaustive ${}^N C_r$ number of times, each time with different unique sets of zealot-influenced nodes (for small N , this search space is feasible). As predicted from Eq. (4.14), the energy cost is quadratic with respect to the zealot's fixed z opinion, which are all validated in Fig. 4.4(b) and Figs. D.1–D.3(b). Unlike its large t_f regime, N drivers counterpart, there is no

Figure 4.4: (a) Continuous-time linear dynamics: Large t_f regime, one driver results in a chain network, where the driver node is located at node 1, and the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.14). (c) shows how the turning points $\log_{10}(|z^*|)$ relate to its associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{large } t_f}(z^*))$. Note that unlike the results from N drivers, the turning points z^* could lie on the negative z region, where a contrarian zealot opinion assists the driver node by reducing energy cost. (d) shows that there is a trend where the turning points z^* tend to decrease when the average path distances from driver node 1 to the zealot-influenced normal agent nodes increase. (e) displays a similar trend, showing that when nodes far away from the driver nodes are influenced by the zealot holding z^* opinion, the energy cost is reduced. Errorbars of $r = 2$ (red right triangles) indicate standard deviations, while the errorbars of $r = 3$ (black squares), and $r = 4$ (purple crosses) are not shown in plots for the purpose of visibility. (f), (g), and (h) respectively indicate that the energy cost scales with length l according to $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$ for various configurations of zealot-influenced nodes when $z = -5$, $z = 5$, and $z = z^*$ when controlling the chain network toward consensus $c = 5$. When $z = -5$ or $z = 5$, almost all configurations of zealot-influenced nodes increase the energy cost (relative to the energy cost needed for controlling the network in the absence of zealots) because the z values are far from their respective turning points z^* . When $z = z^*$, all configurations of zealot-influenced nodes reduce the energy cost

Figure 4.5: (a) Continuous-time linear dynamics: Small t_f regime, one driver results in a chain network, where the driver node is located at node 1, and the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.15). (c) shows the relationship between turning points $\log_{10} |z^*|$ and associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{small } t_f}(z^*))$. (d) shows that as the average path distances from driver node 1 to the zealot-influenced normal agent nodes increase, the turning points z^* decrease. (e) shows that when normal agents furthest away from the driver node are influenced by the zealot holding z^* opinion, the energy cost is most reduced. (f), (g), and (h) respectively indicate that the energy cost scales with its associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$, when $z = -5$, $z = 5$, and $z = z^*$ for various configurations of zealot-influenced nodes.

clear relationship between z^* and $\mathcal{E}(z^*)$, as shown in Figs. 4.4(c), and D.1–D.3(c). However, in all of these, the statistical trend suggests that when $|z^*|$ is small, minima energy costs $\mathcal{E}_1^{\text{large } t_f}(z^*)$ is most reduced relative to the no-zealot energy cost (horizontal green dashed line). Further, it was noticed during computation that when $|z^*|$ is large, the parabola of the $\mathcal{E}_1^{\text{large } t_f}(z)$ curve stretches, and when $|z^*|$ is small, it steepens. Finally, some of the turning points z^* can lie in the negative z region, indicating that despite holding contrarian opinion to the goal of driving the network toward consensus $c = 5$, a contrarian zealot surprisingly assists in the reducing control energy.

The various network properties of configurations of zealot-influenced nodes were measured to find out if any of them can explain the turning points z^* . Unlike its (large t_f regime) N drivers counterparts, node degree $\langle k \rangle$ of zealot-influenced nodes do not predict z^* value for the one driver result. Instead, in Fig. 4.4, for a chain network, where the driver node is located at the root node (node 1), it was found that the average path distances from driver node to zealot-influenced nodes anti-correlate with $\log_{10}(|z^*|)$, where configurations with zealot-influenced nodes further away from the driver node 1 tend to have lower $\log_{10}(|z^*|)$, and lower minima energy cost $\log_{10}(\mathcal{E}_1^{\text{large } t_f}(z^*))$. This is not surprising, considering that it has already been reported that for a chain network, energy cost increases exponentially as the path distances increases linearly [24]. Thus, when zealots influence nodes that are furthest away from the driver node with assisting z^* opinion, the energy cost can be most reduced.

For more complicated simple network topologies in Figs. D.1–D.3, none of the network properties are predictive of $|z^*|$ nor $\mathcal{E}_1^{\text{large } t_f}(z^*)$. However, in all of these, state space trajectory length l remains a strong predictor of energy cost, yielding the scaling law

$$\log_{10}(\mathcal{E}_1^{\text{large } t_f}(l)) \sim \log_{10}(l), \quad (4.22)$$

as evidenced in Figs. 4.4(f)–(h), and Figs. D.1–D.3(d)–(f), respectively for $z = -5$, $z = 5$, and $z = z^*$. Thus, when a particular configuration of zealot-influenced nodes leads to low length l , energy cost is most reduced. When $z = -5$ or $z = 5$, most of the configurations result in increased energy cost relative to no-zealot energy cost (green dashed lines) as the z values are far from their turning points z^* . When $z = z^*$, all configurations lead to reduced energy cost. In star or ring network, the most optimal configurations can reduce the energy cost by 3-4 orders of magnitude. Finally, note that most of the scatter data points overlap, and there is no clear distinction between r values, indicating that for one driver, large t_f regime, the zealot's influencing of 1 normal agent has as much potential in swaying the control action as influencing 4 normal agents. This suggests that, because the one driver state space trajectory is highly circuitous [3], r number of nodes being influenced by the zealot is not predictive of how much the energy cost will change since the circulation of the zealot's influence may move the state space trajectory around through different indirect ways.

One driver: Small t_f regime

The numerical experiments are repeated for the one driver node calculations in the small t_f regime. Respectively, Fig. 4.5, and Figs. E.1–E.3 (Figs. E.1–E.3 are found in **Appendix E**) correspond to chain network with node 1 being the driver node, chain network with node 4 being the driver node, star network, and ring network. Similar to its large t_f counterparts, there is no clear trend between turning points $\log_{10}(|z^*|)$ and minima $\log_{10}(\mathcal{E}_1^{\text{small } t_f}(z^*))$, except that it appears that a smaller $|z^*|$ value tend to have a lower $\mathcal{E}_1^{\text{small } t_f}(z^*)$. Further, the turning points z^* can also lie in the negative z region, indicating that a contrarian zealot can actually assist in lowering the energy cost. Overall, the physical behavior of the one driver node computations

in the small t_f regime is similar to its large t_f regime counterparts, except that the required energy is much larger, which is not surprising, given that the required energy cost decreases as t_f increases [3]. The scaling behavior

$$\log_{10}(\mathcal{E}_1^{\text{small } t_f}(l)) \sim \log_{10}(l) \quad (4.23)$$

are corroborated by Fig. 4.5(f)–(h), and Figs. E.1–E.3(d)–(f).

d drivers

The numerical experiments were repeated for controlling a $N = 200$, $\langle k \rangle = 6$ ER network with continuous-time linear dynamics using 80 control signals. The results, both large t_f and small t_f regimes are plotted in Fig. 4.6. Figs. 4.6(a) and (f) validate Eq. (4.15), showing that the energy cost is quadratic with respect to the zealot's fixed z opinion. Furthermore, Fig. 4.6(f) shows that the turning point z^* lie in the negative z region, although the goal is to drive the state vector towards $c = 5$, indicating that a contrarian zealot is beneficial for control in some zealot-influenced nodes configurations. There is no clear relationship between turning points z^* and their associated minima energy costs $\mathcal{E}(z^*)$, as displayed in Figs. 4.6(b) and (h). This is in contrast to large T_f N drivers, where an increase in z^* leads to a decrease in $\mathcal{E}(z^*)$, and the one driver results, where a decrease in z^* tend to decrease $\mathcal{E}(z^*)$.

Nevertheless, the energy cost of each zealot-influenced nodes configuration can be explained by the state space trajectory length l . l was calculated, along with its associated energy cost, for fixed $z = -5$, $z = 5$, and $z = z^*$, where for each r number of zealot-influenced nodes, from $r = 2$ to $r/N = 60\%$, 20 independent configurations of nodes were randomly selected. For $r = 1$, all independent configurations from node 1 to node N were chosen. From Figs. 4.6(c)–(e) and (h)–(j), it is evident that the energy cost scales as

$$\log_{10}(\mathcal{E}_d(l)) \sim \log_{10}(l), \quad (4.24)$$

where a configuration that causes an increased l would cost the drivers more energy cost. The energy that each configuration costs is relative to their respective turning point z^* and the steepness of the parabola. For the large t_f results, $z = -5$ and $z = 5$ configurations are far from the minima and all configurations have higher energy costs compared to the no-zealot energy cost (green dashed line). At $z = z^*$, all configurations are exactly at their minima, and all configurations reduce energy cost, although by less than one order of magnitude. In the small t_f regime, for $z = -5$, $z = 5$, and $z = z^*$, all configurations are close to or at the minima, and the zealot's influence reduces the energy cost as compared to no-zealot energy cost (not shown in Figs. 4.6(h)–(j)), which was around $\log_{10}(\mathcal{E}) \approx 11.6$.

4.2.2 Discrete-time linear dynamics with conformity behavior model

Next, how zealots affect the energy cost in controlling a complex network with discrete-time linear dynamics and conformity behavior [27] is studied. The dynamics of this model incorporates conformity features, where each member of the network adapts their node state to follow the average of their nearest neighbors over time. Arguably, it is more realistic as it captures conformity behavior, which has been reported in various social systems. For example, evolutionary games [93, 104], learning behaviors [94, 95], and collective movements [96–100] of animals in groups.

An example is given in Fig. 4.7 to introduce this network dynamics, where the node states $x_i(t)$ may represent different opinions on a particular subject, with a positive $x_i(t)$ value indicating support of an idea, and a negative $x_i(t)$ value representing opposition. A minimum

Figure 4.6: Continuous-time linear dynamics: Large t_f (top row) and small t_f (bottom row) d drivers results in a $N = 200$, $\langle k \rangle = 6$ ER network with 80 control signals. (a) and (f) validate Eq. (4.15), showing that the energy cost is quadratic with respect to zealot's z opinion. (b) and (g) show the relationship between turnings z^* and their associated minima energy cost $\mathcal{E}_d(z^*)$. (c)–(e) and (h)–(j) plot the state space trajectory length l that each zealot-influenced nodes configuration takes the system and the corresponding energy cost at $z = -5$, $z = 5$, and at $z = z^*$. They show that the energy cost of each configuration can be explained by length l , where the energy cost scales as $\log_{10}(\mathcal{E}_d(l)) \sim \log_{10}(l)$. In the small t_f regime, all configurations reduce the energy cost relative to no-zealot energy cost (not shown in Figs. (h)–(j)), which was $\log_{10}(\mathcal{E}) \approx 11.6$. Legend in (h) display the marker symbols corresponding to each r number of zealot-influenced normal agents.

of three nodes are needed to showcase the conformity dynamics because otherwise, with two nodes, they will just mimic each other's node states in perpetuity without reaching conformity. In Fig. 4.7(a), there are $N = 3$ normal agents who start with different node states $x_i(\tau = 0)$ drawn from random uniform $[0, 1]$. Over time, each normal agent mimics their nearest neighbors' node states and the network node states reach conformity as demonstrated in Fig. 4.7(b). The 3-dimensional state space trajectory of this time evolution of node states is presented in Fig. 4.7(c). In Fig. 4.7(d), a single control signal $u_1(\tau)$ attaches to node 1 to control the network, driving the node states toward consensus $c = 2$, as shown in Fig. 4.7(e). The state space trajectory of this control action is plotted in Fig. 4.7(f). Finally, in Fig. 4.7(g), a zealot node is introduced into the system, where it holds a fixed $z = -5$ opinion. Fig. 4.7(h) shows the node states evolution of the normal agents being driven toward $c = 2$, along with the zealot's fixed opinion at $z = -5$. Correspondingly, in Fig. 4.7(i), the state space trajectory of this control action, under the influence of the zealot, elongates as the control signal $u_1(\tau)$ now has to consume more energy to overcome the zealot's contrarian opinion. Note that it suffices to display the state space trajectory in 3 dimensions as the zealot's fixed node state, associated with the fourth dimension, would simply shift and constraint $x_1(\tau)$, $x_2(\tau)$, and $x_3(\tau)$ onto the fixed $x_4 = z$ hyperplane.

Networked conformity dynamics is achieved when, at the individual level, each normal agent node updates their node states in the next discrete-time round, $\tau + 1$, by taking the average of their nearest neighbors' in the current round τ [27]. Therefore, for each normal agent node i (for $i = 1, 2, \dots, N$),

$$x_i(\tau + 1) = \frac{1}{\tilde{s}_i} \sum_{j=1}^{\tilde{n}_i} \tilde{a}_{ij} x_j(\tau), \quad (4.25)$$

where node i has $\tilde{n}_i$ nearest normal agent neighbors, $x_j(\tau)$ is the opinion of neighbor j at round

Figure 4.7: (a) Network with conformity behavior in the absence of control signal and zealot. (b) Network node states evolution with conformity in the absence of external influences. (c) State space trajectory of $x_1(\tau)$, $x_2(\tau)$, and $x_3(\tau)$ in the absence of external influences. (d) Network with conformity behavior and control signal $u_1(t)$ attached to node 1. (e) Conformity-based network node states being driven toward consensus $c = 2$. (f) State space trajectory of nodes being driven toward consensus. (g) Network with conformity behavior, control signal $u_1(t)$ attached to node 1, and zealot node influencing node 2. (h) Conformity-based network node states being driven toward consensus, while node 2 is under the influence of the zealot node with fixed opinion $z = -5$. (i) State space trajectory of nodes being driven toward consensus while under the influence of the zealot.

τ , $\tilde{a}_{ij} = \tilde{a}_{ji}$ is the weighted connection between nodes i and j , and $\tilde{s}_i = \sum_{j=1}^{\tilde{n}_i} \tilde{a}_{ij}$ is the strength of node i , obtained by summing over all nearest neighbors' weighted connections. Introducing input control signal terms into Eq. (4.25), the zealot's connections, and rewriting in vector notational form, the system-level dynamics is

$$\begin{aligned} \mathbf{x}(\tau + 1) &= \mathbf{S}^{-1} \mathbf{A} \mathbf{x}(\tau) + \mathbf{B} \mathbf{u}(\tau) \\ &= \mathbf{A} \mathbf{x}(\tau) + \mathbf{B} \mathbf{u}(\tau), \end{aligned} \quad (4.26)$$

where $\mathbf{x}(\tau) = [x_1(\tau), x_2(\tau), \dots, x_N(\tau), z]^T$ is the $n \times 1$ state vector of all N normal agents, and the zealot node (the n -th node, where $n = N + 1$), $\mathbf{u}(\tau) = [u_1(\tau), u_2(\tau), \dots, u_M(\tau)]^T$ is the $M \times 1$ control signals vector, $\mathbf{A}$ is the full $n \times n$ network matrix, which comprises $\tilde{\mathbf{A}}$ as its first $N \times N$ block, where $\tilde{a}_{ij} = \tilde{a}_{ji}$ is non-zero (and non-negative) if normal agents i and j have interactions, otherwise it is zero, and the n -th column describes the zealot's directed link to normal agents, where $a_{in} = 1$ if zealot node influences node i , otherwise it is zero, $\mathbf{B}$ is the control input matrix which describes which nodes are directly controlled by a control signal such that $b_{ij} = 1$ if normal agent node i is controlled by control signal j , otherwise it is zero (in addition, each control signal attaches only to a single normal agent), $\mathbf{S}^{-1} = \text{diag}\{\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_N}, 1\}$ is a $n \times n$ diagonal matrix which holds $\frac{1}{s_i}$ (for $i = 1, 2, \dots, N$) on its main diagonals, with the n -th entry being 1, and $s_i = \sum_{j=1}^{\tilde{n}_i} a_{ij}$ sums over the full $\mathbf{A}$ matrix, inclusive of the zealot node's directed connections. $\mathbf{A} = \mathbf{S}^{-1} \mathbf{A}$ is the full network matrix, where the normal agents evolve in time with conformity, yet the zealot node remains fixed with state z . Note that unlike the continuous-time dynamics, which requires self-links to model system stability, the discrete-time dynamics with conformity model requires that $\tilde{a}_{ii} = 0$ to ensure system stability (see **section 4.4**). Further, the zealot's self-loop $[\mathbf{S}^{-1}]_{nn} = a_{nn} = \underline{a}_{nn} = 1$ is necessary for modelling zealotry, leading to $x_n(\tau + 1) = x_n(\tau) = z$, and the zealot node's opinion remains fixed at z against time. Finally, the non-symmetric full network matrix can be eigen-decomposed as $\mathbf{A} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1}$ and $\mathbf{A}^T = \mathbf{V} \mathbf{D} \mathbf{V}^{-1}$, where $\mathbf{P}$ ($\mathbf{V}$) is the eigenvectors matrices of $\mathbf{A}$ ($\mathbf{A}^T$), and $\mathbf{D}$ is the $n \times n$ diagonal matrix containing the eigenvalues of $\mathbf{A}$ such that $\mathbf{D} = \text{diag}\{\Lambda_1, \Lambda_2, \dots, \Lambda_N, \Lambda_n\}$, where $\Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_N \leq \Lambda_n = 1$. From computation, regardless of network size or topology, the eigenvalues of all normal agents $|\Lambda_i| < 1$, while the zealot node has eigenvalue $\Lambda_n = 1$.

Analytical equations of energy cost

The energy cost required to control the network with discrete-time linear dynamics and conformity behavior is [6, 30, 106, 121]

$$J = \sum_{\tau=0}^{T_f-1} u^T(\tau) u(\tau), \quad (4.27)$$

where T_f is the final control time, which is the amount of time allocated to the control signals to steer the state vector. Minimizing the cost function (Eq. (4.27)), the discrete-time energy-optimal control signal that should be used to control the network is derived [6, 121] as the $M \times 1$ vector

$$\mathbf{u}^*(\tau) = \mathbf{B}^T (\mathbf{A}^T)^{T_f-\tau-1} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} (\mathbf{y}_f - \mathbf{C} \mathbf{A}^{T_f} \mathbf{x}_0), \quad (4.28)$$

where $\mathbf{W} = \sum_{\tau=0}^{T_f-1} \mathbf{A}^{T_f-\tau-1} \mathbf{B} \mathbf{B}^T (\mathbf{A}^T)^{T_f-\tau-1}$ is the $n \times n$ discrete-time controllability Gramian of the conformity-based system, $\mathbf{x}_0 = [x_1(0), x_2(0), \dots, x_N(0), z]^T = [0, 0, \dots, 0, z]^T$ is the $n \times 1$

initial state vector, where the initial node states of the normal agents are assumed to all be zeros, $\mathbf{y}_f = [c, c, \dots, c]^T$ is the $N \times 1$ final output state vector, where normal agents are all controlled toward consensus c , and $\mathbf{C}$ is the $N \times n$ target control matrix which selects the output state vector to be controlled, which in this case are all the N normal agents, so $C_{ii} = 1$ (for $i = 1, 2, \dots, N$), and the n -th final row/column of $\mathbf{C}$ are all zeros.

Thus, substituting Eq. (4.28) into Eq. (4.27), the required energy cost when using $\mathbf{u}^*(\tau)$ to control the network is

$$\mathcal{E} = \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f - 2\mathbf{x}_0^T (\mathbf{A}^T)^{T_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f + \mathbf{x}_0^T (\mathbf{A}^T)^{T_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{CA}^{T_f} \mathbf{x}_0. \quad (4.29)$$

This can be further simplified by noting that [34]

$$(\mathbf{CWC}^T) = \tilde{\mathbf{W}} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{V}}^{-1} = \tilde{\mathbf{P}}[\tilde{\mathbf{Q}} \circ \tilde{\mathbf{F}}]\tilde{\mathbf{V}}^{-1}, \quad (4.30)$$

where the $\mathbf{C}$ and $\mathbf{C}^T$ matrices remove the n -th row/column of $\mathbf{W}$, leading to

$$\tilde{\mathbf{W}} = \sum_{\tau=0}^{T_f-1} \tilde{\mathbf{A}}^{T_f-\tau-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T (\tilde{\mathbf{A}}^T)^{T_f-\tau-1}, \quad (4.31)$$

the $N \times N$ reduced controllability Gramian matrix. $\tilde{\mathbf{A}}$ is the reduced network matrix, obtained by removing the n -th row and column off $\mathbf{A}$, so its eigen-decompositions are $\tilde{\mathbf{A}} = \tilde{\mathbf{P}}\tilde{\mathbf{D}}\tilde{\mathbf{P}}^{-1}$ and $\tilde{\mathbf{A}}^T = \tilde{\mathbf{V}}\tilde{\mathbf{D}}\tilde{\mathbf{V}}^{-1}$, where $\tilde{\mathbf{P}}$ ($\tilde{\mathbf{V}}$) is the eigenvectors matrix of the reduced matrix $\tilde{\mathbf{A}}$ ($\tilde{\mathbf{A}}^T$), with $\tilde{D}_{ii} = \Lambda_i$ corresponding to the eigenvalue of the normal agent node. Unlike its continuous-time counterpart, the reduced network matrix of the discrete-time dynamical system cannot decouple the zealots' connections owing to $\mathbf{S}^{-1}\mathbf{A}$ in Eq. (4.26), and so it must be emphasized that $\tilde{\mathbf{A}}$ is not the same as $\tilde{\mathbf{A}}$. $\tilde{\mathbf{M}}$ is the simplified controllability Gramian matrix of the reduced discrete-time system, which has the Hadamard product analytical form [3–6, 34]

$$\tilde{\mathbf{M}}_{ij} = \tilde{\mathbf{Q}}_{ij}\tilde{\mathbf{F}}_{ij} = [\tilde{\mathbf{P}}^{-1}\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{V}}]_{ij} \left[\frac{1 - (\Lambda_i\Lambda_j)^{T_f}}{1 - \Lambda_i\Lambda_j} \right]. \quad (4.32)$$

Applying the inverse operation,

$$(\mathbf{CWC}^T)^{-1} = \tilde{\mathbf{W}}^{-1} = (\tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{V}}^{-1})^{-1} = \tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}. \quad (4.33)$$

Large T_f regime

The scaling laws of the energy cost needed to control networks with conformity behavior with respect to T_f have been studied in Ref. [34]. Like the continuous-time system [3, 5], the upper bound of the energy cost is characterized into two distinct regimes: The small T_f regime and the large T_f regime. Substituting the eigen-decompositions $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, and Eq. (4.33) into Eq. (4.29), the large T_f energy cost is

$$\begin{aligned} \mathcal{E}^{\text{large } T_f} &= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} + z^2 \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} \\ &= \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} \left\{ c^2 - 2cz + z^2 \right\}, \end{aligned} \quad (4.34)$$

which is quadratic with respect to the zealot's fixed z opinion. The zealot's connections are encoded in $[\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij}$, and the effects of varying the driver nodes connections are encoded in $\tilde{\mathbf{M}}^{-1}$. Note that a minimum of one connection from zealot to normal node must be made in order for Eq. (4.34) to be valid. Since the normal agents have eigenvalues $|\Lambda_i| < 1$ (for $i = 1, 2, \dots, N$), all $(\Lambda_i\Lambda_j)^{T_f}$ terms in Eq. (4.32) vanish in the large T_f limit, and the simplified controllability Gramian becomes

$$\tilde{\mathbf{M}}_{ij} = \frac{[\tilde{\mathbf{P}}^{-1}\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{V}}]_{ij}}{1 - \Lambda_i\Lambda_j}. \quad (4.35)$$

Subsequently, the one driver $\tilde{\mathbf{M}}^{-1}$ follows similarly from Ref. [34], and the one driver energy cost is fully analytical by substituting Eq. (G.1) into Eq. (4.34). When using more than one driver node, $\tilde{\mathbf{M}}^{-1}$ cannot be derived analytically [34], and is computed numerically through the inverse of Eq. (4.35).

Taking the derivative $\frac{\partial \mathcal{E}}{\partial z} = 0$, the optimal zealot opinion that assists in lowering the energy cost is

$$z_{\text{large } T_f}^* = c, \quad (4.36)$$

regardless of number of drivers. This is unsurprisingly, since the networked conformity dynamics (see Fig. 4.7(b)) is global, and the unwavering z opinion forces the adaptive normal agents to take its value. When $z_{\text{large } T_f}^* = c$, the zealot forces the normal agents to have the consensus c opinion, thereby assisting the driver nodes in their tasks in controlling the network. Substituting $z = c$ into Eq. (4.34), it is easy to verify that, in fact, the energy cost $\mathcal{E}^{\text{large } T_f}(z^*)$ is zero.

Small T_f regime

In the small T_f regime, the normal agent nodes have less time to adapt to the zealot's z opinion. Substituting the eigen-decompositions $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, and Eq. (4.33) into Eq. (4.29),

$$\begin{aligned} \mathcal{E}^{\text{small } T_f} &= c^2 \sum_{i,j} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{i,j} [(\mathbf{A}^T)^{T_f}]_{ni} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} \\ &\quad + z^2 \sum_{i,j} [(\mathbf{A}^T)^{T_f}]_{ni} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} [(\mathbf{A})^{T_f}]_{jn} \\ &= c^2 \sum_{i,j} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{i,j,k} v_{ni} [\mathbf{V}^{-1}]_{ij} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \Lambda_i^{T_f} \\ &\quad + z^2 \sum_{i,j,k,l} v_{ni} [\mathbf{V}^{-1}]_{ij} p_{kl} [\mathbf{P}^{-1}]_{ln} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \Lambda_i^{T_f} \Lambda_l^{T_f}, \end{aligned} \quad (4.37)$$

where $\sum_{i,j} = \sum_{i=1}^N \sum_{j=1}^N$, $\sum_{i,j,k} = \sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^N$, $\sum_{i,j,k,l} = \sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N$, with i, j, k, l as the running indices that iterate from 1 to N or 1 to n , and n is the fixed n -th index. The choice of zealot-influenced nodes enters the energy cost equation through the $n \times n$ matrices v_{ni} , $[\mathbf{V}^{-1}]_{ij}$, p_{kl} , $[\mathbf{P}^{-1}]_{ln}$, and the $N \times N$ matrices $\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}$.

Thereafter, the derivative $\frac{\partial \mathcal{E}}{\partial z} = 0$ yields the turning point

$$z_{\text{small } T_f}^* = \frac{c \sum_{i,j,k} v_{ni} [\mathbf{V}^{-1}]_{ij} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \Lambda_i^{T_f}}{\sum_{i,j,k,l} v_{ni} [\mathbf{V}^{-1}]_{ij} p_{kl} [\mathbf{P}^{-1}]_{ln} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \Lambda_i^{T_f} \Lambda_l^{T_f}}, \quad (4.38)$$

which is itself affected by the connections that the zealot node makes. It is difficult to analyze the analytical equations (Eqs. (4.37) and (4.38)) by inspection further, and the use of numerical experiments is needed to understand how the selection of zealot-influenced nodes affect the energy cost.

Numerical experiments

In the results that follow, it can be assumed that the network node states are driven toward consensus value of $c = 5$, while the large T_f value is assumed to be $T_f = 20N$, which is sufficiently large for the network dynamics to stabilize. On the other hand, the small T_f regime of the discrete-time system is relatively short, and decreases with increasing number of drivers [34]. Therefore, it suffices to analyze the small T_f regime energy cost with using one driver node to control the network, setting $T_f = N + 1$, which for one driver, is the smallest possible [34].

Large T_f regime

The large T_f results of the discrete-time system with conformity behavior are plotted respectively in Figs. 4.8, 4.9, and 4.10, corresponding to using N drivers, one driver, and d drivers for control. Figs. 4.8(a) and (c), 4.9(a), and 4.10(a) corroborate the predictions from Eqs. (4.34) and (4.36) that the conformity-based system has the optimal zealot opinion at $z^* = c$, with energy cost of zero (see Fig. 4.8(c) inset), when the zealot's opinion is the same as the consensus c that the state vector is being driven toward. In the large T_f limit, the zealot's fixed z^* opinion steers the state vector of the network and consensus is reached, therefore costing the driver nodes no energy at all. This is regardless of the r number of zealot-influenced nodes, because the conformity mechanism is global (see Fig. 4.7(a)), and in the large time limit, one zealot-influenced node is enough to bring the entire state vector toward consensus c . Contrasting this with the energy cost needed to control the network in the absence of zealots (see Eq. (F.3)), which are respectively, numerically, 1.04, 0.554, 36.6, and 2.55 for Figs. 4.8(a) and (c), 4.9(a), and 4.10(a), suggesting that while the zealots have the potential to lower the energy cost to zero, its improvement over the no-zealot energy cost is marginal as compared to its continuous-time large time limit counterparts which has the potential to reduce energy cost by several orders of magnitudes.

Further, in each of these figures, the r number of zealot-influenced nodes are varied (denoted by the marker symbols therein), as well as the configurations of zealot-influenced nodes. In Figs. 4.8(a) and (c), each r value is represented by 20 independent selections of zealot-influenced nodes chosen at random. For the N drivers large T_f results, the zealot-influenced nodes configurations accounted for small variations in the energy cost, denoted by the error bars plotted. Correspondingly, the relationship between the zealot-influenced state space trajectory length, as exemplified in Fig. 4.7(i), is displayed in Figs. 4.8(b) and (d). In these figures, each r value is represented by the different symbol markers according to legends in Fig. 4.8(a), and the marker colors represent the z value of the influencing zealot node. For each r value, because the energy cost variations due to the different nodes configurations is little, only one random configuration was chosen. It can be seen that generally, the energy cost is proportional to length l :

$$\log_{10}(\mathcal{E}_N^{\text{large } T_f}(l)) \propto l, \quad (4.39)$$

where an increase in r leads to a slight increase in l and steep increase in $\log_{10}(\mathcal{E}_N^{\text{large } T_f}(l))$; further, an increase in z leads to an increase in l and $\log_{10}(\mathcal{E}_N^{\text{large } T_f}(l))$, where the z value that is

Figure 4.8: Conformity dynamics: Large T_f regime, N drivers results in $N = 200$, $\langle k \rangle = 6$ ER and SF networks (top and bottom panels respectively). (a) and (c) validate Eq. (4.34), showing that the energy cost is quadratic with respect to zealot's z opinion. Solid lines and markers are computed as the mean of 20 independent selection of zealot-influenced nodes configurations, with error bars being the standard deviations. (b) and (d) show that the energy cost is proportional to the state space trajectory length l , with varying r and z values. The markers according to legend in (a) display the corresponding r value, and the colors in (b) and (d) indicate the z value of the influencing zealot node. Inset in (c) shows that indeed, $z^* = c = 5$, with energy cost being zero.

furthest away from the turning point $z^* = c$ would cost the most energy, requiring the drivers to steer the state vector back toward consensus c . All in all, the results suggest that when more nodes are influenced by the zealots with strong contrarian opinions, the N drivers have to consume the most amount of energy to steer the state vector back toward consensus c in the large T_f limit. Finally, the discrete-time length l used in these computations is defined as:

$$l = \sum_{\tau=0}^{T_f-1} \sqrt{\sum_{j=1}^N [x_j(\tau) - x_j(\tau+1)]^2}, \quad (4.40)$$

where index j iterates over the dimensionality (or nodes) 1 to N , and length l is computed over the sum of the Euclidean distances of the state vector at time τ and at time $\tau+1$.

When controlling the network with just a single control signal, it is difficult to explain the effects of the zealot in a complex network. Therefore, for this setup, the numerical experiment is performed on a simple chain network with 8 normal agent nodes, with the driver node located at node 1 (see cartoon representation in Fig. 4.11(a)). In Fig. 4.9(a), the different r number of zealot-influenced nodes are denoted by the different markers according to legends therein. The error bars account for the energy cost variations due to the different configurations of zealot-influenced nodes, where an exhaustive ${}^N C_r$ combinations were selected. Much like

Figure 4.9: Conformity dynamics: Large T_f regime, one driver results in a $N = 8$ chain network, with the driver node located at the root node (node 1). (a) validates Eq. (4.34), showing that the energy cost is quadratic with respect to zealot's z opinion. The marker symbols denote the different r number of zealot-influenced normal agents, while solid lines and markers are computed based on the mean of all possible zealot-influenced nodes configurations, with the error bars being the standard deviations. The difference in energy cost within a fixed r value is attributed to where the zealot-influenced nodes are located within the network, with (b) showing that the further the nodes are away from the driver node, the more energy cost is required (assuming fixed $z = -25$). Correspondingly, (c) shows that, at fixed $z = -25$, the energy cost scales as $\log_{10}(\mathcal{E}_1^{\text{large } T_f}(l)) \sim \log_{10}(l)$. Marker symbols relate to legend in (a).

its continuous-time counterparts, the network property that explain the energy cost the best is the average path distances from driver node 1 to the zealot-influenced normal agent nodes. At $z = -25$ (far away from the turning point $z^* = c$), it is shown in Fig. 4.9(b) that an increase in $\langle \text{path distance} \rangle$ of the zealot-influenced nodes leads to an increase in energy cost. Further, an increase in r value leads to an increase in energy cost. This suggests that the driver node has to consume the most amount of energy to steer the zealot-influenced nodes furthest away and bring it back toward consensus $c = 5$. In addition, when more nodes are influenced by the zealot node, the more effort is needed to steer those nodes back toward $c = 5$.

The state space trajectory length l (Eq. (4.40)) is telling of the required energy cost. At fixed zealot opinion $z = -25$, length l scales with the energy cost $\mathcal{E}_1^{\text{large } T_f}(l)$ as

$$\log_{10}(l) \sim \log_{10}(\mathcal{E}_1^{\text{large } T_f}(l)), \quad (4.41)$$

where the configurations of zealot-influenced nodes which leads to an increase in $\log_{10}(l)$ leads to a linear increase in $\log_{10}(\mathcal{E}_1^{\text{large } T_f}(l))$. Further, there is a clear separation of the data points when varying the r , where an increase in r leads to a linear increase translation in the $\log_{10}(l) \sim \log_{10}(\mathcal{E}_1^{\text{large } T_f}(l))$ plots.

The results when using d drivers to control a complex network are similar. While the path distances from driver nodes to their respective nearest zealot-influenced nodes should affect the energy cost, as corroborated in Fig. 4.9(b), this relationship is difficult to show due to many other competing terms in this setup. In Fig. 4.10(a), each r value corresponds to 20 independent realizations of zealot-influenced nodes configurations chosen randomly, and the error bars correspond to the variation in energy cost due to the different nodes configurations. From the same configurations chosen in Fig. 4.10(a), the state space trajectory length l and associated energy cost $\mathcal{E}_d^{\text{large } T_f}(l)$ are plotted in Fig. 4.10(b). Similar to the large T_f one driver results, the energy cost scales with length l as

$$\log_{10}(l) \sim \log_{10}(\mathcal{E}_d^{\text{large } T_f}(l)), \quad (4.42)$$

where a zealot-influenced nodes configuration that leads to increased state space trajectory length l leads to an increase in energy cost at fixed zealot opinion $z = -25$.

Figure 4.10: Conformity dynamics: Large T_f regime, d drivers results in a $N = 200$, $\langle k \rangle = 6$ ER network. (a) validates Eq. (4.34), showing that the energy cost is quadratic with respect to the zealot's fixed z opinion. Markers and solid lines are computed based on the mean of 20 independent selections of zealot-influenced nodes configurations, with the error bars being the standard deviations. Legend therein shows the different marker symbols corresponding to the different r number of zealot-influenced nodes. (b) shows that, at fixed $z = -25$, the configurations which lead to increased state space trajectory length l leads to higher energy cost required, and the energy cost scales as $\log_{10}(\mathcal{E}_d^{\text{large } T_f}(l)) \sim \log_{10}(l)$.

Small T_f regime

The results for controlling a chain network in the small T_f regime using a single control signal are in Fig. 4.11. Fig. 4.11(b) validates Eq. (4.37), showing that the energy cost is quadratic with respect to the zealot's z opinion. In the small T_f regime, the zealot node would not have enough time to affect the entire network sufficiently, and the z^* turning point is not at consensus c , as predicted from Eq. (4.38). Further, as shown in Fig. 4.11(c), a decrease in turning point z^* leads to a decrease in minima energy cost at z^* , although there is no clear distinction between r number of zealot-influenced nodes. This behavior is similar to its continuous-time counterparts, although it is worth pointing out that the turning points z^* of the discrete-time system are all lying in the positive z region. Thus, for the discrete-time dynamical system, a contrarian negative z opinion (relative to positive c) always increases the energy cost. The zealot-influenced nodes configurations were exhaustive with all ${}^N C_r$ combinations selected.

The turning points z^* are related to the average path distances of the zealot-influenced nodes from driver node 1. In Fig. 4.11(d), an increase in $\langle \text{path distance} \rangle$ leads to a decrease in z^* , which correspondingly leads to decrease in associated minima energy cost $\mathcal{E}_1^{\text{small } T_f}(z = z^*)$, as shown in Fig. 4.11(e). Thus, nodes which are far away from the driver nodes which receive influence from zealots with optimal assisting opinion z^* has the most potential to reduce control energy, as compared to the no-zealot energy cost (green dashed line in Fig. 4.11(e)), by up to a few orders of magnitude. This is in contrast to the large T_f discrete-time results, where the reduction in control energy is marginal.

Correspondingly, the state space trajectory length l with different fixed zealot opinions at $z = -5$, $z = 5$, and $z = z^*$ are plotted respectively in Figs. 4.11(f), (g), and (h) against the associated control energy $\mathcal{E}_1^{\text{small } T_f}(l)$. Consequently, the data points show that length l scales with energy cost as

$$\log_{10}(l) \sim \log_{10}(\mathcal{E}_1^{\text{small } T_f}(l)), \quad (4.43)$$

where an increase in $\log_{10}(l)$ leads to an increase in $\log_{10}(\mathcal{E}_1^{\text{small } T_f}(l))$. At $z = -5$, far away from the turning points, all configurations of zealot-influenced nodes increases the control en-

Figure 4.11: Conformity dynamics: Small T_f regime, one driver results in a $N = 8$ chain network, with the driver node located at the root node (node 1). (b) validates Eq. (4.37), showing that the energy cost is quadratic with respect to the zealot's fixed z opinion. For each zealot-influenced nodes configuration, there is a different z^* turning point. Shown in (c), configurations which result in low z^* has associated non-zero low minima energy cost $\mathcal{E}_1^{\text{small}} T_f(z^*)$. The turning points z^* and minima energy cost $\mathcal{E}_1^{\text{small}} T_f(z^*)$ are explained by where in the network are the zealot-influenced nodes located. The further away the influenced nodes are from the driver node, the lower their turning points z^* and minima energy cost, as evidenced respectively in (d) and (e). (f), (g), and (h) show respectively for $z = -5$ (far away from z^*), $z = 5$ (some configurations near z^*), and $z = z^*$ (all configurations at z^*), that the configurations which lead to increased state space trajectory length l leads to increased energy cost. Marker symbols denote r values following legend in (c), while green dashed line indicate the energy cost needed for control in the absence of zealots.

ergy relative to the no-zealot energy cost (green dashed line). At $z = 5$, where some of the configurations are close to their turning points z^* , the zealot's presence assists in reducing the energy cost in some cases. Exactly at $z = z^*$, all configurations assist in reducing energy costs, and the data points are all below the green dashed line.

4.3 Discussion

In this Chapter, how zealots affect the energy cost needed to control complex social networks was examined. To do so, target control [8] was used, where the driver nodes are used to steer N normal agents in a $n = N + 1$ full network. By disallowing the zealot node to receive any directed links (from driver nodes and other normal agents), the complex dynamical system is modelled with N normal interacting nodes, where there are mutual flows of information occurring, as well as a zealot node that stays fixed on their z opinion trying to influence other normal agents. Thus, the energy cost is analytically derived, and it was found that for controlling social networks toward fixed c final consensus state vector, the energy cost is quadratic with respect to the strongness of the zealot's z opinion. In some situations where the zealots hold opinion around $z = z^*$ turning point, the zealots' presence may assist the driver nodes in reducing the control energy as compared to the situation where there were no zealots present. Away from the turning points, the zealots' presence affect the goal of controlling the network negatively and increases energy cost at a rate of z^2 .

Beyond the strongness of the zealot's z opinion, the interplay between number of drivers, final control time regimes, network effects, network dynamics, r number and configurations of zealot-influenced nodes also change the energy cost behavior. For example, when using N drivers to control the network, each normal agent directly receives a control signal, and the indirect manipulations of node states through indirect paths throughout the network is decoupled from the energy cost. In this case, the zealot's influence is linear: As more nodes become influenced by the zealot, the zealot's presence has higher potential to affect the energy cost (increasing or decreasing depending on z) relative to the situation where there were no zealots present. In the large t_f regime, node degree $\langle k \rangle$ of zealot-influenced nodes affect the turning points z^* and associated minima energy cost $\mathcal{E}_N^{\text{large } t_f}(z^*)$. Nodes that have high degree $\langle k \rangle$, when influenced by the zealot helps to propagate the zealot's opinion and confer higher potential to affect the energy cost. Therefore, in SF networks where there are hubs, they are important nodes that the zealot should influence. In the small t_f regime, because there is barely enough time for the zealot's opinion to take root, the choice of which nodes to influence do not matter, and only the r number of nodes to be influenced is of importance.

When using less than N drivers such that some nodes receive node states alteration through indirect pathways, the behavior of how zealots affect the energy cost is less obvious. For example, in some configurations, the optimal zealot opinion z^* could lie in the negative z region when trying to control state vector to consensus $c = 5$, suggesting that a contrarian zealot may actually be beneficial for reducing energy cost. Furthermore, in most cases, the data points corresponding to different r values overlap and so the energy cost is not linear with respect to r . Thus, a particular $r = 1$ configuration with contrarian z^* may have the same potential to affect the energy as much as another $r = 4$ or $r/N = 20\%$ configuration. A simple explanation to understand how the energy cost is affected can be found in a simple chain network where there is only a single control signal located at the root node. For this particular setup, normal agents which are far away from the driver node are important choices for the zealot node to influence, and confer the most potential to affect the energy cost. For other types of networks, where the

networked interactions are more complex, it is difficult to pinpoint any one particular network property that explains the energy cost. Nonetheless, the state space trajectory length l offers a good explanation. Configurations that causes the node states to travel longer l will require higher energy cost.

How zealots affect the energy cost also change when the network dynamics change. When the node states evolve in time with conformity behavior, the most optimal zealot opinion is always $z^* = c$ in the large T_f regime. This is because the conformity behavior of the discrete-time model is global, and so the zealot node can force the entire network to take its opinion. When the zealot's opinion coincides with the control goal of driving state vector toward consensus c , the zealot's forcing helps the driver nodes to achieve their tasks, and they do not have to take any action so no energy cost is required. Regardless of number of drivers, the zealot's influence on the energy cost is always linear with respect to r number of zealot-influenced nodes in the large T_f regime. For example, with any number of drivers, influencing more normal agent nodes will always confer the zealot node higher potential to affect the energy cost. Besides, the data points corresponding to different r values generally do not overlap, with clear segregation of data points according to different r values. In the small T_f regime, however, because there is not enough time for the zealot's opinion to dominate, z^* is not always c and varies as the nodes configuration changes.

All in all, this Chapter explains how zealots affect the energy cost for controlling complex social networks. The results from this Chapter show that the interplay between the different parameters such as the zealots' beliefs, number of drivers, final control time regimes, network effects, network dynamics, number and configurations of nodes influenced by the zealots lead to different complex behavior of energy cost. Thus, when controlling a complex social network, where there may be some members who have their own biases, the energy cost behavior is altered non-trivially, suggesting that the understanding of how to control complex social systems requires a nuanced approach. Within the literature of socio-physics, there may yet exist a wealth of features that could be introduced to the controllability of complex networks that can lead to different rich phenomena. For example, Luddites are agents who actively oppose a particular idea [122], and the Deffuant model [54, 123] is a network whose dynamics are linear, where agents with similar opinions communicate, leading to formation of clusters of agents with the same opinion over time, also known as echo chambers [114].

4.4 Methods

For the continuous-time linear dynamics models, the $N \times N$ network connections matrix of the normal agents, $\tilde{\mathbf{A}}$, are modelled to have stable dynamics by setting self-loops for each node such that [4] $\tilde{a}_{ii} = -\sum_{j=1}^N \tilde{a}_{ij} - \delta$. In this Chapter, $\delta = 0.25$ was chosen, and all eigenvalues λ_i are negative and the continuous-time dynamics is stable. This ensures that the eigenvalues of normal agents network are all distinct and the network is controllable with just a single control signal [4].

On the other hand, the discrete-time linear dynamics with conformity models should have the diagonal entries of the first $N \times N$ block of the $\mathbf{A}$ matrix set to zero, such that $\mathbf{A} = \mathbf{S}^{-1}\mathbf{A}$ has stable dynamics with its eigenvalues $|\Lambda_i| \leq 1$ [34]. The minimum number of drivers needed to ensure controllability for this model is related to network nodes degree $\langle k \rangle$ [27, 34], where networks with sufficiently high $\langle k \rangle$ is controllable with just one driver node. This result is empirical, and high $\langle k \rangle$ is not a guarantee for controllability. Therefore, checks were done to ensure that the conformity-based network is controllable using the exact controllability

method [29] (in this research, the matlab code written for Ref. [116] was used). For example, if the constructed network was found to not satisfy controllability with one driver node, it was discarded and re-created again.

The model networks in this Chapter were constructed by the standard Erdős–Rényi random network algorithm, and the static model [89], and the undirected link weights between normal agents, $\tilde{a}_{ij} = \tilde{a}_{ji}$, were drawn to be random uniform $(0, 1]$. For the simple networks such as chain, ring, and star networks, the links are unweighted with $\tilde{a}_{ij} = \tilde{a}_{ji} = 1$. In addition, checks were done to ensure that all nodes within the constructed network are reachable with at least one link (that is, there are no isolated nodes), thereby ensuring that the networks were generically controllable with a single control signal. Otherwise, the constructed network was discarded and re-created again.

The solutions to the state equations are important for calculating the node states evolution in Figs. 4.1, 4.7, and the state space trajectory lengths Eqs. (4.20) and (4.40). For the continuous-time linear dynamics, the solution is [12]

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 - \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_f-\tau)} d\tau \mathbf{C}^T \hat{\nu}_f, \quad (4.44)$$

where $\hat{\nu}_f = -(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{y}_f - \mathbf{C}e^{\mathbf{A}(t_f-t_0)}\mathbf{x}_0)$. This calculation was done in matlab, where the $e^{\mathbf{A}\tau}$ terms are especially computationally costly if the function `expm(A*tau)` was used. Instead, it can be sped up by eigen-decomposing the $\mathbf{A}$ matrix and then using `P*exp(diag(D*tau))*inv(P)`. Furthermore, the computation $(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}$ is costly, and it is unnecessary to repeat this computation with each t step. Thus, it is better to compute $(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}$ once and store as a variable to be used in each t step of the numerical integration. In some calculations, for example the continuous-time one driver small t_f results, the standard numerical double precision is not enough to compute the controllability Gramian and state evolution properly, thus the numerical precision has to be increased using the matlab toolbox `Advanpix` [115]. For the state equations involving the discrete-time conformity dynamics, the solution is obtained by iterating Eq. (4.26) through τ .

Finally, note that for the continuous-time linear dynamics state space trajectory length l , the calculations were performed with time variable numerically sampled at 100 evenly spaced values. The accuracy of the results may be improved by increasing the number of sampled numerical values. However, there is a trade-off in computation speed. Furthermore, several calculations with 1000 evenly spaced values were computed and compared with those from 100 evenly spaced values, and there were negligible differences.

Chapter 5

Conclusion

This thesis has examined the energy cost needed for controlling complex social networks. Centrally, the aim of the thesis is to check what happens when the canonical complex system of networked controllability is altered slightly with social physics features, such as target control, conformity behavior, or zealotry. Through investigating the control energy of the altered systems, there were three major contributions.

Firstly, in Chapter 2, the selection of target nodes was optimized so as to reduce control energy. It might be useful to find target nodes which are less costly to influence in opinion networks or marketing networks, where the driver nodes are advertisers who wish to understand which customers to target their campaigning effort upon. To do so, the gradient $\frac{\partial \mathcal{E}}{\partial \mathbf{C}}$ was derived, and applied onto the trace-constraint-based projected gradient method (TPGM) [21] to search for $\mathbf{C}^*$ along the solution hyper surface in the direction of the quickest energy cost decrease. From $\mathbf{C}^*$ (all numerical real values), two numerical rounding schemes were proposed, so that $\mathbf{C}_{\text{binary}}^*$ (matrix elements $c_{ij} = \{0, 1\}$) is obtained. The algorithm was corroborated with extensive numerical experiments on various networks to search for optimal target nodes, given that the driver nodes placement ($\mathbf{B}$ matrix) is fixed. Across the board, energy cost reduction was generally a few orders of magnitude, as compared to selecting target nodes randomly. For simple network topologies, it was found that control energy is most reduced when the target nodes are close to the driver nodes. These types of energy-optimal structures turned out to be the same as the energy-optimal structures found from the problem of optimizing driver nodes placement, where the best solutions tend to minimize the length of the longest control chain (LCC). Perhaps the mechanism of minimizing energy cost could be useful for explaining how driver nodes and controlled nodes tend to organize themselves optimally in self-organized adaptive networks.

Secondly, in Chapter 3, the energy cost scaling laws with respect to final control time T_f for controlling complex social networks with conformity behavior was studied. Many types of social systems have conformity features, so understanding the amount of effort needed to control them would be useful. The energy cost lower and upper bound are related to the reciprocal of the absolute largest and smallest eigenvalue of the controllability Gramian. By couching the controllability Gramian into the analytical Hadamard product form, the analytical expression of how the eigenvalues vary when T_f varies can be obtained using an approximate estimation formula. It was found that in the small T_f regime, increasing T_f would reduce energy cost at a rate of $\{\underline{E}, \overline{E}\} \sim T_f^\Theta$, where Θ_1 relating to the one driver node control was numerically found to depend on system size and average network degree: $\Theta_1 \sim \frac{N}{\langle k \rangle}$. Although conformity dynamics do not alter the N_D minimum driver nodes, it was found that the scaling exponent γ of SF networks also tunes N_D , where heterogeneous (lower γ) networks are less controllable, requiring a higher N_D to be controllable. Finally, the large T_f upper bound energy cost $\overline{E}$ for

controlling complex social networks with conformity behavior was compared with those of the canonical continuous-time linear dynamics. Much like the continuous-time linear dynamics, adding more driver nodes reduces the energy cost at an exponential rate $\bar{E} \sim e^{N/M}$, where M is the number of control signals. Further, no matter the amount of driver nodes used, the upper bound energy cost needed for controlling continuous-time networks without conformity is always higher than those with conformity. In addition, a discrete-time network without conformity has unstable linear dynamics, making it practically uncontrollable. However, once this network starts to adopt conformity behavior, the dynamics become stable, and the system becomes controllable. All of these suggest that the mechanism of conformity is useful for making networks more controllable.

Thirdly, in Chapter 4, zealots, agents with unwavering opinions, were introduced to the framework of networked controllability. Studying how zealots affect the energy cost for controlling complex social systems would be interesting, because zealots could represent a realistic feature of social systems, where some people tend to remain firm in their fixed z beliefs. In addition, they alter system behavior, making it fruitful to examine how the system behavior is being altered. Analytically, it was found that the presence of zealots alter the energy cost at a quadratic z^2 rate, assuming that the final state vector remains fixed. To interpret what this means, the energy cost has to be compared with those of controlling networks where there were no zealots present. Further, the z^2 quadratic curve changes as the network, network dynamics, number of driver nodes, final control time, or which normal agents have been influenced by the zealot changes. Of interest was the turning point of the parabola, z^* , which tells that for a fixed configuration, what the zealot's opinion need to be in order to assist in lowering the energy cost. There were some surprising results, for example, for certain configurations, z^* turning point lies in the negative z region in contrast to driving the state vector towards positive $c = 5$, indicating that a contrarian zealot may actually be beneficial for reducing energy cost. Owing to the highly circuitous state space trajectory when a network is being driven from initial state vector to final state vector, it is still unclear which network properties lead to this. However, measuring the state space trajectory length l was illustrative in accounting the energy cost. Heuristically, configurations that cause the system to travel longer l tend to require higher energy cost. For simple network topologies, such as the chain network, it was found that the unweighted path distance from driver node to zealot-influenced nodes give a good explanation on the altered control energy.

It is interesting to draw connections from the theoretical results of the dissertation to real world by means of using empirical data to corroborate the models developed. Recently, a number of social physics research papers have been successful in doing so. For example, Ref. [124] built a pairwise maximum entropy statistical physics model based on empirical data of Rehnquist Court voting behavior. It was uncovered that the model matches well with empirical data and a surprising result was that the model did not have to take into account the justice's political leanings. Ref. [125] disproved the notion that a politician's partisanship (whether they are liberal or conservative) is indicative of how they would vote. Instead, it was found that an approximately century-long consensus within the court is more predictive of voting behavior and this is interesting because the century-long consensus outlasts any justice's tenure (about 40 years at maximum). Besides, justices who served alongside one another tended to vote in a highly correlated manner, revealing the transitivity of voting behavior in social systems. Ref. [126] utilised armed conflict data from news media from 1991–2016 to derive scaling laws. Of particular interest is the fact that battle avalanches occur spatio-temporally, signalling criticality in social systems. Ref. [113] corroborated their opinion formation model which displays the distributions of agree, disagree, or undecided in controversial and non-controversial topics

using poll data, while Ref. [114] studied the emergence of echo chambers in social networks surrounding politically controversial topics using Twitter data. Last, but not least, Ref. [127] used Weibo social media data to study information spreading in social networks under the constraint that each user has limited attention span. While none of the mentioned papers are directly related to the social physics features discussed in the dissertation (namely, target control, conformity behavior and zealotry in social systems), there are some conceptual overlap, and should serve as guides for how to corroborate social physics modelling with empirical data for future research.

Thus, this concludes the thesis.

Appendix A

Chapter 2: Supplementary information

This section details the derivations of the cost function and the cost function gradient information in the chapter main text. Additional information such as the network data set used in the chapter main text, as well as the standard deviations of table 2.1 of the chapter main text can be found in section A.5.

A.1 Definitions and basic results

Controllability gramian definition

The controllability Gramian matrix $W = \int_0^{t_f} e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} dt$ is symmetric. Thus, for Gramian index notation, $W_{ij} = W_{ji}$.

Likewise, the output controllability Gramian $U = (C W C^T)$ is symmetric. Thus, the inverse of output controllability Gramian is also symmetric: $[U^{-1}]^T = U^{-1}$.

Trace properties

The trace of a matrix, written in index notation is: $tr(A) = \sum_a A_{aa} = A_{aa}$. And for trace of matrices multiplication:

$$tr(ABC) = \sum_a \sum_b \sum_c A_{ab} B_{bc} C_{ca} = A_{ab} B_{bc} C_{ca}, \quad (\text{A.1})$$

where the running indices a , b , and c are arbitrarily named, and wherever convenient, the summation operator is not written (for compactness).

Probability theory standard result

Suppose a random variable x , distributed with mean zero and standard deviation 1: $x \sim \mathcal{N}(0, 1)$.

Then a vector $\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^{N \times 1}$, which contains random variable x as its vector element is also distributed as standard normal distribution: $\mathbf{x} \sim \mathcal{N}(0, 1)$.

Since mean is zero, the expectation value of this vector is zero: $\mathbb{E}(\mathbf{x}_0) = [0, \dots, 0]^T$.

Consequently, the covariance matrix of $\mathbf{x}$, follows: $\mathbb{E}(\mathbf{x}_0 \mathbf{x}_0^T) = I_N$, which is a standard result in probability theory, where I_N is an $N \times N$ identity matrix.

A.2 Matrix derivative and index notation

The matrix derivatives with index notation is needed to derive the cost function matrix derivative gradient information. Many of the results are obtained with the help of the matrix cookbook [90], and some supporting lemmas from Ref. [21]. Some definitions are listed as follows.

Basic Derivative. For arbitrary matrix element, X_{kl} , the derivative of X_{kl} with itself, X_{ij} , is given as:

$$\frac{\partial X_{kl}}{\partial X_{ij}} = \delta_{ki} \cdot \delta_{lj} = \delta_{lj} \cdot \delta_{ki}, \quad (\text{A.2})$$

And the matrix transpose derivative with itself is:

$$\frac{\partial X_{kl}^T}{\partial X_{ij}} = \delta_{kj} \cdot \delta_{li} = \delta_{li} \cdot \delta_{kj}, \quad (\text{A.3})$$

where the subscripts kl and ij are used to identify the matrix elements of each of the matrix, and $\delta_{ij} = 1$ iff $i = j$, $\delta_{ij} = 0$ otherwise. The kronecker delta terms δ_{ij} is basically the same as matrix elements of an identity matrix.

Transpose index notation.

$$X_{ij}^T = X_{ji}. \quad (\text{A.4})$$

Dealing with index notation delta terms.

$$\delta_{ki} \cdot \delta_{ij} = \delta_{kj} \quad (\text{A.5})$$

For arbitrary matrix element X_{ij} ,

$$X_{ij} \cdot \delta_{ja} = X_{ia} \quad (\text{A.6})$$

$$\delta_{ia} \cdot X_{ij} = [X^T]_{ji} \cdot \delta_{ia} = [X^T]_{ja} = X_{aj} \quad (\text{A.7})$$

Compatible matrices. Suppose $X = AB$, then matrix element X_{ij} :

$$X_{ij} = \sum_k A_{ik} B_{kj} = A_{ik} \cdot B_{kj}, \quad (\text{A.8})$$

and likewise for compatible matrices $X = ABC$, then matrix element X_{ij} :

$$X_{ij} = [ABC]_{ij} = \sum_k \sum_p [A]_{ak} \cdot [B]_{kp} \cdot [C]_{pj} = [A]_{ak} \cdot [B]_{kp} \cdot [C]_{pj}, \quad (\text{A.9})$$

where for convenience and simplicity, the summation operators are dropped (Einstein summation convention).

Derivative of Inverse

For an invertible matrix X , the derivative of the inverse of X with itself is given by:

$$\frac{\partial [X^{-1}]_{kl}}{\partial [X]_{ij}} = -[X^{-1}]_{ki}[X^{-1}]_{jl}. \quad (\text{A.10})$$

Chain Rule

Suppose of interest is a matrix derivative of a function of matrix, with respect to matrix A , then chain rule is used as follows:

$$\frac{\partial f(A, U(A))}{\partial A} = \frac{\partial f(A, U(A))}{\partial U} \cdot \frac{\partial U(A)}{\partial A}. \quad (\text{A.11})$$

Where $f(\cdot)$ is a function of matrix A and matrix function $U(A)$, where matrix function $U(A)$ is a function of matrix A . For the context of this research, the chain rule function $f(\cdot)$ is a matrix mapping.

Product Rule

Suppose of interest is a series of matrix multiplications. The derivative of them with respect to matrix A follows the product rule as:

$$\begin{aligned} \frac{\partial}{\partial A} [A \cdot f(A) \cdot g(B) \cdot B \cdot A] &= \frac{\partial A}{\partial A} \cdot [f(A) \cdot g(B) \cdot B \cdot A] \\ &+ A \cdot \frac{\partial f(A)}{\partial A} \cdot [g(B) \cdot B \cdot A] \\ &+ [A \cdot f(A) \cdot g(B) \cdot B] \cdot \frac{\partial A}{\partial A} \end{aligned} \quad (\text{A.12})$$

where $f(A)$ is a matrix function of matrix A , and $g(B)$ is a matrix function of matrix B . For arbitrary length of matrices multiplication, product rule holds analogously.

A.3 Energy cost function

To derive the energy cost function, which is the expected value of the energy cost function over all realisations of the initial state vector $\mathbf{x}_0$, picked from standard normal distribution (that is, $\mathbf{x}_0 \sim \mathcal{N}(0, 1)$),

$$\mathcal{E}(C^T) = \mathbb{E} \left[\int_0^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt \right]. \quad (\text{A.13})$$

From Ref. [8], the energy-optimal control signal to target control subset P number of nodes is derived from optimal control theory [62] as:

$$\mathbf{u}^*(t) = B^T e^{A^T(t_f-t)} C^T (C W C^T)^{-1} (\mathbf{y}_f - C e^{A(t_f-t_0)} \mathbf{x}_0). \quad (\text{A.14})$$

Recall from the chapter main text that $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times M}$, $C \in \mathbb{R}^{P \times N}$, $\mathbf{x}(t) \in \mathbb{R}^{N \times 1}$, $\mathbf{y}(t) \in \mathbb{R}^{P \times 1}$, $\mathbf{u}(t) \in \mathbb{R}^{M \times 1}$.

Expanding Eq. (A.14) and setting $t_0 = 0$, it obtains:

$$\mathbf{u}^*(t) = B^T e^{A^T(t_f-t)} C^T (C W C^T)^{-1} \mathbf{y}_f - B^T e^{A^T(t_f-t)} C^T (C W C^T)^{-1} C e^{A t_f} \mathbf{x}_0, \quad (\text{A.15})$$

whose transpose is

$$\mathbf{u}^{*T}(t) = \mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B - \mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B. \quad (\text{A.16})$$

Note that since the $P \times P$ output controllability matrix (CWC^T) is symmetric, real, and non-negative definite [8], then transpose operation has no effect on it, nor its inverse. In other words, $[(CWC^T)^{-1}]^T = (CWC^T)^{-1}$.

Evaluating $\mathbf{u}^{*T}(t)\mathbf{u}^*(t)$,

$$\begin{aligned} \mathbf{u}^{*T}(t)\mathbf{u}^*(t) &= [\mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B - \mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B] \\ &\quad \times [B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f - B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0] \\ &= \mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \\ &\quad - \mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \\ &\quad - \mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0 \\ &\quad + \mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0. \end{aligned} \quad (\text{A.17})$$

Substituting Eq. (A.17) into Eq. (A.13),

$$\begin{aligned} \mathcal{E}(C^T) &= \\ \mathbb{E} \left[\int_0^{t_f} &\mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \right. \\ &- \mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \\ &- \mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0 \\ &\left. + \mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0 dt \right]. \end{aligned} \quad (\text{A.18})$$

Observe the dimensions of each of the terms. The first term $[\mathbf{y}_f^T \dots \mathbf{y}_f]$ has dimensionality 1×1 since $\mathbf{y}_f^T \in \mathbb{R}^{1 \times P}$ and $\mathbf{y}_f \in \mathbb{R}^{P \times 1}$. Likewise, the second term $[\mathbf{x}_0^T \dots \mathbf{y}_f]$, third term $[\mathbf{y}_f^T \dots \mathbf{x}_0]$, and fourth term $[\mathbf{x}_0^T \dots \mathbf{x}_0]$ all have 1×1 dimensionality.

Since Eq. (A.18) is a 1×1 quantity, the trace operator can be directly applied, without changing the meaning:

$$\begin{aligned} \mathcal{E}(C^T) &= \\ \mathbb{E} \left[\int_0^{t_f} &\text{tr} \left(\mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \right) \right. \\ &- \text{tr} \left(\mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \right) \\ &- \text{tr} \left(\mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0 \right) \\ &\left. + \text{tr} \left(\mathbf{x}_0^T e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{At_f} \mathbf{x}_0 \right) dt \right] \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned}
&= \int_0^{t_f} \text{tr} \left(\mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \right) dt \\
&\quad \int_0^{t_f} -\text{tr} \left(\mathbb{E}[\mathbf{x}_0^T] e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} \mathbf{y}_f \right) dt \\
&\quad \int_0^{t_f} -\text{tr} \left(\mathbf{y}_f^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{A t_f} \mathbb{E}[\mathbf{x}_0] \right) dt \\
&\quad \int_0^{t_f} +\text{tr} \left(e^{A^T t_f} C^T (CWC^T)^{-1} C e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} C^T (CWC^T)^{-1} C e^{A t_f} \mathbb{E}[\mathbf{x}_0 \mathbf{x}_0^T] \right) dt
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
&= \text{tr} \left(\mathbf{y}_f^T (CWC^T)^{-1} C \left[\int_0^{t_f} e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} dt \right] C^T (CWC^T)^{-1} \mathbf{y}_f \right) \\
&+ \text{tr} \left(e^{A^T t_f} C^T (CWC^T)^{-1} C \int_0^{t_f} \left[e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} dt \right] C^T (CWC^T)^{-1} C e^{A t_f} \right)
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
&= \text{tr} \left(\mathbf{y}_f^T (CWC^T)^{-1} CWC^T (CWC^T)^{-1} \mathbf{y}_f \right) \\
&+ \text{tr} \left(e^{A^T t_f} C^T (CWC^T)^{-1} CWC^T (CWC^T)^{-1} C e^{A t_f} \right)
\end{aligned} \tag{A.22}$$

$$\therefore \mathcal{E}(C^T) = \text{tr} \left((CWC^T)^{-1} \mathbf{y}_f \mathbf{y}_f^T \right) + \text{tr} \left(C^T (CWC^T)^{-1} C e^{A t_f} e^{A^T t_f} \right). \tag{A.23}$$

Where in Eq. (A.20), the expectation operator $\mathbb{E}[\cdot]$ only acts on random variable matrix, which are mainly $\mathbf{x}_0$ or $\mathbf{x}_0^T$ only. Since the first term does not contain any random variable matrix, the expectation operator does not operate on it. Because $\mathbf{x}_0 \sim \mathcal{N}(0, 1)$, then $\mathbb{E}[\mathbf{x}_0] = [0, \dots, 0]^T$, and thus the second and third terms of matrix multiplication becomes zero. Similarly, it is assumed that $\mathbb{E}[\mathbf{x}_0 \mathbf{x}_0^T] = I_N$, where I_N is an $N \times N$ identity matrix. In the third term, the matrix term $\mathbf{x}_0^T$ is moved to the end using the cyclic permutation property of trace.

Further, in Eq. (A.21), the integration operator only acts on time-dependent terms, while the non time-dependent terms are treated as constants, thus the integration operator is moved within the trace. Observe that the term $[\int_0^{t_f} e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} dt] = W$ is the definition of the controllability Gramian. In Eq. (A.22), observe that the terms $(CWC^T)^{-1}(CWC^T) = I_P$, where I_P is a $P \times P$ identity matrix, since the matrices in the left parenthesis is just the inverse of the matrices in the right parenthesis. Finally, in Eq. (A.23), the matrix terms are moved around using trace cyclic permutation to concatenate the terms which are not related to matrix C for ease of derivation of the energy gradient information.

Thus, this was how energy cost function was derived.

A.4 Energy gradient information

To begin, first re-write the energy cost function, Eq. (A.23) as:

$$\mathcal{E}(C^T) = \text{tr} \left((CW C^T)^{-1} \mathbf{Y}_{f'} \right) + \text{tr} \left(C^T (CW C^T)^{-1} C \mathbf{X}_{f'} \right) \quad (\text{A.24})$$

for convenience. Where $\mathbf{Y}_{f'} = \mathbf{y}_f \mathbf{y}_f^T$ and $\mathbf{X}_{f'} = e^{A t_f} e^{A^T t_f}$ are treated as constant terms which do not contain any matrix C dependence.

Next, the trace terms of Eq. (A.24) is written as a summation with index notation,

$$\begin{aligned} \mathcal{E}(C^T) &= \sum_a \sum_b [C W C^T]_{ab}^{-1} [\mathbf{Y}_{f'}]_{ba} + \sum_c \sum_d \sum_e \sum_f [C^T]_{cd} [(C W C^T)^{-1}]_{de} [C]_{ef} [\mathbf{X}_{f'}]_{fc} \\ &= \sum_a \sum_b [U]_{ab}^{-1} [\mathbf{Y}_{f'}]_{ba} + \sum_c \sum_d \sum_e \sum_f [C^T]_{cd} [U^{-1}]_{de} [C]_{ef} [\mathbf{X}_{f'}]_{fc} \\ &= [U]_{ab}^{-1} [\mathbf{Y}_{f'}]_{ba} + [C^T]_{cd} [U^{-1}]_{de} [C]_{ef} [\mathbf{X}_{f'}]_{fc}, \end{aligned} \quad (\text{A.25})$$

where the output controllability Gramian term $(C W C^T) = U$.

Of interest is the energy gradient, $\frac{\partial \mathcal{E}(C^T)}{\partial C^T}$, so

$$\begin{aligned} \frac{\partial \mathcal{E}(C^T)}{\partial [C^T]_{NP}} &= \frac{\partial}{\partial [C^T]_{NP}} \left\{ [U]_{ab}^{-1} [\mathbf{Y}_{f'}]_{ba} \right\} \\ &\quad + \frac{\partial}{\partial [C^T]_{NP}} \left\{ [C^T]_{cd} [U^{-1}]_{de} [C]_{ef} [\mathbf{X}_{f'}]_{fc} \right\}, \end{aligned} \quad (\text{A.26})$$

which is the derivative of the energy cost function with respect to matrix C^T . Note that the subscripts of $[C^T]_{NP}$ indicates the matrix derivative term index notation, and refers to an arbitrary NP matrix element of the gradient information matrix.

Next,

$$\begin{aligned} \frac{\partial \mathcal{E}(C^T)}{\partial [C^T]_{NP}} &= \frac{\partial [U^{-1}]_{ab}}{\partial [U]_{gh}} \frac{\partial [U]_{gh}}{\partial [C^T]_{NP}} [\mathbf{Y}_{f'}]_{ba} \\ &\quad + \left[\frac{\partial [C^T]_{cd}}{\partial [C^T]_{NP}} [U^{-1}]_{de} [C]_{ef} [\mathbf{X}_{f'}]_{fc} \right. \\ &\quad + [C^T]_{cd} \left\{ \frac{\partial [U^{-1}]_{de}}{\partial [U]_{gh}} \frac{\partial [U]_{gh}}{\partial [C^T]_{NP}} \right\} [C]_{ef} [\mathbf{X}_{f'}]_{fc} \\ &\quad \left. + [C^T]_{cd} [U^{-1}]_{de} \frac{\partial [C]_{ef}}{\partial [C^T]_{NP}} [\mathbf{X}_{f'}]_{fc} \right], \end{aligned} \quad (\text{A.27})$$

where chain rule was used to deal with the inverse term $[U^{-1}]$, and product rule was applied to deal with matrix multiplication terms with matrix dependence on matrix C^T .

At this point, the chain rule part of Eq. (A.27) needs to be solved. In particular, of interest is $\frac{\partial [U^{-1}]_{ab}}{\partial [U]_{gh}} \left(\frac{\partial [U^{-1}]_{de}}{\partial [U]_{gh}} \right)$ and $\frac{\partial [U]_{gh}}{\partial [C^T]_{NP}}$. Thus,

$$\frac{\partial [U^{-1}]_{ab}}{\partial [U]_{gh}} = -[U^{-1}]_{ag} [U^{-1}]_{hb}, \quad (\text{A.28})$$

and

$$\frac{\partial[U^{-1}]_{de}}{\partial[U]_{gh}} = -[U^{-1}]_{dg}[U^{-1}]_{he}, \quad (\text{A.29})$$

where Eq. (A.10) was used.

Next, $\frac{\partial[U]_{gh}}{\partial[C^T]_{NP}}$ needs to be solved. Recall that $U = (CW C^T)$. Thus,

$$\begin{aligned} \frac{\partial[U]_{gh}}{\partial[C^T]_{NP}} &= \frac{\partial}{\partial[C^T]_{NP}} \cdot [CW C^T]_{gh} \\ &= \frac{\partial}{\partial[C^T]_{NP}} \left\{ [C]_{gi}[W]_{ij}[C^T]_{jh} \right\} \\ &= \left[\frac{\partial[C]_{gi}}{\partial[C^T]_{NP}} \right] [W]_{ij}[C^T]_{jh} + [C]_{gi}[W]_{ij} \left[\frac{\partial[C^T]_{jh}}{\partial[C^T]_{NP}} \right] \\ &\quad \text{Denote } [C^T] = [\hat{C}] \text{ and } [C] = [\hat{C}^T] \\ &= \left[\frac{\partial[\hat{C}^T]_{gi}}{\partial[\hat{C}]_{NP}} \right] [W]_{ij}[\hat{C}]_{jh} + [\hat{C}^T]_{gi}[W]_{ij} \left[\frac{\partial[\hat{C}]_{jh}}{\partial[\hat{C}]_{NP}} \right] \\ \therefore \frac{\partial[U]_{gh}}{\partial[C^T]_{NP}} &= \delta_{gP}\delta_{iN}[W]_{ij}[C^T]_{jh} + [C]_{gi}[W]_{ij}\delta_{jN}\delta_{hP}, \end{aligned} \quad (\text{A.30})$$

where in the second line, Eq. (A.9) was used to write out the matrix multiplication terms as separate terms with index notation. The indices i and j were chosen arbitrarily. In the third line, product rule was used. In addition, since the controllability Gramian W does not have C matrix dependence, it was treated as a constant with respect to matrix C . In the fourth line, write C^T as $\hat{C}$, and C as $\hat{C}^T$. Then, apply the basic derivative formula (Eqs. (A.2) and (A.3)). Finally, in the last line, the solution was obtained, and the overhat notation of previous step was undone. Thus, the chain rule part of Eq. (A.27) was solved, and will be substituted back in.

Substituting the chain rule part of the result, Eqs. (A.28), (A.29) and (A.30), into the Eq. (A.27), it obtains:

$$\begin{aligned} &\frac{\partial \mathcal{E}(C^T)}{\partial[C^T]_{NP}} \\ &= -[U^{-1}]_{ag}[U^{-1}]_{hb} \left[\delta_{gP}\delta_{iN}[W]_{ij}[C^T]_{jh} + [C]_{gi}[W]_{ij}\delta_{jN}\delta_{hP} \right] [\mathbf{Y}_f]_{ba} \\ &\quad + \left[\delta_{cN}\delta_{dP}[U^{-1}]_{de}[C]_{ef}[\mathbf{X}_f]_{fc} - [C^T]_{cd} \left\{ [U^{-1}]_{dg}[U^{-1}]_{he} \right\} \right. \\ &\quad \times \left\{ \delta_{gP}\delta_{iN}[W]_{ij}[C^T]_{jh} + [C]_{gi}[W]_{ij}\delta_{jN}\delta_{hP} \right\} [C]_{ef}[\mathbf{X}_f]_{fc} \\ &\quad \left. + [C^T]_{cd}[U^{-1}]_{de}\delta_{eP}\delta_{fN}[\mathbf{X}_f]_{fc} \right], \end{aligned} \quad (\text{A.31})$$

where $[C^T] = [\hat{C}]$, $[C] = [\hat{C}^T]$, Eqs. (A.2) and (A.3) were used to obtain $\frac{\partial[C^T]_{cd}}{\partial[C^T]_{NP}} =$

$$\frac{\partial[\hat{C}]_{cd}}{\partial[\hat{C}]_{NP}} = \delta_{cN}\delta_{dP} \text{ and } \frac{\partial[C]_{ef}}{\partial[C^T]_{NP}} = \frac{\partial[\hat{C}^T]_{ef}}{\partial[\hat{C}]_{NP}} = \delta_{eP}\delta_{fN} \text{ in Eq. (A.31).}$$

By expanding the terms,

$$\begin{aligned}
& \frac{\partial \mathcal{E}(C^T)}{\partial [C^T]_{NP}} \\
&= -[U^{-1}]_{ag}[U^{-1}]_{hb}\delta_{gP}\delta_{iN}[W]_{ij}[C^T]_{jh}[\mathbf{Y}_{f'}]_{ba} \\
&\quad - [U^{-1}]_{ag}[U^{-1}]_{hb}[C]_{gi}[W]_{ij}\delta_{jN}\delta_{hP}[\mathbf{Y}_{f'}]_{ba} \\
&\quad + \delta_{cN}\delta_{dP}[U^{-1}]_{de}[C]_{ef}[\mathbf{X}_{f'}]_{fc} \\
&\quad - [C^T]_{cd}[U^{-1}]_{dg}[U^{-1}]_{he}\delta_{gP}\delta_{iN}[W]_{ij}[C^T]_{jh}[C]_{ef}[\mathbf{X}_{f'}]_{fc} \\
&\quad - [C^T]_{cd}[U^{-1}]_{dg}[U^{-1}]_{he}[C]_{gi}[W]_{ij}\delta_{jN}\delta_{hP}[C]_{ef}[\mathbf{X}_{f'}]_{fc} \\
&\quad + [C^T]_{cd}[U^{-1}]_{de}\delta_{eP}\delta_{fN}[\mathbf{X}_{f'}]_{fc} \\
&= -\delta_{iN}[W]_{ij}[C^T]_{jh}[U^{-1}]_{hb}[\mathbf{Y}_{f'}]_{ba}[U^{-1}]_{ag}\delta_{gP} \\
&\quad - [W]_{Ni}[C^T]_{ig}[U^{-1}]_{ga}^T[\mathbf{Y}_{f'}]_{ab}[(U^{-1})^T]_{bh}\delta_{hP} \\
&\quad + \delta_{cN}[\mathbf{X}_{f'}^T]_{cf}[C^T]_{fe}[(U^{-1})^T]_{ed}\delta_{dP} \\
&\quad - [W]_{Nj}[C^T]_{jh}[U^{-1}]_{he}[C]_{ef}[\mathbf{X}_{f'}]_{fc}[C^T]_{cd}[U^{-1}]_{dg}\delta_{gP} \\
&\quad - [W]_{Ni}[C^T]_{ig}[(U^{-1})^T]_{gd}[C]_{dc}[\mathbf{X}_{f'}^T]_{cf}[C^T]_{fe}[(U^{-1})^T]_{eh}\delta_{hP} \\
&\quad + \delta_{fN}[\mathbf{X}_{f'}]_{fc}[C^T]_{cd}[U^{-1}]_{de}\delta_{eP},
\end{aligned} \tag{A.32}$$

where Eqs. (A.4), (A.5), (A.6), and (A.7) were applied wherever necessary. For example, $[W]_{ij}\delta_{jN} = [W]_{iN}$. Note that the controllability Gramian matrix $\mathbf{W}$ is symmetric, thus taking the transpose has no effect on it: that is, $[W^T]_{iN} = [W]_{Ni} = [W]_{iN}$.

Finally,

$$\begin{aligned}
& \frac{\partial \mathcal{E}(C^T)}{\partial [C^T]_{NP}} \\
&= -[WC^TU^{-1}\mathbf{Y}_{f'}U^{-1}]_{NP} - [WC^TU^{-1}\mathbf{Y}_{f'}U^{-1}]_{NP} + [\mathbf{X}_{f'}C^TU^{-1}]_{NP} \\
&\quad - [WC^TU^{-1}C\mathbf{X}_{f'}C^TU^{-1}]_{NP} - [WC^TU^{-1}C\mathbf{X}_{f'}C^TU^{-1}]_{NP} + [\mathbf{X}_{f'}C^TU^{-1}]_{NP} \\
&= -2[WC^TU^{-1}\mathbf{Y}_{f'}U^{-1}]_{NP} - 2[WC^TU^{-1}C\mathbf{X}_{f'}C^TU^{-1}]_{NP} + 2[\mathbf{X}_{f'}C^TU^{-1}]_{NP} \\
&= -2WC^T(CWC^T)^{-1}\mathbf{y}_f\mathbf{y}_f^T(CWC^T)^{-1} - 2WC^T(CWC^T)^{-1}Ce^{At_f}e^{A^Tt_f}C^T(CWC^T)^{-1} \\
&\quad + 2e^{At_f}e^{A^Tt_f}C^T(CWC^T)^{-1},
\end{aligned} \tag{A.33}$$

where in Eq. (A.33), Eqs. (A.4), (A.5), (A.6), and (A.7) were used wherever necessary. Note that the output controllability Gramian $U = (CWC^T)$, and constant matrices $\mathbf{X}_{f'} = e^{At_f}e^{A^Tt_f}$, $\mathbf{Y}_{f'} = \mathbf{y}_f\mathbf{y}_f^T$ are symmetric. Thus, $(U^{-1})^T = U^{-1}$, $\mathbf{X}_{f'}^T = \mathbf{X}_{f'}$, and $\mathbf{Y}_{f'}^T = \mathbf{Y}_{f'}$. Thus, this was how the energy gradient information was derived.

A.5 Dataset

In the chapter main text Discussion, the condition number of the Gramian matrix was discussed. Sometimes, despite adding more driver nodes, the condition number still remains infeasibly high. To reduce the condition number, the unweighted link weights may be normalised, or changed to be uniform random $[0.5, 1.5]$ if the connection strength is not specified. Further, the network dataset that was used, the source, as well as any other additional information is documented as follows.

Electronic circuits[80]: the electronic network link weights are unspecified. Thus, $\{a_{ij}\} = 1$ if there is a link from node j to node i , and zero otherwise. The dataset is available for download at: <http://www.weizmann.ac.il/mcb/UriAlon/>.

Food web [81–83]: Maspalomas and Rhode networks link weights are specified. However, the condition number of the Gramian can be infeasibly high without normalization. For Maspalomas network, the link weights are normalised by $1e5$, and for Rhode network, the link weights are normalised by $1e4$. StMarks network link weight is unspecified. However, using $\{a_{ij}\} = \{1, 0\}$ will result in the condition number of the Gramian to be high. Thus, for StMarks, set the link weights to be random uniform $[0, 1.5]$ if $\{a_{ij}\}$ is a nonzero link. The dataset can be downloaded at: <http://vlado.fmf.uni-lj.si/pub/networks/data/bio/foodweb/foodweb.htm>.

Physicians[84]: for the physicians networks, the data set represents "person a" trusts "person b." For this reason, similar to ref. [91], control is more sensible when the connection direction is reversed to reflect that "person a" is influenced by "person b." For the physicians networks, link weight is unspecified. Using binary link weights will result in a high condition number of the Gramian, thus use random uniform $[0, 1.5]$ instead. The dataset is available online at: <http://moreno.ss.uci.edu/data.html>.

Teacher-student [85]: the link weight is unspecified, and unweighted link weight was used. The dataset is available online at: <http://moreno.ss.uci.edu/data.html>.

Highschool [86]: the link weight is specified with $\{a_{ij}\} = \{0, 1, 2\}$. They were normalised by 2. The dataset is available online at: http://konect.uni-koblenz.de/networks/moreno_highschool.

Main text table 1 additional information

Additional information of standard deviations to the table 2.1 presented in chapter main text are as follows.

Table A.1: Additional information on table 2.1 in the chapter main text. $\sigma[\mathcal{E}(C_0)]$ refers to the standard deviation of the 10 independent initializations matrix energy cost. $\sigma[\mathcal{E}(C^*)]$ is the standard deviation of the 10 independent converged energy cost. $\mu[\mathcal{E}([C_{\text{binary}}^*]^t)]$ is the mean of the 10 obtained binary optimal candidate solutions. $\sigma[\mathcal{E}(C_{\text{rand}})]$ is the standard deviation of energy cost needed in 100 independent realizations of random selection of target nodes.

Network	N	edges	$\sigma[\mathcal{E}(C_0)]$	$\sigma[\mathcal{E}(C^*)]$	$\mu[\mathcal{E}([C_{\text{binary}}^*]^t)]$	$\sigma[\mathcal{E}([C_{\text{binary}}^*]^t)]$	$\sigma[\mathcal{E}(C_{\text{rand}})]$
Model							
SF300	300	750	4.08E07	2.47E03	7.89E06	5.07E06	1.35E08
ER300	300	750	1.03E06	9.93E01	3.12E05	2.01E05	1.33E06
Electronic circuit							
Circuit-s838	512	819	1.13E06	1.52E02	4.28E05	4.48E05	2.53E06
Circuit-s420	252	399	6.33E04	1.26E01	7.97E04	1.06E05	2.74E06
Circuit-s208	122	189	1.58E04	7.26	6.76E05	2.07E06	6.21E08
Food web							
StMarks	54	356	3.36E03	2.35	4.09E02	9.99E01	3.74E03
Maspalomas	24	82	1.58E03	9.21	6.35E01	1.32E01	2.22E05
Rhode	19	53	3.97E02	0.689	5.06E01	2.06E01	2.24E05
Social Influence							
Phys-discuss-rev	231	565	3.82E03	2.23E01	8.21E03	2.74E03	5.93E05
Teacher-student	60	94	3.47E01	0.526	6.76E01	1.82	1.12E02
Social							
Phys-friend-rev	228	506	2.29E04	3.02E01	8.42E03	3.41E03	2.63E04
Highschool	70	366	5.34E04	8.84	1.77E03	2.77E03	5.07E04

Appendix B

Chapter 3: One driver node analytical equations

(See **Appendix C Supplementary information** for derivation.)

$$\mathbf{M}^{-1}(i, j) = \begin{cases} \frac{(-1)^N \prod_{k=1}^{N-1} (1 - \lambda_i \lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (1 - \lambda_l \lambda_i)}{[\mathbf{P}^{-1}]_{jh} v_{hi} \prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (\lambda_l - \lambda_i)}, & i = j, (N-1) \times (N-1) \text{ block}, \\ \left[\frac{(-1)^{N+1}}{(\lambda_j - \lambda_i) v_{hi} [\mathbf{P}^{-1}]_{jh}} \right] \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_i \lambda_k) \prod_{\substack{k=1 \\ k \neq i}}^{N-1} (1 - \lambda_k \lambda_j)}{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{k=1 \\ k \neq j, i}}^{N-1} (\lambda_k - \lambda_j)} \right], & i \neq j, (N-1) \times (N-1) \text{ block}, \\ \frac{1}{q_{NN} T_f} = \frac{1}{[\mathbf{P}^{-1}]_{Nh} v_{hN} T_f}, & i = j = N, \\ \left[\frac{(-1)^N}{[\mathbf{P}^{-1}]_{jh} v_{hi} (\lambda_j - \lambda_i) T_f} \right] \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_i \lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (1 - \lambda_l \lambda_j)}{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{l=1 \\ l \neq j}}^{N-1} (\lambda_l - \lambda_j)} \right], & i \neq j, N\text{-th row/col.} \end{cases} \quad (\text{B.1})$$

$$\begin{aligned} \underline{\alpha} = \text{trace}[(\mathbf{M}^{-1})^2] &\approx \sum_{i=1}^{N-1} \sum_{\substack{a=1 \\ a \neq i}}^{N-1} [\mathbf{M}^{-1}]_{ia} [\mathbf{M}^{-1}]_{ai} + [\mathbf{M}^{-1}]_{ii} [\mathbf{M}^{-1}]_{ii} \\ &\approx \sum_{i=1}^{N-1} \sum_{\substack{a=1 \\ a \neq i}}^{N-1} \left[\frac{1}{(\lambda_a - \lambda_i) v_{hi} [\mathbf{P}^{-1}]_{ah}} \right] \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_i \lambda_k) \prod_{\substack{k=1 \\ k \neq i}}^{N-1} (1 - \lambda_k \lambda_a)}{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{k=1 \\ k \neq a, i}}^{N-1} (\lambda_k - \lambda_a)} \right] \left[\frac{1}{(\lambda_i - \lambda_a) v_{ha} p_{ih}^{-1}} \right] \\ &\quad \times \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_a \lambda_k) \prod_{\substack{k=1 \\ k \neq a}}^{N-1} (1 - \lambda_k \lambda_i)}{\prod_{\substack{k=1 \\ k \neq a}}^{N-1} (\lambda_a - \lambda_k) \prod_{\substack{k=1 \\ k \neq i, a}}^{N-1} (\lambda_k - \lambda_i)} \right] + \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_i \lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (1 - \lambda_l \lambda_i)}{[\mathbf{P}^{-1}]_{ih} v_{hi} \prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (\lambda_l - \lambda_i)} \right]^2. \end{aligned} \quad (\text{B.2})$$

Appendix C

Chapter 3: Supplementary information

C.1 Deriving energy cost

From [6, 105], discrete-time energy-optimal control signal $\mathbf{u}^*(\tau) = \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{W}^{-1}(\mathbf{x}_f - \mathbf{A}^{T_f}\mathbf{x}_0)$. Let $\mathbf{x}_0 = \mathbf{0}$, then $\mathbf{u}^*(\tau) = \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{W}^{-1}\mathbf{x}_f$, and $(\mathbf{u}^*)^T(\tau) = \mathbf{x}_f^T\mathbf{W}^{-1}\mathbf{A}^{T_f-\tau-1}\mathbf{B}$. Thus, the energy cost with energy-optimal control signal

$$\begin{aligned}\mathcal{E}(T_f) &= \sum_{\tau=0}^{T_f-1} \mathbf{u}^T(\tau)\mathbf{u}(\tau) \\ &= \mathbf{x}_f^T\mathbf{W}^{-1} \left[\sum_{\tau=0}^{T_f-1} \mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1} \right] \mathbf{W}^{-1}\mathbf{x}_f \\ &= \mathbf{x}_f^T\mathbf{W}^{-1}\mathbf{W}\mathbf{W}^{-1}\mathbf{x}_f = \mathbf{x}_f^T\mathbf{W}^{-1}\mathbf{x}_f,\end{aligned}\tag{C.1}$$

where discrete-time controllability Gramian is $\mathbf{W} = \sum_{\tau=0}^{T_f-1} \mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}$. Since $\mathbf{x}_f \in \mathbb{R}^{N \times 1}$, and $\mathbf{W} \in \mathbb{R}^{N \times N}$, it is easy to check that the energy cost, Eq. (C.1), is a 1×1 dimension quantity.

C.2 Showing hadamard product form of gramian

From chapter main text, discrete-time Gramian is $\mathbf{W} = \sum_{\tau=0}^{T_f-1} \mathbf{A}^\tau\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^\tau$, where $\mathbf{A} = \mathbf{PDP}^{-1}$ and $\mathbf{A}^T = \mathbf{VDV}^{-1}$, with $[\mathbf{D}]_{ii} = \lambda_i$ being the eigenvalues of network $\mathbf{A}$ with conformity. Thus substituting eigendecompositions,

$$\begin{aligned}\mathbf{W} &= \sum_{\tau=0}^{T_f-1} \mathbf{PD}^\tau\mathbf{P}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{VD}^\tau\mathbf{V}^{-1} \\ &= \mathbf{P} \left[\sum_{\tau=0}^{T_f-1} \mathbf{D}^\tau\mathbf{P}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{VD}^\tau \right] \mathbf{V}^{-1} \\ &= \mathbf{PMV}^{-1}.\end{aligned}\tag{C.2}$$

Let $\mathbf{M} = \sum_{\tau=0}^{T_f-1} \mathbf{D}^\tau\mathbf{P}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{VD}^\tau$, then

$$\begin{aligned}
\mathbf{M}_{ij} &= \left[\sum_{\tau=0}^{T_f-1} \mathbf{D}^\tau \mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V} \mathbf{D}^\tau \right]_{ij} \\
&= \sum_a \sum_b \sum_{\tau=0}^{T_f-1} [\mathbf{D}^\tau]_{ia} [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ab} [\mathbf{D}^\tau]_{bj} \\
&= \sum_{\tau=0}^{T_f-1} [\mathbf{D}^\tau]_{ii} [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij} [\mathbf{D}^\tau]_{jj} \\
&= [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij} \sum_{\tau=0}^{T_f-1} [\mathbf{D}^\tau]_{ii} [\mathbf{D}^\tau]_{jj} \\
&= [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij} \sum_{\tau=0}^{T_f-1} (\lambda_i \lambda_j)^\tau \\
&= [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij} \frac{1 - (\lambda_i \lambda_j)^{T_f}}{1 - \lambda_i \lambda_j} = \mathbf{Q}_{ij} \frac{1 - (\lambda_i \lambda_j)^{T_f}}{1 - \lambda_i \lambda_j},
\end{aligned} \tag{C.3}$$

where $\mathbf{Q}_{ij} = [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij}$, and because $\mathbf{D}$ is a diagonal matrix, then indices $a = i, b = j$, and the summation is evaluated using geometric series summation formula: $r^0 + r^1 + r^2 + \dots + r^n = \sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r}$, where $r \neq 1$.

C.3 Deriving $\mathbf{Q}_{ij}$

$\mathbf{Q}_{ij} = [\mathbf{P}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{V}]_{ij}$, where $\mathbf{B}$ matrix contains the driver nodes placement, where $\mathbf{B} \in \mathbb{R}^{N \times M}$, and $b_{ij} = 1$ if the j -th driver node is attached to node i , and zero otherwise. Thus, it is easy to see that respectively for $N = 3$, one driver (control signal attached to node 1), and $d = 2$ drivers (driver nodes 1 and 3),

$$\mathbf{B} \mathbf{B}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{J}^{11}, \tag{C.4}$$

and

$$\begin{aligned}
\mathbf{B} \mathbf{B}^T &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \mathbf{J}^{11} + \mathbf{J}^{33},
\end{aligned} \tag{C.5}$$

where $\mathbf{J}^{hh}$ refers to single-entry matrix [90], which has the property that $[\mathbf{J}^{hh}]_{ij}$ is zero for all i, j indices, except when $i = j = h$, where it is one.

Assuming that for one driver, control signal attaches to node h , thus $\mathbf{B} \mathbf{B}^T = \mathbf{J}^{hh}$, and when using d number of drivers (where $1 < d < N$), $\mathbf{B} \mathbf{B}^T = \mathbf{J}^{d_1 d_1} + \mathbf{J}^{d_2 d_2} + \dots + \mathbf{J}^{d_d d_d} = \sum_{k=1}^d \mathbf{J}^{d_k d_k}$,

where d_k represents driver node k for $k = 1, 2, \dots, d$, then respectively

$$\mathbf{Q}_{ij} = [\mathbf{P}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{V}]_{ij} = \sum_a \sum_b [\mathbf{P}^{-1}]_{ia} [\mathbf{B}\mathbf{B}^T]_{ab} [\mathbf{V}]_{bj} = \sum_a \sum_b [\mathbf{P}^{-1}]_{ia} [\mathbf{J}^{hh}]_{ab} [\mathbf{V}]_{bj} = [\mathbf{P}^{-1}]_{ih} [\mathbf{V}]_{hj}, \quad (\text{C.6})$$

$$\begin{aligned} \mathbf{Q}_{ij} &= [\mathbf{P}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{V}]_{ij} = \sum_a \sum_b [\mathbf{P}^{-1}]_{ia} [\mathbf{B}\mathbf{B}^T]_{ab} [\mathbf{V}]_{bj} = \sum_a \sum_b [\mathbf{P}^{-1}]_{ia} \left[\sum_{k=1}^d \mathbf{J}^{d_k d_k} \right]_{ab} [\mathbf{V}]_{bj} \\ &= [\mathbf{P}^{-1}]_{id_1} [\mathbf{V}]_{d_1 j} + [\mathbf{P}^{-1}]_{id_2} [\mathbf{V}]_{d_2 j} + \dots + [\mathbf{P}^{-1}]_{id_d} [\mathbf{V}]_{d_d j} = \sum_{k=1}^d [\mathbf{P}^{-1}]_{id_k} v_{d_k j}. \end{aligned} \quad (\text{C.7})$$

When using N drivers, $\mathbf{B} = \mathbf{I}$, then $\mathbf{Q}_{ij} = [\mathbf{P}^{-1}\mathbf{V}]_{ij}$.

C.4 Deriving one driver $\mathbf{M}^{-1}$

$\mathbf{M}^{-1} = \frac{\mathbf{M}^*}{|\mathbf{M}|}$, where $\mathbf{M}^*$ is the adjoint matrix and $|\mathbf{M}|$ is the determinant of $\mathbf{M}$. Given that $\mathbf{M}_{ij} = \mathbf{Q}_{ij} \frac{1 - (\lambda_i \lambda_j)^{T_f}}{1 - \lambda_i \lambda_j}$, then $\mathbf{M}_{ij} = \mathbf{Q}_{ij} \frac{1}{1 - \lambda_i \lambda_j}$ for large T_f and $i = 1, 2, \dots, N - 1$ and/or $j = 1, 2, \dots, N - 1$ since (as discussed in the chapter main text), $\lambda_i < 1$ for $i = 1, 2, \dots, N - 1$. When $i = j = N$, because $\lambda_N = 1$, $\mathbf{M}_{NN} = \mathbf{Q}_{ij} T_f$, which dominates matrix $\mathbf{M}$.

Starting with 2×2 and assuming that one control signal is attached to node h ,

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad (\text{C.8})$$

with $m_{11} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h1}}{1 - \lambda_1 \lambda_1}$, $m_{12} = \frac{[\mathbf{P}^{-1}]_{2h} v_{h1}}{1 - \lambda_1 \lambda_2}$, $m_{21} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h2}}{1 - \lambda_2 \lambda_1}$, and $m_{22} = \frac{[\mathbf{P}^{-1}]_{2h} v_{h2}}{1 - \lambda_2 \lambda_2}$, then

$$\begin{aligned} |\mathbf{M}| &= \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11} m_{22} - m_{12} m_{21} \\ &= [\mathbf{P}^{-1}]_{1h} v_{h1} [\mathbf{P}^{-1}]_{2h} v_{h2} \frac{T_f}{1 - \lambda_1^2} - \frac{[\mathbf{P}^{-1}]_{2h} v_{h1} [\mathbf{P}^{-1}]_{1h} v_{h2}}{(1 - \lambda_1 \lambda_2)^2} \\ &\approx [\mathbf{P}^{-1}]_{1h} v_{h1} [\mathbf{P}^{-1}]_{2h} v_{h2} \frac{T_f}{1 - \lambda_1^2}, \end{aligned} \quad (\text{C.9})$$

where m_{22} with large T_f dominates the expression and terms not containing T_f are considered negligible.

Similarly, with 3×3 ,

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \quad (\text{C.10})$$

and using Gaussian elimination, $\det(A) = \frac{\prod \text{diag}(B)}{a}$, where a is a scalar quantity subject to the following rules:

1. Swapping two rows multiplies a by -1 ,
2. Multiplying a row by a non-zero scalar multiplies a by that amount,
3. Applying to one row a scalar multiple of another does not change a ,

$$\begin{aligned}
|M| &= \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \xrightarrow{R1=R1 \times \frac{1}{m_{11}}} m_{11} \begin{vmatrix} 1 & \frac{m_{12}}{m_{11}} & \frac{m_{13}}{m_{11}} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \\
&= \begin{matrix} \xrightarrow{R2=R2-m_{21}R1} \\ \xrightarrow{R3=R3-m_{31}R1} \end{matrix} m_{11} \begin{vmatrix} 1 & \frac{m_{12}}{m_{11}} & \frac{m_{13}}{m_{11}} \\ 0 & m_{22} - \frac{m_{12}m_{21}}{m_{11}} & m_{23} - \frac{m_{13}m_{21}}{m_{11}} \\ 0 & m_{32} - \frac{m_{12}m_{31}}{m_{11}} & m_{33} - \frac{m_{13}m_{31}}{m_{11}} \end{vmatrix} \\
&= m_{11} \begin{vmatrix} 1 & \frac{m_{12}}{m_{11}} & \frac{m_{13}}{m_{11}} \\ 0 & \frac{m_{11}m_{22}-m_{12}m_{21}}{m_{11}} & \frac{m_{11}m_{23}-m_{13}m_{21}}{m_{11}} \\ 0 & \frac{m_{11}m_{32}-m_{12}m_{31}}{m_{11}} & \frac{m_{11}m_{33}-m_{13}m_{31}}{m_{11}} \end{vmatrix} \\
&= \xrightarrow{R2=R2 \times \frac{m_{11}}{m_{11}m_{22}-m_{12}m_{21}}} m_{11} \begin{vmatrix} 1 & \frac{m_{12}}{m_{11}} & \frac{m_{13}}{m_{11}} \\ 0 & 1 & \frac{m_{11}m_{23}-m_{13}m_{21}}{m_{11}m_{22}-m_{12}m_{21}} \\ 0 & \frac{m_{11}m_{32}-m_{12}m_{31}}{m_{11}m_{22}-m_{12}m_{21}} & \frac{m_{11}m_{33}-m_{13}m_{31}}{m_{11}m_{22}-m_{12}m_{21}} \end{vmatrix} \\
&= \xrightarrow{R3=R3-R2 \times \frac{m_{11}m_{32}-m_{12}m_{31}}{m_{11}m_{22}-m_{12}m_{21}}} (m_{11}m_{22}-m_{12}m_{21}) \begin{vmatrix} 1 & \frac{m_{12}}{m_{11}} & \frac{m_{13}}{m_{11}} \\ 0 & 1 & \frac{m_{11}m_{23}-m_{13}m_{21}}{m_{11}m_{22}-m_{12}m_{21}} \\ 0 & 0 & \chi \end{vmatrix}, \tag{C.11}
\end{aligned}$$

where $\chi = \left(\frac{m_{11}m_{33}-m_{13}m_{31}}{m_{11}} \right) - \left(\frac{m_{11}m_{23}-m_{13}m_{21}}{m_{11}m_{22}-m_{12}m_{21}} \right) \left(\frac{m_{11}m_{32}-m_{12}m_{31}}{m_{11}} \right)$. Therefore,

$$|M| = (m_{11}m_{22}-m_{12}m_{21}) \left(\frac{m_{11}m_{33}-m_{13}m_{31}}{m_{11}} \right) - (m_{11}m_{23}-m_{13}m_{21}) \left(\frac{m_{11}m_{32}-m_{12}m_{31}}{m_{11}} \right), \tag{C.12}$$

and owing to m_{33} dominance (along with $m_{11} = \frac{[\mathbf{P}^{-1}]_{1h}v_{h1}}{1-\lambda_1\lambda_1}$, $m_{22} = \frac{[\mathbf{P}^{-1}]_{2h}v_{h2}}{1-\lambda_2\lambda_2}$, $m_{12} = \frac{[\mathbf{P}^{-1}]_{2h}v_{h1}}{1-\lambda_1\lambda_2}$, $m_{21} = \frac{[\mathbf{P}^{-1}]_{1h}v_{h2}}{1-\lambda_2\lambda_1}$, $m_{33} = [\mathbf{P}^{-1}]_{3h}v_{h3}T_f$),

$$\begin{aligned}
|M| &\approx (m_{11}m_{22}-m_{12}m_{21}) \left(\frac{m_{11}m_{33}-m_{13}m_{31}}{m_{11}} \right) \approx (m_{11}m_{22}-m_{12}m_{21})m_{33} \\
&= \left[\frac{[\mathbf{P}^{-1}]_{1h}v_{h1}}{(1-\lambda_1\lambda_1)} \frac{[\mathbf{P}^{-1}]_{2h}v_{h2}}{1-\lambda_2\lambda_2} - \frac{[\mathbf{P}^{-1}]_{2h}v_{h1}}{1-\lambda_1\lambda_2} \frac{[\mathbf{P}^{-1}]_{1h}v_{h2}}{1-\lambda_2\lambda_1} \right] [\mathbf{P}^{-1}]_{3h}v_{h3}T_f \\
&= [\mathbf{P}^{-1}]_{1h}v_{h1}[\mathbf{P}^{-1}]_{2h}v_{h2}[\mathbf{P}^{-1}]_{3h}v_{h3}T_f \left[\frac{(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1) - (1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)}{(1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1)} \right] \\
&= [\mathbf{P}^{-1}]_{1h}v_{h1}[\mathbf{P}^{-1}]_{2h}v_{h2}[\mathbf{P}^{-1}]_{3h}v_{h3}T_f (-1) \left(\frac{(\lambda_1-\lambda_2)}{(1-\lambda_1\lambda_1)(1-\lambda_1\lambda_2)} \right) \left(\frac{(\lambda_2-\lambda_1)}{(1-\lambda_2\lambda_1)(1-\lambda_2\lambda_2)} \right). \tag{C.13}
\end{aligned}$$

By inspection, there are four components to $|M|$: the constants, numerator, denominator, and negative sign. For arbitrary N , the constants are always $\prod_{l=1}^N [\mathbf{P}^{-1}]_{lh}v_{hl}T_f$. Let function $f(i, j) = (1 - \lambda_i\lambda_j)$, and it can be seen that $f(i, j)$ repeats itself in the denominator for indices $i, j = 1, 2, \dots, N-1$. The numerator is trickier, but observing the numerators, referencing matrix (C.10), and let $g(i, j) = (\lambda_i - \lambda_j)$, notice that diagonal indices $i = j$ are excluded, in addition, indices (either i or j) containing $N = 3$ are also excluded. Thus, for 3×3 numerators, only $g(1, 2)$ and $g(2, 1)$ are left on the numerator.

Generalizing,

$$|\mathbf{M}| \approx [\mathbf{P}^{-1}]_{Nh} v_{hN} T_f q_1(N) \prod_{l=1}^{N-1} [\mathbf{P}^{-1}]_{lh} v_{hl} \frac{\prod_{\substack{k=1 \\ k \neq l}}^{N-1} (\lambda_l - \lambda_k)}{\prod_{l'=1}^{N-1} \prod_{k=1}^{N-1} (1 - \lambda_{l'} \lambda_k)}, \quad (\text{C.14})$$

where

$$q_1(N) = \begin{cases} -1 & N \text{ is even and } N \bmod 4 = 0, \\ -1 & N \text{ is odd and } N \bmod 4 = 3, \\ +1 & N \text{ is even and } N \bmod 4 = 2, \\ +1 & N \text{ is odd and } N \bmod 4 = 1. \end{cases} \quad (\text{C.15})$$

Function $q_1(N)$ accounts for the sign change and is found and validated empirically with different networks of different size N . Further, analytical $|\mathbf{M}|$, (C.14), has also been validated with numerical $|\mathbf{M}|$.

The process of getting the formulae for matrix adjoints is similar, and for 2×2 , $\mathbf{M}^*(1, 1) = m_{22} = [\mathbf{P}^{-1}]_{2h} v_{h2} T_f$, $\mathbf{M}^*(1, 2) = m_{21} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h2}}{1 - \lambda_2 \lambda_1}$, $\mathbf{M}^*(2, 1) = m_{12} = \frac{[\mathbf{P}^{-1}]_{2h} v_{h1}}{1 - \lambda_1 \lambda_2}$, and $\mathbf{M}^*(2, 2) = m_{11} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h1}}{1 - \lambda_1 \lambda_1}$. For 3×3 , keeping in mind that $m_{33} = [\mathbf{P}^{-1}]_{3h} v_{h3} T_f$ dominates,

$$\mathbf{M}^*(1, 1) = \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} \approx \frac{[\mathbf{P}^{-1}]_{2h} v_{h2} [\mathbf{P}^{-1}]_{3h} v_{h3} T_f}{1 - \lambda_2 \lambda_2}, \quad (\text{C.16})$$

$$\mathbf{M}^*(1, 2) = \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} \approx \frac{[\mathbf{P}^{-1}]_{1h} v_{h2} [\mathbf{P}^{-1}]_{3h} v_{h3} T_f}{1 - \lambda_2 \lambda_1}, \quad (\text{C.17})$$

$$\mathbf{M}^*(2, 1) = \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix} \approx \frac{[\mathbf{P}^{-1}]_{2h} v_{h1} [\mathbf{P}^{-1}]_{3h} v_{h3} T_f}{1 - \lambda_1 \lambda_2}, \quad (\text{C.18})$$

$$\mathbf{M}^*(2, 2) = \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} \approx \frac{[\mathbf{P}^{-1}]_{1h} v_{h1} [\mathbf{P}^{-1}]_{3h} v_{h3} T_f}{1 - \lambda_1 \lambda_1}, \quad (\text{C.19})$$

$$\begin{aligned} \mathbf{M}^*(1, 3) &= \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h2} [\mathbf{P}^{-1}]_{2h} v_{h3}}{(1 - \lambda_2 \lambda_1)(1 - \lambda_3 \lambda_2)} - \frac{[\mathbf{P}^{-1}]_{2h} v_{h2} [\mathbf{P}^{-1}]_{1h} v_{h3}}{(1 - \lambda_2 \lambda_2)(1 - \lambda_3 \lambda_1)} \\ &= [\mathbf{P}^{-1}]_{1h} v_{h2} [\mathbf{P}^{-1}]_{2h} v_{h3} \left[\frac{(1 - \lambda_2 \lambda_2)(1 - \lambda_3 \lambda_1) - (1 - \lambda_2 \lambda_1)(1 - \lambda_3 \lambda_2)}{(1 - \lambda_2 \lambda_1)(1 - \lambda_3 \lambda_2)(1 - \lambda_2 \lambda_2)(1 - \lambda_3 \lambda_1)} \right] \\ &= [\mathbf{P}^{-1}]_{1h} v_{h2} [\mathbf{P}^{-1}]_{2h} v_{h3} \left(\frac{(\lambda_2 - \lambda_1)}{(1 - \lambda_2 \lambda_1)(1 - \lambda_2 \lambda_2)} \right) \left(\frac{(\lambda_3 - \lambda_2)}{(1 - \lambda_3 \lambda_1)(1 - \lambda_3 \lambda_2)} \right), \end{aligned} \quad (\text{C.20})$$

$$\mathbf{M}^*(3, 3) = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h1} [\mathbf{P}^{-1}]_{2h} v_{h2}}{(1 - \lambda_1 \lambda_1)(1 - \lambda_2 \lambda_2)} - \frac{[\mathbf{P}^{-1}]_{2h} v_{h1} [\mathbf{P}^{-1}]_{1h} v_{h2}}{(1 - \lambda_1 \lambda_2)(1 - \lambda_2 \lambda_1)} \quad (\text{C.21})$$

For 4×4 , with dominant $m_{44} = [\mathbf{P}^{-1}]_{4h}v_{h4}T_f$, and $m_{21} = \frac{[\mathbf{P}^{-1}]_{1h}v_{h2}}{(1-\lambda_2\lambda_1)}$, $m_{23} = \frac{[\mathbf{P}^{-1}]_{3h}v_{h2}}{(1-\lambda_2\lambda_3)}$, $m_{31} = \frac{[\mathbf{P}^{-1}]_{1h}v_{h3}}{(1-\lambda_3\lambda_1)}$, $m_{33} = \frac{[\mathbf{P}^{-1}]_{3h}v_{h3}}{(1-\lambda_3\lambda_3)}$,

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}, \quad (\text{C.22})$$

and

$$\begin{aligned} \mathbf{M}^*(1, 2) &= \begin{vmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{vmatrix} \xrightarrow{R1=R1 \times \frac{1}{m_{21}}} m_{21} \begin{vmatrix} 1 & \frac{m_{23}}{m_{21}} & \frac{m_{24}}{m_{21}} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{vmatrix} \\ &= \xrightarrow{R2=R2-m_{31}R1} \xrightarrow{R3=R3-m_{41}R1} m_{21} \begin{vmatrix} 1 & \frac{m_{23}}{m_{21}} & \frac{m_{24}}{m_{21}} \\ 0 & m_{33} - \frac{m_{23}m_{31}}{m_{21}} & m_{34} - \frac{m_{24}m_{31}}{m_{21}} \\ 0 & m_{43} - \frac{m_{23}m_{41}}{m_{21}} & m_{44} - \frac{m_{24}m_{41}}{m_{21}} \end{vmatrix} \\ &= m_{21} \begin{vmatrix} 1 & \frac{m_{23}}{m_{21}} & \frac{m_{24}}{m_{21}} \\ 0 & \frac{m_{21}m_{33}-m_{23}m_{31}}{m_{21}} & \frac{m_{21}m_{34}-m_{24}m_{31}}{m_{21}} \\ 0 & \frac{m_{21}m_{43}-m_{23}m_{41}}{m_{21}} & \frac{m_{21}m_{44}-m_{24}m_{41}}{m_{21}} \end{vmatrix} \\ &= \xrightarrow{R2=R2 \times \frac{m_{21}}{m_{21}m_{33}-m_{23}m_{31}}} m_{21} \frac{m_{21}m_{33}-m_{23}m_{31}}{m_{21}} \begin{vmatrix} 1 & \frac{m_{23}}{m_{21}} & \frac{m_{24}}{m_{21}} \\ 0 & 1 & \frac{m_{21}m_{34}-m_{24}m_{31}}{m_{21}m_{33}-m_{23}m_{31}} \\ 0 & \frac{m_{21}m_{43}-m_{23}m_{41}}{m_{21}m_{33}-m_{23}m_{31}} & \frac{m_{21}m_{44}-m_{24}m_{41}}{m_{21}m_{33}-m_{23}m_{31}} \end{vmatrix} \\ &= \xrightarrow{R3=R3-R2 \times \frac{m_{21}m_{43}-m_{23}m_{41}}{m_{21}m_{33}-m_{23}m_{31}}} (m_{21}m_{33}-m_{23}m_{31}) \begin{vmatrix} 1 & \frac{m_{23}}{m_{21}} & \frac{m_{24}}{m_{21}} \\ 0 & 1 & \frac{m_{21}m_{34}-m_{24}m_{31}}{m_{21}m_{33}-m_{23}m_{31}} \\ 0 & 0 & \chi \end{vmatrix}, \end{aligned} \quad (\text{C.23})$$

where $\chi = \frac{m_{21}m_{44}-m_{24}m_{41}}{m_{21}} - \left(\frac{m_{21}m_{34}-m_{24}m_{31}}{m_{21}m_{33}-m_{23}m_{31}}\right)\left(\frac{m_{21}m_{43}-m_{23}m_{41}}{m_{21}}\right)$.

Therefore,

$$\begin{aligned} \mathbf{M}^*(1, 2) &= (m_{21}m_{33} - m_{23}m_{31}) \left(\frac{m_{21}m_{44} - m_{24}m_{41}}{m_{21}} \right) - (m_{21}m_{34} - m_{24}m_{31}) \left(\frac{m_{21}m_{43} - m_{23}m_{41}}{m_{21}} \right) \\ &\approx m_{21}m_{33}m_{44} - m_{23}m_{31}m_{44} \\ &= \frac{[\mathbf{P}^{-1}]_{1h}v_{h2}}{(1-\lambda_2\lambda_1)} \frac{[\mathbf{P}^{-1}]_{3h}v_{h3}}{(1-\lambda_3\lambda_3)} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f - \frac{[\mathbf{P}^{-1}]_{3h}v_{h2}}{(1-\lambda_2\lambda_3)} \frac{[\mathbf{P}^{-1}]_{1h}v_{h3}}{(1-\lambda_3\lambda_1)} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f \\ &= ([\mathbf{P}^{-1}]_{1h}v_{h2} [\mathbf{P}^{-1}]_{3h}v_{h3} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f) \left(\frac{(1-\lambda_2\lambda_3)(1-\lambda_3\lambda_1) - (1-\lambda_2\lambda_1)(1-\lambda_3\lambda_3)}{(1-\lambda_2\lambda_1)(1-\lambda_3\lambda_3)(1-\lambda_2\lambda_3)(1-\lambda_3\lambda_1)} \right) \\ &= ([\mathbf{P}^{-1}]_{1h}v_{h2} [\mathbf{P}^{-1}]_{3h}v_{h3} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f) \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{(1-\lambda_2\lambda_1)(1-\lambda_3\lambda_3)(1-\lambda_2\lambda_3)(1-\lambda_3\lambda_1)} \\ &= (-1) ([\mathbf{P}^{-1}]_{1h}v_{h2} [\mathbf{P}^{-1}]_{3h}v_{h3} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f) \left(\frac{(\lambda_2 - \lambda_3)}{(1-\lambda_2\lambda_1)(1-\lambda_2\lambda_3)} \right) \left(\frac{(\lambda_3 - \lambda_1)}{(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_3)} \right). \end{aligned} \quad (\text{C.24})$$

Much like $|\mathbf{M}|$, there are the constants, negative sign, numerator, and denominator. For $\mathbf{M}^*(i, j)$,

where $i \neq j$, Constants will have $\prod_{l=1}^N \frac{[\mathbf{P}^{-1}]_{lh}v_{hl}T_f}{v_{hi}[\mathbf{P}^{-1}]_{jh}}$. Let function $f(k, l) = (1 - \lambda_k\lambda_l)$, then it can

be observed that the denominator comprises products of $f(k, l)$, except when $k = i$, and $l = j$. For numerator, let $g(l, k) = (\lambda_l - \lambda_k)$ and observe numerator of last line of (C.24), comparing with first line of (C.23), diagonal indices where $l = k$ are excluded, and $g(l, k)$ with any l or k containing $N = 4$ are also excluded. Further, $g(l = j, k = i)$ is also excluded. This leaves only $g(2, 3)$ and $g(3, 1)$ remaining, which leads to the following generalization (for $i \neq j$ and $i, j < N$)

$$\mathbf{M}^*(i, j) = \left[\frac{q_2(N) T_f \prod_{k=1}^N [\mathbf{P}^{-1}]_{kh} v_{hk}}{(\lambda_j - \lambda_i) [\mathbf{P}^{-1}]_{jh} v_{hi}} \right] \left[\frac{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} \prod_{\substack{l=1 \\ l \neq j, k}}^{N-1} (\lambda_k - \lambda_l)}{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} \prod_{\substack{l=1 \\ l \neq j}}^{N-1} (1 - \lambda_k \lambda_l)} \right], \quad (\text{C.25})$$

where

$$q_2(N) = \begin{cases} +1 & N \text{ is even and } N \bmod 4 = 0, \\ -1 & N \text{ is odd and } N \bmod 4 = 3, \\ -1 & N \text{ is even and } N \bmod 4 = 2, \\ +1 & N \text{ is odd and } N \bmod 4 = 1. \end{cases} \quad (\text{C.26})$$

Function $q_2(N)$ accounts for the sign change and was obtained empirically with different networks of different system size N . It was validated numerically, along with analytical $\mathbf{M}^*(i, j)$, Eq. (C.25).

When $i = j$, then

$$\begin{aligned} \mathbf{M}^*(2, 2) &= \begin{vmatrix} m_{11} & m_{13} & m_{14} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{vmatrix} \xrightarrow{R1=R1 \times \frac{1}{m_{11}}} m_{11} \begin{vmatrix} 1 & \frac{m_{13}}{m_{11}} & \frac{m_{14}}{m_{11}} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{vmatrix} \\ &= \begin{matrix} \xrightarrow{R2=R2-m_{31}R1} \\ \xrightarrow{R3=R3-m_{41}R1} \end{matrix} m_{11} \begin{vmatrix} 1 & \frac{m_{13}}{m_{11}} & \frac{m_{14}}{m_{11}} \\ 0 & m_{33} - \frac{m_{13}m_{31}}{m_{11}} & m_{34} - \frac{m_{14}m_{31}}{m_{11}} \\ 0 & m_{43} - \frac{m_{13}m_{41}}{m_{11}} & m_{44} - \frac{m_{14}m_{41}}{m_{11}} \end{vmatrix} \\ &= m_{11} \begin{vmatrix} 1 & \frac{m_{13}}{m_{11}} & \frac{m_{14}}{m_{11}} \\ 0 & \frac{m_{11}m_{33}-m_{31}m_{13}}{m_{11}} & \frac{m_{11}m_{34}-m_{31}m_{14}}{m_{11}} \\ 0 & \frac{m_{11}m_{43}-m_{41}m_{13}}{m_{11}} & \frac{m_{11}m_{44}-m_{41}m_{14}}{m_{11}} \end{vmatrix} \\ &= \frac{R2=R2 \times \frac{m_{11}}{m_{11}m_{33}-m_{31}m_{13}}}{m_{11}} \frac{m_{11}m_{33}-m_{31}m_{13}}{m_{11}} \begin{vmatrix} 1 & \frac{m_{13}}{m_{11}} & \frac{m_{14}}{m_{11}} \\ 0 & 1 & \frac{m_{11}m_{34}-m_{31}m_{14}}{m_{11}m_{33}-m_{31}m_{13}} \\ 0 & \frac{m_{11}m_{43}-m_{41}m_{13}}{m_{11}} & \frac{m_{11}m_{44}-m_{41}m_{14}}{m_{11}} \end{vmatrix} \\ &= \frac{R3=R3-R2 \times \frac{m_{11}m_{43}-m_{41}m_{13}}{m_{11}}}{m_{11}} (m_{11}m_{33}-m_{31}m_{13}) \begin{vmatrix} 1 & \frac{m_{13}}{m_{11}} & \frac{m_{14}}{m_{11}} \\ 0 & 1 & \frac{m_{11}m_{34}-m_{31}m_{14}}{m_{11}m_{33}-m_{31}m_{13}} \\ 0 & 0 & \chi \end{vmatrix}, \quad (\text{C.27}) \end{aligned}$$

where $\chi = \frac{m_{11}m_{44}-m_{41}m_{14}}{m_{11}} - \left(\frac{m_{11}m_{34}-m_{31}m_{14}}{m_{11}m_{33}-m_{31}m_{13}} \right) \left(\frac{m_{11}m_{43}-m_{41}m_{13}}{m_{11}} \right)$.

Let $m_{11} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h1}}{(1-\lambda_1 \lambda_1)}$, $m_{13} = \frac{[\mathbf{P}^{-1}]_{3h} v_{h1}}{(1-\lambda_1 \lambda_3)}$, $m_{31} = \frac{[\mathbf{P}^{-1}]_{1h} v_{h3}}{(1-\lambda_3 \lambda_1)}$, $m_{33} = \frac{[\mathbf{P}^{-1}]_{3h} v_{h3}}{(1-\lambda_3 \lambda_3)}$, $m_{44} = [\mathbf{P}^{-1}]_{4h} v_{h4} T_f$.

Subsequently,

$$\begin{aligned}
\mathbf{M}^*(2, 2) &= (m_{11}m_{33} - m_{31}m_{13}) \left[\frac{m_{11}m_{44} - m_{41}m_{14}}{m_{11}} - \left(\frac{m_{11}m_{34} - m_{31}m_{14}}{m_{11}m_{33} - m_{31}m_{13}} \right) \left(\frac{m_{11}m_{43} - m_{41}m_{13}}{m_{11}} \right) \right] \\
&\approx m_{21}m_{33}m_{44} - m_{23}m_{31}m_{44} \\
&= \frac{[\mathbf{P}^{-1}]_{1h}v_{h1}}{(1 - \lambda_1\lambda_1)} \frac{[\mathbf{P}^{-1}]_{3h}v_{h3}}{(1 - \lambda_3\lambda_3)} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f - \frac{[\mathbf{P}^{-1}]_{1h}v_{h3}}{(1 - \lambda_3\lambda_1)} \frac{[\mathbf{P}^{-1}]_{3h}v_{h1}}{(1 - \lambda_1\lambda_3)} [\mathbf{P}^{-1}]_{4h}v_{h4}T_f \\
&= ([\mathbf{P}^{-1}]_{1h}v_{h2}[\mathbf{P}^{-1}]_{3h}v_{h3}[\mathbf{P}^{-1}]_{4h}v_{h4}T_f) \left(\frac{(1 - \lambda_3\lambda_1)(1 - \lambda_1\lambda_3) - (1 - \lambda_1\lambda_1)(1 - \lambda_3\lambda_3)}{(1 - \lambda_1\lambda_1)(1 - \lambda_3\lambda_3)(1 - \lambda_3\lambda_1)(1 - \lambda_1\lambda_3)} \right) \\
&= ([\mathbf{P}^{-1}]_{1h}v_{h2}[\mathbf{P}^{-1}]_{3h}v_{h3}[\mathbf{P}^{-1}]_{4h}v_{h4}T_f) \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_1)}{(1 - \lambda_1\lambda_1)(1 - \lambda_3\lambda_3)(1 - \lambda_3\lambda_1)(1 - \lambda_1\lambda_1)} \\
&= (-1)([\mathbf{P}^{-1}]_{1h}v_{h2}[\mathbf{P}^{-1}]_{3h}v_{h3}[\mathbf{P}^{-1}]_{4h}v_{h4}T_f) \left(\frac{(\lambda_1 - \lambda_3)}{(1 - \lambda_1\lambda_1)(1 - \lambda_1\lambda_3)} \right) \left(\frac{(\lambda_3 - \lambda_1)}{(1 - \lambda_3\lambda_1)(1 - \lambda_3\lambda_3)} \right). \tag{C.28}
\end{aligned}$$

Similar as before, it can be generalized

$$\mathbf{M}^*(i, i) = \frac{q_3(N)T_f \prod_{k=1}^N [\mathbf{P}^{-1}]_{kh}v_{hk} \prod_{\substack{k=1 \\ k \neq i}}^{N-1} \prod_{\substack{l=1 \\ l \neq i, k}}^{N-1} (\lambda_k - \lambda_l)}{[\mathbf{P}^{-1}]_{ih}v_{hi} \prod_{\substack{k=1 \\ k \neq i}}^{N-1} \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (1 - \lambda_k\lambda_l)}, \tag{C.29}$$

where

$$q_3(N) = \begin{cases} -1 & \text{N is even and } N \bmod 4 = 0, \\ +1 & \text{N is odd and } N \bmod 4 = 3, \\ +1 & \text{N is even and } N \bmod 4 = 2, \\ -1 & \text{N is odd and } N \bmod 4 = 1. \end{cases} \tag{C.30}$$

Function $q_3(N)$ accounts for sign change, and has been found empirically. It was validated numerically, along with analytical $\mathbf{M}^*(i, i)$, Eq. (C.29).

Next, $\mathbf{M}^*(i, j)$ is derived when index i or j equals N . For $i = j = N$, it is easy to see that $\mathbf{M}^*(N, N)$ is just the same as determinant $|M|$, absent last row/col, which contains $m_{NN} = [\mathbf{P}^{-1}]_{Nh}v_{hN}T_f$. Thus, modifying (C.14),

$$\mathbf{M}^*(N, N) = q_1(N) \prod_{l=1}^{N-1} [\mathbf{P}^{-1}]_{lh}v_{hl} \frac{\prod_{\substack{k=1 \\ k \neq l}}^{N-1} (\lambda_l - \lambda_k)}{\prod_{l'=1}^{N-1} \prod_{k=1}^{N-1} (1 - \lambda_{l'}\lambda_k)}. \tag{C.31}$$

When $i = N$ or when $j = N$, for 3×3 ,

$$\begin{aligned}
\mathbf{M}^*(3, 1) &= \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \\
&= \frac{[\mathbf{P}^{-1}]_{2h}v_{h1} [\mathbf{P}^{-1}]_{3h}v_{h2}}{(1 - \lambda_1\lambda_2)(1 - \lambda_2\lambda_3)} - \frac{[\mathbf{P}^{-1}]_{3h}v_{h1} [\mathbf{P}^{-1}]_{2h}v_{h2}}{(1 - \lambda_1\lambda_3)(1 - \lambda_2\lambda_2)} \\
&= [\mathbf{P}^{-1}]_{2h}v_{h1} [\mathbf{P}^{-1}]_{3h}v_{h2} \frac{(1 - \lambda_1\lambda_3)(1 - \lambda_2\lambda_2) - (1 - \lambda_1\lambda_2)(1 - \lambda_2\lambda_3)}{(1 - \lambda_1\lambda_2)(1 - \lambda_2\lambda_3)(1 - \lambda_1\lambda_3)(1 - \lambda_2\lambda_2)} \\
&= \frac{[\mathbf{P}^{-1}]_{2h}v_{h1} [\mathbf{P}^{-1}]_{3h}v_{h2} (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_2)}{(1 - \lambda_1\lambda_2)(1 - \lambda_2\lambda_3)(1 - \lambda_1\lambda_3)(1 - \lambda_2\lambda_2)}.
\end{aligned} \tag{C.32}$$

The governing rules are the analogous, absent the presence of m_{NN} . For example, the constants are $\frac{\prod_{l=1}^N [\mathbf{P}^{-1}]_{lh}v_{hl}}{v_{hi}[\mathbf{P}^{-1}]_{jh}}$; the denominator comprises products of $f(k, l)$, except when $k = i$, and $l = j$, where function $f(k, l) = (1 - \lambda_k\lambda_l)$; and the numerator is products of $g(l, k) = (\lambda_l - \lambda_k)$, where the indices $l = k, l = j, k = i, l = N, k = N$ should be excluded.

Therefore, for $i = N$ or $j = N$, but not $i = j = N$,

$$\mathbf{M}^*(i, j) = \frac{\prod_{k'=1}^N [\mathbf{P}^{-1}]_{k'h}v_{hk'} \prod_{\substack{k=1 \\ k \neq i}}^N \prod_{\substack{l=1 \\ l \neq k, j}}^N (\lambda_k - \lambda_l)}{[\mathbf{P}^{-1}]_{jh}v_{hi} \prod_{\substack{k=1 \\ k \neq i}}^N \prod_{\substack{l=1 \\ l \neq j}}^N (1 - \lambda_k\lambda_l)}. \tag{C.33}$$

Piecing together all the derived analytical equations, (C.14), (C.25), (C.29), (C.31), (C.33), where $\mathbf{M}^{-1} = \frac{\mathbf{M}^*}{|\mathbf{M}|}$, one driver node h analytical $\mathbf{M}^{-1}$ is obtained as shown in chapter main text Appendix

$$\mathbf{M}^{-1}(i, j) = \begin{cases} \frac{(-1)^N \prod_{k=1}^{N-1} (1 - \lambda_i\lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (1 - \lambda_l\lambda_i)}{[\mathbf{P}^{-1}]_{jh}v_{hi} \prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (\lambda_l - \lambda_i)}, & i = j, (N-1) \times (N-1) \text{ block,} \\ \left[\frac{(-1)^{N+1}}{(\lambda_j - \lambda_i)v_{hi}[\mathbf{P}^{-1}]_{jh}} \right] \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_i\lambda_k) \prod_{\substack{k=1 \\ k \neq i}}^{N-1} (1 - \lambda_k\lambda_j)}{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{k=1 \\ k \neq j, i}}^{N-1} (\lambda_k - \lambda_j)} \right], & i \neq j, (N-1) \times (N-1) \text{ block,} \\ \frac{1}{q_{NN}T_f} = \frac{1}{[\mathbf{P}^{-1}]_{Nh}v_{hN}T_f}, & i = j = N, \\ \left[\frac{(-1)^N}{[\mathbf{P}^{-1}]_{jh}v_{hi}(\lambda_j - \lambda_i)T_f} \right] \left[\frac{\prod_{k=1}^{N-1} (1 - \lambda_i\lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^{N-1} (1 - \lambda_l\lambda_j)}{\prod_{\substack{k=1 \\ k \neq i}}^{N-1} (\lambda_i - \lambda_k) \prod_{\substack{l=1 \\ l \neq j}}^{N-1} (\lambda_l - \lambda_j)} \right], & i \neq j, N^{\text{th}} \text{ row/col,} \end{cases} \tag{C.34}$$

where in (C.34), the sign changes work out to be $\frac{q_2(N)}{q_1(N)} = (-1)^{N+1}$ and $\frac{q_3(N)}{q_1(N)} = (-1)^N$. As before, analytical $\mathbf{M}^{-1}$ has been validated against numerical $\mathbf{M}^{-1}$.

C.5 $\mathbf{M}^{-1}$ with more than one driver

Consider 3×3 $\mathbf{M}$ matrix, using two driver nodes located at node 1, and node 2, where $q_{ij} = [[\mathbf{P}^{-1}]_{j1}v_{1i} + [\mathbf{P}^{-1}]_{j2}v_{2i}]$, then

$$\begin{aligned}
\mathbf{M}^*(3,3) &= \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21} \\
&= \frac{q_{11}}{(1-\lambda_1\lambda_1)} \frac{q_{22}}{(1-\lambda_2\lambda_2)} - \frac{q_{12}}{(1-\lambda_1\lambda_2)} \frac{q_{21}}{(1-\lambda_2\lambda_1)} \\
&= \frac{[[\mathbf{P}^{-1}]_{11}v_{11} + [\mathbf{P}^{-1}]_{12}v_{21}][[\mathbf{P}^{-1}]_{21}v_{12} + [\mathbf{P}^{-1}]_{22}v_{22}]}{(1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)} - \frac{[[\mathbf{P}^{-1}]_{21}v_{11} + [\mathbf{P}^{-1}]_{22}v_{21}][[\mathbf{P}^{-1}]_{11}v_{12} + [\mathbf{P}^{-1}]_{12}v_{22}]}{(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1)} \\
&= \frac{[\mathbf{P}^{-1}]_{11}v_{11}[\mathbf{P}^{-1}]_{21}v_{12} + [\mathbf{P}^{-1}]_{11}v_{11}[\mathbf{P}^{-1}]_{22}v_{22} + [\mathbf{P}^{-1}]_{12}v_{21}[\mathbf{P}^{-1}]_{21}v_{12} + [\mathbf{P}^{-1}]_{12}v_{21}[\mathbf{P}^{-1}]_{22}v_{22}}{(1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)} \\
&\quad - \left[\frac{[\mathbf{P}^{-1}]_{21}v_{11}[\mathbf{P}^{-1}]_{11}v_{12} + [\mathbf{P}^{-1}]_{21}v_{11}[\mathbf{P}^{-1}]_{12}v_{22} + [\mathbf{P}^{-1}]_{22}v_{21}[\mathbf{P}^{-1}]_{11}v_{12} + [\mathbf{P}^{-1}]_{22}v_{21}[\mathbf{P}^{-1}]_{12}v_{22}}{(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1)} \right] \\
&= \frac{[\mathbf{P}^{-1}]_{11}v_{11}[\mathbf{P}^{-1}]_{22}v_{22} + [\mathbf{P}^{-1}]_{12}v_{21}[\mathbf{P}^{-1}]_{21}v_{12}}{(1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)} - \left[\frac{[\mathbf{P}^{-1}]_{21}v_{11}[\mathbf{P}^{-1}]_{12}v_{22} + [\mathbf{P}^{-1}]_{22}v_{21}[\mathbf{P}^{-1}]_{11}v_{12}}{(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1)} \right] \\
&\quad + \frac{[\mathbf{P}^{-1}]_{12}v_{21}[\mathbf{P}^{-1}]_{22}v_{22}[(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1) - (1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)]}{(1-\lambda_1\lambda_1)(1-\lambda_2\lambda_2)(1-\lambda_1\lambda_2)(1-\lambda_2\lambda_1)}.
\end{aligned} \tag{C.35}$$

$\mathbf{M}^*(3,3)$ cannot be factorised, and it is difficult to obtain analytical $\mathbf{M}^*$. Thus, when using more than one driver, numerical $\mathbf{M}^{-1}$ was calculated along with the inverse of Eq. (3.11) of the chapter main text to estimate upper bound control energy, validated against numerical calculation, obtained with the inverse of real absolute minimum eigenvalue of $\mathbf{M}$.

When using N drivers, besides using Eq. (3.11) and numerical $\mathbf{M}^{-1}$, as shown in chapter main text, it was also possible to approximate analytical $\mathbf{M}^{-1}$ as a diagonal matrix such that $\mathbf{M}^{-1}(i,i) = [\mathbf{M}_{ii}]^{-1} = \frac{1-\lambda_i^2}{[\mathbf{P}^{-1}\mathbf{V}]_{ii}}$. This is obtained based on observing numerical $\mathbf{M}^{-1}$, when $\mathbf{B} = \mathbf{I}$, having non-diagonal terms numerically smaller than diagonal terms. This effect compounds with multiplication, thus $[\mathbf{M}^{-1}]^2$ and $[\mathbf{M}^{-1}]^4$ have diagonal terms numerically larger than non-diagonal terms.

C.6 Real networks dataset

The real networks datasets, karate network [111], and office network [110], used in this Chapter were downloaded from [117]. Both network structures, $\hat{\mathbf{A}}$, were undirected and symmetric, with $N = 34$, $\langle k \rangle \approx 4.6$, 78 edges, scale-free degree distribution, and $N = 40$, $\langle k \rangle = 11.9$, 238 edges, bounded degree distribution respectively.

For the karate network, nodes are members of a university karate club, with unweighted network links representing an association or prior interactions between members. For the office network, data was collected by third-party observers on the frequencies of interactions between office workers of a small business. The links are weighted based on the interaction frequency.

Appendix D

Chapter 4: One driver large t_f regime

Figure D.1: Continuous-time linear dynamics: Large t_f regime, one driver results in a chain network, where the driver node is located at node 4, and the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.14). (c) shows the relationship between turning points $\log_{10}(|z^*|)$ and its associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{large } t_f}(z^*))$, and the turning points z^* could lie on the negative z region. Unlike Fig. 4.4, there is no clear relationship between any network node properties such as average path distances from driver to zealot-influenced nodes, average node degree, eigenvalues, or eigencentralities to turning point z^* and its minima energy costs. (d), (e), and (f) respectively indicate that the energy cost scales with associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$ for various configurations of zealot-influenced nodes when $z = -5$, $z = 5$, and $z = z^*$ when controlling the chain network toward consensus $c = 5$.

Figure D.2: Continuous-time linear dynamics: Large t_f regime, one driver results in a star network, where the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.14). (c) shows the relationship between turning points $\log_{10}(|z^*|)$ and its associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{large } t_f}(z^*))$, where the turning points z^* could be negative. (d), (e), and (f) respectively indicate that the energy cost scales with its associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$ for various configurations of zealot-influenced nodes when $z = -5$, $z = 5$, and $z = z^*$ when controlling the star network toward consensus $c = 5$,

Figure D.3: Continuous-time linear dynamics: Large t_f regime, one driver results in a ring network, where the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.14). (c) shows the relationship between turning points $\log_{10}(|z^*|)$ and its associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{large } t_f}(z^*))$, where the turning points z^* could be negative. (d), (e), and (f) respectively indicate that the energy cost scales with its associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$ for various configurations of zealot-influenced nodes when $z = -5$, $z = 5$, and $z = z^*$ when controlling the ring network toward consensus $c = 5$. (f), (g), and (g)

Appendix E

Chapter 4: One driver small t_f regime

Figure E.1: Continuous-time linear dynamics: Small t_f regime, one driver results in a chain network, where the driver node is located at node 4, and the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.15). (c) shows the relationship between turning points $\log_{10} |z^*|$ and associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{small } t_f}(z^*))$, where z^* could lie in the negative region. (d), (e), and (f) respectively indicate that the energy cost scales with its associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$, when $z = -5$, $z = 5$, and $z = z^*$ for various configurations of zealot-influenced nodes.

Figure E.2: Continuous-time linear dynamics: Small t_f regime, one driver results in a star network, where the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.15). (c) shows the relationship between turning points $\log_{10} |z^*|$ and associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{small}} t_f(z^*))$, where z^* could lie in the negative region. (d), (e), and (f) respectively indicate that the energy cost scales with its associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$, when $z = -5$, $z = 5$, and $z = z^*$ for various configurations of zealot-influenced nodes.

Figure E.3: Continuous-time linear dynamics: Small t_f regime, one driver results in a ring network, where the configurations of zealot-influenced nodes are varied. (b) validates Eq. (4.15). (c) shows the relationship between turning points $\log_{10} |z^*|$ and associated minima energy costs $\log_{10}(\mathcal{E}_1^{\text{small}} t_f(z^*))$, where z^* could lie in the negative region. (d), (e), and (f) respectively indicate that the energy cost scales with its associated length l as $\log_{10}(\mathcal{E}(l)) \sim \log_{10}(l)$, when $z = -5$, $z = 5$, and $z = z^*$ for various configurations of zealot-influenced nodes.

Appendix F

Chapter 4: Discrete-time linear dynamics with conformity no-zealot energy cost

In figs. 4.8, 4.9, 4.10, and 4.11, the green dashed lines represent the no-zealot energy cost. This computation is not the same as those in the (4.26) dynamics and taking out the n -th row/column; the no-zealot network cannot be decoupled in this way owing to the $\mathbf{S}^{-1}\mathbf{A}$ coupled matrices. Instead, the no-zealot computation should consider networks in a $N \times N$ matrix, without any directed links from the n -th zealot node (where $n = N + 1$):

$$\begin{aligned}\mathbf{x}(\tau + 1) &= \tilde{\mathbf{S}}^{-1}\tilde{\mathbf{A}}\mathbf{x}(\tau) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(\tau) \\ &= \hat{\mathbf{A}}\mathbf{x}(\tau) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(\tau),\end{aligned}\tag{F.1}$$

where $\mathbf{x}(\tau) = [x_1(\tau), x_2(\tau), \dots, x_N(\tau)]^T$ is a $N \times 1$ vector of the node states of N normal agents, $\tilde{\mathbf{u}}(\tau) = [u_1(\tau), u_2(\tau), \dots, u_M(\tau)]^T$ is the $M \times 1$ vector of input control signals, $\tilde{\mathbf{A}}$ is the network connection matrix where $\tilde{a}_{ij}$ is non-zero if there is a directed link from normal agent node j to node i , otherwise $\tilde{a}_{ij} = 0$, $\tilde{\mathbf{B}}$ describes the where in the network are the control signals attached, with $\tilde{b}_{ij} = 1$ if control signal j attaches to node i , otherwise it is zero. $\tilde{\mathbf{S}}^{-1} = \text{diag}\{\frac{1}{\tilde{s}_1}, \frac{1}{\tilde{s}_2}, \dots, \frac{1}{\tilde{s}_N}\}$, where $\tilde{s}_i = \sum_{j=1}^{\tilde{n}_i} \tilde{a}_{ij}$, and node i has $\tilde{n}_i$ nearest neighbors. Note that $\hat{\mathbf{A}}$ is different from taking the $N \times N$ block from $\mathbf{A}$ in Eq. (4.26).

The energy cost needed to control this network is $J = \sum_{\tau=0}^{T_f-1} \tilde{\mathbf{u}}^T(\tau)\tilde{\mathbf{u}}(\tau)$, and the energy-optimal control signal for controlling this system is [6, 121]

$$\tilde{\mathbf{u}}^*(\tau) = \tilde{\mathbf{B}}^T(\hat{\mathbf{A}}^T)^{T_f-\tau-1}\hat{\mathbf{W}}^{-1}(\mathbf{x}_f - \hat{\mathbf{A}}^{T_f}\mathbf{x}_0),\tag{F.2}$$

where $\mathbf{x}_0 = [x_1(0), x_2(0), \dots, x_N(0)]^T$ is the initial state vector, $\mathbf{x}_f = [x_1(T_f), x_2(T_f), \dots, x_N(T_f)]^T = [c, c, \dots, c]^T$ is the final state vector, where the node states are driven toward consensus c , and

$\hat{\mathbf{W}} = \sum_{\tau=0}^{T_f-1} \hat{\mathbf{A}}^{T_f-\tau-1}\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T(\hat{\mathbf{A}}^T)^{T_f-\tau-1}$ is the controllability Gramian. Assuming $\mathbf{x}_0 = \mathbf{0}$ and

substituting Eq. (F.2) into $J = \sum_{\tau=0}^{T_f-1} \tilde{\mathbf{u}}^T(\tau)\tilde{\mathbf{u}}(\tau)$, the no-zealot energy cost is

$$\mathbf{x}_f^T \hat{\mathbf{W}}^{-1} \mathbf{x}_f.\tag{F.3}$$

In continuous-time linear dynamics, the no-zealot energy cost is the energy cost when $z = 0$. However, for the discrete-time with conformity system, the no-zealot energy cost is not the same as when $z = 0$.

Appendix G

Chapter 4: Conformity dynamics one driver analytical $\tilde{\mathbf{M}}^{-1}$

Assuming that the sole control signal is attached to normal agent node h , then adapting from Ref. [34]:

$$\tilde{\mathbf{M}}^{-1}(i, j) = \begin{cases} \frac{(-1)^{(N+1)} \prod_{k=1}^N (1 - \Lambda_i \Lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^N (1 - \Lambda_l \Lambda_i)}{[\tilde{\mathbf{P}}^{-1}]_{jh} \tilde{v}_{hi} \prod_{\substack{k=1 \\ k \neq i}}^N (\Lambda_i - \Lambda_k) \prod_{\substack{l=1 \\ l \neq i}}^N (\Lambda_l - \Lambda_i)}, & i = j, \\ \left[\frac{(-1)^N}{(\Lambda_j - \Lambda_i) \tilde{v}_{hi} [\tilde{\mathbf{P}}^{-1}]_{jh}} \right] \left[\frac{\prod_{k=1}^N (1 - \Lambda_i \Lambda_k) \prod_{\substack{k=1 \\ k \neq i}}^N (1 - \Lambda_k \Lambda_j)}{\prod_{\substack{k=1 \\ k \neq i}}^N (\Lambda_i - \Lambda_k) \prod_{\substack{k=1 \\ k \neq j, i}}^N (\Lambda_k - \Lambda_j)} \right], & i \neq j \end{cases} \quad (\text{G.1})$$

Appendix H

Chapter 4: Supplementary information

The supplementary material details how the analytical energy cost equations presented in the chapter main text were derived. Recall from the chapter main text that within the full network, there are N normal agents and one zealot node, which is the $n = N + 1$ -th node. Further, most derivations are done with index notation of matrix multiplications, where $[\mathbf{AB}]_{ij} = \sum_a \sum_b [\mathbf{A}]_{ia} [\mathbf{B}]_{aj} = [\mathbf{A}]_{ia} [\mathbf{B}]_{aj}$. Wherever convenient, the index summation notations will be dropped for conciseness.

H.1 Continuous-time linear dynamics

H.1.1 Generic energy cost

The energy cost is defined to be [71]

$$J = \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt, \quad (\text{H.1})$$

whereupon optimization leads to the energy-optimal target control signal [8]

$$\begin{aligned} \mathbf{u}^*(t) &= \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} (\mathbf{y}_f - \mathbf{C} e^{\mathbf{A}(t_f-t_0)} \mathbf{x}_0) \\ &= \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{y}_f - \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t_0)} \mathbf{x}_0, \end{aligned} \quad (\text{H.2})$$

where $\mathbf{W} = \int_{t_0}^{t_f} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} dt$ is the $n \times n$ controllability Gramian of the full network (all N normal agents, and the n -th node, the zealot node), $\mathbf{x}_0 = [0, 0, \dots, 0, z]^T$ is $n \times 1$ initial state vector, and $\mathbf{y}_f = [c, c, \dots, c]^T$ is the $N \times 1$ output final state vector. Respectively, z and c are the zealot node's fixed belief, and c is the consensus opinion that the network is being driven toward. The transpose of (4.3) is

$$(\mathbf{u}^*)^T(t) = \mathbf{y}_f^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} - \mathbf{x}_0^T e^{\mathbf{A}^T(t_f-t_0)} \mathbf{C}^T (\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B}, \quad (\text{H.3})$$

where $(\mathbf{C} \mathbf{W} \mathbf{C}^T)$ is symmetric, real, non-negative, and invertible when the system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is controllable [8], so the transpose of $(\mathbf{C} \mathbf{W} \mathbf{C}^T)^{-1}$ is itself. Substituting Eqs. (H.2) and (H.3) into Eq. (H.1), and setting $t_0 = 0$, the energy cost for control using the energy-optimal control

signals becomes

$$\begin{aligned}
\mathcal{E} &= \int_0^{t_f} \left[\mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} \right] \\
&\quad \times \left[\mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f - \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \right] dt \\
&= \int_0^{t_f} \left\{ \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f \right. \\
&\quad - \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \\
&\quad - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f \\
&\quad \left. + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A}(t_f-t)} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T(t_f-t)} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \right\} dt \\
&= \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} (\mathbf{CWC}^T) (\mathbf{CWC}^T)^{-1} \mathbf{y}_f - \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} (\mathbf{CWC}^T) (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \\
&\quad - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} (\mathbf{CWC}^T) (\mathbf{CWC}^T)^{-1} \mathbf{y}_f \\
&\quad + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} (\mathbf{CWC}^T) (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \\
&= \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f - \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f \\
&\quad + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \\
&= \mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f - 2 \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0,
\end{aligned} \tag{H.4}$$

where $(\mathbf{CWC}^T)^{-1}(\mathbf{CWC}^T) = \mathbf{I}_N$, and $\mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0$ and $\mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f$ are 1×1 scalar quantities, symmetric to each other and numerically the same so $-\mathbf{y}_f^T (\mathbf{CWC}^T)^{-1} \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 - \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f = 2 \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T (\mathbf{CWC}^T)^{-1} \mathbf{y}_f$. Thus, this was how Eq. (4.4) of the chapter main text was obtained.

Because all N normal agents are to be controlled, and the zealot node is not to be controlled, the $N \times n$ $\mathbf{C}$ matrix comprises $N \times N$ identity $\mathbf{I}_N$ on the first $N \times N$ block, with the last column of $\mathbf{C}$ being all zeros. In other words, $[\mathbf{C}]_{ij} = [\mathbf{C}]_{ii} = 1$ for $i = 1, 2, \dots, N$. The transpose, $\mathbf{C}^T$ is $n \times N$, so $[\mathbf{C}^T]_{ij} = [\mathbf{C}^T]_{jj} = 1$ for $j = 1, 2, \dots, N$. Therefore,

$$[(\mathbf{CWC}^T)]_{ij} = [\mathbf{C}]_{ia} [\mathbf{W}]_{ab} [\mathbf{C}^T]_{bj} = [\mathbf{C}]_{ii} [\mathbf{W}]_{ij} [\mathbf{C}^T]_{jj} = \mathbf{W}_{ij} = \tilde{\mathbf{W}}_{ij}, \tag{H.5}$$

for $i, j = 1, 2, \dots, N$, effectively the same as removing n -th row/column off the $n \times n$ controllability Gramian matrix $\mathbf{W}$. This turns out to be the same as the controllability Gramian of the $N \times N$ system without the zealot node (use $\sim$ to denoted reduced matrix), which is

$$\tilde{\mathbf{W}} = \int_0^{t_f} e^{\tilde{\mathbf{A}}(t_f-t)} \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T e^{\tilde{\mathbf{A}}^T(t_f-t)} dt, \tag{H.6}$$

where $\tilde{\mathbf{A}}$ is the same as $\mathbf{A}$ with the n -th row/column removed, and $\tilde{\mathbf{B}}$ is the same as $\mathbf{B}$ with the n -th row removed. Essentially, $(\mathbf{CWC}^T)$ decouples the zealot's connections and calculates the controllability properties of all N normal agents only.

Since $\tilde{\mathbf{A}}$ describes the N normal agents networked interactions, which are undirected links, it is symmetric and eigen-decomposed as $\tilde{\mathbf{A}} = \tilde{\mathbf{P}} \tilde{\mathbf{D}} \tilde{\mathbf{P}}^T$, where $\tilde{\mathbf{P}}$ the eigenvectors matrix is orthonormal such that $\tilde{\mathbf{P}}^{-1} = \tilde{\mathbf{P}}^T$, and $\tilde{\mathbf{D}}_{ii} = \lambda_i$ contains the eigenvalues of the normal agents. Further, assume that the eigenvalues are ordered $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$, and all negative to model

system stability. Substituting the eigen-decompositions into (H.7), the reduced controllability Gramian can be expressed in Hadamard product form [3–5]

$$\begin{aligned}
[\tilde{\mathbf{W}}]_{ij} &= \int_0^{t_f} [\tilde{\mathbf{P}} e^{\tilde{\mathbf{D}}(t_f-t)} \tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}} e^{\tilde{\mathbf{D}}(t_f-t)} \tilde{\mathbf{P}}^T]_{ij} dt \\
&= \int_0^{t_f} [\tilde{\mathbf{P}}]_{ia} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{ab} [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{bc} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{cd} [\tilde{\mathbf{P}}^T]_{dj} dt \\
&= \int_0^{t_f} [\tilde{\mathbf{P}}]_{ia} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{aa} [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ac} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{cc} [\tilde{\mathbf{P}}^T]_{cj} dt \\
&= [\tilde{\mathbf{P}}]_{ia} \left\{ \int_0^{t_f} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{aa} [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ac} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{cc} dt \right\} [\tilde{\mathbf{P}}^T]_{cj},
\end{aligned} \tag{H.7}$$

where, because $\tilde{\mathbf{D}}$ is a diagonal matrix, in $\tilde{\mathbf{D}}_{ab}$, index a must be equal to index b , so $\tilde{\mathbf{D}}_{ab} = \tilde{\mathbf{D}}_{aa}$. Denote the middle portion as $\tilde{\mathbf{M}}_{ac} = \int_0^{t_f} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{aa} [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ac} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{cc} dt$, and since indices a and c are arbitrary, rename them as $a = i$, and $c = j$, it evaluates:

$$\begin{aligned}
\tilde{\mathbf{M}}_{ij} &= \int_0^{t_f} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{ii} [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ij} [e^{\tilde{\mathbf{D}}(t_f-t)}]_{jj} dt \\
&= [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ij} \int_0^{t_f} e^{\lambda_i(t_f-t)} e^{\lambda_j(t_f-t)} dt \\
&= [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ij} \int_0^{t_f} e^{(\lambda_i + \lambda_j)(t_f-t)} dt \\
&= [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ij} \left[\left(\frac{1}{\lambda_i + \lambda_j} \right) e^{(\lambda_i + \lambda_j)t_f} - 1 \right]_{ij} \\
&= \tilde{\mathbf{Q}}_{ij} \tilde{\mathbf{F}}_{ij},
\end{aligned} \tag{H.8}$$

where $\tilde{\mathbf{Q}}_{ij} = [\tilde{\mathbf{P}}^T \tilde{\mathbf{B}} \tilde{\mathbf{B}}^T \tilde{\mathbf{P}}]_{ij}$ and $\tilde{\mathbf{F}}_{ij} = \frac{e^{(\lambda_i + \lambda_j)t_f} - 1}{\lambda_i + \lambda_j}$. Thus, $\tilde{\mathbf{M}} = \tilde{\mathbf{Q}} \circ \tilde{\mathbf{F}}$, where $\circ$ denotes Hadamard product. Piecing (H.7) and (H.8) together, the reduced controllability Gramian in Hadamard product analytical form is thus $\tilde{\mathbf{W}} = \tilde{\mathbf{P}}[\tilde{\mathbf{Q}} \circ \tilde{\mathbf{F}}]\tilde{\mathbf{P}}^T = \tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{P}}^T$.

The inverse of the reduced controllability Gramian is $\tilde{\mathbf{W}}^{-1} = (\tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{P}}^T)^{-1} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T$.

Because $\mathbf{CWC}^T = \tilde{\mathbf{W}}$, $(\mathbf{CWC}^T)^{-1} = \tilde{\mathbf{W}}^{-1} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T$. Therefore, substituting into (H.4):

$$\begin{aligned}
\mathcal{E} &= \mathbf{y}_f^T \tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T \mathbf{y}_f - 2\mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T \tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T \mathbf{y}_f + \mathbf{x}_0^T e^{\mathbf{A}^T t_f} \mathbf{C}^T \tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T \mathbf{C} e^{\mathbf{A} t_f} \mathbf{x}_0 \\
&= [\mathbf{y}_f^T]_{1a} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ab} [\mathbf{y}_f]_{b1} - 2[\mathbf{x}_0^T]_{1a} [e^{\mathbf{A}^T t_f}]_{ab} [\mathbf{C}^T]_{bc} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{cd} [\mathbf{y}_f]_{d1} \\
&\quad + [\mathbf{x}_0^T]_{1a} [e^{\mathbf{A}^T t_f}]_{ab} [\mathbf{C}^T]_{bc} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{cd} [\mathbf{C}]_{de} [e^{\mathbf{A} t_f}]_{ef} [\mathbf{x}_0]_{f1} \\
&= c^2 [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ab} - 2cz [e^{\mathbf{A}^T t_f}]_{nc} [\mathbf{C}^T]_{cc} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{cd} + z^2 [e^{\mathbf{A}^T t_f}]_{nc} [\mathbf{C}^T]_{cc} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{cd} [\mathbf{C}]_{dd} [e^{\mathbf{A} t_f}]_{dn} \\
&= c^2 [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ab} - 2cz [e^{\mathbf{A}^T t_f}]_{nc} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{cd} + z^2 [e^{\mathbf{A}^T t_f}]_{nc} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{cd} [e^{\mathbf{A} t_f}]_{dn} \\
&= c^2 [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} - 2cz [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} + z^2 [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} [e^{\mathbf{A} t_f}]_{jn} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} + z^2 \sum_{i=1}^N \sum_{j=1}^N [e^{\mathbf{A}^T t_f}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} [e^{\mathbf{A} t_f}]_{jn} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [\mathbf{V} e^{\mathbf{D} t_f} \mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} \\
&\quad + z^2 \sum_{i=1}^N \sum_{j=1}^N [\mathbf{V} e^{\mathbf{D} t_f} \mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^T]_{ij} [\mathbf{P} e^{\mathbf{D} t_f} \mathbf{P}^{-1}]_{jn},
\end{aligned} \tag{H.9}$$

recovering Eq. (4.8) in the chapter main text. Because the indices a , b , c , and d are arbitrary, they were renamed to be i and j . $\mathbf{x}_0 = [0, 0, \dots, 0, z]^T$ is a vector containing all zeros except the last n -th element, where it is a constant z . Thus, all terms $[\mathbf{x}_0]_{i1} = 0$ for $i = 1, 2, \dots, N$, except at $i = n$, where it is z . $\mathbf{y}_f = [c, c, \dots, c]^T$ is a vector containing all constant c , and so any $[\mathbf{y}_f]_{i1}$ can just be treated as a constant. Further, for the full $n \times n$ network matrix, which is non-symmetric, the eigen-decompositions $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ were used in (H.9), where $\mathbf{D} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N, 0\}$, and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \leq \lambda_n = 0$ where all λ_i (for $i = 1, 2, \dots, N$) are all negative to model system stability, while the zealot node's eigenvalue $\lambda_n = 0$ to model unchanging opinion z .

Eq. (H.9) is nearly fully analytical, depending on whether or not an analytical $\tilde{\mathbf{M}}^{-1}$ exists, which is dependent on the final control time regime and number of drivers. Wherever possible, analytical $\tilde{\mathbf{M}}^{-1}$ will be replaced in Eq. (H.9) to derive the fully analytical energy cost equations. All derived analytical energy cost equations have been validated against numerical computations, as corroborated in the chapter main text figures.

H.1.2 N drivers

As explained in the chapter main text, when using N drivers to control the network such that all N normal agents each receive a single control signal, then $\tilde{\mathbf{B}} = \mathbf{I}_N$, and $\tilde{\mathbf{Q}} = \mathbf{I}_N$, leading to $\tilde{\mathbf{M}} = \tilde{\mathbf{F}}_{ii}$, a diagonal matrix. Thus, $\tilde{\mathbf{M}}^{-1}$, the inverse of $\tilde{\mathbf{M}}$, is also a diagonal matrix.

Large t_f regime

Given that $\mathbf{D} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N, 0\}$, where all λ_i (for $i = 1, 2, \dots, N$) are negative owing to system stability. Thus $e^{\mathbf{D} t_f} = e^{\lambda_i t_f} = 0$ for the first N main diagonal entries, with the n -th entry being $e^0 = 1$. Thus $e^{\mathbf{D} t_f} = \mathbf{J}^{nn}$, a single-entry matrix [90], which is zero everywhere, except at indices (n, n) , where it is one. Subsequently, $[\mathbf{V} e^{\mathbf{D} t_f} \mathbf{V}^{-1}]_{ni} = [\mathbf{V}]_{na} [\mathbf{J}^{nn}]_{aa} [\mathbf{V}^{-1}]_{ai} = [\mathbf{V}]_{nn} [\mathbf{V}^{-1}]_{ni} = [\mathbf{V}^{-1}]_{ni}$, where it was observed through computation that $\mathbf{V}_{nn}$ is always one, and $[\mathbf{P} e^{\mathbf{D} t_f} \mathbf{P}^{-1}]_{jn} = [\mathbf{P}]_{ja} [\mathbf{J}^{nn}]_{aa} [\mathbf{P}^{-1}]_{an} = [\mathbf{P}]_{jn} [\mathbf{P}^{-1}]_{nn}$. From the chapter main text, in the

large t_f regime, $[\tilde{\mathbf{M}}^{-1}]_{ii} \approx -2\lambda_i$, which when substituted into (H.9) leads to

$$\begin{aligned}
\mathcal{E}_N^{\text{large } t_f} &= c^2 [\tilde{\mathbf{P}}]_{ia} [\tilde{\mathbf{M}}^{-1}]_{aa} [\tilde{\mathbf{P}}^T]_{aj} - 2cz [\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}]_{ia} [\tilde{\mathbf{M}}^{-1}]_{aa} [\tilde{\mathbf{P}}^T]_{aj} \\
&\quad + z^2 [\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}]_{ia} [\tilde{\mathbf{M}}^{-1}]_{aa} [\tilde{\mathbf{P}}^T]_{aj} [\mathbf{P}]_{jn} [\mathbf{P}^{-1}]_{nn} \\
&= c^2 \tilde{p}_{ia} (-2\lambda_a) \tilde{p}_{ja} - 2cz [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ia} (-2\lambda_a) \tilde{p}_{ja} + z^2 [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ia} (-2\lambda_a) \tilde{p}_{ja} p_{jn} [\mathbf{P}^{-1}]_{nn} \\
&= -2c^2 \sum_{i,k,j} \tilde{p}_{ik} \lambda_k \tilde{p}_{jk} + 4cz \sum_{i,k,j} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ik} \lambda_k \tilde{p}_{jk} - 2z^2 [\mathbf{P}^{-1}]_{nn} \sum_{i,k,j} [\mathbf{V}^{-1}]_{ni} \tilde{p}_{ik} \lambda_k \tilde{p}_{jk} p_{jn},
\end{aligned} \tag{H.10}$$

where $\sum_{i,k,j} = \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N$, and because index a is arbitrary, it was renamed to be index k .

Small t_f regime

In the small t_f regime, replace $[\mathbf{V}e^{\mathbf{D}t_f}\mathbf{V}^{-1}]_{ni} = [e^{\mathbf{A}^T t_f}]_{ni}$ and $[\mathbf{P}e^{\mathbf{D}t_f}\mathbf{P}^{-1}]_{jn} = [e^{\mathbf{A} t_f}]_{jn}$ in (H.9). Then, approximate first-order Taylor expansion $[e^{\mathbf{A}^T t_f}]_{ni} \approx [\mathbf{I}_n + \mathbf{A}^T t_f]_{ni}$ and $[e^{\mathbf{A} t_f}]_{jn} \approx [\mathbf{I}_n + \mathbf{A} t_f]_{jn}$. Further, as discussed in chapter main text, $[\tilde{\mathbf{M}}^{-1}]_{ii} \approx t_f^{-1}$, thus the analytical energy cost is

$$\begin{aligned}
\mathcal{E}_N^{\text{small } t_f} &\approx c^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kk} [\tilde{\mathbf{P}}^T]_{kj} - 2cz \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{I}_n + \mathbf{A}^T t_f]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kk} [\tilde{\mathbf{P}}^T]_{kj} \\
&\quad + z^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{I}_n + \mathbf{A}^T t_f]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kk} [\tilde{\mathbf{P}}^T]_{kj} [\mathbf{I}_n + \mathbf{A} t_f]_{jn} \\
&\approx c^2 t_f^{-1} \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{P}} \tilde{\mathbf{P}}^T]_{ij} - 2cz t_f^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{I}_n]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{P}}^T]_{kj} - 2cz \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{A}^T]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{P}}^T]_{kj} (t_f t_f^{-1}) \\
&\quad + z^2 t_f^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{I}_n]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{P}}^T]_{kj} [\mathbf{I}_n + \mathbf{A} t_f]_{jn} + z^2 (t_f^{-1} t_f) \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{A}^T]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{P}}^T]_{kj} [\mathbf{I}_n + \mathbf{A} t_f]_{jn} \\
&\approx c^2 t_f^{-1} \sum_{i=1}^N \sum_{j=1}^N [\mathbf{I}_N]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [\mathbf{A}^T]_{ni} [\mathbf{I}_N]_{ij} + z^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{A}^T]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{P}}^T]_{kj} [\mathbf{I}_N]_{jn} \\
&\quad + z^2 t_f \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [\mathbf{A}^T]_{ni} [\mathbf{I}_N]_{ij} [\mathbf{A}]_{jn} \\
&\approx c^2 t_f^{-1} \sum_{i=1}^N [\mathbf{I}_N]_{ii} - 2cz \sum_{i=1}^N [\mathbf{A}^T]_{ni} [\mathbf{I}_N]_{ii} + z^2 t_f \sum_{i=1}^N [\mathbf{A}^T]_{ni} [\mathbf{I}_N]_{ii} [\mathbf{A}]_{in} \\
&\approx c^2 t_f^{-1} N - 2czr + z^2 r t_f,
\end{aligned} \tag{H.11}$$

where $\sum_{i=1}^N [\mathbf{A}^T]_{ni} = r$ and $\sum_{i=1}^N [\mathbf{A}]_{in} = r$ as they sum over the final row or column denoting the zealot's connections to normal agents, where there are r integer number of zealot-influenced nodes. The cancellations arise from the fact that the running indices i or j iterate from 1 to N , so $[\mathbf{I}_n]_{ni}$ or $[\mathbf{I}_n]_{jn}$ will never satisfy that $i = n$ or $j = n$, and thus $[\mathbf{I}_n]_{ni} = 0$ for all i ; likewise, $[\mathbf{I}_n]_{jn} = 0$ for all j .

H.1.3 One driver, large t_f

When controlling a network with a single control signal, where it is attached to an arbitrary node, node h , $\tilde{\mathbf{B}} \in \mathbb{R}^{N \times 1}$, where $\tilde{b}_{h1} = 1$, leading to $\tilde{\mathbf{Q}}_{ij} = \tilde{p}_{hi}\tilde{p}_{hj}$. For stable system with eigenvalues all negative, such that $\tilde{\mathbf{A}}$ is negative definite (ND), $\tilde{\mathbf{M}}^{-1}$ has an analytical expression in the large t_f limit [5]:

$$\tilde{\mathbf{M}}^{-1}(i, j) = \frac{-4\lambda_i\lambda_j}{\tilde{p}_{hi}\tilde{p}_{hj}(\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k}. \quad (\text{H.12})$$

Substituting (H.12), $[\mathbf{V}e^{\mathbf{D}t_f}\mathbf{V}^{-1}]_{ni} = [\mathbf{V}^{-1}]_{ni}$, and $[\mathbf{P}e^{\mathbf{D}t_f}\mathbf{P}^{-1}]_{jn} = [\mathbf{P}]_{jn}[\mathbf{P}^{-1}]_{nn}$ into (H.9), the analytical energy cost is

$$\begin{aligned} \mathcal{E}_1^{\text{large } t_f} &= c^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kl} [\tilde{\mathbf{P}}^T]_{lj} - 2cz \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N [\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kl} [\tilde{\mathbf{P}}^T]_{lj} \\ &\quad + z^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N [\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kl} [\tilde{\mathbf{P}}^T]_{lj} [\mathbf{P}]_{jn} [\mathbf{P}^{-1}]_{nn} \\ &= c^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kl} [\tilde{\mathbf{P}}]_{jl} - 2cz \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N [\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kl} [\tilde{\mathbf{P}}]_{jl} \\ &\quad + z^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N [\mathbf{V}^{-1}]_{ni} [\tilde{\mathbf{P}}]_{ik} [\tilde{\mathbf{M}}^{-1}]_{kl} [\tilde{\mathbf{P}}]_{jl} [\mathbf{P}]_{jn} [\mathbf{P}^{-1}]_{nn}, \end{aligned} \quad (\text{H.13})$$

For convenience, rename the indices $k \rightarrow i$, $l \rightarrow j$, $i \rightarrow l$, and $j \rightarrow m$, thus

$$\begin{aligned} \mathcal{E}_1^{\text{large } t_f} &= c^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \sum_{m=1}^N \tilde{p}_{li}\tilde{p}_{mj} [\tilde{\mathbf{M}}^{-1}]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \sum_{m=1}^N [\mathbf{V}^{-1}]_{nl} \tilde{p}_{li}\tilde{p}_{mj} [\tilde{\mathbf{M}}^{-1}]_{ij} \\ &\quad + z^2 \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \sum_{m=1}^N [\mathbf{V}^{-1}]_{nl} \tilde{p}_{li}\tilde{p}_{mj} p_{mn} [\mathbf{P}^{-1}]_{nn} [\tilde{\mathbf{M}}^{-1}]_{ij} \\ &= c^2 \sum_{i,j,l,m} \tilde{p}_{li}\tilde{p}_{mj} \frac{-4\lambda_i\lambda_j}{\tilde{p}_{hi}\tilde{p}_{hj}(\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k} \\ &\quad - 2cz \sum_{i,j,l,m} [\mathbf{V}^{-1}]_{nl} \tilde{p}_{li}\tilde{p}_{mj} \frac{-4\lambda_i\lambda_j}{\tilde{p}_{hi}\tilde{p}_{hj}(\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k} \\ &\quad + z^2 \sum_{i,j,l,m} [\mathbf{V}^{-1}]_{nl} \tilde{p}_{li}\tilde{p}_{mj} p_{mn} [\mathbf{P}^{-1}]_{nn} \frac{-4\lambda_i\lambda_j}{\tilde{p}_{hi}\tilde{p}_{hj}(\lambda_i + \lambda_j)} \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\lambda_j + \lambda_k}{\lambda_j - \lambda_k}, \end{aligned} \quad (\text{H.14})$$

where $\sum_{i,j,l,m} = \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \sum_{m=1}^N$, thus recovering Eq. (4.14) of the chapter main text.

H.2 Discrete-time linear dynamics with conformity behavior

H.2.1 Generic energy cost

The energy cost required to control the discrete-time linear dynamics system is by definition [6, 30, 106, 121]

$$J = \sum_{\tau=0}^{T_f-1} u^T(\tau)u(\tau), \quad (\text{H.15})$$

which upon optimizing, leads to the energy-optimal discrete-time target control signal[6, 121] as the $M \times 1$ vector

$$\begin{aligned} \mathbf{u}^*(\tau) &= \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{y}_f - \mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0) \\ &= \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f - \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0, \end{aligned} \quad (\text{H.16})$$

where $\mathbf{W} = \sum_{\tau=0}^{T_f-1} \mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}$ is the $n \times n$ discrete-time full controllability Gramian of the conformity-based system (with all N normal agents, as well as the n -th node, the zealot node), $\mathbf{x}_0 = [x_1(0), x_2(0), \dots, x_N(0), z]^T = [0, 0, \dots, 0, z]^T$ is the $n \times 1$ initial state vector, where the initial node states of the normal agents are assumed to all be zeros, $\mathbf{y}_f = [c, c, \dots, c]^T$ is the $N \times 1$ final output state vector, where normal agents are all controlled toward consensus c . The transpose of Eq. (H.16) is

$$(\mathbf{u}^*)^T(\tau) = \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B} - \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B}, \quad (\text{H.17})$$

where $(\mathbf{C}\mathbf{W}\mathbf{C}^T)$ is invertible if system is controllable. Further, $(\mathbf{C}\mathbf{W}\mathbf{C}^T)$ is symmetric, so the transpose of $(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}$ is itself. Substituting Eqs. (H.16) and (H.17) into Eq. (H.15), the energy cost needed for controlling the conformity network with energy-optimal control signals is

$$\begin{aligned} \mathcal{E} &= \sum_{\tau=0}^{T_f-1} \left[\mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B} - \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B} \right] \\ &\quad \times \left[\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f - \mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 \right] \\ &= \sum_{\tau=0}^{T_f-1} \left\{ \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f \right. \\ &\quad - \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 \\ &\quad - \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f \\ &\quad \left. + \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f-\tau-1}\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^{T_f-\tau-1}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 \right\} \\ &= \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{W}\mathbf{C}^T)(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f - \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{W}\mathbf{C}^T)(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 \\ &\quad - \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{W}\mathbf{C}^T)(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f + \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{W}\mathbf{C}^T)(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 \\ &= \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f - \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 - \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f + \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 \\ &= \mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f - 2\mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f + \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0, \end{aligned} \quad (\text{H.18})$$

where $(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}(\mathbf{C}\mathbf{W}\mathbf{C}^T) = \mathbf{I}_N$, and $\mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0$ and $\mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f$ are 1×1 scalar quantities which are symmetric to each other and numerically the same so $-\mathbf{y}_f^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{C}\mathbf{A}^{T_f}\mathbf{x}_0 - \mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f = -2\mathbf{x}_0^T(\mathbf{A}^T)^{T_f}\mathbf{C}^T(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1}\mathbf{y}_f$. Thus, this was how Eq. (4.29) of the chapter main text was derived.

The $(\mathbf{C}\mathbf{W}\mathbf{C}^T)$ term of (H.18) can be simplified further through expressing it as the Hadamard product analytical form. Because all N normal agents are to be controlled, and the zealot node is not to be controlled, the $N \times n$ $\mathbf{C}$ matrix comprises $N \times N$ identity $\mathbf{I}_N$ on the first $N \times N$ block, with the last column of $\mathbf{C}$ being all zeros. In other words, $[\mathbf{C}]_{ij} = [\mathbf{C}]_{ii} = 1$ for $i = 1, 2, \dots, N$. The transpose $\mathbf{C}^T$ is $n \times N$, so $[\mathbf{C}^T]_{ij} = [\mathbf{C}^T]_{jj} = 1$ for $j = 1, 2, \dots, N$. Therefore,

$$[(\mathbf{C}\mathbf{W}\mathbf{C}^T)]_{ij} = [\mathbf{C}]_{ia}[\mathbf{W}]_{ab}[\mathbf{C}^T]_{bj} = [\mathbf{C}]_{ii}[\mathbf{W}]_{ij}[\mathbf{C}^T]_{jj} = \tilde{\mathbf{W}}_{ij} = \tilde{\mathbf{W}}_{ij}, \quad (\text{H.19})$$

where the $N \times N$ reduced controllability Gramian is obtained by removing the n -th row/column off the full controllability Gramian matrix $\mathbf{W}$. $\sim$ terms denote matrix terms which are reduced. It turns out that, in effect,

$$\tilde{\mathbf{W}} = \sum_{\tau=0}^{T_f-1} \tilde{\mathbf{A}}^{T_f-\tau-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T (\tilde{\mathbf{A}}^T)^{T_f-\tau-1} \quad (\text{H.20})$$

where $\tilde{\mathbf{A}}$ is obtained by taking the n -th row/column off $\mathbf{A}$, and $\tilde{\mathbf{B}}$ is the same as $\mathbf{B}$ with the n -th row removed. It must be emphasized that, unlike the continuous-time dynamical system, where $(\mathbf{C}\mathbf{W}\mathbf{C}^T)$ decouples the zealot's connections and calculates the controllability properties of all N normal agents only, $(\mathbf{C}\mathbf{W}\mathbf{C}^T)$ decouples the n -th row/column but not the zealot's connections. The $N \times N$ conformity-based discrete-time system without zealots is

$$\begin{aligned} \tilde{\mathbf{x}}(\tau+1) &= \tilde{\mathbf{S}}^{-1} \tilde{\mathbf{A}}\tilde{\mathbf{x}}(\tau) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(\tau) \\ &= \hat{\mathbf{A}}\mathbf{x}(\tau) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(\tau), \end{aligned} \quad (\text{H.21})$$

where $\hat{\mathbf{A}} = \tilde{\mathbf{S}}^{-1} \tilde{\mathbf{A}}$ captures the networked interactions with conformity of all N normal agents, $\tilde{\mathbf{x}}(\tau) = [x_1(\tau), x_2(\tau), \dots, x_N(\tau)]^T$ is the reduced state vector, $\tilde{\mathbf{S}}^{-1} = \text{diag}\{\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_N}\}$, with $\tilde{s}_i = \sum_{j=1}^{\tilde{n}_i} \tilde{a}_{ij}$, and node i has $\tilde{n}_i$ nearest neighbors. The reduced $N \times N$ conformity-based discrete-time system with zealots is obtained by taking the n -th row/column off the full $n \times n$ system:

$$\begin{aligned} \mathbf{x}(\tau+1) &= \mathbf{S}^{-1} \mathbf{A}\mathbf{x}(\tau) + \mathbf{B}\mathbf{u}(\tau) \\ &= \hat{\mathbf{A}}\mathbf{x}(\tau) + \mathbf{B}\mathbf{u}(\tau), \end{aligned} \quad (\text{H.22})$$

where $\mathbf{S}^{-1} = \text{diag}\{\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_N}, 1\}$ is a $n \times n$ diagonal matrix which holds $\frac{1}{s_i}$ (for $i = 1, 2, \dots, N$) on its main diagonals, with the n -th entry being 1, and $s_i = \sum_{j=1}^{n_i} a_{ij}$ sums over the full $\mathbf{A}$ matrix, inclusive of the zealot node's directed connections. Thus, the reduced matrix $\tilde{\mathbf{A}}$ is not same as $\hat{\mathbf{A}}$, and $(\mathbf{C}\mathbf{W}\mathbf{C}^T)$ does not decouple the zealot's connections off.

Define the non-symmetric reduced matrix and its eigen-decomposition $\tilde{\mathbf{A}} = \tilde{\mathbf{P}}\tilde{\mathbf{D}}\tilde{\mathbf{P}}^{-1}$, and $\tilde{\mathbf{A}}^T = \tilde{\mathbf{V}}\tilde{\mathbf{D}}\tilde{\mathbf{V}}^{-1}$, where $\tilde{\mathbf{D}}_{ii} = \Lambda_i$ is the $N \times N$ diagonal matrix containing the eigenvalues of the discrete-time network with conformity. With the eigen-decompositions substituted into (H.20), the discrete-time reduced controllability Gramian can be expressed in Hadamard product form

[3–6, 34]:

$$\begin{aligned}
\tilde{\mathbf{W}} &= \sum_{\tau=0}^{T_f-1} (\tilde{\mathbf{P}}\tilde{\mathbf{D}}\tilde{\mathbf{P}}^{-1})^{T_f-\tau-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T (\tilde{\mathbf{V}}\tilde{\mathbf{D}}\tilde{\mathbf{V}}^{-1})^{T_f-\tau-1} \\
&= \sum_{\tau=0}^{T_f-1} \tilde{\mathbf{P}}\tilde{\mathbf{D}}^{T_f-\tau-1} \tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}\tilde{\mathbf{D}}^{T_f-\tau-1} \tilde{\mathbf{V}}^{-1} \\
&= \tilde{\mathbf{P}} \left(\sum_{\tau=0}^{T_f-1} \tilde{\mathbf{D}}^\tau \tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}\tilde{\mathbf{D}}^\tau \right) \tilde{\mathbf{V}}^{-1} \\
&= \tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{V}}^{-1},
\end{aligned} \tag{H.23}$$

where the change of variable $\tau = T_f - \tau - 1$ was used, and define $\tilde{\mathbf{M}} = \sum_{\tau=0}^{T_f-1} \tilde{\mathbf{D}}^\tau \tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}\tilde{\mathbf{D}}^\tau$. Next,

$$\begin{aligned}
[\tilde{\mathbf{M}}]_{ij} &= \sum_a \sum_b \sum_{\tau=0}^{T_f-1} [\tilde{\mathbf{D}}^\tau]_{ia} [\tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}]_{ab} [\tilde{\mathbf{D}}^\tau]_{bj} \\
&= \sum_{\tau=0}^{T_f-1} [\tilde{\mathbf{D}}^\tau]_{ii} [\tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}]_{ij} [\tilde{\mathbf{D}}^\tau]_{jj} \\
&= [\tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}]_{ij} \left(\sum_{\tau=0}^{T_f-1} \Lambda_i^\tau \Lambda_j^\tau \right) \\
&= [\tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}]_{ij} \left(\sum_{\tau=0}^{T_f-1} (\Lambda_i \Lambda_j)^\tau \right) \\
&= [\tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}]_{ij} \frac{1 - (\Lambda_i \Lambda_j)^{T_f}}{1 - \Lambda_i \Lambda_j} \\
&= \tilde{\mathbf{Q}}_{ij} \tilde{\mathbf{F}}_{ij},
\end{aligned} \tag{H.24}$$

where $\tilde{\mathbf{Q}}_{ij} = [\tilde{\mathbf{P}}^{-1} \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T \tilde{\mathbf{V}}]_{ij}$, $\tilde{\mathbf{F}}_{ij} = \frac{1 - (\Lambda_i \Lambda_j)^{T_f}}{1 - \Lambda_i \Lambda_j}$, and the summation is obtained through the standard finite geometric series summation, given that $(\Lambda_i \Lambda_j) < 1$ since all eigenvalues $\Lambda_i < 1$ for $i = 1, 2, \dots, N$ when network has conformity behavior (verified through extensive independent simulations). And thus, the reduced controllability Gramian matrix has the Hadamard product form $\tilde{\mathbf{W}} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{V}}^{-1} = \tilde{\mathbf{P}}[\tilde{\mathbf{Q}} \circ \tilde{\mathbf{F}}]\tilde{\mathbf{V}}^{-1}$, where $\circ$ denotes Hadamard product. Since $(\mathbf{C}\mathbf{W}\mathbf{C}^T) = \tilde{\mathbf{W}} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}\tilde{\mathbf{V}}^{-1}$, $(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1} = \tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}$. Further, the non-symmetric full $n \times n$ network matrix with conformity is eigen-decomposed as $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $\mathbf{A}^T = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where $\mathbf{D} = \text{diag}\{\Lambda_1, \Lambda_2, \dots, \Lambda_N, \Lambda_n\}$, where $\Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_N \leq \Lambda_n = 1$, and $\Lambda_i < 1$ for $i = 1, 2, \dots, N$, and $\Lambda_n = 1$ regardless of network topologies (verified numerically and extensively). These terms will be substituted into (H.18) to derive the analytical energy cost.

H.2.2 Large T_f regime

The next step is to derive an analytical expression for the energy cost in the large T_f regime. In the large T_f limit, because $\mathbf{D} = \text{diag}\{\Lambda_1, \Lambda_2, \dots, \Lambda_N, \Lambda_n = 1\}$, where all $\Lambda_i < 1$ for $i = 1, 2, \dots, N$, $[\mathbf{D}^{T_f}]_{ii} = \Lambda_i^{T_f} = 0$ in the large T_f limit, while $\Lambda_n = 1$ leads to $[\mathbf{D}^{T_f}]_{nn} = \Lambda_n^{T_f} = 1$ in the large T_f limit. Thus, $\mathbf{A}^{T_f} = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^{T_f} = \mathbf{P}\mathbf{D}^{T_f}\mathbf{P}^{-1} = \mathbf{P}\mathbf{J}^{nn}\mathbf{P}^{-1}$, where $\mathbf{J}^{nn}$ is a single-entry matrix [90], which is zero every where, except at indices (n, n) , where it is

one. Similarly, $(\mathbf{A}^T)^{T_f} = (\mathbf{V}\mathbf{D}\mathbf{V}^{-1})^{T_f} = \mathbf{V}\mathbf{D}^{T_f}\mathbf{V}^{-1} = \mathbf{V}\mathbf{J}^{nn}\mathbf{V}^{-1}$. Substituting these, and $(\mathbf{C}\mathbf{W}\mathbf{C}^T)^{-1} = \tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}$ into (H.18),

$$\begin{aligned}
\mathcal{E}^{\text{large } T_f} &= [\mathbf{y}_f^T]_{1a} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ab} [\mathbf{y}_f]_{b1} - 2[\mathbf{x}_0^T]_{1a} [\mathbf{V}]_{ab} [\mathbf{J}^{nn}]_{bc} [\mathbf{V}^{-1}]_{cd} [\mathbf{C}^T]_{de} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ef} [\mathbf{y}_f]_{f1} \\
&\quad + [\mathbf{x}_0^T]_{1a} [\mathbf{V}]_{ab} [\mathbf{J}^{nn}]_{bc} [\mathbf{V}^{-1}]_{cd} [\mathbf{C}^T]_{de} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ef} [\mathbf{C}]_{fg} [\mathbf{P}]_{gh} [\mathbf{J}^{nn}]_{hi} [\mathbf{P}^{-1}]_{ij} [\mathbf{x}_0]_{j1} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2[\mathbf{x}_0^T]_{1a} \left[\sum_{i=1}^n \mathbf{J}^{ni} \right]_{ab} [\mathbf{C}^T]_{bc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{y}_f]_{d1} \\
&\quad + [\mathbf{x}_0^T]_{1a} \left[\sum_{i=1}^n \mathbf{J}^{ni} \right]_{ab} [\mathbf{C}^T]_{bc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{C}]_{de} \left[\sum_{i=1}^n \mathbf{J}^{in} \right]_{ef} [\mathbf{x}_0]_{f1} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2c[\mathbf{x}_0^T]_{1n} \left[\sum_{i=1}^n \mathbf{J}^{ni} \right]_{nc} [\mathbf{C}^T]_{cc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} \\
&\quad + [\mathbf{x}_0^T]_{1n} \left[\sum_{i=1}^n \mathbf{J}^{ni} \right]_{nc} [\mathbf{C}^T]_{cc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{C}]_{dd} \left[\sum_{i=1}^n \mathbf{J}^{in} \right]_{dn} [\mathbf{x}_0]_{n1} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \left[\sum_{i=1}^n \mathbf{J}^{ni} \right]_{nc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} + z^2 \left[\sum_{i=1}^n \mathbf{J}^{ni} \right]_{nc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} \left[\sum_{i=1}^n \mathbf{J}^{in} \right]_{dn} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} + z^2 [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij},
\end{aligned} \tag{H.25}$$

where $[\mathbf{V}]_{ab} [\mathbf{J}^{nn}]_{bc} [\mathbf{V}^{-1}]_{cd} = [\mathbf{V}]_{an} [\mathbf{J}^{nn}]_{nn} [\mathbf{V}^{-1}]_{nd} = [\mathbf{V}]_{an} [\mathbf{V}^{-1}]_{nd} = \sum_{i=1}^n \mathbf{J}^{ni}$. Although it is known that $\mathbf{V}\mathbf{V}^{-1} = \mathbf{I}_n$, it is hard to see analytically that $[\mathbf{V}]_{an} [\mathbf{V}^{-1}]_{nd} = \sum_{i=1}^n \mathbf{J}^{ni}$, a matrix that is zero everywhere, except at the final n -th row, where all elements are one—this was verified to be true computationally, regardless of network topology, or r number of connections (as long as $r \geq 1$) that zealot node makes in the large T_f regime. Similarly, $[\mathbf{P}]_{gh} [\mathbf{J}^{nn}]_{hi} [\mathbf{P}^{-1}]_{ij} = [\mathbf{P}]_{gn} [\mathbf{J}^{nn}]_{nn} [\mathbf{P}^{-1}]_{nj} = [\mathbf{P}]_{gn} [\mathbf{P}^{-1}]_{nj} = \sum_{i=1}^n \mathbf{J}^{in}$, a matrix that is zero everywhere, except at the final n -th column, where all elements are one. In addition, because initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), \dots, x_N(0), z]^T = [0, 0, \dots, 0, z]^T$ is assumed to be such that all normal agents start with $x_i(0) = 0$ beliefs initially, with the zealot node's belief fixed at z , then $[\mathbf{x}_0]_{i1} = 0$ for all $i = 1, 2, \dots, N$, and all $[\mathbf{x}_0]_{i1}$ terms vanish for all $i \neq n$, where it is $[\mathbf{x}_0]_{n1} = z$ at index $i = n$. $\mathbf{y}_f = [c, c, \dots, c]^T$ is the final state vector of the N normal agents, where their node states are driven toward consensus c . Thus, $[\mathbf{y}_f]_{i1} = c$ for all $i = 1, 2, \dots, N$, and is treated like a constant. Further, wherever convenient, the arbitrary running indices were renamed. Lastly, because all N normal agents are to be controlled, and the zealot node is not to be controlled, the $N \times n$ $\mathbf{C}$ matrix comprises $N \times N$ identity $\mathbf{I}_N$ on the first $N \times N$ block, with the last column of $\mathbf{C}$ being all zeros. In other words, $[\mathbf{C}]_{ij} = [\mathbf{C}]_{ii} = 1$ for $i = 1, 2, \dots, N$. The transpose $\mathbf{C}^T$ is $n \times N$, so $[\mathbf{C}^T]_{ij} = [\mathbf{C}^T]_{jj} = 1$ for $j = 1, 2, \dots, N$. Thus, this was how Eq. (4.33) of the chapter main text was derived.

H.2.3 Small T_f regime

In the small T_f regime, the eigenvalues Λ_i cannot be assumed to vanish. Substituting $\underline{\mathbf{A}} = \underline{\mathbf{P}}\underline{\mathbf{D}}\underline{\mathbf{P}}^{-1}$ and $\underline{\mathbf{A}}^T = \underline{\mathbf{V}}\underline{\mathbf{D}}\underline{\mathbf{V}}^{-1}$ into (H.18),

$$\begin{aligned}
\mathcal{E}^{\text{small } T_f} &= [\mathbf{y}_f^T]_{1a} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ab} [\mathbf{y}_f]_{b1} - 2[\mathbf{x}_0^T]_{1a} [(\underline{\mathbf{V}}\underline{\mathbf{D}}\underline{\mathbf{V}}^{-1})^{T_f}]_{ab} [\mathbf{C}^T]_{cb} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{y}_f]_{d1} \\
&\quad + [\mathbf{x}_0^T]_{1a} [(\underline{\mathbf{V}}\underline{\mathbf{D}}\underline{\mathbf{V}}^{-1})^{T_f}]_{ab} [\mathbf{C}^T]_{bc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{C}]_{de} [(\underline{\mathbf{P}}\underline{\mathbf{D}}\underline{\mathbf{P}}^{-1})^{T_f}]_{ef} [\mathbf{x}_0]_{f1} \\
&= c^2 \sum_{a=1}^N \sum_{b=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ab} - 2[\mathbf{x}_0^T]_{1n} [\underline{\mathbf{V}}\underline{\mathbf{D}}^{T_f}\underline{\mathbf{V}}^{-1}]_{nc} [\mathbf{C}^T]_{cc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{y}_f]_{d1} \\
&\quad + [\mathbf{x}_0^T]_{1n} [\underline{\mathbf{V}}\underline{\mathbf{D}}^{T_f}\underline{\mathbf{V}}^{-1}]_{nc} [\mathbf{C}^T]_{cc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{C}]_{dd} [\underline{\mathbf{P}}\underline{\mathbf{D}}^{T_f}\underline{\mathbf{P}}^{-1}]_{dn} [\mathbf{x}_0]_{n1} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz [\underline{\mathbf{V}}\underline{\mathbf{D}}^{T_f}\underline{\mathbf{V}}^{-1}]_{nc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} \\
&\quad + z^2 [\underline{\mathbf{V}}\underline{\mathbf{D}}^{T_f}\underline{\mathbf{V}}^{-1}]_{nc} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\underline{\mathbf{P}}\underline{\mathbf{D}}^{T_f}\underline{\mathbf{P}}^{-1}]_{dn} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{a=1}^n \sum_{c=1}^N \sum_{d=1}^N [\mathbf{V}]_{na} \Lambda_a^{T_f} [\mathbf{V}^{-1}]_{ac} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} \\
&\quad + z^2 \sum_{a=1}^n \sum_{c=1}^N \sum_{d=1}^N \sum_{e=1}^n [\mathbf{V}]_{na} \Lambda_a^{T_f} [\mathbf{V}^{-1}]_{ac} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{cd} [\mathbf{P}]_{de} \Lambda_e^{T_f} [\mathbf{P}^{-1}]_{en} \\
&= c^2 \sum_{i=1}^N \sum_{j=1}^N [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^N [\mathbf{V}]_{ni} \Lambda_i^{T_f} [\mathbf{V}^{-1}]_{ij} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \\
&\quad + z^2 \sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^n [\mathbf{V}]_{ni} \Lambda_i^{T_f} [\mathbf{V}^{-1}]_{ij} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} [\mathbf{P}]_{kl} \Lambda_l^{T_f} [\mathbf{P}^{-1}]_{ln} \\
&= c^2 \sum_{i,j} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{ij} - 2cz \sum_{i,j,k} \underline{v}_{ni} [\mathbf{V}^{-1}]_{ij} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \Lambda_i^{T_f} \\
&\quad + z^2 \sum_{i,j,k,l} \underline{v}_{ni} [\mathbf{V}^{-1}]_{ij} \underline{p}_{kl} [\mathbf{P}^{-1}]_{ln} [\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}]_{jk} \Lambda_i^{T_f} \Lambda_l^{T_f},
\end{aligned} \tag{H.26}$$

where the arbitrary running indices were renamed wherever convenient, and $\sum_{i,j} = \sum_{i=1}^N \sum_{j=1}^N$, $\sum_{i,j,k} =$

$\sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^N$, $\sum_{i,j,k,l} = \sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^n$, with i, j, k, l as the running indices that iterate from 1 to N or 1 to n , and n is the fixed n -th index. The choice of zealot-influenced nodes enters the energy cost equation through the $n \times n$ matrices $\underline{v}_{ni}$, $[\mathbf{V}^{-1}]_{ij}$, $\underline{p}_{kl}$, $[\mathbf{P}^{-1}]_{ln}$, and the $N \times N$ matrices $\tilde{\mathbf{V}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{P}}^{-1}$. Thus, this recovers Eq. (4.37) of the chapter main text.

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