Hyperuniformity and beyond in complex systems

Zheng, Yuanjian

2021

Zheng, Y. (2021). Hyperuniformity and beyond in complex systems. Doctoral thesis, Nanyang Technological University, Singapore. <https://hdl.handle.net/10356/155024>

<https://hdl.handle.net/10356/155024>

<https://doi.org/10.32657/10356/155024>

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**HYPERUNIFORMITY AND BEYOND
IN COMPLEX SYSTEMS**

YUANJIAN ZHENG

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

[2021]

**HYPERUNIFORMITY AND BEYOND
IN COMPLEX SYSTEMS**

YUANJIAN ZHENG

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

A thesis submitted to the Nanyang Technological
University in partial fulfilment of the requirement for the
degree of Doctor of Philosophy

2021

Statement of Originality

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Chapter 3 and Appendix A is published as Y. Zheng, Y.-W. Li, and M. P. Ciamarra, Hyperuniformity and density fluctuations at a rigidity transition in a model of biological tissues. *Soft Matter* **16**, 5942-5950 (2020). DOI: 10.1039/D0SM00776E.

The contributions of the co-authors are as follows:

- A/Prof Yan-Wei Li (Beijing Institute of Technology) developed the numerical tools that were used to obtain energy minimal configurations of the many-body interacting model, which are precursors to the hyperuniform analyses.
- I co-designed the study with A/Prof Massimo Pica Ciamarra (Nanyang Technological University) and performed work involving data collection, analyses and interpretation.
- I prepared the manuscript drafts. The manuscript was revised by A/Prof Massimo Pica Ciamarra.

Chapters 4 and Appendix B is published as Y. Zheng, A. D. S. Parmar, and M. P. Ciamarra, Hidden Order Beyond Hyperuniformity in Critical Absorbing States, *Phys. Rev. Lett.* **126**, 118003 (2021). DOI: 10.1103/PhysRevLett.126.118003.

The contributions of the co-authors are as follows:

- Dr. Parmar (Université de Montpellier) provided insightful comments on the manuscript and helpful discussions.

- I co-designed and developed the study with A/Prof Massimo Pica Ciamarra (Nanyang Technological University) and performed work involving data collection, analyses and interpretation.
- A/Prof Massimo Pica Ciamarra developed a scalable version of the numerical code in simulating random organization process that were used for results from larger system sizes.
- I prepared the manuscript drafts. The manuscript was revised by A/Prof Massimo Pica Ciamarra.

Date _____

Yuanjian Zheng

Acknowledgements

This work would not have been possible if not for the support and goodwill of people. I express my gratitude to all that I have had the great fortune of meeting, and am thankful for each of our interactions, however transient they may be. Here, I briefly mention individuals that have had influence on the outcome of this work, but this list is by no means exhaustive.

Firstly, I am indebted to my advisor Massimo Pica Ciamarra for his guidance and mentorship on all matters, academic and beyond. I am grateful for his receptiveness to the essence of this work while it was still a nascent idea and his continued belief, especially given that the nature and approach involved may not entirely be in the usual style of his work. I am also thankful for his composure and calming presence in the face of ludicrous bureaucracy. Above all, I am most grateful that he reached out at a time when I had needed it most.

I am grateful to Hartmut Löwen and his group for being splendid hosts at the Heinrich Heine University of Düsseldorf. While we worked on unrelated problems in inertial active matter, and introduced the basic description of “Rototaxis”, some inspiration for the more preliminary aspects of this thesis originate from this interaction. I am also appreciative of Dario Poletti at the Singapore University of Technology and Design for being supportive and ever so welcoming despite the many years that have since passed from when I was a member of his group.

Lastly, I thank fellow group members and colleagues from past to present, for the invaluable conversations, discussions and companionships throughout the years. In reverse chronological order: [NTU] - Anshuman Pasupalak, Jing Yang (Emma), Xiuyang Xia, Yan-Wei Li, Shivam Mahajan, Sudhir Pathak, Joyjit Chattoraj, Pin Nie, and Antonio Piscitelli. [SUTD] - Ryan Tan, Chu Guo, Naresh Ragheva and Ulf Bissbort.

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Abstract

Disordered systems and their emergent dynamics are key aspects in understanding non-equilibrium phenomena. However, the inadequacy of mean field approaches below the upper critical dimension, and the general lack of lattice symmetries or conventional measures of long range order, severely limit the available tools at hand to characterize them. In this dissertation, we introduce new theoretical perspectives and statistical tools that capture the subtle presence of hierarchical hidden order in otherwise disordered systems.

In particular, we generalize an existing measure of long-range order known as hyperuniformity, characterized by the anomalous suppression of density fluctuations, and identify diverging length scales that control a rigidity cross-over in a class of many-body interacting, non-integrable models. This result invalidates on-going speculations that the Jamming-hyperuniformity conjecture - an exact duality between hyperuniformity and Jamming derived under rigorous hard sphere assumptions, is generalizable to low-dimensional soft packings.

Next, we extend this notion of hyperuniformity and the use of coarse-grained metrics to the higher moments of probability distributions, and uncover the presence of hidden order beyond hyperuniformity embedded in non-equilibrium states associated to the universality class of conserved directed percolation. These features emerge as characteristic signatures that depart from strong convergence of the central limit theorem and are indication of spatial many-body correlations that we further show to be intimately related to the dynamical correlations of preceding dynamics. Lastly, we speculate on universality, origins and the relevance of such hidden structural order to critical behavior and their potential role in our understanding of complex non-equilibrium phenomena.

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Chapter 1

Introduction

1.1 Disordered Systems

The physics of disorder is central to the study of complex systems [1] and critical phenomena [2–4], for nature is often far removed from the confines of models defined by exact symmetries, homogeneous fields and isotropic interactions [5]. “*Disorder*” in the existing literature thus alludes to several distinct notions that all appeal in some form to the absence of spatial symmetries or deviations from an ordered reference state [5–16]. These descriptions focus on characterizing the presence of localized structural defects that are either introduced explicitly, such as by imposing lattice perturbations [17, 18] in disordered crystals [14–16], or may instead spontaneously emerge in low-temperature liquids [12, 19], where they mediate important dynamical and phase behavior [19, 20].

In this dissertation, we are interested in effects of disorder that are considered *quenched* [21, 22], in that a subset of parameters take on random values that remain constant for all relevant timescales of the system. This static form of disorder manifest in a wide range of physical and model systems that include, disparities in the strength of local interactions between coupled degrees of freedom [9–12, 23, 24], inhomogeneity of global external fields [7, 23], and polydispersity¹ in atomistic descriptions of molecular fluids [25, 26], granular matter [27, 28] or structural glasses [9, 12]. Disordered systems that arise in tangential fields are also increasingly discussed in the language of statistical physics. These include diverse

¹Heterogeneity in the size of particles.

topics that range from the dimensional reducibility of artificial neural networks [29, 30], to the emergent combinatorial complexity of genetic sequences, and the self-assembly of intrinsically disordered proteins [31, 32].

Now, the presence of quenched disorder primarily acts to accentuate the effects of *frustration*, which refers to the inability of a single state to simultaneously minimize local energy between all two-body interactions [21, 23, 33]. This frustration originates from geometrical or topological constraints on the spatial degrees of freedom that leads to competing interactions [24, 34, 35]. Fig. 1.1 illustrates a scenario in which frustration arises in an Ising spin configuration [36, 37] defined on a triangular plaquette [38].

FIGURE 1.1: **Geometrical frustration.** [Left] A frustrated plaquette consisting of anti-ferromagnetic interactions (-). The red arrow represents a frustrated spin that is unable to simultaneously minimize its pair-wise energy with its two neighbors. [Right] Ground state of an un-frustrated plaquette in which all spins are able to locally minimize pair-wise contributions to the global energy.

In the Ising model, spin degrees of freedom s_i are arranged on a given lattice and acquire discrete values $s_i \in \{+1, -1\}$ that represent classical spin orientations $\uparrow, \downarrow$. The hamiltonian - $\mathcal{H} = -\sum_{ij} J_{ij} s_i s_j$, defines the energy of a configuration $\{s_i\}$, and is given by the sum total of pairwise products $s_i s_j$, weighted by the respective coupling strengths $\{-J_{ij}\}$ between interacting spins. Hence, the *sign* of J_{ij} determines if the alignment (i.e $\uparrow\uparrow$) of neighboring spins is to be desired, by determining whether it decreases ($-J_{ij} < 0$) or increases ($-J_{ij} > 0$) local contributions to $\mathcal{H}$, which represent the respective scenarios corresponding to notions of ferro- and anti-ferromagnetic interactions in the literature [36, 39].

In Fig. 1.1, we see that in the absence of disorder, frustration originating from anti-ferromagnetic interactions defined even on a single unit-cell results in the emergence of degenerate ground states (i.e. $\|J_{ij}\| = 1$). This effect is only further enhanced in the thermodynamic limit, allowing for exotic ground states such as spin waves [40]

FIGURE 1.2: **Order vs Disorder.** [Left] Hexagonal (black) and Kagome (blue) lattices illustrative of the physical systems that pose long range translational and rotational order. In contrast, [Right] intrinsically disordered systems lack the discrete symmetries that characterize their crystalline counterparts.

and magnetic monopoles [41] to exist in frustrated magnets and spin-ice [34, 42] that still partially preserve affine symmetries of the lattice. Now, suppose $\{J_{ij}\}$ is instead disordered in that each coupling deviates from the uniform value in a randomized manner^{2 3}, such that translational and rotational symmetries [2] of the system are broken [3] (see Fig. 1.2). In this scenario, the degeneracy of ground states is lifted, resulting in the occurrence of a large number of near-degenerate, low-energy configurations that are characteristic of disordered *glassy* systems [9, 12, 44, 45].

An intuitive understanding of this can be gleaned from the energy landscape perspective of a complex system [46–50], which depicts these states as a collection of disparate local minima living on a rough high-dimensional manifold [46, 47], each with its own basin of attraction in phase space [51] (see Fig. 1.3)⁴. In this picture, a disordered system is unlikely to ever exist in its true ground state simply for entropic reasons. Instead, it is trapped at zero temperature, in a local *metastable* minimum due to a complete absence of quantum or thermal fluctuations and thus do not resemble ground states of frustrated magnets or spin-ice.

² In practice, $\{J_{ij}\}$ is treated as independent and identical random variables drawn from a given distribution (typically gaussian) with a disorder-averaged mean value $\langle J_{ij} \rangle = \bar{J}$, and variance σ^2 that characterizes the strength of disorder [7].

³This model reduces specifically for anti-ferromagnetic nearest neighbor Ising interactions to the Edwards-Anderson (EA) model [43] for spin glasses, for which the fully connected version (Sherrington-Krikpatrick) is analytically solvable using the replica method [22, 24].

⁴Figure partially adapted from [52].

FIGURE 1.3: **Energy landscape picture.** [Left] 2D representation of a high-dimensional manifold of a disordered energy landscape with multiple local minima. [Right] 1D projection along a reaction coordinate (grey-dashed arrow) of the 2D landscape. The near-degenerate metastable states (blue dots) [1], [2] and their respective basins of attraction (a) and (b) are separated by an energy barrier (orange dots). Caged dynamics associated with β -relaxation are characterized by trajectories (red arrows) within a single basin.

At low temperatures, frustrated disordered systems exhibit dynamical heterogeneities⁵ [9, 12, 53–55] and the relaxation dynamics of an abrupt quench⁶ initialized at sufficiently high temperatures⁷ [9, 47, 54] undergo a two-step relaxation⁸ [53, 56]. This involves an initial phase (β -relaxation) that corresponds in the energy landscape picture, to meso-scopically caged dynamical trajectories confined to the basin of a single local minimum, followed by a power-law regime (α -relaxation) that is associated with transitions across energy barriers and the diffusive timescales required to transcend between neighborhoods of different minima, or loosely connected regions of phase space [53]. Hence, relaxation timescales of quenched disordered systems are largely governed by phase space architecture and the scale of fluctuations relative to energy barriers that demarcate basins of attractions [12, 47].

We stress that in practice, these relaxation timescales often vastly exceed observational times, such that a single realization of the disorder degrees of freedom, or otherwise a replica [7, 13], is unable to explore the entirety of its phase space. Such a disordered system is said to be “*ergodicity-breaking*” [24, 57]. Hence, glassy dynamics and their associated metastable states are intrinsically *out-of-equilibrium* phenomena, which stands in contrast to dynamics that occur in the presence of

⁵Co-existence of mesoscopic sub-regions in real space with orders of magnitude difference in their diffusion timescales.

⁶Super-cooled liquids make for good glass formers [12].

⁷ $T(t=0) \gg T_g$ where T_g is the glass transition temperature.

⁸This is most commonly discussed in context of the intermediate scattering function used in defining the dynamical structural factor or in the decay of density correlation functions [9, 12].

annealed disorder, where the system remains at all times in equilibrium, and is characterized by a disorder-averaged canonical partition function [44].

This non-equilibrium nature has a profound consequence on the excitations of disordered steady-state or “frozen” configurations, resulting in a plethora of striking characteristics and phenomena not found in ordered systems. These include the much celebrated Anderson [6, 10] and many-body localization [58, 59] responsible for the existence of non-diffusive waves in the presence of weak disorder, and characteristic features of amorphous solids⁹ such as the Boson peak¹⁰ [61–64] and the universal low-frequency scaling of its density of states¹¹ $D(\omega) \sim \omega^4$ [65, 69–71]. In the macroscopic limit, these features lead to unusual elastic [15, 67, 72–74], rheological [11, 15, 75–79] and thermal [68, 78, 80–82] properties.

Despite its clear importance, disordered systems are notoriously difficult to treat since attributes that result in its most interesting properties are also the very same features that lead to a loss of translational and rotational symmetries (see Fig. 1.2). This in turn prevents the existence of closed form solutions [2, 4], and severely impedes the development of analytical insights; a predicament that is best reflected in the structural behavior of disordered system undergoing a glass transition. As temperature decreases, a super-cooled liquid undergoes a fourteen orders of magnitude increase in the relaxation time and viscosity¹²[9, 15, 83], but yet no discernable features are found in its structure¹³ [44, 83], no commensurate growth in length-scales corresponding to thermodynamic correlations exist [47, 84], and no clear or simple long-range structural order has ever been identified in characterizing states along the glass transition [12, 83, 85]. This remains the single greatest challenge in understanding glassy dynamics of disordered high dimensional systems and has remained an open problem for the past hundred years [86].

⁹Disordered materials that include foams, gels and glass

¹⁰An anomalous excess of modes at intermediate frequencies ω in $D(\omega)/\omega^{d-1}$ where ω^{d-1} is the Debye frequency [60] scaling that characterizes phonon excitations in crystals [2, 4] and d is the physical dimensionality of the system.

¹¹This originates from an excess of low-energy quasi-localized modes [65–69] that do not resemble plane wave excitations in ordered crystals.

¹²This occurs within a 30% variation in the temperature with respect to T_g

¹³In particular, here we refer to two-point correlations such as the radial distribution $g(r)$ and its fourier transform - the structure factor $S(k)$ which we formalize in Ch. 2

1.2 Hyperuniformity

In this section, we introduce a form of long-ranged order known as hyperuniformity [87–89] that is *hidden* in the spatial structure of disordered systems, in that they are often very difficult to identify by a simple visual inspection. This approach relies on characterizing the behavior of real-space *fluctuations* of *coarse-grained* variables and is thus able to reveal the presence of spatial order in disordered configurations [90–99] by circum-navigating the lack of discrete lattice symmetries¹⁴.

To define hyperuniformity, we begin by considering a system consisting of N particles (degrees of freedom) located at positions $\mathbf{r}_i$ defined on a continuous¹⁵ space of linear size L in d spatial dimensions (Fig. 1.4). The number of particles N_R that lie within an observational volume $\Omega_R \sim R^d$ of lengthscale R centred on a point $\mathbf{r}_c$ is a coarse-grained variable defined in terms of a Heaveside function $\Theta(x)$ ¹⁶.

$$N_R \equiv \sum_i^N \Theta(R - \|\mathbf{r}_i - \mathbf{r}_c\|) \quad (1.1)$$

As $\mathbf{r}_c$ is moved to different regions in real space, the value of N_R fluctuates and is a random variable governed by a discrete probability density function $P(N_R)$, where its fluctuations are characterized to lowest order by its second central moment¹⁷, or otherwise known as its variance [100]:

$$\sigma_N^2(R) = \langle N_R^2 \rangle - \langle N_R \rangle^2 \quad (1.2)$$

where $\langle \dots \rangle$ denotes the average over all possible values of $\mathbf{r}_c$ in real space.

For an uncorrelated distribution of points (Fig. 1.4[Left]), $\sigma_N^2(R)$ scales with the size of the coarse-grained volume $\Omega_R \sim R^d$, simply because it can be regarded as the additive sum of $\sigma_N^2(R')$ across uncorrelated sub-regions (i.e. $R' < R$) that constitute Ω_R , such that the composite variance $\sigma_N^2(R)$ is extensive at all lengthscales. On the other hand, a lattice ordered configuration (Fig. 1.4[Right]) is translationally invariant (i.e perfectly correlated in the bulk), such that fluctuations in N_R

¹⁴Note that while hyperuniformity characterizes hidden order in otherwise disordered systems, its various measures detailed in Ch. 2 does not solve the glass transition problem detailed in Sec. 1.1 [12].

¹⁵This is easily generalizable to discrete spaces.

¹⁶A piecewise function where $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$ [100].

¹⁷The m -th central moment of a discrete probability $P(x)$ is given by $\sum_i x_i^m P(x_i)$

FIGURE 1.4: **Point distributions.**[Left] An uncorrelated configuration generated by a Poisson point-process [101]. [Center] Hyperuniform Centroidal Voronoi Tessellations (CVT) [102, 103] generated from Lloyd’s algorithm [104]. Note that the cell boundaries of the voronoi construction (blue) visually indicates that the hyperuniform system is intrinsically disordered. [Right] An ordered triangular lattice with affine symmetries [2]. Hyperuniformity analyses involve understanding how fluctuations in the number of points within a real space volume Ω (red circle) changes with an increasing lengthscale R .

only exists on the boundary that encloses Ω_R . Hence, fluctuations to lowest order, scale as the surface area of the observational window $\sigma_N^2 \sim R^{d-1}$ in ordered systems that posses translational symmetry. Now, in disordered systems, the distribution of points in space are *correlated* to various extents due to the presence of spatially dependent interactions, such that the fluctuations of coarse-grained variables can be in some instances, anomalously suppressed¹⁸. This results in disordered configurations (Fig. 1.4[Center]) that do not resemble random point distributions, and their dominant fluctuations in N_R are instead given by a scaling behavior $\sigma_N^2 \sim R^\epsilon$ where ϵ is bounded by the exponents $d - 1 \leq \epsilon \leq d$ of the two limiting scenarios encountered previously. If this anomalous suppression of fluctuations persist in the infinite lengthscale limit ($R \rightarrow \infty$), the system is said to be *hyperuniform*. Formally, this is given by the following condition:

$$\lim_{\Omega_R \rightarrow \infty} \frac{\sigma_R^2}{\Omega_R} = 0 \quad (1.3)$$

Moreover, Fig. 1.4[Center] represents an “ideal” form of hyperuniformity¹⁹[92], where the bound on the exponent ϵ is seemingly saturated $\epsilon = d - 1$ [87, 88], indicating an extreme suppression in the growth of its fluctuations. This behavior while typical of systems possessing translational symmetry such as crystals [87] or

¹⁸Note that fluctuations can also be anomalously enhanced (i.e. $\epsilon > d$) resulting in *anti-hyperuniformity*[105, 106] in the limit of $R \rightarrow \infty$.

¹⁹This is sometimes referred to as type I hyperuniformity [87, 88].

quasi-crystals [107, 108], is truly remarkable for an intrinsically disordered system, especially considering that for ϵ to be exactly $d-1$, this behavior must be universal across all meta-stable states of all disorder realizations, thus representing a form of self-organized hyperuniformity in the absence of fine-tuning [93, 109].

Analogously, hyperuniformity and fluctuation suppression can also be stated in terms of the local number density $\rho_R \sim N_R/R^d$, which is trivially related to N_R for an isotropic and symmetrically shaped Ω ²⁰ [88, 110], such that $\sigma_\rho^2 = R^{-2d}\sigma_R^2$.

$$\rho_R \sim \begin{cases} R^{-d} & \text{Random} \\ R^{-\epsilon}, d \leq \epsilon \leq d+1 & \text{Hyperuniform} \\ R^{-(d+1)} & \text{Ideal Hyperuniformity} \end{cases} \quad (1.4)$$

Given that hyperuniform point distributions are such unusual states of matter, it is thus surprising that the anomalous suppression of fluctuations are in fact found to be present in a wide array of phenomena. Physical systems where hyperuniformity arises include functional, glassy and amorphous materials [99, 103, 106, 111–117], maximally random jammed (MRJ) sphere packings²¹ [118–121], multi-phase or heterogeneous media [96, 120–125], various random organization models [93, 126–130] and eigenstates of quantum many-body systems [131, 132] or random matrices [133, 134]. In fact, hyperuniformity lies even in our most profound and fundamental descriptions of the physical world, hidden within the distribution of prime numbers [135–137] and the large scale structure of matter in the universe [138].

Furthermore, hyperuniformity is also emergent in a number of natural occurring or model biological systems that range from the suppression of genetic variability in the sequence space of antibody receptors [31, 139], to the spatial distribution of photoreceptors [140] and feather pigmentation [141] in birds, or in models of epithelial tissues [92, 103]. From a more practical point of view, hyperuniform patterns have also been used in the fabrication of disordered materials that display photonic band gaps [142–145] that are comparable in performance to their ordered counterparts, but are more tunable [146], and less sensitive [87] to the presence of defects. Various materials with novel phononic [147, 148], dielectric [149, 150] and spectroscopic [151] properties have also been designed using hyperuniform principles.

²⁰Spherical windows are typically considered without much loss of generality.

²¹A extremely disordered jammed packing that is void of any crystalline features as characterized by the lack of peaks in the pair correlation function at lattice values. [118]

Now, these naturally emergent or programmed features of hyperuniformity in various disordered systems are also found in many instances to be very robust in the presence of considerable thermal fluctuations [94, 152], random noise [126, 127, 153] or point defects [152]. In fact, uncorrelated displacements added to a hyperuniform point distribution can be used to mask the presence of long-range order, such that it no longer appears as a clear consequence of the pair-correlations, while still retaining hyperuniform features [153]. Suppression of density fluctuations are also typically observed in sheared or driven systems [97, 109, 127–130, 154], very much against the naive expectation that internal stresses or strain responses of the system would act to dismantle the hyperuniform structure. In fact, for many such cases, the degree to which density fluctuations are suppressed seem to be even further enhanced by the presence of external shear or oscillatory drives [97, 127, 130, 155].

Despite the ubiquity of hyperuniformity and their vast applications, little is known of its causal origins [87, 93, 93, 116, 127] and its full implications in physical systems are poorly understood [87–89, 93, 135]. In some minimal models, hyperuniformity seem to be the consequence of microscopic constraints that arises either from geometrical or topological frustration [96, 140, 156–158], or the very specific nature of how degrees of freedom couple to random noise [127]. However, no general principle of how or why hyperuniformity emerges exists. There is also some indication that the spatial extent of fluctuation suppression plays a role in characterizing certain phase transitions or dynamical crossover [92–95, 116, 128, 159]. This stands in contrast to the large (density) fluctuations characterized in the standard theory of critical phenomena [160], and are thus sometimes referred to as an inverted critical point [88, 159]. Similarly, such unusual phenomenology is also poorly characterized and understood. Furthermore, our ability to control the fluctuation behavior [90, 93, 153, 161] of a material at various length scales and knowledge of how this relates to macroscopic properties from first principles are also limited.

In this thesis, we address a combination of these fundamental issues regarding disordered hyperuniformity. We establish connections between the nature of fluctuations to critical phenomena, and discuss the spatial or dynamical correlations they encode. This involves identifying disordered systems where behavior of fluctuations are indicative of the thermodynamic phase. In doing so, we introduce statistical measures that extend notions of fluctuations beyond hyperuniformity, and provide new perspectives in characterizing hidden order in disordered systems.

We stress that the statistical framework presented in this thesis is entirely general and can be directly applied to any spatial distribution of points or any given probability measure in real space. This includes any experimental or simulated distribution of point patterns or data set where non-trivial emergent correlations exists. Examples range from the position of particles in complex materials to genetic sequences or spatiotemporal population studies.

1.3 Outline of thesis

Chapter 2. We review the mathematical construction of hyperuniformity, and introduce the structure factor $S(\mathbf{k})$ as a fourier equivalent of the pairwise density correlation function $g(r)$ in characterizing hyperuniformity. We also introduce and quantify useful concepts in the study of anomalous density fluctuations such as *stealthy* and *effective* hyperuniformity.

Chapter 3. In this chapter, we study density fluctuations in a many-body interacting model of biological tissues known to exhibit a rigidity transition. We show that the solid states of this model are not hyperuniform in contrary to existing literature, but are instead, suppressed up to a finite length scale. This lengthscale monotonically increases with the control parameter in the solid phase and diverges at the critical point of the solid-fluid transition, suggesting that the fluid phase is effectively hyperuniform. This establishes a connection between the rigidity transition and hyperuniformity. We discuss implications of our results on generality of the “Jamming-hyperuniform” conjecture.

Chapter 4. This chapter discusses the presence of hidden order in critical absorbing states of active-absorbing phase transitions belonging to the universality class of conserved directed percolation. We show that the lengthscale for which fluctuations are suppressed, grows with a different exponent from previous indications, and is exponentially shorter than the lengthscale identified by the order parameter that characterizes its phase. Instead, we show that the fluctuation suppressed length scale corresponds to hidden dynamical correlations in the freezing time of particles in the absorbing state. In the process, we generalize hyperuniformity

to the higher moments of its coarse-grained density distribution, and discuss general implications of our work in characterizing critical phenomena and uncovering hidden order *beyond* hyperuniformity in disordered complex systems

Chapter 5. We examine the nature of fluctuations in continuum coarse-grained descriptions of motility induced phase separated active matter. In particular, we show that non-integrable field theories and extensions to the Cahn-Hilliard (CH) model through the addition of integrable terms are all hyperuniform despite being heterogeneous and phase separated at shorter lengthscales. In fact, they are indistinguishable from passive CH, and agree to lowest order in their mean-field fluctuations. We further show that hidden order beyond hyperuniformity exists in these systems and can be used to distinguish differences in higher-order correlations of the field equations, that may represent systems with different underlying microscopic interactions.

Chapter 6. In this last chapter, we draw conclusions on how hyperuniformity and the extent to which fluctuations are suppressed can be used to inform on the presence of hidden correlations. We discuss if these features in the higher moments of coarse-grained density are generically found in hyperuniform systems, and suggest problems where hidden order may play a non-trivial role.

Chapter 2

Fluctuations, Correlations and Hidden Order

In Sec. 1.2, we introduced density hyperuniformity as a form of hidden order characterized by the scaling behavior of the variance $\sigma_N^2(R) \sim R^\epsilon$ of a distribution $P(N_R)$ in the thermodynamic limit, where N_R is the coarse-grained density over some finite lengthscale R . Specifically, a disordered system is said to be hyperuniform, if and only if $\epsilon < d$ for $R \rightarrow \infty$.

In this section, we expand on this introduction of hyperuniformity to include formulations of its definition in terms of pair-wise correlations - $g_2(r)$ in direct space (Sec. 2.1) and its fourier equivalent, the structure factor - $S(k)$ (Sec. 2.2). We highlight the connections between various definitions of hyperuniformity, discuss scaling behavior of $S(k)$ in context of Eq. (1.4) and emphasize that while they are equivalent descriptions only in the infinite length-scale limit, behavior of $\sigma^2(R)$ at finite length scales captures non-trivial structural information of disordered states. In addition, we discuss additional concepts regarding hyperuniformity that are essential to understanding the subsequent chapters of this dissertation. These include measures that quantify *effective* hyperuniformity (Sec. 2.3) arising from imperfections in hyperuniform systems, and the existence of a *stealthy* variant of hyperuniformity (Sec. 2.4), that is revealed by examining $S(k)$ in all directions of k -space.

2.1 Density correlation functions

We begin by considering the probability distribution function $\rho_N(\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_N) \equiv \rho_N(\{\mathbf{R}_i\})$ for a general distribution of N points located at $\mathbf{R}_i \in \mathbb{R}^d$ in Euclidean space of some finite volume $V \sim L^d$. Additionally, we assume that the system is translational invariant in that the hamiltonian $\mathcal{H}$ is only dependent on the pairwise distances between particles $\mathbf{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i$, such that $\rho_N(\{\mathbf{R}_i\})$ can be written without loss of generality, in terms of a N -body correlation function that involves only pairwise degrees of freedom $\mathbf{r}_{ij}$,

$$\rho_N(\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_N) = \rho^N g_N(\{\mathbf{R}_i\}) \quad (2.1)$$

where $\rho = N/L^d$ is the *global* number density and $g_N(\{\mathbf{R}_i\})$ is the N -body correlation function of the system. For a perfectly ordered configuration (e.g. crystal) [2, 4], g_N is discontinuous and non-zero only for $\mathbf{r}_{ij}$ that are linear combinations of the primitive lattice vectors. On the other hand, the average distance separation $\langle r_{ij} \rangle$ for an uncorrelated distribution of points is simply ρ^N , such that $g_N = 1$ holds for all values of $\mathbf{r}_{ij}$ in the thermodynamic limit. Hence, we see that in general, the relative value of g_N characterizes if a system is more ($g_N > 1$) or less ($g_N < 1$) correlated than the configuration produced by a random Poisson point process at a given $\mathbf{r}_{ij}$. If we only consider the lowest order contributions to Eq. (2.1), g_N reduces to the *pairwise correlation function* $g_2(\mathbf{r})$ given by

$$g_2(\mathbf{r}) = \frac{1}{\rho} \left\langle \sum_{i \neq 0} \delta(\mathbf{R} - \mathbf{R}_i) \right\rangle \quad (2.2)$$

Note that $g_2(\mathbf{r}) = g_2(-\mathbf{r})$ (i.e $g_2(\mathbf{r})$ is a symmetric function), and is often expressed in relative terms to the uncorrelated value $g_2(\mathbf{r}) = 1$

$$h(\mathbf{r}) = g_2(\mathbf{r}) - 1 \quad (2.3)$$

where $h(\mathbf{r})$ is known as the *total correlation function* in existing literature [87, 88]. $h(\mathbf{r}) \rightarrow 0$ in the limit of $R \rightarrow \infty$ indicates that no long-range order is present.

Now, suppose that the disordered d -dim system is sufficiently isotropic, in that the pair-correlation can be effectively represented by a 1D symmetric function $g_2(R)$ known as the *radial distribution function*. This is done by integrating $g_2(\mathbf{R})$

FIGURE 2.1: **Radial distribution function.** [Left] Evaluating the radial distribution function $g_2(R)$ of a 2D point distribution involves first counting the number of points that lie within an annulus $R < r < R + dr$ centred on each point. The individual counts are then averaged over all points and normalized by $\rho \partial\Omega_r$. [Right] $g_2(R)$ of the meta-stable state of a 2D disordered system typical of super-cooled liquids, glasses and amorphous systems (blue-continuous line). The position of peaks correspond to the typical interparticle spacing a_0 , and the separation to subsequent neighboring shells. This contrasts with the uniform $g_2(r) = 1$ behavior of a point configuration originating from an uncorrelated Poisson point process (red-dashed line).

Eq. (2.2) over the surface of a d -sphere Ω_r defined by a radius r originating from $\mathbf{R}_c$, which eliminates structural and spatial information from $d - 1$ degrees of freedom and thus removes vectorial dependence of the system, such that g_2 is simply characterized by a single pairwise scalar distance r , given by

$$g_2(r) = \frac{1}{\rho \partial\Omega_r} \sum_{r < r' < r+dr} \delta(r - r') \quad (2.4)$$

Here, $\partial\Omega_r$ is the differential form representing the boundary of Ω_r

$$\partial\Omega_r^{(d)} = \frac{2\pi^{d/2} r^{d-1}}{\Gamma(d/2)} \quad (2.5)$$

where $\Gamma(x)$ is the gamma function¹[100], and $\partial\Omega_r^{(2)} = 2\pi r$ and $\partial\Omega_r^{(3)} = 4\pi r^2$ are the familiar expressions for circumference and surface area respectively. Note that this normalization factor $(\rho \partial\Omega_r)^{-1}$ ensures that like its d -dim equivalent Eq. (2.2), $g_2(r) = 1$ for an uncorrelated random distribution of points [87].

¹ $\Gamma(x) = \int_0^\infty x^{z-1} e^{-x} dx$

In Fig. 2.1, we show conceptually how $g_2(r)$ can be evaluated numerically for a 2D point configuration. Note that the computation of $g_2(r)$ scales approximately as N^2 such that in practice, a smaller subset of points are often chosen at random in obtaining the ensemble averaged value of $g_2(r)$, especially for larger system sizes. As with the correlation functions $h(\mathbf{r})$ and $g_N(\mathbf{r})$, the behavior of the radial distribution function - $g_2(r) > 1$ and $g_2(r) < 1$, also indicates respectively, an excess or deficiency in the occurrence of particle pairs at a given r , in comparison to an uncorrelated point configuration.

In addition, we note that $g_2(r)$ shown in Fig. 2.1[Right] is typical of disordered configurations such as glassy or amorphous ground states [12, 44, 83]. In particular, the positions of peaks in $g_2(r)$ correspond to various typical distances between polydisperse particles governed by a microscopic interaction lengthscale a_0 , and appear diffused in nature. This is unlike the discontinuous and discrete peaks found in crystals. Furthermore, $g_2(r)$ decays with r , (typically as power-laws in glasses, but exponentially in crystals [9, 24]), and is thus more adept at capturing the short-range² structural features in disordered systems. Lastly, we also remark that $g_2(r)$ does not appear to display discernable changes to its features at the glass transition temperature T_g [83, 85] which forms part of the difficulty in understanding disordered systems as discussed in Sec. 1.1.

2.2 Structure factor

A complimentary perspective to the pair-correlations (Sec. 2.1) of a disordered system can also be found in the Fourier space representation of the total correlation function $h(\mathbf{r})$ Eq. (2.3) [87, 88]. This is often written in the form of a non-negative spectral function $S(\mathbf{k})$ known as the *static structural factor* given by

$$S(\mathbf{k}) \equiv 1 + \rho \tilde{h}(\mathbf{k}) \quad (2.6)$$

where $\tilde{f}(k) = \int_{\mathbb{R}^d} f(r) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}$ denotes the Fourier transform of $f(r)$ ³.

²Up to lengthscales that are several ($10^0 - 10^1$) multiples of a_0 .

³Note that (analytic continuation) of a Fourier transform of the Dirac-delta function used in defining $g_2(\mathbf{r})$ is defined in closed form $\int_0^\infty \delta(x-a) e^{-kx} dx = e^{-ka}$.

The structure factor for a discrete point distribution can thus also be defined explicitly in terms of the pairwise displacements of particles in direct space⁴.

$$S(\mathbf{k}) = \frac{1}{N} \left\langle \left\| \sum_i^N e^{-i\mathbf{k}\cdot\mathbf{r}_i} \right\|^2 \right\rangle \quad (2.7)$$

Now, in order to relate Eq. (2.7) to the real space fluctuations σ_R^2 that defines hyperuniformity Eq. (1.3), we follow [88] by taking explicitly the averaged-square of Eq. (1.1) for some Ω of lengthscale R .

$$\sigma^2(R) = \langle N_R^2 \rangle - \langle N_R \rangle^2 \quad (2.8)$$

$$= \int_{\mathbb{R}^d} \Theta'_1 d\mathbf{R}_1 + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [\rho_2(\mathbf{R}_1, \mathbf{R}_2) - \rho_1(\mathbf{R}_1)\rho_1(\mathbf{R}_2)] \Theta'_1 \Theta'_2 d\mathbf{R}_1 d\mathbf{R}_2 \quad (2.9)$$

with $\Theta'_i \equiv \Theta(R - \|\mathbf{R}_i - \mathbf{R}_c\|)$. Since the system is translationally invariant and $\int_{\mathbb{R}^d} \Theta'_1 d\mathbf{R}_1 = \langle N_R \rangle$, we can rewrite Eq. (2.9) in terms of the real-space total correlation function by using Eq. (2.1) and Eq. (2.3),

$$\sigma^2(R) = \langle N_R \rangle \left[1 + \rho \int_{\mathbb{R}^d} h(\mathbf{r}) \alpha(\mathbf{r}, \mathbf{R}) d\mathbf{r} \right] \quad (2.10)$$

where $\alpha(\mathbf{r}, \mathbf{R}) = \Omega^{-1} \int_{\mathbb{R}^d} \Theta'_1 \Theta'_2 d\mathbf{R}_c$ is known as the scaled intersection volume in the literature [88] of a given geometry of the window volume Ω . Lastly, we take the Fourier transform of Eq. (2.10) using Parseval's theorem⁵ [100] to obtain an expression for the lowest order real space fluctuations $\sigma^2(R)$, in terms of the correlations in *Fourier* space (i.e. $\tilde{h}(\mathbf{k})$ and $s(\mathbf{k})$).

$$\sigma^2(R) = \langle N_R \rangle \left[1 + \frac{\rho}{(2\pi)^d} \int \tilde{h}(\mathbf{k}) \tilde{\alpha}(\mathbf{k}, \mathbf{R}) d\mathbf{k} \right] \quad (2.11)$$

$$= \langle N_R \rangle \left[\frac{\rho}{(2\pi)^d} \int S(\mathbf{k}) \tilde{\alpha}(\mathbf{k}, \mathbf{R}) d\mathbf{k} \right] \quad (2.12)$$

Here, the second equals sign follows from the relation in Eq. (2.3), and we see that in general, the relation between $S(\mathbf{k})$ and $\sigma^2(r)$ depends on $\alpha(\mathbf{k}, \mathbf{R})$, which relates to the geometrical properties of Ω and shape of $\partial\Omega$.

⁴This expression is often what is used in practice for the numerical evaluation of $S(\mathbf{k})$

⁵ $\sum_{n=-\infty}^{\infty} a_n \tilde{b}_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(x) \tilde{B}(x) dx$, where $A(x), B(x)$ are Fourier series with coefficients a_n, b_n respectively.

For a radially symmetric $\Omega(R)$, the Fourier-transform of $\alpha(r, R)$ is given by [88]

$$\tilde{\alpha}(k, R) = 2^d \pi^{d/2} \Gamma(1 + d/2) [J_{d/2}(kR)]^2 \quad (2.13)$$

where $J_{d/2}$ is a Bessel function of the first kind⁶ [100], which in the $R \rightarrow \infty$ limit results in $\tilde{\alpha} \rightarrow (2\pi)^d \delta(\mathbf{k})$. This implies that the relation between $S(\mathbf{k})$ and σ_N^2 in Eq. (2.12), reduces in the thermodynamic limit to

$$\lim_{R \rightarrow \infty} \frac{\sigma_N^2}{\langle N_R \rangle} = S(\mathbf{k} = \mathbf{0}) \quad (2.14)$$

In other words, the asymptotic behavior of $S(\mathbf{k})$ in the $\mathbf{k} \rightarrow \mathbf{0}$ limit is an equivalent description for the $R \rightarrow \infty$ behavior of the fluctuations. More importantly, since $\langle N_R \rangle \sim R^d$, the hyperuniform condition defined in Eq. (1.4) is equivalent to

$$\lim_{\mathbf{k} \rightarrow \mathbf{0}} S(\mathbf{k}) = 0 \quad (2.15)$$

We emphasize that while Eq. (1.4) and Eq. (2.15) are entirely equivalent descriptions of hyperuniformity in the $R \rightarrow \infty$ limit, features of $S(\mathbf{k})$ at *finite* length scales do not directly correspond to fluctuations in real space at the reciprocal value of R ($k = 2\pi/R$). They are instead coupled by an integral relation (Eq. (2.12)), that may even be influenced by the geometry of Ω and $\alpha(\mathbf{r}, \mathbf{R})$ [110].

The Fourier definition of hyperuniformity Eq. (2.15), also allows for the additional characterization of hyperuniform systems through the scaling behavior of $S(\mathbf{k}) \sim k^\alpha$ in the vicinity of $\mathbf{k} = 0$. This further constraint the behavior of σ_N^2 through Eq. (2.12), such that we can identify three distinct scaling regimes for hyperuniform real space fluctuations [87]

$$\sigma_N^2(R) \sim \begin{cases} R^{d-1} & \alpha > 1 \text{ (Class I)} \\ R^{d-1} \log R & \alpha = 1 \text{ (Class II)} \\ R^{d-\alpha} & 0 < \alpha < 1 \text{ (Class III)} \end{cases} \quad (2.16)$$

As with the pair correlation function $g_2(\mathbf{r})$ Eq. (2.4) discussed earlier in Sec. 2.1, $S(\mathbf{k})$ can similarly also be integrated over a $d-1$ dimensional spherical surface in k -space to arrive at a scalar representation $S(k) = \frac{1}{\partial\Omega} \sum_{\mathbf{k} < \mathbf{k}' < \mathbf{k} + d\mathbf{k}} \delta(k - k')$ for largely

⁶ $J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma(m + \alpha + 1) \left(\frac{x}{2}\right)^{2m+\alpha}$

FIGURE 2.2: **Structure factor.** Schematics of the effectively 1D $S(k)$. Real space fluctuations in the $R \rightarrow \infty$ limit are captured in the asymptotic behavior of $S(k) \sim k^\alpha$ in the $k \rightarrow 0$ limit. [Green] $S(k)$ approaches a non-zero value at $k = 0$ indicating that the system is not hyperuniform. [Red] $\alpha = 1$ Class II hyperuniformity, where fluctuations grow logarithmically. [Blue] $\alpha = 2$ Class I hyperuniform systems, where σ_N^2 scales as a power law in R with index $d - 1$ [Orange] $\alpha = \infty$ Stealthy hyperuniform systems.

isotropic materials. We show in Fig. 3.7 qualitatively different scenarios of $S(k)$ for various point configurations. In particular, we highlight the $S(k)$ in hyperuniform systems indicated by the red, blue and orange curves that all approaches zero in the limit of $k \rightarrow 0$, but with different power-law exponent α . In contrast, $S(k)$ for non-hyperuniform point configurations (green) approach finite non-zero values.

Lastly, we remark that Eq. (2.15) also has physical implications for the bulk properties of disordered materials since the *equilibrium* fluctuations at temperature T of the density ρ at infinite wavelength is related to the isothermal compressibility κ_T of the system [87, 158, 162].

$$S(k \rightarrow 0) = \rho k_B T \kappa_T \quad (2.17)$$

where k_B is the Boltzmann constant. Hence, this result implies that any zero temperature ground state (disordered, crystalline or otherwise) with a finite, positive definite κ_T must be hyperuniform since $S(k)$ is strictly zero as enforced by Eq. (2.17). Conversely, at finite non-zero T , κ_T of *excited* states must be zero, implying that these metastable states are incompressible. This perspective, has led to the search and discovery of exotic and stealthy hyperuniform states with non-trivial spectral properties [87, 163] which we discuss in further detail in Sec. 2.3.

2.3 Effective hyperuniformity

In Sec. 2.2, we established an equivalent definition of hyperuniformity, complementary to real space fluctuations, that is given in terms of the behavior of its structure factor $\lim_{k \rightarrow 0} S(k) = 0$. However, in many physical systems and atomistic numerical studies, the presence of noise such as thermal fluctuations or defects, inevitably obfuscates the features of hyperuniformity, such that $S(k \rightarrow 0)$ approaches a small but non-zero value [92, 98, 99, 103, 115]. Moreover, the asymptotic behavior of the large wavelength limit $k \rightarrow \infty$ in $S(k)$ is often difficult to probe in finite systems, especially with experimental setups [87]. Hence a measure known as the *effective hyperuniformity index* (H) is often used as a measure to characterize the degree to which a system is hyperuniform.

$$H = \frac{S(k \rightarrow 0)}{S(k_{peak})} \quad (2.18)$$

H measures the relative intensity of $S(k)$ in the asymptotic limit of $k \rightarrow 0$ to the value of its structure factor at its first prominent peak $S(k_{peak})$. Typically for an isotropic material, $k_{peak} = 2\pi/a_0$ where a_0 is the characteristic inter-particle lengthscale of the system that is reflected in $g_2(r)$. In practice $S(k \rightarrow 0)$ is often obtained using a linear extrapolation of the scaling behavior $S(k) \sim k^\alpha$ in the vicinity of $k = 0$ [92, 103].

Disordered System	Hyperuniform Index(H)
Amorphous ice	$2 \times 10^{-3} - 4 \times 10^{-3}$ [99]
Amorphous silicon	$4 \times 10^{-3} - 10^{-2}$ [115]
Centroidal Voronoi tessellations (CVT)	$\lesssim 10^{-4}$ [103]
Unjammed Voronoi fluid ($3.813 < p_0 < 4.00$)	$\sim 10^{-4} - 10^{-6}$ [92]
Maximally Random Jammed (MRJ) sphere packings	$\lesssim 10^{-4}$ [98]

TABLE 2.1: **Effective hyperuniformity Index.** Typical values of H for various hyperuniform systems in 2 or 3 dimensions. This is to be contrasted with that of ordinary liquids ≥ 0.02 [162].

In Table 2.1, we list the typical values of H for various systems known to be hyperuniform. While there exists no rigorous threshold based on H that would render a system hyperuniform, H remains a useful measure of effective hyperuniformity in imperfect samples where $H \lesssim 10^{-3}$ is typically considered as evidence of hyperuniformity [92, 103, 163].

2.4 Stealthy hyperuniformity

In this section, we discuss a special case of disordered hyperuniformity where $S(\mathbf{k}) \sim \mathbf{k}^\infty$ in the neighborhood of $\mathbf{k} = \mathbf{0}$ for all $\mathbf{k} \in \mathbb{R}^d$. In other words, $S(\mathbf{k}) = 0$ for $\mathbf{k}$ in all directions of phase space in the region $\|\mathbf{k}\| \leq K$ ⁷. This behavior of $S(\mathbf{k})$ is highly unusual even for hyperuniform systems, and implies that the point configuration is completely transparent at wavelengths $\lambda \geq 2\pi/k$, since all incident wave-vector $\mathbf{k}$ with $0 < \|\mathbf{k}\| \leq K$ are unscattered by the point configuration.

As such, hyperuniform systems that display such a strange $S(\mathbf{k})$ is considered to be “stealthy”⁸ [87, 163–166]. We show the manifestation of such behavior in the scalar form of its structure factor in Fig. 3.7 (Orange-line), represented by the abrupt decrease in the intensity of $S(k)$ at K such that $S(k) = 0$ ⁹ for all $0 < k < K$. In particular, $S(k)$ of stealthy systems assumes the functional form:

$$S(k) = \Theta(k - K)[1 + \tilde{Q}(k)] \quad (2.19)$$

where Θ is the heaveside function, and $\tilde{Q}(k) \geq -1$ is a continuous function in k . This implies that the total correlation function $\tilde{h}(k)$ in Fourier space of a stealthy disordered point configuration is a piecewise discontinuous function - $\rho\tilde{h} = \tilde{f}(k) + \tilde{P}(k)$ ¹⁰ that highly resembles the Mayer f -function which defines the Fourier-transformed total correlation $\tilde{h}_{hs}(k)$ in pseudo-equilibrium hard spheres [96, 156, 163].

This enables a mapping of Fourier space quantities $\tilde{h}(k)$ and $S(k)$, between stealthy disordered systems and hard spheres with small packing fractions in finite dimensions, given that the degrees of freedom in the stealthy system are sufficiently unconstrained¹¹. In fact, the total correlation for stealthy hyperuniform systems can be directly approximated by that of the hard spheres $\tilde{h}_{HS}(k)$ in most equilibrium scenarios where the configurations represent disordered states [163] from a

⁷Stealthy systems thus belong to Class I hyperuniformity Eq. (2.16), and its real space fluctuations scale like the surface area R^{d-1}

⁸A more general definition of “stealthy” includes regions in k that are not necessarily in the neighborhood of $\mathbf{k} = \mathbf{0}$ [164, 165].

⁹Or below a threshold in effectively hyperuniform systems.

¹⁰ $\tilde{f}(k) = -\Theta(K - k)$ and $\tilde{P}(k) = \Theta(k - K)\tilde{Q}(k)$

¹¹See [163] for discussion on the order metric χ for this unconstrained condition to be valid.

canonical ensemble at finite non-zero T , given by

$$h(r) = -\frac{1}{\rho} \left(\frac{K}{2\pi r} \right)^{d/2} J_{d/2}(Kr) \quad (2.20)$$

where $J_{d/2}(Kr)$ represents a Bessel function of the first kind. In the $r \rightarrow \infty$ limit, the correlation function reduces to

$$h(r) \sim -\frac{1}{r^{(d+1)/2}} \cos \left(r - \frac{\pi(d+1)}{4} \right) \quad (2.21)$$

Hence, $h(r) \sim r^{-(d+1)/2}$ in the long wavelength limit for stealthy disordered hyperuniform systems. From Eq. (2.12), we also see that the stealthy scaling behavior of $S(k) \sim k^\infty$ implies that the number fluctuation $\sigma_N^2(R)$ follows

$$\sigma^2(R) = \Lambda(R)^{d-1} + \mathcal{O}(R^{d-3}) \quad (2.22)$$

where $\Lambda(R)$ is an oscillatory function with mean value $\bar{\Lambda}$ that is the parameter which characterizes the relative extent to which fluctuations are suppressed at R in hyperuniform systems. We note that while stealthy hyperuniformity implies $\sigma^2(R) \sim R^{d-1}$ in the $R \rightarrow \infty$ limit, the converse is not necessarily true. This fluctuation behavior exists for a larger class of hyperuniform systems Eq. (2.16), some of which may not be stealthy. Hence, we reiterate that the notion of “stealthiness” is defined on the scattering properties of the point distribution in k -space.

Chapter 3

Hyperuniformity at rigidity and Jamming transitions

In this chapter, we discuss structural correlations $S(\mathbf{k})$ and real space density fluctuations $\langle \delta^2 N \rangle$ ¹ in point configurations that represent disordered metastable states near critical points of a *rigidity transition*² in many-body interacting models of biological tissues³. In particular, we show that density fluctuations are in contrary to existing speculations, suppressed only up to a finite lengthscales $l_{\langle \delta^2 N \rangle}^*$ in the solid phase of this model. More importantly, l^* grows monotonically in the solid and diverges at the critical point of the rigidity transition, suggesting that the fluid state of the model is effectively hyperuniform. We further show that the hyperuniformity observed is stealthy by examining behavior of $S(\mathbf{k})$ in the fluid. Our results suggest that hidden long range order may generically be present, and controls critical phenomena in disordered systems.

In the following, we outline organization of chapter Ch. 3. In Sec. 3.1.1, we present general phenomenology and scaling relations that characterizes the jamming transition for pair-wise interacting models and introduce in Sec. 3.1.2, a constraint counting argument (Maxwell-Calladine) that identifies the coordination number $\langle z \rangle$ ⁴ as a global order parameter that governs the jamming transition. We then

¹ $\langle \delta^2 N \rangle$ corresponds to an alternative notation for the disorder average of $\sigma_N^2(R)$ discussed in Sec. 2.1, which we use interchangeably in this chapter.

²A solid-to-fluid transition characterized by its dynamical relaxation and the shear modulus μ . This is discussed in greater detail in Sec. 3.1

³Results presented in this chapter can be found primarily in our work detailed in [92]

⁴Mean number of pair-wise contacts per particle in the system.

discuss in Sec. 3.1.3, existing indications on the hyperuniformity of jammed pairwise interacting *soft* solids, and their structural or fluctuation behavior as these systems approach the critical J -point at zero temperature in the phase diagram.

In Sec. 3.2, we introduce a many-body interacting model [Voronoi Model (VM)] that exhibits a rigidity crossover⁵ reminiscent of the jamming transition. In particular, we present in Sec. 3.2.1, the many-body energy function (hamiltonian), biological origins, and rheological properties that characterizes VM. We also discuss considerations for the limitations on contact topology arising from the Voronoi construction of real space, and its implications for (Maxwell-Calladine) constraint counting especially in comparison to a similar model known as the vertex model in Sec. 3.2.2

Subsequently, we discuss the behavior of real space density fluctuations in section Sec. 3.3.1. In particular, we extract the lengthscale l^* for which the density is anomalously suppressed and show that it diverges at the critical point p_0^* of the rigidity transition, such that the fluid state is effectively hyperuniform. In Sec. 3.3.2, we examine the complementary picture by looking at the structure factor, and verify our results previously derived based on fluctuations in direct space. Furthermore, by examining the $S(\mathbf{k})$ in 2D, we show that the hyperuniform fluid states are stealthy. Lastly, we conclude this chapter by discussing our results in context of the jamming-hyperuniformity conjecture that applies to hard spheres in maximally random jammed packings in Sec. 3.3.3 .

3.1 Jamming Transition

Amorphous materials such as colloids, foams and gels can simultaneously be viewed as flowing (fluid) or rigid (solid) depending on the observational timescales or mechanical environment of the system [12, 61, 167, 168]. However, unlike the glass transition (see Sec. 1.1) which is a temperature driven phenomena [7, 9, 12, 24, 47], this difference in the dynamic or shear viscosity⁶ is brought about by a change in

⁵This athermal transition shares similar rheological and dynamical properties with jamming across both phases of the transition, but is not typically considered jamming as the system remains confluent at a constant density across the transition.

⁶The resistance of one layer of a fluid over another under shear stress.

the global density $\rho = N/V$ or packing fraction $\phi = \rho v_d^7$ of the system [28, 167, 169–171]. Jamming can thus heuristically be thought of as the zero temperature transition [172] from a fluid like flowing state to a rigid phase of a disordered system. Importantly, no structural order parameter distinguishes configurations from the two states [167, 171]. Thus, the modern perspective of flow-like or rigid behavior in disordered amorphous systems is represented conceptually by a state diagram Fig. 3.1 governed by the independent parameters: (i) Density or packing fraction [167] (ρ or ϕ), (ii) temperature (T) [173] and (iii) shear stress (Σ) [174].

FIGURE 3.1: **Phase diagram of Jamming.** The rheological state of disordered matter is generally dependent on the temperature T , shear stress Σ , and packing fraction ϕ . At low T , Σ and high ϕ , the system is in a solid-like arrested state, while at high T , Σ and low ϕ , it flows like a fluid. The blue curved lines represent phase boundaries of the solid-fluid transition driven by T and Σ that converge on the axes at critical points corresponding respectively to shear-Jamming and the glass transition. At zero-temperatures and in the absence of shear stress, the transition occurs at the Jamming-point or J -point.

In subsequent sections of this chapter, we are interested in the jamming phenomenology at zero temperature and zero shear stress. This transition occurs at the critical point known as the J -point, and is typically the transition referred to in the literature as “Jamming” [167, 171].

⁷ v_d is the d -dimensional volume of each particle. For example, spherical discs in 2d space bounded by a $L \times L$ box would have a packing fraction $\phi = N\pi a_0^2/L^2$ where a_0 is its radius.

3.1.1 Jamming - A mixed order phase transition

In this section we review the Jamming phenomenology ($T = 0$, $\Sigma = 0$) in soft disordered solids and discuss the properties of J -point. Consider N particles of radius σ_i ⁸ distributed in d -dimensional space L^d interacting via soft potentials - pairwise *short-ranged* interactions that are typically modelled by the Weeks-Chandler-Andersen (WCA) potential⁹ [26] or a *purely repulsive* interaction such as the truncated harmonic potential¹⁰ given by

$$V(r_{ij}) = \epsilon_{ij} \left(1 - \frac{r_{ij}}{\sigma_{ij}}\right)^2 \quad \text{for } r_{ij} \leq \sigma_{ij} \quad (3.1)$$

where $\sigma_{ij} = (\sigma_i + \sigma_j)/2$ and $V(r_{ij}) = 0$ for $r_{ij} > \sigma_{ij}$. In the dilute limit, i.e $L \gg a_0 N^{1/d}$, where the packing fraction $\phi = N v_d(a_0)/L^d$ is low, none of the particles are in contact or otherwise at separations where $r_{ij} < \sigma_{ij}$, such that $\sum_{ij} V_{ij} = 0$ and the resulting energy cost for particle re-arrangements is also zero. Hence the system exists in an *un-jammed* fluid state where the pressure (p), bulk (B) and shear modulus (G) are all zero or vanishingly small [167].

Now, suppose that we shrink the size of the bounding box by reducing L such that ϕ increases. At some critical ϕ_c , sufficient number of contacts ($N z_c$) between particles form, and the system is no longer able to re-arrange without resulting in the increase of energy, implying that it abruptly enters a jammed solid state that cannot flow in the absence of thermal fluctuations or external forces. Thus the system acquires finite non-zero values in G , B and p . The reverse picture occurs in the solid phase as $\phi \rightarrow \phi_c^+$ in what is known as an un-jamming transition, which we illustrate schematically for a system of 2D soft discs in Fig. 3.2¹¹.

The critical value ϕ_c at J -point have been studied extensively in various numerical studies [28, 167, 169], and correspond to critical values for random close packings in hard spheres - $\phi_c \sim 0.64$ (3D) and $\phi_c \sim 0.84$ (2D). In the solid phase, bulk

⁸ σ_i is polydisperse in disordered systems and drawn from a probability distribution with finite variance such that $a_0 = \langle \sigma_i \rangle$

⁹ A finite-ranged form of the Lennard-Jones (LJ) potential used in modelling molecular liquids

¹⁰ Generalizations to higher-order power laws (i.e stiffer potentials) such as Hertzian contacts are possible [167, 171]

¹¹ This figure is adapted from [171]

FIGURE 3.2: **Onset of Jamming.** An increase in the packing fraction ϕ beyond critical ϕ_c rigidifies a system. Figure adapted from [171]

properties scale as continuous power-laws in the limit of $\phi \rightarrow \phi_c^+$ ¹² Eq. (3.1), and is given by $B \sim (\phi - \phi_c)^0$ and $G \sim (\phi - \phi_c)^{1/2}$ for harmonic interactions Eq. (3.1), such that the ratio of elastic constants are

$$\frac{G}{B} \sim (\phi - \phi_c)^{1/2} \quad (3.2)$$

We note that $G/B \sim \mathcal{O}(1)$ in ordered crystals, but can vary across several orders of magnitude in jammed solids [15, 167] owing to the presence of short-ranged repulsive interactions of varying non-uniform strengths. Moreover, while bulk properties scale continuously to zero at J -point like second-order phase transitions, the coordination number z changes discontinuously from zero to a finite z_c , as contacts between particles form instantly and percolates the entire jammed solid at ϕ_c ¹³. This behavior of z is thus more reminiscent of a first-order transition, and jamming is therefore considered to be a mixed-order phase transition. We discuss what determines the value of z_c in pair-wise interacting models at the onset of rigidity in Sec. 3.1.2.

Here, we briefly remark that while some properties including the aforementioned rheological behavior can in part be understood by the presence of quasi-localized soft modes [65, 66, 69] that correspond to the anomalous low frequency scaling

¹²This scaling behavior depends in general on the stiffness of potentials. For anharmonic power law interactions of the form $V(r) \sim (1 - r/\sigma)^\alpha$, the elastic constant scale as $B \sim (\phi - \phi_c)^{\alpha-2}$ and $G \sim (\phi - \phi_c)^{\alpha-3/2}$, such that the ratio G/B is independent of the index α

¹³Note that despite Z being discontinuous at ϕ_c , it still scales as a power law in the solid with a similar exponent - $z - z_c \sim (\phi - \phi_c)^{1/2}$

in its vibrational (phonon) density of states [65, 69–71], the exact nature of this phase transition and its critical exponents remains largely unknown. How do these different elastic constants and corresponding lengthscales interact and control the transition at J -point? [175, 176]. There is also little indication of how universal these exponents truly are for jammed systems, or how they are related beyond the use of phenomenological scaling Ansatzs [177].

To further accentuate this problem, it is unclear if the transition at J -point can indeed be considered as the zero temperature limit of a glass transition [178], and how it connects or informs on the long standing debate of what underlies the finite temperature glass transition - Is it one (or more) static diverging lengthscales or a single dynamical lengthscale at play? [179–181]

3.1.2 Maxwell-Calladine criterion for rigidity.

In the previous section (Sec. 3.1.1), we introduced J -point as the critical point at which a pair-wise repulsive system loses mechanical rigidity due to an excessive loss of contacts such that $z \leq z_c$. Here we discuss an argument made by Maxwell [182] and Calladine [183] that sets this threshold of rigidity as $z_c = 2d$. We note that this is derived under the assumptions that interactions are isotropic and the system is un-stressed. We also note that this condition is a *global* criterion for rigidity, and individual particles can be *locally* jammed with fewer contacts than z_c , while still being globally jammed.

We begin by considering an infinitesimal perturbation $\delta \mathbf{R}_i$ to the displacement vector $\mathbf{R}_0$ of N particles in an unstressed state of the system, such that the pairwise potential $V(r_{ij})$ is zero for all r_{ij} (i.e the total energy $E = \sum_{ij} V_{ij} = 0$). To lowest non-trivial order in $\delta \mathbf{R}_i$, this results in an increase of the energy given by

$$\delta E \sim \sum_{ij} \frac{\partial V_{ij}}{\partial R_i \partial R_j} [(\delta \mathbf{R}_i - \delta \mathbf{R}_j) \cdot \mathbf{n}_{ij}]^2 \quad (3.3)$$

where $\mathbf{n}_{ij}$ is the unit vector in the direction of $\mathbf{R}_j - \mathbf{R}_i$. Now for the state to be rigid, we require that $\delta E > 0$ ¹⁴. Hence, since each contribution to the total sum in Eq. (3.3) is non-negative, the condition $\delta E > 0$ implies that each individual

¹⁴ $\delta E \leq 0$ implies that there would be a direction in phase space at $\mathbf{R}_0$ that lowers E , and where the system can flow like a fluid with zero shear modulus.

term must be zero independently. Moreover, the number of ij pairs that appear in Eq. (3.3) is essentially the amount of pairwise contacts N_c in the unstressed state, implying that rigidity places N_c independent constraints on the linear system.

On the other hand, the configuration consists of $N_{dof} = Nd$ degrees of freedom¹⁵. For the system to be rigid, the number of constraints must equal or exceed the degrees of freedom (i.e. $N_c \geq Nd$). Since $N_c = Nz/2$, the Maxwell-Calladine condition for rigidity in an isotropic un-stressed state simplifies to

$$z \geq 2d \quad (3.4)$$

We note that the excess number of contacts in jammed solids away from J -point scale as $z - z_c \sim (\phi - \phi_c)^{1/2}$ [167] and the critical value $z_c = 2d$ is known as the isostatic point. Isostatically jammed solids are marginal systems¹⁶ from a linear stability perspective, implying that they are prone to generating soft modes in their vibrational spectra under topological perturbations or a change to their boundary conditions. These excitations are also typically anisotropic, leading to force propagation along heterogeneous networks known as force chains [170]. As such, isostatic elastic networks and marginally jammed solids are central to the search and design of metamaterials for their various micro-mechanical applications [184–187] and topological properties [188, 189]. In the natural world, functional proteins are also often found at this marginal threshold of stability [190].

In under-constraint systems ($z < z_c$), the presence of internal stresses can also act to rigidify the system. This is due to the emergence of states of self stress [191, 192] which are configurations of bond tensions $\mathbf{f}_{ij} = -\hat{\mathbf{r}}_{ij}\partial V/\partial r_{ij}$ that self-organize such that no resultant force $\mathbf{F}_i = \sum_j \mathbf{f}_{ij} = \mathbf{0}$ is present in equilibrium. This leads to N_{ss} additional constraints from sum rules that do not emerge from energy considerations in Eq. (3.3). Hence, the Maxwell-Calladine counting argument can be written more generally as a linear rank-nullity theorem.

$$N_0 = Nd - \frac{Nz}{2} + N_{ss} \quad (3.5)$$

where N_0 is the number of zero modes or otherwise the dimension of the null space of δE . A system is thus rigid in the absence of soft modes (i.e. $N_0 = 0$). Lastly,

¹⁵In principle $d(d+1)/2$ global rotational and translational symmetries are subtracted from the Nd degrees of freedom. However, their effect is vanishing in the thermodynamic limit.

¹⁶ The condition $N_{dof} = N_c$ in linear dynamical systems

we note that the counting argument can also be extended in certain instances to non-isotropic interactions and frictional contacts [193–195].

3.1.3 Hyperuniformity at J -point

In this section, we discuss historical and on-going debate regarding the nature of long wavelength fluctuations and hidden order at J -point. The fluctuation behavior at large R in mono-disperse, athermal frictionless spheres have long been believed to be hyperuniform [94, 95, 156, 196]. These results were shown numerically based on the direct scaling behavior of $S(k) \sim k$ and $h(r) \sim r^{-4}$ in large systems [156, 196], and indirectly through a dispersion relation to its low-frequency vibration modes. Presence of frictional contacts have also been observed to destroy the behavior of hyperuniformity in ρ [196].

Hence, it came somewhat as a surprise that allowance for bidispersity or polydispersity σ_i in the above systems, resulted in the disappearance of hyperuniformity [197, 198]. These observations further led to the speculation that while they do not exhibit anomalous long-ranged density fluctuations, polydisperse soft spheres in low dimensions are in fact hyperuniform in the fluctuations of their local volume fraction ϕ_R [199] defined as

$$\phi_R = \sum_i v_i \delta(\mathbf{r} - \mathbf{r}_i) \quad (3.6)$$

where v_i is the d -dimensional volume of the particle and $\mathbf{r}_i$ here refers to positions of particles. Hence, hyperuniformity in ϕ_R can be analogously stated as

$$\lim_{\mathbf{q} \rightarrow \mathbf{0}} \chi(\mathbf{q}) = 0 \quad (3.7)$$

where $\chi(\mathbf{q}) = \frac{1}{L^d} \langle \phi_{\mathbf{q}} \phi_{-\mathbf{q}} \rangle$ is the density correlation (structure factor) of $\phi_{\mathbf{q}}$, the Fourier transform of ϕ_R that is given explicitly in 2D by

$$\phi_{\mathbf{q}} = \int_{L^d} d^d r e^{i\mathbf{q} \cdot \mathbf{r}} \phi(\mathbf{r}) \quad (3.8)$$

This perspective that ϕ_R but not ρ is hyperuniform, is further supported by [96, 120] in extensive numerical studies using an alternative definition of ϕ_R

$$\phi_R = \sum_i \Delta_i(\mathbf{r} - \mathbf{r}_i) \quad (3.9)$$

where $\Delta_i(\mathbf{r}) = \Theta(|\mathbf{r} - \mathbf{r}_i| \leq \sigma_i)$ such that $\int_{L^d} \Delta_i(\mathbf{r}_i) d^d r = v_i$. We note that the previous definition of ϕ_R Eq. (3.6) attributes the volume v_i of a particle to just a single point $\mathbf{r}_i$. In contrast, the second definition Eq. (3.9) involves a more diffused association of volume across all points within its interaction radius σ_i .

However, a recent comparison study [95] that examines real and Fourier space fluctuations for both definitions of ϕ_R has cast doubts on this perspective. In particular, Wu and Titel found no convincing signature of hyperuniformity in Fourier space for both definitions, and argued that while a version of real space fluctuations (Eq. (3.6)) appear hyperuniform, the behavior of $\chi(\mathbf{q})$ supersedes $\sigma_\phi^2(R)$ in these systems. The observed hyperuniformity in previous studies were attributed to limitations of small system sizes [199], the specific preparation protocols of Jammed configurations that result in a bias sampling of systems that resemble MRJ configurations of hard spheres known to be hyperuniform [96, 120, 200].

Moreover, there is also fresh indication that hyperuniformity is absent in jammed solids above ϕ_c even in *monodisperse* systems that are sufficiently far away from the J -point [94, 95]. This has led to further speculation that perhaps fluctuations in soft interacting systems are only suppressed up to finite-length scales in the Jammed phase, but grows and diverges on approach to J -point from above [95].

3.2 Model biological tissues

This section introduces minimal models that describe the bio-mechanical behavior and dynamical evolution of biological tissues such as confluent¹⁷ epithelial monolayers. In Sec. 3.2.1, a many-body interacting hamiltonian $\mathcal{H}$ that characterizes the effective free energy of cells is presented [201–203]. $\mathcal{H}$ is a function of geometrical properties of sub-units in a tessellated space [92, 204] and thus cannot be reduced

¹⁷This implies the absence of rattlers (i.e. non-interacting free particles that are disconnected topologically to the bulk of the system.)

to a sum of pair-wise interactions. We discuss the biological origins [201, 205] and dimensionless forms of $\mathcal{H}$, and highlight the role of geometrical frustration [206, 207] to its phase behavior. In Sec. 3.2.2, we examine the rigidity transition that emerges in this model and the various lengthscales involved. We draw parallels and make distinctions between this transition and jamming in soft pair-wise interacting systems (Sec. 3.1), especially in the dense limit. Lastly, we discuss the nature of this transition in context of the Maxwell-Calladine criterion for rigidity (Sec. 3.1.2) and compare the Voronoi and vertex model [205, 208] where the degrees of freedom are defined differently.

3.2.1 Voronoi Model - Many-body interactions

Biological tissues are complex non-equilibrium systems living in noisy environments that behave like viscoelastic materials¹⁸. Solid-like at shorter timescales, their ability to induce correlated cellular re-arrangement [209] at longer timescales to acquire flow like behavior [210, 210] is critical to wound healing [211] and developmental processes such as invagination¹⁹ and metamorphosis [212]. The fluid-like behavior that re-models contact topologies²⁰ in achieving specific packing geometries [206, 209, 213] is also important to growth and pattern formation in biological systems [214, 215]. In particular, the epithelial-to-mesenchymal transition (EMT) [216] in cells is responsible for the emergence of motile and invasive species of cancers [217]. This process is highly reminiscent of the un-jamming transition [210] which renders the tissues rheological properties that are clinically relevant and have profound physiological consequences including the diffusive properties of lung tissues in asthmatic patients [218] and tumor progression [216]. Hence, understanding and controlling the unjamming transition is of tremendous medical importance.

Now, physical modelling of tissues traditionally involves two different approaches. First, a top-down perspective that aims to write down continuous field equations representing the continuum living material such as various active gel theories [219],

¹⁸Matter that behaves both like a viscous fluid and an elastic solid.

¹⁹A mechanism in the early stages of embryo development that involves the folding of an exterior shell of cells into its interior cavity

²⁰This occurs due to either T1 transitions (see Fig. 3.4). The vanishing of an edge length between existing neighbors, followed by the formation of a new edge between adjacent cells that are previously not in contact, or T2 transitions corresponding to out of plane extrusions or cell death. Note that T2 transitions do not conserve number density ρ .

and a bottom-up approach that explicitly models the microscopic degrees of freedom. Describing multi-scale problems in biology such as emergent mechanical or rheological behavior in the un-Jamming transition requires the latter, and thus involves first writing down minimal equations that capture the essential physics of individual cells [204, 205, 208, 220]. Understanding their collective behavior and macroscopic properties is subsequently achieved through the use of numerical simulations of these microscopic descriptions.

In this chapter, we focus on an effective free energy $\mathcal{H} = \sum_{\alpha} \mathcal{H}_{\alpha}$ [201, 208, 221] that models mechanical behavior of cells in a wide range of rheological studies in epithelial tissues [92, 202–204, 210, 218], where $\mathcal{H}_{\alpha}$ is typically given in terms of geometrical properties of 2D projections of the apical layer²¹ associated to the α -th cell in the colony [201, 221]. At sufficiently high density in confluent monolayers, edges of individual cells become oblique under tension and compression (see Fig. 3.3[Left]), such that the shape of cells resemble polygons, and $\mathcal{H}$ can thus be written in terms of the perimeter P_{α} , area A_{α} and direct edge lengths between point vertices l_{ij} ²² of individual cells.

FIGURE 3.3: **Voronoi representation of epithelial tissues.** [Left] Confocal imaging of the apical layer of an epithelial monolayer. The green coloration originates from fluorescent proteins in the apical layer that demarcates cell boundaries. At high densities, the convex polygons (red-lines) of the Voronoi tessellation is good representation of the geometrical boundaries of cells within confluent mono-layers. The blue and green dots represent the vertex and Voronoi degree of freedom respectively. Image courtesy of L. Lin (MBI).

²¹The layer that is closest to the free space.

²²Cell-cell adhesion is in general proportional to the surface area of the lateral walls but enter the 2D hamiltonian effectively as a linear dependence on edge lengths l_{ij} due to generically small fluctuations in cell height h .

In general, such models consider three qualitatively different contributions to $\mathcal{H}_\alpha$, (i) **Area elasticity** $\sim (A_\alpha - A_\alpha^0)^2$ originating from incompressible deformations of the cell and resulting fluctuations in cell height h , where A_α^0 is a *preferred* area of the cell (i.e the area in which an unconstrained isolated cell would occupy), (ii) **Line tensions** $\sim l_{ij}$ that correspond to lateral cell-cell adhesion, where l_{ij} is the edge length connecting vertices i, j , and (iii) **Cell contractility** $\sim P_\alpha^2$ of the actin-myosin ring²³. Collectively, $\mathcal{H}$ of the epithelial monolayer is thus written

$$\mathcal{H} = \sum_{\alpha} \frac{k_{a,\alpha}}{2} (A_\alpha - A_\alpha^0)^2 + \sum_{\langle i,j \rangle} \Lambda_{ij} l_{ij} + \sum_{\alpha} \frac{\Gamma_\alpha}{2} P_\alpha^2 \quad (3.10)$$

where $k_{a,\alpha}$, Λ_{ij} and Γ_α are the respective elastic coefficients that sets the scale of various contributions. It then follows that the force $\mathbf{F}_i$ acting on degrees of freedom $\mathbf{R}_i$ associated to each cell is written

$$\mathbf{F}_i = -\nabla_i \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \mathbf{R}_i} \quad (3.11)$$

Note that several choices for $\mathbf{R}_i$ exists, each with consequence on resultant dynamics and rigidity behavior. In most models, $\mathbf{R}_i$ represent either independent vertices of polygons [201, 202, 208] (i.e *Vertex* model) or the geometrical centres in cells of a given measure [92, 103, 204]. In the following, we focus on a specific partitioning of the space known as *Voronoi* tessellations [92, 204] where points in space are assigned to the corresponding cell with centre $\mathbf{R}_i$ in which they are closest to²⁴ (Fig. 3.3). This results in a confluent tessellation of space where all cells are at all times convex polygons, and thus imposes additional non-holonomic constraints on $\mathbf{R}_i$, not present in the vertex model.

Next, we make an assumption that the tissue is homogenous in that there is no variability between the elastic constants between different cells or edges, such that Eq. (3.10) can be written effectively as

$$\mathcal{H}_\alpha = K_a (A_\alpha - A_0)^2 + \xi P_\alpha^2 + \gamma P_\alpha \quad (3.12)$$

²³Cytoskeletal proteins that line the perimeter of cells on the apical layer.

²⁴The metric here is the Euclidean distance or otherwise 2-norm

for some proportionality constants ξ , γ and K_a . Further rescaling of Eq. (3.12), yields a *non-dimensional* energy function²⁵:

$$\mathcal{H} = \sum_{\alpha} \mathcal{H}_{\alpha} \sim \sum_{\alpha} \left[(a_i - 1)^2 + \frac{(p_i - p_0)^2}{r} \right] \quad (3.13)$$

that is a sum of harmonic contributions from deviations in the rescaled area a_i and perimeter p_i from their respective preferred values. Competition to simultaneously minimize both geometrical contributions to Eq. (3.13) is the source of frustration in metastable states of this model [202, 206, 222]. Now, the control parameter $p_0 \equiv \langle P_0 / \sqrt{A_0} \rangle$ (where $p_0 = \gamma / 2\xi$) that emerges in Eq. (3.13) is regarded as a dimensionless target shape index given by the ratio of a *preferred* perimeter P_0 to an effective length scale $\sqrt{A_0}$ associated to its preferred area a_0 , and controls local stiffness of each particle. In the thermodynamic limit, this parameter p_0 , controls the melting transition induced by thermal noise [222], or by active forces in tissues [223]. On the other hand, $r = K_A A_0 / \xi$ defines the relative contributions to dimensionless $\mathcal{H}$ from respective geometrical components a_i and p_i . In Sec. 3.2.2, we shall examine the role of r and p_0 in the rigidity transition of epithelial tissues.

Lastly, we remark that in this model Eq. (3.13), interaction is truly manybody, in that the mechanical forces ($\mathbf{F}_i = -\nabla \mathcal{H}_i$) acting on the degrees of freedom are not reducible to pairwise forces by virtue of how geometrical quantities such as the area and perimeter are defined in the Voronoi or vertex construction. This is a key feature that is not shared by conventional models used to study the jamming transition (Sec. 3.1), especially in context of hyperuniformity (Sec. 3.1.3).

3.2.2 Rigidity transitions in epithelial tissues

Having defined non-dimensional $\mathcal{H}$ and the Voronoi or vertex constructions in Sec. 3.2.1, here we discuss the p_0 driven rigidity transition at zero temperature [202–204]. To this end, energy metastable states are first obtained by applying a suitable minimization protocol to Eq. (3.13) for a given choice of the degrees of freedom $\mathbf{R}_i$. This can be accomplished either by employing Monte-Carlo techniques such as simulated annealing [224] or conjugate gradient algorithms [92, 222, 223, 225] that

²⁵Note that constant terms that emerge from the rescaling, which have no $\mathbf{R}_i$ dependence, have been neglected in the expression.

are capable in general of reaching deeper energy minima. Conversely, minimization can also be achieved through evolving dissipative equations of motion such as frictional contacts or overdamped dynamics [203, 204] that capture more realistic motions of physical cells. i.e.

$$\frac{d\mathcal{H}}{d\mathbf{R}_i} = \mu \mathbf{F}_i \quad (3.14)$$

where $\mu \geq 0$ is the mobility factor (inverse of the frictional drag) that sets the energy scale of damping. We note that these systems however are more likely to be trapped in minima at higher energies.

In the *vertex* model, topology of contacts and vertices throughout the dynamics are fixed (i.e there are no changes in the identity of neighbors or number of edges in each cell). Hence, un-jamming of the static metastable configuration at mechanical equilibrium ($\sum_{\alpha} \mathbf{F}_{\alpha} = 0$) can only be achieved via cellular re-arrangements known as *T1* transitions (see Fig. 3.4)²⁶. These processes have an associated energy barrier ΔE that quantifies the cost incurred upon quasi-static removal of an edge length l_{ij} between existing contacts, and the subsequent formation of a new edge $l'_{\alpha\beta}$ between adjacent neighbors that are previously unconnected topologically [202, 203].

FIGURE 3.4: **Topological transitions.** T1 transitions in epithelial tissues involves first the vanishing of an existing edge between a pair of particles originally in contact (yellow), followed by the creation of an edge between adjacent new neighbors (green). The degrees of freedom in cellular models are either defined on the vertices (blue) or the Voronoi centres (Black). Such topological transitions can emerge naturally for systems where the degrees of freedom are the Voronoi centers, but needs to be explicitly introduced for Vertex models.

In practice, ΔE is approximated by the maximum increase in $\mathcal{H}$ along a given smooth parameterization of the vanishing edge length l' , and the average of this

²⁶Note that these transitions cannot emerge purely from the dynamical rules Eq. (3.14) of the vertex model and can only be introduced “by hand”, typically using an additional stochastic rule or a threshold for minimum edge lengths.

quantity over all edges determines rigidity of a particular configuration, (i.e. an unjammed state is given by $\overline{\Delta E} \leq 0$). Hence, the ensemble-averaged energy barriers $\langle \overline{\Delta E} \rangle$ over different disorder realizations completely characterize phase behavior for a given p_0 and r at fixed densities ρ in the absence of fluctuations.

From this, a rigidity transition at the critical point $p_0^* = 3.81$ was identified by Bi et al²⁷ in [202, 203] where the system behaves as a solid for $p_0 < p_0^*$ but is fluid-like at $p_0 > p_0^*$. This rigidity transition is density independent unlike j -point, and resembles a second-order phase transition, where a collapse onto distinct branches corresponding to the two rheological phases is observed for a universal scaling function given by:

$$r\overline{\Delta E} = |p_0 - p_0^*|^\beta f_\pm \left(\frac{r}{|p_0 - p_0^*|^\Delta} \right) \quad (3.15)$$

where $f(x)$ is smooth and the critical exponents are reported as $\beta = 1.0 \pm 0.2$ and $\Delta = 4.0 \pm 0.4$. We note that this continuous scaling behavior near criticality exists in both phases of the rigidity transition, and is thus unlike pair-wise jamming, where the continuous behavior is found only in the jammed phase.

Alternatively, the rigidity transition in this model can also be recovered by observing the linear mechanical response such as the shear modulus $\mathcal{G}$ as discussed in Sec. 3.1.1. In systems with periodic boundaries, the elastic coefficient is typically approximated by the principle shear component of the Cauchy stress tensor

$$\mathcal{G} \sim G_{xy} = \frac{1}{L^d} \lim_{\gamma_{xy}} \frac{\partial^2 \mathcal{H}}{\partial \gamma_{x,y}^2} \quad (3.16)$$

for an infinitesimal shear strain $\gamma_{x,y}$ in the form of a Lee-Edwards boundary condition on the bounding box of the system. As with $\langle \overline{\Delta E} \rangle$, the shear modulus $\mathcal{G}$ acquires finite positive definite values at $p_0 < p_0^*$ much like a solid but precipitously decreases towards zero for $p_0 > p_0^*$. $\mathcal{G}$ also scale collapses onto two corresponding rigidity branches with the scaling function

$$r\mathcal{G} = |p_0 - p_0^*|^\beta g_\pm \left(\frac{r}{|p_0 - p_0^*|^\phi} \right) \quad (3.17)$$

but with an additional exponent $\phi = 5.0 \pm 0.5$ that characterizes the presence of a new length scale, such that at p_0^* , the shear modulus scales as $\sim r^{\beta/\phi}$ [203].

²⁷The lengthscale is $\rho^{-1/2}$. i.e. $\rho = 1$ in the numerics

Now on the other hand in the VM, configurations adhere to a symmetry that enforces the voronoi constraint of convex polygons at all times, such that $T1$ transitions naturally emerge from the dynamics, and without having a vanishing edge length. Hence rigidity measures derived from ΔE that are defined based on individual vertices and its resulting scale collapse (Eq. (3.15)) have no equivalent form in the VM. Furthermore, $\mathcal{G}$ (Eq. (3.16)) can transiently attain negative values even for rigid configurations that do not flow at very long timescales τ in the presence of active or thermal fluctuations in the steady state of physically relevant dynamics. Hence, a measure of rigidity that is independent of $\mathcal{H}$ (unlike $\mathcal{G}$ and ΔE), but instead directly dependent on the dynamical behavior of the degrees of freedom is desired. Such a measure is given by an effective diffusion parameter D that characterizes the mean-square displacement $\langle \Delta r^2(t) \rangle$ of $\mathbf{R}_i$.

$$D = \lim_{t \rightarrow \infty} \frac{\Delta r(t)^2}{4t} \quad (3.18)$$

such that $D \gg 0$ in the fluid and $D \sim 0$ in the solid phase of the system. In [204], this measure identified a solid-fluid transition at $p_0^* = 3.81$ that is similar to the vertex model in the absence of fluctuations. We emphasize and clarify that the motion of particles constituting displacements that enter the expression $\Delta r(t)^2$ in Eq. (3.18) can originate solely from the energy minimization protocol itself, and is thus for zero temperature dynamics, not related to thermal noise.

However, we note that there is indication that the observed crossover in VM at p_0^* is unlike a true rigidity transition in the vertex model, in that $\mathcal{G}$ approaches small but distinctively non-zero values. This view is supported by numerical evidence and a Maxwell-Calladine (Sec. 3.1.2) argument presented by Sussman and Merkel [226]. In particular, they noted that the number degrees of freedom in the VM is given by Nd , while the number of quadratic constraints that emerge from $\mathcal{H}$ in Eq. (3.12) is $2N$. This implies that the VM must be isostatic and marginally jammed for all p_0 in two dimensions, and thus no un-jamming transition exists, unlike in the vertex model. Jamming transitions can however exist in VM of higher dimensions. Regardless, $\mathcal{G}$ and other observables of the VM such as $\langle \Delta^2 r \rangle$ and $\langle \mathcal{H} \rangle$ for $p_0 > p_0^*$ are numerically orders of magnitude smaller than their counterparts in the rigid phase $p_0 < p_0^*$ and undergoes a distinct qualitative change at $p_0 = p_0^*$ [92, 204, 226], such that the cross-over is *effectively* an un-jamming transition.

In addition, we also note that the geometrical based interactions in the VM, are not spherically symmetric and this asymmetry is known to influence Maxwell-Calladine counting arguments made for Jammed packings in pairwise interacting systems. Most notably, Elliptic interactions of such "shaped-annealed" packings in soft disks can result in hypostatic rigid structures [193].

Lastly we remark that VM improves on the vertex model in terms of numerical computability by reducing the degrees of freedom for a given number of cells, since the positions of vertices are now uniquely determined by the Voronoi centers that come fewer in number.

3.3 Hidden long-range order at the rigidity transition of the Voronoi Model

In this section, we study the density fluctuations (Ch. 2) of energy minimal configurations at zero-temperatures in model biological tissues (Voronoi model) (Sec. 3.2.1) near the critical point p_0 of rigidity transitions (Sec. 3.2.2), where degrees of freedom $\mathbf{r}_i$ are Voronoi centres of convex polygons that represent cells in the confluent partitioning of $2D$ space.

In particular, we first examine in Sec. 3.3.1, the real space number fluctuations $\delta^2 N(R)$ (Sec. 1.2) at differing coarse-grained length scale R and show that configurations in the solid phase ($p < p_0$) of the model are fluctuation suppressed only up to finite lengthscale $l_{\delta^2 N}^*$. We extract $l_{\delta^2 N}^*$ numerically and show that it monotonically increases with p_0 and diverges at p_0^* with an exponent $\nu \sim 0.83$ that is consistent with $\beta = 1.0 \pm 0.2$ derived from rheological considerations for $\mathcal{G}$ and ΔE as discussed in Sec. 3.2.2.

Next in Sec. 3.3.2, we study the Fourier space representation for the anomalous density fluctuations known as the structure factor $S(k)$. Through quantifying the effective hyperuniformity index (Sec. 2.3), we show that $S(k)$ approaches a finite non-zero value in the $k \rightarrow 0$ limit for configurations in the solid, but undergoes a precipitous decrease at p_0^* such that $S(k) \sim k^4$ in the fluid. This indicates that the system is Class I hyperuniform (Sec. 2) and corroborates the observed power law in the scaled density fluctuations - $\langle \delta^2 N \rangle / R^2 \sim R$ in the limit of $p_0 \rightarrow \infty$. We further show that the system is stealthily hyperuniform by examining $S(\mathbf{k})$ in 2D.

Lastly, in Sec. 3.3.3, we discuss our results in context of the jamming-hyperuniformity conjecture in “strictly”-jammed hard sphere solids, and argue for the perspective that hyperuniformity is not generically found in jammed solids beyond pair-wise interacting hard spheres.

3.3.1 Diverging lengthscales of hyperuniformity

We begin by numerically evolving dimensionless $\mathcal{H}$ (Eq. (3.13)) using a conjugate gradient algorithm [225] starting from random uncorrelated initial conditions of the point distribution²⁸. Unless otherwise stated, each energy minimal state is given by 2D Voronoi tessellations of $N = 16384$ particles in a square box of length $L = \sqrt{N}$ under periodic boundary conditions, such that the system is always confluent with mean area $\bar{A} = 1$. Fluctuation statistics are obtained from averaging 25 unique realizations at any given value of p_0 , and 10^4 observation windows for each value of R of each realization. Note that we have also verified that the qualitative picture presented here, does not change if the energy minimization protocol is performed using overdamped dynamics [204] (i.e directly evolving the equations of motion eq. (2) in [204]).

In Fig. 3.5, we show the scaled density fluctuations $\langle \delta^2 N \rangle / R^2 \sim R^{\epsilon-2}$ of energy minimal configurations of the model specified in sec. 3.2.1 as a function of coarse-grained R for various p_0 in the rigid phase ($p_0 < p_0^*$). In the solid, this system exhibits anomalous suppression of density fluctuations consistent with hyperuniformity ($\epsilon < 2$) at shorter length scales²⁹. However, we see that at some value of R dependent on the value of p_0 , the system exhibits a continuous crossover to a plateau in its scaled fluctuations ($\epsilon = 2$), suggesting that it is in fact not hyperuniform. At even larger R the fluctuations decay due to finite size effects, which we explore in the Appendix A.

Here, we note that the study of hyperuniformity in this model has very recently been examined in context of generating photonic band gaps as an approach to designing novel materials [146]. However, the smaller system sizes considered were insufficient to probe the longer wavelength behavior of the system and thus Li et al. [146] were unable to conclude on the true extent of fluctuation suppression of its density. Indeed, here we find that while density fluctuations are anomalously suppressed at shorter length scales of the system, there exists a crossover of the density fluctuation to a scaling more reminiscent of random configurations ($\langle \delta^2 N \rangle / R^2 \sim R^0$) at larger length scales, representing a significant departure from the key assumption in [146].

²⁸Termination condition for energy minimization is $\Delta E / N < 10^{-8}$ for five consecutive iterations where ΔE is the change in energy per minimization, or after a total of 10^4 steps.

²⁹Oscillations observed at even shorter values of R are due to resonant peaks in $g_2(r)$ corresponding to interparticle lengthscales a_0 .

FIGURE 3.5: **Real space fluctuations $\langle \delta^2 N \rangle$ of the Voronoi Model.** $\langle \delta^2 N \rangle / R^2$ exhibits a continuous crossover from power law behavior at shorter length scales where density fluctuations are strongly suppressed to behavior reminiscent of random configurations at longer length scales. The (solid) lines represent free fits of the fluctuations to Eq. [3.19] over the domain $R_{min} < R < L/2$ which we show for $p_0 = 3.00, 3.20$ and 3.40 . Corresponding chi-square per degree of freedom (χ_ν^2) for the respective fits are also listed. Density fluctuations in the fluid ($p_0 = 4.00$) (black dashed line) are strongly suppressed and the system is effectively hyperuniform. **[Inset]** A fit (blue line) of the density length scale (black circles) to $l_{\langle \delta^2 N \rangle}^* \sim ||p - p_0^*||^{-\nu}$ with $\nu = 0.83$ and $p_0 = 3.813$ fixed at their respective values from the scale collapse. Details of fit parameters and discussion on finite size effects can be found in Appendix A

Now, by assuming the following general Ansatz for the hyperuniformity crossover:

$$\langle \delta^2 N \rangle / R^2 \sim \beta \left(1 + \left(R / l_{\langle \delta^2 N \rangle}^* \right)^{-s} \right)^{\alpha/s} \quad (3.19)$$

we perform a free fit of the region in between cut-off lengthscales indicated by the black-vertical lines, such as to ignore the oscillatory behavior of fluctuations

in the short lengthscale regime and also to mitigate any influence of finite size effects. In doing so, we extract a p_0 dependent lengthscale $l_{\langle \delta^2 N \rangle}^*$ (Fig. 3.5 [Inset]) in addition to the fit parameters β , s and α for which numerical values can be found in Appendix A.

We find that $l_{\langle \delta^2 N \rangle}^*$ is indeed monotonically increasing with p_0 (Fig. Fig. 3.6, and a subsequent fit of $l_{\langle \delta^2 N \rangle}^*$ (blue solid line) to the scaling Ansatz: $l^* \sim ||p_0 - p_0^*||^{-\nu}$ with $p_0^* = 3.813$ and $\nu = 0.83$ held fixed indicates that the length scale extracted from the behavior of $\langle \delta^2 N \rangle$ is consistent with its divergence at the known rigidity transition [203, 204]. Here, we note that slope of the power law regime in fluctuations of rigid states increasingly steepens and approaches that of the fluid (black-dots), as also indicated by the behavior of $\alpha \rightarrow 1$ as $p_0 \rightarrow p_0^*$.

FIGURE 3.6: **Scale collapse of $\langle \delta^2 N \rangle$ in the Voronoi model.** (b) Scale collapse of $\langle \delta^2 N \rangle / R^2$ where $\Delta p_0 \equiv ||p - p_0^*||^{-\nu}$, $p_0^* = 3.813$ and $\nu = 0.83$. Master curve from the scale collapse is given by the following fit parameters of Eq. [3.19] : $\alpha' = 0.98, \beta' = 0.024, s' = 3.19, l'_{\langle \delta^2 N \rangle} = 11.1$ (black line). [Inset] A partial snapshot of a Voronoi tessellation at $p_0 = 3.20$.

The value of ν used in obtaining this fit of the lengthscales come from a scale collapse of the fluctuations that we show in (Fig. 3.6) by plotting $\langle \delta^2 N \rangle \Delta p_0^{-\nu} / R^2$ against $R \Delta p_0^\nu$. In particular, this scale collapse is performed by a scaling of the x - and y - axes of Fig. 3.5 with respect to the distance from the critical point $\Delta p_0 \equiv p_0^* - p_0$, where p_0^* is held fixed at the known value of 3.813. We further remark that $\nu \sim 0.83$ found here is consistent with $\beta = 1.0 \pm 0.2$ in [203], but the uncertainty of β is too large to be conclusive of its coincidence. Furthermore, there is no physical reason to believe that $\nu = \beta$, especially since as we shall show in Ch. 4, this is not the case in hyperuniform absorbing states near criticality of active-absorbing state transitions in random organization models.

A subsequent fit of all scaled quantities for various p_0 in the solid regime to eq. (3.19) also yield parameter estimates for the master curve that the fluctuation profiles seem to collapse on ($\alpha' = 0.98, \beta' = 0.024, s' = 3.19, l'_{\langle \delta^2 N \rangle} = 11.1$). We note that $\alpha' = 0.98$ is in good agreement with the assumption that a single exponent ν controls the scaling behavior near criticality which we had made in obtaining the scale collapse previously in Fig. 3.5, and is also indicative that the fluid state of the system is Class I hyperuniform Eq. (2.16).

Here we also note that $l_{\langle \delta^2 N \rangle}^*$ extracted for larger values of p_0 are greater than the cut-off lengthscale of $L/2$ and are also in a region where energy minimization is known to be difficult [203]. This critical slowing down of the dynamics has been known to influence the correct interpretation of analyses on hyperuniformity [98], but we stress that our results remain robust and consistent with the observed rigidity transition at $p_0^* = 3.81$.

In fact, all state points in the fluid regime results in observations of effective hyperuniformity, even where energy minimization is never truly completed, since $l_{\langle \delta^2 N \rangle}^*$ is expected to have already diverged. As an example, Fig. 3.5 show the behavior of the density fluctuations deep in the fluid phase ($p_0 = 4.00$, black circles). Here we also note that, according to a previous work [157] hyperuniformity is indeed expected in the fluid phase, since both the probability density of the area and perimeter of the cells approach a delta function, as we illustrate in the appendix Fig. A.3. Hence we emphasize that while the variance in the geometrical distributions can be considered an alternative and complementary measure [157] of the length scale $l_{\langle \delta^2 N \rangle}^*$, we show that it is non-trivially related to p_0 and the known rigidity transition of this model.

This in turn represents a qualitative departure from conventional discussions of hyperuniformity at the jamming transition, since hyperuniformity or effective hyperuniformity is not traditionally a behavior associated to the fluid phase of the system. In this regard, it is worth emphasizing that the fluid phase of the Voronoi model is qualitatively distinct from the unjammed phase of sphere packings; While both models have zero shear moduli, unjammed packings of spheres are locally unjammed [227], while the fluid phase of the Voronoi model is locally jammed, as all particles are in a local minimum of the energy.

Lastly, we note that nearly hyperuniform disordered Voronoi tessellations have also recently been reported to be ground states of the Quantizer problem [103]; we remark that this model differs from the Voronoi model for tissues that we consider, not only because it is based on a different energy functional, but most importantly because it does not exhibit a rigidity transition.

3.3.2 The stealthy fluid state

In this section, we study the structure factor $S(k)$ Eq. (2.7) defined in Sec. 2.2, as a complementary perspective to behavior of the real space fluctuations $\langle \delta^2 N(R) \rangle$ found to be exhibiting a hyperuniformity crossover discussed in Sec. 3.3.1.

In fig. Fig. 3.7, we show the k -averaged scalar $S(k)$ as a function of the magnitude k of the wave-vectors for various values of p_0 for the same data used to calculate $\langle \delta^2 N(R) \rangle$ in Sec. 3.3.1. We see that $S(k)$ near p_0^* can be non-monotonic in its approach of $k \rightarrow 0$ for $p_0 < p_0^*$ (solid), and seem to approach a limiting (log-scale) plateau in the large lengthscale limit, indicating that these systems are in fact not hyperuniform as we have also concluded from our results of the density fluctuations. Moreover, we see that this plateau level decreases gradedly with $p_0 \rightarrow p_0^*$, and is $\sim \mathcal{O}(3)$ smaller for the configuration in the fluid phase ($p_0 = 4.00$). Here, notice that beyond a small plateau region of $S(k)$, for which the calculated value is likely overestimating due to the impossibility of perfectly minimizing $\mathcal{H}$ in the fluid phase, the structure factors quickly grows as $S(k) \propto k_k^\alpha$, with $\alpha_k \simeq 4$.

To make this behavior of $S(k)$ more precise, we approximate the limiting behavior $S(k = 0)$, by an average of the low- k plateau regime and evaluated the effective hyperuniformity index $H(p_0)$ [163] which we show in fig. 3.7[Inset] (see Sec. 2.2).

FIGURE 3.7: **Structure factor $S(k)$ in the Voronoi model.** $S(k)$ and corresponding errors as a function of k for various values of p_0 . Monotonically decreasing behavior of $S(k \rightarrow 0)$ as a function of p_0 is captured by the [Inset] effective hyperuniform index $H = S(0)/S(k_{peak})$ obtained from approximation of the plateau level of $S(k)$ that precipitously decreases in the fluid regime.

We see that the value of this index approach the threshold value $\sim 10^{-3}$ for effective hyperuniformity as $p_0 \rightarrow p_0^*$, and precipitously decrease for configurations in the fluid phase. These results are consistent with a picture of a growing lengthscale of fluctuation suppression that diverges at the known rigidity transition ($p_0 = p_0^*$) that we have shown previously in Sec. 3.3.1 and thus indicates that our results derived from real space are robust, as is also the case for jammed soft disk [90]. Moreover, we note that the value of the effective hyperuniformity index in the fluid phase is two orders of magnitude smaller than the smallest reported value of all effectively hyperuniform systems known thus far [103].

Moreover, this dramatic decrease in $S(k)$ near $k \rightarrow 0$ in the fluid phase does not arise from extremal values or edge cases in scattering associated to specific

FIGURE 3.8: **2D $S(\mathbf{k})$ of the Voronoi model.** 2-dimensional $S(\mathbf{k})$ in log-scale for energy minimal configurations in the [Left] rigid ($p_0 = 3.00$), and [Right] fluid ($p_0 = 4.00$) phase of the many-body interacting model. Precipitous decrease by more than 6 orders of magnitude in $S(k)$ in an annulus about $k = 0$ for $p_0 = 4.00$ is indicative that hyperuniformity observed in the fluid phase is *stealthy*

directions of wave vectors in $\mathbf{k}$ -space, as can be seen from the continuous annulus of anomalously low $S(k)$ values in the neighborhood of $\mathbf{k} = 0$ for the 2d structure factor shown in fig. 3.7(c,d). This suggest that the hyperuniformity observed in the fluid phase of this model is not $\mathbf{k}$ -dependent and of the *stealthy* variant [163]. A subset of Class I hyperuniform systems that is consistent with the prior observations of $S(k) \sim k^4$ and $\langle \delta^2 N \rangle / R^2 \sim R^{-1}$ in the fluid phase.

Here we note that the model examined in this work utilizes Voronoi centres as the motional degrees of freedom. Hence, unlike the vertex model [with an identical energy functional, Eq. (3.13)], the present system does not undergo an unjamming transition in that its ground state energy is small but remains non-zero even in the fluid state [226] as discussed in Sec. 3.2.2. This implies that hyperuniformity observed in the fluid state of this model is likely imperfect [103] and is constrained by the finite system size considered. Regardless, we stress that the behavior of this model across the critical point robustly produces drastic changes in dynamical and mechanical observables up to several orders in magnitude [226]. It is this precipitous change in the rigidity properties of the system that we draw connection to, especially given the similarity in behavior of the effective hyperuniformity index

(Eq. (2.18)) observed in Fig. 3.7[Inset]. Hence, this also implies that while our results are shown to be consistent with $p_0^* = 3.813$, the true nature or precision of the critical point remains beyond the sensitivity of the present numerics, adding further to our inability to comment on whether $\nu \sim \beta$ has any physical significance.

3.3.3 Jamming-hyperuniformity conjecture

In Sec. 3.1.3 we discussed speculations and debate over whether jammed soft solids of pair-wise interacting potentials are hyperuniform. This extends from a conjecture that exists for hard sphere interactions [87, 88]

Conjecture 3.1. *Any strictly jammed saturated infinite packing of identical spheres in R^d is hyperuniform.*

where a system is said to be “strictly jammed” if all uniform volume-decreasing strains of the system boundary is disallowed, implying that their bulk and shear moduli are infinitely large [228].

This thus excludes soft interactions where the potentials remain finite for some values of r_{ij} , and also imply that the presence of rattlers may obscure the hyperuniformity present in the thermodynamic limit. Moreover, an analytical tractable model with non-vanishing complexity³⁰ has been found to violate this conjecture [158], even for effectively 1D hard spheres (i.e a system that is confined to a narrow channel slightly larger than the inter-particle lengthscale a_0)³¹. Hence as [158] and [94] conclude, there is simply no reason to believe that perfect hyperuniformity exists in the rigid phase of such systems.

In the Voronoi model, local stiffness of particles monotonically increases as $p_0 \rightarrow 0$ (the limit where cells prefer to be infinitely elongated) in the solid, and the effective interactions become increasingly isotropic since the preferred geometry acts to drive local configurations to unattainable spherical symmetries for $p_0 < \pi$. This limit thus simulate certain behavior of exclusion interactions in jammed hard spheres, but yet are increasingly fluctuation suppressed at shorter length scales.

³⁰The complexity Σ is a Edwards ensemble measure that scales with the logarithm of the number of jammed configurations for a given density ρ and volume V

³¹In principle one could argue that the conjecture is not violated since the confined geometry implies that the system is not infinite in some spatial dimensions.

This indicates that the jammed solid at infinite stiffness is likely not hyperuniform, which at the very least places limitations on the generalizability of the jamming-hyperuniformity conjecture.

Instead our results seem to indicate that density fluctuations in jammed solids are only suppressed up to finite length scales and are thus not hyperuniform. However, these lengthscale grow and diverge at the rigidity transition such that a system at criticality ($p_0 = p_0^*$) is hyperuniform. This is very much like the scenario that has been found for pair-wise jamming and at the critical j -point³² [94, 95] as discussed in Sec. 3.1.3.

In conclusion, we speculate that perfect hyperuniformity could instead be a generic feature only of the critical point at rigidity transitions instead of the entire jammed-phase as proposed by the jamming-hyperuniformity conjecture. Lastly, we also draw attention to a more recent observation that randomly jammed spheres suppresses the fluctuation in their coordination number Z up to a length scale that diverges at the jamming transition [116]. It remains an interesting open question if there indeed exists an analogous scenario in the present model, where enhanced suppression or hyperuniformity beyond density exist for some physical quantities, that may hint at a more fundamental causal mechanism for hyperuniformity.

³²Note however that unlike in the voronoi model, typical pair-wise interacting liquids are not hyperuniform

Chapter 4

Hidden order beyond hyperuniformity

In this chapter, we discuss new forms of hidden order that exists beyond hyperuniformity in disordered complex systems discussed in Ch. 2 and Ch. 3. In particular, we reveal the presence of higher order many-body correlations in the absorbing state of active-absorbing phase transitions¹ in models² belonging to the universality class of conserved directed percolation³. These models are known to be fluctuation suppressed at finite lengthscales in the absorbing state, which grows and diverges at the transition such that the critical point is hyperuniform. We further show that these correlations existing beyond hyperuniformity are also hidden in the higher moments of the probability distribution of the local density $P(\rho_R)$ and extend up to a longer length scale that diverges on approaching the critical point at a rate faster than what is indicated by the hyperuniform lengthscale ξ_h . Our results thus suggest that hidden order beyond hyperuniformity may generically emerge in coarse-grained descriptions of hyperuniform complex systems.

This chapter is organized as follows. In Sec. 4.1, we introduce the non-equilibrium active-absorbing state transition that is observed in models of conserved directed percolation. This involves discussing the Conserved Lattice Gas (CLG) and Random Organization (RO) models in Sec. 4.1.1 and the associated order parameters

¹A non-equilibrium phase transition between a static absorbing states (fixed point in configuration space) and an active steady-state of the system which we discuss in detail in Sec. 4.1

²A class of particulate models in continuous or discrete space, with dynamical rules that are similar to cellular automata [229].

³Minimal models that simulate filtering of fluids through porous materials in a given direction.

and scaling exponents $\nu_\perp$ that govern steady state quantities in the thermodynamic limit (Sec. 4.1.2). We then review the fluctuation behavior of coarse-grained density in the absorbing state, and discuss existing speculations of hyperuniformity and its precise nature at critical points ρ_c or ϕ_c (Sec. 4.1.3).

In Sec. 4.2, we reveal the presence of hidden order beyond hyperuniformity in these models. First we demonstrate in Sec. 4.2.1 that the fluctuation suppressed lengthscale ξ_h grows and diverges at the critical point with an exponent $\nu_h \sim 0.6$ that is against prior assertions, not equivalent to the dynamical exponent $\nu_\perp \sim 0.8$. We consolidate evidence for this contrasting perspective in Sec. 4.2.2 by discussing

- The real-space density correlation function C_{ρ_R} of coarse-grained density ρ_R - $C_{\rho_R} \sim \langle \rho_R(r)\rho_R(0) \rangle - \langle \rho_R \rangle^2$ at $R = \xi_h$, noting that its corresponding length-scale decays exponentially with R , and diverges at the critical point with exponent $\nu_{\rho_R} \sim 0.8$. C_{ρ_R} thus do not scale collapse when plotted against r/ξ_h .
- The correlation function $C_{t_f}(r) \sim \langle t_f(r)t_f(0) \rangle - \langle t_f \rangle^2$ of the mean freezing time t_f defined as the ensemble average final time at which a particle transitions from an active to passive state. As with C_{ρ_R} , we show that C_{t_f} scale collapses with $\xi_{t_f} \sim (\rho - \rho_c)^{0.8}$ but not ξ_h .

The behavior of these correlation lengths indicates that structural correlations beyond hyperuniformity is already present in the variance σ^2 of $P(\rho_R)$ and governs spatial correlations in the dynamics leading to the absorbing state. We further show that these correlations between t_f and ρ_R indicative of hidden order beyond hyperuniformity are non-trivial by considering iso-configurational⁴ averages. From this, we conclude that the observed inhomogeneities at $R \geq \xi_h$ originates from dynamical heterogeneity across real-space sub-regions of the system.

Furthermore, we show in Sec. 4.2.3 that additional hidden order is embedded in the higher centralized moments (Skewness- γ and Kurtosis- κ) of $P(\rho_R)$. This manifest in the form of a non-monotonic convergence of κ and γ to their respective Gaussian values at large R ⁵ that deviates from predictions of the Central Limit Theorem (CLT). Surprisingly, the higher moments scale collapse with exponents that differ

⁴Averaging across the different absorbing states that are accessible by the dynamics from a single seed condition

⁵The Gaussian distribution has a Skewness of $\gamma=0$ and $\kappa = 3$ [100].

from both $\nu_{\perp}$ and ν_h discussed in Sec. 4.2.1 and Sec. 4.2.2. In fact, this non-Gaussian behavior extend up to further distances than ξ_h , implying that while ξ_h indeed diverges at ρ_c and the system is fluctuation suppressed at $r \leq \xi_h$, there exists a hierarchy of scales where the behavior at lengthscales larger than ξ_h is counter-intuitively heterogeneous and non-Gaussian.

In Sec. 4.2.4, we draw analogy between this observed phenomenology in the models for conserved directed percolation to that of the displacement fields in super-cooled liquids, and introduce a similar classification of systems based on the structural correlations and higher moments of coarse-grained density distributions. Lastly, we discuss design elements involved in programming or concealing hidden order beyond hyperuniformity in point patterns (Sec. 4.2.5).

4.1 Active-absorbing state transitions

Active-absorbing state transitions [230–232] are non-equilibrium phase transitions [233, 234] (Sec. 1.1) between *active* states where a finite fraction of particles ρ_a remain ‘active’ (i.e. its position evolves under some dynamical rule) in the steady state at infinite time, and an *absorbing* state where fixed points of dynamics accessible at finite times, represent static configurations where only passive particles exists (i.e. $\rho_a = 0$) [93, 159].

These transitions emerge as descriptions of complex matter that include non-conservative systems such as non-abelian sandpile models used in the study of self-organized criticality⁶[1, 235–238], models of directed bond or site percolation capturing essential physics in phenomena as diverse as the complex flow of fluid through porous materials [239], depinning transitions [240], epidemiology or damage spreading [241], and various abelian variants belonging to the manna universality class [237, 242] of conserved directed percolation [230, 231, 233, 243].

In this chapter, we examine in detail two particular models. Namely, the two-dimensional Conserved Lattice Gas (CLG) [159, 230, 243] and Random Organization models (RO) [128, 130] that exhibit an active-absorbing state transition at a critical density (ρ_c or ϕ_c), where it is believed to be hyperuniform. Such models encode a short ranged repulsion between nearby interacting degrees of freedom which

⁶A dynamical system where the attractor in phase space is a thermodynamical critical point.

mimics the repulsion between colloidal particles in experiments involving sheared suspensions such as [130].

We note that extensions to such models have been made to include the effect and influence of having multiple absorbing sites [244–246], effective caging mechanisms to simulate local jamming-like interactions [247], and generalizations to various lattice symmetries or contact topologies [231]. We remark that the fluctuation behavior and hyperuniformity (or its lack thereof) in these model is largely unknown.

4.1.1 Conserved Lattice Gas (CLG) and Random Organization (RO) Models

Here, we specify dynamics of the Conserved Lattice Gas (CLG) and Random Organization (RO) models that exhibit active-absorbing state transitions belonging to the universality class of conserved directed percolation. We note that while specification of the models in this section and subsequent results are derived predominantly from 2-dimensional systems, these models are easily generalizable to arbitrary dimensions or lattice topologies.

[Conserved Lattice Gas (CLG)] [159, 230, 248, 249] - Particles are placed initially at random on an $L \times L$ periodic lattice with mean density $\rho \equiv N/L^2$. A particle is deemed active if one or more of its immediate neighboring sites is occupied⁷, and active particles with at least one adjacent empty site move in each time step, in parallel⁸, randomly to an adjacent unoccupied site such that the global density ρ of the system is conserved throughout its dynamics. This process mimics short range repulsion in conserved systems.

[Random Organization (RO)] [128, 130, 238, 250] - N monodisperse disks of diameter $\tilde{d}$ occupy a regular simulation domain of side length L , at volume fraction $\phi = \rho\pi(\tilde{d}/2)^2$ where overlapping disks are deemed active. At each time step, active disks move by uncorrelated displacements of random orientation and magnitude d_0

⁷Neighboring site is determined by the lattice contact topology which here we consider to be a square lattice (i.e. $2d$ neighbors per site). Provisions for dynamical transitions between higher ordered neighbors are possible in variants of CLG.

⁸Note that serial update rules were adopted in earlier models of CLG [230] in that a single random active particle is moved in each time step of the dynamics. This results typically in a difference of the critical density ρ_c at the active-absorbing state transition, but retain similar critical exponents that govern diverging length- and timescales.

that is uniformly distributed across the range $[0, \delta]$. Note that subsequent numerical simulations are performed using periodic boundary conditions and with $\tilde{d} = 1$ and $\delta = 0.5$. This RO model is inspired by recent experimental investigations into colloidal suspensions under oscillatory shear [130, 155], that have recently been suggested to be hyperuniform at their active-absorbing transition [129]. Schematics of the two models are as follows in Fig. 4.1

FIGURE 4.1: **Conserved lattice gas and random organization models.** Schematic representing the dynamical rules governing the [Left] Random Organization(RO) and [Right] Conserved Lattice Gas (CLG) models. Particles are considered active (indicated by the red circles for RO and yellow squares for CLG respectively), if they are in contact at time t [Top]. Active particles are displaced in a random direction to new positions at time $t + 1$ [Bottom], at a distance $0 < d_0 < \delta$ in the RO model or to adjacent vacant sites in CLG, such that the number of particles in the system is conserved at all times. Figure partially adapted from [128].

Here, we also note that absorbing states of an isotropically biased variant of the RO model [251] has also recently been found to reproduce characteristic density and structural features of random close packings in jammed configurations.

4.1.2 Critical phenomena and phase transitions

Given the dynamics introduced in Sec. 4.1.1, it is conceivable that the CLG and RO models initialized from random configurations result in the possibility of two distinct outcomes. First, at sufficiently low density ρ or packing fraction ϕ where particles are sparsely located, the fraction of active particles $\rho_a(t)$ rapidly decays to zero, such that the system reaches an *absorbing* state where only passive particles exist. On the other hand, in the high density limit, particles displaced at each time step would still likely be in contact with other particles and hence remain active regardless of the microscopic dynamics that are effectively driving active particles apart. Hence, this system approaches a steady state where a finite fraction of particles ($\rho_a \equiv N_a/N$) remain persistently active and fluctuates about an ensemble-averaged mean value $\langle \rho_a \rangle$ in what is referred to as an *active* state.

The active-absorbing state transition, is thus a density driven phase transition at some critical density ρ_c or packing fraction ϕ_c where the order parameter $\langle \rho_a \rangle$ completely determines phase behavior. The critical points are in general dependent on the dimensionality or symmetries of the given space and further specifics / additional constraints of the model [231, 243]. In particular, $\rho_c \sim 0.2391$ and $\phi_c \sim 0.375$ respectively for the two-dimensional CLG and RO models with the given parameters specified in Sec. 4.1.1 which we consider numerically in this chapter.

Now in the CLG model, the approach to ρ_c from the active phase (i.e. $\rho \rightarrow \rho_c^+$) behaves like a second order phase transition in that the order parameter $\langle \rho_a \rangle$ scales continuously as a power law to zero, i.e.

$$\langle \rho_a^{CLG} \rangle \sim (\rho - \rho_c)^\beta \quad (4.1)$$

with the critical exponent $\beta \sim 0.64$ [230, 231] for all models belonging to the universality class of conserved directed percolation. Next, through scale collapse of ρ_a , in what is typical finite-size analysis of critical phenomena, using the universal scaling function of the form

$$\rho_a(\Delta\rho, L) = L^{-\beta/\nu_\perp} \mathcal{G}(\xi_a^{1/\nu_\perp} \Delta\rho) \quad (4.2)$$

where L is the system size, $\Delta\rho = \rho - \rho_c$ and $\mathcal{G}(x)$ is some continuous real function, an active site - active site correlation length ξ_a is identified [159, 230, 252] and

diverges at the critical point as a power law

$$\xi_a \sim (\rho - \rho_c)^{-\nu_\perp} \quad (4.3)$$

where $\nu_\perp \sim 0.8$ in 2D is often referred to as a dynamical exponent of the active-absorbing phase transition ⁹ [233, 248, 248].

Lastly, since the correlation function of active sites is written $C_a(r) \sim r^{2-d-\eta_\perp}$ ¹⁰, the exponent $\eta_\perp$ relates β and $\nu_\perp$ for all critical active-absorbing state transitions through the equation [243]

$$(2 - d - \eta_\perp)\nu_\perp = -2\beta \quad (4.4)$$

Note that this scenario discussed at the critical point ρ_c of the CLG model, and the entire discussion of the universal scaling exponents also applies to the RO model in the limit of $\phi \rightarrow \phi_c^+$. Furthermore, we note that at this point in our discussion, all exponents, order parameters and properties of critical behavior are derived solely from the steady state in the active phase, and with no reference to the behavior of passive states in the absorbing phase.

In summary, the exponents that characterize the active-absorbing transition relevant to this chapter and our discussion of hidden order, are given in Table 4.1.

Critical exponent	$d = 2$
β	0.64
$\nu_\perp$	0.8
$\eta_\perp$	1.541

TABLE 4.1: **Critical exponents of the active-absorbing transition.** Numerical values of the critical exponents in 2 dimensions

⁹In addition to the exponent $\nu_\parallel$, which we do not discuss here

¹⁰Note that this expression for the correlation function is general, and in fact valid for all translationally invariant systems

4.1.3 Hyperuniformity and anomalous density fluctuations of critical absorbing states

In this section, we turn to the absorbing states of the CLG and RO models ($\rho \leq \rho_c$ or $\phi \leq \phi_c$), where configurations consisting only of passive particles are obtained from the dynamics in finite time. At shorter lengthscales, coarse-grained fluctuations scale characteristically as a power law

$$\sigma^2(R) \sim R^{-\lambda} \quad (4.5)$$

and are ubiquitously observed to be anomalously suppressed up to a finite hyperuniform lengthscale ξ_h , where λ appears to be a *universal*, albeit dimensionally dependent exponent

$$\lambda = d + 2 - \eta_{\perp} \quad (4.6)$$

consistent with the scaling behavior of the translationally invariant total correlation function $h(r)$ discussed in Sec. 2.1. Specifically, we note that $\lambda \sim 2.45$ for $d = 2$. At longer lengthscales ($R \gg \xi_h$), fluctuations scale as $\sigma^2(R) \sim R^{-2}$, which corresponds to scaling behavior indicated by the Central Limit Theorem (CLT) for uncorrelated ρ_R . See Fig. 4.2 for a representative plot of σ_R^2 exhibiting this behavior in an absorbing state of the CLG model. Furthermore, ξ_h have been observed numerically to be monotonically increasing and seemingly divergent, as $\rho \rightarrow \rho_c$ ¹¹. Hence, these active-absorbing state models have been suggested to exhibit a hyperuniformity crossover at the critical point [159] where density fluctuations that were suppressed in the absorbing phase up to a length scale $\xi_h \propto |\rho_c - \rho|^{-\nu_h}$ diverge, implying that configurations at the critical point are perhaps truly hyperuniform.

In [159], it was speculated based on limited empirical observations of fluctuations in relatively small systems¹² that $\nu_h = \nu_{\perp}$. However, as the authors themselves also note, there is no fundamental reason to believe that this equality is exact or anything more than a mere coincidence, especially since $\nu_{\perp}$ is extracted from dynamical behavior of active particles in the active phase of the transition ($\phi \rightarrow \phi_c^+$), while ν_h finds its origins from the static structure of an absorbing state ($\phi \rightarrow \phi_c^-$). In other words, ν_h and $\nu_{\perp}$ are derived from entirely different phases and thus should not at first pass be trivially dependent.

¹¹or equivalently $\phi \rightarrow \phi_c^-$ in the RO model

¹² $L = 1024$ for the CLG model and $N \sim 4 \times 10^4$ particles for the RO model

FIGURE 4.2: $\sigma^2(R)$ and hyperuniform lengthscales ξ_h in absorbing states. Log-Log plot of σ^2 against R in the CLG model for $\rho = 0.2385$. At $R \leq \xi_h$, $\sigma^2 \sim R^{-\lambda}$ is fluctuation suppressed and scales with a universal exponent $\lambda = 2.45$. At longer lengthscale the system recovers central limit theorem scaling of $\sim R^{-2}$ for uncorrelated ρ_R . This is illustrative qualitatively of the typical behavior in the fluctuation profile for configurations in the absorbing phase near the critical point ρ_c .

In fact, as we shall show subsequently in Sec. 4.2, a careful analysis of $\sigma^2(R)$ for significantly larger system sizes in both the RO and CLG models, yield a scale collapse with a universal hyperuniform exponent $\nu_h \sim 0.6 \neq \nu_{\perp} \sim 0.8$.

4.2 Beyond Hyperuniformity

In this section, we reveal the presence of hidden order beyond hyperuniformity in $\sigma^2(R)$ that extends to length scales longer than ξ_h in Sec. 4.2.1 and elucidate their significance to the dynamical origins of absorbing states in the RO and CLG models (Sec. 4.2.2). Additionally, we show in Sec. 4.2.3, that hidden correlations also leave imprints in the higher centralized moments of $P(\rho_R)$, and they scale collapse with

novel exponents that do not correspond to any of the known existing dynamical or static lengthscales governed by $\nu_\perp$, ν_h , β and $\eta_\perp$. Lastly, we propose a classification system for various hyperuniform systems based on the empirical behavior of these higher moments (Sec. 4.2.4), and discuss how hyperuniform point configurations can be tuned to reflect hidden order beyond hyperuniformity in its $P(\rho_R)$.

Numerical results presented in the entirety of this chapter for the CLG model are for systems of linear size $L = 8192$, if not otherwise stated, and $\sim 15 \times 10^6$ particles. These systems are up to two orders of magnitude larger than what has previously been considered in the literature [159, 243]. For the RO model, we consider system sizes of typically $N = 8192 \times 10^3$ disks, which again is well above what has been previously considered [128]. Such large systems are needed due to the strong influence of size effects on the moments of $P(\rho_R)$. Additionally, all values presented are averaged over at least 50 independent realizations.

4.2.1 Hyperuniform scaling exponent

In Fig. 4.3, we examined the behavior of scaled $\sigma^2(R)$ as a function of the coarse-grained lengthscale in the CLG and RO model of systems significantly larger than what have been previously considered¹³. Firstly, we observe that at shorter lengthscales, density fluctuations are anomalously suppressed with $\sigma^2(R) \sim R^{-\lambda}$ where $\lambda = 2.45$ is found to be universal across the broader class of two-dimensional random organization models as discussed in Sec. 4.1.2, and in good agreement with [159]. This suppression extends up to a hyperuniformity length scale $\xi_h \propto \Delta\rho^{-\nu_h}$ consistent with the divergent behavior at known values of the critical points ϕ_c or ρ_c discussed in Sec. 4.1.1, and can be defined by the respective values of R where the scaled fluctuations $R^2\sigma^2(R)$ reaches a minimum.

However, departing from previous suggestions that $\nu_h \simeq \nu_\perp \simeq 0.8$ [159], we find instead that $\nu_h \simeq 0.6$, which in combination with $\lambda \simeq 2.45$, allows for a collapse of the scaled density fluctuation up to the hyperuniformity length scale, as we show in Fig. 4.3(a)-(b). We detail the evaluation of ν_h and of other critical exponents, alongside corresponding information on the pair correlation $g_2(r)$ and structure factor $S(k)$ (Ch. 2), in the appendix Appendix B.

¹³See Sec. 4.2 and Appendix B for details of the numerics

FIGURE 4.3: **Scale collapse of $\sigma^2(R)$ in the CLG and RO models.** The coarse-grained fluctuations $\sigma^2(R)$ at various densities ρ and packing fraction ϕ in the [Left] CLG and [Right] RO models respectively. At shorter lengthscales ($R < \xi_h$), the fluctuations are anomalously suppressed as $\sigma^2 \sim R^{-2.45}$. However, the scaled fluctuations are non-monotonic and the scale collapse of $\sigma^2(R)$ is only achieved for the regions $R \leq \xi_h$. Moreover, the exponent that governs the hyperuniform lengthscale ξ_h is found to be $\nu_h = 0.6$ which disagrees with the speculation that $\nu_h = \nu_\perp$.

Now, the non-monotonic behavior of the scaled density fluctuations, which themselves do not scale collapse for $R > \xi_h$ (see Fig. 4.3), is opposed to the behavior observed in fluctuation suppressed states in models of cell tissues [92] where no minimum appears in the scaled fluctuations $\sigma^2 R^2$ (Sec. 4.2.4). This suggests the presence of additional correlations in the density field at length scales larger than ξ_h . Therefore, we expect $P(\rho_R)$ to be non-Gaussian at the fluctuation suppressed length scales, and only becomes Gaussian (in accordance to CLT) at much larger lengths. Indeed, we observe in Fig. 4.4 a non-Gaussian profile for $P(\rho_R)$ at the hyperuniform length scale ($R = \xi_h$), where a distinctive exponential tail in the low density regime persists.

In particular, we find that $P(\rho_R)$ at $R = \xi_h$ can be described by a universal function,

$$P(\rho_R) \propto \exp\left(A \frac{\rho_R - \rho}{\sigma}\right) \operatorname{erfc}\left(B \frac{\rho_R - \rho}{\sigma}\right), \quad (4.7)$$

for all parameters in both RO and CLG models considered, and where A and B are universal, model independent constants.

FIGURE 4.4: **Universal non-Gaussian distribution at ξ_h .** The (scaled) coarse-grained density distribution $P(\rho_R)$ at the hyperuniform length scale ξ_h follows a universal non-Gaussian form given by Eq. (4.7) where $A = 0.4$ and $B = 0.7$ in both CLG and RO models.

These results indicate that a sufficiently large system ($L \gg \xi_h$) can be seen as a tessellation of subdomains of linear size ξ_h and density ρ_R distributed in accordance to Eq. (4.7), within which density fluctuations are suppressed. Furthermore, the behavior of $P(\rho_R)$ under coarse-graining indicates that the coarse-grained density of these domains is spatially correlated, suggesting a hierarchy of self-organization present at different scales.

To visualize this, we illustrate in Fig. 4.5 a map of the coarse-grained density distribution in a spatial region that spans $100\xi_h \simeq 4200$ for a system of size $L = 8192$ at $\rho = 0.238$ of the CLG model. The observed patches of linear size that extend beyond the coarse-grained lattice spacing ξ_h , confirms existence of correlations beyond hyperuniformity. Here, we also note that size effects affecting the evaluation of ν_h [159], in itself, also indirectly prove the existence of density correlations that extend well beyond ξ_h . This subtle point is deliberated in Sec. 4.2.3, where properties of the higher moments are discussed.

FIGURE 4.5: **Spatial heterogeneity beyond ξ_h** Real space distribution of ρ_R coarse-grained at the hyperuniform lengthscale ξ_h (i.e. $R = \xi_h \sim 44$) in the CLG model for $\rho = 0.238$. Heterogeneity observed at this lengthscale indicates the presence of a hierarchy of scales beyond hyperuniformity.

4.2.2 Timescales, correlations and dynamical origins

In this section we quantify and explore origins of the additional correlations beyond hyperuniformity in the absorbing states that were revealed in Sec. 4.2.1. To do so, we investigate the correlation function of the coarse-grained density at the hyperuniform lengthscale $R = \xi_h$,

$$C_{\rho_R}(r) \propto \langle \rho_R(r) \rho_R(0) \rangle - \langle \rho_R \rangle^2 \quad (4.8)$$

leaving discussion of the general R dependence on $C_{\rho_R}(r)$ to Appendix B.2.

FIGURE 4.6: **Correlation function C_{ρ_R} at $R = \xi_h$.** C_{ρ_R} evaluated at $R = \xi_h$ (Eq. (4.8)) decays exponentially with different exponents for various ρ in the CLG model. The decay length ξ_{ρ_R} is obtained from respective exponential fits (solid line) of the $R \gg \xi_h$ regime

In the CLG model, the correlation decays exponentially, $C_{\rho_R}(r) \propto \exp(r/\xi_{\rho_R})$, after a sharp decline at short length scales (see Fig. 4.6). Furthermore, data corresponding to different densities do not collapse when plotted against r/ξ_h , proving that these observed spatial correlations are a complimentary indication of a hidden additional correlation beyond hyperuniformity. Indeed, we observe in Fig. 4.3 that, while ξ_h diverges with the exponent $\nu_h \simeq 0.6$, ξ_{ρ_R} diverges with the exponent $\nu_{\rho_R} \simeq 0.8$ that is similar to the dynamical exponent $\nu_{\perp}$.

We stress that the radial correlation function and structure factor, which we illustrate in Fig. B.5 of the Appendix B, do not reveal these additional correlations, which implies that they are therefore many-body in nature [161]. These correlations are instead apparent in the two-body correlation function of the coarse-grained quantity $C_{\rho_R}(r)$ as they are encoded in ρ_R , the probability of finding a region of linear size R that contains N_R particles.

Noticing that $\nu_{\rho_R} \simeq \nu_{\perp}$, we investigate the physical origin of this novel length scale focusing on the dynamical process leading to the absorbing states. Specifically, we study the coarse-grained freezing time t_f ¹⁴, defined as the final time at which a particle transitions from an active to passive state. As with the density ρ_R , we coarse-grain the freezing time t_f at the hyperuniform lengthscale ξ_h , which we show in Fig. 4.7 for the CLG model at $\rho = 0.238$, revealing the presence of extended correlations at lengthscales significantly greater than ξ_h .

FIGURE 4.7: **Coarse-grained freezing time t_f of the CLG model.** heat map of the coarse-grained freezing time at length scale ξ_h for $\rho = 0.238$. Presence of spatial structure indicates correlations beyond the hyperuniform length ξ_h

¹⁴We evaluate x_R - the variable x coarse-grained on a length scale R , by dividing the system in $(L/R)^d$ non overlapping square windows, and averaging x within each window.

We further quantify these correlations as with ρ_R by introducing a correlation function for the coarse-grained freezing time at $R = \xi_h$

$$C_{t_f}(r) \propto \langle t_f(0)t_f(r) \rangle - \langle t_f \rangle^2 \quad (4.9)$$

In Fig. 4.8, we see that $C_{t_f}(r)$ of different densities in the absorbing phase of the CLG model do not scale collapse when plotted against r/ξ_h , where a corresponding exponential length ξ_{t_f} is identified through the relation $C_{t_f}(\xi_{t_f}) = e^{-1}$. This behavior is despite $C_{t_f}(r)$ being a correlation of a coarse-grained evaluated at $R = \xi_h$ and indicates that these hidden correlations are as with ρ_R not set by the hyperuniformity length scale.

FIGURE 4.8: **Collapse of the freezing time correlation function C_{t_f} .** $C_{t_f}(r)$ in the CLG model plotted against [Left] r/ξ_{t_f} and [Right] r/ξ_{ρ_h} for various densities considered in Fig. 4.3. The correlation length associated to t_f is defined such that $C(\xi_{t_f}) = e^{-1}$. While C_{t_f} evaluated at $R = \xi_h$ collapses with ξ_{t_f} , it remains uncollapsed with ξ_h , indicating that it originates from hidden correlations beyond hyperuniformity.

Instead, a collapse of the freezing time correlation function C_{t_f} is achieved by rescaling the x-axis by ξ_{t_f} . Furthermore, we find that ξ_{t_f} diverges on approaching the absorbing phase transition as $\Delta\rho^{-0.8}$, as shown in Fig. 4.9, and is therefore proportional to ξ_{ρ_R} . The additional spatial correlations in the density field beyond hyperuniformity, therefore, reflect the spatial correlation in the dynamics leading to the absorbing state.

FIGURE 4.9: **Critical exponents ν_h , ν_{ρ_R} and ν_{t_f}** ξ_{ρ_R} and ξ_{t_f} diverge as power laws in the CLG model at the critical point ρ_c with exponents $\nu_h \approx \nu_{\rho_R} \approx 0.8 \neq \nu_h$, but close to the value of the dynamical exponent $\nu_{\perp}$. This is despite ξ_{ρ_R} and ξ_{t_f} being derived from correlation functions of coarse-grained variables evaluated at $R = \xi_h$.

These results strongly suggest that a single length scale controls the correlation of coarse-grained density ρ_R and that of the dynamical freezing time t_f , representing a novel connection between what is essentially a diverging static lengthscale in its absorbing state, with correlations that develop in its preceding dynamics. This sheds an interesting perspective on an analogous debate regarding the lengthscales that control dynamical behavior of super-cooled glass formers near the glass transition temperature T_c [180, 181]. The correlation between these two quantities, however, is not immediately apparent by a comparison of the coarse-grained density of the final configuration of Fig. 4.5 and that of the coarse-grained freezing time, Fig. 4.7.

To unveil these correlations, we compare the heat maps of the two coarse-grained variables averaged over $N_s = 100$ absorbing states that originate from the same initial configurations. This approach was first introduced by Widmer-Cooper et al. [253] in identifying dynamical heterogeneities and soft spots in super-cooled liquids to eliminate random spatial variation due to intermittent fluctuations.

In the CLG model, these so-called iso-configurational averages mitigate the effect of dynamical noise, and reveal hidden correlations by indicating the *propensity* of a sub-region to acquire particular values under stochastic evolution. The resulting maps, which we show in Fig. 4.10 are clearly similar, where regions with a low averaged ρ_R correspond to the same regions where the typical freezing times t_f are much shorter. Furthermore, by comparing these quantities across sub-regions of the absorbing state in a scatter diagram (Fig. 4.11[Left]), the positive correlation becomes apparent.

FIGURE 4.10: **Propensity and Iso-configurational averages of ρ_R and t_f .** Iso-configurational averages of the coarse-grained variables at $R = \xi_h$ ([Left] $\Delta\rho = \rho_R - \rho$ and [Right] $t_f/\langle t_f \rangle$), over 100 absorbing states that originate from the same initial condition. The regions in sub-space that are under-density (i.e. $\Delta\rho < 0$) in the absorbing state, correlates with the propensity to acquire shorter freezing timescales t_f . Note that the visual similarity in the iso-configurational averages of the two quantities is not due to spurious correlations in the initial conditions of the system, as we show in Fig. 4.11. All data refer to the CLG model at $\rho = 0.238$ and $L = 4096$

Importantly, we remark that these correlations do not originate from fluctuations of the coarse-grained density in the initial configuration. In fact, we show in Fig. 4.11[Right], that the average coarse-grained freezing time does not correlate with the fluctuations of coarse-grained density of the initial configuration. This indicates that the correlations governing hidden order beyond hyperuniformity is non-trivial.

FIGURE 4.11: **Scatter plot of $t_f/\langle t_f \rangle$ vs $\Delta\rho/\sigma$** The scatter distribution of coarse-grained variables at $R = \xi_h$ in the [Left] absorbing states indicate the positive correlation between freezing times and density. [Right] This correlation however is not present in the scenario when t_f is compared to the coarse-grained densities of the initial conditions. Data presented is for $\rho = 0.238$ and $L = 1024$

4.2.3 Non Gaussian behavior of the higher moments

In this section, we discuss correlations and hidden order beyond hyperuniformity embedded in the higher moments of $P(\rho_R)$ that belong to absorbing states of the CLG and RO models. The signature of these higher-ordered correlations manifest as non-monotonic oscillations of the centralized moments, deviating from behavior prescribed by the central limit theorem (CLT) at intermediate lengthscales. Importantly, these correlations are not found in $\sigma^2(R)$, the lowest order contribution to fluctuations. Instead, they diverge at critical points with novel exponents that do not resemble the hyperuniform or dynamical behavior discussed in Sec. 4.2.1.

Specifically, we investigate the approach of $P(\rho_R)$ to the Gaussian limit, focusing on the R dependence of the Skewness

$$\gamma \equiv \langle [(\rho_R - \rho)/\sigma]^3 \rangle \quad (4.10)$$

and the excess kurtosis

$$\kappa_{ex} \equiv \langle [(\rho_R - \rho)/\sigma]^4 \rangle - 3 \quad (4.11)$$

for both the CLG and RO models, which we show in their respective scale collapsed forms in Fig. 4.12.

FIGURE 4.12: **Scale collapse of the higher centralized moments.** [Top] Skewness - γ and [Bottom] excess Kurtosis - κ_{ex} for the [Left] CLG ($L = 8192$) and [Right] RO model (8192×10^3) scale collapse with a universal exponent ~ 0.715 . [Bottom] For κ_{ex} , we show additionally, results of systems with respective parameters closest to the critical point, for system sizes of $L = 2048, L = 4096$ in the CLG and $N = 512 \times 10^3, N = 2048 \times 10^3$ in the RO model. Significant finite size effects are observed in the excess kurtosis at $L = 2048$ for $\rho = 0.2385$

Firstly, we observe that γ and κ_{ex} have non-monotonic dependences on the coarse-grained length scale R , displaying a striking behavior that significantly departs from systems that obey CLT¹⁵ and exhibit respectively, a minimum and maximum at finite length scales ξ_γ and ξ_κ . Note that in general, monotonic convergence in accordance to CLT can be recovered in the higher moments of $P(\rho_R)$ even in hyperuniform systems such as in VM (Ch. 3)[92] and CVT tessellations [103] which we discuss further in Sec. 4.2.4.

¹⁵Consider a random variable x that is drawn from a probability distribution $P(x)$ where the m -th centralized moment exists and is given by $\langle x^m \rangle = \lambda_m$. CLT demands that the distribution governing identical and independent sum $X = \sum_i^N x_i$ (i.e sum of un-correlated x) obeys the relation $\langle X^m \rangle = N^{g(m)} \lambda_m$ where $g(2) = -1/2$, and $g(3) = -2$. Hence γ and κ converge monotonically to their respective Gaussian values as $N \rightarrow \infty$.

More importantly, ξ_γ and ξ_κ diverge at the critical point with a single universal exponent $\simeq 0.715$ (see Appendix B.1 for numerical details) that differs from all critical exponents discussed thus far. This suggests the presence of higher-order correlations beyond hyperuniformity that is not indicated by the behavior of mean-field fluctuations $\sigma^2(R)$. Moreover, R values of extrema in γ and κ_{ex} occurs at distances larger than the hyperuniform lengthscale ξ_h .

These two observations when put together implies a very strange scenario occurring in critical systems of the active-absorbing state transition. While hyperuniform, in that all fluctuations are anomalously suppressed at all finite length scales of the critical system, density distributions are in fact non-Gaussian at all length scales, indicating a persistent heterogeneous behavior in its generalized fluctuations beyond lowest order contributions $\sigma^2(R)$. This also indicates that in finite systems away from the critical point, a sufficiently large system ($L \gg \xi_h$) consists of subdomains of linear size ξ_h that are distributed heterogeneously within which density fluctuations are suppressed.

To demonstrate that these results are not affected by size-effects, we compare data of the kurtosis for systems with density $\rho = 0.2385$ ¹⁶ and $L = 8192$ used in the scale collapse Fig. 4.12[Top], with results at the same density for $L = 4096$ and $L = 2048$ in the CLG model. Fig. 4.12[Bottom] illustrates the good agreement between $L = 8192$ and $L = 4096$, indicating that our results for $L = 8192$ are not affected by size-effects. Conversely, significant deviations are apparent for $L = 2048$. We remark that the ν_h reported in [159] which differs from our results in Sec. 4.2, were derived from systems of size $L = 1000$, and thus may have suffered from significant finite size effects. Similarly, for the RO model, we find an analogous scenario where results from $N = 8192 \times 10^3$ agree with what is obtained from $N = 2048 \times 10^3$ - at the largest considered value of ϕ in Fig. 4.12[Top] and Fig. 4.3. Hence, size effects only become apparent in systems significantly smaller than the respective largest system sizes of both models that are considered in this chapter.

¹⁶Density in the absorbing state closest to the critical point where finite size effects are most apparent

4.2.4 Taxonomy of hyperuniform systems

Observations of the non-Gaussian behavior in the higher moments (Sec. 4.2.3) allow for an intriguing analogy between the behaviour of coarse-grained density distribution $P(\rho_R)$ on increasing R , and that of the displacement field in super-cooled liquids $P(\Delta r_t)$ on increasing the observation time t . In absorbing state models, the fluctuations of ρ_R are suppressed up to a length scale ξ_h that diverges at the active-absorbing phase transition, and the convergence to CLT is recovered gradually at larger length scales. Similarly, the fluctuations of Δr_t are suppressed up to the relaxation time τ that ideally diverges at the glass transition, again with the predictions of CLT being progressively recovered at longer timescales.

Furthermore, in the absorbing states of the CLG and RO models, κ_{ex} peaks at lengthscales that diverge at the transition, where the peak height itself is also diverging. Likewise, in super-cooled liquids, κ_{ex} reaches a maximum at time t that scales with τ , and this peak value diverges at the glass transition [254]. Moreover, the heterogeneities in ρ_R present at $R = \xi_h$ observed in the active-absorbing state models (Fig. 4.5) also resemble that of the displacement field at the relaxation time in super-cooled liquids [54]. Finally, in both cases, the correlations between these spatial heterogeneities and the initial configuration of the system are revealed via isoconfigurational averages [253] Sec. 4.2.2. We note that the two phenomena only differ in their behavior of the Skewness where $\gamma = 0$ at all timescales in super-cooled liquids as a consequence of time reversal symmetry in the equations of motion.

Now in Sec. 4.2, we established the existence of hidden many-body correlations, beyond what is captured by the suppression of density fluctuations in two-dimensional models exhibiting an active-absorbing phase transition ¹⁷. Specifically, density fluctuations are suppressed up to a length scale that diverges at the transition with the exponent $\nu_h \simeq 0.6$ while on the other hand, many-body density correlations extend up to a longer length scale that diverges with the exponent $\nu_\perp \simeq 0.80$. The presence of similar findings beyond the current context is an exciting avenue that demands further investigation. These explorations may, in turn, provide means for further taxonomy and classification of fluctuation suppressed or hyperuniform disordered systems.

¹⁷Qualitatively, the scenario here discussed appears to also hold in three dimensions Appendix B.3.

In particular, we envisage a parallel with the taxonomy recently introduced for diffusive systems [255, 256], where four main classes are identified based on behaviour of the second (Fickian/non-Fickian) and higher moments (Gaussian/non-Gaussian) of the probability distribution governing the displacement field. From this perspective, the CLG and RO models are considered hyperuniform but non-Gaussian systems, random jammed sphere packings [124] discussed in Sec. 3.1.3 are (effectively) hyperuniform and Gaussian, while the Voronoi model for cell tissue [92] which we show in Fig. B.7 and Centroidal Voronoi Tessellations (Fig. 4.13) that emerge from Quantizer problems [103] are (effectively) hyperuniform and Gaussian. Note that uncorrelated point distributions are trivially non-hyperuniform and Gaussian systems. Interestingly, it remains to be discovered if examples of energy minimal states or engineered point configurations belonging to the last category of non-hyperuniform and non-Gaussian systems exists.

Seemingly, non-Gaussian behavior appears to occur in systems exhibiting an active-absorbing transition while Gaussian behavior exists in jammed solids, suggesting that the presence or lack of hidden order beyond hyperuniformity is generically associated to models across universality class of phase transitions. Moreover, the non-monotonic signature in the higher moments, in addition to the scaling behavior of C_{t_f} (Sec. 4.2.2) seem to also indicate that these are signatures of density inhomogeneity or phase separated matter. We test this idea in the subsequent section Sec. 4.2.5. Further work in this direction is certainly needed to uncover the physical significance of such hidden order beyond hyperuniformity. More generally, these analyses based on the higher moments may also provide for additional tools in probing the possible causal role of hyperuniformity in the self-organization of disordered systems by further characterizing the approach to criticality of non-equilibrium phase transitions.

4.2.5 Designing hidden order beyond hyperuniformity in Gaussian-hyperuniform systems

In this section, we discuss how hidden order beyond hyperuniformity can be programmed to exist in Gaussian-hyperuniform systems such as the liquid state of the VM (Ch. 3) or CVTs without destroying the density suppressed behavior at longer lengthscales. Learning how to control and tune the fluctuation behavior

of the higher moments can be instructive towards a general understanding of the physical importance of these hidden correlations. Furthermore, the existence of such a protocol could also serve as the basis for an encoding / decoding system, in which multiple information can be stored and recovered at various coarse-grained lengthscales in the point configurations.

We begin by considering the fluctuation behavior of the liquid phase in the VM discussed in Ch. 3 and show in Fig. 4.13 that the liquid state of the VM ($p_0 = 4.00$) is effectively hyperuniform but does not display the non-monotonic behavior of κ that is present in both models involving active-absorbing state transitions¹⁸. Next, we apply Lloyds algorithm [103] to obtain Centroidal Voronoi Tessellations (CVT) from initial conditions of the liquid phase in the VM.

FIGURE 4.13: **Artificial hidden order beyond hyperuniformity.** [Left] Kurtosis κ and [Right] Scaled fluctuations $\sigma^2 R^3$ of different models considered. Local squeezing ($r_c = 5$) of point configurations belonging to Gaussian-hyperuniform ground states of the liquid (black solid line) in the VM at a shorter lengthscale than ξ_h produces a configuration (grey-dashed lines) that now possess hidden correlations beyond hyperuniformity.

Fluctuation behavior of these two systems indicate that they strongly adhere to their Gaussian-hyperuniform classification¹⁹. Here we attempt to introduce non-monotonic features to κ by introducing density inhomogeneity at a lengthscale

¹⁸Note that the oscillations at small R originate from short range correlations

¹⁹Note that the CVTs appear to have stronger ordering at small R , but behave qualitatively similar to the liquid state of the VM as $R \rightarrow \infty$.

$R \leq \xi_h$, while still preserving the hyperuniform behavior ($R \rightarrow \infty$) of the VM liquid ground-state and the CVTs.

This is done through introducing a *squeezing* operation on the point configuration. In particular, particles within a circular radius r_c of N_c randomly chosen points are affinely displaced *radially* towards the centre $\mathbf{x}_c$ such that their new distance from $\mathbf{x}_c$ is given by $\Delta r' = 0.9\Delta r$ where Δr is the unperturbed separation of particle from $\mathbf{x}_c$. N_c is chosen such $N_c\pi r_c^2/L^2 = \frac{1}{2}$ (i.e. the total perturbed area is about half the system, albeit with significant amount of overlapped and multiply squeezed regions.).

Such a protocol introduces domains of lengthscale r_c in which there exists a density gradient, and introduces under-density regions in its periphery, mimicking the heterogeneous density distributions in the CLG and RO models 4.1.1. So long as $r_c \ll \xi_h$, the system would still appear as hyperuniform or at least fluctuation suppressed at $R = \xi_h$. Indeed, we observe that the squeezed Voronoi configuration now posses a non-monotonic κ that peaks at $R \sim r_c$, and is reminiscent of the absorbing states in the CLG and RO models, while maintaining its hyperuniform behavior at longer lengthscales of $\sigma^2(R)$. We contrast this protocol with random perturbations to the Voronoi liquid, wherein each point of the configuration is displaced by length $a = 0.5$ in random directions drawn from a uniform distribution. Unlike the squeezing protocol, these states destroy their short range correlations as seen from the decrease in amplitude of the oscillations in $\sigma^2(R)$, but do not develop hidden order beyond hyperuniformity.

Hence, this result suggests that directional (non-random) density gradients at sub hyperuniform lengthscales are required to produce the non-monotonic Kurtosis. This further adds to the speculation that hidden order beyond hyperuniformity is generically found in absorbing state models, but absent in jammed solids where such density gradients are missing.

Chapter 5

Beyond Hyperuniformity in Active Matter

In this chapter, we identify the presence of hidden order beyond hyperuniformity in continuum models of active matter. In particular, we study the fluctuation behavior of various active field theories that model collective behavior such as the motility induced phase separation (MIPS) observed in active Brownian particles (ABP). We show that while these minimal models are hyperuniform and all agree in their mean-field fluctuations (i.e $\sigma^2(R)$), features embedded in the higher moments of their coarse-grained distribution¹ discriminates between the various models that may in turn correspond to different underlying microscopic interactions.

In Sec. 5.1, we introduce active matter as a non-equilibrium steady state where unlike systems discussed thus far (Ch. 3 and Ch. 4), entropy is produced locally and time reversal symmetry (TRS) is explicitly broken in the equations of motion for the microscopic degrees of freedom. Specifically, we review the observed phase separation (MIPS) in active matter, where self-propelled particles (ABPs) form finite-sized clusters in the absence of attractive interaction (Sec. 5.1.1). We then present various existing continuum models that collectively aim to describe this phenomenology (Sec. 5.1.2). These include an effective Cahn-Hilliard (CH) equation that extends from its passive counterpart, and various non-integrable field theories such as Active Model B (AMB).

¹In the same spirit to the many body correlations uncovered previously in the CLG and RO models in Ch. 4

We then introduce in 5.2 a generalized field equation, that allows for tunable contributions from both integrable (CH) and non-integrable (AMB) terms. This model result in phase separated steady states where $\sigma^2(R)$ is indistinguishable across all variants, and are hyperuniform in the $R \rightarrow \infty$ limit Sec. 5.2.1. Lastly, we show in Sec. 5.2.2, that despite the ubiquitous hyperuniformity observed, different variants of the field equation can be discriminated through the presence of correlations beyond hyperuniformity. Our results thus indicate that hidden order may generically be used to test and identify effective field theories that are consistent with experimental observations or atomistic models.

5.1 Active matter

Active matter (AM) [257–259] are complex systems driven out-of-equilibrium [260] by the local production of entropy [261–264]. This breaking of TRS in the microscopic equations of motion leads to a plethora of exotic phase and critical behavior that is best exemplified by the striking phenomena collectively known as motility induced phase separation (MIPS) - The active formation of phase-separated density clusters in absence of attractive interactions [265–268]. At the same time, collective behavior in AM also drives self-assembly[269, 270], rendering novel properties and tunability to a fast-growing collection of active-synthetic or living (meta)-materials [271–273]. Hence, active systems form an important paradigm in the understanding of general non-equilibrium phenomena, and also have profound practical and engineering implications.

However, the study of phase behavior in AM is challenging due to the general inaccessibility of equilibrium techniques, lack of conventional measures of order in disordered atomistic systems, and the general inadequacy of mean field approaches for active phase characterization. In agent-based models with complex microscopic interactions [274–278], it is often unclear which features truly originate as a consequence of the underlying activity, and what are minimal models required to reproduce the emergent phenomena. Debate over the applicability of coarse-grained models (scalar field theories) [279, 280] in representing underlying microscopic interactions and the difficulty in distinguishing a large number of active field theories [281–283] further complicates the tractability of the problem. For these reasons, a unified framework of AM remains elusive.

Hence, progress in understanding active matter has largely been through extensive numerical studies of interaction specific agent-based or atomistic models [257, 266, 274, 276, 278, 279], the simplest of which that is known to exhibit MIPS, is known as the model of Active Brownian Particles (ABP) [257, 258, 284, 285] which is considered the microscopic “ideal gas” equivalent for active system. This model simply consists of an additional fluctuating active force $\mathbf{f}_a$ to the overdamped Langevin equation that describes *repulsive* Brownian particles.

$$\gamma \dot{\mathbf{r}}_i = -\nabla_i U(\mathbf{r}_i) + \mathbf{f}_{a,i} + \sqrt{2D} \boldsymbol{\eta}_i \quad (5.1)$$

where $U(\mathbf{r}_i)$ is a purely repulsive potential that is often modelled by the Weeks-Chandler-Andersen (WCA) model ², and the active force of persistent and uniform strength v_0 is given by

$$\mathbf{f}_{a,i} = v_0 \hat{n}_i \quad (5.2)$$

where $\hat{n}_i$ is the unit directional vector often modelled as a phase diffusion of its orientation θ_i given by

$$\dot{\theta}_i = \sqrt{2D_r} \eta_i^R \quad (5.3)$$

where η_i is a stochastic variable that is given by an uncorrelated Gaussian white noise with $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t - t')$, and D and D_r correspond to the linear and rotational diffusion constants respectively ³. The persistence of $\mathbf{f}_{a,i}$ (i.e. D_r is non-zero) is the origins of broken TRS in the system.

5.1.1 Motility Induced Phase Separation (MIPS)

Motility induced phase separation (MIPS) is a hallmark of non-equilibrium AM where phase separation occurs without the need for local attractive or alignment interactions [265, 267, 281, 287]. At low ϕ and v_0 ⁴, the system of ABPs exists

² $V = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^6 \right]$ for $r < 2^{1/6}$. Note that the qualitative phenomenology of MIPS is not sensitive to the exact functional form of $U(\mathbf{r}_i)$ [267].

³Run-and-Tumble particles (RTP) is another popular variant of model active particles that involve a choice for the active force $\mathbf{f}_a$ that has a characteristic persistent time τ in which the orientation θ is fixed, but randomly (stochastically) re-orientates abruptly as $\theta(t)$ because discontinuous in time [284, 286]

⁴More precisely, it is the dimensionless Péclet number $Pe = \frac{v_0}{\sigma D_r}$ that controls the MIPS transition, where σ is the particle interaction radius set by the WCA potential.

in a homogenous gas phase where the distribution of local packing fraction $P(\phi)$ ⁵ is unimodal. However, at sufficiently high ϕ [289], or active velocity ($v_0 \sim \text{Péclet number (Pe)}$), the system of ABPs phase separate into a condensed clustered phase with high local density, and a dilute low density gas phase (see Fig. 5.1). This results in the emergence of a bimodal distribution in local packing fraction $P(\phi)$

FIGURE 5.1: [**Motility Induced Phase Separation (MIPS)**] Phase coexistence of ABPs into dense clusters and dilute gas in the absence of attractive interactions. The color coding indicates the effective Péclet number Figure adapted from [287]

This behavior can be qualitatively understood by emergence of many-body correlations induced by local repulsive interactions, such that an effective caging mechanism result in particle velocities that are spatially, density dependent [265–267, 290]

$$v(\rho) = v_0 - \rho\zeta \quad (5.4)$$

where ζ is the parameter that quantifies imbalances in effective forces experienced by the ABPs [265, 281]. Heuristically, active particles in crowded environments tend to slow down and this promotes cluster formation without the need for additional interactions. Hence, the condition for persistent formation of clusters (i.e the spinodal between MIPS and homogenous phase in the phase diagram), is thus set by a competition between an effective condensation (absorption) rate (i.e inverse

⁵ This is often obtained by a Voronoi construction of the point configuration [288] that ascribes an effective volume (area in 2D) for each particle.

probability of particles entering the cluster) and the evaporation rate of the cluster [267, 291], where the size of these clusters are determined by the flux balanced steady state of these two rates.

Now, by coarse-graining the equations of motion Eq. (5.1) using the phenomenological ansatz for the velocity Eq. (5.4), an effective hydrodynamic description [281, 291] for the ABPs is given in terms of a scalar density $\rho(\mathbf{r}, t)$ and a polarization (orientation) field $\mathbf{p}(\mathbf{r}, t)$

$$\partial_t \rho = -\nabla \cdot [v(\rho)\mathbf{p} - D\nabla \rho] \quad (5.5)$$

$$\partial_t \mathbf{p} = \frac{1}{2}\nabla [v(\rho)\rho] + D\nabla^2 \mathbf{p} - D_r \mathbf{p} \quad (5.6)$$

Upon a linear perturbative expansion of Eq. (5.6) above the mean field (i.e. the transformation $\rho \rightarrow \bar{\rho}(1 + \delta\rho)$, $\mathbf{p} \rightarrow \bar{\rho}\mathbf{p}$, and keeping terms to lowest order in $\delta\rho$), an effective scalar field equation is obtained [281] and is given by

$$\partial_t \phi = \sigma_1 \nabla^2 \phi - \nabla^4 \phi - 2g_0 \nabla \cdot (\phi \nabla \phi) \quad (5.7)$$

where σ_1 and g_0 are model specific parameters. Importantly, Eq. (5.7) can be written in free energy form: $\partial_t \phi = \nabla^2 \frac{\partial F}{\partial \phi}$, where

$$F = \int d\mathbf{r} \left[\frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} \sigma_1 \phi^2 - \frac{1}{3} g_0 \phi^3 \right] \quad (5.8)$$

which implies that Eq. (5.7) is effectively a Cahn-Hilliard (CH) equation that governs passive phase separation [292]. It is important to note that by virtue of Eq. (5.8), this emergent mean field equation is effectively in *equilibrium*, while the underlying microscopic dynamics governed by Eq. (5.1) is *non-equilibrium* in nature. Hence, the validity and extent at which the continuum theory describes the non-equilibrium behavior of active particles in MIPS is not universally accepted [279, 280, 283] with discrepancies notably in higher order behavior of the dynamics.

Here we note that the passive CH equation responsible for phase separation is stabilized by the presence of a quartic term in the free energy and also marked by absence of the cubic term. Explicitly it can be written as

$$F = \int d\mathbf{r} \left[\frac{1}{2} |\nabla \phi|^2 + \frac{\gamma_0}{4} (\phi^2 - 1)^2 \right] \quad (5.9)$$

where γ_0 is a free parameter of the model. Interestingly, we note that this passive CH model is known to be hyperuniform in its $R \rightarrow \infty$ limit for parameter ranges where it is in its two-phase media form [123]. This is surprising, considering that the short length scale behavior is heterogeneous (by virtue of phase separation).

FIGURE 5.2: **[Phase separation in continuum models]** Cahn-Hilliard model for phase separation in passive systems. $L = 256$, $D = 0.01$, $\gamma_0 = 2$, and $t = 10^5$. Note that this model is equivalent to $g = 0$ in Eq. (5.14).

5.1.2 Active field theory

In Sec. 5.1.1, we discussed the effective CH equation Eq. (5.8) as a continuum description of MIPS. This involved a bottom-up approach that involves coarse-graining of the microscopic equations of motion [281] that resulted in an equilibrium (integrable) description of the critical phenomena. Here, we discuss an alternative top-down approach that aims at writing down various minimal field equations that are consistent with the presence or breaking of relevant symmetries, without explicitly regard for the microscopic interactions. These models can thus accommodate non-integrable descriptions of the scalar field ϕ , and typically involve additional terms to the field equations that define passive Brownian motion known as (passive) “Model B” [18, 282–284, 293], that destroys detailed balance in the dynamical equations of motion of the scalar field ϕ .

Explicitly, Model B is given by

$$\dot{\phi} = -\nabla \cdot \mathbf{J} \quad (5.10)$$

$$\mathbf{J} = -\nabla \mu + \mathbf{\Lambda} \quad (5.11)$$

$$\mu = -\phi + \phi^3 - \nabla^2 \phi \quad (5.12)$$

Eq. (5.10) is the continuity equation that conserves the field ϕ with flux $\mathbf{J}$ and where μ (Eq. (5.12)) is a chemical potential similar to the CH equation discussed earlier, and thus drives the flow of mean current $\mathbf{J}$ through its gradient. Importantly, μ implies that Model B is a typical symmetric ϕ^4 field theory. The term $\mathbf{\Lambda}$ is the (field) stochastic contribution to Brownian motion that correspond to the term $\sqrt{2D}\boldsymbol{\eta}$ in Eq. (5.1) and is thus modelled as uncorrelated Gaussian white noise. In the absence of this noise term (i.e. $\mathbf{\Lambda} = 0$), Model B reduces to passive CH equations, and CH is thus considered the mean field limit of Model B.

To break TRS in Model B, non-integrable terms are then added to the “chemical potential” μ ⁶. This has resulted in a large variety of active field theories that include: Active Model B (AMB) [282], Active Model B+ (AMB+) that includes $\nabla\phi$ terms [262] and momentum conserving Active Model H (AMH) [294], each with their own subtle nuances in the dynamics. Here we focus specifically on AMB that minimally break TRS. This model introduces an additional non-integrable term $\sim (\nabla\phi)^2$ to the chemical potential in Eq. (5.12).

$$\mu_B = -\phi + \phi^3 - \nabla^2 \phi + \lambda(\nabla\phi)^2 \quad (5.13)$$

This non-equilibrium field theory (AMB), has several qualitative differences from the CH equations. Most importantly, unlike CH, it does not admit static domain walls between the two bulk phases as a solution [282].

5.2 Hidden order in Active Matter

In this section, we explore the presence of hidden order, hyperuniformity and beyond, in continuum models of AM discussed in Sec. 5.1.2. To this end, we introduce

⁶This in principle would cause the system to be non-equilibrium, and μ is thus strictly not a chemical potential. However, the role of μ in driving the flux $\mathbf{J}$ through its gradient is unchanged.

a minimal model that combines non-integrable contributions from AMB to the CH equations in a single dynamical equation for the scalar field ϕ , which reads

$$\frac{\partial \phi}{\partial t} = D \nabla^2 \mu - g \nabla \cdot (\phi \nabla \phi) + \lambda \nabla^2 [(\nabla \phi)^2] \quad (5.14)$$

where $\mu = \phi^3 - \phi - \gamma_0 \nabla^2 \phi$ is identical⁷ to the chemical potential of Model B Eq. (5.12) besides a scale factor γ_0 to the term $\nabla^2 \phi$. And the coefficients g and λ controls respectively, the relative strength of the integrable and non-integrable active contributions, such that $g = \lambda = 0$ reverts to the passive CH equation.

In Fig. 5.3, we illustrate phase separation (in the steady state) observed in 2D integrable active models (i.e. $\lambda=0$) from an initial state where the binary volume fraction is 1:1, commonly known as the critical quench condition [123] under periodic boundary conditions. While phase separation is readily observed across all integrable variants, visual inspection suggests that the local structure beyond two point correlations must be manifestly different. In fact, $g = 0.5$ is surprisingly reminiscent of the micro-phase separation that has been conjectured to emerge in critical quenches only in presence of non-integrable terms [282] and are unstable solutions owing to a mechanism that is similar to Ostwald Ripening⁸.

In the remainder of this chapter, we show that hidden correlations observed in the spatial distribution of ϕ in Fig. 5.4 and Fig. 5.2 can be made mathematically precise by employing techniques typically used to study Hyperuniformity and its various extensions (Ch. 2 and Ch. 4).

5.2.1 Hyperuniformity in phase separated matter

In Fig. 5.4, we show the fluctuations $\sigma_{\phi_R}^2$ for various models considered at different parameter values of the generalized CH equation Eq. (5.14). We find that all models considered (including non-integral AMB) exhibits a cross-over from an uncorrelated

⁷The prefactor γ_0 appears in the original formulation of the passive CH equation and controls the timescale of dynamics

⁸Coarsening of the heterogeneous behavior due to the dissolution of a smaller but higher concentrated regions of the solute (ϕ) and the subsequent re-emergence through precipitation in extended regions of large clusters.

FIGURE 5.3: **[Active field theories of MIPS]** Generalized Cahn-Hilliard model for MIPS. $L = 256$, $D = 0.01$, $\gamma_0 = 2$, and $t = 10^5$. Increasing strength of the active contributions - $\nabla \cdot (\phi \cdot \nabla \phi)$, promotes micro phase-separation. [Left] $g = 0.2$, [Right] $g = 0.5$.

$\sim R^{-2}$ scaling at shorter length scales⁹ to a hyperuniform $\sim R^{-3}$ behavior at some hyperuniform lengthscale ξ_h (see Fig. 5.4).

While it is known that the passive CH model initialized from a critical quench is indeed hyperuniform [123], this observation in itself is surprising since the underlying field is phase separated, and thus resembles a scenario found in hyperuniform circle-swimmers that exhibit giant fluctuations at much shorter length scales [295]. Moreover, the close resemblance of $\sigma_{\phi_R}^2$ across all models and at all length scales suggest the presence of universal hyperuniform fluctuations across a large family of active and passive phase separation. On the other hand, owing to the lack of momenta conservation and neglect of noise terms, there are no fundamental reasons to believe that any of the generic continuum models exhibiting MIPS are truly hyperuniform unlike in [127].

More importantly, there has also been, to the best of our knowledge, no discussion on hyperuniformity or anomalous fluctuations that is derived from atomistic models of AM. Hence, are ABPs truly hyperuniform? Do these minimal continuum descriptions of MIPS actually represent the static structure of phase separated atomistic models (ABP)? These questions are pertinent especially considering that spontaneous ordering of the (total) velocity field, which cannot be captured by a scalar order theory, has been surprisingly observed recently in ABPs [279]. An explicit coarse-graining of ABPs would thus serve as a litmus test for many field

⁹This correspond to the lengthscale of the heterogeneous clusters.

FIGURE 5.4: [**Active hyperuniformity**] Variance $\sigma^2(R)$ of coarse grained scalar density ϕ_R at length scales R for systems with $L = 1024$, $D = 0.01$, $\gamma_0 = 2$, and $t = 10^5$. $\sigma^2(R)$ (Mean-field fluctuations) at all length scales are largely indistinguishable for various models of MIPS, with Hyperuniform fluctuations universally emerging in the limit of $R \rightarrow \infty$. The green-dashed line indicates the fluctuation profile of non-integrable AMB.

theories, where the presence or lack there-of long range order in atomistic models would shed light on their validity and applicability. This certainly represents an important avenue for rigorous exploration that demands further attention.

5.2.2 Discriminating active field theories from hidden order beyond hyperuniformity

While the variances $\sigma^2_{\phi_R}$ of the various continuum models are largely indistinguishable at all length scales and are seemingly hyperuniform Fig. 5.4, we show in this section that the higher moments (Skewness - γ Eq. (4.10)) and (Kurtosis - κ Eq. (4.11)) (see Fig. 5.5[Left],[Right] respectively) indicate that marked differences exist across the different models.

Notably, γ and κ deviate significantly from their respective Gaussian values (exhibited by the passive CH model at all lengthscales) and is non-monotonic at

FIGURE 5.5: **[Beyond hyperuniformity in active matter]** [Left] Skewness γ and [Right] *Kurtosis* of coarse grained scalar density ϕ_R at length scales R for systems with $L = 1024$, $D = 0.01$, $\gamma_0 = 2$, and $t = 10^5$. While σ^2 across various continuum active models are indistinguishable, characteristic features of many-body correlations are instead hidden in the higher moments of the density distributions, and persist at length scales larger than the hyperuniform length ξ_h indicated by their variances. The green dashed line indicates the higher moments of non-integrable AMB.

intermediate lengths. Moreover, unlike the RO and CLG models, κ is Leptokurtic (i.e $\kappa_{ex} < 0$) at short lengthscales, and both γ and κ are oscillatory beyond the hyperuniform length scales ξ_h . Furthermore, non-integrable Active Model B (Green-dashed line), unlike their integrable counterparts also has a pronounced negative skew at much shorter length scales. Importantly, we note that these deviations of the respective moments from their Gaussian values persists at $\xi_h > h$, suggesting that these characteristic signatures of the various models originate from many-body correlations and hidden order that extends *beyond* hyperuniformity.

This thus provide an avenue for investigating correspondence between dynamical and structural features of MIPS, allowing for means to further distinguish between the large number of continuum models, and make comparison to various microscopic interactions they represent. Moreover, hidden order beyond hyperuniformity may also be used to characterize potential field theories that completely abandon the scalar ϕ^4 form in attempt to explain the higher order correlations found in atomistic simulations of ABP such as those observed in [279, 280].

Chapter 6

Conclusion

In this dissertation, we uncovered the generic presence of hidden order, both hyperuniformity and beyond, in otherwise disordered complex systems. This hidden order is revealed by examining the behavior of sequential orders of centralized moments in the probability distribution $P(\rho_R)$ of coarse-grained density ρ_R at increasing lengthscale R . In particular, we formalize the characterization of hidden order beyond hyperuniformity and show that it manifests as violations of the Central Limit Theorem (CLT) for uncorrelated random variables, emerging as a non-monotonic convergence in R of the higher moments beyond its variance $\sigma^2(R)$. Importantly, these features are indicative of hidden many-body correlations that are not captured by two-point functions such as the static structure factor $S(\mathbf{k})$ or pair-correlation (radial distribution) functions $g_2(r)$.

Additionally, we show that these hidden correlations play a highly non-trivial role in a wide variety of critical phenomena with vastly different microscopic interactions and macroscopic phase behavior. In particular, this include

- Ch. 3 The rigidity transition in systems mediated by intrinsically many-body interactions and at the zero-temperature Jamming transition.
- Ch. 4 Active-absorbing state transitions in the Random Organization (RO) and Conserved Lattice Gas (CLG) models that belong to the universality class of conserved directed percolation.
- Ch. 5 Active field theories for the Motility induced phase separation (MIPS) in continuum active matter.

In these transitions, the anomalous behavior in the mean-field fluctuations categorically extend to some finite length scale (ξ_h), which we show for the first two transitions, monotonically increases with R and seemingly diverge at the respective critical point of the system. On the other hand, in the various continuum models for MIPS, ξ_h is always found to be divergent, and is thus universally hyperuniform regardless of the status of its phase separation or the presence of activity. However, we speculate that this may be due to inadequacy of the scalar order active field theories themselves, and an explicit coarse-graining of atomistic models may prove otherwise. Collectively, results in this dissertation hints at a possible universality of the hyperuniform lengthscale ξ_h in controlling continuous phase transitions.

Secondly, our results indicate that the absence and presence of hidden order beyond hyperuniformity is in fact transition dependent, and its origins are specific to the dynamical behavior of the complex system. Moreover, we also show that this dependency can in fact be used to distinguish variants of active field theories that are otherwise indistinguishably hyperuniform. Collectively, these works indicate that hidden order beyond hyperuniformity can be used as a tool to distinguish and categorize different critical phenomena and bulk phase behavior. In systems, where such correlations beyond hyperuniformity exist, they can also reveal hidden multi-scale behavior that extend beyond fluctuation suppression. We remark that knowledge of the minimal conditions required or mechanisms involved in driving a system towards hyperuniformity and beyond is sorely missing, and there is a need for a better understanding of how hidden order beyond hyperuniformity generically emerge. The speculated existence of non-hyperuniform, non-Gaussian systems in our taxonomy presented (Ch. 3) has also yet been verified.

Finally, we briefly discuss several open problems, additional considerations and potential future directions of this work.

- Hidden order beyond hyperuniformity is revealed by the higher moments in this dissertation. These moments however are not independent of each other and are subjected to various physical and statistical constraints. We seek a single characterization of the departure from CLT induced by many-body correlations that couples to all moments of $P(\rho_R)$. Such a measure can be given by using a recently introduced Gaussian distance metric and related bounds derived in [161].

- An alternative approach to arrive at a single fluctuation based order parameter, could also be to consider the information entropic based Kullback-Leibler divergence (D_{KL}) that measures the relative entropy of $P(\rho_R)$ from a reference distribution $P(\psi_R)$ for un-correlated scalar ψ_R . It would also be interesting to investigate how D_{KL} relates to hidden production of thermodynamic entropy and large deviations of work fluctuations as discussed in [264], or various entropic measures of non-equilibrium hidden order that are not based on hyperuniformity analyses [296].
- Does hidden order either in the form of hyperuniformity or beyond extend to generalized fluctuations of arbitrary scalar fields defined on an underlying density field? While there is limited precedence that complex systems self-organize to suppress fluctuations in scalar fields beyond its density [116], this concept has never been established as a paradigm for self organization. At the same time, many of the novel phenomena in AM including MIPS originate from spontaneous velocity correlations [279, 280] that are extended spatially up to sub-system length scales. Hence it is likely that structural correlations alone are insufficient at fully characterizing AM. By examining the fluctuation behavior of equivalent coarse-grained momentum variables (active and total contributions) analogous to ϕ_R (which depart from scenarios described by scalar order theories), it may be possible to capture hidden order beyond pairwise velocity correlations typically considered in the study of self-propelled active matter.
- Can the emergent hidden order beyond hyperuniformity in complex environments be used to identify semantic states? If correlations in dynamical histories of complex systems are generically captured in the intermediate-to-long range order of their transient or steady-state configurations, then it may present an intriguing possibility for investigation of multi-scale biological problems. In essence, hidden order present in combinations of coupled scalar fields associated to mechanical, chemical or geometrical properties, may possibly be used to identify disease states, reveal bio-chemical histories, or characterize developmental phases.

Appendix A

Quantifying fluctuations and geometry in the Voronoi model

A.1 Estimating the hyperuniform lengthscale $l_{\delta^2 N}^*$

In this appendix we present numerical estimates of the hyperuniform lengthscales $l_{\langle \delta^2 N \rangle}$ from a fit to the ansatz (Eq. (3.19)). Results and quality of these fits can be found in Table A.1.

p_0	α	β	s	$l_{\langle \delta^2 N \rangle}$	χ_ν^2
3.00	0.801	0.0193	3.57	18.3	0.600
3.20	0.869	0.0156	3.19	19.6	0.217
3.40	0.915	0.0115	3.11	24.3	0.525
3.50	0.971	0.0073	2.04	34.7	0.493
3.60	0.973	0.0047	1.89	53.4	0.729
3.70	0.951	0.0031	1.99	87.9	1.754

TABLE A.1: Numerical fits of the fluctuations to the ansatz: $\langle \delta^2 N \rangle / R^2 = \beta \left[1 + (R/l_{\langle \delta^2 N \rangle})^{-s} \right]^{\alpha/s}$ for $N_s = 25$ samples of $N = 16384$ particles. Fits are performed over the domains $R_{min} < R < R_{max}$ such that $R_{min} \sim 5.2$ $R_{max} = L/2$, where $L = \sqrt{N} = 128$.

We note the monotonically increasing $l_{\langle \delta^2 N \rangle}$ and monotonically decreasing β with p_0 , indicating a growing lengthscale and a decreasing pleateau height at which fluctuations are no longer suppressed as $p_0 \rightarrow p_0^*$, as we argue for in the main text. In Fig. A.1, we show the numerical fits in table Table A.1 overlayed on to data for

FIGURE A.1: **Least-square fits of $\langle \delta^2 N \rangle$.** $\langle \delta^2 N \rangle / R^2$ as a function of R at various values of p_0 for $N_s = 25$ realizations of a system with $N = 16384$ particles (blue). The error bars represent standard deviation across N_s and characterizes the typical variation of the fluctuations at lengthscale R . The red curve represents fits of the data to (Eq. (3.19)) with parameter values listed in table A.1, over the fit region $L_{max} < R < L_{min}$ where $L_{min} \sim 5.2$ and $L_{max} = L/2$ are shown as vertical black dashed lines. The inserts show a magnified version of the fit region.

$\langle \delta^2 N \rangle / R^2$ with corresponding errorbars, and show that they are in good agreement as indicated by the goodness of fit characterized by the reduced chi-square (χ_ν^2)

$$\chi_\nu^2 \equiv \frac{N_s}{\nu} \sum_i \frac{(O_i - E_i)^2}{\sigma_{N,i}^2} \quad (\text{A.1})$$

where $\sigma_{N,i}^2$ is variance in $\langle \delta^2 N \rangle / R^2$ at a given lengthscale R .

A.2 Finite-size effects on $l_{\delta^2 N}^*$

In this appendix, we study finite size effects of $\langle \delta^2 N \rangle / R^2$ by considering fluctuations between configurations with $N = 4096$, $N=8192$ and $N = 16384$ particles. Specifically, we focus here on finite-size effects on the hyperuniformity length scale, which is at which the rescaled fluctuations cross-over from power law to plateau . Thus, to this end of elucidating the effect of system size on the cross-over, we first fit the power law regime at intermediate lengthscales ($L_{min} < R < L_{max}$) of the scaled fluctuation $\langle \delta^2 N \rangle / R^2$ to a single exponent γ (i.e $\langle \delta^2 N \rangle / R^2 \sim \tilde{k} R^\gamma$). Details of the fits are listed in Table A.2.

p_0	N	γ	$\tilde{k}$	χ_ν^2
3.00	4096	-0.778	0.192	0.533
	8192	-0.762	0.188	1.878
	16384	-0.780	0.192	2.029
3.20	4096	-0.841	0.199	0.508
	8192	-0.854	0.203	2.765
	16384	-0.833	0.195	1.129
3.40	4096	-0.911	0.212	1.370
	8192	-0.916	0.214	2.330
	16384	-0.894	0.205	2.554
3.50	4096	-0.937	0.217	1.707
	8192	-0.938	0.218	0.585
	16384	-0.931	0.216	2.002
3.60	4096	-0.939	0.216	1.484
	8192	-0.939	0.216	3.858
	16384	-0.940	0.216	2.701
3.70	4096	-0.949	0.218	2.309
	8192	-0.940	0.216	4.264
	16384	-0.938	0.215	3.152
3.80	4096	-0.964	0.230	1.622
	8192	-0.967	0.230	2.421
	16384	-0.959	0.227	1.826

TABLE A.2: Fit parameters of the power law regime observed in the fluctuation profiles to the functional form: $\langle \delta^2 N \rangle / R^2 \sim \tilde{k} R^\gamma$ for systems comprising of $N = 4096, 8192$, and 16384 particles. $N_s \geq 15$ configurations were considered for each N , and at each value of p_0 considered. Goodness of fit is characterized by the reduced chi-square : χ_ν^2 defined in Appendix A.1

From this we see that the parameters obtained are largely independent of system sizes ($N > 4096$) for all values of p_0 in the solid regime, indicating that the power

law behavior and the inferred anomalous fluctuation suppression are robust. Next, to elucidate the effect of system size on the crossover, we rescale $\langle \delta^2 N \rangle / R^2$ by the strict power laws obtained earlier with the fits in Fig. A.2 such that the power law regime is normalized to one.

FIGURE A.2: **Finite-size effects.** Scaled number fluctuations $\langle \delta^2 N \rangle / \tilde{k} R^{2+\gamma}$ and corresponding errors for various values of p_0 for system size of $N = 4096$ (blue), $N = 8192$ (red) and $N = 16384$ (orange) particles. Dashed line indicates a threshold of 5% deviation from the power law behavior.

Hence, any deviation from unity of the rescaled fluctuations is a percentage measure of departure from the power law behavior at smaller lengthscales. From these plots, we see that the point of departure for the mean values at each R is independent of N for a wide range of p_0 , indicating that the crossover is relatively free of system size effects for $N > 4096$. In fact, they are largely consistent even for a relatively large deviation threshold value of $5\% \sim 2\sigma_{i,N}$ indicated by the black-dashed lines in fig. A.2. We note that the statistics obtained at $p_0 = 3.70$ for the smallest system ($N = 4096$) (i.e. fig. A.2(f)), is significantly affected by finite size effect at the 5% threshold level, although the two larger system sizes remain robust.

A.3 Geometrical distributions

Probability density functions of area and perimeter for energy minimal configuration as seen in Fig. A.3 approaches a delta function as p_0 increases and approaches the fluid regime.

FIGURE A.3: **Geometrical distributions** (a,b) Probability density of the area $P(A)$, and perimeter $P(P)$ for the energy minimal configurations at various p_0 in the solid regime. (c,d) Inserts show the distributions in relation to that of the fluid at $p_0 = 4.00$ which is strongly peaking at $A = 1$ and $P = p_0$.

Note that $P(A)$ appears to be bi-modal deep in the solid (i.e $p_0 \ll p_0^*$), but progressively becomes uni-modal as $p_0 \rightarrow p_0^*$ (i.e towards the fluid) as $\langle \mathcal{H} \rangle \rightarrow 0$ and the system becomes increasingly unfrustrated.

Appendix B

Numerical details of hidden order in absorbing states

In this appendix, we provide additional figures and supporting information that compliment various results contained in Ch. 4 regarding the presence and significance of hidden order beyond hyperuniformity in absorbing states of the Conserved Lattice Gas (CLG) and Random Organization (RO) models.

B.1 Determination of the critical exponents

In this section, we detail evaluation of the exponent ν_h that controls divergence of the hyperuniform length scale ξ_h at the critical point ρ_c from the absorbing phase (i.e. quantifying the scaling behavior $\xi_h \sim (\rho_c - \rho)^{-\nu_h}$). We have previously claimed in the main text that $\nu_h \simeq 0.60$, a value that significantly differs from $\nu_h \simeq 0.80$, which is what has been previously reported in [159].

Now, since $\sigma^2(R) \propto R^{-\lambda}$ where $\lambda \simeq 2.45$ in the fluctuation suppressed regime, we introduce the scaling ansatz - $\sigma^2(R)\Delta\rho^{-\nu_h\lambda} \propto (R/\Delta\rho^{-\nu_h})^{-\lambda}$ (similar to Eq. (4.2)) as is also considered by [159]. The value of ν_h can then be estimated from the scale collapse of $\sigma^2(R)$ for various values of the control parameters (ϕ and ρ). In Fig. B.1(a) and (b) we show that while a reasonable data collapse is obtained for $\nu_h \simeq 0.8$ in the fluctuation suppressed region, for both models, $\nu_h \simeq 0.6$ (Panels (c) and (d)) provide a much cleaner collapse. This suggests that $\nu_h \sim 0.8$ estimated in

FIGURE B.1: **The hyperuniform critical exponent (ν_h)** Scaling of $\sigma^2(R)$, for the [Left] CLG and the [Right] RO models. [Top] $\nu_h = 0.8$ and [Bottom] $\nu_h = 0.6$. Note the visibly better scale collapse obtained for $\nu_h = 0.6$

[159] might not be accurate especially considering the significantly smaller systems involved, and the presence of strong finite size effects.

To quantitatively evaluate the critical exponents without resorting to the visual impressions provided by a data collapse, we investigate the coarse-grained dependence of the scaled fluctuations, $R^2 \sigma^2(R)$, as well as the skewness, γ , and the kurtosis, κ . It is instructive to focus on $R^2 \sigma^2(R)$ as this quantity exhibits a minimum at the R value at which the density fluctuations are maximally suppressed, signalling the crossover from the suppressed fluctuation region, $\sigma^2(R) \propto R^{-\lambda}$, to the CLT scaling, $\sigma^2(R) \propto R^{-2}$. The non-monotononic dependence on R of these quantities, illustrated in Fig. B.2, allows the unambiguous identification of the R values corresponding to their extrema, which we indicate as ξ_x where $x = \sigma^2, \gamma$ or κ . Specifically, we first fit the respective profiles to fourth-order polynomial

FIGURE B.2: **Extracting the lengthscales ξ_x .** Dependence of the scaled fluctuations $\sigma^2 R^2$, skewness γ and kurtosis κ of the probability distribution for the coarse grained density ρ_R in the CLG and RO models. We estimate the length scales ξ_h , ξ_γ and ξ_κ at which these quantities attain their extreme values via polynomial fits that is shown as black solid lines in each panel.

functions close to their extreme values as illustrated in the figure, and then analytically evaluate ξ_x by evaluating positions of the appropriate turning points in these smooth functions.

Fig. B.3 shows conclusively that for both the RO and CLG models, the length scale ξ_h diverges at the critical point with exponent $\nu_h \simeq 0.6$ and not with the previously claimed value of $\nu_h = \nu_\perp \simeq 0.8$. Furthermore, the length scales ξ_γ and ξ_κ diverge with critical exponent $\simeq 0.715$, that further supports the claim that $\xi_h \neq \xi_\gamma$ or $\xi_h \neq \xi_\kappa$ in the main text .

FIGURE B.3: **Power-law divergences of ξ_x near criticality** Lengthscales obtained from the extreme values of the scaled fluctuations $\sigma^2 R^2$, skewness, γ , and kurtosis κ in the [Left] CLG and [Right] RO models. Power-law fits to these length scales are indicated by the black dashed lines.

B.2 Correlation of the coarse-grained density

In Sec. 4.2.2, we extracted a length scale from the decay of the correlation function $C_{\rho_R}(R)$ of the density coarse-grained over a length scale $R = \xi_h$. Here examine how C_{ρ_R} behaves at lengthscales above and below ξ_h by comparing heat maps of the density coarse-grained at different length scales, which we show in Fig. B.4. We note that heterogeneity and amplitude of fluctuations from the mean density ρ is enhanced at shorter lengthscales ($R < \xi_h$) as seen in Fig. B.4(a). Furthermore, note that spatial structures corresponding to density fluctuations extending well beyond the coarse-grained lattice size exists in Fig. B.4(c) at $R \gg \xi_h$, consistent with the presence of hidden correlations beyond hyperuniformity.

In Fig. B.4(d), we show that the decay of C_{ρ_R} at various coarse-grained lengthscales do not critically depend on R (i.e they have the same exponential scaling behavior in the log-linear plot). Hence a single lengthscale that is not ξ_h controls C_{ρ_R} . However, we note that the amplitude of C_{ρ_R} does depend on R , in that the amplitude vanishes as R decreases. Hence, the visual difference in the heterogeneity observed at various R is due to differences in the magnitude of the correlation and not due to changes in the correlation length.

FIGURE B.4: **Coarse-grained density correlations** C_{ρ_R} . spatial distribution of of the density field ρ_R coarse-grained at length scale $R = \xi_h/2$ (a), $R = \xi_h$ (b) and $R = 2\xi_h$ (c), for the CLG model at $\rho = 0.237$, where $\xi_h \simeq 29$ lattice sites. (d) A single length scale controls the decay of correlation function $C_{\rho_R}(R)$ at large distances.

We stress that the correlation length of the coarse-grained density does not reflect pair-wise correlations in the radial distribution function $g_2(r)$ or structure factor $S(k)$ Sec. 2.1 (see Fig. B.5[Left]). Indeed, the radial distribution function $g_2(r)$ attains its limiting uncorrelated value at a length scale much smaller than ξ_h . Similarly, the structure factor $S(k)$, as illustrated in Fig. B.5[Right], suggests that suppression of density fluctuations increases on approaching the critical point through the hyperuniform lengthscale, and do not indicate the existence of any other characteristic length scales. We note that these results for the structure factor are analogous to those previously observed in [159].

FIGURE B.5: **Pair correlations in the CLG and RO models** [Left] Radial distribution function $g_2(r)$ of the CLG model at $\rho = 0.237$ approaches the uncorrelated ideal gas limit ($g_2(r) = 1$) at a length scale much shorter than the hyperuniform length ξ_h , which at this density is $\xi_h \simeq 29$. [Right] The structure factor $S(k)$ at various densities indicate a diverging lengthscale in the direction of $\rho \rightarrow \rho_c$ and corroborate the universal scaling behavior $S(k) \sim k^{\lambda-2}$ in the fluctuation suppressed regime.

B.3 Hidden order beyond hyperuniformity of absorbing states in higher dimensions

Here we explore the dimensional dependence of our findings by performing simulations of the CLG model in three spatial dimensions:

FIGURE B.6: **Central moments of $P(\rho_R)$ for $d = 3$ in the CLG model** (a) scaled density fluctuations $\sigma^2 R^3$, (b) skewness γ and (c) excess kurtosis κ_{ex} of $P(\rho_R)$ at a given length scale R , for the CLG model in three spatial dimensions. Symbols and color identify system sizes and densities according to legend in (a).

In particular, Fig. B.6 illustrate the dependence of the respective central moments on R . Results are qualitatively analogous to those observed in two dimensions suggesting that the general phenomenology remains the same. We are unable to perform a detailed investigation of the three dimensional results due to their huge computational cost. In fact, the noisy data shown in Fig. B.6 already involves simulations of systems with 100 millions particles.

B.4 Gaussian hyperuniform systems

We illustrate in Fig. B.7 the dependence of κ on the coarse graining length scale for the Voronoi model of cell tissue Ch. 3 . In this model, the parameter p_0 drives a transition/crossover from a solid to a fluid phase at $p_0^* \simeq 3.81$, in the direction of increasing p_0 . The system is known to be effectively hyperuniform at the critical point and in the fluid phase [92]. In the figure, we observe a rapid monotonic decay of the kurtosis to its Gaussian value independent of its distance from the critical point. The kurtosis does not indicate the existence of additional correlations beyond hyperuniformity. The Voronoi model, therefore, is an example of a effectively hyperuniform and Gaussian systems.

FIGURE B.7: **Gaussian hyperuniform Voronoi model** Monotonic and p_0 independent decay of the kurtosis to Gaussian values at very short length scales in the solid phase of the voronoi tissue model which is known to be effectively hyperuniform at $p_0^* \simeq 3.81$ and above.

Bibliography

- [1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
- [2] N. Ashcroft and N. Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976).
- [3] L. Landau, Zh. Eksp. Teor. Fiz. **7**, 19 (1937).
- [4] C. Kittel, *Introduction to Solid State Physics* (John Wiley & Sons, Inc., New York, 1986).
- [5] P. W. Anderson, Science **177**, 393 (1972).
- [6] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
- [7] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
- [8] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
- [9] S. Sastry, P. G. Debenedetti, and F. H. Stillinger, Nature **393**, 554 (1998).
- [10] D. J. Thouless, Physics Reports **13**, 93 (1974).
- [11] M. F. Thorpe, Journal of Non Crystalline Solids **57**, 355 (1983).
- [12] L. Berthier and G. Biroli, Rev. Mod. Phys. **83**, 587 (2011).
- [13] T. Castellani and A. Cavagna, J. Stat. Phys. **2005**, P05012 (2005).
- [14] H. Tong, P. Tan, and N. Xu, Sci. Rep. **5** (2015).
- [15] C. P. Goodrich, A. J. Liu, and S. R. Nagel, Nature Physics **10**, 578 (2014).
- [16] P. Acharya, S. Sengupta, B. Chakraborty, and K. Ramola, Phys. Rev. Lett. **124**, 168004 (2020).
- [17] R. Phillips, *Crystals, Defects and Microstructures: Modeling Across Scales* (Cambridge University Press, Cambridge, 2001).
- [18] C. P. M. and T. C. Lubensky, *Principles of condensed matter physics* (Cambridge University Press, Cambridge, 2000).
- [19] D. R. Nelson, Phys. Rev. B **27**, 2902 (1983).

- [20] B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. **41**, 121 (1978).
- [21] H. Nishimori, *Statistical physics of spin glasses and information processing* (Oxford University Press, Oxford, 2001).
- [22] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin glasses and beyond; an introduction to the replica method and its applications* (World Scientific, Singapore, Singapore, 1986).
- [23] Y. Imry and S. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
- [24] G. Parisi, *Statistical Field Theory* (Addison-Wesley, Singapore, 1988).
- [25] W. Kob and H. C. Andersen, Phys. Rev. E **51**, 4626 (1995).
- [26] D. Chandler and H. C. Andersen, J. Chem. Phys. **57**, 1930 (1972).
- [27] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
- [28] C. O'Hern, L. Silbert, A. Liu, and S. Nagel, Phys. Rev. E **68**, 011306 (2003).
- [29] Y. Bahri, J. Kadmon, J. Pennington, S. Schoenholz, J. Sohl-Dickstein, and S. Ganguli, Annu. Rev. Cond. Matter Phys. **11**, 501 (2020).
- [30] G. Carleo, I. Cirac, K. Cranmer, L. Daudet, M. Schuld, N. Tishby, L. Vogt-Maranto, and L. Zdeborová, Rev. Mod. Phys. **91**, 045002 (2019).
- [31] T. Mora, A. M. Walczak, W. Bialek, and J. Callan, C. G., Proc. Natl. Acad. Sci. U.S.A. **107**, 5405 (2010).
- [32] F. Morcos, A. Pagnani, B. Lunt, A. Bertolino, D. S. Marks, C. Sander, R. Zecchina, J. Onuchic, T. Hwa, and M. Weigt, Proc. Natl. Acad. Sci. U.S.A. **108**, E1293 (2011).
- [33] D. R. Nelson, Phys. Rev. B **28**, 5515 (1983).
- [34] L. Balents, Nature **464**, 199 (2010).
- [35] K. Kim, M. S. Chang, S. Korenblit, R. Islam, E. E. Edwards, J. K. Freericks, G. D. Lin, L. M. Duan, and C. Monroe, Nature **465**, 590 (2010).
- [36] E. Ising, Zeitschrift fur Physik **31**, 253 (1925).
- [37] L. Onsager, Phys. Rev. **65**, 117 (1944).
- [38] G. H. Wannier, Phys. Rev. **79**, 357 (1950).
- [39] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941).
- [40] M. Plihal, D. L. Mills, and J. Kirschner, Phys. Rev. Lett. **82**, 2579 (1999).
- [41] C. Castelnovo, R. Moessner, and S. Sondhi, Nature **451**, 42 (2008).

- [42] S. T. Bramwell and M. J. P. Gingras, *Science* **294**, 1495 (2001).
- [43] S. F. Edwards and P. W. Anderson, *J. Phys. F: Met. Phys.* **5** (1975).
- [44] L. F. Cugliandolo, *Dynamics of glassy systems* (Springer, Les Houches, 2003).
- [45] A. Ikeda, L. Berthier, and P. Sollich, *Phys. Rev. Lett.* **109**, 018301 (2012).
- [46] P. Charbonneau, J. Kurchan, G. Parisi, P. Urbani, and F. Zamponi, *Nat Commun* **5** (2014).
- [47] P. G. Debenedetti and F. H. Stillinger, *Nature* **410**, 259 (2001).
- [48] T. Aspelmeier, R. A. Blythe, A. J. Bray, and M. A. Moore, *Phys. Rev. B* **74**, 184411 (2006).
- [49] A. J. Ballard, R. Das, S. Martiniani, D. Mehta, L. Sagun, J. D. Stevenson, and D. J. Wales, *Phys. Chem. Chem. Phys.* **19**, 12585 (2017).
- [50] B. Zhang and P. G. Wolynes, *Proc. Natl. Acad. Sci. U.S.A.* **112**, 6062 (2015).
- [51] S. H. Strogatz, *Nonlinear dynamics and chaos* (Westview Press, Boulder, 2001).
- [52] M. Jansen, *Pure and Appl. Chem.* **86**, 883 (2014).
- [53] M. D. Ediger, *Annu. Rev. Phys. Chem.* **51** (2000).
- [54] L. Berthier, *Physics* **4** (2011).
- [55] K. W., D. C., P. S. J., P. P. H., and G. S. C., *Phys. Rev. Lett.* **79**, 2827 (1997).
- [56] C. Donati, S. C. Glotzer, P. H. Poole, W. Kob, and S. J., *Phys. Rev. E* **60** (2000).
- [57] B. V. Venkataraman G., Sahoo D., *Ergodicity Breaking. Beyond the Crystalline State, Springer Series in Solid-State Sciences, vol 84.* (Springer, Berlin, Heidelberg, 1989).
- [58] R. Nandkishore and D. A. Huse, *Annu. Rev. Condens. Matter Phys.* **6**, 15 (2015), arXiv:1404.0686 [cond-mat.stat-mech] .
- [59] A. Pal and D. A. Huse, *Phys. Rev. B* **82**, 174411 (2010).
- [60] P. Debye, *Annalen der Physik* **344**, 789 (1912).
- [61] W. A. Phillips, *Amorphous Solids: Low-Temperature Properties* (Springer, Berlin, 1981).
- [62] N. Xu, M. Wyart, A. J. Liu, and S. R. Nagel, *Phys. Rev. Lett.* **98** (2007).
- [63] M. L. Manning and A. J. Liu, *EPL (Europhysics Letters)* **109**, 36002 (2015).

- [64] M. Baggioli and A. Zaccane, *Phys. Rev. Lett.* **122**, 145501 (2019).
- [65] E. Lerner, G. Düring, and E. Bouchbinder, *Phys. Rev. Lett.* **117**, 035501 (2016).
- [66] B. B. Laird and H. R. Schober, *Phys. Rev. Lett.* **66**, 636 (1991).
- [67] K. González-López, S. Mahajan, Y. Zheng, M. P. Ciamarra, and E. Lerner, *Phys. Rev. E* **103**, 022605 (2021).
- [68] K. González-López, S. Mahajan, Y. Zheng, M. P. Ciamarra, and E. Lerner, *Phys. Rev. E* **103**, 022606 (2021).
- [69] D. Richard, K. González-López, G. Kapteijns, R. Pater, T. Vaknin, E. Bouchbinder, and E. Lerner, *Phys. Rev. Lett.* **125**, 085502 (2020).
- [70] M. Shimada, H. Mizuno, L. Berthier, and A. Ikeda, *Phys. Rev. E* **101**, 052906 (2020).
- [71] C. Rainone, E. Bouchbinder, and E. Lerner, *Proc. Natl. Acad. Sci. U.S.A.* **117**, 5228 (2020).
- [72] J. P. Wittmer, A. Tanguy, J.-L. Barrat, and L. Lewis, *Europhysics Letters (EPL)* **57**, 423 (2002).
- [73] A. Tanguy, J. P. Wittmer, F. Leonforte, and J.-L. Barrat, *Phys. Rev. B* **66**, 174205 (2002).
- [74] F. Leonforte, R. Boissière, A. Tanguy, J. P. Wittmer, and J.-L. Barrat, *Phys. Rev. B* **72**, 224206 (2005).
- [75] M. Wyart and M. E. Cates, *Phys. Rev. Lett.* **112**, 098302 (2014).
- [76] H. A. Barnes, *J. Rheol.* **33**, 329 (1989).
- [77] P. Coussot and N. Wagner, *Rheologica Acta* **48**, 827 (2009).
- [78] J. Chattoraj, C. Caroli, and A. Lemaître, *Phys. Rev. Lett.* **105**, 266001 (2010).
- [79] M. Ozawa, L. Berthier, G. Biroli, A. Rosso, and G. Tarjus, **115**, 6656 (2018).
- [80] R. C. Zeller and R. O. Pohl, *Phys. Rev. B* **4**, 2029 (1971).
- [81] J. J. Freeman and A. C. Anderson, *Phys. Rev. B* **34**, 5684 (1986).
- [82] R. O. Pohl, X. Liu, and E. Thompson, *Rev. Mod. Phys.* **74**, 991 (2002).
- [83] G. Tarjus, S. A. Kivelson, Z. Nussinov, and P. Viot, *Journal of Physics: Condensed Matter* **17**, 1143 (2005).
- [84] M. D. Ediger, C. A. Angell, and S. R. Nagel, *J. Phys. Chem.* **100**, 13200 (1996).

- [85] G. Tarjus and D. Kivelson, *Jamming and Rheology: Constrained dynamics on microscopic and macroscopic scales* (Taylor and Francis, London, 2001).
- [86] S. Napolitano, E. Glynos, and N. B. Tito, Reports on Progress in Physics **80**, 036602 (2017).
- [87] S. Torquato, Phys. Rep. **745**, 1 (2018).
- [88] S. Torquato and F. H. Stillinger, Phys. Rev. E **68**, 041113 (2003).
- [89] S. Torquato, Phys. Rev. E **94**, 022122 (2016).
- [90] A. T. Chieco, M. Zu, A. J. Liu, N. Xu, and D. J. Durian, Phys. Rev. E **98**, 042606 (2018).
- [91] A. T. Chieco and D. J. Durian, Phys. Rev. E **103**, 062609 (2021).
- [92] Y. Zheng, Y.-W. Li, and M. P. Ciamarra, Soft Matter **16**, 5942 (2020).
- [93] Y. Zheng, A. D. S. Parmar, and M. P. Ciamarra, Phys. Rev. Lett. **126**, 118003 (2021).
- [94] A. Ikeda and L. Berthier, Phys. Rev. E **92**, 012309 (2015).
- [95] Y. Wu, P. Olsson, and S. Teitel, Phys. Rev. E **92**, 052206 (2015).
- [96] C. E. Zachary, Y. Jiao, and S. Torquato, Phys. Rev. Lett. **106**, 178001 (2011).
- [97] J. H. Weijs, R. Jeanneret, R. Dreyfus, and D. Bartolo, Phys. Rev. Lett. **115**, 108301 (2015).
- [98] S. Atkinson, F. H. Stillinger, and S. Torquato, Phys. Rev. E **94**, 032902 (2016).
- [99] F. Martelli, S. Torquato, N. Giovambattista, and R. Car, Phys. Rev. Lett. **119**, 136002 (2017).
- [100] W. Appel, *Mathematics for Physics and Physicists* (Princeton university press, New Jersey, 2007).
- [101] J. F. C. Kingman, *Poisson Processes* (Clarendon Press, Oxford, 1992).
- [102] Q. Du, V. Faber, and M. Gunzburger, SIAM Rev. **41**, 637?676 (1999).
- [103] M. Klatt, J. Lovrić, D. Chen, S. C. Kapfer, F. M. Schaller, P. Schönhöfer, B. S. Gardiner, A. Smith, G. E. Schröder-Turk, and S. Torquato., Nat. Commun. **10**, 811 (2019).
- [104] S. Lloyd, IEEE Trans. Inf. Theory **28**, 129 (1982).
- [105] E. C. Oğuz, J. E. S. Socolar, P. J. Steinhardt, and S. Torquato, Acta Crystallogr A Found Adv. **75**, 3 (2019).

- [106] D. Chen, Y. Zheng, C.-H. Lee, S. Kang, W. Zhu, H. Zhuang, P. Y. Huang, and Y. Jiao, *Phys. Rev. B* **103**, 224102 (2021).
- [107] E. C. Oğuz, J. E. S. Socolar, P. J. Steinhardt, and S. Torquato, *Phys. Rev. B* **95**, 054119 (2017).
- [108] C. Lin, P. J. Steinhardt, and S. Torquato, *J. Condens. Matter Phys.* **29**, 204003 (2017).
- [109] J. Wang, J. M. Schwarz, and J. D. Paulsen., *Nat. Commun.* **9** (2018).
- [110] J. Kim and S. Torquato, *J. Stat. Mech. Theory Exp.* **2017**, 013402 (2017).
- [111] G. Zhang, F. Stillinger, and S. Torquato, *Sci. Rep.* **6**, 36963 (2016).
- [112] Y. Zheng, L. Liu, H. Nan, Z.-X. Shen, G. Zhang, D. Chen, L. He, W. Xu, M. Chen, Y. Jiao, and H. L. Zhuang, *Sci. Adv.* **6** (2020).
- [113] J. Ricouvier, R. Pierrat, R. Carminati, P. Tabeling, and P. Yazhgur, *Phys. Rev. Lett.* **119**, 208001 (2017).
- [114] D. Chen, Y. Zheng, L. Liu, G. Zhang, M. Chen, Y. Jiao, and H. Zhuang, *Proc. Natl. Acad. Sci. U.S.A.* **118** (2021).
- [115] R. Xie, G. G. Long, S. J. Weigand, S. C. Moss, T. Carvalho, S. Roorda, M. Hejna, S. Torquato, and P. J. Steinhardt, *Proc. Natl. Acad. Sci. U.S.A.* **110**, 13250 (2013).
- [116] D. Hexner, A. J. Liu, and S. R. Nagel, *Phys. Rev. Lett.* **121**, 115501 (2018).
- [117] A. Chremos and J. F. Douglas, *Phys. Rev. Lett.* **121**, 258002 (2018).
- [118] S. Atkinson, F. H. Stillinger, and S. Torquato, *Proc. Natl. Acad. Sci. U.S.A.* **111**, 18436 (2014).
- [119] Y. Jiao and S. Torquato, *Phys. Rev. E* **84**, 041309 (2011).
- [120] C. E. Zachary, Y. Jiao, and S. Torquato, *Phys. Rev. E* **83**, 051308 (2011).
- [121] C. E. Zachary, Y. Jiao, and S. Torquato, *Phys. Rev. E* **83**, 051309 (2011).
- [122] J. Kim and S. Torquato, *Phys. Rev. E* **103**, 012123 (2021).
- [123] Z. Ma and S. Torquato, *J. Appl. Phys.* **121**, 244904 (2017).
- [124] M. A. Klatt and S. Torquato, *Phys. Rev. E* **94**, 022152 (2016).
- [125] R. Dreyfus, Y. Xu, T. Still, L. A. Hough, A. G. Yodh, and S. Torquato, *Phys. Rev. E* **91**, 012302 (2015).
- [126] D. Hexner, P. M. Chaikin, and D. Levine, *Proc. Natl. Acad. Sci. U.S.A.* **114**, 4294 (2017).

- [127] D. Hexner and D. Levine, Phys. Rev. Lett. **118**, 020601 (2017).
- [128] E. Tjhung and L. Berthier, Phys. Rev. Lett. **114**, 148301 (2015).
- [129] S. Wilken, R. E. Guerra, D. J. Pine, and P. M. Chaikin, Phys. Rev. Lett. **125**, 148001 (2020).
- [130] L. Corté, P. M. Chaikin, J. P. Gollub, and D. J. Pine, Nat. Phys. **4**, 420 (2008).
- [131] S. Torquato, A. Scardicchio, and C. Zachary., J. Stat. Mech: Theory Exp. **11**, 11019 (2008).
- [132] R. P. Feynman and M. Cohen, Phys. Rev. **102**, 1189 (1956).
- [133] F. J. Dyson, Comm.Math.Phys. **19**, 235 (1970).
- [134] P. M. Bleher, F. J. Dyson, and J. L. Lebowitz, Phys. Rev. Lett. **71**, 3047 (1993).
- [135] G. Zhang, F. Martelli, and S. Torquato, J. Phys. A Math. Theor. **51**, 115001 (2018).
- [136] S. Torquato, G. Zhang, and M. de Courcy-Ireland, J. Stat. Phys. **2018**, 093401 (2018).
- [137] S. Torquato, G. Zhang, and M. de Courcy-Ireland, J. Phys. A Math. Theor. **52**, 135002 (2019).
- [138] A.Gabrielli, F.S.Labini, M.Joyce, and L.Pietronero, *Statistical Physics for Cosmic Structures* (Springer-Verlag, NewYork, 2005).
- [139] A. Mayer, V. Balasubramanian, T. Mora, and A. M. Walczak, Proc. Nat. Acad. Sci. U.S.A. **112**, 5950 (2015).
- [140] Y. Jiao, T. Lau, H. Hatzikirou, M. Meyer-Hermann, J. C. C., and S. Torquato, Phys. Rev. E **89**, 022721 (2014).
- [141] H. Noh, S. F. Liew, V. Saranathan, S. G. J. Mochrie, R. O. Prum, E. R. Dufresne, and H. Cao., Adv. Mater. **22**, 2871 (2010).
- [142] W. Man, M. Florescu, E. P. Williamson, Y. He, S. Hashemizad, B. Y. C. Leung, D. R. Liner, S. Torquato, P. M. Chaikin, and P. J. Steinhardt, Proc. Natl. Acad. Sci. **110**, 15886 (2013).
- [143] W. Man, M. Florescu, K. Matsuyama, P. Yadak, G. Nahal, S. Hashemizad, E. Williamson, P. Steinhardt, S. Torquato, and P. Chaikin, Opt. Express **21**, 19972?19981 (2013).
- [144] J. Haberko, N. Muller, and F. Scheffold, Phys. Rev. A **88**, 043822 (2013).

- [145] T. Ma, H. Guerboukh, M. Girard, A. D. Squires, R. A. Lewis, and M. Skrobogatiy, *Adv. Optical Mater.* **4**, 2085?2094 (2016).
- [146] X. Li, A. Das, and D. Bi, *Proc. Natl. Acad. Sci. U.S.A.* **115**, 6650 (2018).
- [147] G. Gkantzounis, T. Amoah, and M. Florescu, *Phys. Rev. B* **95**, 094120 (2017).
- [148] J. Kim and S. Torquato, *New J. Phys.* **22**, 123050 (2020).
- [149] D. Chen and S. Torquato, *ActaMater.* **142**, 152 (2018).
- [150] M. A. Klatt and S. Torquato, *Phys. Rev. E* **97**, 012118 (2018).
- [151] C. D. Rosa, F. Auriemma, C. Diletto, R. D. Girolamo, A. Malafronte, P. Morvillo, G. Zito, G. Rusciano, G. Pesce, and A. Sasso, *Phys. Chem. Chem. Phys.* **17**, 8061 (2015).
- [152] J. Kim and S. Torquato, *Phys. Rev. B* **97**, 054105 (2018).
- [153] M. A. Klatt, J. Kim, and S. Torquato, *Phys. Rev. E* **101**, 032118 (2020).
- [154] S. Mitra, A. D. S. Parmar, P. Leishangthem, S. Sastry, and G. Foffi, *J. Stat. Mech.* **2021**, 033203 (2021).
- [155] D. J. Pine, J. P. Gollub, J. F. Brady, and A. M. Leshansky, *Nature* **438**, 997 (2005).
- [156] A. Donev, F. H. Stillinger, and S. Torquato, *Phys. Rev. Lett.* **95**, 090604 (2005).
- [157] A. Gabrielli, M. Joyce, and S. Torquato, *Phys. Rev. E* **77**, 031125 (2008).
- [158] M. J. Godfrey and M. A. Moore, *Phys. Rev. Lett.* **121**, 075503 (2018).
- [159] . Hexner and D. Levine, *Phys. Rev. Lett.* **114**, 110602 (2015).
- [160] M. E. Fisher, *Reports on Progress in Physics* **30**, 615 (1967).
- [161] S. Torquato, J. Kim, and M. A. Klatt, *Phys. Rev. X* **11**, 021028 (2021).
- [162] J. Hansen and I. McDonald, *Theory of Simple Liquids, with Applications to Soft Matter* (Academic Press, 1976).
- [163] S. Torquato, G. Zhang, and F. H. Stillinger, *Phys. Rev. X* **5**, 021020 (2015).
- [164] R. D. Batten, F. H. Stillinger, and S. Torquato, *J. Appl. Phys.* **104**, 033504 (2008).
- [165] R. D. Batten, F. H. Stillinger, and S. Torquato, *Phys. Rev. Lett.* **103**, 050602 (2009).
- [166] F. H. S. R. D. Batten and S. Torquato, *Phys. Rev. E* **80**, 031105 (2009).

-
- [167] S. R. N. A. J. Liu, *Ann. Rev. Condens. Matter Phys.* **1**, 347 (2010).
- [168] H. Scher and E. W. Montroll, *Phys. Rev. B* **12**, 2455 (1975).
- [169] C. S. O'Hern, S. A. Langer, A. J. Liu, and S. R. Nagel, *Phys. Rev. Lett* **88**, 075507 (2002).
- [170] M. E. Cates, J. P. Wittmer, J. P. Bouchaud, and P. Claudin, *Phys. Rev. Lett.* **81**, 1841 (1998).
- [171] M. van Hecke, *J. Phys. Condens. Matter* **22**, 033101 (2010).
- [172] D. Bi, S. Henkes, K. E. Daniels, and B. Chakraborty, *Annu. Rev. Cond. Matter Phys.* **6**, 63 (2015).
- [173] E. DeGiuli, E. Lerner, and M. Wyart, *J. Chem. Phys.* , 164503 (2015).
- [174] D. Bi, J. Zhang, B. Chakraborty, and R. P. Behringer, *Nature* , 355 (2011).
- [175] C. E. Maloney and A. Lemaître, *Phys. Rev. E* **74**, 016118 (2006).
- [176] D. L. Malandro and D. J. Lacks, *J. Chem. Phys.* **110** (1999).
- [177] C. P. Goodrich, A. J. Liu, and J. P. Sethna, *Proc. Natl. Acad. Sci. U.S.A.* **113**, 9745 (2016).
- [178] Z. Zhang, N. Xu, D. Chen, P. Yunker, A. Alsayed, K. Aptowicz, P. Habdas, A. J. Liu, S. R. Nagel, and A. Yodh, *Nature* **459**, 07998 (2009).
- [179] J. Russo and H. Tanaka, *Proc. Natl. Acad. Sci.* **22**, 6920 (2015).
- [180] W. Matthieu and M. E. Cates, *Phys. Rev. Lett.* **119**, 195501 (2017).
- [181] L. Berthier, G. Biroli, J.-P. Bouchaud, and G. Tarjus, *J. Chem. Phys.* **150**, 094501 (2019).
- [182] J. C. Maxwell, *Phil. Mag.* **27**, 294 (1864).
- [183] C. R. Calladine, *Int. J. Solids. Struct.* **14**, 161 (1978).
- [184] J. W. Rocks, N. Pashine, I. Bischofberger, C. P. Goodrich, A. J. Liu, and S. R. Nagel, *Proc. Natl. Acad. Sci. U.S.A.* **114**, 2520 (2017).
- [185] D. R. Reid, N. Pashine, J. M. Wozniak, H. M. Jaeger, A. J. Liu, S. R. Nagel, and J. J. de Pablo, *Proc. Natl. Acad. Sci. U.S.A.* **115**, E1384 (2018).
- [186] J. W. Rocks, H. Ronellenfitsch, A. J. Liu, S. R. Nagel, and E. Katifori, *Proc. Natl. Acad. Sci. U.S.A.* **116**, 2506 (2019).
- [187] D. Z. Rocklin, *New J. Phys.* **19**, 065004 (2017).
- [188] D. M. Sussman, O. Stenull, and T. C. Lubensky, *Soft Matter* **12**, 6079 (2016).

- [189] D. Z. Rocklin, B. G. Chen, M. Falk, V. Vitelli, and T. C. Lubensky, *Phys. Rev. Lett.* **116**, 135503 (2016).
- [190] D. M. Taverna and R. A. Goldstein, *Proteins: Structure, Function, and Bioinformatics* **46**, 105 (2002).
- [191] D. M. Sussman, C. P. Goodrich, and A. J. Liu, *Soft Matter* **12**, 3982 (2016).
- [192] A. Bose, M. F. J. Vermeulen, C. Storm, and W. G. Ellenbroek, *Phys. Rev. E* **99**, 023001 (2019).
- [193] M. Mailman, C. F. Schreck, C. S. O'Hern, and B. Chakraborty, *Phys. Rev. Lett.* **102**, 255501 (2009).
- [194] C. F. Schreck, M. Mailman, B. Chakraborty, and C. S. O'Hern, *Phys. Rev. E* **85**, 061305 (2012).
- [195] K. Liu, S. Henkes, and J. M. Schwarz, *Phys. Rev. X* **9**, 021006 (2019).
- [196] L. E. Silbert and M. Silbert, *Phys. Rev. E* **80**, 041304 (2009).
- [197] R. Kurita and E. R. Weeks, *Phys. Rev. E* **82**, 011403 (2010).
- [198] N. Xu and E. S. C. Ching, *Soft Matter* **6**, 2944 (2010).
- [199] L. Berthier, P. Chaudhuri, C. Coulais, O. Dauchot, and P. Sollich, *Phys. Rev. Lett.* **106**, 120601 (2011).
- [200] P. Rissone, E. I. Corwin, and G. Parisi, *Phys. Rev. Lett.* **127**, 038001 (2021).
- [201] R. Farhadifar, J.-C. Röper, B. Aigouy, S. Eaton, and F. Jülicher, *Current Biology* **17**, 2095 (2007).
- [202] D. Bi, J. H. Lopez, J. M. Schwarz, and M. L. Manning, *Soft Matter* **10**, 1885 (2014).
- [203] D. Bi, J. H. Lopez, J. M. Schwarz, and M. L. Manning, *Nat. Phys.* **11**, 1074 (2015).
- [204] D. Bi, X. Yang, M. C. Marchetti, and M. L. Manning, *Phys. Rev. X* **6**, 021011 (2016).
- [205] S. Alt, P. Ganguly, and G. Salbreux, *Philos. Trans. R. Soc. B* **372**, 20150520 (2017).
- [206] M. J. B. M. Moshe and M. C. Marchetti, *Phys. Rev. Lett.* **120**, 268105 (2018).
- [207] L. A. et. al., *Nat. Phys.* **14**, 613 (2018).
- [208] A. G. Fletcher, M. Osterfield, R. Baker, and S. Shvartsman, *Biophysical Journal* **106**, 2291 (2014).
- [209] L. Yan and D. Bi, *Phys. Rev. X* **9**, 011029 (2019).

- [210] S. Garcia, E. Hannezo, J. Elgeti, J.-F. Joanny, P. Silberzan, and N. S. Gov, Proc. Natl. Acad. Sci. U.S.A. **112**, 15314 (2015).
- [211] R. J. Tetley and e. a. M. F. Staddon, D. Heller, Nat. Phys. **15**, 1195 (2019).
- [212] A. Mongera, P. Rowghanian, H. J. Gustafson, E. Shelton, D. A. Kealhofer, E. K. Carn, F. Serwane, A. A. Lucio, J. Giammona, and O. Campàs, Nature **561**, 401 (2018).
- [213] X. Yang, D. Bi, M. Czajkowski, M. Merkel, M. L. Manning, and M. C. Marchetti, Proc. Natl. Acad. Sci. U.S.A. **114**, 12663 (2017).
- [214] A. M. Turing, Philos. Trans. R. Soc. B **237**, 37 (1952).
- [215] A. J. Koch and H. Meinhardt, Reviews of Modern Physics **66**, 1481 (1994).
- [216] R. Kalluri and W. R. A, J. Clin. Invest. **119**, 1420 (2009).
- [217] L. Oswald, S. Grosser, D. M. Smith, and J. A. Käs, J. Phys. D Appl. Phys. **50**, 483001 (2017).
- [218] J.-A. P. et a., Nat. Mat. **14**, 1040?1048 (2015).
- [219] J. Prost, F. Jülicher, and J. F. Joanny, Nat. Phys. **11**, 111 (2015).
- [220] T. E. Angelini, E. Hannezo, X. Trepát, M. Marquez, J. J. Fredberg, and D. A. Weitz, Proc. Natl. Acad. Sci. U.S.A. **108**, 4714 (2011).
- [221] D. B. Staple, R. Farhadifar, J.-C. Röper, B. Aigouy, S. Eaton, and F. Jülicher, Eur. Phys. J. E **33**, 117 (2010).
- [222] Y.-W. Li. and M. P. Ciamarra, Phys. Rev. Mat. **2**, 045602 (2018).
- [223] A. Pasupalak, Y.-W. Li, R. Ni, and M. P. Ciamarra, Soft Matter **16**, 3914 (2020).
- [224] K. Binder, J. Baschnagel, C. Bennemann, and W. Paul, J. Condens. Matter Phys. **11** (1999).
- [225] M. Galassi, *GNU scientific library: reference manual* (Network Theory Ltd., 2009) p. 574.
- [226] D. M. Sussman and M. Merkel, Soft Matter **14**, 3397 (2018).
- [227] S. Torquato and F. H. Stillinger, Rev. Mod. Phys. **82**, 2633 (2010).
- [228] S. Torquato, A. Donev, and F. Stillinger, International Journal of Solids and Structures **40**, 7143 (2003).
- [229] S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983).
- [230] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).

-
- [231] S. Lübeck, Int. J. Mod. Phys. B **18**, 3977 (2004).
- [232] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998).
- [233] H. Hinrichsen, Advances in Physics **49**, 815 (2000).
- [234] M. Henkel, H. Hinrichsen, S. Lübeck, and M. Pleimling, *Non-Equilibrium Phase Transitions* (Springer, 2008).
- [235] D. L. Turcotte, Rep. Prog. Phys. **62**, 1377 (1999).
- [236] D. Dhar, Physica A Statistical Mechanics and its Applications **369**, 29 (2006).
- [237] S. S. Manna, J. Phys. A: Math. Gen. **24**, L363 (1991).
- [238] L. Corté, S. J. Gerbode, W. Man, and D. J. Pine, Phys. Rev. Lett. **103**, 248301 (2009).
- [239] S. R. Broadbent and J. M. Hammersley, Proceedings of the Cambridge Philosophical Society **53**, 629 (1957).
- [240] P. Le Doussal and K. J. Wiese, Phys. Rev. Lett. **114**, 110601 (2015).
- [241] P. Grassberger, J. Stat. Phys. **79** (1995).
- [242] M. Basu, U. Basu, S. Bondyopadhyay, P. K. Mohanty, and H. Hinrichsen, Phys. Rev. Lett. **109**, 015702 (2012).
- [243] S. Lübeck, Phys. Rev. E **64**, 016123 (2001).
- [244] M. J. de Oliveira, Phys. Rev. E **71**, 016112 (2005).
- [245] S. B. Lee, Phys. Rev. E **89**, 060101 (2014).
- [246] F. van Wijland, Phys. Rev. Lett. **89**, 190602 (2002).
- [247] C. Ness and M. E. Cates, Phys. Rev. Lett. **124**, 088004 (2020).
- [248] S. Lübeck, Phys. Rev. E **64**, 016123 (2003).
- [249] S. Lübeck and P. C. Heger, Phys. Rev. E **68**, 056102 (2003).
- [250] Z. Ma and S. Torquato, Phys. Rev. E **99**, 022115 (2019).
- [251] S. Wilken, R. E. Guerra, D. Levine, and P. M. Chaikin, Phys. Rev. Lett. **127**, 038002 (2021).
- [252] R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998).
- [253] A. Widmer-Cooper, P. Harrowell, and H. Fynewever, Phys. Rev. Lett. **93**, 135701 (2004).

- [254] L. Wang, N. Xu, W. H. Wang, and P. Guan, *Phys. Rev. Lett.* **120**, 125502 (2018).
- [255] B. Wang, J. Kuo, S. C. Bae, and S. Granick, *Nat. Mater.* **11**, 481 (2012).
- [256] A. V. Chechkin, F. Seno, R. Metzler, and I. M. Sokolov, *Phys. Rev. X* **7**, 021002 (2017).
- [257] C. Bechinger, H. Di Leonardo, R. and Löwen, C. Reichhardt, G. Volpe, and G. Volpe, *Rev. Mod. Phys.* **88**, 045006 (2016).
- [258] S. Ramaswamy, *Annu. Rev. Condens. Matter Phys.* **1**, 323 (2010).
- [259] M. C. Marchetti, J.-F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, *Rev. Mod. Phys.* **85**, 1143 (2013).
- [260] E. Fodor, C. Nardini, M. E. Cates, J. Tailleur, P. Visco, and F. van Wijland, *Phys. Rev. Lett.* **117**, 038103 (2016).
- [261] D. Mandal, K. Klymko, and M. R. DeWeese, *Phys. Rev. Lett.* **119**, 258001 (2017).
- [262] C. Nardini, E. Fodor, E. Tjhung, F. Van Wijland, J. Tailleur, and M. E. Cates, *Physical Review X* **7**, 021007 (2017).
- [263] L. Dabelow, S. Bo, and R. Eichhorn, *Physical Review X* **9**, 021009 (2019).
- [264] S. Shankar and M. C. Marchetti, *Phys. Rev. E* **98**, 020604 (2018).
- [265] M. E. Cates and J. Tailleur, *Annu. Rev. Condens. Matter Phys.* **6**, 219 (2015).
- [266] Y. Fily and M. C. Marchetti, *Phys. Rev. Lett.* **108**, 23572 (2012).
- [267] G. S. Redner, M. F. Hagan, and A. Baskaran, *Phys. Rev. Lett.* **110**, 055701 (2013).
- [268] I. Buttinoni, J. Bialké, F. Kümmel, H. Löwen, C. Bechinger, and T. Speck, *Phys. Rev. Lett.* **110**, 238301 (2013).
- [269] A. Kabir and A. Kakugo, *Polymer Journal* **50**, 1139 (2018).
- [270] B. Zhang, K. Martens, L. Kneer, T. Funck, L. Nguyen, R. Berger, M. Dass, S. Kempter, J. Schmidtke, T. Liedl, and et al., *Nanomaterials* **10**, 1695 (2020).
- [271] D. Banerjee, V. Vitelli, F. Jülicher, and P. Surówka, *Phys. Rev. Lett.* **126**, 138001 (2021).
- [272] K. Bertoldi, V. Vitelli, J. Christensen, and M. Van Hecke, *Nat. Rev. Mater.* **2**, 1 (2017).
- [273] B. C. van Zuiden, J. Paulose, W. Irvine, D. Bartolo, and V. Vitelli, *Proc. Natl. Acad. Sci. U.S.A.* **113**, 12919 (2016).

- [274] C. Tung, J. Harder, C. Valeriani, and A. Cacciuto, *Soft Matter* **12**, 555 (2016).
- [275] T. Bäuerle, A. Fischer, T. Speck, and C. Bechinger, *Nat. Commun.* **9**, 1 (2018).
- [276] T. Markovich, E. Fodor, E. Tjhung, and M. E. Cates, *Phys. Rev. X* **11**, 021057 (2021).
- [277] M. C. Marchetti, Y. Fily, S. Henkes, A. Patch, and D. Yllanes, *Current Opinion in Colloid & Interface Science* **21**, 34 (2016).
- [278] S. A. Mallory, C. Valeriani, and A. Cacciuto, *Annu. Rev. Phys. Chem.* **69**, 59 (2018).
- [279] L. Caprini, U. M. B. Marconi, and A. Puglisi, *Phys. Rev. Lett.* **124**, 078001 (2020).
- [280] L. Caprini, U. M. B. Marconi, C. Maggi, M. Paoluzzi, and A. Puglisi, *Phys. Rev. Research* **2**, 023321 (2020).
- [281] T. Speck, J. Bialké, A. M. Menzel, and H. Löwen, *Phys. Rev. Lett.* **112**, 218304 (2014).
- [282] R. Wittkowski, A. Tiribocchi, J. Stenhammar, R. J. Allen, D. Marenduzzo, and M. E. Cates, *Nat. Commun.* **5**, 1 (2014).
- [283] E. Tjhung, C. Nardini, and M. E. Cates, *Phys. Rev. X* **8**, 031080 (2018).
- [284] M. E. Cates and J. Tailleur, *EPL (Europhysics Letters)* **101**, 20010 (2013).
- [285] J. Stenhammar, D. Marenduzzo, R. J. Allen, and M. E. Cates, *Soft matter* **10**, 1489 (2014).
- [286] A. P. Solon, M. E. Cates, and J. Tailleur, *Eur. Phys. J. Spec. Top.* **224**, 1231 (2015).
- [287] A. Wysocki, R. G. Winkler, and G. Gompper, *Europhys. Lett.* **105**, 48004 (2014).
- [288] C. Rycroft, *Chaos* **19** (2009).
- [289] J. Zhang, R. Alert, J. Yan, N. S. Wingreen, and S. Granick, *Nature Physics*, 1 (2021).
- [290] M. E. Cates, D. Marenduzzo, I. Pagonabarraga, and J. Tailleur, *Proc. Natl. Acad. Sci. U.S.A.* **107**, 11715 (2010).
- [291] P. Nie, J. Chattoraj, A. Piscitelli, P. Doyle, R. Ni, and M. P. Ciamarra, *Phys. Rev. Research* **2**, 023010 (2020).

-
- [292] J. W. Cahn and J. E. Hilliard, The Journal of chemical physics **28**, 258 (1958).
 - [293] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
 - [294] A. Tiribocchi, R. Wittkowski, D. Marenduzzo, and M. E. Cates, Phys. Rev. Lett. **115**, 188302 (2015).
 - [295] R. N. Q.-L. Lei, M. P. Ciamarra, Sci. Adv. **5**, eaau7423 (2019).
 - [296] S. Martiniani, P. M. Chaikin, and D. Levine, Phys. Rev. X **9**, 011031 (2019).

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- **Yuanjian Zheng**, Anshul Parmar and Massimo Pica Ciamarra*, Physical Review Letters 126, 118003 (2021). Hidden order beyond Hyperuniformity in critical absorbing states.
- **Yuanjian Zheng**, Shivam Mahajan, Joyjit Chattoraj and Massimo Pica Ciamarra*, Frontiers in Physics, 9, 665391 (2021). Designing Phononic Band Gaps with Sticky Potentials.
- Karina González-López, Shivam Mahajan, **Yuanjian Zheng**, Massimo Pica Ciamarra and Edan Lerner*, Physical Review E 103, 022606 (2021). Mechanical disorder of sticky-sphere glasses II. Thermo-mechanical inannealability.
- Karina González-López, Shivam Mahajan, **Yuanjian Zheng**, Massimo Pica Ciamarra and Edan Lerner*, Physical Review E 103, 022605 (2021). Mechanical disorder of sticky-sphere glasses. I. Effect of attractive interactions.
- **Yuanjian Zheng***, Yan-Wei Li and Massimo Pica Ciamarra, Soft Matter 16, 5942 (2020). Hyperuniformity and density fluctuations at a rigidity transition in a model of biological tissues.
- Anshuman Pasupalak, Shawn Khuhan Samidurai, Yanwei Li, **Yuanjian Zheng**, Ran Ni, Massimo Pica Ciamarra*, Soft Matter 17, 7708-7713 (2021). Unconventional rheological properties in systems of deformable particles.
- **Yuanjian Zheng***, and Hartmut Löwen, Physical Review Research 2, 023079 (2020). Rototaxis: localization of active motion under rotation.
- **Yuanjian Zheng**, and Dario Poletti*, Physical Review E 98, 052140, (2018). Irreversible work reduction by disorder in many-body quantum system.
- **Yuanjian Zheng**, Steve Campbell, Gabriele De Chiara, and Dario Poletti*, Physical Review A 94, 042132 (2016). Cost of counterdiabatic driving and work output.
- **Yuanjian Zheng**, Peter Hänggi and Dario Poletti*, Physical Review E 94, 012137 (2016). Occurrence of discontinuities in the performance of finite-time quantum Otto cycles.
- **Yuanjian Zheng**, and Dario Poletti*, Physical Review E 92, 012110 (2015). Quantum statistics and the performance of engine cycles.
- **Yuanjian Zheng***, and Dario Poletti, Physical Review E 90, 012145 (2014). Work and efficiency of quantum Otto cycles in power-law trapping potentials.