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Perfect quantum state transfer on Cayley graphs over semi-dihedral groups

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Abstract

Perfect quantum state transfer plays a crucial role in quantum information processing and quantum computation. There have been extensive study of perfect quantum state transfer on Cayley graphs over abelian groups. In this paper, we consider the existence of perfect quantum state transfer on Cayley graphs over semi-dihedral groups which are nonabelian groups. Using the representations of semi-dihedral groups, we provide some necessary and sufficient conditions for Cayley graphs over semi-dihedral groups admitting perfect quantum state transfer. By those conditions, we present examples of perfect quantum state transfer on Cayley graphs over semi-dihedral groups. In addition, we propose results about whether some new Cayley graphs over nonabelian groups admit perfect quantum state transfer.

Keywords: Perfect quantum state transfer, Cayley graph, spectrum, semi-dihedral group.

1 Introduction

In a physical quantum computing protocol, the accurate transfer of quantum states between processers and registers of a quantum computer is a crucial ingredient for the short distance communication. Perfect state transfer (PST for short), introduced by Bose [10], addresses this task perfectly. More precisely, the output state from the receiver at some time t is, with probability equal to one, identical up to complex modulus to the input state of the sender at time $\tau = 0$.

By modeling various quantum networks on finite graphs, one can solve the problem of quantum networks in mathematical perspectives. Let $G(V, E)$ be an undirected simple graph and A be the adjacency matrix of $G(V, E)$. The transfer matrix is given by

$$H(t) = H_G(t) = \exp(-itA) = \sum_{k=0}^{+\infty} \frac{(-itA)^k}{k!} = (H_{a,b}(t))_{a,b\in V},$$

where $t \in \mathbb{R}$, $i = \sqrt{-1}$ and $H_{a,b}(t)$ stands for the (a, b)-entry of the matrix $H(t)$. Then, for $a, b \in V$, the graph $G(V, E)$ is said to admit PST from a to b at time $t(> 0)$ if the absolute value of $H_{a,b}(t)$ is equal to 1. When the previous condition holds for $a = b$, $G(V, E)$ is termed periodic at a with period

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$t(>0)$. Furthermore, we say that $G(V,E)$ is periodic if it is periodic at every vertex a with period $t(>0)$.

Searching for graphs exhibiting PST is a significant work since it has wide utilization in quantum information processing and cryptography [2–5, 9, 13, 17, 18, 30]. Consequently, much effort has been undertaken to investigate the existence of PST on all sorts of graphs. Christandl et al. [17] proved that paths and hypercubes admit perfect quantum state transfer. The existence of PST on certain graph operations has been considered in [1, 11, 19]. Godsil [22–24] provided a survey on perfect state transfer, which states the applications of algebraic combinatorics such as spectrum of adjacency matrix and association schemes to the existence of perfect state transfer on certain graphs. Notably, Godsil has completely characterized perfect state transfer using simple connected graphs. For some classes of special graphs such as distance-regular graphs, complete bipartite graphs, Hadamard diagonalizable graphs and so on, the problem of exhibiting PST was studied in [20, 21, 27, 30, 31]. Cayley graphs, as a well known class of vertex transitive graphs, have employed frequently in quantum communication networks since they possess marvelous algebraic structure. Research on which classes of Cayley graphs admitting PST has been presented in [6–9, 14, 16, 22–24, 33]. In these results, Bašić [6–8], Cheung et al. [16] and Bernasconi et al. [9] employed circulant graphs and cubelike graphs to study the existence of PST. A more general result about the existence of PST on Cayley graphs over abelian groups was provided in [33]. However, relatively little research has been carried out on Cayley graphs over non-abelian groups having PST. Precisely speaking, the only construction of PST using Cayley graphs over dihedral groups was proposed by Cao and Feng [14] very recently.

The semi-dihedral group, as a well-known non-abelian group, is a hot topic in group theory and has applications in graph theory [28] and symmetry classes of tensors [25, 26]. In this paper, we consider the existence of PST on Cayley graphs over semi-dihedral groups. Using semi-dihedral groups and their representations, several necessary and sufficient conditions for Cayley graphs over semi-dihedral groups admitting PST are carried out. We also prove that a Cayley graph over the semi-dihedral group is periodic if and only if it is integral, i.e., the eigenvalues of the graph are integers. Furthermore, we discuss the difference between the construction of PST on Cayley graphs over semi-dihedral groups and that on Cayley graphs over dihedral groups introduced in [14]. As a result, we can indeed obtain the existence and non-existence of PST on some new Cayley graphs.

An outline of this paper is as follows. Section 2 is devoted to some definitions and results about the representations of semi-dihedral groups and spectrum of Cayley graphs. In Section 3, we present necessary and sufficient conditions for the existence of PST on Cayley graphs over semi-dihedral groups. In Section 4, we discuss the integrality of Cayley graphs over semi-dihedral groups. In Section 5, we provide comparisons and examples of PST on Cayley graphs over semi-dihedral groups. Section 6 concludes the paper.

2 Preliminaries

In this section, we recall some basic definitions and results about the representations of semi-dihedral groups and spectrum of Cayley graphs.

2.1 The representations of semi-dihedral groups

For a finite group G and a complex vector space V with dimension $r>1$, a homomorphism τ from G to $GL(V)$ is called a representation of G and the degree of τ is r. Suppose that W is a non-zero complex vector space with finite dimension and ς is a representation of G mapping G to $GL(W)$. If there is an isomorphism $T: V \to W$ satisfying $\tau(a) = T^{-1}\varsigma(a)T$ for any $a \in G$, then τ and ς are said to be equivalent. For any representation $\tau: G \to GL(V)$ of G, we define the corresponding character of χ_τ by

$$\chi_\tau : G \to \mathbb{C}$$

such that $\chi_\tau(a) = \mathrm{tr}(\tau(a))$, where $a \in G$ and $\mathrm{tr}(\tau(a))$ stands for the trace of the representation matrix with regard to a basis of V.

For a subspace U of V, it is termed G-invariant subspace if $\tau(a)\theta \in U$ for any $a \in G$ and $\theta \in U$. Evidently, V and $\{0\}$ are G-invariant subspaces, which are named trivial subspaces. If V only has trivial G-invariant subspaces, then τ and χ_τ are called irreducible representation and irreducible character of G, respectively.

Assume that $n \geq 2$ is an integer. Define the semi-dihedral groups by

$$\begin{aligned}\mathrm{SD}_{8n} &= \langle u, v \mid u^{4n} = v^2 = 1, vuv = u^{2n-1}\rangle \\ &= \{1, u, u^2, \cdots, u^{4n-1}, v, vu, vu^2, \cdots, vu^{4n-1}\}.\end{aligned}$$

Throughout this paper, let Q_1, Q_2 and Q_3 be sets of integers such that

$$Q_1 = \{2, 4, \cdots, 2n-2\}, \tag{1}$$
$$Q_2 = \{1, 3, \cdots, n-1\} \cup \{2n+1, 2n+3, \cdots, 3n-1\}, \tag{2}$$
$$Q_3 = \{1, 3, \cdots, n-2\} \cup \{2n+1, 2n+3, \cdots, 3n-2\}. \tag{3}$$

Hormozi and Rodtes [25] deduced the representations and characters of SD_{8n} in the following lemma.

Lemma 2.1. [25] *Let $n \geq 2$ be an integer and $\omega = \exp(\frac{\pi i}{2n})$. If n is even, the representations and characters of SD_{8n} are provided in Table 1 and Table 2, respectively. If n is odd, we list the representations and characters of SD_{8n} in Table 3 and Table 4, respectively.*

Table 1: Representations of SD_{8n} for an even n

	u^ℓ $(0 \leq \ell \leq 4n-1)$	vu^ℓ $(0 \leq \ell \leq 4n-1)$
σ_1	1	1
σ_2	1	-1
σ_3	$(-1)^\ell$	$(-1)^\ell$
σ_4	$(-1)^\ell$	$(-1)^{\ell+1}$
ρ_h, $h \in Q_1 \cup Q_2$	$\begin{pmatrix} \omega^{h\ell} & 0 \\ 0 & \omega^{(2n-1)h\ell} \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^{(2n-1)h\ell} \\ \omega^{h\ell} & 0 \end{pmatrix}$

Table 2: Character table of SD_{8n} for an even n

	u^ℓ $(0 \leq \ell \leq 4n-1)$	vu^ℓ $(0 \leq \ell \leq 4n-1)$
φ_1	1	1
φ_2	1	-1
φ_3	$(-1)^\ell$	$(-1)^\ell$
φ_4	$(-1)^\ell$	$(-1)^{\ell+1}$
χ_h, $h \in Q_1$	$2\cos(\frac{h\pi\ell}{2n})$	0
χ_h, $h \in Q_2$	$\exp(\frac{h\pi i\ell}{2n}) + (-1)^\ell \exp(-\frac{h\pi i\ell}{2n})$	0

2.2 Cayley graphs basics

Suppose that G is a finite group and S is a non-empty subset of G such that $1_G \notin S$ and $S = S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph $\mathrm{Cay}(G, S)$ with the connection set S is the graph whose vertex set is G in which two vertices u and v are adjacent iff $uv^{-1} \in S$. Here we consider the condition $G = \langle S \rangle$, which says that $\mathrm{Cay}(G, S)$ is a connected graph. The adjacency matrix of $\mathrm{Cay}(G, S)$ is given by $A = (\alpha_{u,v})_{u,v \in G}$, where

$$\alpha_{u,v} = \begin{cases} 1, & \text{if } uv^{-1} \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Table 3: Representations of SD_{8n} for an odd n

	u^ℓ $(0 \le \ell \le 4n-1)$	vu^ℓ $(0 \le \ell \le 4n-1)$
σ_1	1	1
σ_2	1	-1
σ_3	$(-1)^\ell$	$(-1)^\ell$
σ_4	$(-1)^\ell$	$(-1)^{\ell+1}$
σ_5	$(i)^\ell$	$(i)^\ell$
σ_6	$(i)^\ell$	$(i)^{\ell+2}$
σ_7	$(-i)^\ell$	$(-i)^\ell$
σ_8	$(-i)^\ell$	$(-i)^{\ell+2}$
ρ_h $h \in Q_1 \cup Q_3$	$\begin{pmatrix} \omega^{h\ell} & 0 \\ 0 & \omega^{(2n-1)h\ell} \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^{(2n-1)h\ell} \\ \omega^{h\ell} & 0 \end{pmatrix}$

Table 4: Character table of SD_{8n} for an odd n

	u^ℓ $(0 \le \ell \le 4n-1)$	vu^ℓ $(0 \le \ell \le 4n-1)$
φ_1	1	1
φ_2	1	-1
φ_3	$(-1)^\ell$	$(-1)^\ell$
φ_4	$(-1)^\ell$	$(-1)^{\ell+1}$
φ_5	$(i)^\ell$	$(i)^\ell$
φ_6	$(i)^\ell$	$(i)^{\ell+2}$
φ_7	$(-i)^\ell$	$(-i)^\ell$
φ_8	$(-i)^\ell$	$(-i)^{\ell+2}$
χ_h $h \in Q_1$	$2\cos(\frac{h\pi\ell}{2n})$	0
χ_h $h \in Q_3$	$\exp(\frac{h\pi i\ell}{2n}) + (-1)^\ell \exp(-\frac{h\pi i\ell}{2n})$	0

Clearly, A is a real symmetric matrix and the eigenvalues of A are real numbers. Furthermore, if the eigenvalues of the adjacency matrix A are integers, then $\mathrm{Cay}(G,S)$ is called an integral graph.

In order to determine the eigenvalues and eigenvectors of the adjacency matrix A of the Cayley graph $\mathrm{Cay}(G,S)$, we need the following lemma.

Lemma 2.2. [32, pp. 69-70] *Assume that G is a finite group with order n. Let $\tau^{(1)}, \cdots, \tau^{(r)}$ be a complete set of unitary representatives of the equivalent classes of irreducible representations of G. Let χ_j be the corresponding character of the representation $\tau^{(j)}$ with degree d_j. Suppose that S is a non-empty subset of G satisfying $S = S^{-1}$ and $aSa^{-1} = S$ for any $a \in G$. Then the eigenvalues of the Cayley graph $\mathrm{Cay}(G,S)$ are*

$$\lambda_j = \frac{1}{d_j} \sum_{h \in S} \chi_j(h)$$

with multiplicity d_j^2, where $1 \le j \le r$. Furthermore, the vectors $v_{xy}^{(j)}$ $(1 \le x, y \le d_j)$ form an orthonormal basis for the eigenspace V_{λ_j}, where $v_{xy}^{(j)} = \sqrt{\frac{d_j}{n}} \left(\tau_{xy}^{(j)}(a)\right)_{a \in G}^t$, $\tau_{xy}^{(j)}(a)$ is the xy-th entry of the matrix $\tau^{(j)}(a)$.

3 Perfect state transfer on the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$

In this section, some necessary and sufficient conditions for the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ having PST are presented. We begin with a basic result about Hermitian matrices.

Assume that S is a subset of SD_{8n} such that $1_{\mathrm{SD}_{8n}} \notin S$ and $S = S^{-1} = \{s^{-1} : s \in S\}$. Let A be the adjacency matrix of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ and λ_j $(1 \le j \le 8n)$ its eigenvalues. Then there exits an unitary matrix $Q = (q_1, \cdots, q_{8n})$ to make the matrix A diagonal, where q_j is an eigenvector of A corresponding to λ_j ($1 \le j \le 8n$). So we obtain the following spectral decomposition of A:

$$A = \lambda_1 E_1 + \cdots + \lambda_{8n} E_{8n},$$

where $E_j = q_j q_j^*$ ($1 \le j \le 8n$) satisfy

$$E_\ell E_j = \begin{cases} E_\ell, & \text{if } \ell = j, \\ 0, & \text{otherwise.} \end{cases}$$

It is apparent from the spectral decomposition of A that the transfer matrix of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ has the following decomposition

$$H(t) = \exp(-i\lambda_1 t)E_1 + \cdots + \exp(-i\lambda_{8n} t)E_{8n}. \tag{4}$$

Since $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is vertex-transitive, one can get that $H(t) = \xi P$ for a unit norm number ξ and a permutation matrix P. Thus, Cao et al. [15] and Godsil [22] presented the following result.

Lemma 3.1. [15, 22] *Let S be a non-empty subset of SD_{8n} satisfying $gSg^{-1} = S$ for any $g \in \mathrm{SD}_{8n}$. Let $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ be a simple connected Cayley graph with the connection set S. If $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ exhibits PST from x to $y(\neq x)$, then yx^{-1} lies in the center of SD_{8n} and the order of yx^{-1} is 2.*

Clearly, the centers of SD_{8n} are $\{1, u^{2n}\}$ if n is even and $\{1, u^n, u^{2n}, u^{3n}\}$ if n is odd. Hence, PST occurs on $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ between any two distinct vertices x, y such that $y^{-1}x = u^{2n}$.

To describe the necessary and sufficient conditions for the existence of PST on the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$, we introduce the 2-adic exponential valuation of rational numbers. Let

$$\upsilon_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\},$$

such that $\upsilon_2(0) = \infty, \upsilon_2(2^k \frac{a}{b}) = k$, where $a, b, k \in \mathbb{Z}$ and $2 \nmid ab$. Suppose that $\infty + \infty = \infty + k = \infty$ and $\infty > k$ for any $k \in \mathbb{Z}$. Then, for any $\alpha, \beta \in \mathbb{Q}$, υ_2 has two properties as follows:

- $\upsilon_2(\alpha\beta) = \upsilon_2(\alpha) + \upsilon_2(\beta)$;
- $\upsilon_2(\alpha + \beta) \ge \min\{\upsilon_2(\alpha), \upsilon_2(\beta)\}$ and the equality holds if $\upsilon_2(\alpha) \neq \upsilon_2(\beta)$.

Note that

$$\begin{aligned} \mathrm{SD}_{8n} &= \langle u, v \mid u^{4n} = v^2 = 1, vuv = u^{2n-1} \rangle \\ &= \{1, u, u^2, \cdots, u^{4n-1}, v, vu, vu^2, \cdots, vu^{4n-1}\}. \end{aligned}$$

For simplicity, we view the element u^a of SD_{8n} as a if $0 \le a \le 4n-1$ and the element vu^a of SD_{8n} as a if $4n \le a \le 8n-1$ in the sequel. Below, we will discuss the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ admitting PST in the following two cases.

3.1 The case that n is odd

In this subsection, we study the case where n is odd. From the irreducible representations of SD_{8n} and Lemma 2.2, we obtain the main result as follows.

Theorem 3.2. *Suppose that $n > 1$ is an odd number and S is a subset of SD_{8n} such that the cardinality of S is $d > 0$ and $gSg^{-1} = S$ for any $g \in \mathrm{SD}_{8n}$. Let $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ be a simple connected Cayley graph with the connection set S. Let Q_1 and Q_3 be the sets defined by (1) and (3), respectively. Then $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ has eight (not necessarily distinct) eigenvalues $\lambda_1 = d, \lambda_2, \cdots, \lambda_8$ which correspond to the representations σ_1 to σ_8 of degree one, respectively, and $2n-2$ eigenvalues δ_j ($j \in Q_1 \cup Q_3$) with multiplicity 4 corresponding to the representations ρ_j of degree two, respectively. Furthermore, if $\kappa = \gcd(\lambda - d : \lambda \in \mathrm{Spec}(\mathrm{Cay}(\mathrm{SD_{8n}}, \mathrm{S})) \setminus \{\lambda_1\})$, then*

(1) the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ *is periodic with minimum period* $\frac{2\pi}{\kappa}$ *if and only if it is an integral graph.*

(2) the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ *has PST from* a *to* b *at time* t *if and only if*

(2i) the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ *is integral;*

(2ii) $a - b = 2n$ *or* $a - b = -2n$ *with* $0 \le a, b \le 4n-1$ *or* $4n \le a, b \le 8n-1$*;*

(2iii) For each $j \in Q_3$ *and* $z = 5, 6, 7, 8$, $\upsilon_2(\delta_j - d) = \upsilon_2(\lambda_z - d) = r$ *and* $\upsilon_2(\lambda - d) > r$ *for any other eigenvalues* $\lambda \ne \delta_j$ *with* $j \in Q_3$ *and* $\lambda \ne \lambda_z$ *with* $z = 5, 6, 7, 8$.

In addition, the minimum time $t = \frac{\pi}{\kappa}$.

Proof. Since $gSg^{-1} = S$ for any $g \in \mathrm{SD}_{8n}$, it follows from Lemma 2.1 and Lemma 2.2 that the eigenvalues of the adjacency matrix A of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ are $\lambda_1 = d, \lambda_2, \cdots, \lambda_8$ having multiplicity 1 and δ_j ($j \in Q_1 \cup Q_3$) with multiplicity 4. Let $\omega = \exp(\frac{\pi i}{2n})$. The vectors $q_1, q_2, \cdots, q_8, q_j^{(1)}, q_j^{(2)}, q_j^{(3)}, q_j^{(4)}$ ($j \in Q_1 \cup Q_3$) form an orthonormal basis for $\mathbb{C}^{8n}$, where

$$\begin{aligned} q_1 &= \tfrac{1}{\sqrt{8n}}(1, 1, \cdots, 1)^t, & q_3 &= \tfrac{1}{\sqrt{8n}}(1, -1, 1, -1, \cdots, 1, -1)^t, \\ q_2 &= \tfrac{1}{\sqrt{8n}}(1, \cdots, 1, -1, \cdots, -1)^t, & q_4 &= \tfrac{1}{\sqrt{8n}}(1, -1, \cdots, 1, -1, -1, 1, \cdots, -1, 1)^t, \\ q_5 &= \tfrac{1}{\sqrt{8n}}(\{i^k\}_{k=0}^{4n-1}, \{i^k\}_{k=0}^{4n-1})^t, & q_6 &= \tfrac{1}{\sqrt{8n}}(\{i^k\}_{k=0}^{4n-1}, \{-i^k\}_{k=0}^{4n-1})^t, \\ q_7 &= \tfrac{1}{\sqrt{8n}}(\{i^{3k}\}_{k=0}^{4n-1}, \{i^{3k}\}_{k=0}^{4n-1})^t, & q_8 &= \tfrac{1}{\sqrt{8n}}(\{i^{3k}\}_{k=0}^{4n-1}, \{-i^{3k}\}_{k=0}^{4n-1})^t, \end{aligned}$$

and for $j \in Q_1 \cup Q_3$,

$$\begin{aligned} q_j^{(1)} &= \tfrac{1}{\sqrt{4n}}(\{\omega^{jk}\}_{k=0}^{4n-1}, 0)^t, & q_j^{(2)} &= \tfrac{1}{\sqrt{4n}}(0, \{\omega^{(2n-1)jk}\}_{k=0}^{4n-1})^t, \\ q_j^{(3)} &= \tfrac{1}{\sqrt{4n}}(0, \{\omega^{jk}\}_{k=0}^{4n-1})^t, & q_j^{(4)} &= \tfrac{1}{\sqrt{4n}}(\{\omega^{(2n-1)jk}\}_{k=0}^{4n-1}, 0)^t. \end{aligned}$$

Then the first eight corresponding projective matrices $E_\ell = q_\ell q_\ell^*$ ($1 \le \ell \le 8$) of order $8n$ are

$$E_1 = \frac{1}{8n} J_{8n}, \qquad E_2 = \frac{1}{8n} \begin{pmatrix} J_{4n} & -J_{4n} \\ -J_{4n} & J_{4n} \end{pmatrix}, \tag{5}$$

$$E_3 = \frac{1}{8n}((-1)^{a+b})_{0 \le a,b \le 8n-1}, \qquad E_4 = \frac{1}{8n}(f_4(a,b))_{0 \le a,b \le 8n-1}, \tag{6}$$

$$E_5 = \frac{1}{8n}(i^{a-b})_{0 \le a,b \le 8n-1}, \qquad E_6 = \frac{1}{8n}(f_6(a,b))_{0 \le a,b \le 8n-1}, \tag{7}$$

$$E_7 = \frac{1}{8n}(i^{3(a-b)})_{0 \le a,b \le 8n-1}, \qquad E_8 = \frac{1}{8n}(f_8(a,b))_{0 \le a,b \le 8n-1}, \tag{8}$$

where J_m is the all-one matrix of order m and

$$\begin{aligned} f_4(a,b) &= \begin{cases} (-1)^{a+b}, & 0 \le a, b \le 4n-1 \text{ or } 4n \le a, b \le 8n-1, \\ (-1)^{a+b+1}, & \text{otherwise}, \end{cases} \\ f_6(a,b) &= \begin{cases} i^{a-b}, & 0 \le a, b \le 4n-1 \text{ or } 4n \le a, b \le 8n-1, \\ -i^{a-b}, & \text{otherwise}, \end{cases} \\ f_8(a,b) &= \begin{cases} i^{3(a-b)}, & 0 \le a, b \le 4n-1 \text{ or } 4n \le a, b \le 8n-1, \\ -i^{3(a-b)}, & \text{otherwise}. \end{cases} \end{aligned}$$

For $j \in Q_1 \cup Q_3$, the other corresponding projective matrices $E_j^{(k)} = q_j^{(k)} q_j^{(k)*}$ ($1 \le k \le 4$) of order $8n$ are

$$E_j^{(1)} = \frac{1}{4n}\left(e_j^{(1)}(a,b)\right)_{0 \le a,b \le 8n-1}, \qquad E_j^{(2)} = \frac{1}{4n}\left(e_j^{(2)}(a,b)\right)_{0 \le a,b \le 8n-1}, \tag{9}$$

$$E_j^{(3)} = \frac{1}{4n}\left(e_j^{(3)}(a,b)\right)_{0 \le a,b \le 8n-1}, \qquad E_j^{(4)} = \frac{1}{4n}\left(e_j^{(4)}(a,b)\right)_{0 \le a,b \le 8n-1}, \tag{10}$$

where

$$e_j^{(1)}(a,b)=\begin{cases}\omega^{j(a-b)}, & 0\le a,b\le 4n-1,\\ 0, & \text{otherwise},\end{cases}\qquad e_j^{(2)}(a,b)=\begin{cases}\omega^{(2n-1)j(a-b)}, & 4n\le a,b\le 8n-1,\\ 0, & \text{otherwise},\end{cases}$$

$$e_j^{(3)}(a,b)=\begin{cases}\omega^{j(a-b)}, & 4n\le a,b\le 8n-1,\\ 0, & \text{otherwise},\end{cases}\qquad e_j^{(4)}(a,b)=\begin{cases}\omega^{(2n-1)j(a-b)}, & 0\le a,b\le 4n-1,\\ 0, & \text{otherwise}.\end{cases}$$

Hence, we deduce the following spectral decomposition

$$A=\sum_{z=1}^{8}\lambda_z E_z+\sum_{j\in Q_1\cup Q_3}\delta_j(E_j^{(1)}+E_j^{(2)}+E_j^{(3)}+E_j^{(4)}),$$

and the transfer matrix

$$H(t)=\sum_{z=1}^{8}\exp(-i\lambda_z t)E_z+\sum_{j\in Q_1\cup Q_3}\exp(-i\delta_j t)(E_j^{(1)}+E_j^{(2)}+E_j^{(3)}+E_j^{(4)}). \tag{11}$$

If we plug (5) to (10) back into (11), then we derive the (a,b)-th entry of the transfer matrix as follows:

(i) if $0\le a,b\le 4n-1$ or $4n\le a,b\le 8n-1$,

$$\begin{aligned}(H(t))_{a,b} &= \frac{1}{8n}\Big(\exp(-i\lambda_1 t)+\exp(-i\lambda_2 t)+(-1)^{a+b}\left(\exp(-i\lambda_3 t)+\exp(-i\lambda_4 t)\right)\Big)\\ &\quad+\frac{1}{8n}\Big(i^{a-b}(\exp(-i\lambda_5 t)+\exp(-i\lambda_6 t))+i^{3(a-b)}\left(\exp(-i\lambda_7 t)+\exp(-i\lambda_8 t)\right)\Big)\\ &\quad+\frac{1}{4n}\sum_{j\in Q_1\cup Q_3}\Big(\omega^{j(a-b)}\exp(-i\delta_j t)+\omega^{(2n-1)j(a-b)}\exp(-i\delta_j t)\Big);\end{aligned} \tag{12}$$

(ii) if $0\le a\le 4n-1, 4n\le b\le 8n-1$ or $4n\le a\le 8n-1, 0\le b\le 4n-1$,

$$\begin{aligned}(H(t))_{a,b} &= \frac{1}{8n}\Big(\exp(-i\lambda_1 t)-\exp(-i\lambda_2 t)+(-1)^{a+b}\left(\exp(-i\lambda_3 t)-\exp(-i\lambda_4 t)\right)\Big)\\ &\quad+\frac{1}{8n}\Big(i^{a-b}(\exp(-i\lambda_5 t)-\exp(-i\lambda_6 t))\Big)\\ &\quad+\frac{1}{8n}\Big(i^{3(a-b)}\left(\exp(-i\lambda_7 t)-\exp(-i\lambda_8 t)\right)\Big).\end{aligned} \tag{13}$$

Obviously, in the case (ii), PST cannot occur due to $|H(t))_{a,b}|\le\frac{1}{n}$. It suffices to prove the theorem in the case that $0\le a,b\le 4n-1$, and the case that $4n\le a,b\le 8n-1$ can be proved similar to the previous comment. It follows from (12) that

$$|H(t)_{a,b}|\ \le\ \frac{8}{8n}+\frac{4n-4}{4n}=1.$$

Therefore, $|H(t)_{a,b}|=1$ if and only if

$$\begin{cases}\exp(-i\lambda_1 t)=\exp(-i\lambda_2 t),\\ \exp(-i\lambda_1 t)=(-1)^{a+b}\exp(-i\lambda_3 t),\\ \exp(-i\lambda_1 t)=(-1)^{a+b}\exp(-i\lambda_4 t),\\ \exp(-i\lambda_1 t)=i^{a-b}\exp(-i\lambda_5 t),\\ \exp(-i\lambda_1 t)=i^{a-b}\exp(-i\lambda_6 t),\\ \exp(-i\lambda_1 t)=i^{3(a-b)}\exp(-i\lambda_7 t),\\ \exp(-i\lambda_1 t)=i^{3(a-b)}\exp(-i\lambda_8 t),\\ \exp(-i\lambda_1 t)=\omega^{(a-b)j}\exp(-i\delta_j t),\\ \exp(-i\lambda_1 t)=\omega^{(2n-1)j(a-b)}\exp(-i\delta_j t),\end{cases} \tag{14}$$

for all $j \in Q_1 \cup Q_3$.

On one hand, if $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ has PST from a to b ($b \neq a$), from Lemma 3.1, we have $|a-b| = 2n$. Let $t = 2\pi T$ for $T \in \mathbb{R}$. Then $|H(t)_{a,b}| = 1$ if and only if

$$\begin{cases} (\lambda_2 - \lambda_1)T \in \mathbb{Z}, \\ (\lambda_3 - \lambda_1)T \in \mathbb{Z}, \\ (\lambda_4 - \lambda_1)T \in \mathbb{Z}, \\ (\delta_j - \lambda_1)T \in \mathbb{Z}, \text{ for any } j \in Q_1, \\ (\lambda_5 - \lambda_1)T - \frac{1}{2} \in \mathbb{Z}, \\ (\lambda_6 - \lambda_1)T - \frac{1}{2} \in \mathbb{Z}, \\ (\lambda_7 - \lambda_1)T - \frac{1}{2} \in \mathbb{Z}, \\ (\lambda_8 - \lambda_1)T - \frac{1}{2} \in \mathbb{Z}, \\ (\delta_j - \lambda_1)T - \frac{1}{2} \in \mathbb{Z}, \text{ for any } j \in Q_3. \end{cases} \tag{15}$$

Since the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is a simple graph, we get that $\sum_{z=1}^{8} \lambda_z + 4\sum_{j \in Q_1 \cup Q_3} \delta_j = 0$. It follows from (15) that

$$T\left(\sum_{z=2}^{8} \lambda_z - 7\lambda_1 + 4\sum_{j \in Q_1 \cup Q_3} (\delta_j - \lambda_1)\right) \in \mathbb{Z},$$

which is equivalent to $8n\lambda_1 T \in \mathbb{Z}$. Note that $\lambda_1 = d$ is a positive integer. Then T is a rational number. According to (15), λ_z ($z = 2, \cdots, 8$) and δ_j ($j \in Q_1 \cup Q_3$) are rational numbers. By Lemma 2.2 and the definition of characters over finite groups, we know that λ_z ($z = 2, \cdots, 8$) and δ_j ($j \in Q_1 \cup Q_3$) are algebraic integers which implies that λ_z ($z = 2, \cdots, 8$) and δ_j ($j \in Q_1 \cup Q_3$) are integers. Using the same argument as in the proof of Theorem 2.4 in [33], we can derive that $\upsilon_2(\delta_j - d) = \upsilon_2(\lambda_z - d) = r$ for each $j \in Q_3$, $z = 5, 6, 7, 8$ and $\upsilon_2(\lambda - d) > r$ for any other eigenvalues $\lambda \neq \delta_j$ with $j \in Q_3$ and $\lambda \neq \lambda_z$ with $z = 5, 6, 7, 8$.

One the other hand, applying the conditions ($2i$), ($2ii$) and ($2iii$) to (14), it is easy to check that (14) holds which implies that $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ admits PST from a to b at time $t \in \{\frac{\pi}{\kappa} + \frac{2\pi}{\kappa}s : s = 0, 1, 2, \cdots\}$.

If $a = b$, proceeding as in the proof above, we can show that the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is periodic with minimum period $\frac{2\pi}{\kappa}$ if and only if it is an integral graph. The proof of the theorem is now completed. □

3.2 The case that n is even

In this subsection, we prove the existence of PST on the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ for the case that n is even.

Theorem 3.3. *Assume that $n > 0$ is an even number and S is a subset of SD_{8n} such that the cardinality of S is $d > 0$ and $gSg^{-1} = S$ for any $g \in \mathrm{SD}_{8n}$. Let $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ be a simple connected Cayley graph with the connection set S. Let Q_1 and Q_2 be the sets defined by (1) and (2), respectively. Then $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ has four (not necessarily distinct) eigenvalues $\lambda_1 = d, \lambda_2, \lambda_3, \lambda_4$ corresponding to the representations σ_1 to σ_4 of degree one, respectively, and $2n-1$ eigenvalues δ_j ($j \in Q_1 \cup Q_2$) with multiplicity 4 which correspond the representations ρ_j of degree two, respectively. Furthermore, if $\kappa = \gcd(\lambda - d : \lambda \in \mathrm{Spec}(\mathrm{Cay}(\mathrm{SD_{8n}}, \mathrm{S})) \setminus \{\lambda_1\})$, then*

(1) the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is periodic with minimum period $\frac{2\pi}{\kappa}$ if and only if it is an integral graph.

(2) the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ has PST from a to b at time t if and only if

(2i) the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is integral;

(2ii) $a - b = 2n$ or $a - b = -2n$ with $0 \leq a, b \leq 4n-1$ or $4n \leq a, b \leq 8n-1$;

(2iii) For each $j \in Q_2$, $\upsilon_2(\delta_j - d) = r$ and $\upsilon_2(\lambda - d) > r$ for any other eigenvalues $\lambda \neq \delta_j$ with $j \in Q_2$.

Additionally, the minimum time $t = \frac{\pi}{\kappa}$.

Proof. This theorem can be proved in a similar method as shown in Theorem 3.3 and we omit the proof here. □

From Theorem 3.2 and Theorem 3.3, we see that the graph $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is an integral graph if it exhibits PST. Hence, we will discuss characterizations of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ being integral in the next section.

4 The integrality of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$

The integrality of graphs has attracted a great deal of attention in the past four decades. For general graphs, characterizing the integrality is extremely hard. Bridges and Mena [12] gave a complete characterization of Cayley graphs over abelian groups. Some necessary and sufficient conditions were presented for the integrality of Cayley graphs over dihedral groups in [29]. In order to construct PST on Cayley graphs over semi-dihedral groups by Theorem 3.2 and Theorem 3.3, we carry out a characterization of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ being integral under the case that $gSg^{-1} = S$ for any $g \in \mathrm{SD}_{8n}$.

Now we recall some basic definitions on cyclotomic fields. Let $\omega = \exp(\frac{\pi i}{2n})$ and denote by $K = \mathbb{Q}(\omega)$ the cyclotomic field. Then the Galois group of $K/\mathbb{Q}$ is

$$\mathrm{Gal}(K/\mathbb{Q}) = \{\epsilon_m : m \in \mathbb{Z}_{4n}^*\} \cong \mathbb{Z}_{4n}^*,$$

where $\mathbb{Z}_{4n}^* = \{m \in \mathbb{Z}_{4n} : \gcd(m, 4n) = 1\}$ is the unit group of the ring $\mathbb{Z}_{4n} = \mathbb{Z}/4n\mathbb{Z}$ and ϵ_m is defined by $\epsilon_m(\omega) = \omega^m$. Basing on the parity of n, we divide the study of the integrality of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ into the following two cases.

Proposition 4.1. *Let S be a non-empty subset of $\mathrm{SD}_{8n} = \langle u, v \mid u^{4n} = v^2 = 1, vuv = u^{2n-1}\rangle$ such that $gSg^{-1} = S$ for all $g \in \mathrm{SD}_{8n}$, where $n > 1$ is even. Put $S_1 = S \cap \langle u\rangle$, $S_2 = S \cap v\langle u\rangle$. Suppose that $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is a simple connected Cayley graph with respect to S. Then $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is an integral graph if and only if $S_1^m = S_1$ for all $m \in \mathbb{Z}_{4n}^*$, where $S_1^m = \{a^m : a \in S_1\}$.*

Proof. According to Lemma 2.1 and Lemma 2.2, the eigenvalues of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ are λ_z $(z = 1, 2, 3, 4)$ and δ_j $(j \in Q_1 \cup Q_2)$, where $\lambda_z = \sum_{a\in S} \varphi_z(a)$ and $\delta_j = \frac{1}{2}\sum_{a\in S_1} \chi_j(a)$. Evidently, λ_z are integers and δ_j are algebraic integers. Therefore, $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is an integral graph if and only if $\chi_j(S_1) = \sum_{a\in S_1} \chi_j(a) \in \mathbb{Q}$ for all $j \in Q_1 \cup Q_2$. For $j \in Q_1$ and $m \in \mathbb{Z}_{4n}^*$, we get that

$$\epsilon_m(\chi_j(S_1)) = \sum_{a\in S_1} \epsilon_m(\omega^{j\log_u a} + \omega^{-j\log_u a}) = \sum_{a\in S_1} (\omega^{mj\log_u a} + \omega^{-mj\log_u a}) = \chi_j(S_1^m).$$

For $j \in Q_2$ and $m \in \mathbb{Z}_{4n}^*$, we get that

$$\epsilon_m(\chi_j(S_1)) = \sum_{a\in S_1} \epsilon_m(\omega^{j\log_u a} + (-1)^{\log_u a}\omega^{-j\log_u a}) = \sum_{a\in S_1} (\omega^{mj\log_u a} + (-1)^{m\log_u a}\omega^{-mj\log_u a}) = \chi_j(S_1^m).$$

Thus $\chi_j(S_1) \in \mathbb{Z}$ if and only if $\chi_j(S_1) = \chi_j(S_1^m)$ for all $m \in \mathbb{Z}_{4n}^*$. Hence the Cayley graph $\Gamma = \mathrm{Cay}(SD_{8n}, S)$ is integral if and only if $S_1 = S_1^m$ for all $m \in \mathbb{Z}_{4n}^*$, which follows from the inverse formula for the group ring $\mathbb{C}[\mathrm{SD}_{8n}]$. The desired result follows. □

Proposition 4.2. *Let S be a non-empty subset of SD_{8n} such that $gSg^{-1} = S$ for any $g \in \mathrm{SD}_{8n}$ and $n > 1$ is odd. Assume that $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is a simple connected Cayley graph with respect to S. Then $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ is an integral graph if and only if $S^m = S$ for all $m \in \mathbb{Z}_{4n}^*$.*

Proof. According to Lemma 2.1 and Lemma 2.2, the eigenvalues of $\mathrm{Cay}(\mathrm{SD}_{8n}, S)$ are λ_z ($z = 1,2,3,4,5,6,7,8$) and δ_j ($j \in Q_1 \cup Q_3$), where $\lambda_z = \sum_{a\in S}\varphi_z(a)$ and $\delta_j = \frac{1}{2}\sum_{a\in S_1}\chi_j(a)$. Since S is a conjugate-closed subset of SD_{8n}, we assume that $S_1 = \{u^{\pm k_1}, u^{\pm k_1(2n-1)}, \ldots, u^{\pm k_r}, u^{\pm k_r(2n-1)}\}$ and S_2 is an empty set or a set consisting of an element belonging to

$$\{[v], [vu^2], [vu] \cup [vu^3], [v] \cup [vu^2], [v] \cup [vu] \cup [vu^3], [vu] \cup [vu^2] \cup [vu^3], v\langle u\rangle\}.$$

Then $\lambda_z (z = 1, 2, \ldots, 8)$ are integers and δ_j are algebraic integers. Proceeding as in the proof of Proposition 4.1, we can get the desired results. □

5 Comparisons and examples

In this section, we compare our constructions of PST with those in [14] and indicate that our constructions can generate PST on new Cayley graphs.

Cao and Feng [14] investigated the existence of PST on Cayley graphs over dihedral groups $D_{2m} = \langle a, b \mid a^m = b^2 = 1, bab = a^{-1}\rangle$. If m is odd, they pointed out that these is no PST on Cayley graphs over D_{2m} between any two distinct vertices. If m is even, the following fact was deduced.

Lemma 5.1. [14, Theorem 3.2] *Assume that $k > 1$ is an integer and $m = 2k$. Let $\mathrm{Cay}(D_{2m}, S)$ be a connected Cayley graph with regard to S satisfying $xSx^{-1} = S$ for each $x \in D_{2m}$. Then $\mathrm{Cay}(D_{2m}, S)$ has four (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and some multiple eigenvalues $\mu_h (1 \le h \le m-1)$. Furthermore,*

(1) If k is even, $\mathrm{Cay}(D_{2m}, S)$ has PST between two distinct vertices α and β iff

- *(1i) $\mathrm{Cay}(D_{2m}, S)$ is an integral graph;*
- *(1ii) $\alpha\beta^{-1} = a^k$;*
- *(1iii) $\upsilon_2(\mu_{2j'-1} - \lambda_1) = r$ for any $1 \le j' \le k/2$ and $\upsilon_2(\lambda_z - \lambda_1) > r$, $\upsilon_2(\mu_{2\widetilde{j}} - \lambda_2) > r$ for all $z = 1, 2, 3, 4$ and $1 \le \widetilde{j} \le (k-2)/2$.*

(2) If k is odd, $\mathrm{Cay}(D_{2m}, S)$ has PST between two distinct vertices α and β iff

- *(2i) $\mathrm{Cay}(D_{2m}, S)$ is an integral graph;*
- *(2ii) $\alpha\beta^{-1} = a^k$;*
- *(2iii) $\upsilon_2(\lambda_3 - \lambda_1) = \upsilon_2(\lambda_4 - \lambda_1) = \upsilon_2(\mu_{2h'-1} - \lambda_1) = s$ and $\upsilon_2(\lambda_2 - \lambda_1) > s, \upsilon_2(\mu_{2h'} - \lambda_1) > s$ for all $1 \le h' \le \frac{k-1}{2}$.*

Next, we consider the dihedral group D_{8n}. For a finite group G and $\gamma \in G$, define the conjugacy class of $\gamma \in G$ by

$$[\gamma] = \{g\gamma g^{-1} : g \in G\}.$$

Then, it is easy to check that D_{8n} has $2n + 3$ conjugacy classes as follows:

- $[1_{D_{8n}}] = \{1_{D_{8n}}\}$, $[a^{2n}] = \{a^{2n}\}$;
- $[a^s] = \{a^s, a^{-s}\}$, $s = 1, 2, \cdots, 2n-1$;
- $[b] = \{ba^{2e} : e = 0, 1, \cdots, 2n-1\}$, $[ba] = \{ba^{2e+1} : e = 0, 1, \cdots, 2n-1\}$.

In Lemma 5.1, since the subset S satisfies $gSg^{-1} = S$ for all $g \in D_{8n}$, the set S must be one of the conjugacy classes or some union of conjugacy classes of D_{8n}. Thanks to the connection set of the graph $\mathrm{Cay}(D_{8n}, S)$, i.e., $D_{8n} = \langle S\rangle$, S is consisted of at least one of the conjugacy classes $[b]$ and $[ba]$. Therefore, the cardinality of S is greater than $2n - 1$. That is to say, $\mathrm{Cay}(D_{8n}, S)$ is a d-regular graph, where $d > 2n-1$. According to Lemma 5.1, Cao and Feng [14] explored the existence of PST on d-regular graphs $\mathrm{Cay}(D_{8n}, S)$. However, for $d \leq 2n-1$, there is no result about the problem whether $\mathrm{Cay}(D_{8n}, S)$ admits PST due to the limitation of the set S.

Now we turn to our constructions of PST on the graph $\mathrm{Cay}(SD_{8n}, S)$. All conjugacy classes of SD_{8n} are introduced in the following proposition.

Proposition 5.2. [25] *Assume that $n > 1$ is an integer. Let Q_1, Q_2 and Q_3 be the sets defined by (1), (2) and (3), respectively.*
(1) If n is even, then SD_{8n} has $2n+3$ conjugacy classes which are given by

- $[1_{SD_{8n}}] = \{1_{SD_{8n}}\}$, $[u^{2n}] = \{u^{2n}\}$;
- $[u^s] = \{u^s, u^{(2n-1)s}\}$, $s \in Q_1 \cup Q_2$;
- $[v] = \{vu^{2e} : e = 0, 1, \cdots, 2n-1\}$, $[vu] = \{vu^{2e+1} : e = 0, 1, \cdots, 2n-1\}$.

(2) If n is odd, then SD_{8n} has $2n+6$ conjugacy classes as follows:

- $[1_{SD_{8n}}] = \{1_{SD_{8n}}\}$, $[u^n] = \{u^n\}$, $[u^{2n}] = \{u^{2n}\}$, $[u^{3n}] = \{u^{3n}\}$;
- $[u^s] = \{u^s, u^{(2n-1)s}\}$, $s \in Q_1 \cup Q_3$;
- $[v] = \{vu^{4e} : e = 0, 1, \cdots, n-1\}$, $[vu] = \{vu^{4e+1} : e = 0, 1, \cdots, n-1\}$, $[vu^2] = \{vu^{4e+2} : e = 0, 1, \cdots, n-1\}$, $[vu^3] = \{vu^{4e+3} : e = 0, 1, \cdots, n-1\}$.

The above proposition shows that the numbers of conjugacy classes of SD_{8n} are greater than those of D_{8n} if n is odd. It follows from Proposition 5.2, Theorem 3.2 and Theorem 3.3 that we have more choices of the connection set S than those in Lemma 5.1. Notably, one can check out the existence of PST on k-regular graphs $\mathrm{Cay}(SD_{8n}, S)$ for some $k > n-1$. Leaving the possible isomorphisms between $\mathrm{Cay}(SD_{8n}, S)$ and $\mathrm{Cay}(D_{8n}, S)$ aside, the existence of PST on some new Cayley graphs can be determined by Theorem 3.3. Below, we present several examples by designing the connection set S from Theorem 3.2 and Theorem 3.3.

Example 1. *Let $n > 1$ be an odd number. Assume that $S = \{u^n, u^{3n}\} \cup \{vu^{4e} : e = 0, 1, \cdots, n-1\}$. Then $\mathrm{Cay}(SD_{8n}, S)$ is a simple connected graph and exhibits no PST between any two distinct vertices. Furthermore, $\mathrm{Cay}(SD_{8n}, S)$ is periodic with minimum period 2π.*

Proof. By the definition of S, it is easy to verify that $\mathrm{Cay}(SD_{8n}, S)$ is a simple connected graph. It follows from Proposition 4.2 that $\mathrm{Cay}(SD_{8n}, S)$ is integral. Applying Lemma 2.1 and Lemma 2.2, we obtain the spectrum of $\mathrm{Cay}(SD_{8n}, S)$ as follows:

$$\mathrm{Spec}(\mathrm{Cay}(SD_{8n}, S)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \delta_j \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix},$$

where

$$\begin{aligned}
&\lambda_1 = n+2, \lambda_2 = 2-n, \lambda_3 = n-2, \lambda_4 = -2-n, \\
&\lambda_5 = \lambda_7 = n, \lambda_6 = \lambda_8 = -n, \\
&\delta_j = \begin{cases} 2, & j \in Q_1, j \equiv 0 \pmod 4, \\ -2, & j \in Q_1, j \equiv 2 \pmod 4, \\ 0, & j \in Q_3. \end{cases}
\end{aligned}$$

Since $\upsilon_2(\delta_j - \lambda_1) \neq \upsilon_2(\lambda_5 - \lambda_1)$ for any $j \in Q_3$, the desired result follows from Theorem 3.2. $\square$

Remark 1. *The graph* $\mathrm{Cay}(SD_{8n}, S)$ *defined in Example 1 is* $(n+2)$*-regular. Note that the existence and non-existence of PST on the d-regular graphs* $\mathrm{Cay}(D_{8n}, S)$ *have been proved in [14] for any* $d \geq 2n$*. Therefore, Example 1 has given the non-existence of PST on a new Cayley graph. In general, by Theorem 3.2, one can obtain the existence and non-existence of PST on more new Cayley graphs.*

Example 2. *Assume that* $n > 1$ *is an odd number. Let* $S = \{u^n, u^{3n}\} \cup \{vu^e : e = 0, 1, \cdots, 4n-1\}$*. Then* $\mathrm{Cay}(SD_{8n}, S)$ *is a simple connected graph and admits PST from* $v^\ell u^j$ *to* $v^\ell u^{j+2n}$ *with the minimum time* $t = \frac{\pi}{2}$ *for all* $\ell = 1, 2$ *and* $j = 0, 1, \cdots, 4n-1$*. Furthermore,* $\mathrm{Cay}(SD_{8n}, S)$ *is periodic with minimum period* π*.*

Proof. From the definition of S, $\mathrm{Cay}(SD_{8n}, S)$ is a simple connected graph. It follows from Proposition 4.2 that $\mathrm{Cay}(SD_{8n}, S)$ is integral. By Lemma 2.1 and Lemma 2.2, the spectrum of $\mathrm{Cay}(SD_{8n}, S)$ is

$$\mathrm{Spec}(\mathrm{Cay}(SD_{8n}, S)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \delta_j \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix},$$

where

$$\begin{aligned} &\lambda_1 = 4n+2, \lambda_2 = 2-4n, \lambda_3 = \lambda_4 = -2, \\ &\lambda_5 = \lambda_7 = \lambda_6 = \lambda_8 = 0, \\ &\delta_j = \begin{cases} 2, & j \in Q_1, j \equiv 0 \pmod 4, \\ -2, & j \in Q_1, j \equiv 2 \pmod 4, \\ 0, & j \in Q_3. \end{cases} \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} &\upsilon_2(\delta_j - \lambda_1) = \upsilon_2(\lambda_z - \lambda_1) = 1, \text{for any } j \in Q_3 \text{ and } z = 5, 6, 7, 8, \\ &\upsilon_2(\lambda_2 - \lambda_1) = 3, \upsilon_2(\lambda_3 - \lambda_1) = \upsilon_2(\lambda_4 - \lambda_1) \geq 3, \\ &\upsilon_2(\delta_j - \lambda_1) = 2, \text{for any } j \in Q_1, j \equiv 0 \pmod 4, \\ &\upsilon_2(\delta_j - \lambda_1) \geq 3, \text{for any } j \in Q_1, j \equiv 2 \pmod 4. \end{aligned}$$

Thanks to Theorem 3.2, $\mathrm{Cay}(SD_{8n}, S)$ has PST from $v^\ell u^j$ to $v^\ell u^{j+2n}$ with the minimum time $t = \frac{\pi}{2}$ for all $\ell = 1, 2$ and $j = 0, 1, \cdots, 4n-1$. □

Example 3. *Suppose that* $n > 1$ *is an integer. Let* $S = \{u^{2n}\} \cup \{vu^e : e = 0, 1, \cdots, 4n-1\}$*. If* n *is even, then* $\mathrm{Cay}(SD_{8n}, S)$ *has PST between from* $v^\ell u^j$ *to* $v^\ell u^{j+2n}$ *with the minimum time* $t = \frac{\pi}{2}$ *for all* $\ell = 1, 2$ *and* $j = 0, 1, \cdots, 4n-1$*. If* n *is odd, then* $\mathrm{Cay}(SD_{8n}, S)$ *exhibits no PST between any two distinct vertices.*

Proof. **Case** 1: When n is even, according to Lemma 2.1 and Lemma 2.2, the spectrum of $\mathrm{Cay}(SD_{8n}, S)$ is

$$\mathrm{Spec}(\mathrm{Cay}(SD_{8n}, S)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \delta_j \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix},$$

where

$$\begin{aligned} &\lambda_1 = 1+4n, \lambda_2 = 1-4n, \lambda_3 = \lambda_4 = 1, \\ &\delta_j = \begin{cases} 1, & j \in Q_1, \\ -1, & j \in Q_2. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} &\upsilon_2(\delta_j - \lambda_1) = 1, \text{for all } j \in Q_2, \\ &\upsilon_2(\lambda_2 - \lambda_1) \geq 4, \\ &\upsilon_2(\lambda_3 - \lambda_1) = \upsilon_2(\lambda_4 - \lambda_1) = \upsilon_2(\delta_j - \lambda_1) \geq 3, \text{for all } j \in Q_1. \end{aligned}$$

From the definition of S, $\mathrm{Cay}(SD_{8n}, S)$ is a simple connected graph. Then the desired result follows from Theorem 3.3.

Case 2: When n is odd, using Lemma 2.1 and Lemma 2.2, the spectrum of $\mathrm{Cay}(SD_{8n}, S)$ is

$$\mathrm{Spec}(\mathrm{Cay}(SD_{8n}, S)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \delta_j \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix},$$

where

$$\lambda_1 = 4n+1, \lambda_2 = 1-4n, \lambda_3 = \lambda_4 = 1,$$
$$\lambda_5 = \lambda_7 = \lambda_6 = \lambda_8 = -1,$$
$$\delta_j = \begin{cases} 1, & j \in Q_1, \\ 0, & j \in Q_3. \end{cases}$$

Observe that $\upsilon_2(\delta_j - \lambda_1) \neq \upsilon_2(\lambda_5 - \lambda_1)$ for any $j \in Q_3$. Then it follows from Theorem 3.2 that $\mathrm{Cay}(SD_{8n}, S)$ has no PST between any two distinct vertices. □

Remark 2. *Note that* $D_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{-1}\rangle$. *Assume that* $W = \{a^k : k \in W_1\} \cup \{ba^\ell : \ell \in W_2\}$ *is a subset of* D_{8n} *and* $Y = \{u^k : k \in Y_1\} \cup \{vu^\ell : \ell \in Y_2\}$. *We say that* W *and* Y *have similar structures if* $W_1 = Y_1$ *and* $W_2 = Y_2$. *Cao et al. [14] has verified that the graph* $\mathrm{Cay}(D_{8n}, \widehat{S})$ *admits PST, where* $\widehat{S} = \{a^{2n}\} \cup \{ba^e : e = 0, 1, \cdots, 4n-1\}$ *(see Example 5.3 in [14]). Observe that the set* S *defined by Example 3 and* $\widehat{S}$ *have similar structures. However,* $\mathrm{Cay}(SD_{8n}, S)$ *admits PST if* n *is even and has no PST if* n *is odd. By Proposition 5.2, besides the conjugacy classes* $[a^s]$ *of* D_{8n} *and* $[u^s]$ *of* SD_{8n} *for any odd* s*, the conjugacy classes of* D_{8n} *and conjugacy classes of* SD_{8n} *have similar structures for an even* n*. If* n *is odd,* SD_{8n} *and* D_{8n} *have different conjugacy classes. Therefore, we conjecture that* $\mathrm{Cay}(SD_{8n}, S)$ *admits PST if and only if* $\mathrm{Cay}(D_{8n}, \widehat{S})$ *has PST if* n *is even and the sets* S *and* $\widehat{S}$ *have similar structures. Note that* $\mathrm{Cay}(D_{8n}, \widehat{S})$ *of Example 5.3 in [14] has PST and* $\mathrm{Cay}(SD_{8n}, S)$ *of Example 4 admits no PST. In addition, the graph* $\mathrm{Cay}(SD_{8n}, R)$ *defined by Example 2 admits PST and* $\mathrm{Cay}(D_{8n}, \widehat{R})$ *defined in Example 5.1 of [14] has PST, where* R *and* $\widehat{R}$ *have similar structures. Consequently, we are more convinced that the existence of PST on* $\mathrm{Cay}(SD_{8n}, S)$ *has no direct link to the existence of PST on* $\mathrm{Cay}(D_{8n}, \widehat{S})$ *if* n *is odd and the sets* S *and* $\widehat{S}$ *have similar structures.*

Since the dihedral groups and semi-dihedral groups can be decomposed as semi-direct products of two cyclic groups, an interesting question is raised.

Open question 1. Let G_i be a finite group which can be decomposed as semi-direct products of two cyclic groups G_{i1} and G_{i2}, where $i = 1, 2$. Suppose that G_{1j} and G_{2j} have the same order for $j = 1, 2$. Assume that S_1 and S_2 with the similar structures are subsets of G_1 and G_2, respectively. If the conjugacy classes of G_1 and G_2 have similar structures, determine whether $\mathrm{Cay}(G_1, S_1)$ admits PST if and only if $\mathrm{Cay}(G_2, S_2)$ has PST.

6 Concluding remarks

By determining the eigenvalues and corresponding eigenvectors of $\mathrm{Cay}(SD_{8n}, S)$, we deduced some necessary and sufficient conditions for $\mathrm{Cay}(SD_{8n}, S)$ exhibiting PST. With those conditions, we provided some examples about the existence and non-existence of PST on $\mathrm{Cay}(SD_{8n}, S)$. A comparison between PST on Cayley graphs on dihedral groups and PST on Cayley graphs on semi-dihedral groups has been made. Notably, basing on our necessary and sufficient conditions for PST on $\mathrm{Cay}(SD_{8n}, S)$, we can provide the existence and non-existence of PST on some new Cayley graphs that has never been considered in [14]. By setting a subset S of SD_{8n} and a subset $\widehat{S}$ of D_{8n} with similar structures, we conjecture the link between the existence of PST on $\mathrm{Cay}(SD_{8n}, S)$ and the existence of PST on $\mathrm{Cay}(D_{8n}, \widehat{S})$. A challenge question (Open question 1) is posed and our future work will focus on this question.

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Data Availability Statements

All data included in this paper are available upon request by contact with the corresponding author.

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