Efficient methodology and algorithms for resilience analysis of critical infrastructure systems subjected to natural hazards

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**EFFICIENT METHODOLOGY AND ALGORITHMS FOR
RESILIENCE ANALYSIS OF CRITICAL INFRASTRUCTURE
SYSTEMS SUBJECTED TO NATURAL HAZARDS**

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**INTERDISCIPLINARY GRADUATE SCHOOL
INSTITUTE OF CATASTROPHE RISK MANAGEMENT**

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RESILIENCE ANALYSIS OF CRITICAL INFRASTRUCTURE
SYSTEMS SUBJECTED TO NATURAL HAZARDS**

HAO CHANGYU

INSTITUTE OF CATASTROPHE RISK MANAGEMENT

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

2021

Statement of Originality

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I have reviewed the content and presentation style of this thesis and declare it is free of plagiarism and of sufficient grammatical clarity to be examined. To the best of my knowledge, the research and writing are those of the candidate except as acknowledged in the Author Attribution Statement. I confirm that the investigations were conducted in accord with the ethics policies and integrity standards of Nanyang Technological University and that the research data are presented honestly and without prejudice.

5 March, 2021

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Law Wing Keung, Adrian

Authorship Attribution Statement

This thesis contains materials from a paper accepted and in press in the following peer-reviewed journal and where I was the first author.

Chapter 6 and some parts of Chapter 4 are accepted as C. Hao, and S. H. Cheung, “Bayesian updating of failure probability curves with multiple performance functions of nonlinear structural dynamic systems”, now in press with journal ‘Computer Methods in Applied Mechanics and Engineering’.

The contributions of the co-authors are as follows:

- Prof. CHEUNG Sai Hung provided the research directions, the basic ideas and revised the manuscript.
- I wrote the codes, designed and executed the examples, and prepared the manuscript drafts.

2 December, 2021

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Date

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Hao Changyu

Abstract

Critical infrastructure systems (CIS) are highly connected to human life and industrial productions. Well protected and resilient critical infrastructure systems are the key components of modern smart cities.

In this research, we answer the following key questions on resilience of critical infrastructure systems (CIS): (1) how to characterize the multi-criteria system resilience behaviour under uncertainties? (2) how to efficiently simulate the system performance under a large number of uncertain scenarios? (3) how can we optimize the system design for better resilience performance under uncertainties? (4) how do we systematically improve and update our knowledge about the system?

A multiple Complementary Cumulative Distribution Function (CCDF)-based stochastic resilience assessment method is proposed. Critical infrastructure systems can be subjected to stochastic environmental interruptions, and given an interruptive event, the system's behaviours are often uncertain. To take such uncertainties into consideration, the proposed resilience assessment method quantifies the system resilience using multiple metrics, each having explicit physical meanings and expressed as a complimentary cumulative distribution function (CCDF). The proposed method provides explicit information of the system behaviour when the system is subjected to a broad spectrum of stochastic events with different degrees of rarity and extremeness. The proposed method can provide decision makers a bird's eye view of the system resilience performance under the attacks of future uncertain events and guide the allocation of recovery resources as well as system enhancement strategies.

Resilience assessment with uncertainties being considered will inevitably bring about higher computation requirements if a simulation-based method is adopted to estimate the CCDF curves, especially when the tail parts of the CCDF curves are of interest. It is a non-trivial or even computationally prohibitive task if brute-force Monte Carlo simulation is adopted. In this research, the state-of-the-art stochastic simulation methods are adapted to facilitate the analysis of the system resilience considering the uncertainty involved. A new scheme of how the samples are used is proposed to improve the estimation of CCDF curves. In our illustrative example, about 10 times of computational savings are achieved with our improved method

compared with the brute-force Monte Carlo simulation.

Resilience optimization in the framework of multi-objective optimization is studied. We propose a framework optimizing the exceedance probabilities directly, where the objectives are probability integrals. We solve the computational burden by an off-line strategy: preparing approximations to the objectives before the multi-objective optimization starts. These approximations are efficiently obtained by adapting and improving a density estimation based approximation method. A realistic urban drainage system (UDS) is shown as an illustration of the multi-objective optimization problem with resilience as objectives.

We proposed a novel efficient Bayesian model updating method for systematically improving and updating our knowledge about the system. We observed significant computational saving with our proposed Bayesian model updating method, and the method is also more robust to smaller sample size. With this method, stakeholders can quickly obtain the state-of-the-art parameter of the system and be more confident in making decisions.

To summarize, this thesis provides a series of tools for (1) resilience assessment under a whole range of uncertain events, (2) efficient simulation method for resilience analysis, (3) design optimization with resilience as the objectives, (4) system state updating and improving our knowledge about the system.

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List of Abbreviations

ARS	Acceptance-Rejection Sampling
CDF	Cumulative Distribution Function
CCDF	Complementary Cumulative Distribution Function, CCDF is the complimentary of CDF.
CIS	Critical Infrastructure Systems
DOPAS	Design Optimization in the Partitioned Design Space
ISS	Improved Subset Simulation
LSF	Limit-State Function
MCMC	Markov Chain Monte Carlo
MCS	brute-force Monte Carlo Simulation method
MHS	Metropolis-Hastings Sampling
MOO	Multi-Objective Optimization
PDF	Probability Density Function
RI	Resilience Index
SS	Subset Simulation
UDS	Urban Drainage System
WDS	Water Distribution System

Chapter 1

Introduction

1.1 Background

1.1.1 Critical Infrastructure Systems (CIS)

Critical infrastructure systems (CIS) refer to the infrastructure systems whose disfunction or failure will result in severe consequences on the society and human life [1]. Rinaldi et al. [2] identifies critical nation infrastructures as the systems that “affect all areas of daily life, include electric power, natural gas and petroleum production and distribution, telecommunications (information and communications), transportation, water supply, banking and finance, emergency and government services, agriculture, and other fundamental systems and services that are critical to the security, economic prosperity, and social well-being of the nation.” The Cyber Security Agency of Singapore (CSA) identifies 11 Critical Information Infrastructure (CII) sectors as “essential services that Singapore relies on”. They are: “(1) Government, (2) Infocomm, (3) Energy, (4) Aviation, (5) Maritime, (6) Land Transport, (7) Healthcare, (8) Banking and Finance, (9) Water, (10) Security and Emergency, and (11) Media.” [3]

This thesis aims at a subset of critical infrastructure systems that are with strong engineering characteristics and physical topologies. Such systems for example, include electric power supply system (EPSS), urban drainage systems (UDS), water distribution systems (WDS), gas supply networks, telecommunication systems, land & air transportation, civil structures, etc. The

performances of these systems are critical to the government, the economy as well as the health of human beings. A brief illustration of the selected CIS's relation to people's daily lives is depicted in Fig. 1-1 [4]. This thesis will not cover the other critical infrastructure systems whose delivery of services are highly affected by human factors, such as government, health, food, finance, security and emergency services, etc. These systems are very much depend on social activities and it is almost impossible to be simulated purely by a physical model. Some different studies methods may be required. Therefore, they are out the the scope of this thesis.

Figure 1-1: An illustration of the critical infrastructure systems and their connections to human life.

Figure from [4].

1.1.2 Risks of Critical Infrastructure Systems

Critical infrastructure systems are prone to many types of risks, including natural hazards, attacks, ageing components, etc. There are several definitions of risk. T. Kovalenko and D. Sornette [5] take risk as “the triplet of (i) probability/uncertainty, (ii) potential loss and (iii) mitigation techniques, i.e., counter-measures to reduce vulnerability of a system”. Varnes [6] defines the risk as a function of the hazards, the exposure and the vulnerability . The probability of the hazards happening at a specific level is a random variable. Vulnerability refers to the susceptibility and the sensitivity to the natural hazards [7]. Exposure refers to the

total quantified potential loss. It is worth noting that in some other definitions, the probability and vulnerability may be combined as probability and is defined as: the probability a system fails to some extent, and the exposure is defined as vulnerability, which refers to how severe the consequence can be, it is often measured by the system performance or financial loss. By any way of definition, it can be seen that the risk analysis is highly depend on probabilistic methods. Actually, probability-based risk analysis has a relatively long history, it dates back to the 1950s, when reliability appeared among nuclear engineers. Critical infrastructure systems operate in uncertain environments, it has been a widely accepted to consider uncertainties in risk analysis of CIS.

1.1.3 Concept of Resilience and Resilience of Critical Infrastructure Systems

The word ‘resilience’ means ‘bounce back’ literally. In year 1807, Thomas Young introduced the modulus of elasticity (also named Young’s modulus) to describe the elastic ‘bounce back’ of materials. There are several interpretations of resilience from different viewpoints. US National Academy of Sciences (NAS) gives the following definition of resilience: “the ability to prepare and plan for, absorb, recover from, and more successfully adapt to adverse events.” Haines [8] pointed out that “The resilience of a system is a manifestation of the states of the system. Perhaps most critically, it is a vector that is time dependent.” And he gives the following definition: (Resilience is) “the ability of the system to withstand a major disruption within acceptable degradation parameters and to recover within an acceptable time and composite costs and risks.” L. Flax et al. [9] define resilience as: “the capacity of individuals, communities, institutions, businesses, and systems within a city to survive, adapt, and grow no matter what kinds of chronic stresses and acute shocks they experience.” The ASME (American Society of Mechanical Engineers) [10] considers resilience as “the ability to sustain disruptions without interruption of system performance, and if the system is interrupted, to recover rapidly and fully”. The NIAC (National Infrastructure Advisory Council) [11] suggests the definition to be “the ability of the system to predict, absorb, adapt, and quickly recover from disruptions”.

It is impossible to list all definitions of resilience here. Resilience research is spreading in such a broad disciplines that we need to narrow down to the target systems of this thesis. For the subset of critical infrastructure systems to be studied in this thesis, the definition by

Vugrin, E. D. et al. [12] may be more suitable. They examined several definitions of resilience and proposed the following definition: “Given the occurrence of a particular disruptive event (or set of events), the resilience of a system to that event (or events) is the ability to efficiently reduce both the magnitude and duration of the deviation from targeted system performance levels”.

1.2 Motivations

Resilience of critical infrastructure systems has attracted a lot of attentions of researchers. It is an important concept and guiding idea in the future system design and operations. In traditional engineering design, the resistance level is usually adopted, where the resistance level refers to the ability of the system to withstand the impact of hazards [13]. However, the low-probability events but causing disastrous consequences which exceed the pre-designed resistance level of the system can be expected. The ‘black swan’ [14] is a metaphor of such highly improbable events but with major impacts. Building a resilient system is thus necessary in the uncertain environment. From the definition of resilience, we can see a resilient infrastructure system is not only robust but also reconfigurable. A robust system is insensitive to changes of operating environment and design assumption, this is a definition adapted from Saleh et al. [15]. A reconfigurable system is capable of absorbing the impact of changes in operating environment (for example, disruptive events), and it has the ability to reorganized to keep delivering services, as well as recover quickly. Resilience analysis is a tool to capture the characteristics of critical infrastructure systems (CIS) during disruptions.

Resilience is highly connected to risk analysis. I. Linkov et al. [16] compared the similarity and difference between risk management and resilience analysis. They pointed out that due to the difficulties of quantifying uncertain events, resilience quantification is not as mature as risk management. T. Kovalenko and D. Sornette [5] pointed out that resilience and risk are complementary and “resilience plays a distinct and crucial role in uncertain environments”. They also argued the importance of considering the extreme risks in resilience analysis. Considering uncertainties has been widely accepted as natural in risk analysis. It is also important that resilience analysis of critical infrastructure systems incorporates the uncertainties. In a very recent review paper on the resilience of critical infrastructure networks by Liu and Song [17],

uncertainty was identified as one of the research challenges in resilience research. Besides, due to the changing environment (for example the climate change), extreme events that seldom happened before may become more frequent. However, existing resilience analysis methods fails to characterize the resilience of systems under a whole range of uncertain disruptive events. This thesis aims at filling this gap. We can see from the literature (an in-depth literature review will be given in Chapter 2), while the resilience theory has been gradually expanding and containing more methods and implications, some key questions are not perfectly answered. These questions include: (1) how to characterize the multi-criteria system resilience behaviour under uncertainties? (2) how to efficiently simulate the system performance under a large number of uncertain scenarios? (3) how can we optimize the system design for better resilience performance under uncertainties? (4) how do we systematically improve and update our knowledge about the system?

In this thesis, question (1) is addressed in Chapter 3; question (2) is addressed in Chapter 4; question (3) is answered in Chapter 5; and Chapter 6 shows a solution to question (4).

1.3 Main contributions

This thesis provides an overall framework on the resilience assessment and optimization of critical infrastructure systems. The contribution of this thesis can be generally divided into four parts. In brief, these four contributions are: (1) resilience quantification, (2) resilience simulation, (3) resilience optimization and (4) system updating. The four contributions are connected with each other: (1) and (3) are methodologies, and (2) and (4) are algorithms supporting the application of the methodologies. A detailed introduction of these four contributions are as follows:

1. A multiple CCDF (Complementary Cumulative Distribution Function) based resilience assessment method for critical infrastructure systems is proposed. The multiple CCDF method provides an overall resilience quantification method by describing system resilience in multiple resilience dimensions in a decoupled way. The proposed method is capable of capturing the uncertainties in both the system and the environment. Introducing the CCDF into resilience assessment eliminates the phenomenon that the same resilience

index corresponds to different system performances. (This phenomenon will be further elaborated in Chapter 2.) Another merit of the proposed multiple CCDF-based resilience assessment method is that it is capable of capture the rare cases. While the necessity of rare event simulation has been well accepted in reliability analysis, it is not commonly seen resilience analysis methods.

2. The propose resilience assessment method requires realistic simulation of the system performance under uncertainties, which is usually computationally expensive and even prohibitive if rare events are considered. To address this, an efficient simulation method called Improved Subset Simulation (ISS) method [18] is applied to the proposed resilience assessment method. The original ISS method has a problem of discontinuous estimation: there are sudden jumps at the intersection of simulation levels on the CCDF curves. We further improve the method by proposing a new scheme on how the samples are used in estimating the CCDF curves. The improved ISS method is capable of estimating multiple CCDF curves simultaneously. It avoids running the stochastic simulation multiple times. In this sense, the simulation method enables a faster assessment of resilience of the critical infrastructure systems. In our example on urban drainage system, we obtained about 10 times of computational savings over brute-force Monte Carlo simulation while keeping the same level of coefficient of variations of the estimators.
3. We show an application of the proposed resilience assessment method in the perspective of optimal system design under uncertainties. We propose a resilience optimization method within the multi-objective optimization framework. With the objective functions being probability integrals, massive computational burden is to be alleviated. To solve this problem, an advanced exceedance probability approximation method is introduced to alleviate the high computational requirement. Realistic critical infrastructure system optimization is shown as an illustration.
4. Both the proposed resilience assessment and resilience optimization relies on a good knowledge about the system and an accurate model of the system performance. To address this problem, a novel efficient Bayesian model updating method is proposed. We show that the proposed efficient Bayesian model updating method obtains significant computational savings. Besides, compared with existing method, the proposed method is more robust

with reduced number of model evaluations. A fast model updating can be helpful in guiding the recovery decision making.

The first contribution (resilience quantification method) has been formed a journal paper pending for review. The second contribution (resilience simulation) and the fourth contribution (resilience updating) are combined into another journal paper which is in press. And the third contribution (resilience optimization) is being polished as the third journal paper.

To summarize, this thesis provides a series of methods on resilience assessment and resilience optimization of critical infrastructure systems under a whole range of disruptive events and uncertainties in system behaviours. To support the propose resilience assessment method and resilience optimization method, efficient algorithms are improved or proposed to obtain higher efficiency and accuracy in analyzing system resilience.

1.4 Organization of the thesis

This thesis is consists of 7 chapters and is organized as follows:

1. Chapter 1 introduces the background, the motivations and the main contributions of this research work. A summary of the thesis organization is presented.
2. Chapter 2 summarizes existing literature on four topics: (1) Resilience assessment and quantification methods for critical infrastructure systems; (2) Stochastic simulation methods and efficient sampling methods; (3) Multi-objective optimization methods; (4) Bayesian model updating methods. At the end of this chapter, research gaps are identified.
3. Chapter 3 presents the proposed multiple CCDF-based resilience assessment method. The physical meaning of the method is discussed. The merits of introducing CCDF is shown. Two examples are given as illustrations.
4. Chapter 4 presents an efficient simulation method that enables a faster assessment of system resilience. We improved an efficient simulation method and adapt it to the resilience assessment of two realistic CIS examples. The first example is a revisit to example 2 in Chapter 3 and the second example is an urban water distribution system (WDS) subjected to future uncertain earthquakes.

5. In Chapter 5, a multi-objective optimization based resilience optimization framework is illustrated. The special part of this optimization problem is that the objectives involves functions of exceedance probability, which requires evaluating the computationally-expensive probability integrals. The computation issue is addressed by introducing an advanced exceedance probability approximation method into the multi-objective optimization framework.
6. Chapter 6 proposes a novel Bayesian model updating method, which enables us a quick updating of our knowledge about the system. The proposed method achieves significant efficiency improvement compared with the existing ones.
7. This thesis is summarized in Chapter 7. The future research suggestions are provided.

Chapter 2

Literature Review

This chapter provides a summary of the state-of-the-art research works in the literature that are related to this thesis. This chapter is divided into the following four sections:

- 1. Resilience assessment and quantification methods for critical infrastructure systems;*
- 2. Stochastic simulation methods and efficient sampling methods;*
- 3. Multi-objective optimization;*
- 4. Bayesian model updating.*

These four sections serves as the background and motivations to the following Chapters. The first section reviews the existing resilience assessment method, research gaps are identified. In Chapter 3, we propose our simulation-based resilience assessment method. The second section is on efficient simulation methods. We identify the research gap and proposed our solution for efficient resilience assessment in Chapter 4. In the third section we review multi-objective optimization methods. This is the background of Chapter 5, where we proposed a multi-objective optimization-based resilience optimization framework. And finally, In the fourth part we review Bayesian model updating, an approach to improve our knowledge about the system and the accuracy of our models. In Chapter 6, we propose our efficient Bayesian model updating method.

2.1 Resilience Assessment and Quantification Methods

In this section, we present a survey of literature on resilience assessment and quantification methods. In Subsection 2.1.1, we start with the generic resilience assessment metrics. We can see that resilience consists of multiple dimensions and the assessment of resilience should be conducted with multiple criteria. In Subsection 2.1.2, we review the existing resilience quantification methods from deterministic perspective. And in Subsection 2.1.3, existing methods considering resilience of systems under uncertainties are presented. Finally, in Subsection 2.1.4, we summarize the literature survey and identify the research gaps.

2.1.1 Generic Resilience Metrics

A generic resilience metric refers to a framework of conceptual resilience assessment method. It is a step forward from the concept of resilience and can be seen as a detailed interpretation of the concept. A generic resilience metric may not target at a certain type of system, but it provides a general guidance on resilience assessment.

Bruneau et al. [19] identify a R4 framework of resilience which includes Robustness, Redundancy, Resourcefulness, and Rapidity. A brief summary of these four concept of their framework is as follows: Robustness is the ability of a system to withstand disasters and to keep its performance; Redundancy is to what extent the system can maintain its function if functional loss is inevitable; Resourcefulness is the ability and available resources to detect and solve the problem; And Rapidity is how fast the system function can be recovered. Heinimann proposed an eight-functions-based resilience measurement framework for social-technical infrastructure systems [20], as shown in Fig. 2-1. This framework is also called AR^6A for short (see Fig. 2-1 for the components of AR^6A). This AR^6A framework considered a system over an extended time scale, where resilience is not limited to robustness and restoration. It also covers the pre-event activities such as the preparedness and attentiveness, and post-event behaviours such as strengthening, remembering and adaptive functions for future events.

The generic resilience metrics provide a guiding framework for resilience analyses. However, for a specific realistic critical infrastructure system, measuring, analyzing and managing the resilience of the system requires much more work and considerations of the characteristics of

Figure 2-1: The eight generic functions concept by Hans Heinimann, figure from [20].

the system. For example, when we consider a power supply system, we have to think about taking what parameter as the indicator of system performance, is it continuous or discrete? And how to model the system, etc. More importantly, when we say a system is resilient or another system is not resilient, we need a comparable basis. This requires us to quantify system resilience. Resilience quantification methods are thus proposed by researchers.

2.1.2 Resilience Quantification Methods for Critical Infrastructure Systems

Mendonca [21] highlighted the importance of establishing a resilience assessment framework that can be used to measure the factors that contribute to resilience. He also pointed out that a comprehensive single resilience measurement is not an easy task. Tamvakis et al. [22, 23] reviewed existing resilience quantification methodologies and suggested a conceptual system-entropy-based resilience evaluation framework. In recent years, a number of resilience quantification methods have been proposed by researchers. Resilience quantification requires the system performance being measurable with the evolution of time so that a time history curve of the system performance can be obtained. An example of the system performance curve is shown in Fig. 2-2. Such a curve is also referred to as “resilience curve” in the literature. It is worth noting that a system can have multiple measures of performance (MOPs) and resilience analysis may base on any number of the MOPs subjected to the interest of decision makers.

Figure 2-2: An illustration of the system performance curve based on one system measure of performance (MOP) when the system is subjected to a single disruptive event.

Bruneau et al. [19] proposed to measure the resilience by loss of resilience. They measure resilience as the integral of the performance loss from the time of disruption to the time of full recovery. Fig. 2-3 shows the ‘resilience triangle’ proposed by Bruneau et al. [19] which assumes the system performance level experiences a sudden drop to the bottom. Mathematically, their resilience metric is written as:

$$R = \int_{t_0}^{t_1} [100 - Q(t)] dt, \quad (2.1)$$

where $Q(t)$ is the performance of infrastructure, it is normalized as the percentage of service, with $Q(t) = 100$ represents for no degradation of the system. A larger value of R indicate more loss of resilience, thus the system is referred to as less resilient. This resilience assessment method is “area-based”: it calculates some area on the performance curve v.s. time plot. This resilience metric considers both the system performance level and the time factor. It is also easy to apply and provides a comprehensive measurement of the overall resilience level of a system. However, there is a the problem with this method (and some other area-based method we will see later): the same value of the resilience metric may correspond to different system performances. For example, in Fig. 2-4, all the four performance curves have the same area of the resilience loss, but the system behaviours are quite different. In other words, some key characteristics of the system performance may be hidden if we consider only one single measurement by the area-based method.

Francis and Bekera [24] identifies three resilience capacities: absorptive capacity, adaptive

Figure 2-3: The resilience triangle and loss of resilience measurement. Figure adapted from [19].

Figure 2-4: An example of systems with same Loss of Resilience but different behaviors.

capacity, and restorative capacity. They proposed a comprehensive resilience metric considering the tree resilience capacities and the time to recovery. The resilience metric ρ_i is as follows:

$$\rho_i(S_p, F_r, F_d, F_0) = S_p \frac{F_r}{F_0} \frac{F_d}{F_0}, \quad (2.2)$$

where ρ_i is the resilience factor for event i , S_p is the speed recovery factor, F_0 is the system performance level without disruptions, F_d is the performance level immediately after the disruption happens, F_r is the post-event fully recovered performance level. This resilience metric is deterministic, the author further incorporates the fragility and probability of events, expanding

the method for a series of uncertain event. The resilience metric ζ is as follows:

$$\zeta = \sum_i Pr[D_i] \cdot f(\mu|z_i) \cdot \rho_i(S_p, F_r, F_d, F_0), \quad (2.3)$$

where D_i represents the i th event, $f(\mu|z_i)$ is the fragility function, z is a parameter vector. This method has two drawbacks: (1) the physical meaning of the resilience metric ρ_i is implicit. It neither comes with a meaningful unit nor within the range $[0, 1]$; (2) the resilience metric is a weighted combination of several components, it fails to solve the problem that the same resilience value corresponds to multiple system resilience performances.

Tang et al. [25] proposed a function-based resilience metric. The key idea is to measure the Resistance R_m , Re-stabilization R_e , Rebuilding R_d , and Reconfiguration R_s separately. There are too many definitions to list, so we explain their method with the assistance of Fig. 2-5, where P stands for performance level, ET is called Elasticity Threshold. t_{pre} is the time a trigger event happens and the system performance starts to drop, t_{event} is the time when the system performance level drops to its minimum, t_{post} is the time when the system is completely recovered.

Figure 2-5: The resilience curve by Tang et al. Figure is adapted from [25].

Resistance is measured as

$$R_m = P(t_{event}) + RR, \quad (2.4)$$

where RR is the width of Robust Range. Re-stabilization is given by

$$R_e = \frac{ET - P(T_{\text{event}})}{P(t_{\text{pre}}) - P(t_{\text{event}})} \quad \text{if } ET > \min[P(t)]. \quad (2.5)$$

Rebuilding is measured by

$$R_d = \frac{S_r}{S_f}, \quad (2.6)$$

where,

$$S_f = \frac{P(t_{\text{pre}}) - P(t_{\text{event}})}{t_{\text{event}} - t_{\text{pre}}}; \quad (2.7)$$

$$S_r = \frac{P(t_{\text{post}}) - P(t_{\text{event}})}{t_{\text{post}} - t_{\text{event}}}. \quad (2.8)$$

And Reconfiguration is as follows:

$$R_s = \frac{P(t_{\text{post}}) - P(t_{\text{event}})}{P(t_{\text{pre}}) - P(t_{\text{event}})}. \quad (2.9)$$

Finally, the resilience indicator is:

$$RI(T) = R_m \times (1 - R_e) \times R_d \times R_s. \quad (2.10)$$

Similar to Francis and Bekera [24], this method combines the individual resilience component into a single resilience index. The drawback is two folds: (1) the combination result in the loss of many details of the system resilience performance. The resistance, re-configuration, etc. is readily available, but such a combination hides these details. (2) the overall resilience index can be dominated by some components. For example, in the method of Tang et al., Rebuilding is measured by the ratio of the slopes of the two sections of the resilience curve. This value can be very large compared with the other resilience components. In such cases, Rebuilding would dominate the overall resilience index.

We categorize the methods in [24] and [25] as equally-weighted methods, as actually each resilience component is assigned a weight equals to 1 in the combination step. There also exist methods that combine the resilience components with different weights. Munoz and Dunbar [26] studied the resilience of supply chain. They measure the resilience from the manufacturer side (whose resilience is represented by R_m), and the retailer side (whose resilience is represented

by R_r). These two resilience indices are calculated as:

$$R_m = w_{1,m}(Recovery_m) + w_{2,m}(Impact_m) + w_{3,m}(PerformanceLoss_m) + w_{4,m}(ProfileLength_m) + w_{5,m}(Weighted - sum_m), \quad (2.11)$$

and

$$R_r = w_{1,r}(Recovery_r) + w_{2,r}(Impact_r) + w_{3,r}(PerformanceLoss_r) + w_{4,r}(ProfileLength_r) + w_{5,r}(Weighted - sum_r), \quad (2.12)$$

respectively. where $w_{i,m}$ and $w_{i,r}$ are weights for each dimension of resilience on manufacturer side and retailer side, respectively. And the overall resilience is given by:

$$R = R_m + R_r \quad (2.13)$$

There are resilience assessment methods that look at the resilience dimensions without combining them into a single resilience index. Panteli et al. [27] proposed a resilience quantification method for power system named $\Phi AEII$ (pronounce like “FLEP”), where resilience is considered in three phases and multiple metrics. These metrics are as follows: Φ - the speed of dropping of resilience, A - how low resilience level can be, E - the extensiveness of the degradation, and II - the speed of recovery. They also have the fifth index defined as the area between the performance curve and the time axis, which is similar to Eq. 2-3 but it is the complimentary of the area by Eq. 2-3. Besides the multiple dimensional resilience quantification method, another contribution in [27] is that the method is capable of measuring both infrastructure resilience and operational resilience. The resilience analysis result are given in five separate tables, and each table shows the values of the corresponding resilience metric. This method has two drawbacks: (1) the metrics Φ and II are slopes of degrading and recovering respectively, which relies on an assumption that the degrading and recovering are both linear. (2) The method is for one-scenario-based: although the method considers probability of failure with fragility functions, the resilience assessment result is for one scenario. It can be used for analysing only limited number of disruptive scenarios.

Shen et al. [28] proposed a multi-index resilience measurement framework considering the reliability, robustness, recovery and reconfigurability, and multiple criteria decision analysis

(MCDA) was suggested. Their framework seeks to represent resilience in the range $[0,1]$ (or in the range $[0, \infty)$, when the slope of the performance curve is incorporated), the physical meanings of the indices are implicit. Their paper focused more on conceptual resilience assessment framework with no realistic application examples shown.

It is worth mentioning that there are resilience metrics being proposed for a specific type of critical infrastructures. These resilience assessment methods may not be generalized to other systems but they are capable of characterizing the unique and important behaviours of the specific infrastructure system. Cardoni et al. [29] considered the resilience of a power distribution system as the integral over time of four metrics: the transformer restoration rapidity, the percentage of undamaged substations, the presence of alternative lines, and the availability of portable generators.

2.1.3 Resilience Assessment Methods Considering Uncertainties

Critical infrastructure systems operate under various of uncertainties. These uncertainties include (1) the uncertain system state and system performance level, (2) the uncertain disruptive events, in both the magnitude and the frequency, (3) the topological dependencies (4) uncertainties in operating and users. The first uncertainty is within the system, it exists even there is no disruptive event. The second uncertainty is from the operating environment. The third uncertainty exists in interdependent networks and the fourth uncertainty is commonly seen in systems relying on human operating. The research works reviewed in this part mainly focus on the first two types of uncertainties. These two types of uncertainties are commonly considered in the literature and they are the also the main uncertainties studied in this thesis.

M. Ouyang et al. proposed an expected annual resilience metric [30]. It is a time-dependent metric given by a ratio of two areas. The numerator is the area between actual performance curve and the time axis, and the denominator is the area between the target performance curve

and the time axis of a whole year:

$$R = E \left[\frac{\int_0^T P(t)dt}{\int_0^T TP(t)dt} \right] = E \left[\frac{\int_0^T TP(t)dt - \sum_{n=1}^{N(T)} AIA_n(t_n)}{\int_0^T TP(t)dt} \right], \quad (2.14)$$

where T is the one year time interval, $P(t)$ is the actual performance curve, $TP(t)$ represents the target performance curve, $N(T)$ is the total number of events during time interval T , t_n is the time that the n th event happens, and AIA_n is the area between the actual performance curve and the target performance curve for the n th event. This metric considers the uncertain events as t_n is random, and AIA_n is affected by the hazard types and hazard level. This resilience metrics is capable of considering a series of events and gives the overall estimation of resilience. However, it is expectation-based. Multiple performance curves may result in the same expectation. Therefore, we argue that this metric may miss some resilience performance information.

Chang and Shinozuka [31] considers resilience from two dimensions with uncertainties. In their paper, the system performance is represented by r , and time is represented by t . Two acceptable thresholds are predefined: a performance loss standard that is acceptable as r^* , and maximum acceptable disruption time t^* . These two dimensions are measured by two criteria: (1) loss of performance, and (2) length of recovery. Their resilience index is a joint probability that these two criteria are not meet:

$$R = P(A|i) = P(r_0 < r^* \quad \text{and} \quad t_1 < t^*), \quad (2.15)$$

where R is the resilience index, A is the predefined standards, i is the i th event, r_0 represents the lowest performance level during an event, an t_1 is the time of full recovery. They also pointed out that this method can be expanded consider a broad range of possible events. There are two limitations of this method: (1) a predefined standard is required, (2) the resilience index is a joint probability, which is actually a mix of two dimensions, i.e. performance level and recovery time. One of the criteria may dominate the resilience index. For example, a system performance curve with very significant drop but recovers very fast, and another system has mild performance drop but the interruption lasts for very long time. Both cases may result in a low exceedance probability R . This method may not be able to distinguish these cases.

Ayyub [32] proposed a comprehensive resilience measurement method that integrates robustness, redundancy and recoverability. The resilience index is written as:

$$R_e = \frac{T_i + F\Delta T_f + R\Delta T_r}{T_i + \Delta T_f + \Delta T_r}, \quad (2.16)$$

where T_i is the time interval from time t_0 to the time an incident happens t_i , where t_0 is the starting time to be determined; T_f is the time interval from time t_0 to t_f , representing the time when the system fails; T_r is the time to recovery, it is defined as the time interval t_0 to the time instant that the system is fully recovered t_r ; $\Delta T_f = t_f - t_i$ is called failure duration; ΔT_r is the recovery duration; $\Delta T_d = T_r - T_i$ is the disruption duration. F is the failure profile, R is the recovery profile, they are given as:

$$F = \frac{\int_{t_i}^{t_f} f dt}{\int_{t_i}^{t_f} Q dt}; \quad (2.17)$$

$$R = \frac{\int_{t_f}^{t_r} r dt}{\int_{t_f}^{t_r} Q dt}, \quad (2.18)$$

where Q is the system performance, f and r are trajectories of failure and recovery, respectively. This is a very comprehensive single-index-based resilience quantification method. Even though this method considers many factors of system resilience, the output R_e blends all information together. This is similar to the “area-based” method and it is an integrated-resilience-component methods. Besides, the author shows no realistic application of the method, the complexity of this method may limit its practicability.

Pant et al. [33] proposed a method to measure system resilience by the percentage of recovery as a function of time. They called the method “Time to $\alpha\%$ -Resilience”. They represent system performance curve by $\phi(t)$, where t is time. For a given disruptive event e^j , where j represents the j th event, the lowest performance level is $\phi(t_d)|e^j$. Fig. 2-6 shows the definitions of variables in their method.

They measure resilience by the percentage of recovery at time t_r , where t_r is some time in the recovery process: $t_d < t_r < t_f$, where t_d is the time that recovery starts, and t_f is the time of full recovery. The resilience index is:

$$R_\phi(t_r|e^j) = \frac{\phi(t_r|e^j) - \phi(t_d|e^j)}{\phi(t_0) - \phi(t_d|e^j)} \quad (2.19)$$

Figure 2-6: Resilience curve and definitions of variables in the method by Pant et al. Figure from [33].

As the events e^j is uncertain, this method is capable of considering uncertainties in disruptive events. The drawback of this method is that the resilience index is normalized by the maximum performance drop (the denominator in Eq. 2.19). If a system's maximum performance drop is slight, the time to, for example 50% recovery, can be much shorter than another system with severe performance drop. It is non-trivial to say which system is more resilient by this metric.

From the perspective of reliability, Hashimoto et al. [34] proposed to measure resilience as the average recovery rate. They first define two system states, S : safe state, and F : failure state. The system's output status at time t is represented by X_t . In their method, resilience is equal to the reciprocal of the mean length of time that the system remains in failure state. First, they consider reliability as the probability that the system status is in safe state:

$$\alpha = \text{Prob}\{X_t \in S\}, \quad (2.20)$$

where α is the reliability of the system. Next, they define a probability that the system transit from safe state to failure state:

$$\rho = \text{Prob}\{X_t \in S, X_{t+1} \in F\}. \quad (2.21)$$

And finally, the resilience of the system is:

$$R = \frac{1}{E(T_F)} = \frac{\rho}{1 - \alpha} = \frac{Prob\{X_t \in S, X_{t+1} \in F\}}{Prob\{X_t \in F\}}, \quad (2.22)$$

where T_F is the mean length of time that the system remains in failure state, $E(T_F)$ is its mean. This method is from the perspective of reliability thus it naturally considers the uncertainties in disruptive events. The drawback of this method is two folds: (1) a mean value is used as the resilience index; (2) only the dimension of time is considered, the performance level of the degraded state is not included.

There are also research works focus on a specific type of infrastructure and consider the uncertainties. Mean value is commonly used as the representation of system resilience under uncertain scenarios.

Yang et al. [35] proposed a resilience index considering the duration of extreme event (RICD), which is based on the expectation value of the product of two ratios. Fig. 2-7 is a schematic illustration of the definitions. The first ratio is the area of the shape formed by the actual performance level and the time axis divided by the area of the rectangular formed by the ideal performance level and the time axis.

$$R_1 = \frac{\int_{t_0}^{t_5} LR(t)dt}{\int_{t_0}^{t_5} LI(t)dt}, \quad (2.23)$$

where t_0 is the time when the typhoon attacks, and t_5 is the time of fully recovery. The second ratio is the duration of typhoon divided by the time interval from typhoon attacks to the time of fully recovery.

$$R_2 = \frac{t_4 - t_0}{t_5 - t_0}, \quad (2.24)$$

where t_4 is the time when the typhoon ends.

The RICD index is given by:

$$R_{RICD} = E[R_1 \times R_2], \quad (2.25)$$

where E represents for taking expectation. They adopt Monte Carlo simulation for simulating multiple typhoons.

Figure 2-7: A schematic illustration of the definitions in the framework of Yang et al. [35].

Cicilio et al. [36] measured the resilience of an electrical grid by the total load loss (MW), their resilience index is adapted from Bruneau et al. [19], which we have reviewed in Section 2.1.2. They adopt Monte Carlo simulation to simulate the uncertainties events. The resilience of the power system is represented by an expectation of multiple failure scenarios. Liu et al. [37] calculated the mean value of the resilience metric of a power system assuming the failure rate of the system components following some predefined distribution. While the mean value represents part of the plausible system resilience performance, it cannot effectively characterize the system performance under rare or extreme events.

As a literature review on such an active research area, it is by no means exhaustive. Some comprehensive reviews are recommended: Yodo and Wang [38] reviewed the engineering resilience quantification methods. A comprehensive review of definitions and system resilience assessment methods is given by Hosseini et al. [39]. Shen et al. [28] also provides a valuable survey of system resilience.

2.1.4 Conclusive Remarks

There are numerous resilience assessment methods proposed by researchers from multiple disciplines. Most of them measure resilience as a single resilience index: either defining resilience

as an overall measurement or combining the multiple resilience components to one index by weights. We argue that a single-index-based resilience assessment method is inadequate to character the resilience performance of systems. Single-index-based resilience assessment methods suffer the problem that the same resilience index may correspond to totally different resilience curves, as we have illustrated in Fig. 2-4. We also reviewed multiple-indices-based resilience assessment methods. Although multiple-indices-based methods is not as concise as the single-index based methods, they keep the important characteristics of the system behaviour, which can provide decision makers more information to support their judgements, as well as making us get more knowledge on our critical infrastructure systems.

Resilience assessment methods can also be divided into two categories: deterministic approach and probabilistic approach. Deterministic approach considers only a limited number of disruptive events. For probabilistic approaches, there are two types of uncertainties: (1) the uncertainty of system performances, which refers to the that phenomenon that the system performance is not deterministic even if the disruptive event is deterministic (2) the uncertain disruptive events (such as hazards and attacks), where uncertainties are both in terms of intensity and probability of happening. Researches tend to measure system resilience under uncertainties by taking the expectation of some scenarios. The expectation-based approach suffer from the similar problem with single-indices-based methods: the same expectation value (as measurement of resilience) can correspond to multiple resilience performance distributions. For example, a distribution with high peak but short tail and another distribution with lower peak but fat long tail may have the same expectations. Fig. 2-8 is such an example. Here we show two lognormal distributions for illustrative purpose. The parameters of these two lognormal distribution are as follows: $\mu_1 = 3.1$, $\sigma_1^2 = 0.05$; $\mu_2 = 2.25$, $\sigma_2^2 = 1.75$. Both the probability distribution functions and the complementary cumulative distribution functions (CCDF) are shown in Fig. 2-8 . These two distributions have the same mean value 22.7599. Actually, as the mean of lognormal distribution is analytically available as $\exp(\mu + \sigma^2/2)$, we can have infinite number of combinations of the parameters that make the distributions having the same mean value, as long as $\exp(\mu_1 + \frac{\sigma_1^2}{2}) = \exp(\mu_2 + \frac{\sigma_2^2}{2})$. More importantly, using a mean value fails to effectively characterize the system performance under rare or extreme events. This is because the mean of a distribution can be dominated by the more frequent part, making the long fat tail part unimportant. This is what we have to avoid when analysis resilience in an uncertain

environment, especially when rare but high impact events exist. CCDF curves is a good choice to see the hidden long tail of a distribution. As in Fig. 2-8, distribution 2 is with a relatively long tail which is not very significant in the left PDF (probability distribution function) plot, but it is obvious in the CCDF plot at right hand side. We will discuss CCDF curves again in Chapter 3.

Figure 2-8: An illustration of two distributions with same mean value.

To summarize, we can find that there are few resilience quantification methods meet the following criteria at the same time: (1) multiple criteria-based (2) consider both the uncertainty of disruptive events and uncertainties of the system performance (3) capable of considering a whole range of possible disruptive events. Chapter 3 aims at filling this research gap.

2.2 Stochastic Simulation Methods for the Estimation of Exceedance Probabilities

This section reviews the stochastic simulation methods and sampling methodologies. To consider uncertainties, simulation-based method is widely adopted as it has solid physical basis. This section serves as the background of Chapter 4, where we adapt and improve an efficient simulation method (ISS method) for resilience assessment under a whole range of uncertainties.

2.2.1 Brute-force Monte Carlo Method and Exceedance Probability Estimating

The Classical Monte Carlo Integration solves the following problem: suppose we want to estimate the mean of some function $g(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a random variable vector following distribution $p(\boldsymbol{\theta})$. The expectation of $g(\boldsymbol{\theta})$ is given by:

$$\mu = E[g(\boldsymbol{\theta})] = \int g(\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}. \quad (2.26)$$

This expectation can be estimated by the brute-force Monte Carlo Method (also called crude Monte Carlo Simulation, we will refer to it as Monte Carlo simulation for short and abbreviate it as MCS for simplicity) [40, 41]. Suppose we generate n random samples $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(n)}$ from $p(\boldsymbol{\theta})$, the mean value in Eq. 2.26 is given by:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n g(\boldsymbol{\theta}^{(i)}), \quad (2.27)$$

where $\hat{\mu}$ is an unbiased estimator of the true μ by Monte Carlo simulation. The variance of this estimator is analytically available: [41]

$$\text{Var}(\hat{\mu}) = \frac{1}{n^2} \sum_{i=1}^n [g(\boldsymbol{\theta}^{(i)}) - \hat{\mu}]^2. \quad (2.28)$$

Due to the fact that $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(n)}$ are independent and identically distributed (i.i.d.) samples, this variance of the estimator can be calculated as: [42]

$$\text{Var}[\hat{\mu}] = \frac{\text{Var}[g(\boldsymbol{\theta})]}{n}. \quad (2.29)$$

The coefficient of variation (c.o.v.) is also a metric that is usually adopted. It is computed as:

$$\delta_n[\hat{\mu}] = \frac{\sqrt{\text{Var}[\hat{\mu}]}}{E[\hat{\mu}]} = \frac{1}{E[g(\boldsymbol{\theta})]} \times \frac{\sqrt{\text{Var}[g(\boldsymbol{\theta})]}}{\sqrt{n}}. \quad (2.30)$$

We can see that the c.o.v. of the estimator decrease at a rate proportional to the reciprocal of the square root of sample size n .

In reliability analysis, we frequently encounter this type of question: what is the probability that a system fails? This question can be interpreted as what is the probability that the output of some system performance function $g(\boldsymbol{\theta})$ exceeds some threshold b ? Mathematically, we write as:

$$P[g(\boldsymbol{\theta}) > b], \quad (2.31)$$

where if threshold b is the boundary of system failure, we say $\boldsymbol{\theta} \in F$ if $g(\boldsymbol{\theta}) > b$, where F represents the failure region.

Eq. 2.31 can be estimated using Monte Carlo simulation by replacing the function $g(\boldsymbol{\theta})$ in Eq. 2.26 by an indicator function:

$$h(\boldsymbol{\theta}) = \mathbf{I}_g(\boldsymbol{\theta}) = \mathbf{I}[g(\boldsymbol{\theta}) > b] = \begin{cases} 1, & \boldsymbol{\theta} \in F; \\ 0, & \text{otherwise.} \end{cases} \quad (2.32)$$

With Monte Carlo simulation, we can estimate the failure probability as follows:

$$\hat{p}_F = \frac{1}{n} \sum_{i=1}^n \mathbf{I}_g(\boldsymbol{\theta}^{(i)}), \quad (2.33)$$

where p_F is the unbiased estimator given by Monte Carlo simulation. Correspondingly, the c.o.v. of this estimator is:

$$\delta_n[\hat{p}_F] = \frac{1}{E[h(\boldsymbol{\theta})]} \times \frac{\sqrt{\text{Var}[h(\boldsymbol{\theta})]}}{\sqrt{n}} = \sqrt{\frac{1 - p_F}{p_F \times n}}, \quad (2.34)$$

where n is the number of i.i.d. samples. We can see that $\delta_n[\hat{p}_F] \propto \frac{1}{\sqrt{n}}$. The c.o.v. of the estimator by Monte Carlo simulation is analytically available, and this is an important merit of the method. In addition, Monte Carlo simulation is insensitive to the dimension of random variables.

But the efficiency of brute-force Monte Carlo Simulation may be low when simulating rare events [42, 43], where the events happening probability $p_F \ll 1$. In this case, from Eq. 2.34, we have:

$$\delta_n[\hat{p}_F] = \sqrt{\frac{1 - p_F}{p_F \times n}} \approx \frac{1}{\sqrt{p_F \times n}}. \quad (2.35)$$

It is clear that the c.o.v. increases very fast with the decreasing of failure probability p_F .

There is an example in [42] to illustrate this: if a 30% c.o.v. is desired and the failure probability is $p_F = 0.0001$, to simulate 100 failure events, on average, the number of samples will have to be around 1 million. This is a huge number of samples to simulate, and the computation time may not be affordable.

2.2.2 Acceptance-Rejection Method

Acceptance-Rejection Method, also called Accept-Reject Method or Rejection Sampling, is a method to sample from a distribution $p(\boldsymbol{\theta})$ which cannot be derived analytical and thus is hard to sample from directly. The idea is to sample from another proposal distribution $\pi(\boldsymbol{\theta})$ instead, where samples $\boldsymbol{\theta} \sim \pi(\boldsymbol{\theta})$ can be sampled directly. Suppose we have n samples from $\pi(\boldsymbol{\theta})$: $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(n)}$, the target samples $\boldsymbol{\theta}' \sim p(\boldsymbol{\theta})$ can be obtained by the following process:

Algorithm 1: Acceptance Rejection Method

Input: $\boldsymbol{\theta}^{(i)} \sim \pi(\boldsymbol{\theta})$, $i = 1, 2, \dots, n$, target distribution $p(\boldsymbol{\theta})$, constant M

Output: $\boldsymbol{\theta}' \sim p(\boldsymbol{\theta})$

```

1 for  $i = 1$  to  $n$  do
2   simulate random number  $u \sim U(0, 1)$ ;
3   if  $u < \frac{p(\boldsymbol{\theta}^{(i)})}{M\pi(\boldsymbol{\theta}^{(i)})}$  then
4     | Accept  $\boldsymbol{\theta}^{(i)}$  into  $\boldsymbol{\theta}'$ ,
5   else
6     | Reject  $\boldsymbol{\theta}^{(i)}$ ;
7   end
8 end
9
```

A schematic diagram is shown as Fig. 2-9. In the Acceptance-Rejection Method, a key step is to choose the constant M . The principle is that this M has to be equal or larger than the ratio of $p(\boldsymbol{\theta})/\pi(\boldsymbol{\theta})$, i.e. in Fig. 2-9, the outer PDF $M\pi(\boldsymbol{\theta})$ should be an envelope of the target PDF $p(\boldsymbol{\theta})$. To guarantee this, the proposal distribution $\pi(\boldsymbol{\theta})$ has to fulfill the following requirements:

$$\forall \boldsymbol{\theta}, \quad \text{if } p(\boldsymbol{\theta}) > 0, \quad \pi(\boldsymbol{\theta}) > 0. \quad (2.36)$$

Figure 2-9: A schematic diagram of Acceptance-Rejection Method.

2.2.3 The difficulties in generating failure samples

In estimating the exceedance probability as given in Eq. 2.31, we are interested in how likely the failure scenarios may happen. From the perspective of simulation, this question can be answered by generating samples that correspond to failure scenarios. Let's use θ to represent the vector of random samples, and $p(\theta)$ is the PDF of θ . Our target samples that are conditional on failure follow $q(\theta|F)$, where F represents "failure scenarios". If we define failure as some performance function $g(\theta)$ exceeding some threshold b , i.e., $\{g(\theta) > b\}$, we say a sample θ is a failure sample and note it as $\theta \in F$. We want to generate enough samples following PDF $q(\theta|F)$, which is given by:

$$q(\theta|F) = \frac{I(\theta)p(\theta)}{P(F)}, \quad (2.37)$$

where $I(\theta)$ is some indicator function as follows:

$$I(\theta) = \begin{cases} 1, & \theta \in F; \\ 0, & \text{otherwise.} \end{cases} \quad (2.38)$$

The samples following distribution $q(\theta|F)$ usually cannot be simulated efficiently as F is a rare event. Alternatively, we can simulate samples following $q(\theta|F)$ by simulating candidate samples $\tilde{\theta}$ following $p(\theta)$, and we keep the candidate sample $\tilde{\theta}$ as a failure sample if $\tilde{\theta} \in F$, and discard it otherwise. This is the actually the approach of brute-force Monte Carlo simulation. When considering scenarios with small failure probability, i.e., $P(F)$ is very small, we need to generate

a large number of candidate samples and check if $\tilde{\theta} \in F$ or not. This explains the efficiency problem of the brute-force Monte Carlo simulation we have introduced with Eq. 2.35.

To simulate samples following $q(\theta|F)$ directly, we can use the Markov Chain Monte Carlo (MCMC) simulation method. According to Gamerman and Lopes [44], Markov Chain Monte Carlo refers to “techniques enable simulation from a distribution by embedding it as a limiting distribution of a Markov chain and simulating from the chain until it approaches equilibrium.” We refer this equilibrium state as a state that the samples converge to a stationary distribution. It is worth mentioning that not all Markov chains have stationary distributions.

Before we go into the details of the MCMC method, some definitions on Markov chain are introduced as follows. A descriptive definition of Markov chain is given in [41] as: “A Markov chain is a sequence of random variables that can be thought of as evolving over time, with probability of a transition depending on the particular set in which the chain is.” Given a sequence of **random variables** denoted as $X^{(0)}, X^{(1)}, \dots, X^{(t)}, X^{(t+1)}$, the distribution of $X^{(t+1)}$ is assumed to depend only on the previous state $X^{(t)}$:

$$P\left(X^{(t+1)} = x | X^{(0)}, X^{(1)}, \dots, X^{(t)}\right) = P\left(X^{(t+1)} = x | X^{(t)}\right). \quad (2.39)$$

If this is true, this sequence of random variables is said to form a **Markov chain**. The conditional probability $P\left(X^{(t+1)} = x\right) = P\left(x | X^{(t)}\right)$ is called the transition distribution.

A very basic MCMC algorithm is the Metropolis algorithm. Metropolis et al. [45] proposed the Simulated Annealing (SA) algorithm by introducing the random walk with transition probability of $\exp(-\Delta E/kT)$, where ΔE is the change of system energy, k is Boltzmann’s constant, and T is the temperature. This work is believed to be a pioneer work of MCMC method. The pseudo-code of Metropolis algorithm is given as Algorithm 2. Conducting the “for-loop” of Metropolis algorithm, the output samples θ form a Markov chain with the stationary PDF to be our target distribution $q(\theta)$, and these samples will be at least asymptotically following $q(\theta)$.

The proposal distribution $p^*(\cdot|\theta)$ in Metropolis algorithm should be symmetric and i.i.d. samples following this proposal distribution should be able to be generated efficiently. There are various choices of proposal distributions as long as they satisfy these requirements. For

Algorithm 2: Metropolis algorithm

Input: target distribution $q(\boldsymbol{\theta})$, which is non-trivial to be directly sampling from;
 proposal distribution $p^*(\cdot|\boldsymbol{\theta})$, constant n
Output: $\boldsymbol{\theta} : \{\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(n)}\} \sim q(\boldsymbol{\theta})$

```

1 for  $i = 1$  to  $n - 1$  do
2   (1) generate a candidate sample  $\boldsymbol{\theta}_c$  from  $p^*(\cdot|\boldsymbol{\theta}^{(i)})$ ;
3   (2) calculate the ratio  $r = \min\left(1, \frac{q(\boldsymbol{\theta}_c)}{q(\boldsymbol{\theta}^{(i)})}\right)$ ;
4   (3) simulate random number  $u \sim U(0, 1)$  and compare  $u$  and  $r$  as follows:
5   if  $u < r$  then
6     | Accept  $\boldsymbol{\theta}_c$  and update  $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}_c$ ,
7   else
8     | Reject  $\boldsymbol{\theta}_c$  and set  $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)}$ ;
9   end
10 end
11
```

example, we can use Gaussian distribution as the proposal distribution, which is given by:

$$p^*(\boldsymbol{\theta}_c|\boldsymbol{\theta}) = \frac{\exp\left(-\frac{1}{2}(\boldsymbol{\theta}_c - \boldsymbol{\theta}^{(i)})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}_c - \boldsymbol{\theta}^{(i)})\right)}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}}, \quad (2.40)$$

where $\boldsymbol{\theta}^{(i)}$ is the i th sample and is taken as the current sample, $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2)$ is the k -dimensional covariance matrix, supposing the dimension of $\boldsymbol{\theta}$ is k . The values of $\sigma_1, \sigma_2, \dots, \sigma_k$ describe the distance of the candidate sample $\boldsymbol{\theta}_c$ away from the current sample $\boldsymbol{\theta}^{(i)}$, i.e., the step length of the random walk. If the step length is very small, the value of r as computed in step (2) of Algorithm 2 will be close to 1 (as the candidate sample $\boldsymbol{\theta}_c$ will be “close to” the current sample $\boldsymbol{\theta}^{(i)}$). Therefore, there is a high probability that the candidate sample $\boldsymbol{\theta}_c$ be accepted as the next sample $\boldsymbol{\theta}^{(i+1)}$. However, too small step length will result in low efficiency in the exploration of the sample space and the samples will be highly correlated due to their proximity. On the contrary, if this step length is set to be very large, the acceptance rate in step (3) of Algorithm 2 can be very low, and in many cases we reject $\boldsymbol{\theta}_c$ and set $\boldsymbol{\theta}^{(i+1)}$ to be $\boldsymbol{\theta}^{(i)}$. In this way, there may exist many repeated samples, resulting in highly correlated samples because of repeating. It is worth mentioning that setting $\boldsymbol{\theta}^{(i+1)}$ to be current sample $\boldsymbol{\theta}^{(i)}$ when the candidate sample is rejected in step (3) of Algorithm 2 is compulsory to make the Markov chain have the correct stationary distribution. In choosing the proposal distribution, there is

a trade-off between increasing the acceptance rate and decreasing the correlation among the samples. A proper choice of the step length should take both factors into considerations.

Hastings [46] generalized Metropolis's work [45] and proposed a simulation method referred to as Metropolis-Hastings Sampling (MHS) method. Instead of requiring the proposal distribution to be symmetric as in Metropolis algorithm, Metropolis-Hastings Sampling allows a non-symmetric proposal distribution.

Algorithm 3: Metropolis Hastings Sampling

Input: target distribution $q(\boldsymbol{\theta})$, proposal distribution $p^*(\cdot|\boldsymbol{\theta})$, constant n
Output: $\boldsymbol{\theta} : \{\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(n)}\} \sim q(\boldsymbol{\theta})$

```

1 for  $i = 1$  to  $n - 1$  do
2   (1) generate a candidate sample  $\boldsymbol{\theta}_c$  from  $p(\cdot|\boldsymbol{\theta}^{(i)})$ ;
3   (2) calculate the ratio  $r = \min\left(1, \frac{q(\boldsymbol{\theta}_c)}{q(\boldsymbol{\theta}^{(i)})} \cdot \frac{p(\boldsymbol{\theta}^{(i)}|\boldsymbol{\theta}_c)}{p(\boldsymbol{\theta}_c|\boldsymbol{\theta}^{(i)})}\right)$ ;
4   (3) simulate random number  $u \sim U(0, 1)$ 
5   if  $u < r$  then
6     | Accept  $\boldsymbol{\theta}_c$  and update  $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}_c$ ,
7   else
8     | Reject  $\boldsymbol{\theta}_c$  and set  $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)}$ ;
9   end
10 end
11
```

We have shown that MCMC method enables the generating of samples from a distribution $q(\boldsymbol{\theta})$ that is non-trivial to be sampled from directly. The distribution of failure samples ($q(\boldsymbol{\theta}|F)$) as given in Eq. 2.37 is such a distribution that usually cannot be easily sampled from efficiently. Given a failure sample $\boldsymbol{\theta}^{(i)}$ located in the failure region F , i.e., $\boldsymbol{\theta}^{(i)} \in F$, we can generate more failure samples following $q(\boldsymbol{\theta}|F)$ with a modification in the process of Metropolis algorithm by adding a failure check step in step (3) of Algorithm 2, and this failure sample $\boldsymbol{\theta}^{(i)}$ is taken as the starting point of a Markov chain (or a 'seed' sample).

It is worth mentioning that a similar process with a non-symmetric proposal distribution can be conducted for generating failure samples based on Metropolis-Hastings Sampling method. There are two problems remaining: (1). If the failure probability is very small, the initial failure samples $\boldsymbol{\theta}$ which serve as the starting point of the Markov chains, are still non-trivial to obtain by random sampling method (for example brute-force Monte Carlo simulation); (2). If $\boldsymbol{\theta}$ is high-dimensional, the acceptance rate in MCMC method can be very low. Subset Simulation [43]

Algorithm 4: Metropolis Sampling with failure check

Input: target distribution $q(\boldsymbol{\theta}|F)$; proposal distribution $p^*(\cdot|\boldsymbol{\theta})$, where $\boldsymbol{\theta} \in F$; constant n

Output: $\boldsymbol{\theta} : \{\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(n)}\} \sim q(\boldsymbol{\theta}|F)$

```

1 for  $i = 1$  to  $n - 1$  do
2   (1) generate a candidate sample  $\boldsymbol{\theta}_c$  from  $p^*(\cdot|\boldsymbol{\theta}^{(i)})$ ;
3   (2) calculate the ratio  $r = \min\left(1, \frac{q(\boldsymbol{\theta}_c)}{q(\boldsymbol{\theta}^{(i)})}\right)$ ;
4   (3) simulate random number  $u \sim U(0, 1)$  and compare  $u$  and  $r$  as follows:
5   if  $u < r$  and  $\boldsymbol{\theta}_c \in F$  then
6     | Accept  $\boldsymbol{\theta}_c$  and update  $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}_c$ ,
7   else
8     | Reject  $\boldsymbol{\theta}_c$  and set  $\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)}$ ;
9   end
10 end
11
```

provides a solution to these problems.

2.2.4 Subset Simulation and Its Recent Developments

To solve the low efficiency problem of crude Monte Carlo Simulation when simulating rare events, Au and Beck [43] proposed the Subset Simulation (SS). Subset Simulation is an advanced simulation methodology that is robust and efficient, especially in simulating rare events. Subset Simulation originally aims at estimating the reliability of structures, and it exhibits advantages in high-dimensional problems [43].

The basic idea of Subset Simulation is to avoid unnecessarily repeated sampling from the high probability region. To estimate the failure probability efficiently, instead of directly aiming at estimating the ultimate failure probability, Subset Simulation seeks a series of intermediate

failure regions $F_1, F_2, \dots, F_m$ to write the ultimate failure probability $P(F)$ as:

$$\begin{aligned}
 P(F) &= P(F_m) \\
 &= P(F_m | \cap_{i=1}^{m-1} F_i) \\
 &= P(F_m | F_{m-1}) \times P(\cap_{i=1}^{m-2} F_i) \\
 &= P(F_m | F_{m-1}) \times P(F_{m-1} | F_{m-2}) \times \dots \times P(F_2 | F_1) \\
 &= P(F_1) \prod_{i=1}^{m-1} P(F_{i+1} | F_i),
 \end{aligned} \tag{2.41}$$

where $F_i, i = 1, \dots, m$ are a sequence of failure regions with $F_m \subset F_{m-1} \subset \dots \subset F_1$. $P(F_i)$ is the i th failure probability subjected to failure threshold b_i , i.e. $P(F_i) = P(g(\boldsymbol{\theta}) > b_{i-1})$, and $b_{i-1} < b_i$ for $i = 0, 1, \dots, m-1$.¹ In this way, Subset Simulation converts the problem of estimating small exceedance probability $P(F)$ into a problem of estimating an exceedance probability F_1 and a series of conditional probabilities $P(F_{i+1}|F_i)$. Due to the fact that $F_m \subset F_{m-1} \subset \dots \subset F_1$, failure event F_1 is no longer rare, and $P(F_1)$ is no longer very small probability. Therefore, failure samples following $q(\boldsymbol{\theta}|F_1)$ are much more easier to be obtained by Monte Carlo simulation methods compared with the ultimate failure samples which follow $q(\boldsymbol{\theta}|F)$. Based on samples following $q(\boldsymbol{\theta}|F_1)$, we can obtain failure samples following $q(\boldsymbol{\theta}|F_2)$ with the following process: (1). We first take samples following $q(\boldsymbol{\theta}|F_1)$ as seeds and use MCMC methods (for example, the algorithm described in the previous section) to generate more new failure samples following $q(\boldsymbol{\theta}|F_1)$; (2). Using the seeds and the new samples generated, we identify samples that are located in the failure region F_2 by checking if $g(\boldsymbol{\theta}) > b_2$. Repeating this process on F_{i-1} and F_i , we can generate and identify failure samples following $q(\boldsymbol{\theta}|F_i)$ from samples following $q(\boldsymbol{\theta}|F_{i-1})$, $i = 2, 3, \dots, m$.

It is worth mentioning that the generation of new failure samples is necessary because F_2 is a subset of F_1 . If we use the initial samples (which follow $p(\boldsymbol{\theta})$) without generating new failure samples, there will be fewer failure samples following $q(\boldsymbol{\theta}|F_2)$ than failure samples following $q(\boldsymbol{\theta}|F_1)$. With the process going on, sooner or later at some level, there will be few or even no samples following $q(\boldsymbol{\theta}|F_l)$, where $l \geq 2$.

¹We use b_{i-1} to represent the threshold corresponding to F_i , this is to imply threshold b_{i-1} is identified in the $(i-1)$ th simulation level. At the same time, we make the notations consistent with our improved ISS method in Chapter 4, where this b_{i-1} notation facilitates the illustration of the algorithms.

Based on the failure samples obtained, the probability $P(F_1)$ in Eq. 2.41 is estimated by:

$$P(F_1) \approx \frac{1}{N} \sum_{\iota=1}^N I_1(\boldsymbol{\theta}^{(\iota)}), \quad (2.42)$$

where $\boldsymbol{\theta} \sim p(\boldsymbol{\theta})$, $\boldsymbol{\theta}^{(\iota)}$ is the ι th instance of $\boldsymbol{\theta}$ ², and I_1 is an indicator function given by:

$$I_1(\boldsymbol{\theta}) = \begin{cases} 1, & \boldsymbol{\theta} \in F_1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.43)$$

The conditional probabilities $P(F_{i+1}|F_i)$ in Eq. 2.41 is estimated by:

$$P(F_{i+1}|F_i) \approx \frac{1}{N} \sum_{\iota=1}^N I_{i+1}(\boldsymbol{\theta}^{(\iota)}), \quad (2.44)$$

where $\boldsymbol{\theta}$ follows $q(\boldsymbol{\theta}|F_i)$ and $\boldsymbol{\theta}^{(\iota)}$ is the ι th instance of $\boldsymbol{\theta}$. I_{i+1} is an indicator function given by:

$$I_{i+1}(\boldsymbol{\theta}) = \begin{cases} 1, & \boldsymbol{\theta} \in F_{i+1}; \\ 0, & \text{otherwise.} \end{cases} \quad (2.45)$$

Substitute Eq. 2.42 and Eq. 2.44 into Eq. 2.41, the target probability $P(F)$ can be estimated.

The intermediate thresholds $b_1, b_2, \dots, b_m$ are suggested to be determined adaptively [43]: threshold b_i is set to be the $(p_0 N)$ th largest value of $g(\boldsymbol{\theta})$, where $g(\boldsymbol{\theta})$ is the performance function we are considering, and p_0 is a pre-fixed positive value smaller than 1. It is suggested $p_0 = 0.1 \sim 0.3$ [42]. It is worth mentioning that the failure regions $F_i : g(\boldsymbol{\theta}) > b_{i-1}$, for $i = 0, 1, \dots, m-1$ are known implicitly, i.e., we can only tell if a sample is located in a failure region F_i or not by failure checking. [42]. Based on this adaptively determining thresholds approach, we can conduct Subset Simulation in the following two steps:

1. First level simulation (level $i = 1$):

²Different from previous sections of this chapter, In this Section 2.2.4, we will use ι to represent the instance of samples instead of i , which is already taken to represent the simulation level of Subset Simulation.

- (a) Do random sampling with Monte Carlo Simulation to get N samples of the first level;
 - (b) Define a ratio p_0 , sort the samples in increasing order with respect to the value of $g(\boldsymbol{\theta})$ and keep the last p_0N samples as seeds.
2. Simulation of subsequent levels:
- For level $i = 2, 3, \dots$:

- (a) Sampling more failure samples using the seed of the previous level, MCMC method is commonly used in this step. The number of new samples to be simulated is $N - p_0N$, which guarantees the total number of failure samples in this level (including p_0N seeds from the previous level) is still N .
- (b) Sort the obtained N samples in increasing order with respect to $g(\boldsymbol{\theta})$ and keep the last p_0N samples as seeds of next level.

It is worth mentioning that Subset Simulation also solves the problem of high-dimensional cases by the Modified Metropolis algorithm [43] and Modified Metropolis Hastings Sampling (MMHS) algorithm [47] which enable generating candidate samples a component by a component. For simplicity, Algorithm 5 shows a special case of MMHS algorithm by taking each dimension as a component. We can also combine some of the dimensions as one component, this group proposal is presented in [47]. The MMHS algorithm guarantees the new samples $\boldsymbol{\theta}$ falling into the failure region, i.e. $\boldsymbol{\theta} \in F_i$. An illustrative example on how Subset Simulation works and how the failure samples are generated is shown in Appendix A. on Page 169.

In conducting Subset Simulation, a key step is to choose the proposal distribution. While some not well-chosen proposal distribution may affect the efficiency of Subset Simulation and result in high correlation between the samples, in realistic applications, unless some very special problems (for example, the random vector is very high dimensional and there exist some unclear high-order dependence between the random variables), most proposal distributions (such as, Gaussian distribution, uniform distribution) would work well. We can also use the adaptive sampling method proposed by Papaioannou et al. [51], which enables adjusting the parameters of the proposal distribution automatically with the information collected from the first few

transitions.

The c.o.v. of Subset Simulation is [42]:

$$\delta_{SS} \propto (\ln p_F^{-1})^r, \quad (2.46)$$

where, $1 \leq r \leq 1.5$. Comparing with Eq. 2.34, it is much more efficient, especially in simulating rare events, where $p_F \ll 1$.

Algorithm 5: Modified Metropolis Hastings Sampling for sampling in high dimensions

Input: target distribution $q(\boldsymbol{\theta}|F_i)$, proposal distribution $p^*(\cdot|\boldsymbol{\theta})$, dimension of $\boldsymbol{\theta}$: n_{dim} ,
target number of samples $N - p_0N$,

Output: $\boldsymbol{\theta} : \{\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(N-p_0N)}\} \sim q(\boldsymbol{\theta}|F_i)$

```

1 for  $\iota = 1$  to  $N - p_0N$  do
2   for  $k = 1, 2, \dots, n_{\text{dim}}$  do
3     (1) generate a candidate sample  $\theta_c^{(\iota+1)}$  from  $p(\cdot|\boldsymbol{\theta}_k^{(\iota)})$ , (where  $\boldsymbol{\theta}_k^{(\iota)}$  represents the
        $k$ th dimension of  $\boldsymbol{\theta}$  of the  $\iota$ th sample  $\boldsymbol{\theta}^{(\iota)}$ );
4     (2) calculate the ratio  $r = \min \left( 1, \frac{q(\theta_c^{(\iota+1)})}{q(\theta_k^{(\iota)})} \cdot \frac{p(\theta_k^{(\iota)}|\theta_c^{(\iota+1)})}{p(\theta_c^{(\iota+1)}|\theta_k^{(\iota)})} \right)$ ;
5     (3) simulate random number  $u \sim U(0, 1)$ 
6     if  $u < r$  then
7       | set  $\tilde{\boldsymbol{\theta}}_k = \theta_c^{(\iota+1)}$ ,
8     else
9       | set  $\tilde{\boldsymbol{\theta}}_k = \boldsymbol{\theta}_k^{(\iota)}$ ;
10    end
11  end
12  (4) checking failure
13  if  $\tilde{\boldsymbol{\theta}} \in F_i$  then
14    | Accept  $\tilde{\boldsymbol{\theta}}$  and update  $\boldsymbol{\theta}^{(\iota+p_0N)} = \tilde{\boldsymbol{\theta}}$ ,
15  else
16    | Reject  $\tilde{\boldsymbol{\theta}}$  and set  $\boldsymbol{\theta}^{(\iota+p_0N)} = \boldsymbol{\theta}^{(\iota)}$ ;
17  end
18 end
19
```

Subset Simulation has been an attractive research and many improvements to Subset Simulation can be found in the literature. Au, S. K. and Beck, J. L. [47] improved the MCMC algorithms to increase the efficiency of Subset Simulation. Ching et al. [48] developed an alterna-

tive Subset Simulation method without the Markov Chain. This alternative Subset Simulation does not rely on the proposal distribution and the rejection rate is much smaller. A hybrid Subset Simulation method was proposed by Ching et al. [49], which is less sensitive to the proposal PDF. Auxiliary Domain Method (ADM) was proposed by Katafygiotis et al. [50]. Compared with standard Subset Simulation, the ADM uses the samples in auxiliary failure domain to avoid the Monte Carlo simulation step in standard Subset Simulation. It is shown that the ADM is an improvement of the standard Subset Simulation, particularly in simulating rare events. Papaioannou et al. [51] reviewed the MCMC algorithms used in Subset Simulation and proposed a new adaptive sampling method that enables sampling from conditional normal distribution.

These improvements focus on more efficient sampling and alternative sampling directions. They fail to solve one important drawback of Subset Simulation: it is not capable of considering the case with multiple performance functions. Subset Simulation considers the exceedance probability $P(h(\boldsymbol{\theta}) > b)$, where b is some threshold. Let us use Y to represent $h(\boldsymbol{\theta})$. Y is also called “driving variable” in literature [52, 53]. A problem with multiple performance function is: we want to estimate the multiple individual failure probability $P(Y_j > b_j) = P(h_j(\boldsymbol{\theta}) > b)$, where Y_j is the j th driving variable, $h_j(\boldsymbol{\theta})$ is the j th performance function, and correspondingly, b_j is the j th threshold. Unlike brute-force Monte Carlo Simulation, Subset Simulation searches the feasible domain with a direction: it moves to the lower probability region level by level. Such a direction is determined by the driving variable. Therefore, to estimate the multiple failure probability, we have to conduct multiple runs of Subset Simulation to estimate $P(Y_j > b_j)$, $j = 1, 2, \dots$, one by one. It is worth noting that brute-force Monte Carlo Simulation is not limited to one performance function, as it searches the feasible domain randomly, i.e. without direction. And also because of this random searching, its efficiency in estimating rare events is lower than Subset Simulation.

To solve this limitation to the original Subset Simulation, H.S. Li et al. [54] developed a Generalized Subset Simulation that enables the estimation of multiple stochastic responses in one simulation run. Bansal, S. and Cheung, S. H. [18] proposed an algorithm that enables the estimation of multiple failure probability curves in one run. The basic idea of these two papers are similar: define a driving variable Y as a union of the multiple driving variables, $Y = Y_1 \cup Y_2 \cup \dots \cup Y_j \cup \dots \cup Y_m$, where m is the total number of performance functions.

In this part we introduce these two methods together and identify their difference after that. For simplicity, we introduce the basic idea of the methods with two performance functions, i.e. $j = 1, 2$. It is worth noting that both methods do not limit the number of performance functions.

The driving variable is $Y = Y_1 \cup Y_2$, and an unified intermediate event of simulation level i is $F_i = F_{i,1} \cup F_{i,2} = \{Y_1 > b_{i,1} \cup Y_2 > b_{i,2}\}$, where $b_{i,1}$ is the threshold of performance function 1 of the i th simulation level, $b_{i,2}$ has the similar meaning. With simulation being conducted from level to level, our target is to reach the target thresholds b_1 and b_2 . The target failure probability of performance function 1 is given as:

$$P_1 = P(F_{m,1}) = P(F_{m,1}|F_{m-1}) \prod_{i=2}^{m=1} P(F_i|F_{i-1})P(F_1), \quad (2.47)$$

where $F_1 \supset F_2 \dots \supset F_{m-1} \supset F_{m,1}$ is a sequence of failure events. And the target failure probability of performance function 2 is:

$$P_2 = P(F_{m,2}) = P(F_{m,2}|F_{m-1}) \prod_{i=2}^{m=1} P(F_i|F_{i-1})P(F_1), \quad (2.48)$$

similarly, $F_1 \supset F_2 \dots \supset F_{m-1} \supset F_{m,2}$ is a sequence of failure events. The main difference of [54] and [18] is as follows: In the paper by Li et al. [54], the number of samples in level $i + 1$ is controlled by the union driving variable $\{(Y_1 > b_{i,1}) \cup (Y_2 > b_{i,2})\}$ and is given by:

$$N_{i+1} \approx N_i \times P(F_{i+1}|F_i), \quad (2.49)$$

where N_i is the number of samples at simulation level N_i . The drawback of this method is that it requires the performance functions to about equally share the failure region identified by the union driving variable. If not, for example, in simulation level i , if $\{Y_1 > b_{i,1}\}$ dominates the failure domain identified by $\{Y_1 > b_{i,1} \cup Y_2 > b_{i,2}\}$, most of the N_i samples would fall in $\{Y_1 > b_{i,1}\}$, the c.o.v. of estimator on failure probability F_2 can be out of control. In contrast, in Bansal and Cheung's paper [18], the number of samples in level $i + 1$ is controlled by each of the performance function:

$$N_{i+1} = n_{i+1,1} + n_{i+1,2}, \quad (2.50)$$

where $n_{i+1,1}$ is the number of samples at the $(i+1)$ th simulation level and belonging to performance function 1, $n_{i+1,2}$ is the number of samples at the $(i+1)$ th simulation level and belonging to performance function 2, and

$$n_{i+1,1} = n_{i,1} \times P(F_{i+1,1}|F_{i,1}), \quad (2.51)$$

where $n_{i,1}$ is the number of samples at the i th level and belonging to performance function 1, and

$$n_{i+1,2} = n_{i,2} \times P(F_{i+1,2}|F_{i,2}), \quad (2.52)$$

where $n_{i,2}$ is the number of samples at the i th level and belonging to performance function 2. In this way, for performance function 1, there are always around $n_{1,1}$ samples belonging to it at any simulation level; and for performance function 2, there are always around $n_{1,2}$ samples belonging to it at any simulation level. The number of samples belonging to each performance function is guaranteed. From the perspective of each performance function, it is equivalent to running the original Subset Simulation. It is worth noting that both methods' computational savings come from the overlapping between the failure domains defined by each performance function.

As the method by Bansal and Cheung is capable of controlling the c.o.v. of each failure probability estimator, we will adapt their method to resilience analysis. For clarity, this method is mentioned as “ISS method” hereafter, ISS represents “Improved Subset Simulation”. However, ISS method is not perfect, it suffers discontinuity problem at the thresholds. We will address this problem in Chapter 4.

2.3 Multi-objective Optimization Problem

Multi-objective optimization (MOO) is a method that solves the optimization problem with more than one objective. Mathematically, a MOO problem can be written as:

$$\begin{aligned}
 & \text{minimize} && f_i(\boldsymbol{\Phi}), && i = 1, 2, \dots, N_{\text{obj}}; \\
 & \text{subjected to} && g_j(\boldsymbol{\Phi}) \leq 0, && j = 1, 2, \dots; \\
 & && h_k(\boldsymbol{\Phi}) = 0, && k = 1, 2, \dots; \\
 & && \boldsymbol{\Phi} \in [\boldsymbol{\Phi}^{(L)}, \boldsymbol{\Phi}^{(U)}],
 \end{aligned} \tag{2.53}$$

where $f_i(\cdot)$ are the objective functions, $\boldsymbol{\Phi}$ is the design variable vector, $g_j(\cdot)$ and $h_k(\cdot)$ are boundary conditions, $\boldsymbol{\Phi}^{(L)}$ and $\boldsymbol{\Phi}^{(U)}$ are the lower and upper bounds of $\boldsymbol{\Phi}$, and N_{obj} represents the total number of objectives.

If all the objectives $f_i(\boldsymbol{\Phi})$, $i = 1, 2, \dots, N_{\text{obj}}$ are linear, the problem is called Multi-objective Linear Programming (MOLP); if any of the objective function is nonlinear, it is a nonlinear multi-objective optimization. We call the space of design variable $\boldsymbol{\Phi}$ the design variable space Φ , and the space of objectives $f_i(\boldsymbol{\Phi})$ the Pareto space Ω (also called solution space). If both spaces are convex, the problem is a convex MOO problem.

2.3.1 Pareto Optimality

In a typical MOO problem, the objectives are conflicting, i.e., no single solution can be optimal for all the objectives. Therefore, multiple optimal solutions exist, representing different trade-offs between the objectives. Such trade-offs in the objectives result in Pareto optimality, which is also called Pareto efficiency, is named after an engineer and economist, Vilfredo Pareto, who adopted this concept in his studies. Compared to a single objective optimization, in an MOO problem the optimal solutions lie on a frontier called the Pareto Front. Moving towards the Pareto Front is a process of optimization (literally called Pareto Improvement) and moving along the front will optimize some objective functions but at the expense of at least one of the other objective functions. A graphical illustration of the Pareto optimality is shown in Fig.

2-10. Suppose we have two objectives to be minimized f_1 and f_2 . All the points on the Pareto front are solutions to this MOO problem. If we move from A to B , we decrease (optimize) f_2 at the expense of increasing f_1 , and vice versa. We say A and B are not dominated by each other, and such moves are not called Pareto improvements. In contrast, if we move from C to B , both f_1 and f_2 become smaller, so we say such a move is a Pareto improvement. The target of MOO is to find such a Pareto front that tells us what trade-offs we can make.

Figure 2-10: An illustration of Pareto front (black curve) and Pareto improvement.

2.3.2 Multi-objective Optimization Algorithms

There are various methods for solving the MOO problems defined in Eq. 2.53. A straight forward method is the weighted-sum method, which consider a single objective equals to the weighted sum of the multiple objectives:

$$f_w = \sum_{i=1}^{N_{obj}} w_i f_i(\Phi), \quad i = 1, 2, \dots, N_{obj}, \quad (2.54)$$

where w_i is the weight corresponding to the i -th objective function. f_w is the converted single objective function. The MOO problem in Eq. 2.53 is thus converted into a single objective

optimization

$$\begin{aligned}
 & \text{minimize} && f_w = \sum_{i=1}^{N_{\text{obj}}} w_i f_i(\boldsymbol{\Phi}), && i = 1, 2, \dots, N_{\text{obj}}; \\
 & \text{subjected to} && g_j(\boldsymbol{\Phi}) \leq 0, && j = 1, 2, \dots; \\
 & && h_k(\boldsymbol{\Phi}) = 0; && k = 1, 2, \dots; \\
 & && \boldsymbol{\Phi} \in [\boldsymbol{\Phi}^{(L)}, \boldsymbol{\Phi}^{(U)}],
 \end{aligned} \tag{2.55}$$

The benefit of weighted-sum method is that it can significantly reduce the number of function evaluations. However, it requires a predefined weight w_i to each of the objectives, which is non-trivial in many cases. Eq. 2.55 shows only the linear combinations. In realistic applications, other combinations exist, such as quadratic, cubic, etc. The choosing of weights requires knowledge and experiences on the system performance, which limits the application of this method. Another limitation to the weighted-sum method is that if the MOO problem is nonconvex, the method is not capable of finding all Pareto-optimal solutions [55].

Multi-objective Evolutionary Algorithms (MOEAs) have been one of the most popular approaches in solving MOO problems. MOEAs have fewer limitations on the characteristics of the objective functions as no gradient information is required and scalarization is not compulsory. The NSGA-II algorithm proposed by Deb et al. [56] may be one of the most popular MOEA. NSGA-II algorithm is a generic algorithm (GA) which simulates the population evolution procedure. It is based on fast non-dominated sorting and elitist-preserving. The general flowchart of the NSGA-II algorithm is shown in Fig. 2-11.

Most criticisms on NSGA-II focus on two aspects: first, it does not guarantee the solutions to be global optimal; second, it requires a large number of evaluations. The time complexity of the NSGA-II is $\Omega(MN^2)$, where M is the total number of objectives, N represents the size of population. Although NSGA-II is often criticized by the above two points, it is still one of the most commonly used algorithms in multi-objective optimization. In this thesis, we will adopt NSGA-II algorithm for resilience optimization and alleviate the computational burden by replacing the evaluation of physical model by surrogate model.

Figure 2-11: The simplified flow chart of the NSGA-II algorithm, $\phi^{(k)}$ represents the k -th design of design variable ϕ .

2.3.3 Multi-objective Optimization with Surrogate Model

The NSGA-II (and similarly other MOO algorithms) requires a huge number of model evaluations. This computational burden is non-trivial for most problems. The use of a surrogate model is a possible solution to address the computational problem. Surrogate model is a technique that predicts the outputs of more complex physical model [57], by establishing a mapping from the inputs to the outputs [58], and it is usually computationally much more ‘cheaper’ to evaluate compared with the physical model. Surrogate model is also referred as response surfaces, meta-model, emulators etc. [59]. To construct a surrogate model, we need to prepare the support points. A support point is a pair of input and output of the physical model. Fig. 2-12 is a typical process of building a surrogate model.

In optimization problems, a surrogate model has been commonly adopted to avoid repeatedly running the computationally-expensive physical model. Han et al. [60] adopted the surrogate model to replace the computationally expensive-to-evaluate physical model in the optimization

Figure 2-12: An illustration of the process of constructing a surrogate model.

of the airfoil. Gong et al. [61] introduced a Multi-Objective Adaptive Surrogate Modelling-Based Optimization (MO-ASMO) algorithm and test the proposed algorithm on benchmark problems. The MO-ASMO methodology incorporates the NSGA-II algorithm and the surrogate model (Gaussian Process). A 90% to 95% of computing time saving was gained compared with pure NSGA-II optimization in which the evaluation of the function is calculated by simulating the original physical model. However, all the benchmark problems in [61] are of low dimension and the sampling space is limited.

2.3.4 Adaptive Surrogate Model for Extremely-Low Exceedance Probability

When rare events are of interest, the tail parts of the distributions of the design and random variables play an important role. If a surrogate model is built with random sampling, the accuracy in the rare tail parts is not guaranteed. This problem is commonly seen in reliability assessment. To overcome this problem, advanced stochastic sampling methodologies (for example, the Subset Simulation) can be integrated with the surrogate model. Z. Shao [62] proposed an adaptive Subset Simulation-Moving Least Square (SS-MLS) algorithm and the adaptive Subset Simulation-Gaussian Process (SS-GP) algorithm for the reliability assessment of structures subjected to tsunamis. Efficiency improvement was shown compared with the Subset Simulation without surrogate model. Keeping the same level of accuracy of the estimators, in their example, Subset Simulation takes 1500 evaluations of the physical model, SS-MLS takes 581, and SS-GS takes 666 evaluations. H. Jensen et al. [63] proposed an adaptive scheme for

constructing a surrogate model for reliability-based global design optimization problem with the assistance of Transitional Markov Chain Monte Carlo (TMCMC) [64] algorithm and convex hull. The target of reliability-based global design optimization is to optimize some failure probability function. A typical mathematical representation of the problem they are targeting at is:

$$\begin{aligned} & \text{minimize} && P_F(\boldsymbol{\Phi}), \\ & \text{subjected to} && \boldsymbol{\Phi} \in [\boldsymbol{\Phi}^{(L)}, \boldsymbol{\Phi}^{(U)}], \end{aligned} \tag{2.56}$$

where $\boldsymbol{\Phi}$ represents design variables, $P_F(\boldsymbol{\Phi})$ is the failure probability.

The method by H. Jensen et al. [63] aims at one objective probability function. The method is also limited by the usage of convex hull, where a convex hull is the convex envelop contains the design variables in Euclidean space. When the design variable space is of medium-to-high dimensional, the efficiency of the method drops.

2.3.5 Conclusive Remarks

In this section, multi-objective optimization problems are introduced. We also reviewed two of the most commonly used MOO algorithms. We show that NSGA -II algorithm has much fewer limitations than weighted-sum method. However, NSGA -II usually requires very large number of evaluation of the physical model. To alleviate the heavy computational burden, researchers replace the physical model with surrogate model, which usually computes very fast. In some cases, we may want to optimize the exceedance probability that this system performance $g(\boldsymbol{\Phi})$ is larger than some threshold b , i.e., $P(g(\boldsymbol{\Phi}) > b) = \int I(g(\boldsymbol{\Phi}) > b) d\boldsymbol{\Phi}$ is taken as the objective, where $I(g(\boldsymbol{\Phi}) > b)$ is some indicator function same to Eq. 2.38. The problem in Eq. 2.56 belongs to this type. However, it is only a single objective optimization problem. If we have more than one probability integrals as the objectives in an MOO problem, it become even more difficult. The difficulties are twofold: (1) the MOO algorithm need to evaluate very large number designs before convergence; (2) given one design parameter $\boldsymbol{\Phi}$, the target function is a probability integral, which is non-trivial to evaluate, especially when we consider the low-probability region (i.e. $P(g(\boldsymbol{\Phi}) > b) \ll 1$). In Chapter 5, we will show how to build a special surrogate model for the multi-objective optimization with multiple probability integrals as the resilience objectives. And we will illustrate the multi-objective resilience optimization

with a realistic example.

2.4 Bayesian Model Updating

A system will not always operates as designed due to the following reasons: (1) the built system may be biased from the designed one (2) there are uncertainties in the performance of system components (3) the system performance may deteriorate due to aging or partially broken by hazards. Our “a priori” information on the system is usually inadequate, and our knowledge on the system thus need to be updated. Bayesian model updating provides such an approach to update the system’s “a posteriori” information.

Bayesian updating is a widely used and robust method for model calibration and model selection. Let’s denote the collected data as $D : \{\mathbf{x}\}$, where D represents the whole data set and $\mathbf{x}$ is the data. The model is represented by $\mathcal{M} : g(\boldsymbol{\theta})$, where $\mathcal{M}$ represents the model assumptions, g is some performance function, and $\boldsymbol{\theta}$ are the variables to be updated. Each value of $\boldsymbol{\theta}$ represents a possible model. The variable $\boldsymbol{\theta}$ is assumed to follow some prior distribution specified by $\mathcal{M}$: $\boldsymbol{\theta} \sim p(\boldsymbol{\theta}|\mathcal{M})$, it represents the plausibility of each model. We want to know the updated distribution of $\boldsymbol{\theta}$ with the data D , i.e., $p(\boldsymbol{\theta}|D)$. From Bayes’ theorem: [65]

$$p(\boldsymbol{\theta}|D, \mathcal{M}) = \frac{p(D|\boldsymbol{\theta}, \mathcal{M})p(\boldsymbol{\theta}|\mathcal{M})}{p(D|\mathcal{M})}, \quad (2.57)$$

where

$$p(D|\mathcal{M}) = \int_{\boldsymbol{\theta}} p(D|\boldsymbol{\theta}, \mathcal{M})p(\boldsymbol{\theta}|\mathcal{M})d\boldsymbol{\theta}, \quad (2.58)$$

is the normalizing constant.

Eq. 2.57 can be very useful in resilience analysis. Let’s consider the question represented by Eq. 2.31, (the question is repeated as: “what is the probability that output of the system performance function $g(\boldsymbol{\theta})$ exceeds some threshold b ?”). If we explicitly show the assumptions $\mathcal{M}$, Eq. 2.31 can be written as:

$$P(g(\boldsymbol{\theta}|\mathcal{M}) > b) = \int g(\boldsymbol{\theta})p(\boldsymbol{\theta}|\mathcal{M})d(\boldsymbol{\theta}). \quad (2.59)$$

With Eq. 2.57, we can make use of the measured data D and estimate the following updated probability:

$$P(g(\boldsymbol{\theta}|D, \mathcal{M}) > b) = \int g(\boldsymbol{\theta})p(\boldsymbol{\theta}|D, \mathcal{M})d(\boldsymbol{\theta}). \quad (2.60)$$

Eq. 2.60 provides a robust exceedance probability [66]. The robustness comes from the incorporation of measured data D .

Except for the difficulties in calculating the integral in Eq. 2.59 and 2.60 (as we have discussed in Section 2.2.1), the updated (“posterior”) distribution $p(\boldsymbol{\theta}|D, \mathcal{M})$ in Eq. 2.57 is non-trivial to obtain. A direct application of the Acceptance-Rejection Method as described in Section 2.2.2 can be intractable as the posterior PDF is usually sharply peaked, which takes a very small region in the space of variable $\boldsymbol{\theta}$ (a visualization is given in Fig. 6-1 in Section 6.3.5). This will result in an extremely low acceptance rate.

To solve this problem, J. L. Beck and L. S. Katafygiotis [67, 68] proposed Laplace’s method of Asymptotic Approximation [69] for updating a structural model with uncertainties. Their method is very efficient it but has two major limitations [70]: first, the dataset D should be large enough for an accurate approximation of the posterior distribution; second, the number of parameters to be updated should be relatively small, otherwise, the updated distributions of some model parameters may not be peaked at their optimal values, this is the so-called unidentifiable problem.

Beck and Au [65] proposed a Bayesian updating method that can be applied to both model updating and reliability updating using MCMC. Instead of finding the posterior PDF $p(\boldsymbol{\theta}|D)$ directly, Beck and Au sought to find a series of PDFs $p_1, p_2, \dots, p_m = p(\boldsymbol{\theta}|D, \mathcal{M})$ to iteratively approach the updated PDF. Metropolis-Hastings Sampling (MHS) is adopted in their method. With the help of these intermediate PDFs p_i , where $i = 1, 2, \dots, p_{m-1}$ ($p_0 = p(\boldsymbol{\theta}|\mathcal{M})$ is the prior PDF), the problem is divided into multiple steps, and by each step the PDF moves from p_i to p_{i+1} . If the intermediate PDFs are well chosen, the change from p_i to p_{i+1} would be small, thus the acceptance rate in MHS can be guaranteed. However, the method highly depends on well-constructed intermediate PDFs.

Ching et al. [64] proposed the TMCMC method for Bayesian model updating. Similar to the spirit of Beck and Au [65], TMCMC method also sought to find the posterior distribution

with the help of a series of intermediate PDFs $p(D|\boldsymbol{\theta}, \mathcal{M}_j)^{\beta_j}$:

$$p_j(\boldsymbol{\theta}) \propto p(D|\boldsymbol{\theta}, \mathcal{M}_j)^{\beta_j} p(\boldsymbol{\theta}, \mathcal{M}_j), \quad (2.61)$$

where $0 = \beta_0 < \beta_1 < \dots < \beta_j < \dots < \beta_m = 1$ is a series of increasing parameters. With this method, when the number of uncertain model parameters increases, the performance of the method might deteriorate.

Bansal and Cheung [71] integrated the Gibbs sampler [72] they proposed for Bayesian structural dynamic model updating and the Subset Simulation for evaluating the failure probability curves for linear dynamic systems and nonlinear dynamic systems, respectively, to propose approaches for conditional sampling referred to in their papers as ‘Constrained Metropolis within Gibbs sampling’ algorithm and ‘Constrained multilevel Metropolis within Gibbs sampling’ algorithm, respectively. Though the approaches are robust to the number of uncertain parameters, random variables, and the amount of dynamic response data, the approaches are only applicable to cases where the response data are measured when the structural dynamic systems are in the linear response regime.

D. Straub and I. Papaioannou [73] proposed the Bayesian updating with structural reliability methods (BUS). In BUS method the likelihood function is converted to an artificial limit state function by introducing an auxiliary uniformly distributed random variable u . The failure domain corresponds to the limit state function F_D is given by (for simplicity, we leave the model assumption $\mathcal{M}$ implicit):

$$F_D = \{u \leq cL(\boldsymbol{\theta}|D)\}, \quad (2.62)$$

where $L(\boldsymbol{\theta}|D) = p(D|\boldsymbol{\theta}, \mathcal{M})$ is the likelihood function, c is some multiplier constant not pre-known. The value of c is to meet the following requirement to ensure the updated PDF is unbiased:

$$c \leq \frac{1}{L_{\max}}, \quad (2.63)$$

where is the maximum of likelihood function. It is safe to choose very small c such that Eq. 2.63 is guaranteed, but the smaller c value is, the lower efficiency is expected. From Eq. 2.63, the ideal value of c would be $c_{\max} = \frac{1}{L_{\max}}$. However, it is non-trivial to get the $L_{\max}$ value because the likelihood is usually a very peaked distribution, which takes a very rare part of the

prior.

DiazDelaO et al. [53] proposed an alternative formulation of BUS (referred to reformulated BUS in the following context) by taking log on both sides of Eq. 2.62 and move the term with c to the right-hand side:

$$F_D = \{\ln \frac{L(\boldsymbol{\theta}|D)}{u} > -\ln c\}, \quad (2.64)$$

this is in the form of Eq. 2.31 by taking $Y = \ln \frac{L(\boldsymbol{\theta}|D)}{u}$ as the driving variable. In the reformulated BUS method, the unknown multiplier c is not in the driving variable, Subset Simulation is thus applicable to get the updated (posterior) PDF. The thresholds of Subset Simulation will approaching $c_{\max}$ iteratively. DiazDelaO et al. [53] also proposed a stopping criteria to judge when the $c_{\max}$ is reached. The main problem of the reformulated BUS method is that when the posterior PDF is very peak, it requires too many levels of Subset Simulation to converge to the posterior PDF. We will further show this point and propose our efficient Bayesian updating method in Chapter 6.

2.5 Conclusions

In this chapter, I reviewed four research topics, resilience assessment methods, efficient stochastic simulation methods, MOO methods, and Bayesian model updating. A quick summarizing is listed as follows.

1. State-of-the-art resilience assessment methods are reviewed. Most of the current resilience quantification methods represent resilience by a single resilience index by either representing the system resilience with one comprehensive metric, or by combining multiple resilience indices into one. We argue that a single resilience index is inadequate as the same resilience index may correspond to multiple system behaviours. We also reviewed resilience assessment methods considering uncertainties. Most literature represent system resilience under uncertain scenarios by an expectation value. We argue that this expectation-based approach suffers from the similar problem with single-indices-based methods: the same expectation value (as measurement of resilience) can correspond to multiple probability distributions. More importantly, using the expectation-based approach, we are likely to miss the high-impact, low-probability events, as those rare events

can easily be overwhelmed by the more frequent ones. We identify the research gap as follows: there is few resilience quantification method that is (1) multiple-indices based (2) capable of considering both the uncertainty of disruptive events and uncertainties of the system performance (3) capable of considering a whole range of possible disruptive events. We will address this research gap in Chapter 3.

2. Stochastic simulation methods are reviewed. We started from the brute-force Monte Carlo Simulation and discussed its efficiency problem in simulation rare events. We reviewed Subset Simulation and its developments. We reviewed the ISS method and introduced how it estimates multiple exceedance probabilities in a single run. We also pointed out the discontinuity problem of the ISS method. In Chapter 4, we will improve the ISS method by solving its discontinuity problem and apply the efficient improved ISS method to resilience assessment.
3. Multi-objective optimization problems and some most popular MOO algorithms are reviewed. We also reviewed the surrogate model assisted MOO. We identify a computationally difficult problem: multi-objective optimization with probability integrals as the objectives. In Chapter 5, we will address this problem and solve the computational difficulties by constructing a special surrogate model.
4. Bayesian model updating methods are reviewed. We illustrate that Bayesian model updating is very useful in resilience analysis as it provides a robust model if the system response data is available. We reviewed several Bayesian model updating methods and pointed out that in some cases, the efficiency of existing Bayesian model updating methods are not satisfying. In Chapter 6, we will propose an efficient Bayesian updating method that updates the system status and the multiple CCDF curves in one run.

Chapter 3

Proposed Multi-CCDF based Stochastic Resilience Assessment Method

3.1 Introduction

In this Chapter, a realistic stochastic resilience quantification method is proposed. The proposed method quantifies the system resilience using multiple physically meaningful metrics and each metric is a complimentary cumulative distribution function (CCDF) of the system performance. In addition, the proposed method enables the assessment of system resilience under a broad spectrum of events with different degrees of rarity or extremeness. In Section 3.2, we present the proposed stochastic realistic resilience quantification method. Then in Section 3.3, we illustrate the proposed method using two examples. Example 1 is a hypotheticalal model. In example 2, an urban drainage system (UDS) stressed by heavy rainfalls with return period from 10 years to 1000 years is considered. Return period here is referred as the reciprocal of average frequency of occurrence of an event. Different resilience enhancement strategies are compared using the proposed resilience assessment method.

3.2 The Proposed Stochastic Resilience Quantification Method

We propose to measure resilience using multiple metrics while considering the stochastic nature of the system in an uncertain environment. We consider the cases that the system is subjected to a single uncertain event at a time. i.e., the system can fully recover before the next event comes. For cases with continuous or multiple successive events are to be considered in our future work.

Random variables are introduced to the stochastic modelling. The randomness comes from two sources: (1) the uncertain future events, in terms of both the happening probability and the uncertain intensity; (2) the uncertainty in system performance. In this thesis, a vector of random variables is represented by θ , whose feasible space is Θ . Due to the stochastic nature of the environment and the system, multiple realizations of θ have to be considered and the performance curve is no longer a single deterministic curve but a bunch of curves with each curve corresponding to a realization of the random variables. An illustrative example of the system's performance curves under uncertainties are shown in Fig. 3-1, where the subscript i in the performance related quantities $T_{d,i}$, $T_{r,i}$, DT_i , and g_i , etc. corresponds to the i th performance curve obtained considering the i th realization of the random variables.

Figure 3-1: An illustration of the system performance curves considering stochastic features.

In our proposed method, the system resilience is measured by five components. Three of them are time related, the fourth one is the worst performance related and the last one is an overall description of the system performance loss. Referring to Fig. 3-1, the robust range

is defined first. Inspired by Tang et al. [25], we define the robust range as an allowance of fluctuations of the system performance, within which we suppose it is acceptable. It is bounded by an upper limit $g_0 + \epsilon_g$ and a lower limit $g_0 - \epsilon_g$. The existence of robust range eliminates the resilience metrics being affected by the random drop of performance due to the stochastic nature of the system. In addition, for some systems, very tiny drops of performance may cause little loss. The existence of the robust range makes the resilience analysis insensitive to these minor system performance drops. It is worth mentioning that the width ϵ_g of the robust range is to be decided by the decision makers. If the width ϵ_g is set to be zero, there is no robust range, which will not affect the effectiveness of the proposed method. The resilience components are described as follows:

1. deterioration time interval, T_d

T_d is the duration from the time t_1 that the system performance drops out of the robust range to the time instant $t_{\min}$ that the minimum performance is observed, $T_d = t_{\min} - t_1$;

2. recovering time interval, T_r

T_r is the duration from the minimum performance to the time t_2 when the system recovers back to the robust range. $T_r = t_2 - t_{\min}$;

3. down time, DT

DT is defined as the duration from the time that the performance drops out of the robust range to the time instant that the performance recovers back to the robust range. $DT = t_2 - t_1$;

4. maximum performance drop $\max\Delta g$

$\max\Delta g$ reflects the worst performance of the system during a disruptive event. It is defined to be the difference between the normal performance level and the minimum performance level. $\max\Delta g = g_0 - g_{\min}$;

5. Loss of resilience, LoR

LoR represents the overall system performance loss. Following the similar definition as in [19, 74], here LoR is defined as the area enclosed by the normal (ideal) performance line

and the actual performance curve. As the resilience curves are nonlinear in most cases, this LoR metric cannot be eliminated even we have the maximum performance drop and the downtime information.

We avoid using the recovery speed or recovery rate, or the slope of the recovery phase as a resilience component. The reason is as follows: For different degrees of damage, the recovery speed is not a good measure the recoverability of the system. The recovery speed comparison is meaningful only when the degrees of damages are of the same level. In the resilience components considered in this chapter, the recovery is decoupled into two dimensions: the recovery time and maximum performance drop.

The system performance g is a function of system design variables $\boldsymbol{\phi}$, random variables $\boldsymbol{\theta}$ and time t . The design variables $\boldsymbol{\phi}$ include both system parameters and resilience enhancement strategies. With reference to Fig. 3-1, let us denote:

1. $t_1(\boldsymbol{\phi}, \boldsymbol{\theta}, \epsilon_g)$ as the time instant that the system performance $g(\boldsymbol{\phi}, \boldsymbol{\theta}, t)$ drops out of the robust range for the first time. It is a function of variables $\boldsymbol{\phi}, \boldsymbol{\theta}$ and a pre-defined half-width of the robust range ϵ_g ;
2. Similarly, let us denote $t_2(\boldsymbol{\phi}, \boldsymbol{\theta}, \epsilon_g)$ as the time instant that the system performance $g(\boldsymbol{\phi}, \boldsymbol{\theta}, t)$ finally returns back to the robust range;
3. Let $t_{\min}(\boldsymbol{\phi}, \boldsymbol{\theta})$ denote the time instant that the system performance reaches its global minimum. i.e.

$$t_{\min} = \arg \min_t g(\boldsymbol{\phi}, \boldsymbol{\theta}, t). \quad (3.1)$$

3.2.1 The Proposed Stochastic Resilience Assessment Metrics

The proposed resilience metrics are expressed as the probability that each resilience component exceeds some threshold. To consider the uncertainty, each resilience metric is expressed as the probability that the corresponding resilience component (T_d , T_r , DT , $\max \Delta_g$, LoR) exceeds

some threshold, in which the threshold is treated as a variable. For simplicity, Let's use $M_j, j = 1, 2, \dots, 5$ to represent the resilience components: $M_1 = T_d$, $M_2 = T_r$, $M_3 = DT$, $M_4 = \max \Delta_g$, $M_5 = LoR$, and the corresponding thresholds are b_1, b_2, b_3, b_4, b_5 , respectively. The resilience metric corresponding to M_i is expressed as a function of the threshold variable $b_j, j = 1, 2, \dots, 5$:

$$R_{M_j}(b_j) = P(M_j > b_j). \quad (3.2)$$

Eq. 3.2 can be written as a probability integral of the system performance related function as

$$\begin{aligned} R_{M_j}(b_j) &= P(M_j > b_j) \\ &= \int_{\Theta} I_j(\Phi, \Theta, b_j) p(\Theta) d\Theta. \end{aligned} \quad (3.3)$$

where I_j is an indicator function given as follows:

For $j = 1$, the indicator function I_1 corresponding to T_d is

$$I_1(\Phi, \Theta, b_1) = \begin{cases} 1, & \text{if } t_{\min}(\Phi, \Theta) - t_1(\Phi, \Theta, \epsilon_g) > b_1; \\ 0, & \text{otherwise.} \end{cases} \quad (3.4)$$

For $j = 2$, the indicator function I_2 corresponding to T_r is

$$I_2(\Phi, \Theta, b_2) = \begin{cases} 1, & \text{if } t_2(\Phi, \Theta, \epsilon_g) - t_{\min}(\Phi, \Theta) > b_2; \\ 0, & \text{otherwise.} \end{cases} \quad (3.5)$$

For $j = 3$, the indicator function I_3 corresponding to DT is

$$I_3(\Phi, \Theta, b_3) = \begin{cases} 1, & \text{if } t_2(\Phi, \Theta, \epsilon_g) - t_1(\Phi, \Theta, \epsilon_g) > b_3; \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

For $j = 4$, the indicator function I_4 corresponding to $\max \Delta_g$ is

$$I_4(\Phi, \Theta, b_4) = \begin{cases} 1, & \text{if } g_0(\Phi) - \min g(\Phi, \Theta, t) > b_4; \\ 0, & \text{otherwise.} \end{cases} \quad (3.7)$$

For $j = 5$, the indicator function I_5 corresponding to LoR is

$$I_5(\boldsymbol{\Phi}, \boldsymbol{\Theta}, b_5) = \begin{cases} 1, & \text{if } \int_{\tau_1(\boldsymbol{\Phi}, \boldsymbol{\Theta})}^{\tau_2(\boldsymbol{\Phi}, \boldsymbol{\Theta})} [g_0(\boldsymbol{\Phi}) - g(\boldsymbol{\Phi}, \boldsymbol{\Theta}, t)] dt > b_5; \\ 0, & \text{otherwise.} \end{cases} \quad (3.8)$$

where $\tau_1(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ and $\tau_2(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ are the time instants the system performance $g(\boldsymbol{\Phi}, \boldsymbol{\Theta}, t)$ starts to drop and goes back to the normal performance level g_0 , respectively. A summary of the proposed resilience indices is given as in Table 3-1.

Table 3-1: A summary of the proposed stochastic-based resilience assessment method, where $t_1(\boldsymbol{\Phi}, \boldsymbol{\Theta}, \epsilon_g)$, $t_2(\boldsymbol{\Phi}, \boldsymbol{\Theta}, \epsilon_g)$, $t_{min}(\boldsymbol{\Phi}, \boldsymbol{\Theta})$ and $g(\boldsymbol{\Phi}, \boldsymbol{\Theta}, t)$ are system performance related functions, b_1, b_2, b_3, b_4, b_5 are moving thresholds.

Resilience component	expression	condition of indicator = 1
Time to bottom	$R_{Td}(b_1) = \int_{\boldsymbol{\Theta}} I_1(\boldsymbol{\Phi}, \boldsymbol{\Theta}, b_1) p(\boldsymbol{\Theta}) d(\boldsymbol{\Theta})$	$t_{min}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) - t_1(\boldsymbol{\Phi}, \boldsymbol{\Theta}, \epsilon_g) > b_1$
Time recovery	$R_{Tr}(b_2) = \int_{\boldsymbol{\Theta}} I_2(\boldsymbol{\Phi}, \boldsymbol{\Theta}) p(\boldsymbol{\Theta}) d(\boldsymbol{\Theta})$	$t_2(\boldsymbol{\Phi}, \boldsymbol{\Theta}, \epsilon_g) - t_{min}(\boldsymbol{\Phi}, \boldsymbol{\Theta}) > b_2$
Down time	$R_{DT}(b_3) = \int_{\boldsymbol{\Theta}} I_3(\boldsymbol{\Phi}, \boldsymbol{\Theta}, b_3) p(\boldsymbol{\Theta}) d(\boldsymbol{\Theta})$	$t_2(\boldsymbol{\Phi}, \boldsymbol{\Theta}, \epsilon_g) - t_1(\boldsymbol{\Phi}, \boldsymbol{\Theta}, \epsilon_g) > b_3$
Max performance drop	$R_{\max \Delta g}(b_4) = \int_{\boldsymbol{\Theta}} I_4(\boldsymbol{\Phi}, \boldsymbol{\Theta}, b_4) p(\boldsymbol{\Theta}) d(\boldsymbol{\Theta})$	$g_0(\boldsymbol{\Phi}) - \min g(\boldsymbol{\Phi}, \boldsymbol{\Theta}, t) > b_4$
Loss of Resilience	$R_{LoR}(b_5) = \int_{\boldsymbol{\Theta}} I_5(\boldsymbol{\Phi}, \boldsymbol{\Theta}, b_5) p(\boldsymbol{\Theta}) d(\boldsymbol{\Theta})$	$\int_{\tau_1(\boldsymbol{\Phi}, \boldsymbol{\Theta})}^{\tau_2(\boldsymbol{\Phi}, \boldsymbol{\Theta})} [g_0(\boldsymbol{\Phi}) - g(\boldsymbol{\Phi}, \boldsymbol{\Theta}, t)] dt > b_5$

3.2.2 Discussion on the Proposed Resilience Metrics and the Corresponding Physical Meanings

Among the above proposed resilience metrics, the first three are time related. The down time is the overall duration that the system is interrupted by the disruptive event. The time to bottom (R_{T_d}) describes how soon the system performance drops to the minimum. However, we avoid judging the resilience performance by this R_{T_d} because in different systems, the decision makers may prefer this deterioration period to be longer or shorter. For example, if the performance drops too fast, it leaves little time for people to react. A representative example is the evacuation in an earthquake. If the performance keep dropping for too long time, it is psychologically not acceptable. R_{T_d} is not a reflection of the performance drop rate. Instead, it describes the percentage of time among the total downtime. To consider the performance drop rate, the maximum performance drop $\max \Delta_g$ has to be considered together. R_{T_r} tells us how soon the system recovers from the minimum performance to an acceptable normal state. Usually we prefer such a time to be as short as possible. R_{DT} tells us how long the disruption lasts. It is worth mentioning that $DT = T_d + T_r$ holds for a single performance curve only. If we consider a whole bunch of system performance curves, we cannot infer R_{DT} from R_{T_d} and R_{T_r} , as all of them are probability integrals. Therefore, we need to keep all of these three time-related resilience metrics instead two of them.

The maximum performance drop $\max \Delta_g$ reflects the worst performance of the system during the event. Based on this decision makers can decide whether enhancements should be placed. The last measurement is the overall loss of resilience (LoR). It is a combination of the performance and the time dimensions. The physical meaning of this LoR depends on the system being studied. We will show its physical meaning in the realistic example. In addition to the explicit physical meanings, the proposed method has two more merits: (1) the problem that the same value of the resilience metric corresponding to multiple system behaviors is alleviated to a significant extent. (2) the proposed method explores the system's resilience performance under a spectrum of events including rare or extreme events and avoids solely looking at the mean value. These are achieved by expressing the system resilience as the exceedance probabilities which are functions of variable thresholds, and the resilience assessment results are shown as multiple CCDF curves.

The proposed resilience assessment method aims to give decision makers or system managers a “bird’s-eye view” of the key system performance characteristics when the system is subjected to uncertain future events. It is worth mentioning that the proposed resilience assessment method can be easily expanded to include more metrics, for example, the post-event adaptability.

3.3 Illustrative Examples

3.3.1 Generic Hypothetical Systems

For illustrative purpose, let us consider a hypothetical system. This hypothetical system is adapted from the work by Cassottana et al. [75], where they propose a family of recovery functions which are analytically available to model the performance of critical infrastructure under disruptive events. Examples show that the analytical function fits realistic system performance well. The system performance is a function of time t and can be represented by the gamma family of recovery functions with tunable parameters [75]

$$g(t) = g_0 - a\{\Gamma[d+1] - \Gamma[d + f(t|b, c)]\}, \quad a, b, c, d \geq 0, \quad (3.9)$$

where $f(t|b, c)$ is a monotonic non-increasing function. Here we choose $f(t|b, c)$ to be a stretched exponential function:

$$f(t|b, c) = \exp(-ct^b) \quad (3.10)$$

We simulate two system configurations as summarized in Table 3-2. Among the four tunable parameters, a controls the maximum drop of performance level, the larger a will result in larger performance drop; b and c are the shape and scale parameter, respectively. These two parameters control the time to recovery. d is related to the level of the post-event enhancement. As we are not considering the post-event enhancement effect, $d_{\max} = 1$ is chosen for both configurations.

The system performances as functions of time are shown in Fig. 3-2. This is a hypothetical system, in which the system performance is given by mathematical expressions, so the computational burden is light. Therefore, brute-force Monte Carlo simulation is adopted. A total

Table 3-2: Parameters of the two configurations.

parameter	config-1		config-2	
	distribution	range	distribution	range
a	Truncated lognormal ($\mu = 1.5, \sigma = 0.3$)	[2, 8]	Truncated lognormal ($\mu = 0.6, \sigma = 0.4$)	[2, 5.5]
b	Uniform	[0.73, 0.93]	Uniform	[0.57, 0.77]
c	Uniform	[0.25, 0.45]	Uniform	[0.31, 0.41]
d	Deterministic	1	Deterministic	1

number of 50000 independent performance curves are generated for each system, and 200 out of the 50000 curves are picked randomly and plotted in Fig. 3-2.

Figure 3-2: The performance curves of two hypothetical systems, 200 simulations out of 50000 are randomly selected and plotted

Resilience analysis results from our proposed stochastic resilience assessment method are shown in Fig. 3-3. All the resilience measurements are expressed as CCDF curves. The benefit of using a CCDF curve is that the system resilience can be explicitly seen at all simulated probability levels. For example, comparing the last column in Fig. 3-3, when LoR is smaller than 3 (correspond to the exceeding probability from 2×10^{-2} to 1), the CCDF curve of config-1 is almost coincided with config-2. However, when looking at the lower probability part, it can be seen that config-2 can have higher probability for larger LoR .

The mean LoR value of config-1 is 0.1526 and the mean LoR of config-2 is 0.1509. The gap

between these two values is quite small. If the decision maker judges the resilience with the mean *LoR* only, config-2 may be preferred. This is contradict with our judgement based on the rare tail parts on the CCDF curve.

Although the *LoR* of the two configurations are similar, the other system performances are quite different. From Fig. 3-3, column 3 and column 4, it can be seen that config-1 has shorter downtime but larger maximum performance drop than that of config-2, which is consistent with Fig. 3-2.

The 1st and 2nd columns of Fig. 3-3 provide details of the “skewness” of the performance curves. These two curves are to be considered together with the downtime. In this example, the “skewness” of the performance curves of both configurations are similar, the time going down is smaller than the time to recover, indicating that the system performance curves are “left-skewed” within the down time period.

Figure 3-3: Resilience analysis of the two hypothetical systems. The first row shows the CCDF curves, and the second row shows the corresponding c.o.v.

3.3.2 Urban Drainage System (UDS) Subjected to Heavy Rainfall Events

In this second example, urban drainage systems subjected to heavy rainfall events are studied. Unlike an earthquake or a tsunami, a rainfall event stresses the system for a duration of time. The urban drainage system is modelled as a one-dimensional drainage network. This example is adapted from [76]. The location of the study area is located in the city Porto Alegre, Brazil. The total area being studied is around 6.88 km². The performance of the drainage network is simulated using a physics-based model called Storm Water Management Model (SWMM) [77] by the U.S. Environmental Protection Agency (EPA). In the context hereafter, EPA SWMM or SWMM for short, refers to the SWMM package by the EPA. Fig. 3-5 is a screen shot of the SWMM model being studied. The network consists of 23 junction nodes, 22 conduits, 17 subcatchments, and 1 outlet. N2, N3, N4 in Fig. 3-5 are three of the 23 junction nodes. In the following context, junction nodes are also referred as nodes for simplicity. Conduits are pipelines or channels that drains the water. Subcatchments are land surface area which receive the precipitations. The outlet is the special node where the water leaves the drainage system.

This area is subjected to heavy rainfalls. The hyetographs are generated by the alternating block method [78] based on the Intensity-Duration-Frequency (IDF) curve. The IDF relationship adopted in this example is given as

$$Intensity = 826.8 \frac{RP^{0.143}}{(T + 13.3)^{0.79}}, \quad (3.11)$$

where T is the rainfall duration, RP is the return period. The *Intensity* has a unit of mm/min if duration is in minutes. Return period here is referred as the reciprocal of average frequency of occurrence of an event. For example, an event with $RP = 10$ represents that an event of this level averagely happens once every ten years. In this study, return periods ranging from 10 years to 1000 years are considered. This is because the drainage system of this area is capable of draining the water effectively in precipitation events with return period less or equal than 10 years. The 1000 years return period events are so rare that it is a conservative upper bound. The rainfall duration is assumed to be uniformly distributed with a minimum duration of 30 minutes and maximum duration of 180 minutes. The return period and the duration are the random variables considered in this problem, i.e., $\theta = [RP, T]$. The simulation

lasts 300 minutes (5 hours) to make sure all surface water drains out before the simulation ends. The time step in the simulation is 2 minutes. The performance of the drainage system is measured by the sum of overflow rate at all the nodes: Q with unit being CMS (cubic meter per second). The system performance is considered ideal when $Q(t) = 0$, i.e., there is no flooding. A larger $Q(t)$ indicates worse performance. The maximum performance drop is thus calculated as $\max(Q(t))$. The robust range tolerance ϵ_Q is taken as 20 CMS, which is relatively a small value. The existence of this tolerance is due to the fact that a very small amount of overflow rate may not affect normal life, and thus is considered as acceptable. It is worth mentioning that our proposed framework allows a tolerance of 0 CMS. The local government can tell the modeler how much overflow is acceptable in this area for the modeler to choose the tolerance. If no such information is available, the modeler can just set the tolerance to 0. In this example, the system performance curve is represented by the sum of nodal overflow rates-time relationship shown in Fig. 3-4. The curves look different from Fig. 3-1: a going up curve represents the deterioration of system performance.

Figure 3-4: The performance curve of the drainage system is represented by the sum of nodal overflow rates - time relationship.

3.3.2.1 Resilience analysis

We model three drainage systems and compare their resilience performances. The detailed configurations of these three systems are as follows. The first system serves as the benchmark. There are 23 junction nodes, 22 conduits, and 17 subcatchments and 1 outlet. The second system enhances the drainage system by building two detention tanks. The location of these two detention tanks follows the work by Maharjan et al. [76], where four detention tanks are identified. In this research, it is simplified to two detention tanks, which are the two more critical locations identified by Maharjan et al. The third system is also built based on the benchmark system, but with several critical pipelines enlarged. A comparison of the three systems is given in Table 3-3.

Figure 3-5: Drainage system configuration 1.

Monte Carlo simulation is adopted to simulate the system performance under uncertain events. A total number of 10^6 simulations with random combinations of return period and rainfall durations are performed. The five resilience metrics are computed and plotted in the

Figure 3-6: Drainage system configuration 2, with two detention tanks.

Table 3-3: The detailed configuration of the three drainage systems. “RECT_CLOSED” represents rectangular cross section, closed pipe, “1*1” indicates the size. Similarly, “CIRCULAR” represents a pipe with circular cross section, “d1.2” is the diameter. The unit is meter (m).

System #	Configuration details	Corresponding Fig.
system 1	benchmark	Fig. 3-5
system 2	benchmark + detention tanks J1 and J2	Fig. 3-6
system 3	benchmark replacing pipes	Fig. 3-7
	T5: from “RECT_CLOSED 1*1” to “RECT_CLOSED 2.5*2.5”	
	T4: from “CIRCULAR d1.2” to “CIRCULAR d2.0”	
	T91: from “RECT_OPEN 1*1” to “RECT_OPEN 2*2”	
	T92: from “RECT_CLOSED 1*1” to “RECT_CLOSED 2*2”

Figure 3-7: Drainage system configuration 3, with critical pipelines (T4, T5, T91, T92) enlarged.

CCDF curves as shown in Fig. 3-8. The second row shows the c.o.v. of the estimators. The third row gives the unit c.o.v. of each estimator. The unit c.o.v. Δ is computed as $\Delta = \delta \cdot \sqrt{N}$, [79] where δ is the c.o.v. of the estimator, and N represents the number of samples. It can be seen that for system 1, the overflow time can be more than 1 hour in some extreme cases (Probability $< 10^{-4}$). The total overflow volume can be as large as $4000 m^3$. From Fig. 3-8, we can see that both adding detention tanks and enlarging the critical pipelines help decreasing the downtime and reducing the peak overflow rate. System 2 (Adding detention tanks) seems more efficient in reducing the peak overflow rate. This is due to the fact that the detention tanks reserve the precipitation water, which reduces the water to be drained to downstream. This is somewhat similar to an effect of peak-load shifting. The effect is equivalent to reducing the precipitation intensity.

Figure 3-8: Resilience analysis result of the three drainage systems configurations, subjected to rainfall events of return period from 10 years to 1000 years, with various rainfall durations. It is simulated using the Monte Carlo method with $N = 10^6$ samples (runs). The second and third row plots show the c.o.v. and the unit c.o.v. of the Monte Carlo estimators, respectively.

The last column of Fig. 3-8 is Loss of Resilience (LoR). In this example, as the sum of overflow rate at all nodes is taken as a measure of the performance of the drainage system, the physical meaning of LoR is thus the total overflow volume. The mean $LoRs$ corresponding to

Table 3-4: A comparison of the *LoR* (represents the total overflow volume in this example) of the three system configurations.

System config. #	Remark	mean <i>LoR</i> (total overflow volume)	c.o.v.
System 1	Benchmark	1545.7 m ³	0.31
System 2	Add detention tanks	1076.9 m ³	0.29
System 3	Enlarge critical pipelines	945.8 m ³	0.42

the three systems are given in Table 3-4. Let's focus on the effect of the different improvement strategies. As our target here is to compare the resilience of the three configurations from the technical perspective. So let's put aside the cost of detention tanks and the enlargement of pipes at the moment. We will discuss the trade-off between the improvements and the economic considerations in Chapter 5. If the decision makers use the *LoR* value to choose the strategy to be adopted to enhance the resilience of the drainage system, system 3 may be preferred due to the minimum *LoR* (total overflow volume). However, from columns 3 and 4 in the first row of Fig. 3-8, the resilience performance of system 3 is worse than system 2 in terms of the maximum overflow rate, while the downtime of system 3 and system 2 are of similar level. From the last column in the first row of the figure, it can also be seen that the *LoR* of system 2 outperforms system 3 in rare events. The smaller value of the mean *LoR* of system 3 is mainly due to the smaller *LoR* at more frequent events. In other words, the mean *LoR* is dominated by the high probability events. We can see that with the proposed resilience analysis method, decision makers can get more information on how system configurations change the system behavior in rare events. Compared to the commonly used single resilience index, the system resilience is represented by five different metrics, and each metric is expressed as the probability of exceeding some variable thresholds. The proposed method enables the consideration of such stochastic feature of the engineering system in resilience analysis.

3.4 Conclusions

We propose a method to measure the system resilience using multiple resilience components, and at the same time incorporating stochastic features of the systems and the uncertain disruptive events. The proposed method does so by expressing resilience as multiple CCDF (complementary cumulative distribution function) curves for a broad spectrum of events with different

degrees of rarity or extremeness. It provides stakeholders and decision makers a bird's-eye view of how the system behaviors and the probability that the system resilient performance falls below certain desired standard, especially the rare or extreme future events are considered. This is an improvement over the common practice of using the mean value of some resilience measurement as the resilience metric (or resilience index), where the same value of the resilience metric can correspond to different system behaviors.

Each resilience metric in our proposed method comes with its own physical meaning. For example, the first three metrics are time related, whereas the fourth metric is directly computed from the minimum system performance and hence it is with the unit of system performance measurement. The last resilience metric is a comprehensive one. For different systems, these metrics have different physical meanings. It is important to provide decision makers resilience metrics with explicit physical meaning. It is worth mentioning that although in the UDS example, we focus on the resilience of the whole system, the propose method is capable of measuring the resilience of a sub-system or a single component. All we need to do is to replace the overall system performance measurement by a sub-system based one.

There are some limitations of the proposed CCDF-based resilience measurement method. (1) the proposed method focuses on the impact of one-time shock of uncertain future events. i.e., the systems have enough time to fully recover before the next event comes. The proposed method is not capable of tackling with scenarios with continuous shocks by secondary events before system recovery. For such scenarios, some substantial improvements to the proposed resilience quantification method are to be made in future research. (2) the costs of enhancement practices are currently not considered. We will partially address the cost factor in Chapter 5. (3) as a simulation-based resilience measurement method, it requires a large number of evaluations of the physical model. The computation time is not negligible in some cases, especially when rare events are taken into consideration. In Chapter 4, we will improve an efficient algorithm and apply it to alleviate the heavy computational burden.

Chapter 4

Efficient Simulation Method for the Proposed Resilience Assessment Method

The computation efficiency of applying Monte Carlo simulation in the proposed resilience assessment framework is addressed in this chapter. A Subset Simulation based multiple exceedance probability estimation method, i.e ISS method is improved and adapted to the resilience assessment. The discontinuity in the original ISS method is alleviated. We revisit the urban drainage system (UDS) in chapter 3. Besides, a water distribution network (WDS) which suffers damages due to earthquakes is shown as another illustrative example. The effectiveness on system resilience improvements of different number of repair teams is illustrated.

4.1 Introduction

For many critical infrastructure systems, the simulation under uncertainties requires a large number of evaluations of the model, which is usually time-consuming. V. Guinot [80] pointed out that the computational expenses of a dynamic model that traces the real-time physical process of the urban flooding are almost unaffordable. Gibson et al. [81] argue that the assessing of the risk of urban flooding, thousands of simulations may be required. In simulating the rare

events, as the example [42] we have discussed following Eq. 2.35, if we use the brute-force Monte Carlo simulation and expected a medium-rare event with probability 10^{-4} , we can expect an estimator with 30% c.o.v. at the expense of 1 million samples. It is non-trivial to evaluate the physical model 1 million times. For example, a realistic 2D flood model of the same area as in Section 3.3.2 by DHI MIKE URBAN would take 20 minutes for one single run with 28 cores 56 threads CPU running on its full capacity of all cores (i.e. already paralleled). An estimation of the running time for exceedance probability estimation is given in 4-1. For a model that requires 1 min for one run (this is a very conservative runtime for a realistic engineering software), the computation is almost intractable as it requires almost 1.9 years to finish. Even if we run 600 instances at the same time (like “multi-processing”) so that we averagely finish one run in 1 second, it still requires 11.57 days. In reality, we would encounter some limitations in running the model in a “multi-processing” way. For example, a commercial software usually has a limitation on the license.

Table 4-1: An illustration of the computation requirement.

run time	Number of evaluations	Total run time
1 min per instance	1 000 000 (Monte Carlo)	About 19 years
1 s per instance	1 000 000 (Monte Carlo)	11.57 days

It is worth noting that the Table 4-1 is for resilience assessment under uncertainties. Decision makers may also be interested in the resilience-based optimization problem, where the computational expenses will be even higher. For example, if the model runs 1s for each instance of metric, and 100 candidate designs are to be considered, the total running time will be 1157 days (3 years), which is by no means affordable. Therefore, efficient simulation and optimization methods is to be proposed to alleviate the computational burden. We will address the computational burden in optimization in Chapter 5. In this Chapter, we will adapt a Subset Simulation-based method to resilience assessment under uncertainties to take place the Monte Carlo simulation for higher simulation efficiency.

As we have discussed in Section 2.2.4, Subset Simulation [43] is a stochastic simulation-based method that is capable of simulating rare extreme event in an efficient way. However, in the original Subset Simulation, a single “driving variable” is considered. As we have five measurements and each corresponds to a threshold (b_1, b_2, b_3, b_4, b_5 as in Eq. 3.4, 3.5, 3.6,

3.7, and 3.8 correspondingly), if Subset Simulation is used, five independent runs have to be performed. Based on the fact that the five performance functions are from the same simulated system, the ISS method [18] is adapted to the resilience assessment problem in this chapter to simultaneously estimate the multiple CCDF curves. This chapter will be organized as follows. We first give a brief review of the ISS method in Section 4.2. The discontinuity problem of the original ISS method is discussed in Section 4.2.4. In Section 4.3, we propose an improved ISS method to alleviate the discontinuity problem. We show the application of the proposed improved ISS method in resilience analysis with two examples in Section 4.4.

4.2 The Original ISS Method

Our target exceedance probabilities are $P(F_{m,j})$, where j represents the j th performance function, m represents the m th level of Subset Simulation. With the ISS method, this exceedance probability is computed as:

$$P(F_{m,j}) = P(F_{m,j}|F_{m-1}) \prod_{i=2}^{m-1} P(F_i|F_{i-1})P(F_1), \quad (4.1)$$

where $F_1 \supset F_2 \cdots \supset F_{m-1} \supset F_{m,j}$ is a sequence of failure events. The failure region F_i is given by the union of each failure region $F_{i,j}$:

$$F_i = F_{i,1} \cup F_{i,2} \cup \cdots \cup F_{i,R}, \quad (4.2)$$

where $F_{i,j}$ is the failure region in terms of the performance function j in level i , and R represents the total number of performance functions.

4.2.1 Level 0 Simulation

At level 0 in the original ISS, Monte Carlo simulation was adopted to generate N_0 samples $\boldsymbol{\theta}$. By evaluating the performance functions at the N_0 samples $\boldsymbol{\theta}^{(k)}$, $k = 1, 2, \dots, N_0$, the vector $\mathbf{G}_{0,j}$, $j = 1, 2, \dots, R$ which includes a total number of N_0 output values can be obtained for

each performance function:

$$\mathbf{G}_{0,j} = [g_j(\boldsymbol{\theta}^{(1)})g_j(\boldsymbol{\theta}^{(2)}) \dots g_j(\boldsymbol{\theta}^{(N_0)})]^T, \quad (4.3)$$

where g_j is the j th performance function. It is assumed that for a given $\boldsymbol{\theta}^{(k)}$, the R performance functions output vector $\mathbf{g}_j$, $j = 1, 2, \dots, R$, can be calculated in a single dynamic analysis. We need to sort vector $\mathbf{G}_{0,j}$. Let's use $\mathbf{S}_{0,j}$ to represent the vectors obtained by sorting the elements of $\mathbf{G}_{0,j}$ in ascending order ($\mathbf{S}_{0,j} = \text{sort}(\mathbf{G}_{0,j})$). Let $s_{0,j}^{(n)}$ be the n th element of vector $\mathbf{S}_{0,j}$, $n = 1, 2, \dots, N_0$. The intermediate threshold with regard to the j th performance function is determined as $b_{0,j} = s_{0,j}^{(N_0 - p_0 N_0)}$. The intermediate threshold refers to the special cross-level threshold where the last Subset Simulation level ends and the next subset simulation level starts. This threshold is chosen so that there are $p_0 N_0$ samples satisfying $g_j(\boldsymbol{\theta}) > b_{0,j}$. As the elements in $\mathbf{S}_{0,j}$ are sorted in ascending order, we have

$$\begin{aligned} s_{0,j}^{(1)} &\leq s_{0,j}^{(\iota)} \leq s_{0,j}^{(N_0 - p_0 N_0)} = b_{0,j}, \\ 1 - \frac{1}{N_0} &\geq P(g_j(\boldsymbol{\theta}) > s_{0,j}^{(\iota)}) = 1 - \frac{\iota}{N_0} \geq p_0, \end{aligned} \quad (4.4)$$

where $\iota = 1, \dots, N_0 - p_0 N_0$. Eq. 4.4 gives the section of the failure probability curves from $P(g_j(\boldsymbol{\theta}) > s_{0,j}^{(N_0 - p_0 N_0)}) = p_0$, to $P(g_j(\boldsymbol{\theta}) > s_{0,j}^{(1)}) = 1 - \frac{1}{N_0}$ for $j = 1, \dots, R$. An illustration of how Eq. 4.4 represents the points on CCDF curve at level 0 is as Fig. 4-1.

With all the R thresholds $b_{0,j}$ and performance function vectors $\mathbf{S}_{0,j}$ determined above, the failure probability at the cross-level threshold $P(F_1)$ can be readily estimated by:

$$P(F_1) \approx \hat{P}(F_1) = \frac{1}{N_0} \sum_{k=1}^{N_0} I_{F_1}(\boldsymbol{\theta}^{(k)}), \quad (4.5)$$

where I_{F_1} is an indicator function as follows:

$$I_{F_1}(\boldsymbol{\theta}) = \begin{cases} 1, & \text{if } \boldsymbol{\theta} \in F_1; \\ 0, & \text{otherwise,} \end{cases} \quad (4.6)$$

where $F_1 = \cup_{j=1}^R F_{1,j}$ and $F_{1,j}$ is the failure domain determined by the intermediate cross-level thresholds $b_{0,j}$, i.e., $F_1 = \{\boldsymbol{\theta} : g_j(\boldsymbol{\theta}) > b_{0,j}\}$.

Figure 4-1: A schematic illustration of the CCDF segment of the j th performance function represented by Eq. 4.4.

4.2.2 Subsequent Levels ($i \geq 1$)

From level $(i - 1)$, a total number of $N_{i-1}P(F_i|F_{i-1})$ samples that follow $p(\boldsymbol{\theta}|F_i)$ can be obtained where $N_0P(F_1|F_0) = N_0P(F_1)$. These samples are used as seeds to generate $N_i - N_{i-1}P(F_i|F_{i-1})$ additional samples distributed as $p(\boldsymbol{\theta}|F_i)$ so that there will be N_i samples that follow the distribution $p(\boldsymbol{\theta}|F_i)$ in the i th level. The modified Metropolis Hastings algorithm we reviewed in Section 2.2.4 is one of the algorithms that generates the additional samples. The determination of the value of N_i is discussed in section 4.2.3. While obtaining the N_i samples $\boldsymbol{\theta}^{(k)} \sim p(\boldsymbol{\theta}|F_i)$, $k = 1, 2, \dots, N_i$, the corresponding performance functions $\mathbf{G}_{i,j} = [g_j(\boldsymbol{\theta}^{(1)}), g_j(\boldsymbol{\theta}^{(2)}), \dots, g_j(\boldsymbol{\theta}^{(N_i)})]^T$, $j = 1, 2, \dots, R$, have already been evaluated. Let's use $\mathbf{S}_{i,j}$ to represent the vector obtained by sorting those $N_{i,j}$ elements which are larger than $b_{i-1,j}$ in $\mathbf{G}_{i,j}$ in ascending order. Let $s_{i,j}(n)$ be the n th element of vector $\mathbf{S}_{i,j}$, thus $n = 1, 2, \dots, N_{i,j}$. The intermediate cross-level failure threshold is $b_{i,j} = s_{i,j}(N_{i,j} - p_0 N_{i,j})$. This value is chosen so that there are $p_0 N_{i,j}$ samples satisfying $g_j(\boldsymbol{\theta}) > b_{i,j}$ and $\hat{P}(F_{i,j}|F_{i-1,j}) = p_0$ where $\hat{P}(F_{1,j}|F_{0,j}) = \hat{P}(F_{1,j})$.

Base on the samples obtained, this section of the failure probability curves are given by:

$$\begin{aligned} s_{i,j}^{(1)} \leq s_{i,j}^{(\iota)} \leq s_{i,j}^{(N_{i,j}-p_0N_{i,j})} = b_{i,j}, \\ \frac{N_{i,j}-1}{N_i} \hat{P}(F_i) \geq P(g_j(\boldsymbol{\theta}) > s_{i,j}^{(\iota)}) = \frac{N_{i,j}-\iota}{N_i} \hat{P}(F_i) \geq p_0 \frac{N_{i,j}}{N_i} \hat{P}(F_i), \end{aligned} \quad (4.7)$$

where $\iota = 1, 2, \dots, (N_{i,j} - p_0N_{i,j})$ and since $\hat{P}(F_{i,j}|F_i) = \frac{N_{i,j}}{N_i}$, the failure probabilities $P(g_j(\boldsymbol{\theta}) > s_{i,j}^{(\iota)})$ can be estimated as $\frac{N_{i,j}-\iota}{N_{i,j}} \times \hat{P}(F_{i,j}|F_i) \times \hat{P}(F_i) = \frac{N_{i,j}-\iota}{N_i} \times \hat{P}(F_i)$. An illustration of how Eq. 4.7 represents the points on CCDF curve at level 1 and 2 are as Fig. 4-2 and 4-3, respectively. Similar plots can be drawn for level 3 and on, we omit here for conciseness.

Figure 4-2: A schematic illustration of the CCDF segment of the j th performance function represented by Eq. 4.7 (level 1).

With all the R thresholds $b_{i,j}$ and performance function vectors $\mathbf{S}_{i,j}$ determined above, the conditional failure probability at the cross-level threshold $P(F_{i+1}|F_i)$ can be readily estimated by:

$$P(F_{i+1}|F_i) \approx \hat{P}(F_{i+1}|F_i) = \frac{1}{N_i} \sum_{k=1}^{N_i} I_{F_{i+1}}(\boldsymbol{\theta}_i^{(k)}), \quad (4.8)$$

where $I_{F_{i+1}}$ is an indicator function as follows:

$$I_{F_{i+1}}(\boldsymbol{\theta}) = \begin{cases} 1, & \text{if } \boldsymbol{\theta} \in F_{i+1}; \\ 0, & \text{otherwise,} \end{cases} \quad (4.9)$$

Figure 4-3: A schematic illustration of the CCDF segment of the j th performance function represented by Eq. 4.7 (level 2).

where $F_{i+1} = \cup_{j=1}^R F_{i+1,j}$ and $F_{i+1,j}$ is the failure domain determined by the intermediate cross-level thresholds $b_{i,j}$, i.e., $F_{i+1,j} = \{\theta : g_j(\theta) > b_{i,j}\}$. With the above, $P(F_{i+1}) = P(F_{i+1}|F_i)P(F_i) \approx \hat{P}(F_{i+1}|F_i)\hat{P}(F_i)$.

4.2.3 The Determination of N_i

At the i th level, there will be N_i samples follow the distribution $p(\theta|F_i)$. This number N_i is adaptively chosen based on the following principles. For the j th performance function, where $j = 1, 2, \dots, R$, there are around N_0 samples following $p(\theta|F_{i,j})$:

$$P(F_{i,j}|F_i) = \frac{P(F_{i,j})}{P(F_i)} \approx \frac{N_0}{N_i}, \forall j = 1, \dots, R. \quad (4.10)$$

Thus, N_i can be determined by the estimation of $P(F_{i,j})$ and $P(F_i)$. It should be noted that for each performance function j :

$$P(F_{i,j}) = P(F_{i,j}|F_{i-1,j}) \times P(F_{i-1,j}|F_{i-2,j}) \times \dots \times P(F_{1,j}), \quad (4.11)$$

$P(F_{i,j}|F_{i-1,j}), i = 1, \dots, m-1$ is set to be p_0 for each performance function. Therefore,

$\hat{P}(F_{i,j}) = p_0^i$. Thus,

$$N_i \approx N_0 \frac{\hat{P}(F_i)}{\hat{P}(F_{i,j})} = N_0 \frac{\hat{P}(F_i)}{p_0^i}, \quad (4.12)$$

where $\hat{P}(F_i)$ can be estimated by the method presented in section 4.2.2

4.2.4 Discontinuity Problem with the Original ISS Method

In the ISS method by Bansal and Cheung [18], there are two approaches to estimate the failure probabilities of intermediate thresholds at the i th level and j th performance function (or Limit State Function, LSF). One is given by:

$$\begin{aligned} P(F_{i,j}) &= P(F_{i,j}|F_{i-1,j}) \times P(F_{i-1,j}|F_{i-1}) \times P(F_{i-1}) \\ &\approx p_0 \frac{N_{i-1,j}}{N_{i-1}} \hat{P}(F_{i-1}). \end{aligned} \quad (4.13)$$

The other estimation is given by:

$$\begin{aligned} P(F_{i,j}) &= P(F_{i,j}|F_i) \times P(F_i|F_{i-1}) \times P(F_{i-1}) \\ &\approx \frac{N_{i,j}}{N_i} \times p_0 \frac{N_i}{N_{i-1}} \times \hat{P}(F_{i-1}) \\ &\approx p_0 \frac{N_{i,j}}{N_{i-1}} \hat{P}(F_{i-1}), \end{aligned} \quad (4.14)$$

where the approximation of $P(F_i|F_{i-1})$ is obtained by Eq. 4.12 . The above two estimations can yield different failure probabilities at the same threshold. Therefore, discontinuities of the failure probability curves occurs at the intermediate thresholds. In addition, because the estimator at the intermediate thresholds differs, the failure probability (exceedance probability) at the other thresholds (non-intermediate thresholds) are also affected. This effect decreases with the increasing of distance from the intermediate thresholds. This is shown in Fig. 4-9 and 4-10. We will explain these two figures in Section 4.3, where we illustrate the discontinuities and our improvement method with an example. Before that, Let's take a look at the example settings and the discontinuities in this example.

4.2.5 An Example on the Discontinuity

An example of four limit state functions (LSF) is selected as an illustration, with the four LSFs defined as follows:

$$\begin{aligned} g_1 &= 3 - \frac{(x_1 - x_2)^2}{10} + \frac{x_1 + x_2}{\sqrt{2}}, \\ g_2 &= 3 + \frac{(x_1 - x_2)^2}{10} - \frac{x_1 + x_2}{\sqrt{2}}, \\ g_3 &= -x_1 - \frac{5}{\sqrt{2}}, \\ g_4 &= x_1 + \frac{5}{\sqrt{2}}. \end{aligned} \tag{4.15}$$

Each limit state function corresponds to one system performance function. The definition of a failure event is that the system performance exceeds some predefined threshold.

$$F : g_j > b_j, \tag{4.16}$$

where $j = 1, 2, 3, 4$ represents the j th performance function. The corresponding exceedance probability is calculated as:

$$P(F) = P(g_j > b_j), \tag{4.17}$$

The original ISS method is adopted to estimate the exceedance probability curves. The results are shown in Fig. 4-4, where the first row shows the overall exceedance curves, the second and third rows shows the zoomed in curves at exceedance probabilities $P \approx 0.1$ and $P \approx 0.01$, respectively. It can be seen that discontinuities of the curves appear at the intermediate thresholds, i.e., at the neighbourhood of $P \approx 0.1$ and $P \approx 0.01$. In the following levels, i.e., at $P \approx 10^{-3}$ and $P \approx 10^{-4}$, similar discontinuities are expected. This discontinuity problem has to be solved because (1) the CCDF curves should be continuous (2) it will not only result in discontinuous estimators but also introduce extra c.o.v. to the estimator.

4.3 Proposed improved ISS method

In this section, we propose a new approach to resolve this discontinuity. The basic idea is: when estimating the failure probability at the thresholds located in the i th Subset Simulation level,

Figure 4-4: An illustration of the discontinuity in ISS method corresponds to the example in Eq. 4.15. The first row plots show the overall CCDFs; the second and third row show the sections on the CCDF curves which are zoomed in at around exceedance probability of 10^{-1} and 10^{-2} , respectively.

instead of using the samples belonging to the i th level only (as did in the original ISS method by Bansal and Cheung [18]), the samples belonging to the $(i + 1)$ th level and even all available samples can be utilized. A detailed description of this new approach is presented as follows.

We extend the ISS framework by Bansal and Cheung [18], for the j th performance function, by the Law of Total Probability, we have

$$\begin{aligned}
 P(g_j(\boldsymbol{\theta}) > b_{i,j}^*) &= P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | \bar{F}_1) P(\bar{F}_1) + \sum_{m=1}^{i-1} P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_m, \bar{F}_{m+1}) P(F_m, \bar{F}_{m+1}) \\
 &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_i, \bar{F}_{i+1}) P(F_i, \bar{F}_{i+1}) \\
 &\quad + \sum_{m=1}^{N-i-1} P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+n}, \bar{F}_{i+n+1}) P(F_{i+n}, \bar{F}_{i+n+1}) \\
 &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_N) P(F_N),
 \end{aligned} \tag{4.18}$$

where $b_{i-1,j} \leq b_{i,j}^* \leq b_{i,j}$, $F_{i,j} = \{\boldsymbol{\theta} : g_j(\boldsymbol{\theta}) > b_{i-1,j}\}$, $F_{i+1,j} = \{\boldsymbol{\theta} : g_j(\boldsymbol{\theta}) > b_{i,j}\}$, for

$i = 1, 2, \dots, N$, and N represents the total levels simulated; F_i represents failure domain defined by the threshold of the i th level, it is given by $F_i : \{g_1(\boldsymbol{\theta}) > b_{i,1}\} \cup \dots \cup g_j(\boldsymbol{\theta}) > b_{i,j}\} \cup \dots \cup g_{N_j}(\boldsymbol{\theta}) > b_{i,N_j}\}$, where N_j represents the number of performance functions; $\bar{F}_i$ is the non-failure domain defined by the threshold of the i th level, F_i and $\bar{F}_i$ are complementary. A schematic illustration of these definitions are given in Fig. 4-5.

Figure 4-5: An illustration of the definitions of $F_{i,j}$, $F_{i,j+1}$, F_i , and $\bar{F}_i$, two dimensional variable space (θ_1, θ_2) is shown for the convenience of visualization.

We have:

$$P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_m, \bar{F}_{m+1}) = 0, \quad (4.19)$$

for $m = 1, 2, \dots, i-1$, and

$$P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | \bar{F}_1) = 0, \quad (4.20)$$

due to the fact that

$$\bar{F}_{m+1} \subset \bar{F}_{m+1,j} \subseteq \bar{F}_{i,j}, \quad (4.21)$$

and

$$\bar{F}_{i,j} \subseteq \{\boldsymbol{\theta} : g_j(\boldsymbol{\theta}) > b_{i,j}^*\}, \quad (4.22)$$

where $b_{i,j}^*$ is some threshold for the j th performance function larger than $b_{i,j}$.

Eq. 4.21 and 4.22 is the prove to Eq. 4.19, and Eq. 4.20 is a special case of Eq. 4.19. This prove can also be seen in Fig. 4-6, where obviously we cannot find any sample $\boldsymbol{\theta}$ that belongs

to region $(F_m, \bar{F}_{m+1})$ and satisfy $g_j(\boldsymbol{\theta}) > b_{i,j}^*$. Therefore, the first two terms of Eq. 4.18 are equal to zero. $P(g_j(\boldsymbol{\theta}) > b_{i,j}^*)$ can be obtained:

Figure 4-6: A schematic illustration for Eq. 4.19 to 4.22, $m+1 \leq i$ as given in Eq. 4.18. Two dimensional variable space (θ_1, θ_2) is shown for the convenience of visualization.

$$\begin{aligned}
 P(g_j(\boldsymbol{\theta}) > b_{i,j}^*) &= P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_i, \bar{F}_{i+1}) P(F_i, \bar{F}_{i+1}) \\
 &+ \sum_{n=1}^{N-i-1} P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+n}, \bar{F}_{i+n+1}) P(F_{i+n}, \bar{F}_{i+n+1}) \\
 &+ P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_N) P(F_N),
 \end{aligned} \tag{4.23}$$

where $P(F_i, \bar{F}_{i+1}) = P(F_i) - P(F_{i+1})$ and $P(F_{i+n}, \bar{F}_{i+n+1}) = P(F_{i+n}) - P(F_{i+n+1})$. In Eq. 4.23, the samples from the i th level to the N th level are all used to estimate $P(g_j(\boldsymbol{\theta}) > b_{i,j}^*)$ on the failure probability curves. A simplified version of the method is to use only the samples in the i th level and the level $(i+1)$, and the failure probability is:

scheme “**i plus 1**”

$$\begin{aligned}
 P(g_j(\boldsymbol{\theta}) > b_{i,j}^*) &= P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_i, \bar{F}_{i+1}) P(F_i, \bar{F}_{i+1}) \\
 &+ P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+1}) P(F_{i+1}).
 \end{aligned} \tag{4.24}$$

A schematic illustration of this is given in Fig. 4-7. In the original ISS method by Bansal and Cheung [18], $P(g_j(\boldsymbol{\theta}) > b_{i,j}^*)$ is estimated by the samples of the i th level where $\boldsymbol{\theta}$ belongs to F_i , including those serving as the seeds of the next level. In comparison, in the proposed

Figure 4-7: A comparison of how samples are used in original ISS and improved ISS. Left: Failure estimation by Bansal and Cheung [18], where the exceedance probabilities are given by Eq. 4.7; right: proposed "i plus 1" scheme, where the exceedance probabilities are given by Eq. 4.24. For the convenience of visualization, both methods show the two dimensional variable space (θ_1, θ_2) case.

improved method, the same failure probability is estimated by samples in the i th level and the $(i + 1)$ th level. To be specific, it is estimated by samples located in the region $F_i, \bar{F}_{i+1}$ and region F_{i+2} . Similarly, we can also use two levels of samples beyond level i and three levels of samples beyond level i :

scheme "i plus 2"

$$\begin{aligned} P(g_j(\boldsymbol{\theta}) > b_{i,j}^*) &= P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_i, \bar{F}_{i+1}) P(F_i, \bar{F}_{i+1}) \\ &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+1}, \bar{F}_{i+2}) P(F_{i+1}, \bar{F}_{i+2}) \\ &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+2}) P(F_{i+2}). \end{aligned} \quad (4.25)$$

A schematic illustration of this i plus 2 scheme is given in Fig. 4-8.

scheme "i plus 3"

$$\begin{aligned} P(g_j(\boldsymbol{\theta}) > b_{i,j}^*) &= P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_i, \bar{F}_{i+1}) P(F_i, \bar{F}_{i+1}) \\ &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+1}, \bar{F}_{i+2}) P(F_{i+1}, \bar{F}_{i+2}) \\ &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+2}, \bar{F}_{i+3}) P(F_{i+2}, \bar{F}_{i+3}) \\ &\quad + P(g_j(\boldsymbol{\theta}) > b_{i,j}^* | F_{i+3}) P(F_{i+3}). \end{aligned} \quad (4.26)$$

scheme "use all" If we use as many levels of samples as possible, we have the "use all" scheme, which is the case given by Eq. 4.23.

Figure 4-8: A schematic illustration of the proposed "i plus 2" scheme, where the exceedance probabilities are given by Eq. 4.25. For the convenience of visualization, both methods show the two dimensional variable space (θ_1, θ_2) case.

To illustrate the improvements of the proposed schemes compared with the original ISS method, example in Section 4.2.5 is revisited here. We conduct four schemes: "i plus 1" (Eq. 4.24), "i plus 2" (Eq. 4.25), "i plus 3" (Eq. 4.26) and "use all" (Eq. 4.23). It is worth noting that the improvement schemes do not require extra evaluations of the model, we are using the same samples as the original ISS method. The difference is in how the samples are used. The results are shown in Fig. 4-9 and Fig. 4-10. We also show the curve by the original ISS method for comparison. In Fig. 4-9, the second row shows the zoomed-in segments at exceedance probability $\approx 10^{-1}$ and the third row shows the zoomed-in segments at exceedance probability $\approx 10^{-2}$. Fig. 4-10 shows segment from exceedance probability $\approx 10^{-1}$ to $\approx 10^{-2}$ of the second performance function, which corresponds to the second column of Fig. 4-9. In the second row of Fig. 4-10, the left sub-plot is the segment zoomed in around exceedance probability $\approx 10^{-1}$, the right sub-plot is the segment zoomed in around exceedance probability $\approx 10^{-2}$. From Fig. 4-10 we can see that the curve by the original ISS method are consistent with the proposed schemes when it is away from the thresholds (for example, from exceedance probability 2×10^{-2} to 9×10^{-2}), and we can see discrepancies between the curve by the original ISS method and our proposed schemes (for example at exceedance probabilities around 10^{-2} and 10^{-2} , which are zoomed-in in the second row of Fig. 4-10). The CCDF curve should be non-increasing with growing thresholds, but we can see there is a sudden 'jump up' on the curve by the original ISS

method (as shown in the lower right sub-plot in Fig. 4-10). These discontinuities are alleviated with the proposed improved ISS method.

The differences between the four improvement schemes are not significant. We suggest that the “i plus 1” scheme is sufficient in terms of alleviating the discontinuity. Some other examples we have tested support this suggestion too.

Figure 4-9: The exceedance probability curves with improved ISS method, several schemes are shown, with comparison with the original ISS.

In this chapter, we use the simplest scheme, i.e., the “i plus 1” scheme, which appears to be good enough for the illustrative examples considered in section 3. It resolves the discontinuity problem encountered in the original ISS [18]. The other schemes show no obvious improvement compared with the simplest scheme for the examples shown in this section. In the remaining part of this thesis, the original ISS method in combination with the exceedance probabilities estimation schemes proposed in this section is referred to as “improved ISS method”.

Figure 4-10: The exceedance probability curves with improved ISS method, with comparison with the original ISS. Zoomed-in at simulation level 0 for the 2nd performance function.

4.3.1 A Visualization of the Propagating of Samples in Brute-force Monte Carlo, Subset Simulation and (Improved) ISS Method

A simple hypothetical problem with two limit state functions is shown as an example to compare the simulation process of the ISS method (Fig. 4-11), Subset Simulation (Fig. 4-12) and Monte Carlo simulation (Fig. 4-13). This example has two performance functions given as follows:

$$\begin{aligned} g_1 &= -0.5\theta_1^2 + 3.5\theta_1 - \theta_2 + 8; \\ g_2 &= -\theta_1^2 + 8\theta_1 + \theta_2 - 20, \end{aligned} \quad (4.27)$$

where θ_1 and θ_2 follow standard normal distribution. The number of limit state functions is represented as $nLSF$. The same $p_0 = 0.1$ is set for both ISS method and Subset Simulation. ISS method starts with 500 samples at level zero, and the total number of samples to simulate until level 4 is 3173. Subset Simulation starts with the same sample size of 500, the total number of samples used is ' $nLSF \times [500 + 500 \times (1 - p_0) \times 4] = 4600$ '. The computational savings of the ISS method is that it makes use of the samples located in the region where the

limit state functions (LSFs) overlap. No repeated simulation and computation of these samples in overlapping regions is required. It is to note that if the number of limit state functions increases, a higher chance that more overlap between the domains exists, thus ISS method can be more efficient. As a comparison, 5000 Monte Carlo samples are simulated (Fig. 4-13). It can be seen that almost no samples located in the 4th level region defined by Subset Simulation or ISS method.

Figure 4-11: An illustration of ISS Simulation method, sample scatter plot are shown from level 0 to level 4 with both limit state functions (LSF).

In this chapter, the five resilience components are considered as artificial “failure modes”, and the corresponding resilience metrics are serving as the driving variables in the simulation with ISS method or Subset Simulation.

Figure 4-12: An illustration of Subset Simulation, samples scatter from level 0 to level 4 are shown, SS simulate one limit state function (LSF) in one run.

Figure 4-13: The scatter plot of 5000 samples generated by Monte Carlo Simulation.

4.4 Improved ISS Method in the Proposed Multiple CCDF-based Resilience Analysis Method

With the improved ISS method, the multiple CCDF-based resilience analysis method can be done in an efficient way. We first revisit the urban drainage system illustrated in Chapter 3, where the improved ISS method is adopted as the simulation method. The computational savings are shown compared with the brute-force Monte Carlo method adopted in Chapter 3. In the second example, we adopt the proposed improved ISS method to analyse a water distribution network which is subjected to the future uncertain earthquakes.

4.4.1 Revisiting the Urban Drainage System Resilience Analysis

We first revisit the urban drainage system (UDS) in Section 3.3.2. The same drainage systems are modelled to compare their resilience performances when subjected to uncertain rainfalls. The improved ISS method illustrated in Section 4.3 is adopted to assess the resilience. In the Monte Carlo level of ISS (i.e., level 0), an initial sample size of $N = 500$ is chosen. 50 independent runs of ISS are conducted for each system. With the simulation going from one level to the following level, the number of samples will increase. Here we take the benchmark system 1 as an example. A summary of the sample size at each level is shown in Table 4-3. It is worth noting that the values in Table 4-3 is the mean sample size of 50 independent runs. Table 4-3 gives the information how the sample size increases from level to level and the overlap of samples corresponding to each resilience component. $\sum N_{i,j} > N_i$ indicates overlaps exist. The larger difference between $\sum N_{i,j}$ and $\sum N_i$, the higher computation efficiency can be expected. It is to note that the numbers in Table 4-3 are the number of samples including the seeds. Considering the fact that seeds are kept to the next level in improved ISS method, the actual number of simulated samples are actually smaller than those numbers shown in Table 4-3.

To show the efficiency of the improved ISS method, Subset Simulation and Monte Carlo simulation are also performed. A comparison is given in Fig. 4-14. Both the results obtained by Monte Carlo simulation with sample size of $N = 5 \times 10^4$ and $N = 1 \times 10^6$ are shown. The c.o.v. of the exceedance probability curves are plotted in the second row. Let's consider a threshold of $b =$

[12, 50, 62, 130, 4000] corresponding to the five resilience components and yielding exceedance probability around 10^{-4} . (These thresholds are chosen based on two principles: (1) In each figure, the exceedance probabilities corresponding to the threshold is firstly larger than 10^{-4} and secondly as much close to 10^{-4} as possible; (2) The thresholds are kept as integers, for example, 12 is chosen instead of 12.1 as the first threshold). With the ISS method, to get the exceedance probability $P(M_j > b_j)$, the total number of evaluation of the physical model is $\sum N_{ISS} = 3602$. If the original Subset Simulation is used, a total number of evaluations will be $\sum N_{SS} = 9052$. The improved ISS method has a similar magnitude of c.o.v. with the original Subset Simulation and the Monte Carlo simulation with sample size $N_{MC} = 5 \times 10^4$. At this threshold, the computational saving of the improved ISS method over the original Subset Simulation is $\frac{\sum N_{SS}}{\sum N_{ISS}} = 2.51$. The computational saving of the improved ISS method over brute-force Monte Carlo simulation is $\frac{N_{MC}}{\sum N_{ISS}} = 13.88$. We also compare the run times of each method on the same hardware platform. The hardware is as follows: the CPU is AMD RyzenTM R7-5800X, 8 cores 16 threads with base clock of 3.8 GHz and maximum boost clock of 4.7 GHz; Dual-channel 16 GB DDR4 memory running at 3600 MHz is installed. The operation system is Ubuntu 18.04 LTS. The codes of all the three methods are written in MATLAB. The python multiple processing module is integrated to compute the model in parallel. The run times (average of 10 independent runs) of each method is as follows: the run time of Monte Carlo is 44.23 minutes, the run time of the original Subset Simulation is 12.82 minutes, and the run time of the improved ISS method is 4.32 minutes. Table 4-2 gives the run time of each run and the standard deviations of the mean run times. The ratio of saving of the improved ISS method over the original Subset Simulation is $\frac{t_{SS}}{t_{ISS}} = 2.97$, where t represents for run time. This ratio is higher than the ratio given by number of evaluations of physical model (which is $\frac{\sum N_{SS}}{\sum N_{ISS}} = 2.51$). The reason is that in the improved ISS method, we can evaluate more samples than the original Subset Simulation method in one batch of run. For example, as shown in Table 4-3, with the improved ISS method, in level 3, we can simulate 109 (this is an average of 50 runs, which is given by $1094/10$, the division by 10 is because we have $p_0 = 0.1$, thus 9 transitions are to be calculated, plus one batch of the seeds) samples in one batch of run. In comparison, with the original Subset Simulation, we can only evaluate 50 (which is given by $500/10$ for the same reason of $p_0 = 0.1$). A smaller number of samples in one batch requires more io (input output) operation, especially the SWMM model we used is highly depend on

Table 4-2: Run time of the three methods reaching the exceedance probability level of 10^{-4} at the same c.o.v. level. Unit of run time is minutes.

	Improved ISS	Subset Simulation	Monte Carlo
run 1	4.32	12.50	44.36
run 2	4.64	13.36	44.43
run 3	4.03	12.76	44.34
run 4	4.39	12.88	44.10
run 5	4.47	12.80	44.24
run 6	3.80	12.72	44.09
run 7	3.87	12.80	44.28
run 8	3.70	12.93	44.14
run 9	4.26	12.72	44.22
run 10	3.85	12.74	44.17
mean run time	4.32	12.82	44.23
standard deviation	0.32	0.22	0.12

reading and writing files on hard disks. This also indicates that the CPU is powerful enough to compute this simple model of the sample size around 500. If the model is more complicated or the sample size is larger, the time spent on io operation is negligible compared with evaluating the physical model, and the ratio of savings give by run time and number of evaluations would be closer. With the same reason, the ratio of saving of the improved ISS method over the Monte Carlo simulation is $\frac{t_{MC}}{t_{ISS}} = 10.23$, which is smaller than the ratio given by number of evaluations of the physical model ($\frac{N_{MC}}{\sum N_{ISS}} = 13.88$). To further illustrate the efficiency of the improved ISS method, the unit covs of the exceedance probability estimators are also calculated. The unit c.o.v. (denoted by Δ) is calculated as $\Delta = \delta \cdot \sqrt{N}$, where δ represents the c.o.v. of the estimator of the exceedance probability. N is the sample size. A lower unit c.o.v. represents higher efficiency. From the last row of Fig. 4-14, the efficiency of the improved ISS method is significant. Similarly, Table 4-4 and Fig. 4-4 show the number of samples and a comparison of the CCDF curves for system configuration #2. Table 4-5 and Fig. 4-5 show the number of samples and a comparison of the CCDF curves for system configuration #3.

Table 4-3: Sample sizes at level i , the total sample size (N_i) and number of samples belonging to each resilience driving variable ($N_{i,1}$ to $N_{i,5}$), for system configuration 1.

level i	N_i	$N_{i,1}$	$N_{i,2}$	$N_{i,3}$	$N_{i,4}$	$N_{i,5}$
0 (Monte Carlo level)	500	500	500	500	500	500
1	872	509	512	516	503	510
2	994	466	504	508	503	518
3	1094	472	499	499	508	527
4	1552	416	450	459	563	463

* average number of 50 independent runs

Table 4-4: Sample sizes at level i , the total sample size (N_i) and number of samples belonging to each resilience driving variable ($N_{i,1}$ to $N_{i,5}$), for system configuration 2.

level i	N_i	$N_{i,1}$	$N_{i,2}$	$N_{i,3}$	$N_{i,4}$	$N_{i,5}$
0 (Monte Carlo level)	500	500	500	500	500	500
1	782	491	503	501	501	504
2	1019	476	518	518	511	518
3	1343	435	490	528	525	535
4	1610	380	452	509	521	539

* average number of 50 independent runs

Table 4-5: Sample sizes at level i , the total sample size (N_i) and number of samples belonging to each resilience driving variable ($N_{i,1}$ to $N_{i,5}$), for system configuration 3.

level i	N_i	$N_{i,1}$	$N_{i,2}$	$N_{i,3}$	$N_{i,4}$	$N_{i,5}$
0 (Monte Carlo level)	500	500	500	500	500	500
1	902	506	501	498	503	500
2	1057	429	505	504	503	500
3	994	261	506	506	506	505
4	1118	41	486	498	493	508

* average number of 50 independent runs

Figure 4-14: Comparison of resilience analysis results of configuration #1, obtained by improved ISS method, original Subset Simulation, and Monte Carlo simulation; in Monte Carlo simulation, both results with sample size $N_{MC} = 5 \times 10^4$ and $N_{MC} = 1 \times 10^6$ are shown. The unit c.o.v. of Subset Simulation is not plotted as Subset Simulation is a method for single performance function.

Figure 4-15: Comparison of resilience analysis results of configuration #2, obtained by improved ISS method, Subset Simulation, and Monte Carlo simulation; in Monte Carlo simulation, both results with sample size $N_{MC} = 5 \times 10^4$ and $N_{MC} = 1 \times 10^6$ are shown. The unit c.o.v. of Subset Simulation is not plotted as Subset Simulation is a method for single performance function.

Figure 4-16: Comparison of resilience analysis results of configuration #3, obtained by the improved ISS method, Subset Simulation, and Monte Carlo simulation; in Monte Carlo simulation, both results with sample size $N_{MC} = 5 \times 10^4$ and $N_{MC} = 1 \times 10^6$ are shown. The unit c.o.v. of Subset Simulation is not plotted as Subset Simulation is a method for single performance function.

4.4.2 Urban Water Supply Network

A water distribution system (WDS) subjected to future uncertain earthquakes is studied. The layout of the WDS is shown in Fig. 4-17. There are 92 junctions, 3 tanks, 117 pipes, 2 reservoirs (Lake, River) and 2 pumps. The pump at the lake operates at regular intervals daily, and the operation of the pump near the river is controlled by the water level in Tank 1. A sensor is placed in Tank 1 to detect the water level. If the water level is below a threshold, the pump will start working until the water level reaches a certain threshold. The center of the network is defined in Fig. 4-17. The position of the earthquake epicenter is uncertain. The relative position of the center of network and the epicenter is described by two random variables: distance d and angle α , where α is assumed to be uniformly distributed in the range $[0, 2\pi]$. and distance d is assumed to follow a triangular distribution

$$p(d) = \frac{2d}{d_{\max}^2}, \quad (4.28)$$

where $p(d)$ is the PDF of d , $d_{\max} = 150$ km is the maximum distance considered in this simulation. The range of d is thus $d \in [0 \text{ km}, 150 \text{ km}]$. The distribution of uncertain earthquake magnitude is modelled by the bounded Gutenberg-Richter recurrence law [82]. The minimum and maximum earthquake magnitudes ($M_{\min}$ and $M_{\max}$ respectively) are defined with the PDF of the magnitude M given by

$$p(M) = \frac{\beta \exp(-\beta M)}{\exp(-\beta M_{\min}) - \exp(-\beta M_{\max})}, \quad (4.29)$$

where $\beta = 2.303$, $M_{\min} = 6.5$ and $M_{\max} = 8.0$. The random vector $\boldsymbol{\theta}$ in this example consists of three variables, i.e., the earthquake magnitude M , the distance of the epicenter to the center of the network d , and the angle α . i.e., $\boldsymbol{\theta} = [M, d, \alpha]$. As an illustration, we consider the failure of pipelines. The repair rate is a function of the Peak Ground Velocity (PGV). The PGV at a specific pipeline when it is subjected to an earthquake of magnitude M and a distance to the epicenter d is estimated by an attenuation relationship [83,84]

$$PGV = e^{c_1 + c_2 M + c_3 \log(d + d_0)}, \quad (4.30)$$

Figure 4-17: The water distribution network and the earthquake locations.

where c_1 , c_2 , c_3 , d_0 are coefficients. As an illustrative case, we assume soil site condition is considered. For soil site condition, $c_1 = -0.285$, $c_2 = 0.711$, $c_3 = -1.851$, $R_0 = 17$ km. [83]. The repair rate of a pipe is assumed to be a linear function of the PGV at the center of this pipe [85]:

$$RR = 0.0002415 \times PGV, \quad (4.31)$$

where RR is the repair rate in 1/m, PGV is of unit m/s. The fragility curve shown in Fig. 4-20 is adapted from the WNTR package [84], where a major leak refers to a pipe lost all its function of conveying water, and a minor leaked pipe keeps conveying water but at its degraded state. The package WNTR (Water Network Tool for Resilience) [84] is used in this example. When the water distribution system fails to fulfil the requirement, the actual demand is a variable related to the expected full servicing pressure P_f and minimum service pressure P_0 . If the actual pressure at the user end p is less than P_0 , no water service will be available [86]. The

actual demand D is given by:

$$D = \begin{cases} 0, & \text{if } p \leq P_0; \\ D_f \sqrt{\frac{p - P_0}{P_f - P_0}}, & \text{if } P_0 < p < P_f; \\ D_f, & \text{if } p \geq P_f, \end{cases} \quad (4.32)$$

where D_f is the expected (full) demand. This expected demand at each node follows a demand pattern that fluctuates within a day. To illustrate the patterns, we show the expected demand of 15 nodes in the network. These nodes are selected by picking the nodes with the smallest 15 node labels. The position of these nodes are shown in Fig. 4-18. In Fig. 4-19, the left-hand-side plot is the expected demand, which is the ideal demand without disruptive events. It can be seen that the demand patterns have a period of 24 hours. And the right-hand-side plot is the actual demand under a disruptive event given by Eq. 4.32.

Figure 4-18: The nodes whose expected demand are shown to illustrate the demand patterns.

We assume that the time that a specific pipe fails after the earthquake attacks is

$$t_i = \frac{\sqrt{d_i^2 + h^2}}{v_s}, \quad (4.33)$$

Figure 4-19: The demand patterns of the 15 nodes. Left: the expected demand; Right: the actual demand under disruptive event.

Figure 4-20: The fragility curves for the pipelines (adapted from WNTR package [84]).

where i represents the i th pipeline, d_i is the distance from the midpoint of the i th pipeline to the epicentre, h is the depth of the earthquake, we consider this depth to be 10 km, v_s is the speed of the earthquake S-wave. Manpower is arranged to fix the leaked pipes. The failed pipes are sorted based on the leaked demand, and the repair priority gives to pipes with larger leaked demand. Assume there are n repair teams that can working simultaneously on the one leaked pipeline, and each team requires 5 hours to fix a pipeline. The performance of this water distribution network is measured by the water service availability [86]. The water service availability ω is a function of time t :

$$\omega(t) = \frac{\sum_{j=1}^N V_j(t)}{\sum_{j=1}^N \hat{V}_j(t)}, \quad (4.34)$$

where N represents the number of network nodes. $V_j(t)$ is the actual volume of water received per unit time at node j at time t , it has a unit of m^3/s , $\hat{V}_j(t)$ is the expected water service volume (also of unit time) at node j at time t .

4.4.2.1 Simulation Method and Resilience Analysis

50 independent runs of the ISS method with initial sample size of $N_0 = 500$ at level 0 (Monte Carlo level) are performed. It is worth noting that at least $p_0 N_0$ samples should yield non-zero response for each of the five metrics, where p_0 is the level to level conditional probability. If there are not enough samples fulfilling this requirement, more samples will be generated. In this example $p_0 = 0.1$ is adopted and the results are shown in Fig. 4-21.

Two repair manpower arrangement plans are compared. One has 5 teams of available crews for pipes repairing, and the other has 8 teams of available crews. A team of repair crews take repair practice together, i.e. a team cannot be divided to repair at two locations at the same time. In the fourth column of Fig. 4-21, we can see there is a high probability that the system lost most of its performance (we observe the probability that the maximum performance drop exceeds 95% is 4×10^{-2}), which indicates that this network is not well protected against seismic hazards, and enhancements on resistance to the seismic hazards are suggested. We can also see from Fig. 4-21 that the extra repair manpower reduces the downtime significantly. The time deteriorating (first column) also differs in the tail part. This is because there are too

Figure 4-21: Resilience analysis results of the water distribution network, 50 independent runs of ISS method are performed. Two cases are simulated, with 5 teams and 8 teams. The effectiveness of adding additional pipe-repair crews is shown.

many broken pipelines for 5 repair teams to handle in one round of repairing work and there are fluctuations in the demands. However, the more manpower requires higher budget. There is a trade-off between the budget and the speed of the recovery. Whether the extra repair manpower worthy or not is a question left to the decision makers. The Loss of Resilience (LoR) in this case represents the average level of water service availability. By dividing the LoR value by the total time period being considered, we will get the average reduction in water service level during this period. As expected, the reduction of LoR with extra repair crews is mainly due to the shortened downtime.

4.4.2.2 Extreme Hazards Analysis

We continue the simulation to the next level to capture the extreme events. The samples at simulation level 4 of the improved ISS method for the case with 5 repair crews are extracted and analysed. These samples fulfil the following criteria: $DT > b_{4,3}$, $\max \Delta_g > b_{4,4}$ and $LoR > b_{4,5}$, corresponding to an exceedance probability smaller than 10^{-4} for each of the three metrics.

These samples are classified into three categories:

1. samples belonging to $DT > b_{4,3}$;
2. samples belonging to $\max \Delta_g > b_{4,4}$;
3. samples belonging to $LoR > b_{4,5}$.

Here $b_{4,3} = 111.68$, $b_{4,4} = 0.9957$, and $b_{4,5} = 84.80$. The pairplots of these samples are given in Fig. 4-22. The pairplots are 3-by-3 because the samples contain three aspects of information: (1) The Earthquake magnitude; (2) The distance from the epicentre to the centre of the water distribution network; (3) The direction of the epicentre. In the upper-left plot, we can see that the peak for maximum performance drop (the red curve, at around magnitude 8.42) is different from the peak corresponding to the longest downtime (the blue curve, at around magnitude 8.23). This indicates that the earthquakes that have the largest impact on the resilience components (for example, the downtime) of this network may not be those with the largest magnitudes. It is affected by several comprehensive random variables including the distance and α . Therefore, we argue that resilience analyses assuming earthquakes with one fixed magnitude may be misleading, even if the magnitude is fixed at a very large value.

We also visualize the overlaps of samples as in Fig. 4-23, where the samples are classified into 7 categories, 1. samples belong to $DT > b_{4,3}$ only, legend as R3 only; 2. samples belong to $\max \Delta_g > b_{4,4}$ only, legend as R4 only; 3. samples belong to $LoR > b_{4,5}$ only, legend as R4 only; 4. samples belongs to $DT > b_{4,3}$ and $\max \Delta_g > b_{4,4}$ but not $LoR > b_{4,5}$, legend as R34; 5. samples belong to $DT > b_{4,3}$ and $LoR > b_{4,5}$ but not $\max \Delta_g > b_{4,4}$, legend as R35; 6. samples belong to $\max \Delta_g > b_{4,4}$ and $LoR > b_{4,5}$ but not $DT > b_{4,3}$, legend as R45, 7. samples belong to $DT > b_{4,3}$ and $LoR > b_{4,5}$ and $\max \Delta_g > b_{4,4}$, legend as R345. R35, R45 and R345 are the overlap samples, which is the reason that the improve ISS achieves the computational savings over Subset Simulation.

Figure 4-22: Samples at level 4, classified into three categories: 1. samples belong to $DT > b_{4,3}$, 2. samples belong to $\max \Delta g > b_{4,4}$, 3. samples belong to $LoR > b_{4,5}$. The diagonal plots show the density distributions of samples belong to each category. The cases with 5 crews is illustrated

Figure 4-23: Samples at the 4th level, classified into 7 categories: 1. samples belong to $DT > b_{4,3}$ only, legend as R3 only; 2. samples belong to $\max \Delta g > b_{4,4}$ only, legend as R4 only; 3. samples belong to $LoR > b_{4,5}$ only, legend as R5 only; 4. samples belong to $DT > b_{4,3}$ and $LoR > b_{4,5}$ but not $\max \Delta g > b_{4,4}$, legend as R35; 5. samples belong to $\max \Delta g > b_{4,4}$ and $LoR > b_{4,5}$ but not $DT > b_{4,3}$, legend as R45; 6. samples belong to $DT > b_{4,3}$ and $LoR > b_{4,5}$ and $\max \Delta g > b_{4,4}$, legend as R345. There is no sample belongs to $DT > b_{4,3}$ and $\max \Delta g > b_{4,4}$ but not $LoR > b_{4,5}$ at this level, i.e. R34 is empty. The diagonal plots show the density distributions of samples belong to each category.

4.5 Conclusions

We alleviate the computational burden in resilience assessment by introduce Subset Simulation-based methods to the proposed multiple CCDF curves-based resilience assessment method. We first elaborate the needs of introducing more efficient simulation methods for resilience assessment. Following that, the ISS method is introduced, which is an efficient method capable of evaluating multiple CCDF curves in one run. But ISS method is not perfect, it has a discontinuity problem. We proposed a solution to solved the discontinuity by new schemes in making use of the samples. In our proposed improved ISS scheme, no extra simulation is required compared with the original ISS method. We apply the improved ISS method to two resilience assessment examples. Example 1 is urban drainage system. We compare the results with Subset Simulation and Monte Carlo simulation. Computational savings are reported. In example 2, we apply the improved ISS method to water supply network, where different rescue resources levels are compared. The extreme hazards are analysed and we argue the necessity of considering hazards with a whole range of uncertainties.

Chapter 5

Resilience Optimization in the Multi-Objective Optimization Framework

In this chapter, we address the problem of finding the optimal designs with resilience considerations. Multi-objective optimization (MOO) is considered. MOO is a mathematical optimization process that tackles with more than one objective function at the same time. In a multi-objective optimization, the objective functions are usually conflicting, i.e., the decrease of one objective function comes with the increase of another. The set of solutions to an MOO problem is called the Pareto optimal solutions. In this chapter, the proposed multiple resilience metrics and the costs are taken as objectives to be optimized. Our target is to find the optimal design that minimizes the multiple exceedance probabilities and the cost of the corresponding design. The main challenge in this MOO problem is that the objective functions being probability integrals, their evaluations require a huge number of analyses of the physical models. Therefore, brute-force optimization with the multi-objective optimization methods reviewed in Section 2.3 is often computationally prohibitive. In this chapter, a recently proposed design optimization in the partitioned design space (DOPAS) method by Liu and Cheung [87] is adapted to incorporate the adaptive rejection sampling method by Jia et al. [88] to address the computation issue in the multi-objective resilience optimization. An optimization example of a drainage network is shown for illustrative purposes.

5.1 Introduction

With the resilience assessment framework defined in Chapter 3 and efficient assessment method described in Chapter 4, we consider a resilience optimization problem within the multi-objective optimization framework in this chapter. In Table 3-1, we have five resilience metrics given as in Eq. 3.3 which is repeated here:

$$\begin{aligned} R_{M_j}(b_j) &= P(M_j > b_j) \\ &= \int_{\Theta} I_j(\Phi, \theta, b_j) p(\theta) d\theta, \end{aligned} \quad (5.1)$$

where I_j is an indicator function given by Table 3-1.

The optimization problem is as follows:

$$\begin{aligned} &\text{optimize} \quad R_{M_j}(b_j) = \int_{\Theta} I_j(\Phi, \theta, b_j) p(\theta) d\theta, \quad j = 1, 2, \dots; \\ &\text{minimize} \quad f_C(\Phi), \\ &\text{subjected to} \quad g_i(\Phi) \leq 0, \quad i = 1, 2, \dots; \\ &\quad \quad \quad h_k(\Phi) = 0, \quad k = 1, 2, \dots; \\ &\quad \quad \quad \Phi \in [\Phi^{(L)}, \Phi^{(U)}], \end{aligned} \quad (5.2)$$

where Φ are design variables, $f_C(\Phi)$ is the cost function, $g_i(\Phi) \leq 0$, $i = 1, 2, \dots$ and $h_k(\Phi) = 0$, $k = 1, 2, \dots$ are the inequality and equality conditions respectively, and $[\Phi^{(L)}, \Phi^{(U)}]$ define the boundaries of the design variables. In summary, our target is to find the optimal design Φ_{opt} that optimize the resilience metrics (which are expressed as the exceedance probabilities), and the cost of the corresponding design. As discussed in Section 2.3.2, we will use the NSGA-II algorithm to find the Pareto front.

5.2 Comparing Surrogate Models Constructed at Different Levels

The Evolutionary Algorithms and other heuristic multi-objective optimization algorithms are capable of finding the Pareto front in one run. However, they often require a huge number of evaluations of the models, which is usually computationally prohibitive for many real-life problems. In the resilience optimization problem in this chapter, the objective functions contain probability integrals. This makes the computation even more challenging. In resilience optimization, we care about the time-history of the system's behaviour. This requires the system performance be a function of time. Correspondingly, in modelling a system, the outputs of a physical model should be time series that represent the system performance during and after the shocks of natural hazards. Suppose we have a system model “ M ” with design variables $\boldsymbol{\phi}$, and random variables $\boldsymbol{\theta}$. If the hazards are also in the form of time series (for example, the earthquake citations in the form of ground acceleration records), let $X_t(\boldsymbol{\theta})$ represent the input time series. Let's represent the output time series of the physical model (for example, the performance curve) as $Y_t(\boldsymbol{\phi}, \boldsymbol{\theta})$. Suppose the NSGA-II algorithm is adopted. The resilience optimization in the multi-objective optimization framework with physical model evaluations is illustrated as the flowchart in Fig. 5-1. This approach is also referred to as “double-loop” approach, where the part at the left side (enveloped in dashed rectangular) in Fig. 5-1 is the inner loop, and the part at the right side is the optimization outer loop. It can be seen that for any given design $\boldsymbol{\phi}$, many loops of evaluations with different random variables $\boldsymbol{\theta}$ are required to calculate the resilience metrics $R_i(\boldsymbol{\phi})$. As the resilience metrics are integrals of the product of the indicator function and a probability distribution function, the computation is not trivial for a single design $\boldsymbol{\phi}$, let alone a large number of designs. Even if Subset Simulation is used instead of brute-force Monte Carlo simulation to estimate the probability integrals in the inner loop, the computation is still very challenging.

To replace the time-consuming evaluation of the physical model, researchers have proposed to use surrogate models instead. We have discussed researches on surrogate model assisted multi-objective optimization in Section 2.3.3. The uniqueness of the resilience optimization problem in this chapter is two folds: (1) we have time series included, this is because the

Figure 5-1: Double loop resilience optimization with physical model.

system performance curve is a function of time and we are interested in resilience indices which are measured on time scale; (2) The objective functions are probability integrals: $R_{M_j}(b_j) = \int_{\Theta} I_j(\phi, \theta, b_j) p(\theta) d\theta$, $j = 1, 2, \dots$

We have several options in building surrogate models at different levels, where by ‘levels’, we refer to the layer of the support points. Fig. 5-2 shows two choices of surrogate models. The first choice is to construct a surrogate model on the input time series and output time series. In this case, the support points of the surrogate model are $(X_t(\theta), Y_t(\theta|\phi))$, i.e. for each time instant, there is a set of support points. This is actually finding a mapping $X_t(\theta) \rightarrow Y_t(\theta|\phi)$ for every design ϕ . To establish such a mapping, a good choice is the Long Short-Term Memory (LSTM) [89], which is a deep learning algorithm that is widely used to forecast time series. We see its application in engineering problems frequently in the past few years. Weg et al. [90] applied the LSTM to mimic the process of fracture response, Zhang et al. [91] constructed a LSTM surrogate model to study the responses of caisson foundations. Most research works adopt LSTM for forecasting and predicting instead of incorporating it with optimization. It is a possible solution for persistent hazards like the heavy rainfall, but may not a good choice for hazards like the earthquake, which happens in a much shorter time than the recovery of the system. Another option is to construct a surrogate at a higher level, i.e., from random variable θ to the key resilience performance functions $M_j(\phi, \theta, b_i)$, where M_j represents a

measure of the j th resilience component as given in Section 3.2.1. This mapping relationship is $\boldsymbol{\theta} \rightarrow M_i(\boldsymbol{\phi}, \boldsymbol{\theta}, b_i)$. This surrogate model avoids building surrogate models on the time series. However, it may require more support points to achieve a good quality as in this way we actually ignore the information hidden the time series. Also, we need to build a separate surrogate for each design $\boldsymbol{\phi}$, this is because we are building surrogate within the double-loop framework. Li and Wang [92] proposed an adaptive hybrid low and high fidelity surrogate model for Reliability-based optimization problem (RBDO), their surrogate works on this level, despite the fact that in their method, instead of building an independent new surrogate model for a new design $\boldsymbol{\phi}^{(\text{new})}$, they update the surrogate model to incorporate this new design.

Figure 5-2: Surrogate models within the framework of double-loop approach.

These two choices of surrogate models share the similar difficulty: the repeated surrogate building (or updating) for each design $\boldsymbol{\phi}$. This is because we are working with a double loop program: the outer loop enumerates design variable $\boldsymbol{\phi}$, the inner loop enumerates every random variable $\boldsymbol{\theta}$. This is usually not an efficient way. We can consider the augmented variable space method proposed by Au [93], where the design variables $\boldsymbol{\phi}$ are treated as random variables $\boldsymbol{\theta}$

artificially . We can evaluate

$$\begin{aligned} R_{M_j}(b_j) &= P(M_j > b_j) \\ &= \int_{\Phi} \int_{\Theta} I_j(\Phi, \theta, b_j) p(\Phi) p(\theta) d\Phi d\theta, \end{aligned} \quad (5.3)$$

where M_j is the performance measurement, Φ is the design variable space, θ is the random variable, Θ is the random variable space, I_j is a indicator function, b_j is the corresponding threshold for the j th resilience metric. We can consider construct surrogates in the augmented variable space. As shown in Fig. 5-3, one of the choices is building a surrogate from the augmented variable (Φ, θ) to system performance measurement $M_i(\Phi, \theta, b_i)$. Such surrogate model is seen in the literature of reliability engineering. Dubourg et al. [94] construct a kriging surrogate model for RBDO problem. Their method belongs to this type of surrogate model, despite their performance measurement is not based on time series. This approach is limited by the number of variables and it requires large number of support points for medium-to-high dimensional problem. We also have another choice: to directly approximate the exceedance probability $R_i(\Phi, \theta, b_i)$, which are probability integrals of the indicator functions over the augmented variables (Φ, θ) . The method by Jensen et al. [63], which we have reviewed in Section 2.3.4, belongs to this type. As we pointed out there, their method is limited to low-to-medium dimensional variable space. Beside, the using of a convex hull may not be the best choice if the samples are distributed in a region not much like a convex one.

5.3 Proposed Multi-objective Resilience Optimization with the assistance of Probability Density Estimation

We strive to find the mapping from design variable Φ to the resilience metrics $R_{M_j}(b_j)$, which is given as:

$$\begin{aligned} R_{M_j}(\Phi, b_j) &= P(M_j > b_j | \Phi) \\ &= \int_{\Theta} I_j(\Phi, \theta, b_j) p(\theta) d\theta, \end{aligned} \quad (5.4)$$

where M_j is the performance measurement, θ is the random variable, Θ is the random variable space, I_j is a indicator function, b_j is the corresponding threshold for the j th resilience metric.

Figure 5-3: Surrogate models in the augmented space.

This is actually the same equation with Eq. 3.3, but in Eq. 3.3, as a resilience analysis problem, the design variable Φ are treated as fixed. The left-hand-side of Eq. 3.3 is equivalent to $P(M_j > b_j | \Phi = \Phi_0)$, for simplicity, we omit the condition $\Phi = \Phi_0$ there.

We optimize resilience metrics R_{M_j} in an ‘off-line’ way: we first construct the mapping $\Phi \rightarrow R_{M_j}(\Phi, b_j)$ for every design Φ , and the MOO algorithms is called to find the Pareto optimal with the support points $\Phi, R_{M_j}(\Phi, b_j)$. In this way, we decoupled the optimization with the evaluation of physical model by a special surrogate model, where the output support points are actually probability integrals as $R_{M_j}(\Phi, b_j) = P(M_j > b_j | \Phi)$.

Now the problem is how we evaluate Eq. 5.4 efficiently for every design Φ under a whole range of random variable Θ . Our target $R_j(\Phi, b_j)$ can be expressed as follows according to Bayes’ Theorem: [95,96]

$$R_{M_j}(\Phi, b_j) = P(M_j > b_j | \Phi) = \frac{p(\Phi | M_j > b_j)P(M_j > b_j)}{p_{\Phi}(\Phi)}. \quad (5.5)$$

$P(M_j > b_j)$ is given by the augmented variable space method [93] as in Eq. 5.3. The denominator in Eq. 5.5 is the PDF of predefined distribution of Φ and know beforehand. The term $p(\Phi | M_j > b_j)$ is a PDF of the design variables Φ conditional on the event $M_j > b_j$. If this PDF is known, the target resilience metrics $R_{M_j}(\Phi, b_j)$ is obtained for all the designs Φ .

Now the problem is how to estimate $p(\boldsymbol{\Phi}|M_j > b_j)$. We can adapt a method called DOPAS (Design Optimization in the PArtitioned Design Space), which was proposed by Liu and Cheung [87], for solving RBDO (Reliability Based Design Optimization) problem. In DOPAS, this PDF $p(\boldsymbol{\Phi}|M_j > b_j)$ is approximated by density estimation over the samples of $\boldsymbol{\Phi}$ which are located in the region $M_j(\boldsymbol{\Phi}) > b_j$.

In this way, the resilience assessment problem is converted to a density estimation problem. It is worth mentioning that, DOPAS [87] and its generalization DOPAS_m [97] are methods aiming at solving reliability-based design optimization (RBDO) problems, where the probabilistic exceedance functions (reliability functions) are in the constraints instead. In RBDO problem, this constraint is written as $p_F(\boldsymbol{\Phi}) \leq [P_F]$ [97], where $[P_F]$ is the required failure probability threshold, $p_F(\boldsymbol{\Phi})$ is the failure probability. In DOPAS method, $\hat{p}_F(\boldsymbol{\Phi})$ is an approximation of $p_F(\boldsymbol{\Phi})$. We call the region that guarantees $\hat{p}_F(\boldsymbol{\Phi}) \leq [P_F]$ a feasible region. In RBDO problem, it requires a high-accuracy in separating the feasible region from the non-feasible region. In other words, as long as the classification of feasible regions versus non-feasible regions is accurate enough, the quality of the optimization result is guaranteed. And it will not be affected by a low accuracy of density estimation in regions that are some distance away from the threshold which separates the feasible region from non-feasible region. In contrast, in our proposed resilience optimization framework, the probabilistic exceedance functions $R_{M_j}(\boldsymbol{\Phi}, b_j)$ are no longer in the constraints, they are our objectives instead. In our resilience optimization framework, the estimation of probabilistic exceedance functions must be accurate enough at all probability levels. This sets a higher requirement on the accuracy than in the RBDO problem. Fig. 5-4 illustrates of resilience optimization with a surrogate model that approximates the resilience metrics directly via a density estimation method.

Figure 5-4: Resilience optimization with a surrogate model built on the resilience metrics.

5.3.1 Iterative Density Estimation with Adaptive Rejection Sampling

In the DOPAS method [87], the density $p(\boldsymbol{\Phi}|M_j > b_j)$ is estimated iteratively. This is because if we estimate the density in one step, for example we use brute-force Monte Carlo simulations, there would be low-probability regions with inadequate samples. The density estimation in these low-probability regions can be very poor. In the DOPAS [87] framework, new samples are generated in the low-probability region with the existing samples as the seeds. Thereafter, various density estimation methods, for example, the kernel density estimation (KDE) [98,99] and its multivariate generalization [100], can be applied to estimate the density function $\hat{p}(\boldsymbol{\Phi}|M_j > b_j)$. In this thesis, we adopt the Bayesian Sequential Partition (BSP) method [101], which is a recently proposed method for multivariate density estimation. This is because BSP method is capable of giving more accurate estimates if the dimension of the sample space is moderate or high [101] and we want to keep the method's capability in solving higher dimensional problems. In order to get i.i.d. samples based on the seeds, the adaptive rejection sampling method using AKSD (Adaptive Kernel Sampling Densities) proposed by Jia et al. [88] is adopted. To reduce the number of evaluations of the physical model, we follow the spirit of the modified Metropolis-Hastings sampling [43] and the modified rejection sampling [88], to evaluate the physical model at the last possible step. This is shown in the pseudo-code Algorithm 6. Unlike the modified Metropolis-Hastings [43] and the modified rejection sampling [88], in this research we need to take care of one more criterion required by the BSP method, i.e., the samples must be located in the corresponding partitions, as in Step 2 of the pseudo-code Algorithm 6.

Algorithm 6: Modified Rejection Sampling with AKSD for DOPAS.

Input: Proposal distribution $\pi(\mathbf{x})$ and constant M by AKSD;
 target (prior) distribution $p(\mathbf{x})$, number of samples to be generated N_{target} ;
 current threshold b ;
 current partition Par

Output: $\mathbf{x}_{\text{new}} \sim p(\mathbf{x})$ && $\mathbf{x}_{\text{new}}$ in Par

```

1 initialization;
2  $cnt = 0$  # reset the counter
3 while  $cnt < N_{\text{target}}$  do
4     # Step 1 generate candidate
5     simulate a pre-candidate  $\mathbf{x}_{\text{pre}} \sim \pi(\mathbf{x})$ 
6     simulate random number  $u \sim U(0, 1)$ 
7     if  $u < \frac{p(\mathbf{x}_{\text{pre}})}{M\pi(\mathbf{x}_{\text{pre}})}$  then
8         | Accept  $\mathbf{x}_{\text{pre}}$  as candidate  $\mathbf{x}_{\text{cand}}$ , go to Step 2
9     else
10        | Reject  $\mathbf{x}_{\text{pre}}$  and return to Step 1
11    end
12    # Step 2 check the candidate in the partition, if so, accept it as effective candidate
    and go to physical model evaluation
13    if  $\mathbf{x}_{\text{cand}} \in Par$  then
14        | Accept  $\mathbf{x}_{\text{cand}}$  as effective candidate  $\mathbf{x}_{\text{eff\_cand}}$ , proceed to Step 3
15    else
16        | Reject  $\mathbf{x}_{\text{cand}}$  and return to Step 1
17    end
18    # Step 3 evaluate the physical model, decide if  $\mathbf{x}_{\text{eff\_cand}}$  is to be accepted as  $\mathbf{x}_{\text{new}}$ 
    Evaluate  $\mathbf{x}_{\text{eff\_cand}}$ , get the model output  $g(\mathbf{x}_{\text{eff\_cand}})$ 
19    if  $g(\mathbf{x}_{\text{eff\_cand}}) > b$  then
20        | Accept  $\mathbf{x}_{\text{eff\_cand}}$  as new sample  $\mathbf{x}_{\text{new}}$ ,  $cnt = cnt + 1$ 
21    else
22        | Reject  $\mathbf{x}_{\text{eff\_cand}}$  and return to step 1
23    end
24 end
25
```

5.4 Multi-objective Resilience Optimization of Urban Drainage Network Pipelines

To illustrate the above method we have described, we adapt the urban drainage system (UDS) model shown in Chapter 3 and 4. In Chapter 3 and 4, the dimension of the pipelines are assumed to be fixed. In this example, we relax the dimension restrictions on some of the pipelines. Our target is to find the optimal dimension of the pipelines that maximize the resilience performance and the cost. We identify two critical pipelines $T92$ and $T91$, to which the up stream nodes connected have the most severe flooding. The dimensions of pipelines $T92$ and $T91$ are treated as the design variables $\boldsymbol{\phi}$. The cross-section of these two pipelines are assumed to be a square. Therefore, we have only 2 design variables. These design variables and their corresponding physical meanings are as follows:

- $\boldsymbol{\phi}_1$: width and height of pipeline $T92$,
- $\boldsymbol{\phi}_2$: width and height of pipeline $T91$.

The bounds for the design variables $\boldsymbol{\phi}$ are $1 \leq \boldsymbol{\phi}_i \leq 3$, for $i = 1, 2$. The cost function is the sum of the cross-sectional areas of the 2 pipelines, as the cross sections of these two pipelines are both of squared shape:

$$f_C(\boldsymbol{\phi}) = \phi_1^2 + \phi_2^2. \quad (5.6)$$

The multi-objective optimization problem is

$$\begin{aligned} & \text{optimize} && R_{M_j}(b_j) = \int_{\boldsymbol{\Theta}} I_j(\boldsymbol{\phi}, \boldsymbol{\theta}, b_j) p(\boldsymbol{\theta}) d\boldsymbol{\theta}, && j = 1, 2; \\ & \text{minimize} && f_C(\boldsymbol{\phi}), && \\ & \text{subjected to} && \boldsymbol{\phi} \in [\boldsymbol{\phi}^{(L)}, \boldsymbol{\phi}^{(U)}]. \end{aligned} \quad (5.7)$$

In this example, from the five resilience metrics proposed in Chapter 3, we select two resilience metrics to optimize. This is for the convenience of visualizing the Pareto front in a 3-dimensional space. The downtime related resilience metric R_{DT} and the worst performance resilience metric $R_{\max \Delta_g}$ are chosen as the objectives. These two resilience metrics, as discussed in Chapter 3, section 3.2.2, are the most critical ones out of the five metrics. The thresholds are set as

$b_3 = 40$, $b_4 = 90$, i.e., we want to minimize the probability that the system downtime is longer than 40 minutes and the probability that the maximum discharge exceeds 90 CMS. It is worth mentioning that these thresholds adopted here are for illustrative purpose. In realistic optimization, the thresholds are up to decision makers. With these two resilience objectives and the objective of cost function as the third objective, we have three optimization objectives in total.

5.4.1 Optimization Results and Discussions

The NSGA-II algorithm is adopted in the MOO. The optimization results are shown in Fig. 5-5. The upper-left subplot in is the pair plot of the optimal design variables with the diagonal showing the histogram of each design variable. The upper-right subplot is the corresponding pair plot of the Pareto front. The lower-left subplot is the parallel plot of the three objectives. To illustrate the trend, some solutions out of the 1500 combinations are plotted in orange color. These solutions are picked so that they are evenly distributed according to objective 3. The plot in the lower right corner is a 3D scatter of the Pareto front. A linear interpolation mesh plot is also shown. It should be noted that the mesh plots are based on the interpolation with the identified optimal points on the Pareto front. It is worth noting that this is to help show the value of the third objective and illustrate the trends of the three objectives on the Pareto front only. In the regions without support points, the mesh plot may not be accurate.

From Fig. 5-5, it can be seen that the objective 3, which is the cost function, competes against the other two objectives. A larger value of objective 3 means larger cross-sectional area, i.e., higher investment. With increasing cost, the corresponding exceedance probability of both objectives 1 (R_{DT}) and 2 ($R_{\max \Delta_g}$) decreases. Since the resilience is defined as the exceedance probability, a smaller exceedance probability is identified as more resilient. From the second column of the upper-right plot (the Pareto front), we see that objectives 1 and 2 show different sensitivities with regard to the cost function. When the cost function (cross-sectional area) increases from the lowest value, objective 2 improves slowly until a threshold where objective 3 reaches 10 m^2 . After this threshold, objective 2 improves more quickly. Similarly, a threshold exists for objective 1: when the cross-sectional area exceeds 15 m^2 , improvements in objective 1 become less significant. In this example, objectives 1 and 2 are not fully competing

with each other, but they have different sensitivity levels with regard to the cost function. To illustrate this point, parallel plots are shown in Fig. 5-6. In this figure, the grey lines are all the combinations on the Pareto front. The left subplot highlights the top 5 design combinations for objective 1. The right subplot highlights the top 5 design combinations for objective 2.

Figure 5-5: Resilience optimization results. The upper-left is the pairplot of design variables; the upper-right is the pairplot of the Pareto fronts; the lower-right is a 3D plot of the Pareto fronts; The lower-left shows the solutions in parallel plot.

The optimization results can help us improve system resilience in the design process. For example, if the decision maker decides to choose a design with cost function $c \approx 12 \text{ m}^2$, multiple candidate designs can be identified. Table 5-1 shows the candidate optimal design where objective 3 is evaluated in the neighbour of 12 m^2 . These design combinations are also shown as blue lines in Fig. 5-7. Consider one of these candidate design combinations, for example, $\phi_1 = 2.4763$, $\phi_2 = 2.4242$, the objectives are estimated as $Obj_1 = 0.0165$, $Obj_2 = 0.0111$, and $Obj_3 = 12.0087$. In realistic design, a rounded number is often preferred. So we may choose $\phi_1 = 2.5 \text{ m}$, $\phi_2 = 2.5 \text{ m}$ as the optimal design. It is computationally prohibitive to verify all the points on the Pareto front, we do a simple verification at picked points using by an efficient simulation method (improved ISS method) illustrated in Chapter 4. Taking this optimal design combination with $\phi_1 = 2.4763$, $\phi_2 = 2.4242$ as an example, the exceedance probabilities

Figure 5-6: Resilience optimization results plotted in parallel plots, the left plot highlights the top 5 designs with regards to Objective 1 only, the right plot highlights the top 5 designs with regards to Objective 2 only.

calculated by the improved ISS method (with initial sample size at level 0 being 5000) are: $Obj_1 = 0.0136$, $Obj_2 = 0.0105$. Similarly, several selected designs are verified. These selected designs are picked at different levels of objective 3, evaluated in the neighbourhood of 4, 8, 12, and 16 m², respectively. The parallel plots of these selected designs are shown in Fig. 5-7 as red lines. The verification results are given in Table 5-2 compared to benchmark exceedance probabilities given by the ISS method with a sample size of 5000 at level 0.

Table 5-1: Candidate design combinations where objective 3 is evaluated in the neighbourhood of 12 m², while design variables ϕ_1 and ϕ_2 are rounded to four decimal places.

design number	ϕ_1	ϕ_2	objective 1	objective 2	objective 3
1	2.4789	2.4385	0.0164	0.0110	12.09
2	2.4776	2.4311	0.0164	0.0111	12.05
3	2.1850	2.6966	0.0170	0.0111	12.05
4	2.4763	2.4242	0.0165	0.0111	12.01
5	2.3559	2.5199	0.0165	0.0112	11.90
6	2.5769	2.3171	0.0173	0.0114	12.01
7	2.1810	2.7055	0.0171	0.0111	12.08

Figure 5-7: Design optimization in the framework of multi-objective resilience optimization, visualized as a parallel plot. In this figure, blue lines show candidate designs where objective 3 is evaluated in the neighbourhood of 12 m², red lines show the selected points for verification at different values of objective 3.

Table 5-2: Verification at selected design combinations.

ϕ_1	ϕ_2	Obj_3	Obj_1	Obj_1 benchmark	Obj_2	Obj_2 benchmark
1.6282	1.1712	4.0229	0.0643	0.0777	0.0187	0.0165
2.1365	1.8530	7.9982	0.0307	0.0201	0.0168	0.0128
2.4763	2.4242	12.0087	0.0165	0.0136	0.0111	0.0105
2.7932	2.8628	15.9973	0.0129	0.0141	0.0054	0.0049

5.4.1.1 Computational Efficiency

In this example, the initial sample sizes for both objectives are set as 2500. The evaluations of the physical model come from two steps: (1) generate enough samples by Subset Simulation such that the performance (for example, the downtime or maximum performance drop) of these samples exceeds the corresponding thresholds ([40, 90] in this example); (2) partition the design variable space into high-probability regions and low-probability regions, generate additional samples in the low-probability regions with the adaptive rejection sampling method with AKSD [88]. We keep iterating this process until the stopping criteria are met. These criteria are defined as either the expected minimum exceedance probability level is reached, or the predefined maximum number of iterations is reached. The process can also be stopped if the BSP algorithm fails to find a new low-probability region. This happens when the region being searched is almost uniform at a certain iteration.

The total number of evaluations of the physical model in the whole optimization is given in Table 5-3.

Table 5-3: A summary of the number of evaluations of the physical model.

objective #	subset simulation	iteration 1	iteration 2	iteration 3	iteration 4	total
obj 1	7 000	6 000	8 000	6 500	7 500	35 000
obj 2	5 500	5 500	8 000	7 500	12 500	39 000
total number of evaluations						74 000

5.5 Conclusions

In this chapter, we study resilience optimization within the multi-objective optimization framework. To enable a computationally tractable optimization process, surrogate model assisted multi-objective optimization method is adopted. Surrogate models can be constructed at different levels. The different levels of surrogate model are discussed and compared. In the resilience optimization framework proposed in this chapter, the special part of the problem is that the resilience objectives are expressed as the exceedance probabilities, which requires computing the probability integrals. To calculate the resilience objectives by stochastic simulation methods, a huge number of evaluations are required. In addition, the calculation of the CCDF requires an accurate estimation on the rare tail parts. All these factors make surrogate models constructed on the physical model itself not applicable for the multi-objective resilience optimization problem. Therefore, we directly approximate the exceedance probabilities. The exceedance probabilities are approximated by estimating the probability density functions of the correspondence samples. The DOPAS method [87] is adapted for constructing the surrogate model in this multi-objective resilience optimization problem. A modification to the sampling part of the original DOPAS is made, where the modified rejection sampling method with AKSD [88] is incorporated for generating samples. An example of a realistic drainage system optimization problem is shown as an illustration of the multi-objective resilience optimization framework.

With an illustrative example, as expected, we found that the cost objective is contradict with the exceedance probability of the two resilience metrics: the downtime and maximum performance drop. We identified the different sensitivities of the resilience objectives on the cost function. The proposed method enables the optimization of exceedance probabilities under a range of levels of economic investigations. Decision makers can see the trade-offs between cost and the system resilience performance under uncertainties. In this example case, we set a group of threshold of $[40, 90]$. However, we can actually consider multiple groups of thresholds in one run with the same spirit of ISS method. We can do this by adding more objective functions corresponding to other thresholds. This will address the case when decision makers are not sure which threshold they want to see or they are interesting in multiple threshold levels.

The accuracy of the optimization result are verified by several selected points equidistantly distributed over objective 3. The quality of the approximation of exceedance probabilities are highly depend on the effectiveness of the sampling method and density estimation. While the adapted the AKSD method show good performance in generate i.i.d. samples and the Bayesian Sequential Partition method explores the feasible spaces well in this example, studies on the accuracy of the approximation at every exceedance performance level is to be done in our near future research plan. Following that, we will further apply the method to higher dimensional cases and considering multiple thresholds at the same time.

Chapter 6

Efficient Bayesian Updating of Multiple CCDF Curves

The resilience assessment method and resilience optimization illustrated in the previous chapters have a pre-requisite: a reliable and high-quality system model. If the model fails to emulate the system behaviour, the resilience analysis and optimization results can be biased or even yield wrong results. While no model is perfect, in this chapter, we develop an efficient Bayesian model updating method which enables the system model to be updated given observed data. In this way, we may rectify the model to the maximum extent based on the available observations. In this chapter, Bayesian model updating is performed via structural reliability methods. A novel Bayesian updating framework is proposed and tested to be efficient compared with existing method.

6.1 Introduction

This Chapter answers the last question identified in Section 1.2: how do we systematically improve and update our knowledge about the system? System parameters may bias from their designed values, and the system may subjected to performance deterioration due to aging or partially broken by hazards. Therefore, we need to update our knowledge about the system. Bayesian model updating provides such an approach.

There is a need in both model calibration for more accurate resilience analysis and the research topic of failure probability curve updating in reliability research. The failure probability curve is a CCDF curve describing the failure probability of a system with regards to a series of thresholds. There are similarities between the resilience metrics in Chapter 3 and the failure probability curves: both are in the form of CCDF curves and related to the system performance. However, the resilience metrics in Chapter 3 are different from failure probability curves in that the system performance in term of time series are analysed. In contrast, the failure probability curve is usually estimated by the extreme value of the system performance. A system with multiple system performances can have multiple limit state functions, and multiple failure probability curves can be estimated. In the resilience assessment method proposed in Chapter 3, there are also multiple CCDF curves. Due to these similarities, in this chapter, we show the system model updating in the background of failure probability curves updating topic with application examples on structural analysis.

For Bayesian updating of multiple CCDF curves based on measured system response data collected, ISS is not directly applicable as it requires the probability distribution of the underlying random variables to be in the form that allows direct sampling in level 0 and effective sampling from $p(\boldsymbol{\theta}|F_i)$ (as in Chapter 4) in subsequent levels i . In this Chapter, a new approach based on an extension of the ISS is proposed, so that multiple failure probability curves can be simultaneously updated considering all the performance functions in one run of exceedance probability analysis. The remaining parts of this Chapter are arranged as follows. In Section 6.2, we illustrate the multiple CCDF curves updating framework. In Section 6.3 a new efficient approach for Bayesian updating is proposed. In Section 6.4, the proposed method is illustrated in two examples, highlighting the method's effectiveness and efficiency. Finally, this chapter is summarized in Section 6.5.

6.2 Multiple CCDF Curves Updating Framework

Many nonlinear structural dynamic systems have multiple 'failure' states. Correspondingly, multiple performance functions can be identified. Suppose there are R performance functions $g_j(\boldsymbol{\theta})$, $j = 1, \dots, R$, where the random variables vector $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbb{R}^{N_{\boldsymbol{\theta}}}$ has the model input random variables or uncertain parameters as its components. $\boldsymbol{\Theta}$ is the space of $\boldsymbol{\theta}$, $N_{\boldsymbol{\theta}}$ represents

for the dimension of random variables $\boldsymbol{\theta}$. The failure probability curves corresponding to each failure state F_j with the corresponding failure threshold b_j are given by

$$P_{F_j}(b_j) = P(g_j(\boldsymbol{\theta}) > b_j) = \int_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} I_{F_j}(\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}, \quad (6.1)$$

where $I_{F_j}(\boldsymbol{\theta})$ is the indicator function of the j -th failure state. $p(\boldsymbol{\theta})$ is the joint distribution which the random variables follow. It is given as $I_{F_j}(\boldsymbol{\theta}) = 1$ if $g_j(\boldsymbol{\theta}) > b_j$ and 0 otherwise. If $p(\boldsymbol{\theta})$ is in the form that allows direct sampling, stochastic simulation-based methods can be applied to estimate Eq. 6.1. Brute-force Monte Carlo simulation (MCS) is capable of solving this problem, but as we discussed in Section 2.1.3, its efficiency is of concern in simulating rare event. Some more efficient methods exist, such as importance sampling [102, 103], ISEE [102] and its variant, Subset Simulation (SS) [43]. However, these efficient methods are for estimating the failure probability of one performance function. When there are R performance functions, it is required to apply these methods R times to estimate all the failure probability curves. To avoid repetitive applications, parallel Subset Simulation (pSS) was proposed by Hsu and Ching [52]. By introducing a “principle variable” that correlates all the performance functions, the pSS method enables the estimation of all the failure probabilities corresponding to each system performance function over a single time run of the Subset Simulation. The drawback of the pSS method is that it requires the performance functions to be sufficiently linearly correlated. Bansal and Cheung [18] proposed a stochastic simulation-based method, referred to as ISS here. The ISS method is a generalization of the Subset Simulation (SS) [43]. It is an improvement of the Generalized Subset Simulation (GSS) method proposed by Li et al. [54], which is another approach derived from the SS method. The novelty of ISS lies in the selection and propagation of samples in the intermediate levels. ISS can evaluate the multiple exceedance probability curves simultaneously in a single run and is efficient in problems with small failure probabilities without requiring the correlation condition between the different performance functions. The ISS method is robust to the number of random variables, model complexity, the number of performance functions, and the degree of nonlinearity, and we have solved its discontinuity problem in Chapter 4. However, the original ISS method and our improved ISS method is for exceedance probabilities estimation, our question is how we update the exceedance probabilities with measured system response data D , (for example, the acceleration data of a

building structure recorded by accelerometers during an earthquake).

As we have discussed in Section 2.4, the multiple failure probability curves of the system can be updated for $j = 1, \dots, R$ as follows (here we omit the model assumption $\mathcal{M}$ compared with Eq. 2.57):

$$P_{F_j|D}(b_j) = P(g_j(\boldsymbol{\theta}) > b_j|D) = \int_{\boldsymbol{\theta} \in \Theta} I_{F_j}(\boldsymbol{\theta}) p(\boldsymbol{\theta}|D) d(\boldsymbol{\theta}), \quad (6.2)$$

where by Bayes' Theorem, the posterior PDF $p(\boldsymbol{\theta}|D)$ of $\boldsymbol{\theta}$ is equal to the product of the likelihood function $p(D|\boldsymbol{\theta})$ and the prior PDF $p(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$, and normalized by $p(D)$, which is the evidence [68, 104] :

$$p(\boldsymbol{\theta}|D) = \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(D)}. \quad (6.3)$$

We have reviewed several Bayesian model updating methods to obtain the posterior PDF $p(\boldsymbol{\theta}|D)$ and the updated exceedance probability $P_{F_j|D}(b_j)$ in Section 2.4. Each method has some limitations and may not be suitable for this multiple CCDF problem as they are aiming at problems with single performance functions. ISS method is efficient in estimating Eq. 6.1, but it is not directly applicable to estimate Eq. 6.2 as the posterior PDF $p(\boldsymbol{\theta}|D)$ cannot be directly sampled. In this chapter, we propose an extension to ISS, so that the method can update multiple failure probability curves by simultaneously considering all performance functions in one run.

6.3 Proposed Efficient Multiple CCDF Curves Updating Method

As an extension to the formulation proposed by Bansal and Cheung [18], when the observation dynamic response data D is available, the posterior (updated) failure probabilities can be written as:

$$P(F_{m,j}|D) = P(F_{m,j}|F_{m-1}, D) \prod_{i=2}^{m-1} P(F_i|F_{i-1}, D)P(F_1|D), \quad (6.4)$$

where $F_1 \supset F_2 \supset \dots \supset F_{m-1} \supset F_{m,j}$ is a sequence of failure events conditioned on data D . The region F_i is the union of each failure region $F_{i,j}$:

$$F_i = F_{i,1} \cup F_{i,2} \cup \dots \cup F_{i,R}, \quad (6.5)$$

where $F_{i,j}$ is the failure region in terms of the performance function j in the level i , and R represents total number of performance functions.

6.3.1 Level 0 Simulation

At level 0 simulation, in ISS method by Bansal and Cheung [18], the samples follow some pre-defined distribution $p(\boldsymbol{\theta})$. In the proposed method, samples for $\boldsymbol{\theta}$ are required to be distributed as $p(\boldsymbol{\theta}|D)$. Suppose we have N_0 samples. We evaluate the performance functions of these N_0 samples $\boldsymbol{\theta}^{(k)}$ (where $k = 1, 2, \dots, N_0$). For each performance function, the following vector $\mathbf{G}_{0,j}$, $j = 1, 2, \dots, R$, can be obtained:

$$\mathbf{G}_{0,j} = \left[g_j(\boldsymbol{\theta}^{(1)}), g_j(\boldsymbol{\theta}^{(2)}), \dots, g_j(\boldsymbol{\theta}^{(N_0)}) \right]^T, \quad (6.6)$$

where g_j is the j -th performance function. It is assumed that for a given $\boldsymbol{\theta}^{(k)}$, the R performance functions output vectors $g_j(\boldsymbol{\theta}^{(k)})$, $k = 1, 2, \dots, N_0$ can be calculated in a single dynamic analysis. Let $\mathbf{S}_{0,j} = \text{sort}(\mathbf{G}_{0,j})$ denote the vector obtained by sorting the elements of $\mathbf{S}_{0,j}$ in ascending order. Let $s_{0,j}^{(n)}$ be the n -th element of vector $\mathbf{S}_{0,j}$, $n = 1, 2, \dots, N_0$. The intermediate threshold with regard to the j -th performance function is determined as $b_{0,j} = s_{0,j}^{(N_0 - p_0 N_0)}$. The intermediate threshold refers to the special cross-level threshold where the last subset simulation level ends and the next subset simulation level starts. This threshold is chosen so that there are $p_0 N_0$ samples satisfying $g_j(\boldsymbol{\theta}) > b_{0,j}$. As the elements in $\mathbf{S}_{0,j}$ are sorted in ascending order, we have

$$\begin{aligned} s_{0,j}^{(1)} &\leq s_{0,j}^{(\iota)} \leq s_{0,j}^{(N_0 - p_0 N_0)} = b_{0,j}, \\ 1 - \frac{1}{N_0} &\geq P(g_j(\boldsymbol{\theta}) > s_{0,j}^{(\iota)} | D) = 1 - \frac{\iota}{N_0} \geq p_0, \end{aligned} \quad (6.7)$$

where $\iota = 1, \dots, N_0 - p_0 N_0$. Eq. 6.7 gives the section of the failure probability curves from $P(g_j(\boldsymbol{\theta}) > s_{0,j}^{(N_0 - p_0 N_0)} | D) = p_0$, for $j = 1, \dots, R$. With all the R thresholds $b_{0,j}$ and performance function vectors $\mathbf{S}_{0,j}$ determined above, the failure probability at the cross-level threshold $P(F_1|D)$ can be readily estimated by:

$$P(F_1|D) \approx \hat{P}(F_1|D) = \frac{1}{N_0} \sum_{k=1}^{N_0} I_{F_1}(\boldsymbol{\theta}^{(k)}), \quad (6.8)$$

where I_{F_1} is an indicator function as follows:

$$I_{F_1}(\boldsymbol{\theta}) = \begin{cases} 1, & \text{if } \boldsymbol{\theta} \in F_1; \\ 0, & \text{otherwise,} \end{cases} \quad (6.9)$$

where $F_1 = \cup_{j=1}^R F_{1,j}$ and $F_{1,j}$ is the failure domain determined by the intermediate cross-level thresholds $b_{0,j}$, i.e., $F_1 = \{\boldsymbol{\theta} : g_j(\boldsymbol{\theta}) > b_{0,j}\}$.

6.3.2 Subsequent Levels ($i \geq 1$)

From level $(i-1)$, a total number of $N_{i-1}P(F_i|F_{i-1}, D)$ samples that are distributed as $p(\boldsymbol{\theta}|D)$ can be obtained where $N_0P(F_1|F_0, D) = N_0P(F_1|D)$. These samples are used as seeds to generate $N_i - N_{i-1}P(F_i|F_{i-1}, D)$ additional samples based on the method presented in Section 6.3.4. These samples should be distributed as $p(\boldsymbol{\theta}|F_i, D)$. There should be N_i samples following $p(\boldsymbol{\theta}|F_i, D)$ in the level i . The determination of the value of N_i is discussed in Section 6.3.3. During the process of obtaining a total number of N_i samples $\boldsymbol{\theta}^{(k)} \sim p(\boldsymbol{\theta}|F_i, D)$, $k = 1, 2, \dots, N_i$, the corresponding performance functions $\mathbf{G}_{i,j} = [g_j(\boldsymbol{\theta}^{(1)}), g_j(\boldsymbol{\theta}^{(2)}), \dots, g_j(\boldsymbol{\theta}^{(N_i)})]^T$, $j = 1, \dots, R$, have already been evaluated. Let $\mathbf{S}_{i,j}$ denote the vector obtained by sorting those $N_{i,j}$ elements which are larger than $b_{i-1,j}$ in $\mathbf{G}_{i,j}$ in ascending order. Let $s_{i,j}^{(n)}$ be the n -th element of vector $\mathbf{S}_{i,j}$, thus $n = 1, 2, \dots, N_{i,j}$. The intermediate cross-level failure threshold is $b_{i,j} = s_{i,j}^{(N_{i,j}-p_0N_{i,j})}$. This value is chosen so that there will be a $p_0N_{i,j}$ samples satisfying $g_j(\boldsymbol{\theta}) > b_{i,j}$ and $P(F_{i,j}|F_{i-1,j}, D) = p_0$, where $P(F_{1,j}|F_0,j, D) = P(F_{1,j}, D)$. Based on these samples, this section of the failure probability curves can be calculated as follows:

$$\begin{aligned} s_{i,j}^{(1)} &\leq s_{i,j}^{(\iota)} \leq s_{i,j}^{(N_{i,j}-p_0N_{i,j})} = b_{i,j}, \\ \frac{N_{i,j}-1}{N_i} \hat{P}(F_i|D) &\geq P(g_j(\boldsymbol{\theta}) > s_{i,j}^{(\iota)}|D) = \frac{N_{i,j}-\iota}{N_i} \hat{P}(F_i|D) \geq p_0 \frac{N_{i,j}}{N_i} \hat{P}(F_i|D), \end{aligned} \quad (6.10)$$

where $\iota = 1, 2, \dots, (N_{i,j} - p_0N_{i,j})$ and since $P(F_{i,j}|F_i, D) = N_{i,j}/N_i$, the failure probabilities $P(g_j(\boldsymbol{\theta}) > s_{i,j}^{(\iota)}|D)$ can be estimated as

$$\frac{N_{i,j}-\iota}{N_{i,j}} \times \hat{P}(F_{i,j}|F_j, D) \times \hat{P}(F_i|D) = \frac{N_{i,j}-\iota}{N_i} \times \hat{P}(F_i|D). \quad (6.11)$$

With all the R thresholds $b_{i,j}$ and performance function vectors $\mathbf{S}_{i,j}$ determined above, the conditional failure probability at the cross-level threshold $P(F_{i+1}|F_i, D)$ can be readily estimated by:

$$P(F_{i+1}|F_i, D) \approx \hat{P}(F_{i+1}|F_i, D) = \frac{1}{N_i} \sum_{k=1}^{N_i} I_{F_{i+1}}(\boldsymbol{\theta}_i^{(k)}), \quad (6.12)$$

where $I_{F_{i+1}}$ is an indicator function as follows:

$$I_{F_{i+1}}(\boldsymbol{\theta}) = \begin{cases} 1, & \text{if } \boldsymbol{\theta} \in F_{i+1} \text{ \& } \boldsymbol{\theta} \in F_D; \\ 0, & \text{otherwise,} \end{cases} \quad (6.13)$$

where $F_{i+1} = \cup_{j=1}^R F_{i+1,j}$ and $F_{i+1,j}$ is the failure domain determined by the intermediate cross-level thresholds $b_{i,j}$, i.e., $F_{i+1,j} = \{\boldsymbol{\theta} : g_j(\boldsymbol{\theta}) > b_{i,j}\}$. With the above, $P(F_{i+1}) = P(F_{i+1}|F_i)P(F_i) \approx \hat{P}(F_{i+1}|F_i)\hat{P}(F_i)$.

6.3.3 The Determination of N_i

At the i th level, there are N_i samples following $p(\boldsymbol{\theta}|F_i)$. This number N_i is adaptively chosen based on the following principles: For the performance function j , where $j = 1, 2, \dots, R$, there should be approximately N_0 samples following $p(\boldsymbol{\theta}|F_{i,j})$:

$$P(F_{i,j}|F_i, D) = \frac{P(F_{i,j}|D)}{P(F_i|D)} \approx \frac{N_0}{N_i}, \forall j = 1, \dots, R. \quad (6.14)$$

Thus, N_i can be determined by the estimation of $P(F_{i,j}|D)$ and $P(F_i|D)$. Note that for each performance function j :

$$P(F_{i,j}|D) = P(F_{i,j}) \times P(F_{i-1,j,D}|F_{i-2,j,D}) \times \dots \times P(F_{1,j}, D), \quad (6.15)$$

$P(F_{i,j}|F_{i-1,j}, D), i = 1, \dots, m-1$ is set to be p_0 for each performance function. Therefore, $\hat{P}(F_{i,j}|D) = p_0^i$. Thus,

$$N_i \approx N_0 \frac{\hat{P}(F_i|D)}{\hat{P}(F_{i,j}|D)} = N_0 \frac{\hat{P}(F_i|D)}{p_0^i}, \quad (6.16)$$

where $\hat{P}(F_i|D)$ can be estimated by the method presented in section 6.3.2

6.3.4 Simulation of the Posterior Samples $\boldsymbol{\theta} \sim p(\boldsymbol{\theta}|D)$ and “Failure” Samples

$$\boldsymbol{\theta} \sim p(\boldsymbol{\theta}|F_i, D)$$

In this section, the method to generate the initial samples in level 0 with $\boldsymbol{\theta}$ following $p(\boldsymbol{\theta}|D)$ and the samples in the other levels with $\boldsymbol{\theta}$ following $p(\boldsymbol{\theta}|F_i, D)$ is discussed. The original Accept-Reject method by Neumann [105] for simulating samples following any target distribution is a classical and robust algorithm [41]. Our target distribution is the posterior distribution $p(\boldsymbol{\theta}|D)$ and let's denote $\pi(\boldsymbol{\theta})$ as the proposal distribution. If $\boldsymbol{\theta}$ is simulated from $\pi(\boldsymbol{\theta})$ and u is simulated from a uniform distribution $p(u)$ between 0 and 1, $\boldsymbol{\theta}^{(i)}$ is accepted as a sample distributed as $p(\boldsymbol{\theta}|D)$ if $u < r(\boldsymbol{\theta}^{(i)})$. Otherwise, $\boldsymbol{\theta}^{(i)}$ is rejected. The acceptance ratio $r(\boldsymbol{\theta})$ is given by:

$$r(\boldsymbol{\theta}) = \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M\pi(\boldsymbol{\theta})}, \quad (6.17)$$

where M is the upper bound for the ratio

$$\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\pi(\boldsymbol{\theta})}. \quad (6.18)$$

For Bayesian updating problem involving dynamic data, M is unknown a priori and the log likelihood function is often considered instead of the likelihood function to avoid the underflow problem. Thus, $u < r(\boldsymbol{\theta})$ can be written in terms of the log likelihood function, log prior PDF and $\log \pi(\boldsymbol{\theta})$ as follows:

$$y(\boldsymbol{\theta}, u) = \ln p(D|\boldsymbol{\theta}) + \ln p(\boldsymbol{\theta}) - \ln \pi(\boldsymbol{\theta}) - \ln(u) > \ln(M). \quad (6.19)$$

From the above, it can be seen that $\boldsymbol{\theta}$ following the posterior distribution $p(\boldsymbol{\theta}|D)$ is equivalent to $\boldsymbol{\theta}$ following $\pi(\boldsymbol{\theta}|F_D)$ (i.e., $\pi(\boldsymbol{\theta}|F_D) = p(\boldsymbol{\theta}|D)$) which is the marginal distribution of $\pi(\boldsymbol{\theta}, u|F_D) \propto \pi(\boldsymbol{\theta})p(u)I_{F_D}(\boldsymbol{\theta}, u)$ with $F_D = (\boldsymbol{\theta}, u) : y(\boldsymbol{\theta}, u) > \ln(M)$. Subset simulation can be used here to solve the failure probability estimation problem $P(y(\boldsymbol{\theta}, u) > \ln(M^*))$ to find $\ln(M_1), \ln(M_2), \dots, \ln(M_i)$ where $M_1 < M_2 < \dots < M_i$. The series of M_i values, where $i = 1, 2, \dots$ is the thresholds at the i th level of Subset Simulation. The question is when to stop, i.e., to decide when the value M^* reaches the maximum ratio M and how much time it takes will depend on $P(y(\boldsymbol{\theta}, u) > \ln(M))$ which can be shown to depend on the evidence $P(D)$ and M as

follows.

$$\begin{aligned}
 P(y(\boldsymbol{\theta}, u) > \ln(M^*)) &= P\left(\ln \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{u\pi(\boldsymbol{\theta})} > \ln(M^*)\right) \\
 &= P\left(u < \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})}\right) \\
 &= P\left(u < \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \mid \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \leq 1\right) P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \leq 1\right) \\
 &\quad + P\left(u < \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \mid \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} > 1\right) P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} > 1\right) \\
 &= \left[\int_{\boldsymbol{\Theta}} \frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \pi(\boldsymbol{\theta}) d(\boldsymbol{\theta})\right] P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \leq 1\right) + P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} > 1\right) \\
 &= \frac{\int_{\boldsymbol{\Theta}} p(D|\boldsymbol{\theta})p(\boldsymbol{\theta}) d(\boldsymbol{\theta})}{M^*} P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \leq 1\right) + P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} > 1\right) \\
 &= \begin{cases} \frac{p(D)}{M^*} P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} \leq 1\right) + P\left(\frac{p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})}{M^*\pi(\boldsymbol{\theta})} > 1\right) & \text{if } M^* < M \\ \frac{p(D)}{M^*} & \text{if } M^* \geq M \end{cases}
 \end{aligned} \tag{6.20}$$

From the above, for $M^* \geq M$, by taking the logarithm, we have

$$\ln P(y(\boldsymbol{\theta}, u) > \ln(M^*)) = \ln p(D) - \ln(M^*). \tag{6.21}$$

The novel work of DiazDelaO et al. [53] for Bayesian model updating and model class selection can be viewed as a special case of the method presented here for obtaining the posterior samples following $p(\boldsymbol{\theta}|D)$ with $\pi(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$. Similar to the stopping criteria proposed by DiazDelaO et al. [53], when the $\ln P(y(\boldsymbol{\theta}, u) > \ln(M^*))$ against $\ln(M^*)$ plot reaches the slope of -1 , $\ln(M^*)$ reaches $\ln(M)$ and the $\boldsymbol{\theta}$ samples follow $\pi(\boldsymbol{\theta}|F_D)$ and thus follow the posterior distribution $p(\boldsymbol{\theta}|D)$.

In the original Accept-Reject method, the support of $\pi(\boldsymbol{\theta})$ should cover the support of the target distribution, i.e., $p(\boldsymbol{\theta}|D)$ in this chapter. To satisfy this, an easy choice of the proposal distribution would be the prior distribution, i.e., $\pi(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$. The problem with such a selection is the poor efficiency in a lot of applications (such as the examples presented in Section 6.4). From Eqs. 6.17 and 6.18, the more $\pi(\boldsymbol{\theta})$ resembles the posterior distribution ($p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})$), the more the acceptance ratio $r(\boldsymbol{\theta})$ will be close to 1, and the value of M is smaller.

Consider Eq. 6.20, a smaller M leads to a larger $\ln P(y(\boldsymbol{\theta}, u) > \ln(M))$. In the perspective of Subset Simulation, the ‘task’ of finding M is a less rare event. The number of simulation levels and samples in the Subset Simulation can thus be reduced. Even though the construction of such $\pi(\boldsymbol{\theta})$ often requires the additional dynamic analyses, it is still worth the effort as the overall number of dynamic analyses required for the whole process is reduced as shown in the illustrative examples. While there are unlimited choices of the proposal distribution, for illustrative purpose, a simple version of the Laplace asymptotic approximation assisted proposal distribution is considered.

Two choices of the proposal distribution are compared. One is taking the prior as the proposal, denote as π_0 , where $\pi_0(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$. The other choice is called π_1 here which is obtained using Laplace’s asymptotic approximation [68]. $\boldsymbol{\theta}$ are the model parameters transformed from the original model parameters through Rosenblatt and Nataf transformations [98, 106]. This is for the convenience of proposing samples in MCMC method. In π_1 , those model parameters that are identifiable are assumed to follow a Gaussian distribution with mean equal to one of the peaks of $\ln p(D|\boldsymbol{\theta}) + \ln(\boldsymbol{\theta})$, and the covariance matrix is the inverse of Hessian matrix of the function $-\ln p(D|\boldsymbol{\theta}) - \ln(\boldsymbol{\theta})$ multiplied by a factor larger than 1 to ensure π_1 covering the support of the posterior distribution. In π_1 , those model parameters that are unidentifiable are assumed to follow their prior distribution.

The samples following $p(\boldsymbol{\theta}|F_i, D)$ can be obtained by simulating samples following $\pi(\boldsymbol{\theta}|F_D, F_i)$ which is the marginal distribution of $\pi(\boldsymbol{\theta}|F_D, F_i) \propto \pi(\boldsymbol{\theta})p(u)I_{F_D}(\boldsymbol{\theta}, u)I_{F_i}(\boldsymbol{\theta})$. In this chapter, the modified Metropolis-Hastings approach [43] is used to generate the samples which satisfy both $\{\boldsymbol{\theta}, u\} \in F_D$ and $\boldsymbol{\theta} \in F_i$. Other MCMC methods and other sampling methods presented in the literature can also be used.

6.3.5 A Visualization of the Multiple CCDF Curves Updating Framework

In this section, we show a schematic illustration of the exceedance probabilities estimation process with ISS method, as a summary of the multiple failure curves updating framework. The process is illustrated by a toy problem with one system variable θ_s and one random environmental variable θ_e , u is a random variable introduced in the Acceptance-Rejection sampling framework. The prior distribution of θ_s is standard normal. θ_e is a random variable also follows

standard normal distribution. We assume a data size of $N = 300$. The synthetic data y is analytically expressed as $y_i = 2a_i + 5 + \epsilon$, $i = 1, 2, \dots, N$, where a_i is some parameter in the likelihood function, we can adopt $\mathbf{a} : \{a_i\}, i = 1, 2, \dots, N$ as a series of random integers. ϵ is standard Gaussian noise as a mimic to the measurement error. The likelihood function is as follows:

$$P(D|\theta_s) = \frac{1}{(2\pi)^{\frac{N}{2}}} \exp\left(-\frac{1}{2} \sum_{i=1}^N (a_i \theta_s - y_i)\right). \quad (6.22)$$

The updated posterior samples of θ_s accumulated between the range $[2.023, 2.0265]$. The prior and posterior samples of θ_s is visualized in Fig. 6-1. As this toy problem is very simple, we adopt the π_0 proposal, i.e. the proposal distribution is the prior distribution (standard normal, as shown in the left-hand size plot in Fig. 6-1). Two performance functions are considered in this simple example, both limit state functions are assumed to be quadratic functions of system variable θ_s . Then by the multiple failure curves estimation method, these samples propagate to the rarer domains from one level to another, with the driving variable being the union of both limit state function 1 and limit state function 2. As shown in Fig. 6-2, as the samples moving from one level to another, they are always enclosed under the posterior boundary (i.e., posterior samples), which is ensured by the dropping the samples located outside the boundary.

Figure 6-1: A visualization of the prior and posterior samples of θ_s .

Figure 6-2: An illustration of the process of multiple failure probability curves estimation with posterior boundary. The “target” is the analytical boundary of posterior samples.

6.4 Illustrative Examples

6.4.1 Example 1, 4-DOF Nonlinear Degradation Model

A 4-storey building subjected to earthquake excitations is considered here. The building is modelled by a 4-DOF nonlinear degradation shear building model modified from the one presented in Ching et al. [107]. Fig. 6-3 shows the building model. Each floor corresponds to one degree of freedom. This model is called simplified time varying nonlinear degradation model according to Ching et al. [107]. Degradation refers to the behavior that the stiffness decreases with the deformation, and the damping increases with the deformation. To consider the degradation, the stiffness and the damping coefficient of the viscous damper depend on the drift ratio as follows [107]:

$$k_{j,t} = k_{j,0} \cdot e^{-\gamma \mu_{j,t}^\rho}, \quad (6.23)$$

$$c_{j,t} = \delta + \lambda \mu_{j,t}, \quad (6.24)$$

where $k_{j,t}$ is the stiffness of storey j at time t , $k_{0,t}$ is the initial stiffness of storey i at time $t = 0$. $c_{j,t}$ is the damping coefficient of the viscous damper for storey j at time t , λ and ρ are stiffness related parameters, δ and λ are damping related parameters. $\mu_{j,t}$ is the maximum inter-storey drift ratio of storey i considering the drift ratio from the start up to time t :

$$\begin{aligned} \mu_{1,t} &= \max_{0 \leq \tau \leq t} (|x_{1,\tau}|/H_1), \\ \mu_{j,t} &= \max_{0 \leq \tau \leq t} (|x_{j,\tau} - x_{j,\tau-1}|/H_j), \end{aligned} \quad (6.25)$$

where H_j is the height of storey j and $x_{j,\tau}$ is the displacement of the lumped mass of storey j at time τ of the shear building model. Fig. 6-3 shows the building model. In this example, four performance functions are considered. Each performance function corresponds to one of the four maximum inter-storey drift ratios. The failure probability curve of the j th performance function is defined as:

$$P_{F_j|D}(b_j) = P(g_j(\boldsymbol{\theta}) = \mu_{j,T}(\boldsymbol{\theta}) > b_j|D), \quad j = 1, \dots, 4. \quad (6.26)$$

Figure 6-3: The 4-storey building model with 4 degrees of freedom.

In this example, the lumped mass, ρ and δ are assumed to be known with $\rho = 0.4$, $\delta = 0.4$ MN/m. The initial stiffness $k_{j,0}$, λ , and γ are parameters to be updated. The prior distribution of $k_{j,0}$ is assumed to follow a lognormal distribution with mean 5.19 and standard deviation 0.5 for $\ln k_{j,0}$. The prior distribution of γ and λ is assumed to be normally distributed with a mean of 5 kN and 50 kNs/m, respectively. The prior cov of γ and λ are both 20%. A summary of the parameters is given in Table 6-1.

Table 6-1: The prior distribution of model inputs.

Parameters(j= 1, ..., 4)	Distribution	Interval/Value	μ	σ
$m_1/m_2/m_3/m_4$ (tons)	deterministic	120/100/100/100	-	-
$k_{j,0}$ (MN/m)	lognormal	-	5.19	0.5
γ (kN)	Gaussian	-	5	1
ρ	deterministic	0.4	-	-
δ (kN/s)	deterministic	0.5	-	-
λ (kNs/m)	Gaussian	-	50	10

The data used for updating are synthetic accelerometer data. The data are simulated assuming the building is subjected to the 1999 Chi-Chi earthquake (Mw 7.6 and ML 7.3). 25 seconds of the ground motion record with a sampling time interval of 0.02 second is considered as the ground excitation. To imitate the gap between the realistic data and simulated data, the acceleration data are contaminated by a noise of 50% rms noise-to-signal ratio. The accelerometer

data collected at the base floor and the top floor (Fig. 6-4) are used for the failure probability updating purpose. Let N_0 denote the number of degrees of freedom with data measured and collected. In this example, $N_0 = 2$. Let N_T denote the length of the discrete time series data. With a total length of 25 seconds and time interval of 0.02 seconds, $N_T = 1250$. The collected accelerometer data is denoted by $\hat{y}_n(t_m)$. The absolute acceleration yielded by the model is denoted by $y_n(t_m|\boldsymbol{\theta})$. The prediction errors are computed as $\epsilon_n(t_m) = \hat{y}_n(t_m) - y_n(t_m|\boldsymbol{\theta})$, $n = 1, \dots, N_0$, and $m = 1, \dots, N_T$. These errors are modelled by independently and identically distributed Gaussian distribution with mean equal to zero and some standard deviation σ . This σ is assumed unknown and is updated together with the other model parameters. The likelihood function is given by:

$$P(D|\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{\frac{N_0 N_T}{2}}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=1}^{N_0} \sum_{m=1}^{N_T} [\hat{y}_n(t_m) - y_n(t_m|\boldsymbol{\theta})]^2\right) \quad (6.27)$$

A point-source model [108] is used to generate the future earthquakes. The parameters of the point-source model are the moment magnitude (denoted by M) and the distance (represented by r) from the building to the epicenter [109]. The distribution of moment magnitude M of future earthquakes is modelled the by the bounded Gutenberg-Richter recurrence law [82]. Given the minimum and maximum earthquake magnitude ($M_{\min}$ and $M_{\max}$ respectively), the PDF of magnitude M is

$$p(M) = \frac{\beta \exp(-\beta M)}{\exp(-\beta M_{\min}) - \exp(-\beta M_{\max})} \quad (6.28)$$

where $\beta = 2.303$ and $[M_{\min}, M_{\max}]$ are the lower and upper bounds of the magnitude of future earthquakes being considered. In this study, $M_{\min}$ is set to be 5.0 and the upper bound $M_{\max}$ is set to be 8.0. The distance to the epicenter (r) is modelled by a triangular distribution

$$p(r) = \frac{2r}{r_{\max}^2} \quad (6.29)$$

where $r_{\max} = 50$ km is assumed. Given a specific magnitude M and distance r , to generate the ground motion, the time-domain simulation method [108] is adopted to simulate the future earthquakes ground motions. In this example, the duration of the future earthquakes is set to be $T = 25$ s and the sampling time interval is $\Delta t = 0.02$ s. Correspondingly, a white noise

sequence of length 1250 is generated by the method proposed by Boore [108].

Figure 6-4: The recorded floor acceleration data of 4-storey building

The failure probability curves of the four performance functions are updated with the proposed method considering the two proposal distributions (π_1 and π_0) as discussed in Section 6.3.4. The proposed improved ISS method is adopted to estimate the corresponding failure probability curves. To compare the effectiveness of the two proposal distributions, various sample sizes are considered. It should be noted that the sample size here refers to the number of samples per Subset Simulation level in the updating phase, and it is also the number of samples at level zero. Three cases of sample size including 500, 1000, and 2000 are considered. For each sample size and each of the two proposal distributions, 50 independent runs are performed. The results of the updated failure probability curves are shown in Fig. 6-5. Results obtained with π_0 and 5000 sample size are also shown for reference. A comparison of the number of evaluations of the nonlinear dynamic model by the proposed method using π_0 and π_1 is given in Table 6-2. The number in the bracket shows the average number of levels till the stopping criteria (Section 6.3.4) is reached. A target maximum drift ratio threshold [0.014, 0.0085, 0.008, 0.005]

(corresponding to storey 1 to storey 4, respectively) is set to count the number of evaluations of the nonlinear dynamic model in the failure probability estimation stage by the improved ISS method, and it is assumed two evaluations of the nonlinear dynamic model is required per sample, one is for failure checking, the other is for computing the likelihood function as in Eq. 6.27. Referring to Table 6-2 where the number of evaluations is shown for the same target threshold, the proposed method with π_1 requires smaller number of evaluations of the nonlinear dynamic model than that with π_0 . In Fig. 6-5, it can be seen that with the same sample size per level, the cov of the proposed method using the proposal distribution π_1 is lower than that of using the proposal distribution π_0 at most failure probability levels of each performance function. Besides, the proposed method with π_1 is more robust than that with π_0 in terms of the sample size. i.e., when the sample size decreases, the proposed method with π_1 maintains consistent updated failure probability curves. In contrast, the updated failure probability curves with π_0 and smaller sample sizes obviously deviate from the curves given by the larger sample size. A comparison of the performance of the two proposal distributions at the threshold [0.014, 0.0085, 0.008, 0.005] is shown in Table 6-3. The proposed method with π_1 gives estimators of lower cov for most performance functions and sample sizes.

Table 6-2: Number of evaluations of dynamic model by the proposed method with π_1 and π_0 for different sample size per level (Average of results of 50 independent runs of the proposed method with proposal distributions, π_1 and π_0).

Sample Size	π_1				π_0		
	Laplace + Hessian	Subset Simulation (levels)	failure probability estimation (improved ISS)	total	Subset Simulation (levels)	failure estimation (improved ISS)	total
2000	492+21	10568(6)	26065	37146	27704(15)	25116	52820
1000	492+21	4852(5)	12478	17843	14536(16)	12522	27158
500	492+21	2408(5)	6330	9251	8267(18)	6201	14468

To illustrate the efficiency of the improved ISS method, a comparison is made between the improved ISS method and the original subset simulation. The improved ISS method can estimate all the four failure probability curves in one run. With the original subset simulation, each failure probability curve estimation requires an independent run. In this comparison, both the improved ISS and the original subset simulation are based on the posterior samples

Figure 6-5: Comparison of updated failure probability curves using the proposed approach with π_1 and π_0 as the proposal distribution with different sample size per level. The plots in first row are the CCDF curves, the plots in the second row shows the corresponding c.o.v. of the estimators.

obtained by the proposed method with the proposal distribution of π_1 . The sample size of the posterior samples is 2000. The failure probability curves are updated till around $P_f = 10^{-4}$. The updated failure probability curves are shown in Fig. 6-6. The first row of Fig. 6-6 shows the mean failure probability curve obtained by the results of 50 independent runs for both methods. It can be seen that the cures obtained by the improved ISS method is consistent with that of original subset simulation. As shown in the second row of Fig. 6-6, the cov of the improved ISS is of the same level as the original subset simulation. The improved ISS method yields the result with a smaller number of evaluations of the nonlinear dynamic model. To illustrate this, the ‘unit cov’ of the estimator is calculated and compared. The unit cov (denote by Δ hereafter) is calculated as $\Delta = \delta \cdot \sqrt{N_s}$ [79], where δ is the cov, N_s is sample size used.

Table 6-3: Failure probability updating result comparison of the proposed method with two different proposal distributions π_1 and π_0 , values are the average of results from 50 independent runs of the proposed method considering each proposal and target maximum drift ratio threshold of [0.014, 0.0085, 0.008, 0.005].

Performance function	Proposal distribution		Sample size =2000	Sample size =1000	Sample size =500
g_1	π_1	mean	1.35	1.62	1.61
		cov	0.27	0.63	0.56
	π_0	mean	1.24	1.20	1.00
		cov	0.32	0.51	0.7
g_2	π_1	mean	1.18	1.29	1.33
		cov	0.32	0.55	0.59
	π_0	mean	1.26	1.32	1.56
		cov	0.55	0.55	0.79
g_3	π_1	mean	1.40	1.64	1.76
		cov	0.36	0.57	0.75
	π_0	mean	1.54	1.67	1.66
		cov	0.59	0.83	0.85
g_4	π_1	mean	1.35	1.56	1.54
		cov	0.47	0.59	0.67
	π_0	mean	2.31	3.21	4.88
		cov	0.69	0.73	0.87

Smaller Δ represents higher computational efficiency. The last row of Fig. 6-6 is the unit cov of both improved ISS method and the original subset simulation. The unit cov of the proposed method can be seen to be much less than that of the original subset simulation. Consider the target maximum inter-storey drift ratio threshold of [0.014, 0.0085, 0.008, 0.005], from the 1st floor to 4th floor, respectively. The updated failure probabilities, the cov of the estimator and the unit cov of both the improved ISS method and the original Subset Simulation are shown in Table 6-4. It can be seen that both methods yield similar failure probability estimates at the thresholds. While the cov of the estimator by the improved ISS method is at the same level with that of the original subset simulation, the unit cov of the improved ISS method is significantly lower making it a more efficient approach than the original subset simulation. The last row of Table 6-4 shows the number of evaluations of the nonlinear dynamic model required to obtain the failure probability estimation at the corresponding drift ratio thresholds. The

improved ISS method requires 26065 evaluations for failure probability curves estimation and 61936 evaluations are required by the original subset simulation. In this example, the improved ISS method obtains a ratio of saving of 2.38 over the original subset simulation. Referring to Fig. 6-6, the results at different failure probability levels would give similar results.

Figure 6-6: Updated failure probability curves of example 1 obtained by both the proposed method with the proposed improved ISS method and the original subset simulation (Both methods use the same posterior samples generated by the proposed approach considering π_1 and sample size of 2000 per level)

6.4.2 Example 2, Nonlinear Shear Building Model with a Large Number of Uncertain Parameters

In this example, a 5-storey building subjected to earthquake excitations is considered. The building is modelled by a 5-DOF nonlinear shear building model modified from [110]. Fig.

Table 6-4: A comparison of the failure probability updating results at target threshold $[0.014, 0.0085, 0.008, 0.005]$, obtained using the proposed method considering the proposal distribution π_1 with the proposed improved ISS method and the original subset simulation LSF Proposal distribution π_1 with improved ISS Proposal distribution π_1 with original Subset Simulation.

LSF	Proposal distribution π_1 with improved ISS			Proposal distribution π_1 with original Subset Simulation		
	$PF(\times 10^{-4})$	δ	Δ	$PF(\times 10^{-4})$	delta	Delta
g_1	1.35	0.27	26.34	1.61	0.37	62.87
g_2	1.18	0.32	31.48	1.35	0.37	63.47
g_3	1.40	0.36	35.68	1.79	0.44	75.27
g_4	1.35	0.47	46.68	1.41	0.46	79.95
Neval	26065			61936		
Ratio of saving	2.38					

6-7 shows the building model. Each floor corresponds to one degree of freedom. Newmark's Method is adopted to compute the structural responses. The lumped mass of each floor is 45 000 tons. The height of each storey is 2.5 m. The parameters to be updated are the stiffness of each floor k_j , the yield limits V_j the modal damping ratios ζ_j , and the post-yield stiffness ratios α_j , $j = 1, \dots, 5$. This post-yield stiffness ratio is computed as $\alpha_j = k_{y,j}/k_j$, where $k_{y,j}$ and k_j are the post-yield stiffness and stiffness of the linear stage of storey j , respectively. The prior distributions of the stiffnesses and yielding limits are assumed to follow lognormal distribution with the mean and standard deviation of their logarithm given in Table 6-5 while those of the post-yield stiffness ratios and damping ratios are assumed to be uniformly distributed. The prior distributions of these parameters are summarized in Table 6-5

Table 6-5: The prior distribution of model parameters.

Parameters($j= 1, \dots, 4$)	Distribution	Interval/Value	μ	σ
m_j (tons)	deterministic	45 000	-	-
k_j (MN/m)	lognormal	-	3.76	0.7
V_j (kN)	lognormal	-	3.76	0.7
α_j	uniform	[0, 0.2]	-	-
ζ_j	uniform	[0, 0.2]	-	-

The failure event is defined as the maximum inter-storey drift ratios exceeding the given

Figure 6-7: The 5-storey building model with 5 degrees of freedom

threshold. The corresponding failure probability is defined as:

$$P_{F_j|D}(b_j) = P(g_j(\boldsymbol{\theta}) = \mu_j(\boldsymbol{\theta}) > b_j|D), \quad j = 1, \dots, 5 \quad (6.30)$$

A total number of 5 failure probability curves are to be updated in this example. D is the structural response data collected for updating purposes. μ_j is the maximum inter-storey drift ratio of storey j , which is given as follows:

$$\begin{aligned} \mu_1 &= \max_{0 \leq \tau \leq t_{max}} (|x_{1,\tau}|/H_1) \\ \mu_j &= \max_{0 \leq \tau \leq t_{max}} (|x_{j,\tau} - x_{j,\tau-1}|/H_j) \end{aligned} \quad (6.31)$$

where $x_{j,\tau}$ denotes the displacement of the j th floor relative to the ground at time instant τ , and t_{max} is the total time simulated and H_j is the height of the j th storey. Similar to example 1, the data used for updating is synthetic accelerometer data. The 1999 Chi-Chi earthquake is taken as the excitation to simulate the data. Again, a truncated ground motion records of length of 25 seconds is considered. The sampling time interval is 0.02s. The absolute acceleration of the first and top floor are collected. The absolute acceleration data are contaminated by a noise of

50 rms noise-to-signal ratio to consider the gap between the simulated data and the realistic data. Fig. 5 shows the accelerometer data used for updating in this example. Let N_0 be the number of degrees of freedom with data collected. We use the data collected on the first and top floor, thus $N_0 = 2$. Let N_T represents the length of the discrete time series data. In this example $N_T = 1250$. The collected accelerometer data is denoted by $\hat{y}_n(t_m)$. The absolute acceleration output of the nonlinear shear building model is denoted by $y_n(t_m|\boldsymbol{\theta})$. Again the prediction errors ($\epsilon_n(t_m) = \hat{y}_n(t_m) - y_n(t_m|\boldsymbol{\theta})$, $n = 1, \dots, N_0$, and $m = 1, \dots, N_T$) are modelled as independently and identically distributed Gaussian distribution with mean at zero and some standard deviation σ . This σ is the additional parameter to be updated with the other 20 uncertain model parameters. The likelihood function is:

$$P(D|\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{\frac{N_0 N_T}{2}}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=1}^{N_0} \sum_{m=1}^{N_T} [\hat{y}_n(t_m) - y_n(t_m|\boldsymbol{\theta})]^2\right) \quad (6.32)$$

The future uncertain earthquake excitations are generated using the same method as in example 1. The details are not repeated here. The minimum and maximum magnitude are set to be 5.0 and 8.0 respectively. The maximum distance to the epicenter is $r_{max} = 50$ km. The duration of future earthquakes is assumed to be 25s, and the sample time is 0.02s. The five failure probability curves (Eq. 6.30) are updated with the proposed method. The failure probabilities are estimated from $P_f = 1$ to $P_f = 10^{-6}$. Both the two proposal distributions (π_1 and π_0 as presented in Section 6.3.4) are adopted and compared. The failure probability curves estimation is done by the improved ISS method as illustrated in Chapter 4, Section 4.3.

Similar to example 1, multiple sample sizes are considered. This sample size refers to the number of samples per subset simulation level in the updating phase, as well as the number of sample at level zero as in Section 6.3.1. In this example, the tested sample sizes are 500, 1000, 2000, and 5000. A total number of 50 independent runs are performed per sample size, per proposal distribution (i.e., π_1 and π_0). The failure probability curves updating results are shown in Fig. 6-9. With the same sample size, the cov of updated failure probability with the proposal distribution π_1 is lower than that of π_0 at most failure probability levels. Also, the proposed method with π_1 is more robust to the decreasing sample size, as the failure probability curves updated with π_1 and sample size of 500 and 1000 are still consistent with the failure probability curves yielded by larger sample size. This is more obvious for the failure probability

Figure 6-8: The recorded floor acceleration data of the 5-storey building

curves for storey 4 and storey 5. The number of evaluations of dynamic model is summarized in Table 6-6. It should be noted that in the column “failure curves estimation by improved ISS”, the numbers are collected with a target maximum drift ratio threshold of $[0.048, 0.026, 0.014, 0.012, 0.008]$, from storey 1 to storey 5, respectively. Again, it is assumed two evaluations of the nonlinear dynamic model is required per sample with the improved ISS method. The mean and cov of the updated failure probability estimations at the target threshold is collected as shown in Table 6-7. It shows that the proposed method with π_1 proposal distribution yields a lower cov estimator in most cases. Referring to Table 6-6, the proposed method with π_1 proposal distribution yields the estimates with fewer evaluations.

To further illustrate the efficiency of the proposed method, a comparison is made with multiple runs of subset simulation. In this comparison, posterior samples obtained with the proposal distribution π_1 are used and the posterior sample size is 5000. The failure probability curves are calculated till the failure probability level of $P_f = 10^{-6}$ by both the proposed method with the improved ISS and original subset simulation. 50 independent runs of both methods are

Table 6-6: Number of evaluations of dynamic model by the proposed method with π_1 and π_0 for different sample size per level (Average of results of 50 independent runs of the proposed method with proposal distributions, π_1 and π_0).

Sample Size	π_1				π_0		
	Laplace + Hessian	Subset Simulation (levels)	failure probability estimation (improved ISS)	total	Subset Simulation (levels)	failure estimation (improved ISS)	total
5000	1944+595	53060(12)	113970	169569	140090(31)	116620	256710
2000	1944+595	19100(11)	47871	69510	56036(32)	48379	104415
1000	1944+595	8812(10)	24183	35534	28774(32)	24606	53380
500	1944+595	3938(9)	12587	19064	15827(35)	NA	NA

NA indicates in the 50 independent runs, at least one run fails to reach the corresponding target threshold

performed. The mean estimated failure probability curves, as well as the cov and unit cov of the estimators are shown in Fig. 6-10. The failure probabilities curves obtained by the proposed method with improved ISS are consistent with those by the original Subset Simulation. The cov of the estimators by both methods are of similar levels. The unit cov of both methods are given in the last row of Fig. 6-10. The unit cov of the proposed method with improved ISS method is significant less than that of the original Subset Simulation. For a direct comparison, an analysis at the target maximum drift ratio threshold of [0.048,0.026, 0.014,0.012, 0.008] is given in Table 6-8. At this threshold, the estimated failure probabilities and the cov of the failure probability estimators obtained by the improved ISS method and the original subset simulation are quite close to each other. With this target threshold, the ratio of computational saving of the improved ISS method compared with the subset simulation is 2.18.

Figure 6-9: Comparison of updated failure probability curves using the proposed approach with π_1 and π_0 for different sample size per level. The plots in first row are the CCDF curves, the plots in the second row shows the corresponding c.o.v. of the estimators.

Table 6-7: Failure probability updating result comparison of the proposed method with two different proposal distributions π_1 and π_0 , values are the average of results from 50 independent runs of the proposed method with each proposal considering the target maximum drift ratio threshold of [0.048,0.026, 0.014,0.012, 0.008]

Performance function	Proposal distribution		Sample size = 5000	Sample size = 2000	Sample size = 1000	Sample size = 500
g1	π_1	mean	0.17	0.15	0.18	0.12
		cov	0.61	0.73	1.01	1.49
	π_0	mean	0.17	0.21	0.17	0.16
		cov	0.57	1.46	1.30	1.55
g2	π_1	mean	0.19	0.14	0.16	0.12
		cov	0.75	0.69	0.90	0.91
	π_0	mean	0.21	0.24	0.21	NA
		cov	0.47	1.18	1.08	NA
g3	π_1	mean	4.97	4.14	4.63	3.95
		cov	0.30	0.54	0.98	0.78
	π_0	mean	5.13	6.13	6.69	NA
		cov	0.49	0.61	1.06	NA
g4	π_1	mean	6.92	5.68	6.51	6.06
		cov	0.30	0.55	0.65	0.69
	π_0	mean	7.99	8.54	1.17	NA
		cov	0.36	0.45	1.02	NA
g5	π_1	mean	3.55	3.23	3.00	2.60
		cov	0.36	0.52	0.63	0.88
	π_0	mean	3.22	2.47	1.70	NA
		cov	0.59	0.72	2.12	NA

Figure 6-10: Updated failure probability curves of example 2. The curves are obtained by the proposed method with the proposed improved ISS method and the original subset simulation. Both use the posterior samples that are generated by the proposed approach with π_1 and a sample size of 5000 per level.

Table 6-8: A comparison of the failure probability updating results at target threshold $[0.048, 0.026, 0.014, 0.012, 0.008]$, obtained using the proposed method considering the proposal distribution π_1 with the proposed improved ISS method and the original Subset Simulation.

LSF	Proposal distribution π_1 with improved ISS			Proposal distribution π_1 with original Subset Simulation		
	$PF(\times 10^{-6})$	delta	Delta	$PF(\times 10^{-6})$	delta	Delta
g_1	0.17	0.61	135.36	0.17	0.53	197.32
g_2	0.19	0.75	165.75	0.24	0.77	284.30
g_3	4.97	0.30	55.13	5.23	0.33	111.76
g_4	6.93	0.30	55.42	7.75	0.31	105.12
g_5	3.55	0.36	66.18	3.67	0.33	110.36
Neval	113970			248180		
Ratio of saving	2.18					

6.5 Conclusions

In this chapter, we proposed a new method to perform Bayesian updating of multiple CCDF curves (failure probability curves) based on an extension of the ISS method and our improved ISS method in Chapter 4. We propose an approach integrating the classical Accept-Reject method, Laplace's asymptotic approximation and Subset Simulation facilitate the process of sampling the posterior samples and conditional failure samples. This approach opens up a new opportunity which allows us to have different choices of $\pi(\boldsymbol{\theta})$ to improve the efficiency of Bayesian model updating. The computational savings of the proposed method are twofold: we consider multiple performance functions simultaneously in one run; we reduce the computations significantly by the proposed analytical approximation (Laplace's asymptotic approximation) assisted Subset Simulation method. The examples show that the proposed method is not only efficient but also robust to smaller sample size.

We illustrated two choices of proposal distributions ($\pi_0(\boldsymbol{\theta})$ and $\pi_1(\boldsymbol{\theta})$). However, there are numerous difference choices to be explored. Future research direction may focus on other proposal distributions, further improvements of the updating efficiency can be expected.

The proposed method provides us an efficient way of systematically updating our knowledge about the system. It is meaningful to resilience managers: a fast and robust Bayesian updating method enables the decision makers take actions with the up-to-date system status. If condition allows, we can even develop a near real-time updating scheme, which can further be applied to digital-twins, where systems are emulated by a digital model. The proposed method yields the updated multiple CCDF curves which are in the same form of our proposed resilience assessment method in Chapter 3. Actually, we can apply the proposed Bayesian updating method to a wide range of problems defined in the form of Eq. 2.60, Eq. 3.3 and Eq. 6.2.

Chapter 7

Concluding Remarks and Future Works

7.1 Conclusions

In this thesis, we answered the following questions: (1) how to characterize the multi-criteria system resilience behaviour under uncertainties? (2) how to efficiently simulate the system performance under a large number of uncertain scenarios? (3) how can we optimize the system design for better resilience performance under uncertainties? (4) how do we systematically improve and update our knowledge about the system?

To answer these questions, we proposed a novel stochastic resilience assessment method for critical infrastructure systems. We adapted and improved the ISS method for a more efficient simulation of systems. We presented a multi-objective optimization-based resilience optimization method with computational burden alleviated by preparing a surrogate model before the optimization starts. We also proposed an efficient Bayesian updating framework to facilitate the modelling and model calibration, which enables a fast updating of our knowledge about the system. The proposed methodologies and algorithms are distributed in Chapter 3 to Chapter 6.

Chapter 3 illustrates the proposed the multi-CCDF curves-based resilience assessment method. It has several benefits:

- The proposed resilience quantification method is capable of considering the uncertainties in system performance and a whole range of uncertain events, which gives decision makers a bird's-eye view of the system behaviour at different exceedance probability levels including rare or extreme events;
- The proposed resilience metrics have explicit physical meaning and they can be easily interpreted and linked to the actual system behaviours;
- With the proposed resilience assessment method, the problem that the same value of resilience metric corresponds to different system performances is alleviated.

Chapter 4 addresses the efficient simulation method for resilience assessment. The ISS method is improved to solve its discontinuity problem. A comparison of the improved ISS method with the original Subset Simulation and Monte Carlo simulation is made in terms of resilience assessment. Physical model-based examples are shown.

In Chapter 5, multi-objective optimization-based resilience optimization is presented where the resilience metrics are taken as the objectives. Surrogate models are considered to alleviate the high computational requirements in the multi-objective optimization. Different from common surrogate models which approximate the outputs of the physical models, in this chapter, we directly approximate the resilience metrics, which are probability integrals of system performance functions.

Chapter 6 presents the Bayesian model updating framework for improving and updating our knowledge about the system. The proposed Bayesian updating method is capable of updating multiple CCDFs in one run. Besides, with the help of Laplace's method of Asymptotic Approximation, the proposed Bayesian updating method shows significant improvement in terms of computational efficiency and is also more robust to smaller sample size.

7.2 Future Works

This section discusses some of the future works that can be considered but not covered in this thesis due to the limitation of time. These future works are listed as follows.

7.2.1 Time-Series Data-based Surrogate Model Assisted Resilience Assessment and Optimization

As discussed in Chapter 5, different levels of surrogate models are discussed, and we choose to build surrogates on the resilience exceedance performance function directly. It is worth studying surrogate models constructed on the time series. As resilience is time-sensitive, surrogate models that yields system performance curves is pretty interesting and valuable. Machine learning methods such as Artificial Neural Networks, Recurrent Neural Network, are possible to be applied. Also, Polynomial Chaos Expansions [111], Stochastic Collocation methods [112] are also interesting methods that can be adapted to the stochastic based resilience assessment method.

7.2.2 Interdependencies in Critical Infrastructure Systems

We identified four types of uncertainties in resilience research in Section 2.1.3: (1) the uncertain system state and system performance level, (2) the uncertain disruptive events, in both the magnitude and the frequency, (3) the topological dependencies (4) uncertainties in operating and users. We addressed the first two types. The third type of uncertainty corresponds to the case where there are interdependencies between the critical infrastructure systems. When one critical infrastructure system is attacked by natural hazards, the malfunction of this system may have a secondary effect on other critical infrastructure systems. For example, when urban flooding happens, the traffic system may also be affected. In such cases, the traffic system may operate with a lower efficiency than the normal state. A reduction in the efficiency of the traffic system may in turn affect the delivery of fuel, the emergency rescue, as well as the drainage dredge and recovery works. Thus interdependency is a complex but interesting topic that will be considered in future research. This research will be a very important expansion to

the current work.

A typical interdependency of the critical infrastructure systems is demonstrated in Fig. 7-1.

Figure 7-1: Secondary impact of the hazards and interdependency of the critical infrastructure systems (The direction of the arrows means the flow of the impact and the size of the arrows denotes the level of impact)

The interdependency research is commonly seen in statistical physics and the researches on networks topologies. Buldyrev et al. [113] studied the cascade failures of interdependent networks based on percolation theory. Hu et al. [114] studied the interdependent and interconnected networks from the perspective of percolation theory. Rosato et al. [115] modelled the interdependent power supply system and the internet network, a network coupling parameter is introduced to consider the geographical interdependencies. However, for realistic physical networks, the characteristics of each system are by no means fully represented by theoretical topologies. To simulate the behaviour of such systems and study the risk propagation mechanisms between infrastructure systems, physical models are suggested to be adopted.

7.2.3 Resilience Assessment of Critical Infrastructure Systems Subjected to Successive Multiple Events

As illustrated in Chapter 3, the proposed resilience assessment method in this research work is limited to one single disruptive event at a time, i.e., the system is assumed to have enough

time to fully recover before the next event occurs. In reality, disruptive events may come one after another in a very short time. In a more severe case, the system may be attacked by multiple disruptive events at the same time. There is a need to extend the proposed resilience assessment methods to analysis such cases with multiple disruptive events. The method by Ouyang et al. [30] which we have reviewed in Chapter 2 is a possible approach, but their method is ‘Area-based’. How to consider a series of events with multiple criteria resilience metrics is a topic worth further exploration.

References

- [1] M. Ouyang, “Review on modeling and simulation of interdependent critical infrastructure systems,” *Reliability Engineering & System Safety*, vol. 121, pp. 43–60, January 2014.
- [2] S. M. Rinaldi;, J. P. Peerenboom;, and T. K. Kelly, “Identifying, understanding, and analyzing critical infrastructure interdependencies,” *IEEE Control Systems*, vol. 21, no. 6, pp. 11–25, December 2001.
- [3] “Cybersecurity act,” The Cyber Security Agency of Singapore (CSA), Tech. Rep., 2018. [Online]. Available: <https://www.csa.gov.sg/Legislation/Cybersecurity-Act>
- [4] K. Furuta, R. Komiyama, T. Kanno, H. Fujii, S. Yoshimura, and T. Yamada, “Resilience analysis of critical infrastructure,” in *Recent Advances in Information Science*, Barcelona, Spain, February 2016, pp. 21–28.
- [5] T. K. D. Sornette, *Risk and resilience management in social-economic*, F. M.-V and L. I., Eds. Lausanne: Resource Guide on Resilience. Lausanne: EPFL International Risk Governance Center. v29-07-2016, 2016.
- [6] D. J. Varnes, *Landslide Hazard Zonation: A Review of Principles and Practice*. Unesco (Paris), 1984, no. 3.
- [7] A. Van Der Veen and C. Logtmeijer, “Economic hotspots: Visualizing vulnerability to flooding,” *Natural Hazards*, vol. 36, no. 1-2, pp. 65–80, 2005.
- [8] Y. Y. Haimes, “On the definition of resilience in systems,” *Risk Analysis*, vol. 29, no. 4, pp. 498–501, April 2009.
- [9] L. Flax, A. Armstrong, and L. Yee, *Measuring urban resilience as you build it - insights from 100 resilient cities*, F. M.-V and L. I., Eds. Lausanne: Resource Guide on Resilience. Lausanne: EPFL International Risk Governance Center. v29-07-2016, 2016.
- [10] American Society of Mechanical Engineers ASME-ITI, *All Hazards Risk and Resilience: Prioritizing Critical Infrastructure Using the RAMCAP Plus*. ASME, January 2009.
- [11] National Infrastructure Advisory Council (NIAC), “Critical Infrastructure Resilience Final Report and Recommendations,” Tech. Rep., 2009.

REFERENCES

- [12] E. D. Vugrin, D. E. Warren, M. A. Ehlen, and R. C. Camphouse, “A framework for assessing the resilience of infrastructure and economic systems,” in *Sustainable and Resilient Critical Infrastructure Systems: Simulation, Modeling, and Intelligent Engineering*, K. Gopalakrishnan and S. Peeta, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 77–116. [Online]. Available: https://doi.org/10.1007/978-3-642-11405-2_3
- [13] D. A. Nethercot, “Design of building structures to improve their resistance to progressive collapse,” *Procedia Engineering*, vol. 14, pp. 1–13, 2011. [Online]. Available: <http://dx.doi.org/10.1016/j.proeng.2011.07.001>
- [14] N. N. Taleb, *The Black Swan: The Impact of the Highly Improbable*. Random House (New York), 2007.
- [15] J. H. Saleh, D. E. Hastings, and D. J. Newman, “Flexibility in system design and implications for aerospace systems,” *Acta Astronautica*, pp. 927–944, December 2003.
- [16] I. Linkov, B. D. Trump, and C. Fox-Lent, *Resilience: approach to risk analysis and governance*, F. M.-V and L. I., Eds. Lausanne: Resource Guide on Resilience. Lausanne: EPFL International Risk Governance Center. v29-07-2016, 2016.
- [17] W. Liu and Z. Song, “Review of studies on the resilience of urban critical infrastructure networks,” *Reliability Engineering & System Safety*, vol. 193 art. 106617, January 2020.
- [18] S. Bansal and S. H. Cheung, “On the evaluation of multiple failure probability curves in reliability analysis with multiple performance functions,” *Reliability Engineering & System Safety*, vol. 167, no. December 2016, pp. 583–594, Nov. 2017. [Online]. Available: <http://dx.doi.org/10.1016/j.ress.2017.07.010><http://linkinghub.elsevier.com/retrieve/pii/S0951832016309425>
- [19] M. Bruneau, S. E. Chang, R. T. Eguchi, G. C. Lee, T. D. O’Rourke, A. M. Reinhorn, M. Shinozuka, K. Tierney, W. A. Wallace, and D. Von Winterfeldt, “A framework to quantitatively assess and enhance the seismic resilience of communities,” *Earthquake Spectra*, vol. 19, no. 4, pp. 733–752, 2003.
- [20] H. R. Heinimann, *A generic framework for resilience assessment*, F. M.-V and L. I., Eds. Lausanne: Resource Guide on Resilience. Lausanne: EPFL International Risk Governance Center. v29-07-2016, 2016.
- [21] D. Mendonca, “Measures of resilient performance,” in *Resilience Engineering Perspectives, Volume 1: Remaining Sensitive to the Possibility of Failure*, E. H. Christopher P. Nemeth, Ed., vol. 1. CRC Press (Boca Raton, FL), 2008, pp. pp. 43–62.
- [22] P. Tamvakis and Y. Xenidis, “Resilience in transportation systems,” *Procedia - Social and Behavioral Sciences*, vol. 48, pp. 3441–3450, 2012. [Online]. Available: <http://linkinghub.elsevier.com/retrieve/pii/S1877042812030510>

-
- [23] P. Tamvakis and Y. Xenidis, "Comparative evaluation of resilience quantification methods for infrastructure systems," *Procedia - Social and Behavioral Sciences*, vol. 74, pp. 339–348, 2013. [Online]. Available: <http://linkinghub.elsevier.com/retrieve/pii/S187704281300459X>
- [24] R. Francis and B. Bekera, "A metric and frameworks for resilience analysis of engineered and infrastructure systems," *Reliability Engineering & System Safety*, vol. 121, pp. 90–103, 2014. [Online]. Available: <http://dx.doi.org/10.1016/j.res.2013.07.004>
- [25] J. Tang, H. Heinimann, and L. Khoja, "Quantitative evaluation of consecutive resilience cycles in stock market performance: A systems-oriented approach," *Physica A: Statistical Mechanics and its Applications*, vol. 532 art. 121794, 2019. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S0378437119310350>
- [26] A. Munoz and M. Dunbar, "On the quantification of operational supply chain resilience," *International Journal of Production Research*, vol. 53, no. 22, pp. 6736–6751, 2015. [Online]. Available: <https://doi.org/10.1080/00207543.2015.1057296>
- [27] M. Panteli, P. Mancarella, D. N. Trakas, E. Kyriakides, and N. D. Hatziaargyriou, "Metrics and quantification of operational and infrastructure resilience in power systems," *IEEE Transactions on Power Systems*, vol. 32, no. 6, pp. 4732–4742, November 2017.
- [28] L. Shen, B. Cassottana, H. R. Heinimann, and L. C. Tang, "Large-scale systems resilience: A survey and unifying framework," *Quality and Reliability Engineering International*, vol. 36, no. 4, pp. 1386–1401, June 2020.
- [29] A. Cardoni, G. Cimellaro, M. Domaneschi, S. Sordo, and A. Mazza, "Modeling the interdependency between buildings and the electrical distribution system for seismic resilience assessment," *International Journal of Disaster Risk Reduction*, vol. 42 art. 101315, 2020. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S2212420919306296>
- [30] M. Ouyang, L. Dueñas-Orsorio, and X. Min, "A three-stage resilience analysis framework for urban infrastructure systems," *Structural Safety*, vol. 36-37, pp. 23–31, May 2012.
- [31] S. E. Chang and M. Shinozuka, "Measuring improvements in the disaster resilience of communities," *Earthquake Spectra*, vol. 20, no. 3, pp. 739–755, August 2004.
- [32] B. M. Ayyub, "Systems resilience for multihazard environments: definition, metrics, and valuation for decision making," *Risk Analysis*, vol. 34, no. 2, pp. 340–355, July 2013.
- [33] R. Pant, K. Barker, J. E. Ramirez-Marquez, and C. M. Rocco, "Stochastic measures of resilience and their application to container terminals," *Computer & Industrial Engineering*, vol. 70, pp. 183–194, April 2014.
- [34] T. Hashimoto, J. R. Stedinger, and D. P. Loucks, "Reliability, resiliency, and vulnerability criteria for water resource system performance evaluation," vol. 18, no. 1, pp. 14–20, February 1982.

REFERENCES

- [35] Y. Yang, W. Tang, Y. Liu, Y. Xin, and Q. Wu, "Quantitative resilience assessment for power transmission systems under typhoon weather," *IEEE Access*, vol. 6, pp. 40 747–40 756, 2018.
- [36] P. Cicilio, L. Swartz, B. Vaagensmith, C. Rieger, J. Gentle, T. McJunkin, and E. Cotilla-Sanchez, "Electrical grid resilience framework with uncertainty," *Electric Power Systems Research*, vol. 189 art. 106801, December 2020.
- [37] X. Liu, K. Hou, H. Jia, J. Zhao, L. Mili, Y. Mu, J. Rim, and Y. Lei, "A resilience assessment approach for power system from perspectives of system and component levels," *International Journal of Electrical Power & Energy Systems*, vol. 118 art. 105837, June 2020.
- [38] N. Yodo and P. Wang, "Engineering resilience quantification and system design implications: A literature survey," *Journal of Mechanical Design*, vol. 138 art. 111408, no. 11, September 2016.
- [39] S. Hosseini, K. Barker, and J. E. Ramirez-Marquez, "A review of definitions and measures of system resilience," *Reliability Engineering & System Safety*, vol. 145, pp. 47–61, 2016.
- [40] N. Metropolis and S. Ulam, "The monte carlo method," *Journal of the American Statistical Association*, vol. 44, no. 247, pp. 335–341, 1949, PMID: 18139350. [Online]. Available: <https://www.tandfonline.com/doi/abs/10.1080/01621459.1949.10483310>
- [41] C. P. Robert and G. Casella, *Monte Carlo Statistical Methods*. Springer (New York, USA), 2004.
- [42] S.-K. Au and Y. Wang, *Engineering Risk Assessment with Subset Simulation*. John Wiley & Sons (Singapore), 2014.
- [43] S. K. Au and J. L. Beck, "Estimation of small failure probabilities in high dimensions by subset simulation," *Probabilistic Engineering Mechanics*, vol. 16, no. 4, pp. 263–277, 2001.
- [44] D. Gamerman and H. F. Lopes, *Markov Chain Monte Carlo, Stochastic Simulation for Bayesian Inference*. Chapman and Hall/CRC (Boca Raton, FL), May 2006.
- [45] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, "Equation of state calculations by fast computing machines," *The Journal of Chemical Physics*, vol. 21, no. 6, pp. 1087–1092, June 1953.
- [46] W. K. Hastings, "Monte Carlo sampling methods using markov chains and their applications," vol. 57, no. 1, pp. 97–109, April 1970.
- [47] S. K. Au and J. L. Beck, "Subset simulation and its application to seismic risk based on dynamic analysis," *Journal of Engineering Mechanics*, vol. 129, no. 8, pp. 901–917, 2003. [Online]. Available: [http://ascelibrary.org/doi/10.1061/\(ASCE\)0733-9399\(2003\)129:8\(901\)](http://ascelibrary.org/doi/10.1061/(ASCE)0733-9399(2003)129:8(901))

-
- [48] J. Ching, S. K. Au, and J. L. Beck, "Reliability estimation for dynamical systems subject to stochastic excitation using subset simulation with splitting," *Computer Methods in Applied Mechanics and Engineering*, vol. 194, no. 12-16, pp. 1557–1579, 2005.
- [49] J. Ching, J. L. Beck, and S. K. Au, "Hybrid Subset Simulation method for reliability estimation of dynamical systems subject to stochastic excitation," *Probabilistic Engineering Mechanics*, vol. 20, no. 3, pp. 199–214, 2005.
- [50] L. Katafygiotis, T. Moan, and S. H. Cheung, "Auxiliary domain method for solving multi-objective dynamic reliability problems for nonlinear structures," *Structural Engineering and Mechanics: An International Journal*, vol. 25, no. 3, pp. 347–363, 2007.
- [51] I. Papaioannou, W. Betz, K. Zwirgmaier, and D. Straub, "MCMC algorithms for Subset Simulation," *Probabilistic Engineering Mechanics*, vol. 41, pp. 89–103, 2015. [Online]. Available: <http://dx.doi.org/10.1016/j.pro bengmech.2015.06.006>
- [52] W.-C. Hsu and J. Ching, "Evaluating small failure probabilities of multiple limit states by parallel subset simulation," *Probabilistic Engineering Mechanics*, vol. 25, no. 3, pp. 291–304, July 2010.
- [53] F. DiazDelaO, A. Garbuno-Inigo, S. Au, and I. Yoshida, "Bayesian updating and model class selection with subset simulation," *Computer Methods in Applied Mechanics and Engineering*, vol. 317, pp. 1102–1121, April 2017.
- [54] H. S. Li, Y. Z. Ma, and Z. Cao, "A generalized Subset Simulation approach for estimating small failure probabilities of multiple stochastic responses," *Computers and Structures*, vol. 153, pp. 239–251, 2015.
- [55] K. Miettinen, *Nonlinear Multiobjective Optimization*. Kluwer Academic Publishers (Norwell, Massachusetts, USA), 1998.
- [56] K. Deb, A. Pratap, S. Agarwal, and T. Meyarivan, "A fast and elitist multiobjective genetic algorithm: NSGA-II," *IEEE Transactions on Evolutionary Computation*, vol. 6, no. 2, pp. 182–197, 2002.
- [57] Z.-H. Han, K.-S. Zhang *et al.*, "Surrogate-based optimization," *Real-world Applications of Genetic Algorithms*, vol. 343, 2012.
- [58] C. Wang, Q. Duan, W. Gong, A. Ye, Z. Di, and C. Miao, "An evaluation of adaptive surrogate modeling based optimization with two benchmark problems," *Environmental Modelling & Software*, vol. 60, pp. 167–179, October 2014.
- [59] B. Williams and S. Cremaschi, "Selection of surrogate modeling techniques for surface approximation and surrogate-based optimization," *Chemical Engineering Research and Design*, vol. 170, pp. 76–89, June 2021.
- [60] Z. Han, K. Zhang, W. Song, and Z. Qiao, "Optimization of Active Flow Control over an Airfoil Using a Surrogate-Management Framework," *Journal of aircraft*, vol. 47, no. 2, pp. 603–612, 2010.

REFERENCES

- [61] W. Gong, Q. Duan, J. Li, C. Wang, Z. Di, A. Ye, C. Miao, and Y. Dai, “Multiobjective adaptive surrogate modeling-based optimization for parameter estimation of large, complex geophysical models,” *Water Resources Research*, vol. 52, no. 3, pp. 1984–2008, 2016.
- [62] Z. Shao, “Reliability assessment of structural dynamic systems subject to earthquake-induced tsunamis,” Ph.D. dissertation, School and Civil and Environmental Engineering, Nanyang Technological University, 2016.
- [63] H. Jensen, D. Jerez, and M. Valdebenito, “An adaptive scheme for reliability-based global design optimization: A Markov chain Monte Carlo approach,” *Mechanical Systems and Signal Processing*, vol. 143106836, September 2020.
- [64] J. Ching and Y.-C. Chen, “Transitional markov chain monte carlo method for bayesian model updating, model class selection, and model averaging,” *Journal of Engineering Mechanics*, vol. 133, no. 7, pp. 816–832, July 2007.
- [65] J. L. Beck and S. K. Au, “Bayesian updating of structural models and reliability using Markov Chain Monte Carlo simulation,” *Journal of Engineering Mechanics*, vol. 128, no. 4, pp. 380–391, 2002.
- [66] C. Papadimitriou, J. Beck, and L. Katafygiotis, “Updating robust reliability using structural test data,” *Probabilistic Engineering Mechanics*, vol. 16, no. 2, pp. 103–113, April 2001.
- [67] J. L. Beck and L. S. Katafygiotis, “Updating of a model and its uncertainties utilizing dynamic test data,” in *Computational Stochastic Mechanics*, P. D. Spanos and C. A. Brebbia, Eds. Dordrecht: Springer Netherlands, 1991, pp. 125–136. [Online]. Available: https://doi.org/10.1007/978-94-011-3692-1_12
- [68] J. L. Beck and L. S. Katafygiotis, “Updating models and their uncertainties. i: Bayesian statistical framework,” *Journal of Engineering Mechanics*, vol. 124, no. 4, pp. 455–461, April 1998.
- [69] C. Papadimitriou, J. L. Beck, and L. S. Katafygiotis, “Asymptotic expansions for reliability and moments of uncertain systems,” *Journal of Engineering Mechanics*, vol. 123, no. 12, pp. 1219–1229, December 1997.
- [70] L. S. Katafygiotis and J. L. Beck, “Updating models and their uncertainties. II: model identifiability,” *Journal of Engineering Mechanics*, vol. 124, no. 4, pp. 463–467, April 1998.
- [71] S. Bansal and S. H. Cheung, “A new stochastic simulation algorithm for updating robust reliability of linear structural dynamic systems subjected to future gaussian excitations,” *Computer Methods in Applied Mechanics and Engineering*, vol. 326, pp. 481–504, November 2017.

-
- [72] S. Bansal and S. H. Cheung, “Stochastic simulation algorithm for robust reliability updating of structural dynamic systems based on incomplete modal data,” *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, vol. 3 art. 04017008, no. 4, December 2017.
- [73] D. Straub and I. Papaioannou, “Bayesian updating with structural reliability methods,” *Journal of Engineering Mechanics*, vol. 141 art. 04014134, no. 3, March 2015.
- [74] M. Ouyang and L. Dueñas-Osorio, “Time-dependent resilience assessment and improvement of urban infrastructure systems,” *Chaos*, vol. 22 art. 033122, no. 3, 2012.
- [75] B. Cassottana, L. Shen, and L. C. Tang, “Modeling the recovery process: A key dimension of resilience,” *Reliability Engineering & System Safety*, vol. 190 art. 106528, 2019. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0951832018313152>
- [76] M. Maharjan, A. Pathirana, B. Gersonius, and K. Vairavamoorthy, “Staged cost optimization of urban storm drainage systems based on hydraulic performance in a changing environment,” *Hydrology and Earth System Sciences*, vol. 13, no. 4, pp. 481–489, April 2009.
- [77] L. Rossman, “Storm water management model (swmm version 5.0) user’s manual,” *United States Environment Protection Agency*, 2004.
- [78] V. Chow, D. Maidment, and L. Mays, *Applied Hydrology*. McGraw-Hill (Singapore), 1988.
- [79] S. K. Au, J. Ching, and J. L. Beck, “Application of subset simulation methods to reliability benchmark problems,” *Structural Safety*, vol. 29, no. 3, pp. 183–193, 2007.
- [80] V. Guinot, “Multiple porosity shallow water models for macroscopic modelling of urban floods,” *Advances in Water Resources*, vol. 37, pp. 40–72, 2012. [Online]. Available: <http://dx.doi.org/10.1016/j.advwatres.2011.11.002>
- [81] M. J. Gibson, D. A. Savic, S. Djordjevic, A. S. Chen, S. Fraser, and T. Watson, “Accuracy and computational efficiency of 2D urban surface flood modelling based on cellular automata,” *Procedia Engineering*, vol. 154, pp. 801–810, 2016. [Online]. Available: <http://dx.doi.org/10.1016/j.proeng.2016.07.409>
- [82] S. L. Kramer, *Geotechnical Earthquake Engineering*. Pearson Education (India), 1996.
- [83] Y. X. Yu and C. Y. Jin, “Empirical peak ground velocity attenuation relations based on digital broadband records,” in *World Conference on Earthquake Engineering*, Beijing, China, October 12-17 2008.
- [84] K. A. Klise, M. Bynum, D. Moriarty, and R. Murray, “A software framework for assessing the resilience of drinking water systems to disasters with an example earthquake case study,” *Environmental Modelling & Software*, vol. 95, pp. 420–431, 2017. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S1364815216309501>

REFERENCES

- [85] A. L. Alliance, “Seismic fragility formulations for water systems, part 1,” American Lifelines Alliance, Tech. Rep., 2001.
- [86] M. Wagner Janet, S. Uri, and H. Marks David, “Water distribution reliability: Simulation methods,” *Journal of Water Resources Planning and Management*, vol. 114, no. 3, pp. 276–294, May 1988. [Online]. Available: [https://doi.org/10.1061/\(ASCE\)0733-9496\(1988\)114:3\(276\)](https://doi.org/10.1061/(ASCE)0733-9496(1988)114:3(276))
- [87] W.-S. LIU and S. CHEUNG, “Reliability based design optimization with approximate failure probability function in partitioned design space,” *Reliability Engineering & System Safety*, vol. 167, pp. 602–611, November 2017. [Online]. Available: <http://linkinghub.elsevier.com/retrieve/pii/S0951832016300813>
- [88] G. Jia, A. A. Taflanidis, and J. L. Beck, “A new adaptive rejection sampling method using kernel density approximations and its application to subset simulation,” *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, vol. 3 art. D4015001, no. 2, pp. 1–12, June 2017.
- [89] S. Hochreiter and J. Schmidhuber, “Long Short-Term Memory,” *Neural Computation*, vol. 9, no. 8, pp. 1735–1780, November 1997. [Online]. Available: <https://doi.org/10.1162/neco.1997.9.8.1735>
- [90] B. van de Weg, L. Greve, M. Andres, T. Eller, and B. Rosic, “Neural network-based surrogate model for a bifurcating structural fracture response,” *Engineering Fracture Mechanics*, vol. 241 art. 107424, January 2021.
- [91] P. Zhang, Z.-Y. Yin, Y. Zheng, and F.-P. Gao, “A LSTM surrogate modelling approach for caisson foundations,” *Ocean Engineering*, vol. 204, p. 107263, May 2020.
- [92] M. Li and Z. Wang, “Reliability-based multifidelity optimization using adaptive hybrid learning,” *ASCE-ASME J Risk and Uncert in Engrg Sys Part B Mech Engrg*, vol. 6, no. 2, 2020.
- [93] S. K. Au, “Reliability-based design sensitivity by efficient simulation,” *Computers and Structures*, vol. 83, no. 14, pp. 1048–1061, 2005.
- [94] V. Dubourg, B. Sudret, and J. M. Bourinet, “Reliability-based design optimization using kriging surrogates and subset simulation,” *Structural and Multidisciplinary Optimization*, vol. 44, no. 5, pp. 673–690, 2011.
- [95] J. Ching and Y.-H. Hsieh, “Approximate reliability-based optimization using a three-step approach based on subset simulation,” *Journal of Engineering Mechanics*, vol. 133, no. 4, pp. 481–493, April 2007.
- [96] A. A. Taflanidis and J. L. Beck, “Stochastic subset optimization for reliability optimization and sensitivity analysis in system design,” *Computers & Structures*, vol. 87, no. 5-6, pp. 318–331, March 2009.

-
- [97] W. Liu, “Optimal design of stochastic complex systems with applications to deep excavations,” Ph.D. dissertation, School of Civil and Environmental Engineering, Nanyang Technological University, 2018.
- [98] M. Rosenblatt, “Remarks on some nonparametric estimates of a density function,” *The Annals of Mathematical Statistics*, vol. 27, no. 3, pp. 832–837, 1956. [Online]. Available: <https://doi.org/10.1214/aoms/1177728190>
- [99] E. Parzen, “On estimation of a probability density function and mode,” *The Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 1065–1076, 1962. [Online]. Available: <http://www.jstor.org/stable/2237880>
- [100] T. Cacoullos, “Estimation of a multivariate density,” *Annals of the Institute of Statistical Mathematics*, vol. 18, no. 1, pp. 179–189, December 1966.
- [101] L. Lu, H. Jiang, and W. H. Wong, “Multivariate density estimation by bayesian sequential partitioning,” *Journal of the American Statistical Association*, vol. 108, no. 504, pp. 1402–1410, July 2013. [Online]. Available: <https://doi.org/10.1080/01621459.2013.813389>
- [102] S. Au and J. Beck, “First excursion probabilities for linear systems by very efficient importance sampling,” *Probabilistic Engineering Mechanics*, vol. 16, no. 3, pp. 193–207, July 2001.
- [103] R. Y. Rubinstein and D. P. Kroese, *Simulation and the Monte Carlo Method*. John Wiley & Sons, Inc. (Hoboken, New Jersey, USA), 2016.
- [104] J. L. Beck and K.-V. Yuen, “Model selection using response measurements: Bayesian probabilistic approach,” *Journal of Engineering Mechanics*, vol. 130, no. 2, pp. 192–203, February 2004.
- [105] J. Von Neumann, “Various techniques used in connection with random digits,” *National Bureau of Standards Applied Mathematic Series*, vol. 3, pp. 36–38, 1963.
- [106] A. Nataf, “Détermination des distribution de probabilités dont les marges sont données,” *Comptes Rendus de l’Académie des Sciences*, vol. 225, pp. 42–43, 1962.
- [107] J. Ching, J. L. Beck, K. A. Porter, and R. Shaikhutdinov, “Bayesian state estimation method for nonlinear systems and its application to recorded seismic response,” *Journal of Engineering Mechanics*, vol. 132, no. 4, pp. 396–410, 2006.
- [108] D. M. Boore, “Simulation of ground motion using the stochastic method,” *Pure and Applied Geophysics*, vol. 160, no. 3, pp. 635–676, March 2003.
- [109] G. M. Atkinson and W. Silva, “Stochastic modeling of California ground motions,” *Bulletin of the Seismological Society of America*, vol. 90, no. 2, pp. 255–274, 2000.
- [110] A. K. Chopra, *Dynamics of structures: Theory and Applications to Earthquake Engineering*, 4th ed. Prentice-Hall (Upper Saddle River ,NJ, USA), 2012.

REFERENCES

- [111] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach*. Springer (New York, USA), 1991.
- [112] D. Xiu and J. S. Hesthaven, “High-order collocation methods for differential equations with random inputs,” *SIAM Journal on Scientific Computing*, vol. 27, no. 3, pp. 1118–1139, 2005.
- [113] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, “Catastrophic cascade of failures in interdependent networks,” *Nature*, vol. 464, no. 7291, pp. 1025–1028, April 2010. [Online]. Available: <http://dx.doi.org/10.1038/nature08932><http://www.nature.com/doifinder/10.1038/nature08932>
- [114] Y. Hu, B. Kshirim, R. Cohen, and S. Havlin, “Percolation in interdependent and interconnected networks: Abrupt change from second- to first-order transitions,” *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, vol. 84, no. 6, pp. 1–6, 2011.
- [115] V. Rosato, L. Issacharoff, F. Tiriticco, S. Meloni, S. D. Porcellinis, and R. Setola, “Modelling interdependent infrastructures using interacting dynamical models,” *International Journal of Critical Infrastructures*, vol. 4, no. 1/2, pp. 63–79, 2008.

Appendix A.

An Illustrative Example on How Subset Simulation Works

This example illustrates how Subset Simulation “pushes” the samples from a higher probability region to the lower probability region from one level to the next level, as a supplement to Section 2.2.3 and 2.2.4, where we introduced Subset Simulation. The problem settings are as follows. We want to estimate the exceedance probability $P(F) = P(g(\boldsymbol{\theta}) > b)$, where F is a failure event, g is some performance function, $\boldsymbol{\theta}$ is a vector of random variables, b is some threshold. In this illustrative example, let us consider $\boldsymbol{\theta}$ to be two-dimensional: $\boldsymbol{\theta} = \{\theta_1, \theta_2\}$, where θ_1 and θ_2 are independent and identically distributed as standard Gaussian distribution, i.e., $\theta_i \sim N(0, 1)$, for $i = 1, 2$. The performance function g is given as ¹:

$$g(\boldsymbol{\theta}) = \theta_1 + \theta_2. \quad (\text{A..1})$$

We perform Subset Simulation according to the steps listed in the paragraph below Eq. 2.45 (on Page 34)) with the following settings: (1). The ratio p_0 is set to be 0.1; (2). Initial sample size $N = 200$;² (3). The proposal distribution is chosen to be a uniform distribution with its center located at a seed $\boldsymbol{\theta}^{(i)}$, and the lower and upper bound are $\boldsymbol{\theta}^{(i)} - \boldsymbol{\sigma}$ and $\boldsymbol{\theta}^{(i)} + \boldsymbol{\sigma}$, respectively, where $\boldsymbol{\sigma}$ is the standard deviation of all the seeds of this simulation level.

Let’s consider a target threshold $b = 6$, i.e., $P(g(\boldsymbol{\theta}) > 6)$. We give the scatter plot of samples in each level as in Fig. A.-1 directly. We will focus on showing how the seeds are used to generate new failure samples by Algorithm 5.

¹This problem can be easily solved analytically, but we still use Subset Simulation for the purpose of illustration.

²This is to consider the clearance of visualization, a larger sample size will result in heavy overlap of samples in the scatter plots. In realistic applications, especially when simulating rare events, this initial sample size can be much larger. For example, in this example, if we consider a target threshold $b = 6$, i.e., $P(g(\boldsymbol{\theta}) > 6)$, with $N = 200$, we can expect a very poor estimator (the c.o.v. can be reaching 3) and increasing N to be 1000 would be more appropriate.

Figure A.-1: Scatter plot of samples in each simulation levels, the initial sample size in the 1st level is $N = 200$.

A..1 Visualization of Generating New Failure Samples

Let's take the 2nd simulation level as an illustration. The other levels just repeat this process as we have introduced in the paragraph following Eq. 2.45 (item “2. Subsequent levels of simulation”, on Page 35). The 1st level is Monte Carlo simulation, which provides us with the seeds taken as the 10% largest samples in terms of the performance function $g(\boldsymbol{\theta})$. The samples in this 1st level and the seeds are plotted in Fig. A.-2.

Using these seeds, we can generate new failure samples following Algorithm 5. We have 20 seeds, and we need to recover the sample size to 200 again. We do MCMC with 20 chains and the length of each chain being 10 (i.e., we need 9 transitions to generate 180 samples). Fig. A.-3 shows the first three transitions. In the first row of Fig. A.-3, we show the first transition, where the seeds are taken as the starting points of Markov chains. The first column shows the seeds. The second column shows the accepted candidates $\hat{\boldsymbol{\theta}}$ (represented by green dots). The dash line connects each candidate with its seed. This column corresponds to line 6 to line 10 in Algorithm 5. In the third column, we do line 13 to line 17 in Algorithm 5: if the candidate

Figure A.-2: The samples in the 1st level by Monte Carlo simulation, the seed identified are represented by *. The initial sample size $N = 200$, and we have 20 seeds.

$\tilde{\boldsymbol{\theta}} \in F_1$, we accept this candidate and move to it as the next state (as shown by the black arrows). Otherwise, we reject this candidate and set the next state to be the corresponding seed (i.e., repeat $\boldsymbol{\theta}^{(i)}$). The last column shows the seeds and the newly generated failure samples. The second and the third row show the 2nd and 3rd transitions and the processes are similar to the 1st transition. The only difference is that in the first column, the newly generated samples (instead of the seeds identified in the 1st level) are used as current samples to propose new candidate samples (as shown in line 1 of Algorithm 5). We can see from the last column that the sample size is growing as the process goes on. For completeness, the 4th to 6th transitions are shown in Fig. A.-4, and the 7th to 9th transitions are shown in Fig. A.-5. The last column in the last row of Fig. A.-5 includes all the samples in this level (the 2nd level). We can identify the 10% largest samples in terms of the performance function $g(\boldsymbol{\theta})$ as the seeds to generate new failure samples in the 3rd simulation level. To show this, we re-plot these 2nd level samples in uniform style as in Fig. A.-6.

Now we have successfully finished the 2nd level of simulation. For the subsequent levels, the procedures are similar and are not repeated here.

Figure A.-3: The first three transitions for generating new failure samples using Algorithm 5. The first row shows the 1st transition, and the second and third row show the 2nd and 3rd transition, respectively.

Figure A.-4: The 4th to 6th transitions for generating new failure samples using Algorithm 5. The first row shows the 4th transition, and the second and third row show the 5th and 6th transition, respectively.

Figure A.-5: The 7th to 9th transitions for generating new failure samples using Algorithm 5. The first row shows the 7th transition, and the second and third row show the 8th and 9th transition, respectively.

Figure A.-6: The samples in the 2nd level. These samples are the same with those in the last column and last row of Fig. A.-5, but re-plotted in uniform style (blue circles). The seeds to the next level are identified and plotted as *.

A..2 A Comparison with Monte Carlo Simulation

As a closure of this appendix, we compare Subset Simulation with brute-force Monte Carlo simulation in terms of the computational efficiency in estimating very low probability events. We can later see that Subset Simulation “pushes” the samples to the low probability region. Let’s consider estimating the target $P(g(\boldsymbol{\theta}) > 6)$. As we have explained in the footnote 2 on Page 169, our previous settings for Subset Simulation are for better visualization of the arrows and samples only. Here we increase the initial sample size N to be 1000 and keep the other settings unchanged. With these settings, we need on average 5068 evaluations of the performance functions to get an estimator of $P(g(\boldsymbol{\theta}) > 6)$ and the c.o.v. of the estimator is about 0.77. (This can be seen in Fig. A.-8, where $P(g(\boldsymbol{\theta}) > 6) \approx 10^{-5}$, and in the second row, we can see that the c.o.v. of exceedance probability estimator at 10^{-5} is around 0.77). This number is estimated based on 100 independent Subset Simulation. If we use the brute-force Monte Carlo simulation to get an estimator with c.o.v. of 0.77, the estimated number of evaluations is given by Eq. 2.34, where $\delta_n[\hat{p}_F] = 0.77$, $p_F \approx 10^{-5}$:

$$n = \frac{1 - p_F}{\delta_n^2[\hat{p}_F] p_F} \approx 130000. \quad (\text{A..2})$$

Fig. A.-7 shows both the samples of Subset Simulation and the 130000 samples of Monte Carlo simulation. We can see that the samples of Subset Simulation “move with a direction” to the low probability region. On the contrary, the samples of Monte Carlo simulation are randomly generated and spreading to all directions.

Figure A.-7: The samples with Subset Simulation (left) and Monte Carlo simulation (right) for estimating $P(g(\boldsymbol{\theta}) > 6)$. For Subset Simulation, the initial sample size at the first level is $N = 1000$; For Monte Carlo simulation, the total sample size is 130000.

Figure A.-8: The exceedance probability curves (top subplot) and the corresponding c.o.v. of the estimators (bottom subplot) by Subset Simulation and Monte Carlo simulation. For Subset Simulation, the initial sample size $N = 1000$, the mean exceedance probability curve is given by 100 independent runs of Subset Simulation; For Monte Carlo simulation, the total sample size is 130000.