Reduced aperture radar signal processing

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REDUCED APERTURE RADAR SIGNAL PROCESSING

CHIA LEK GUAN

School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Master of Engineering

2021

Statement of Originality

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Chia Lek Guan

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Summary

Antenna arrays are an important component of radar systems. As computational power and efficiency increase over the years, the trend towards adopting digital signal processing has increasingly spurred the development of novel and unconventional processing techniques for antenna arrays. In this thesis, the two following unconventional topics are investigated:

1. Explore the Infinite Impulse Response (IIR) array concept to further develop understanding on the topic and establish a novel application to Synthetic Aperture Radar (SAR).
2. Introduce a novel Multi-Frequency Virtual Array (MFVA) technique, explore, and validate its properties through simulations and an experiment.

Due to the equivalence that exists between conventional antenna arrays and Finite Impulse Response (FIR) filters, there has been an interest in the equivalent array structure corresponding to the IIR filter, the IIR array. While previous literature has established the basics of this array structure, its properties are not well explored. In this thesis, the properties of the IIR array are further explored, and a novel application to SAR is introduced. Analysis against the FIR array for unfocused SAR through application of the IIR array structure to two real datasets suggests that the IIR array structure can significantly reduce computational power requirements.

In the subsequent chapter, a novel technique that combines signals from multiple frequencies in a coherent manner to form an extended virtual array, MFVA, is introduced. Due to its multi-frequency nature, this array structure also introduces a beam pattern that varies with range, and with it an increased sensitivity to phase errors,

which manifests itself when the signal has bandwidth, violating the assumption when deriving the MFVA. However, phase modulated signals are shown to be fairly robust against such an issue. Finally, some other practical issues such as virtual antenna element redundancy in MFVA design and coherence of the different carriers propagating through a physical medium are also briefly discussed within the scope of this thesis.

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List of Abbreviations

ADC	Analog-to-Digital Converter
BIBO	Bounded-Input Bounded-Output
CS	Compressive Sensing
DOA	Direction-Of-Arrival
DSP	Digital Signal Processor
FIR	Finite Impulse Response
FPGA	Field-Programmable Gate Array
HF	High Frequency
IIR	Infinite Impulse Response
LFM	Linear Frequency Modulation
MFVA	Multi-Frequency Virtual Array
MSE	Mean-Squared Error
PSLR	Peak-to-Sidelobe Ratio
RAR	Real Aperture Radar
RF	Radio Frequency
SAR	Synthetic Aperture Radar
SLAR	Side-Looking Airborne Radar
SNR	Signal-to-Noise Ratio
TDL	Tapped Delay-Line
UAV	Unmanned Aerial Vehicle
ULA	Uniform Linear Array

List of Symbols

a	Filter feedback coefficient
$\mathbf{a}(\theta)$	Array vector for a single source impinging from angle θ
$\mathbf{A}$	Matrix of array steering vectors towards multiple angles
b	Filter feedforward coefficient
B	Bandwidth of signal
C	Speed of light
d	Array inter-element spacing
$\mathbf{d}(\theta)$	Array delay vector corresponding to a source impinging from angle θ
D	Array directivity
$\mathbf{D}$	Matrix of array delay vectors
e	Euler's number
f	Frequency
h	Filter impulse response
H	Filter transfer function
k	Frequency factor for MFVA
$m(t)$	Modulating signal at time t
N	Number of array elements
R	Range of target from transmitter/receiver
$\mathbf{r}$	Position vector
$s(t)$	Transmitted signal at time t
$\mathbf{s}(t)$	A vector of transmitted signal at time t
t	Time from a given reference (where $t = 0$)
T	Period
$u(t)$	Transmit signal at time t , time-delayed from source to array reference element

v	Speed of SAR platform along track
w	Array element weighting
$\mathbf{w}$	Array weighting vector
W	Array element weighting defined in the frequency domain
$x(t)$	Signal at array/filter element at time t
$\mathbf{x}(t)$	Vector of signal across array/filter elements at time t
$y(t)$	Weighted array/filter output signal at time t
$\mathbf{y}(t)$	Weighted array/filter output vector at time t
z	Unit vector in the Z-transform domain
α	Dolph-Chebyshev equation parameter
β	Dolph-Chebyshev equation parameter
δ	SAR image resolution
θ	Angle as measured clockwise from the x-axis
λ	Wavelength
τ	Time constant or time delay
$\phi(t)$	Signal phase at time t

Chapter 1

Introduction

This thesis presents an investigation into Infinite Impulse Response (IIR) arrays as well as a novel virtual array technique which we term Multi-Frequency Virtual Array (MFVA). The purpose of this chapter is to put the techniques introduced into the context of the practical world, to highlight the reasons for their use, and to clearly define the scope which these issues are addressed within this thesis by means of a concise description of the motivation, objectives, and major contributions of the thesis.

1.1 Background and Motivation

Radar has come a long way since its principle was first demonstrated in 1904 by Christian Huelsmeyer. Radars have been widely deployed in many military and civilian applications [1], and its technology is becoming increasingly mature. Along with the trend towards autonomous vehicles, radars are also set to become of increasing importance in the consumer market [2]. Already, radar has started making its presence felt in this space, having more recently appeared in commercially available products such as mobiles phones [3], fall detection systems for homes [4], amongst others. However, there is still much room for further adoption of radar technologies. Typically, cost effectiveness and design flexibility are important traits in consumer products, hence enabling radar technologies such as those which allow

computationally simpler signal processing and the ability to work around physical constraints will find great use in this market. Hence, it is likely that the trend towards simple and low-cost radar will find applications for some time to come.

One of the most important components of the radar is its antenna array. Antenna arrays are antenna systems which utilise multiple, often identical, and evenly spaced array elements to achieve increased directivity and provide spatial filtering capabilities to the system such as the ability to perform direction finding. Complex weightings can be assigned to the individual array elements to achieve a desired beam pattern and look direction. Essentially, antenna arrays are filters, but which exist in the spatial domain instead of the frequency domain. Instead of frequency selectivity, antenna arrays are selective to the directions of arrival of signals. Consequently, many filtering concepts can be directly applied to antenna arrays. This correspondence between time and spatial domain brings with it numerous opportunities for applying filtering and beamforming concepts in the other domain, as well as understanding concepts from a different perspective.

Signal processing has been an integral part of antenna array systems, and over the years, as technology has increasingly moved towards the digital domain, there has been increasing freedom in the design of processing schemes for antenna arrays. Increasing computational power and efficiency has brought along with them not only the ability to process increasing amounts of data and to use more complex processing to increase performance of antenna arrays, but also the opportunity to apply digital processing techniques in increasingly lower budget systems due to the increasing accessibility of cheap and ubiquitous digital processing capabilities. This opens doors to more unconventional array processing techniques. Riding on this digitalisation

trend that has been going on for years, this thesis explores two processing techniques to exploit such opportunities, with a focus on possible radar applications.

The first technique is the IIR array, a structure which has been proposed [5, 6] as a spatial equivalent to the conventional array structure, in an attempt to answer the question of whether arrays can have an IIR structure and if so, would they similarly inherit the advantages brought about by the IIR structure which is recursive in nature? The first question of the existence of such structure has been affirmed in the earlier literature, with the structures proposed and simulated, but a more thorough investigation into the second question is largely yet to be done. More recently, novel IIR spatial processing techniques have exploited the stationarity between signal transmitter, receiver, and target, to achieve augmentation of array through an IIR structure concept. This is done by continuously re-transmitting waveforms modified with a weighted and summed receive signal [7], forming a delay and sum feedback loop, hence emulating an IIR array. This thesis is an extension of the original proposed structure which does not exploit re-transmission, by proposing and applying the structure in Synthetic Aperture Radar (SAR) processing.

The second technique is a novel technique which attempts to address the question of whether it would be possible to replace physical antennas by substituting with additional frequencies. Motivated by numerous papers in the recent years on virtual arrays [8-12], there is desire to attempt to formulate an analytical method to generate virtual array elements for beamforming by utilising additional frequency components.

The motivation to explore these unconventional topics in array processing culminates in the explorations that is contained within this thesis.

1.2 Objectives

The objectives set out in this thesis are as follows:

1. To explore IIR arrays to establish possible applications for the array processing structure and to understand its limitations.
2. To introduce a novel Multi-Frequency Virtual Array technique, explore and validate its properties through simulations and experiments.

1.3 Major Contribution of The Thesis

The main contribution of this thesis are as follows:

1. Established the equivalence between a zero-initialised IIR array and an FIR filter of the same length, hence showing a greater similarity between the FIR array and IIR array than previously thought.
2. Applied the IIR array structure on SAR and validated its performance with 2 real datasets.
3. Developed a novel Multi-Frequency Virtual Array concept and validated it through numerous simulations and an experiment.

1.4 Organisation of Thesis

This thesis is organised into 5 chapters. In Chapter 2, some of the array beamforming concepts are reviewed. The optimality of array spacing for a Uniform Linear Array (ULA) is also briefly discussed, along with presenting the analytic method to calculate the optimal spacing for a Dolph-Chebyshev weighted broadside ULA. Chapter 3 reviews the relevant digital filtering concepts and the equivalence between

array spatial processing and digital filtering is established. Chapter 4 focuses on introducing the IIR array concept and a novel application of the concept to SAR processing, with analysis of the results from application to 2 real datasets, as well as on exploring some IIR filter properties which are of relevance to the IIR array. Chapter 5 introduces the novel MFVA concept and explores simulations and experiments performed to verify and understand properties of the MFVA. Finally, Chapter 6 concludes the thesis and provides recommendations for future works.

Chapter 2

Antenna Array Concept Review

In this chapter, a review of relevant antenna array concepts is presented. The Dolph-Chebyshev weighting which is frequently used in this thesis is also introduced in more details, in addition to an analytic method of calculating the optimal spacing for a Dolph-Chebyshev weighted broadside Uniform Linear Array (ULA).

2.1 Coordinate System

The coordinate system used in this thesis for array processing is illustrated in Figure 1. Any point in the coordinate system can be defined by the vector $\mathbf{r}$, where $|\mathbf{r}|$ represents the distance from the origin to the point and $\angle \mathbf{r}$ is the angle between the vector $\mathbf{r}$ and the x-axis, measured in the clockwise direction. Hence, all angular measurements are hence defined w.r.t. to the array axis.

Figure 1: Two-dimensional coordinate system.

2.2 Array Beampattern

Consider a single far-field narrowband signal impinging on an N-element linear array with elements uniformly spaced at a distance d apart, from an angle θ as measured w.r.t. the array axis as shown in Figure 2.

Figure 2: Structure of a ULA.

It can be observed that the path difference between the signal received at the n^{th} element with respect to the first element is $d_n \cos \theta$, where d_n is the physical distance (in meters) from the n^{th} element to the first element, $n=2,3,\dots,N$, and $d_1 = 0$. Denoting the signal received by the first element at time t by $s(t)$, the signal received at the n^{th} element can then be written as

$$x_n(t) = s(t)e^{-j2\pi f\left(\frac{d_n \cos \theta}{c}\right)}. \quad (2.1)$$

Noting that the difference between the signals of the elements differs only by a complex phase term, the 1^{st} element of the array can be arbitrarily set as the reference element and the array factor can then be defined as

$$\begin{aligned}
AF &= 1 + e^{-j2\pi f \left(\frac{d_2 \cos \theta}{c} \right)} + e^{-j2\pi f \left(\frac{d_3 \cos \theta}{c} \right)} + \dots \\
&= \sum_{n=1}^N e^{-j2\pi f \left(\frac{d_n \cos \theta}{c} \right)}.
\end{aligned} \tag{2.2}$$

By plotting the magnitude of the array factor across different angles, the beam pattern of the array can then be obtained. In a more general case, multiple far-field signals can impinge on the N -element ULA simultaneously, and the signal sampled across the array can then be written in a more general form,

$$\mathbf{x}(t) = \mathbf{s}^T(t) \mathbf{D}, \tag{2.3}$$

where $\mathbf{s}(t)$ is a vector representing the signal from I different sources and $\mathbf{D}$ is a matrix with I vectors of $\mathbf{d}(\theta_i)$, each representing an array vector of delays for a single source impinging from angle θ_i as measured from the array axis. The vectors are given by the equations

$$\mathbf{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_I(t) \end{bmatrix}, \tag{2.4}$$

$$\mathbf{D} = [\mathbf{d}(\theta_1) \quad \mathbf{d}(\theta_2) \quad \dots \quad \mathbf{d}(\theta_I)], \tag{2.5}$$

$$\mathbf{d}(\theta_i) = \begin{bmatrix} 1 \\ e^{-j2\pi f_0 \left(\frac{d_2 \cos \theta_i}{c} \right)} \\ \vdots \\ e^{-j2\pi f_0 \left(\frac{d_N \cos \theta_i}{c} \right)} \end{bmatrix}, \tag{2.6}$$

where f_0 is the carrier frequency and d_n is the physical distance (in meters) from the n^{th} element to the first element, $n=2,3,\dots,N$.

2.3 Array Weighting

Conventionally, beamforming is performed on the received signal of an array by weighting and the signal of each element and summing the results together, or as expressed in equation,

$$\mathbf{y}(t) = \mathbf{w}^H \mathbf{x}(t) , \quad (2.7)$$

where vector $\mathbf{w}$ is given by the equation

$$\mathbf{w} = [w_1, w_2, w_3, \dots, w_N]^T . \quad (2.8)$$

Figure 3: Beampattern of unweighted ULA.

In the case of an n -element, isotropic ULA with elements spaced $\lambda/2$ apart where the weighting values are uniform, or all having a unity value, plotting the array factor results in an unweighted array beampattern as shown in Figure 3. The width of the mainlobe is dictated by the number of elements, and the first sidelobe peak, which is also the highest sidelobe level, is at 13.26dB below the peak of the mainlobe, giving a Peak-to-Sidelobe Ratio (PSLR) of 13.26dB.

The PSLR of a ULA can be modified and controlled through array weighting. Numerous weighting functions exist, including the Dolph-Chebyshev weighting function [13], which gives the optimal mainlobe width or beamwidth for a given peak sidelobe level, keeping the sidelobe levels constant as shown in Figure 4.

Figure 4: Dolph-Chebyshev weighted array beampatterns with differing designed sidelobe levels.

The Dolph-Chebyshev weighting is more conveniently described in the frequency domain by its N -point Discrete Fourier Transform given by the real value of the equations

$$W[k] = \frac{T_{N-1}(\beta \cos \frac{\pi k}{N})}{10^\alpha}, \quad (2.9)$$

$$T_n(x) = \begin{cases} \cos(n \cos^{-1}(x)) & \text{if } -1 \leq x \leq 1 \\ \cosh(n \cosh^{-1} x) & \text{if } x \geq 1 \\ (-1)^n \cosh(n \cosh^{-1}(-x)) & \text{if } x \leq -1, \end{cases} \quad (2.10)$$

$$\beta = \cosh\left[\frac{\cosh^{-1}(10^\alpha)}{N}\right], \quad (2.11)$$

where $k = 0, 1, \dots, N-1$ and α is related to the side lobe level, SLL , by the following expression

$$\alpha = -\frac{SLL}{20}. \quad (2.12)$$

2.4 Array Directivity

The directivity of an unweighted ULA is given by the equation

$$D = \frac{2Nd}{\lambda}, \quad (2.13)$$

where d is the inter-element spacing and λ is the operating wavelength of the array, if $d \ll \lambda$ and $Nd \gg \lambda$. It can be observed that when the inter-element spacing is set as the commonly designed $\lambda/2$, the directivity simplifies to the number of elements in the array. Furthermore, it can also be observed that increasing inter-element

spacing within the valid range of values increases the directivity of the ULA, assuming grating lobes do not appear in the beampattern.

Past the value of $\lambda/2$ and as the value of d approaches the value of λ , the angle of beamsteering required before the 1st grating lobe appears decreases. At an inter-element spacing of λ , the peak value of the 1st grating lobe appears exactly at the endfire directions of the array as shown in Figure 5. Therefore, it follows that if no beamsteering is desired, the optimal inter-element spacing for the ULA to maximise array directivity is then a value between $\lambda/2$ and λ , but much closer to λ .

Figure 5: Unweighted broadside array beampatterns with differing inter-element spacing.

In the case of a Dolph-Chebyshev weighted ULA, this value can be conveniently derived analytically. Given that the level of the 1st grating lobes exceeds the designed

sidelobe level when $x < -1$, the maximum inter-element spacing where the sidelobe level does not exceed the designed sidelobe level occurs when $x = -1$. Observing equations (2.9) - (2.11), it can be derived that this maximum spacing is given by

$$d_{max} = \frac{\cos^{-1}(-\frac{1}{\beta})}{\pi} . \quad (2.14)$$

Applying this maximised inter-element spacing can yield a noticeable increase in the directivity of the array, such as a significant reduction in beamwidth for a given peak sidelobe level, with the same number of array elements, as shown in Figure 6.

Figure 6: Comparison of Dolph-Chebyshev weighted array beampattern with $\lambda/2$ spacing and with optimal spacing for broadside array.

In practical designs, other factors may influence the most desirable inter-element

spacing such as the requirement for beamsteering, where in the case of the requiring steering to nearly the endfire direction will require an inter-element spacing of nearly $\lambda/2$ to prevent grating lobes from appearing, or such as in the case of the use of directional antenna elements, the inter-element spacing can be further increased as long as the grating lobes can be mitigated due to being suppressed by the antenna element's radiation pattern.

Chapter 3

Array-Filter Equivalence

In signal processing, there is a direct correspondence that exists between a Uniform Linear Arrays (ULA) and Finite Impulse Response (FIR) filtering. In this chapter, the basics of FIR and IIR filtering are introduced, and the correspondence between FIR filters and ULAs are established, which paves the way for the introduction of the Infinite Impulse Response (IIR) array in the following chapter.

3.1 Digital Frequency Filters

In signal processing, digital filters can be characterised into two classes by their impulse response behaviour, namely FIR and IIR filters. The impulse response fully characterises the performance and response of a filter, and in general can be expressed by the transfer function in z-domain,

$$H(z) = \frac{\sum_{i=0}^I b_i z^{-i}}{\sum_{k=0}^K a_k z^{-k}}, \quad (3.1)$$

where a_k and b_i are the coefficients of the filter, for which $k=0,1,\dots,M$ and $i=0,1,\dots,N$.

3.2 FIR Filters

An FIR filter operates on a delay-and-sum principle and its direct implementation is that of a tapped delay line as shown in Figure 7.

Figure 7: Structure of an FIR filter.

With no recursion or feedback, the transfer function of the N^{th} order FIR filter, can be simplified by setting $a_0 = 1$ and $a_k = 0$ in equation (3.1) for all $k=1,2,\dots,M$, giving the equation

$$H_{FIR}(z) = \sum_{i=0}^N b_i z^{-i} , \quad (3.2)$$

or as equivalently expressed in the time domain,

$$y_{FIR}[n] = \sum_{i=0}^M b_i x[n-i] . \quad (3.3)$$

3.3 IIR Filters

To understand IIR arrays, it is imperative to first discuss its equivalent structure in

the frequency domain, the IIR filter. Contrary to FIR filters, IIR filters have an impulse response of infinite length. The infinite length of the impulse response is a result of recursion, an additional tapped delay feedback path in the filter structure. With reference to equation (3.1), IIR filters are generally designed with $a_0 = 1$, and often, but not always, an N^{th} order IIR filter can be expressed by setting $M=N$, or as expressed in the Z-transform domain,

$$H_{IIR}(z) = \frac{\sum_{i=0}^N b_i z^{-i}}{1 + \sum_{k=1}^M a_k z^{-k}} . \quad (3.4)$$

Equivalently, this can be expressed in the time domain as

$$y_{IIR}[n] = \sum_{i=0}^N b_i x[n - i] + \sum_{k=1}^M a_k y[n - k] . \quad (3.5)$$

Figure 8: Direct-form I structure of an IIR filter.

The introduction of the feedback path in the IIR filter structure results in numerous implementation structures which have different tradeoffs. Of these structures, the most direct implementation of an IIR filter, and the structure most relevant to the discussion of IIR arrays, is the Direct-Form I, which is a direct translation of an IIR filter's equation into a filter structure as shown in Figure 8.

3.4 IIR Filter Stability and Transience

The addition of the feedback path in IIR filters bring about issues such that of ensuring the stability of the filter design, that any bounded input to the filter should result in a bounded output. It follows therefore that the impulse response of a Bounded-Input Bounded-Output (BIBO) stable IIR filter should not be increasing with time. In practice, IIR filters are designed with the poles lying strictly within the unit circle, which is a necessary and sufficient condition for the stability of a casual IIR filter, and which guarantees that the impulse response of the filter decays over time. However, even this criterion presents itself as a practical design challenge, especially as the IIR filter order gets higher and the location of the filter poles become increasingly sensitive to quantisation errors. One common way to mitigate this issue is to design high-order IIR filters as a cascade of lower order filters, whose coefficients are less sensitive to quantisation errors, and are easier to guarantee stability.

The addition of the feedback path also results in an infinite length in the impulse response. This results in the IIR filter output never reaching a perfect steady state response. In practice, the output is taken to be at steady state once sufficient time

delay has occurred, such that the output is similar to that which would be present if the input signal was present since time $t = -\infty$. For an IIR filter with an exponentially decaying impulse response such as a single-pole IIR filter, the steady state response can be conveniently taken to be a multiple of its time constant. For the exponential filter, which can be described as an IIR filter with $b_0 = b$, $a_1 = 1 - b$, and $0 < b < 1$, or equivalently by the equation,

$$y_{exp}[n] = bx[n] + (1 - b)y_{exp}[n - 1] . \quad (3.6)$$

Its time constant can be expressed in number of samples by the equation

$$\tau = -\frac{1}{\log(1-b)} , \quad (3.7)$$

for which its impulse response value reduces from its original value by a multiple of $1/e$ after every time constant. Consequently, the output of the filter will reach over 99% of its final value in less than 5 time-constants and can therefore be considered to practically be in steady state.

3.5 Array-filter Equivalence

Consider an arbitrary i^{th} signal impinging on the array. The signal captured by the array can be equivalently written as a vector of different time delayed versions of the signal at the reference element, $s_i(t)$. The vector is given by

$$x_i(t) = \begin{bmatrix} s_i(t) \\ s_i(t - \frac{d_2 \cos \theta_i}{c}) \\ \vdots \\ s_i(t - \frac{d_N \cos \theta_i}{c}) \end{bmatrix}, \quad (3.8)$$

and conventionally, the output of the array after beamforming, as a result of the i^{th} signal impinging on the array, can thus be written as

$$y_i(t) = \sum_{k=0}^N w_k s(t - \frac{d_k \cos \theta_i}{c}) . \quad (3.9)$$

It should be noted from the equation that both the linear array spacing and the angle from which the signal impinges on the array have an impact on the time delay between samples of the transmitted signal across the array. This can be interpreted by first setting $\cos \theta_i$ to the maximum value of 1. By noting that

$$c = \frac{\lambda}{T_0}, \quad (3.10)$$

where T_0 is the period corresponding to the array operating frequency of f_0 , it can be observed that by constraining the elements to be spaced uniformly $\frac{\lambda}{2}$ apart, the transmitted signal is sampled at the Nyquist rate. Therefore, it can be said that the distance between linear array elements corresponds to the time delay between samples of a temporal-frequency filter.

A second observation can be made from the presence of $\cos \theta_i$ in the equation. This implies that there is a correspondence between the temporal frequency in digital filtering and the cosine of the direction of arrival of a narrowband signal impinging

on the array, as shown in the results simulated in Figure 9. Therefore, a narrowband beamformer is in fact, rigorously, a frequency filter.

Figure 9: Frequency component of signal received across array elements.

Finally, drawing upon the above observations and from equations (3.3) and (3.9), it can be noted that this method of weighting received signals across an array, in particular a Uniform Linear Array, has a one-to-one correspondence to FIR filtering. This equivalence is well established in array signal processing [14]. For clarity, this conventional array beamforming structure will be referred to as the “FIR array”, in contrast to the “IIR array” concept which will be discussed in the next chapter.

Chapter 4

IIR Spatial Arrays

The Infinite Impulse Response (IIR) array structure has been established in earlier papers [5, 6], but its properties and advantages are not well studied. In this chapter, the IIR structure is reviewed, with more in depth understanding of its limitations. With better understanding, a novel application of the IIR spatial structure on Synthetic Aperture Radar (SAR) processing is introduced to exploit its advantages, with application to 2 real datasets. Finally, analysis of the results from these 2 datasets and some other relevant properties of the IIR arrays are explored towards the end of the chapter.

4.1 IIR Array Concept

The conventional form of array weighting on a Uniform Linear Array (ULA) is a spatial equivalence of the Finite Impulse Response (FIR) filter and is thus also known as the FIR array. Due to the array-filter equivalence discussed at the end of the previous chapter, an FIR array can directly be weighted with the coefficients of an FIR filter, with windowing functions commonly being used as the weighting functions. In a single array snapshot, a filter with length equivalent to the number of array elements is applied on the spatial data to perform spatial filtering on the received signals, resulting in a multi-input, single output data structure. Contrary to the FIR array, the equivalent array structure for the IIR filter is not well studied.

Figure 10: IIR subarray processing.

While the length of FIR filters and the length of its impulse response are equivalent, due to recursion, the length of the IIR filter's feedforward and feedback elements are far shorter than its impulse response, which stretches to infinity. Furthermore, feedback is necessary for the IIR array to be structurally similar to an IIR filter. Therefore, the only logical structure for an IIR array is the subarray processing concept as shown in Figure 10, which can equivalently be considered in the time domain as an IIR filter with data input longer than the IIR filter's feedforward and feedback network. This is in contrast to the FIR array where the temporal equivalence would be an FIR filter operating on data equivalent in length to the filter length.

Consequently, while a parallel can be drawn between the output of an FIR array and an FIR filter in steady state, the filter equivalence for an IIR array is that of an IIR filter in a transient state. However due to the nature of the array length being fixed, the nature of the transience is fully predictable and can be given by an equivalent FIR array of length equivalent to the number of array elements. Consider an IIR filter of order N as described in equation (3.4), but for which there exists the

constraint $M=N$. The transfer function of the filter can then be expressed by the equation

$$H(z) = \frac{\sum_{k=0}^N a_k z^{-k}}{1 - \sum_{i=1}^N b_i z^{-i}}, \quad (4.1)$$

where a_k and b_i are the coefficients of the filter, for which $k=0,1,\dots,N$ and $i=1,\dots,N$.

The equation can be expanded and written in a non-recursive form,

$$\begin{aligned} H(z) = & \sum_{m=0}^N \left(\sum_{n=0}^m a_n c_{m-n,1} \right) z^{-m} \\ & + \sum_{m=N+1}^{\infty} \left(\sum_{n=0}^N a_n c_{N,m-n+1} \right) z^{-m}, \end{aligned} \quad (4.2)$$

where $c_{k,j} = \sum_{l=1}^k b_l^{k-l+j}$, $k \geq 1$ and $c_{0,j} = 1$, which is in essence an FIR filter of infinite length. It can be shown that due to the convolution process, with an input signal $x[n]$ of finite length $M+1$, where $M > N$, the output is only affected by the first $M+1$ coefficients. Hence the transfer function of the IIR filter is effectively truncated to length $M+1$ as described by the equation

$$\begin{aligned} H_M(z) = & \sum_{k=0}^N \left(\sum_{n=0}^k a_n c_{k-n,1} \right) z^{-k} \\ & + \sum_{k=N+1}^M \left(\sum_{n=0}^N a_n c_{N,k-n+1} \right) z^{-k}. \end{aligned} \quad (4.3)$$

Essentially, given $h[n]$ and $h_M[n]$, the impulse response of the filters $H(z)$ and $H_M(z)$, the convolution process can be written as

$$y[n] = (h * x)[n] \quad (4.4)$$

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= \sum_{m=0}^{\infty} h[m]x[n-m] \\
&= \sum_{m=0}^M h[m]x[n-m] \\
&= \sum_{m=-\infty}^{\infty} h_M[m]x[n-m] \\
&= (h_M * x)[n] .
\end{aligned}$$

Such a filtering process can equivalently be described in a simplified manner by an FIR filter of order M , as given by the transfer function

$$H_M(z) = a_0 + \sum_{k=1}^M a_k z^{-k} . \quad (4.5)$$

Given that at any time instance t , all inputs of the array snapshot are available instantaneously, and as noted earlier, only a single output, $y_n(t)$, is a useful output of the IIR array, it is therefore logical to conclude that the advantages of the IIR structure over the FIR structure cannot apply to a standard antenna array.

4.2 Unfocused Synthetic Aperture Radar

Radar imaging uses information from radio waves reflected off surfaces which are typically, but not always, generated by the radar, to form images [15, 16]. Airborne radars, typically operating on aircrafts or satellites, typically have antennas which point to the side, perpendicular to the direction of motion of the platform. Hence these radars are known as Side-Looking Airborne Radars (SLAR). Due to their operation in the Radio Frequency (RF) band, these radars possess certain advantages over

optical systems, including the ability to remain operational regardless of optical visibility range, especially when operated at a suitably low frequency. Broadly, these radars are categorised into Real Aperture Radars (RAR) and Synthetic Aperture Radars (SAR), both of which exploit range (cross track) resolution and azimuthal (along track) resolution to form 2-dimensional images. An illustration of the SAR image data collection process is shown in Figure 11 and described in further details later in the section.

Figure 11: Illustration of the synthesis of an aperture from numerous returns as a plane moves along a given direction.

The synthesis of the range resolution utilises the same principle in which most radar systems rely upon. This resolution is synthesized in the direction perpendicular to the direction of motion and utilises the coherency of radar systems to measure the time delay between the transmission and reception of the reflected signal off a target, a concept shown in Figure 12. After pulse compression, the range resolution is given in relation to the bandwidth B of the signal transmitted by the equation

$$\delta_{range} = \frac{c}{2B}, \quad (4.6)$$

where c is the speed of light. This can be equivalently expressed for a pulsed signal as,

$$\delta_{range} = \frac{c\tau_{pulse}}{2}, \quad (4.7)$$

given that the bandwidth of a rectangular pulse can be given by

$$B_{pulse} = \frac{1}{\tau_{pulse}}. \quad (4.8)$$

Figure 12: Transmitted and received pulse for ranging.

The resolution of the pulse can be equivalently thought of as the time it takes for a pulse to travel twice the width of the pulse, given that the distance of the target from a monostatic radar (whose transmitter and receivers are co-located), R , is also given by the equation

$$R = \frac{c\Delta t}{2}, \quad (4.9)$$

where Δt is the time it takes for the signal to travel from the radar to the target located at distance R away, and back, or twice the distance from the radar to the target. Hence, to achieve a higher resolution, the pulse width must be reduced. This also reduces the energy of the signal within the pulse, for a given transmission power, reducing the Signal-to-Noise Ratio (SNR), and hence the operating range of the radar. One method to mitigate this issue is to integrate returns from numerous pulses transmitted over time. This integration is most effective when the returns can be coherently integrated, which broadly speaking, occurs if the targets remain within the same range cell of the radar throughout the whole integration time, although techniques do exist [17-20] to coherently integrate longer than such a limit.

Another method to mitigate this problem is through a technique known as pulse compression, which is applied to signals whose frequency content varies in time, often through a matched filtering process, to obtain a pulse-compressed signal. Most commonly, the signal used is a Linear Frequency Modulation (LFM) signal, also known as a chirp signal. This allows for the signal to simultaneously have high energy, but a short pulse duration, which after pulse compression is just dictated by the inverse proportion of its bandwidth, or as expressed in equation,

$$T_{compressed} = \frac{1}{B}. \quad (4.10)$$

In the azimuth direction, orthogonal to the range direction, SAR systems derives its resolution in a manner that differs from conventional RAR systems. While a RAR has its azimuthal resolution defined by the physical beamwidth of the antenna or

antenna array, SAR systems coherently combine signal returns along the flight path to synthesize a long antenna aperture, hence resulting in an azimuthal resolution that can be of orders of magnitude higher than the antenna beamwidth. Conceptually, this can be thought of as a long antenna array imaging a given area, with the assumption that the subjects of the imaging are largely stationary from the start to end of the imaging, which often holds true for imaging land masses, although techniques do exist to compensate for moving targets [21-23].

Given an antenna of aperture D , the beamwidth can be expressed as a function of its wavelength, λ , and aperture by the relationship

$$Beamwidth = \frac{\lambda}{D}, \quad (4.11)$$

and the one-way real aperture resolution at range R from the platform operating can be obtained after applying the small angle approximation to be

$$\delta_a = \frac{\lambda R}{D}. \quad (4.12)$$

For a point target as marked by a cross in Figure 11, the maximum distance the platform can travel while still keeping the target within the beamwidth is equivalent to the one-way real aperture resolution δ_a . Hence, taking into account the two-way path transmission, the resolution of a stripmap SAR system is

$$\begin{aligned}\delta_{focused} &= \frac{\lambda R}{2R_a} \\ &= \frac{D}{2},\end{aligned}\tag{4.13}$$

which is in fact, independent of range. This resolution can be further improved by applying the spotlight-mode SAR where the antennas are steered to focus on a given target so it can be illuminated for a longer travel distance, at the expense of coverage.

In practice, to approach the focused SAR resolution, numerous compensation techniques usually need to be applied to compensate for the non-linear motion of platform as well as range migration or range walk due to the requirement of integrating over a long distance of travel. Depending on the way the motions involved deviate from the ideal situation, and the parameters of the SAR system, these processes can be computationally intensive. In recent years, however, as there has been an increasing interest in SAR systems being deployed on smaller and lighter platforms, including Unmanned Aerial Vehicle (UAV) platforms such as multi-rotor drones [24-27], the limited computational resources on these platforms, render such complex processing schemes unfeasible, especially if offline processing of data is not desirable, due to reasons such as real-time requirements or lack of adequate storage space. Therefore, simpler processing schemes such as unfocused SAR has found uses in some of these systems [28-31].

The typical assumption in deriving the resolution of the unfocused SAR is that the acceptable deviation of the distance of every array point in the synthetic aperture to an arbitrary fixed location along any given downrange is no more than $\pm \frac{\lambda}{8}$. This corresponds to a phase deviation of no more than $\frac{\pi}{4}$. In this case, the limit of the

unfocused cross-range or azimuthal resolution of the unfocused SAR can be obtained by the equation [32]

$$\delta_{crossrange} = \sqrt{\frac{R\lambda}{2}}, \quad (4.14)$$

where R is the slant-range, and λ is the wavelength of the signal. Therefore, unfocused SAR systems are more desirable in systems with smaller slant-range such as those mounted on smaller UAV platforms.

4.3 IIR Array for Unfocused SAR

Figure 13: SAR image processing scheme.

In a real antenna array, there is no data overlap from a single snapshot to the subsequent snapshot in time. In a SAR array, however, the array snapshot forming a single azimuth pixel and the snapshot forming the subsequent azimuth pixel overlaps significantly. From a filtering equivalence perspective, this is equivalent to filtering with a sliding window across a long sampling window. Hence, although the SAR system is structurally an antenna array performing spatial filtering, unlike real antenna arrays, its input and output data streams continuously. The implication of this is that some advantages which apply to an IIR filter can therefore also directly apply in SAR processing. One of these advantages is the significantly lower IIR order required in comparison to an FIR to meet a given set of filter specifications. Therefore, for SAR unfocused processing, it is possible to replace a given FIR array structure with an IIR array structure of lower order to achieve similar array performance. Due to the lower filter order and smaller number of historical data points required, this results in the IIR SAR array having lower computational and memory requirements.

In order to evaluate the effectiveness of this proposed technique, a basic SAR signal processing chain is applied to the experimental data in this thesis as shown in Figure 13. Firstly, range compression or matched filtering is performed, followed by azimuth compression for the focused SAR image. For the unfocused SAR image, a simple broadside beamforming is performed in place of azimuth compression. In the following sections, the proposed IIR array processing scheme will be compared with the conventional unweighted FIR array processing for unfocused SAR. A Dolph-Chebyshev weighted FIR array will also be included for comparison due to its optimality in ensuring the smallest beamwidth, hence the highest resolution for a given sidelobe level. Due to the relatively higher resolution of the focused SAR

image, it can serve the purpose of being the reference image for comparison between the different unfocused SAR methods. The unfocused images' similarity to this "ground truth" image can be approximately quantified through a mean-square error comparison. For the dataset with considerable amount of speckle noise in the image, a multilook processing step will also be performed before the images are shown to enhance the perceptible details for visual comparison. This is done by a simple averaging of every x cells in the azimuth direction, where x is the chosen multilook factor.

The PC used to process the data and as a benchmark for the processing load is a fairly standard, although somewhat dated, desktop PC equipped with Intel Xeon E3-1225 V2 processor. A comparison will also be made on the number of multiply and addition operations per output pixel, a metric which would be important if the processing is performed on a Digital Signal Processor (DSP), Field-Programmable Gate Array (FPGA) or other specialised platforms instead.

In designing an IIR filter, and by extension, IIR array weighting, ensuring stability is an important consideration. As unfocused SAR processing is often applied to systems that prioritise lower computation and ease of design, a straightforward design method and low IIR order is also desirable. It is empirically found that the above objectives can be achieved while obtaining a good response with an IIR filter described by the transfer function

$$H(z) = \frac{b+bz^{-1}}{1+(1-2b)z^{-1}}, \quad (4.15)$$

where $2b < 1$. This simple 1st order IIR filter is also guaranteed to be stable.

4.4 ERS Satellite Dataset

The online tutorial platform, EO College, provides data from a European ERS satellite mission scene in raw data format [33], with the possibility to perform range and azimuth compression to obtain a focused image and is therefore suitable for evaluation of the relative performance of the proposed IIR unfocused SAR method in comparison to the conventional FIR unfocused SAR method. Based on the relevant parameters of the dataset summarised in Table 1 and equation (4.14), the unfocused resolution is estimated to be 155m.

Figure 14: Focused SAR image with no multilook.

Figure 15: Focused SAR image with multilook factor of 5.

Parameter	Value
Operating wavelength	5.656cm
Range to target at $t=0$	852358.15m

Table 1: Relevant ERS satellite dataset parameters.

A crop of the focused SAR image is shown in Figure 14. To aid visual comparison of the results, a multilook factor of 5 is applied across the azimuth direction, resulting in an image with more visually discernable details as shown in Figure 15. This is especially helpful when images are scaled and displayed. Upon careful comparison, it can be noted that the details present in the image with multilook are mostly visible in the image without multilook, although somewhat obscured. For quantitative comparison, no multilook will be applied in order to preserve the maximum azimuthal resolution of the processed data.

Applying the moving average filter to form an unfocused SAR image and minimising for the Mean-Square Error (MSE) of the output amplitude between the unfocused image and the focused image, the ideal length of the FIR array was found to be $L=53$, corresponding to a phase deviation of the signal in the azimuth direction by approximately 0.52π radians. The resulting image is shown in Figure 16. It can be noted that the resolution of the unfocused image is considerably lower than that of the focused image and fine details are mostly lost. This can be largely attributed to the long range of the satellite from the imaged area, resulting in a low unfocused SAR resolution.

Figure 16: Optimised unfocused, unweighted FIR SAR image.

Figure 17: Optimised unfocused, Dolph-Chebyshev weighted FIR SAR image.

Figure 18: Optimised unfocused, IIR SAR image.

Array Type/weighting	FIR Unweighted L=53	FIR Dolph-Chebyshev (L=54, R=22.3dB)	IIR (ord=1)
Mean Squared Error	10577526	10644145	10819454
Runtime in Matlab	0.38s	0.38s	0.13s
Multiply/add operations per pixel	52 Add	53 Add, 54 Multiply	3 Multiply, 3 Add
Memory slots required (excluding current output)	53	54	3

Table 2: Comparison of optimal unfocused processing algorithms.

Optimising the IIR Array weighting with the constraint as shown in equation (4.15), the lowest MSE value is obtained with b-value at 0.0137. A comparison of the runtime and MSE of the unfocused image using both IIR and FIR Array structure is presented in Table 2. In order to ensure that the FIR Array is not disadvantaged due to the high Peak-to-Sidelobe Ratio (PSLR) of the uniformly weighted array response, the Dolph-Chebyshev array weighting is introduced, with the resulting image shown in Figure 17, as another basis for comparison. As can be seen from the results, however, optimizing the sidelobe ratio of the Dolph-Chebyshev weighting to minimise MSE did not yield any improvement over the unweighted FIR array. Upon close visual comparison, it can be noted that the results from the FIR arrays are virtually indistinguishable from one another while the IIR array produced the result shown in Figure 18, which is a nearly identical, but slightly fuzzier image.

While the IIR array was not able to match the MSE performance of the FIR arrays, it can be noted that the image produced is largely similar, at a significantly lower Matlab processing load. In order to have a fair comparison between the FIR and IIR arrays, additional comparisons will be shown below. In the first comparison, the MSE values

are matched as closely as possible, and in the second comparison, the Matlab runtime is matched as closely as possible. It should be noted that the cost of multiplication in comparison to addition depends highly on the hardware, and the metric for comparison, for example in most DSP chips, multiplication and addition can both be done in a single instruction cycle, but multiplication on FPGA can cost up to orders of magnitude more gates than addition, especially if floating point calculation is involved. Therefore, direct comparison for this metric is difficult, if not impossible, without any knowledge of the exact hardware, and the number of multiplication and addition cycles will be included only for reference.

Figure 19: Unweighted FIR SAR image with MSE matched to IIR SAR.

Figure 20: Dolph-Chebyshev weighted FIR SAR image with MSE matched to IIR SAR.

Figure 21: Optimised unfocused, IIR SAR image.

Array Type/weighting	FIR Unweighted L=33	FIR Dolph-Chebyshev (L=34, R=18.6dB)	IIR (ord=1)
Mean Squared Error	10826373	10824652	10819454
Runtime in Matlab	0.32s	0.32s	0.13s
Multiply/add operations per pixel	32 Add	33 Add, 34 Multiply	3 Multiply, 3 Add
Memory slots required (excluding current output)	33	34	3

Table 3: Comparison of MSE-matched unfocused processing algorithms.

As it is not possible to exactly match the MSE values, in the following comparison, the lengths of the FIR arrays are chosen to be as small as possible while not having an MSE smaller than the IIR array. The results are tabulated in Table 3 and shown in Figure 19 to Figure 21 (Figure 21 is a repeat of Figure 18, reproduced for ease of comparison). As it can be seen from the SAR images, the results are nearly indistinguishable from one another, but the FIR methods incur a significantly larger processing load on Matlab, and the unweighted FIR array would likely incur higher processing load on most hardware than the IIR array.

In the final comparison, the lengths of the FIR arrays are shortened until the Matlab processing times are just below that of the IIR arrays. Due to the way the processing time decreases in steps, it is not possible to exactly match the processing time of all the methods in comparison, but the lengths of the FIR arrays are chosen such that the processing times are as closely matched as possible. The results are tabulated in Table 4 and the FIR array images shown in Figure 22 and Figure 23. It can be observed that in this comparison, the FIR arrays have significantly higher MSE values, and visually, the images are significantly worse in quality than the IIR array.

Figure 22: Unweighted FIR SAR image with runtime matched to IIR SAR.

Figure 23: Dolph-Chebyshev weighted FIR SAR image with runtime matched to IIR SAR.

Array Type/weighting	FIR Unweighted L=6	FIR Dolph-Chebyshev (L=6, R=18.6dB)	IIR (ord=1)
Mean Squared Error	11841377	11795643	10819454
Runtime in Matlab	0.12s	0.12s	0.13s
Multiply/add operations per pixel	5 Add	5 Add, 6 Multiply	3 Multiply, 3 Add
Memory slots required (excluding current output)	6	6	3

Table 4: Comparison of runtime-matched unfocused processing algorithms.

The overall results from the ERS satellite dataset suggest that while the unfocused SAR processing do produce results that have discernibly less details than the focused SAR processing, the structure can approximate the best of the FIR array, but at a significantly reduced processing load. Furthermore, it is observed that the Dolph-Chebyshev windowing does not appear to increase the quality of the unfocused SAR image.

4.5 Unmanned Aerial Vehicle SAR Dataset

As described by equation (4.14), higher operating frequency and shorter distance to the imaging scene both can contribute to an increase in the unfocused SAR resolution. In this following UAV SAR dataset, the operating frequency is higher, and the range of the scene is orders of magnitude nearer, hence the estimated achievable unfocused resolution is much higher at 1.9m. The relevant parameters are tabulated in Table 5 and further details of the platform and dataset, which was collected at Chinese Garden in Singapore, can be found in the original paper [34].

Parameter	Value
Operating wavelength	2cm
Range to centre of SAR image	343m

Table 5: Relevant parameters of UAV SAR platform.

The focused image is shown in Figure 24. Due to the nature of this dataset, speckle is low and no multilook averaging in the azimuthal direction is required. This dataset was processed on a different PC, a laptop with an Intel i5-1135g7 processor. Due to the differences in the architecture of the processor amongst other factors, the ratio of the processing load between the IIR and FIR arrays will differ.

Applying the moving average filter to form an unfocused SAR image and minimising the MSE of the amplitude between the unfocused image and the focused image, the ideal length of the FIR array was found to be $L=470$, with results shown in Figure 25, and the optimal IIR Array weighting was found to have a b-value of 0.0017 with results shown in Figure 26. A comparison of the run-time and MSE of the unfocused image using both IIR and FIR array structures is presented in Table 6.

Figure 24: Focused SAR image for UAV flight.

Figure 25: UAV flight SAR image for optimal unfocused FIR array.

Figure 26: UAV flight SAR image for optimal unfocused IIR array.

Figure 27: UAV flight SAR image for unfocused FIR array with runtime matched to IIR array.

Array Type/weighting	FIR Unweighted L=460	FIR Unweighted L=20	IIR (ord=1)
Mean Squared Error	24.98	88.07	28.89
Runtime in Matlab	9.7s	6.2s	6.0s
Multiply/add operations per pixel	469 Add	19 Add	3 Multiply, 3 Add
Memory slots required (excluding current output)	460	20	3

Table 6: Comparison of Unfocused UAV SAR image processing.

As observed from the results, the unfocused images are not significantly different from the focused image and contain large amounts of detail, nearly comparable with the focused image, unlike in the satellite dataset. This can be attributed to the significantly higher unfocused resolution of the UAV dataset. It also appears that similar to the satellite dataset, even though the optimal FIR array and IIR array MSE values somewhat differ, the visual difference between both images is minimal, with the IIR array processing requiring significantly less processing to achieve the image. Also similar to the satellite dataset, when the runtime between IIR and FIR arrays are approximately matched in Matlab, the image quality for the FIR array, as shown in Figure 27, suffers significantly and has barely any details in the image.

The result from this dataset shows the possibility of using the IIR array structure for unfocused SAR processing to reduce the azimuth processing load, while simultaneously maintaining relative high fidelity in the output image, especially if the range to the target is small, such as in the case of SAR images from a multirotor UAV.

4.6 Real Data Results Discussion

For any given point on the ground, the symmetrical phase variation of the return signal due to the variation in distance of the platform can be expressed by the equation

$$\phi(t) = e^{-j2\pi\frac{v^2 t^2}{\lambda R_0}}, \quad (4.16)$$

where R_0 represents the shortest downrange at reference time $t = 0$ and v represents the speed of the SAR platform along track. The minimum MSE values for the ERS dataset was obtained when the phase deviation across the integration interval differs by no more than 0.5π while the phase in the UAV dataset differs by no more than 0.52π . The 2 values are in close agreement, although somewhat larger and not in exact agreement to the value assumed when deriving equation (4.14).

Figure 28: Phase of azimuth chirp with line in red indicating the length of integration for the unfocused ERS satellite SAR data.

Figure 29: Phase of azimuth chirp with line in red indicating the length of integration for the unfocused UAV SAR.

Through application of the IIR array processing on 2 real datasets, it has been shown that it is possible for an unfocused SAR system with IIR array structure to achieve similar results as that processed using a conventional FIR array structure. If the unfocused SAR system is implemented on a hardware through fixed point multiplication where the hardware only has built-in adders, a single multiplication will require $N-1$ additions, where N is the number of bits. However, most FPGAs do also have dedicated multipliers built in, therefore, depending on the processing load, the use of these multipliers may not incur significant processing cost. For floating point operation, there are implementations where the total area and speed of the adders and multipliers are actually comparable [35]. Therefore, depending on the hardware system which the technique is applied to, the savings brought about by the IIR array processing can be significant. This is especially beneficial in small, low-

cost, shorter-range platforms where the resolution loss due to moving to an unfocused system may be small and that the additional processing power saved by applying the IIR Array technique can have significant benefits in terms of cost and weight savings to the system.

4.7 Higher Order IIR Array Weighting

Although the use of unfocused processing often implies that processing load is likely to be of greater concern than image quality, the results from the above datasets do suggest that the use of the IIR array structure in unfocused processing is suboptimal compared to the unweighted FIR array, even if the penalty is minimal. There is therefore at least some value in exploring if a higher IIR order can lead to a more optimal IIR array design. This section attempts to address the issue by exploring the quality of approximation of the minimum MSE FIR array (unweighted array with $L=53$ for ERS satellite dataset) with increasing IIR filter order by adopting a least-square approach [36].

Figure 30: MSE vs IIR Order for IIR approximating FIR filter.

By applying the IIR filter as designed by this least-square method on the ERS satellite dataset as described in Section 4.4, the MSE value converges to approximately the FIR array value as the IIR filter order increases, as shown in Figure 30. Therefore, it shows that it is possible to use a sufficiently high order IIR to approximate the FIR array, where in this case the IIR array of order 4 starts to approach the FIR array's MSE value, but the practical applications may be limited given computational power penalty involved with minimal improvement in image quality.

4.8 Approximating FIR Arrays with IIR Arrays

In a scenario largely unique to the IIR array, there is the question of whether it would be possible to design an IIR array weighting which approximates an optimal FIR array weighting, such as the Dolph-Chebyshev array. In essence, the aim is to design an IIR filter whose transient response after a fixed input signal length resembles the Dolph-Chebyshev weighted FIR filter of equivalent length.

IIR filter type	Chebyshev type-II
Normalised passband frequency edge	0.0005
Normalised stopband frequency edge	0.1
Passband ripples (dB):	0.01
Stopband ripples (dB)	25.27

Table 7: Parameters for designing a Dolph-Chebyshev-like IIR array weighting for array $L=26$.

Figure 31: Comparison between limited-input IIR and FIR filters.

By adopting the design parameters as shown in Table 7, a 2nd order IIR filter design is obtained. Applying the weighting to an IIR array of length $L=26$, or equivalently truncating the impulse response of the designed filter at the given length, the resulting beampattern is shown in Figure 31, which appears to closely confirm to the Dolph-Chebyshev weighted array with designed sidelobe level of -23.7dB.

4.9 IIR Initialisation

As IIR filters are typically designed for their steady state response, it is possible for the transient response to be poor and has traditionally been a concern for IIR filters, especially those used in clutter removal [37], which is a type of highpass filtering. In the standard implementation of the IIR filter, no *a priori* information on the signal is assumed, and the best unbiased estimator for the output from the IIR filter at time $t < 0$, given that the signal has been present since $t = -\infty$, is the value 0. Hence, IIR

filters are, by default, initialised with the '0' value in its feedback taps. This method is also known as zero-initialisation. To mitigate the poor clutter rejection during transience, and hence increase the transient performance of the IIR filters for clutter rejection, step initialisation [38] and projection initialisation [39] are some of the initialisation techniques which have been proposed. In this section, an attempt is made to establish if such methods can bring improved performance to the IIR array, which in contrast to clutter rejection, is in fact a form of lowpass filtering.

Figure 32: Effect of step initialisation.

The idea of step initialisation is to estimate the 0Hz or DC value of the steady state filtered signal and use it as a value to initialise the IIR feedback taps to allow the value to be as close to the steady state as possible early in the transience. In step initialisation, all past input values are assumed to be the same value as the first input to the filter, and this value is then repeated to a sufficiently long history to initialise the filter. The net effect of this when applied on a broadside IIR beamformer, as

shown in Figure 32, is a significant increase in the sidelobe level due to the residual value from a high initial feedback value to the filter, which severely degrades the performance of the IIR array.

Projection initialisation is an initialisation method which claims superiority over step initialisation. The advantage of this initialisation method allows the ‘zeros’ of the filter to converge quickly at the expense of the passband. Upon implementation of the technique, the effect found on an array response is that the first few sidelobes is somewhat reduced. However, due to the proximity of the zeros to the 0 frequency as shown in Figure 35, as it always would be in an IIR broadside beamformer, the mainlobe is also attenuated, as shown in Figure 33. The net effect, as shown in Figure 34, is a decrease in the PSLR and a broadening of the mainlobe, both of which are undesirable.

Figure 33: Effect of projection initialisation, unnormalized plot.

Figure 34: Effect of projection initialisation, normalized plot.

Figure 35: Pole-zero plot of IIR array weights.

Therefore, the IIR array, with the investigated initialisation methods, do not perform better than an equivalent FIR filter designed with the optimal Dolph-Chebyshev weightings. Other initialisation methods do still exist [40-42] and it may be worth investigating if they can possibly bring about any improvement over the optimal FIR filter, hence increasing the performance of the IIR array.

Chapter 5

Multi-Frequency Virtual Array

In this chapter, a novel concept of exploiting multiple frequency, termed Multi-Frequency Virtual Array (MFVA), to augment an antenna array with virtual elements for beamforming applications is explored. This results in a range-dependent beampattern, which is verified in both simulations and an audio experiment. Finally, more practical considerations such as sensitivity of the proposed system to deviation from the pure-tone assumption in derivation of the MFVA, and redundancy of the designed MFVA are briefly explored.

5.1 Array Aperture Limitation

In narrowband beamforming, the performance of the array is largely limited by the number of elements available in the array. This constraint is particularly limiting in systems which operate at lower frequencies. The FM system is one such example. Due to the carrier signal having a relatively large wavelength of 3m, achieving high angular resolution with a FM-based radar system requires placing multiple array elements across a physically large aperture. Furthermore, to achieve a beampattern with no grating lobes and a with a large Peak-to-Sidelobe Ratio (PSLR) after weighting, it is often desirable to space the array elements uniformly at $\lambda/2$ spacing. In practice however, such physical requirements can be difficult to realise.

It has always been the aim of array processing to increase the performance of antenna arrays through techniques which can better exploit the data collected from antenna arrays. One of such a method is the virtual array. The concept of virtual arrays is not new. Numerous techniques have used the concept of virtual arrays, such as to reduce the number of required array elements for frequency-invariant beamforming without tapped delay-line (TDL) filters[8], or in aiding beampattern design for circular arrays [9]. In recent years, there has also been an increasing interest in exploiting multiple frequencies to create virtual arrays in order to enhance array performance in direction-of-arrival (DOA) estimation [10, 11] without utilising more physical elements.

In addition, a Multi-Carrier Frequency virtual array method has been explored [12] for High Frequency (HF) arrays, by utilising Compressive Sensing (CS) to reconstruct the samples at the virtual elements. As the computational load for utilising CS can be quite significant, in this section, we introduce an alternative method of utilising phase compensation to reconstruct the virtual array, which we term Multi-Frequency Virtual Array (MFVA). The benefit over using CS is that the compensation is low in computation and is therefore desirable for real-time and low complexity systems.

5.2 Extending the Array Aperture with Virtual Array

Consider an N -element uniform linear array (ULA) receiving I far-field sources simultaneously from differing directions, where the angle of signal arrival is denoted by angle $\theta_i (i = 1, 2, \dots, I)$ as measured w.r.t. the array axis. At a given time t , the signals across the array can be described by the equation

$$\mathbf{x}(t) = \mathbf{s}^T(t)\mathbf{A} , \quad (5.1)$$

where $\mathbf{s}(t)$ is a vector representing the signals from I different sources and $\mathbf{A}$ is a matrix with I vectors $\mathbf{a}(\theta_i)$, each representing an array vector for the I sources, as given by the equations

$$\mathbf{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_I(t) \end{bmatrix} , \quad (5.2)$$

$$\mathbf{A} = [\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2) \quad \cdots \quad \mathbf{a}(\theta_I)] , \quad (5.3)$$

$$\mathbf{a}(\theta_i) = \begin{bmatrix} 1 \\ e^{-j2\pi f_0 \left(\frac{d \cos \theta_i}{c} \right)} \\ \vdots \\ e^{-j2\pi f_0 \left(\frac{Nd \cos \theta_i}{c} \right)} \end{bmatrix} , \quad (5.4)$$

where f_0 is the carrier frequency and d is the inter-element spacing. Consider that the sources also simultaneously emit signals from multiple frequencies which can be described as a multiple of an arbitrary reference frequency as given by the equation

$$f_j = k_j f_1 , \quad (5.5)$$

we can then rewrite equations (5.1) - (5.4) as

$$\mathbf{x}(f_j, t) = \mathbf{s}^T(f_j, t)\mathbf{A}(f_j) , \quad (5.6)$$

$$\mathbf{s}(f_j, t) = \begin{bmatrix} s_1(f_j, t) \\ s_2(f_j, t) \\ \vdots \\ s_L(f_j, t) \end{bmatrix}, \quad (5.7)$$

$$\mathbf{A}(f_j) = [\mathbf{a}(f_j, \theta_1) \quad \mathbf{a}(f_j, \theta_2) \quad \cdots \quad \mathbf{a}(f_j, \theta_L)], \quad (5.8)$$

$$\mathbf{a}(f_j, \theta_i) = \begin{bmatrix} 1 \\ e^{-j2\pi f_j \left(\frac{d \cos \theta_i}{c} \right)} \\ \vdots \\ e^{-j2\pi f_j \left(\frac{Nd \cos \theta_i}{c} \right)} \end{bmatrix}. \quad (5.9)$$

Noting Equation (5.5), the relative phase of the n^{th} element in the array vector in a single snapshot, which is expressed as

$$a_n(f_j, \theta_i) = e^{-j2\pi f_j \left(\frac{nd \cos \theta_i}{c} \right)}, \quad (5.10)$$

can also be written as

$$a_{k_j n}(f_1, \theta_i) = e^{-j2\pi f_1 \left(\frac{k_j n d \cos \theta_i}{c} \right)}. \quad (5.11)$$

Therefore, virtual elements at locations $k_j n$ from the reference element can be obtained by processing the additional frequencies with the MFVA technique. First consider the case where all sources in the system are cooperative and synchronised. A radar system closely resembles this case. For simplicity of illustration, we can set an arbitrary reference time such that at $t = 0$, for a given desired source at the reference array element $n = 0$, the phases of the carrier frequency components of all signals are at 0. Given that the signal vector at the reference element is as expressed in Equation (5.7), the signal of the i^{th} source at time t can also be written as a time-

delayed version of the signal at the source, $u_i(f_j, t)$, given by the equation

$$s_i(f_j, t) = u_i(f_j, t) \tau_\phi(f_j, R_i), \quad (5.12)$$

where $\tau_\phi(f_j, R_i)$ denotes the time delay due to the signal travelling distance R_i from the source location to the reference element of the array, as represented by the equation

$$\tau_\phi(f_j, R_i) = e^{-j \frac{2\pi f_j R_i}{c}}. \quad (5.13)$$

It can be derived that at any given time t , the phase compensation required to form the virtual array for signals corresponding to f_j is given by the equation

$$x_c(f_j, t) = \frac{e^{j2\pi f_1 t} \tau_\phi(f_1, R_i)}{e^{j2\pi f_j t} \tau_\phi(f_j, -R_i)}, \quad (5.14)$$

for signals of frequency other than f_1 . This can be extended to the case of any arbitrary reference time by further compensating for the phase of the signals of different frequencies at the source location, at the reference time $t = 0$, as shown in the equation

$$x'_c(f_j, t) = x_c(f_j, t) e^{-j\phi_{f_j}}, \quad (5.15)$$

where $e^{j\phi_{f_j}}$ is the phase of the signal at frequency f_j at the source location at the arbitrary reference time $t = 0$.

Figure 36: System-level block diagram of a 2-real, 3-virtual element MFVA.

In performing the above processing, we obtain the MFVA such as in the example of a 2-element real array receiving signals from a far-field source transmitting at f_1 and f_2 coherently, where $f_2 = f_1/2$, as illustrated in Figure 36. After MFVA processing, a 3-element MFVA is formed, receiving signals which have been converted to a single reference frequency, f_1 . Conceptually, stretching or compressing a signal in time alters its frequency. Consequently, by making use of the temporal and spatial equivalence, we see that modifying the time information of the signal is equivalent to affecting the antenna location in space. Because the signal at frequency f_2 can be treated as a signal of frequency f_1 by changing the way the time indexes are interpreted, this, in a spatial sense affects the location of the antenna in which the signal is received from, creating a virtual antenna element. As inferred from Figure 37, the output of the signal at f_2 due to the 2nd element is exactly the output that would be expected from f_1 if there were a real element in between these 2 elements.

Figure 37: Single snapshot of a double-frequency virtual array, with $f_2 = \frac{f_1}{2}$.

5.3 Range-Dependant Beampattern

The processing to combine multiple frequencies into a single array snapshot gives rise to range-dependency in the beampattern of the MFVA. Consider a pseudo-monostatic radar scenario where a single transmit antenna transmits 2 continuous complex sinusoids at frequencies f_1 and $f_2 = 2f_1$, and a 2-element receive MFVA spaced $\lambda_1/2$ apart as shown in Figure 38.

Figure 38: Receiving antenna array of beampattern simulation.

Ignoring range ambiguity, the transmitted signal and the time the transmitted signal has been in propagation is known at any given time. To eliminate the inherent assumptions involved in simulations involving array steering vectors, the propagation

of the complex sinusoid through space to the elements individually is simulated using first principles through time-delay calculations, hence generating the receive signals at each of the 2 array elements. An exact phase compensation was calculated from Equation (5.14) for the simulated range corresponding to the sum of the distance to and from the target. From the results shown in Figure 39, a beampattern resembling the theoretical beampattern of a 3-element real linear array is achieved. While the receive power at the desired look range is similar to that of the larger virtual array, the power does not remain constant as the range of the receive signal varies. As the frequency of a signal is, by definition, the rate of change of its phase, the phase between the different frequencies at the reference element changes from snapshot to snapshot in a deterministic manner. In order to correctly determine the phase compensation, we need to know t , the amount of time elapsed from the reference time where $t = 0$, and R , the distance between the source and the reference element exactly, and an offset from this value will result in a beampattern differing from that designed 3-element beampattern. This results in a range-dependant beamformer as shown in the simulation result in Figure 40.

Figure 39: Beampattern at desired look range after phase compensation.

Figure 40: Simulated 2-dimensional beampattern.

By denoting the impinging signal as having an arbitrary range of r , and making the assumption that the signals $u_i(f_j, t)$ are pure-tone sinusoidal signals, it can be derived that the signal at the i^{th} array element, $x_i(f_j, t)$, after compensation is performed for a range R_i , is given by

$$\begin{aligned}
 x(f_j, t)x_c(f_j, t) &= s_i(f_j, t)a_i(f_j) \frac{e^{j2\pi f_1 t} \tau_\phi(f_1, R_i)}{e^{j2\pi f_j t} \tau_\phi(f_j, -R_i)} \\
 &= u_i(f_j, t) \tau_\phi(f_j, r) a_i(k_j f_1) \frac{u_i(f_1, t) \tau_\phi(f_1, R_i)}{u_i(f_j, t) \tau_\phi(f_j, -R_i)} \\
 &= u_i(f_1, t) e^{\frac{j2\pi f_j r}{c}} a_{k_{ji}}(f_1) e^{\frac{-j2\pi f_j R_i}{c}} e^{\frac{j2\pi f_1 R_i}{c}} \\
 &= u_i(f_1, t) e^{\frac{j2\pi f_1 r}{c}} a_{k_{ji}}(f_1) e^{\frac{j2\pi [f_j(r-R_i) - f_1(r-R_i)]}{c}} \\
 &= s_i(f_1, t) a_{k_{ji}}(f_1) e^{\frac{j2\pi [f_j(r-R_i) - f_1(r-R_i)]}{c}},
 \end{aligned} \tag{5.16}$$

where $a_i(f_j)$ is the steering exponent for the i^{th} array element at frequency f_j . The range dependency can be seen from the final term of the equation and is observed to be repetitive in range. This repetition corresponds to the highest common factor of all the frequencies in the system.

5.4 Audio Experiment Results

As the MFVA concept is not restricted to any specific type of array and can apply to radio frequency or acoustics-based array alike, an acoustics-based experiment was set-up to verify the theoretical concept.

A 2-element calibrated microphone array, spaced 10.5cm apart was placed 120cm away from the loudspeaker producing 2 pure-tone sine waves, with $f_1 = 1620\text{Hz}$ and $f_2 = 3240\text{Hz}$, hence producing a MFVA similar to that shown in Figure 38. Acoustically, the microphone array is in the far-field region of the speaker. The microphone array is placed on a turntable which allows for precise angular facing of the array setup. The signals from the microphones are fed through a microphone pre-amplifier into a Data Translation DT9873C pre-amp/Analog-to-Digital Converter (ADC) box, which sends digitised signals to a PC through USB. The PC also generates the dual-tone signal which is sent to the loudspeaker via an analog connection. The overall setup is placed in an acoustically treated sound room and summarised in the schematic setup diagram in Figure 41, with photos of the setup shown in Figure 42. It should be noted that a 3rd microphone was also connected in the array but was used only for debugging purposes.

Figure 41: Schematic of experimental setup.

(a)

(b)

(c)

Figure 42: Photos of experimental setup. (a): ADC/pre-amp box. (b) Overview. (c) Microphone array mounted on turntable.

A recording was conducted with the microphone array scanned from 0° to 180° w.r.t. the array axis, in steps of 0.5° clockwise rotation. The signals were bandpass filtered and the first 1500 samples and samples after the 2400th samples were all discarded. Complex data for each channel was obtained through a Hilbert transform, and the phase was compensated at each snapshot in accordance with Equation (5.14) by assuming an arbitrary range to form a beampattern at an arbitrary look range. The range is progressively increased in small steps until the 2-dimensional beampattern in Figure 43 is obtained. It can be observed that the beampattern obtained resembles the simulated plot in Figure 40 and confirms the validity of the MFVA concept. A cut of the 2-dimensional beampattern with highest peak value is selected and plotted in Figure 44, showing its resemblance to the ideal 3-element beampattern. It should be noted that an offset of positive 3.5° was noted in all data and plots but this does not affect the inference of the experimental results.

Figure 43: 2-dimensional beampattern obtained from audio experiment.

Figure 44: 1-dimensional beampattern cut obtained from audio experiment.

5.5 Signal with Bandwidth

In the earlier sections, an assumption was made in deriving the MFVA that the signals are purely sinusoidal. In conventional array beamforming, the narrowband assumption holds true even for signals with relatively large bandwidths, allowing array processing techniques to function in a close-to-ideal manner even under such conditions. Due to the nature of the MFVA exploiting the phase of signals, it is expected that it would have a higher sensitivity to deviation from this ideal assumption. In this section we explore the effects of signals with bandwidth and show that there exists a near-ideal modulation scheme for the MFVA.

The ideal MFVA modulation scheme exists if such a scheme ensures that the

modulating signal is not altered by the phase compensation process. Consider the case of a phase-modulated signal as given by the equation

$$u_i(f_j, t) = e^{j(2\pi f_j t + m(t))}, \quad (5.17)$$

where $m(t)$ denotes the modulating signal. Due to the addition of bandwidth, $u_i(f_j, t) \neq e^{j2\pi f_j t}$ and the derivations in equation (5.14) are no longer exact. Simplifying the equations for illustration, but without loss of generality, we can assume that the signal is co-located with the reference element. Then, the signal across the array due to the signal from angle θ can be written as

$$x_i(\theta, t) = e^{j2\pi\left(f_j\left(t - \frac{id \cos \theta}{c}\right) + m\left(t - \frac{id \cos \theta}{c}\right)\right)}. \quad (5.18)$$

Performing MFVA phase compensation, or in this case multiplying by $e^{-j2\pi(f_j(t) - f_1(t))}$, the compensated signal at the array is given by the equation

$$x'_i(\theta, t) = e^{j2\pi\left(f_1\left(t - \frac{k_j id \cos \theta}{c}\right) + m\left(t - \frac{id \cos \theta}{c}\right)\right)}. \quad (5.19)$$

The implication is that when the phase modulation scheme is adopted, the location of the carrier can be perfectly virtualized, but the modulating signal remains at the location of the physical element, and this will introduce a small distortion to the beampattern. However, this distortion due to the displacement of the location of the modulating signal from the virtual array element location to the physical location is minimal, and it can be shown that this holds true even in the case of a larger 10%

bandwidth. This is illustrated in the simulation result shown in Figure 45. Hence this modulation scheme, while not perfectly ideal, can be considered to be near-ideal.

Figure 45: Beampattern of signal with and without 10% bandwidth at desired look range, phase modulated.

Conversely, because the phase compensation involves a frequency multiplication, if the modulation scheme is a frequency modulation scheme, the phase compensation process will not only alter the frequency of the carrier, but also the frequency content of the modulating signal. This results in distortion of the phase-compensated array snapshot, and this can have a significant impact on the bandwidth tolerance of the MFVA. Figure 46 and Figure 47 show the simulation of the effect of a Digital Video Broadcast – Terrestrial (DVB-T) modulated signal received with the 2-real 3-element MFVA, which uses orthogonal frequency-division multiplexing, a type of frequency modulation. At a bandwidth of 1%, the distortion to the beampattern is much larger

than the 10% bandwidth of the phase-modulated signal. At 10% bandwidth with the DVB-T modulation, the beampattern is noticeably degraded. Hence, this shows a limitation of the MFVA system being sensitive to signals with bandwidth, but a phase modulated signal does increase the robustness of the system against such deviation from the pure-tone assumption.

Figure 46: Beampattern of signal with and without 3% bandwidth at look range.

Figure 47: Beampattern of signal with and without 10% bandwidth at look range.

5.6 Beyond Two Frequencies

Figure 48: 2D beampattern of MFVA with 3 frequencies.

Figure 49: 2D beampattern of MFVA with 4 frequencies.

In deriving the MFVA, no assumptions were made as to the number of frequencies used for generating the virtual array, hence the number of frequencies can be as many as physical limits would allow. By having more higher frequency components, the beampattern narrows in a 2-dimensional sense, giving higher directivity in both range and angle, as illustrated by the simulations shown in Figure 48 and Figure 49.

5.7 Virtual Element Redundancy

In designing the MFVA, if a requirement is set on having a set of continuous array elements, there will exist some virtual array elements which are unused. This can be illustrated with the example shown in Figure 50. In this example, 2 frequencies are used, with the virtual elements formed using a frequency double of the reference

frequency. The unutilised elements numbers 0, 14 and 18 are circled in red. Element 0 is a repeated element while the lack of elements 11, 12, 13, 15, 16 and 17 results in elements 14 and 18 being non-adjacent to the previous element of the MFVA. Hence, in this example, 11 of the 14 available elements are utilised to synthesize the virtual array.

Figure 50: 11-element MFVA designed with 7 real elements and 2 frequency components.

It is therefore of interest to investigate if certain number of frequencies can bring about an increase in the efficiency of the MFVA in terms of minimising the element redundancy. A design efficiency metric can be defined by

$$Design\ Efficiency = \frac{(synthesised\ array\ length)}{(no.of\ physical\ elements) \times (no.of\ frequencies)}, \quad (5.20)$$

where the numerator term represents the number of elements used for the synthesis of the ULA while the denominator term, $(no.of\ physical\ elements) \times (no.of\ frequencies)$, denotes the total number of real and virtual elements available, or in short, available elements. An exhaustive search can be performed by varying the number of frequencies used in designing the MFVA. The results of such an exhaustive search are shown in the semilog plot in Figure 51.

Figure 51: Semilog plot of the ratio of the number of physical elements required to the effective size of the synthesised MFVA.

The results suggest that the minimum redundancy of the MFVA design is largely dictated by the number of frequencies used and follows a relatively constant trend up to 28 elements. Also, the ratio of utilised elements reduces as the number of frequency increases in an approximately logarithmic manner.

5.8 Propagation Coherence

A practical consideration for implementation of the MFVA is the coherence of the signal propagation between signals spaced apart in frequency. As the frequency separation between carriers increases, so will the decoherence between these carriers while propagating in space. This is a property that is widely quantified by the

coherence bandwidth [43]. While numerous techniques to equalise the frequency spectrum do exist [44, 45], the resultant signal still suffers from relative signal strength fluctuation between the different frequencies. The MFVA technique is sensitive to phase, as evidenced in the results in Section 5.5 on signal with bandwidth, and distortions to the phase of the signal can cause a large impact on the ability for the MFVA to conform to its ideal beampattern. Therefore, practically, this may limit the application of the MFVA to more specific scenarios such as in a larger sparse array where the virtual elements due to closer-spaced frequencies will still result in meaningful gains, while having fewer redundant virtual elements.

Chapter 6

Conclusions and Future Work

6.1 Summary and Conclusions

In this thesis, two unconventional array processing techniques have been explored with a focus targeted at radar applications. On the first topic of IIR arrays, a more thorough understanding on the topic than previously available has been established, leading to the investigation of application of the IIR array structure in unfocused SAR processing. In this investigation, the technique has been applied to 2 datasets, and it has been shown on both occasions that the results from the IIR array structure approaches that of the FIR array structure for unfocused SAR processing, while requiring significantly less processing power. When the processing requirement in Matlab is matched, the FIR array performed significantly worse than the IIR array, resulting in images that are fuzzy, and which have lost most of the details. Furthermore, in the UAV dataset where the range to the scene is shorter, the unfocused SAR images approach the quality generated by focused SAR processing, hence demonstrating the potential of unfocused SAR with IIR array structure for such short-range applications on low-cost, lightweight platforms such as multi-rotor UAVs.

In Chapter 5, a novel MFVA technique has been introduced, and the results from an audio experiment performed to verify the concept agreed closely with the results obtained from simulations generated through first principles. Due to the phase

compensation involved, the beampattern of this technique is range dependant, which has also been shown in the audio experiment to be in agreement to the 2-dimensional beampattern generated in simulations. The phase compensation also implies sensitivity to the relative phase between the frequencies. This manifests itself in 2 issues. The first is the question of the sensitivity of the derived technique to bandwidth is due to assumptions of pure-tone signals in derivation. Due to the phase compensation, it is expected that the bandwidth of signals would distort the relative phases between different frequencies, leading to poor performance. Simulations verifies that in a DVB-T based modulation, the beampattern started to deviate in as little as 1% bandwidth, and at 10% bandwidth, the performance was poor. However, phase modulation is also found to significantly preserve this phase relationship and is able to maintain a relatively low distortion in the designed beampattern of the virtual array even at 10% bandwidth. However, the issue of phase coherence in actual systems over long distances due to large separation of the carrier wave frequencies is something that has been brought up in the thesis but requires further investigation to ascertain the actual effects and possible ways to mitigate it in real applications.

6.2 Recommendations for Future Work

While there has been some investigation to the IIR array and MFVA topics, more investigations are still required to better understand these topics.

For IIR array, on its application to SAR processing, while a simple and convenient low order IIR filter has been chosen for the investigation, the question of what the optimal order is remains. Furthermore, the current IIR array processing scheme only

utilizes real-valued weightings, however it may be possible to also include phase as part of the weighting design to improve upon the IIR array performance. Also, while it has been found that the best results are obtained when the filter length corresponds to a phase deviation in the azimuth of 0.5π for the 2 datasets in this thesis, more datasets will be required to confirm if this value is generally applicable in most, or all datasets. If this phase deviation is found to be a fixed value, then a data-free near-optimal IIR array weighting becomes a possibility, enabling real-time processing of unfocused SAR images on lightweight, low-cost aerial platforms. Other IIR array topics including initialisation, approximating specific FIR arrays were also introduced but still await further investigations to better understand the topics. For example, a more rigorous exploration of the initialisation issue may lead to the conclusion if there might be any ways to improve on IIR array performance beyond the optimal FIR array such as the Dolph-Chebyshev weighted FIR array.

Finally, MFVA is a novel concept introduced in this thesis. Although some initial verification of its performance has been done in simulation and through an audio experiment, more research will be required to understand the practical limitations, including issues such as propagation coherence, and possible ways to mitigate such issues. Also, the subject of range-dependant beamforming is one which could lend itself well to designing beampattern in the range domain, or even joint range and angular 2-dimensional beampattern design. This is something that is worth further investigation into. Although the MFVA is developed as a general array processing technique, it requires in its current form a cooperative transmitter and has limitations on the type waveforms used. This precludes its practical application in many communication scenarios without further development to overcome these limitations. Radar, with its coherent nature and especially with the co-location of transmitter and

receiver in a monostatic radar, is an ideal potential application of MFVA. Although MFVA has been validated experimentally in a close-range audio experiment, a monostatic radar experiment is likely the next step in exploring physical effects, such as propagation coherence, in real-world applications. At the current state of development, a phase modulated signal would be the ideal choice of radar waveform, although further investigation into the possibility of adopting other waveforms will greatly increase the versatility of applying the MFVA technique. Lastly, it is believed that a more thorough treatment of the MFVA redundancy topic will ultimately lead to more efficient MFVA designs.

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