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2022

https://hdl.handle.net/10356/162428

https://doi.org/10.1109/CDC51059.2022.9992991

Gradient-Free Nash Equilibrium Seeking in N-Cluster Games with Uncoordinated Constant Step-Sizes

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Abstract— This work investigates a problem of simultaneous global cost minimization and Nash equilibrium seeking, which commonly exists in N-cluster non-cooperative games. Specifically, the players in the same cluster collaborate to minimize a global cost function, being a summation of their individual cost functions, and jointly play a non-cooperative game with other clusters as players. For the problem settings, we suppose that the explicit analytical expressions of the players' local cost functions are unknown, but the function values can be measured. We propose a gradient-free Nash equilibrium seeking algorithm by a synthesis of Gaussian smoothing techniques and gradient tracking. Furthermore, instead of using the uniform coordinated step-size, we allow the players across different clusters to choose different constant step-sizes. When the largest step-size is sufficiently small, we prove a linear convergence of the players' actions to a neighborhood of the unique Nash equilibrium under a strongly monotone game mapping condition, with the error gap being propotional to the largest step-size and the smoothing parameter. The performance of the proposed algorithm is validated by numerical simulations.

Index Terms— Nash equilibrium seeking, gradient-free methods, non-cooperative games.

I. INTRODUCTION

The research on cooperation and competition across multiple interacting players has been extensively studied in recent years, especially on distributed optimization and Nash equilibrium (NE) seeking in non-cooperative games. Specifically, distributed optimization deals with a cooperative minimization problem among a network of players. On the other hand, NE seeking in non-cooperative games is concerned with a number of players, who are self-interested to minimize their individual cost given the other players' decisions.

To simultaneously model the cooperative and competitive behaviors in networked systems, an N-cluster game is formulated. This game is essentially a non-cooperative game played among N interacting clusters with each cluster being a virtual player. In each cluster, there are a group of players who collaboratively minimize a cluster-level cost function given by a summation of their individual local cost functions. With these features, the N-cluster game naturally accommodates both collaboration and competition in a unified framework, which motivates us to study and propose

solutions to find its NE. In this paper, we consider such an N-cluster non-cooperative game. Moreover, we further suppose that the explicit analytical expressions of the players' local cost functions are unknown, but the function values can be measured.

A substantial works on NE seeking algorithms for non-cooperative games have been reported in the recent literature, including [1]–[8], to list a few. The focus of the aforementioned works is mainly on the competitive nature in the non-cooperative games. Different from that, the works in [9], [10] considered two sub-networks zero-sum games, where each subnetwork owns an opposing cost function to be cooperatively minimized by the players in the corresponding subnetwork. Then, an extension of such problems to N subnetworks was firstly formulated in [11], which is known as an N-cluster (or coalition) game. Then, this problem has received a high research interest recently, which includes [12]–[17]. Most of the above works focus on continuous-time based methods, such as gradient play [12], [13], subgradient dynamics [14], projected primal-dual dynamics [15], and extremum-seeking techniques [16]. Our previous work in [17] introduced a discrete-time NE seeking strategy based on a synthesis of gradient-free and gradient-tracking techniques. This paper revisits the N-cluster game, and aims to extend the results to uncoordinated step-sizes across different clusters.

Contributions: As compared to the aforementioned relevant works, the contributions of this work can be summarized as follows. 1) In contrast to the problem setups in [11]–[15], we limit the players on the access to the cost functions: no explicit analytical expressions but only the values of the local cost functions can be utilized in the update laws. In this case, no gradient information can be directly utilized in the design of the algorithm. Hence, gradient-free techniques are adopted in this work. 2) As compared to our prior work [17], we extend the gradient-tracking method to allow uncoordinated constant step-sizes across different clusters, which further reduces the coordination among players from different clusters. 3) The technical challenges of the convergence analysis brought by gradient tracking methods in games, and uncoordinated step-sizes are addressed in this work. For the convergence results: we obtain a linear convergence to a neighborhood of the unique NE with the error being proportional to the largest step-size and a smoothing parameter under appropriate settings.

Notations: $\mathbf{1}_m$ denotes the vector of dimension m with all elements being 1, and I_m denotes the $m \times m$ identity matrix. For a vector u, we use $[u]_i$ for its i-th element, and $\mathrm{diag}(u)$

for a diagonal matrix formed by the elements of u. For a (square) matrix A, the spectral norm is denoted by $\|A\|$, and the spectral radius is represented by $\rho(A)$.

II. PROBLEM STATEMENT

A. Problem Formulation

An N-cluster game is defined by $\Gamma(\mathcal{N}, \{f^i\}, \{\mathbb{R}^{n_i}\})$, where each cluster, indexed by $i \in \mathcal{N} \triangleq \{1, 2, \ldots, N\}$, consists of a group of players, denoted by $\mathcal{V}^i \triangleq \{1, 2, \ldots, n_i\}$. Denote $n \triangleq \sum_{i=1}^N n_i$. These players aim to minimize a cluster-level cost function $f^i : \mathbb{R}^n \to \mathbb{R}$, defined as

$$f^i(\mathbf{x}^i, \mathbf{x}^{-i}) \triangleq \frac{1}{n_i} \sum_{j=1}^{n_i} f_j^i(\mathbf{x}^i, \mathbf{x}^{-i}), \quad \forall i \in \mathcal{N},$$

where $f_j^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is a local cost function of player j in cluster i, $\mathbf{x}^i \triangleq [x_1^{i\top}, \ldots, x_{n_i}^{i\top}]^\top \in \mathbb{R}^{n_i}$ is a collection of all players' actions in cluster i with $x_j^i \in \mathbb{R}$ being the action of player j in cluster i, and $\mathbf{x}^{-i} \in \mathbb{R}^{n-n_i}$ denotes a collection of all players' actions except cluster i. Denote $\mathbf{x} \triangleq (\mathbf{x}^i, \mathbf{x}^{-i}) = [\mathbf{x}^{1\top}, \ldots, \mathbf{x}^{N\top}]^\top$.

Definition 1: (NE of N-Cluster Games). A vector $\mathbf{x}^* \triangleq (\mathbf{x}^{i*}, \mathbf{x}^{-i*}) \in \mathbb{R}^n$ is said to be an NE of the N-cluster non-cooperative game $\Gamma(\mathcal{N}, \{f^i\}, \{\mathbb{R}^{n_i}\})$, if and only if

$$f^i(\mathbf{x}^{i*}, \mathbf{x}^{-i*}) \leq f^i(\mathbf{x}^i, \mathbf{x}^{-i*}), \quad \forall \mathbf{x}^i \in \mathbb{R}^n, \quad \forall i \in \mathcal{N}.$$

Within each cluster $i \in \mathcal{N}$, there is an underlying directed communication network, denoted by $\mathcal{G}_i(\mathcal{V}^i, \mathcal{E}^i)$ with an adjacency matrix $\mathcal{A}^i \triangleq [a_{jk}^i] \in \mathbb{R}^{n_i \times n_i}$. In particular, $a_{jk}^i > 0$ if player j can directly pass information to player k, and $a_{jk}^i = 0$ otherwise. We suppose $a_{jj}^i > 0, \forall j \in \mathcal{V}^i$. Regarding the communication network, the following standard assumption is supposed.

Assumption 1: For $i \in \mathcal{N}$, the digraph $\mathcal{G}_i(\mathcal{V}^i, \mathcal{E}^i)$ is strongly connected. The adjacency matrix $\mathcal{A}^i$ is doubly-stochastic.

Noting that $\sigma_{\mathcal{A}^i} \triangleq \|\mathcal{A}^i - \frac{1}{n_i} \mathbf{1}_{n_i} \mathbf{1}_{n_i}^\top\| < 1$ [18, Lemma 1], we define $\bar{\sigma} \triangleq \max_{i \in \mathcal{N}} \sigma_{\mathcal{A}^i}$ and $\varsigma \triangleq \max_{i \in \mathcal{N}} (1 + \sigma_{\mathcal{A}^i}^2)/(1 - \sigma_{\mathcal{A}^i}^2)$.

Moreover, we consider the scenario where the explicit analytical expressions of the players' local cost functions are unknown, but the function values can be measured, similar to the settings in [7], [16], [17], [19]. Regarding the cost function, the following standard assumption is supposed.

Assumption 2: For each $j \in \mathcal{V}^i, i \in \mathcal{N}$, the local cost function $f_j^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is convex in $\mathbf{x}^i$, and continuously differentiable in $\mathbf{x}$. The total gradient $\nabla f_j^i(\mathbf{x})$ is L-Lipschitz continuous in $\mathbf{x}$, *i.e.*, for any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, $\|\nabla f_j^i(\mathbf{x}) - \nabla f_j^i(\mathbf{x}')\| \leq L \|\mathbf{x} - \mathbf{x}'\|$.

The game mapping of $\Gamma(\mathcal{N}, \{f^i\}, \{\mathbb{R}^{n_i}\})$ is defined as

$$\Phi(\mathbf{x}) \triangleq [\nabla_{\mathbf{x}^1} f^1(\mathbf{x})^\top, \ldots, \nabla_{\mathbf{x}^N} f^N(\mathbf{x})^\top]^\top.$$

The following standard assumption on the game mapping $\Phi(\mathbf{x})$ is supposed.

Assumption 3: The game mapping Φ of game Γ is strongly monotone with a constant $\chi > 0$, *i.e.*, for any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have $\langle \Phi(\mathbf{x}) - \Phi(\mathbf{x}'), \mathbf{x} - \mathbf{x}' \rangle \geq \chi \|\mathbf{x} - \mathbf{x}'\|^2$.

Remark 1: It is known that under Assumptions 2 and 3, game Γ admits a unique NE.

B. Preliminaries

This part presents some preliminary results on gradient-free techniques based on Gaussian smoothing [20].

For $j \in \mathcal{V}^i$, $i \in \mathcal{N}$, a Gaussian-smoothed function of the local cost function $f_j^i(\mathbf{x})$ can be defined as

$$f_{j,\mu}^i(\mathbf{x}) \triangleq \mathbb{E}_{\zeta \sim \mathcal{N}(\mathbf{0}_n, I_n)}[f_j^i(\mathbf{x} + \mu\zeta)], \tag{1}$$

where ζ is generated from a Gaussian distribution $\mathcal{N}(\mathbf{0}_n, I_n)$, and $\mu \geq 0$ is a smoothing parameter.

For each cluster $i \in \mathcal{N}$, the randomized gradient-free oracle of $f_j^i(\mathbf{x})$ for player j with respect to player k, $j, k \in \mathcal{V}^i$, $i \in \mathcal{N}$ at time step $t \geq 0$ is developed as

$$g_{jk}^i(\mathbf{x}_t) \triangleq \frac{f_j^i(\mathbf{x}_t + \mu\zeta_{j,t}^i) - f_j^i(\mathbf{x}_t)}{\mu} [\zeta_{j,t}^i]_k^i, \tag{2}$$

where $[\zeta_{j,t}^i]_k^i$ denotes the $(\sum_{l=0}^i n_l + k)$-th element of $\zeta_{j,t}^i$ with $n_0 = 0$, and $\zeta_{j,t}^i$ being player j's own version of ζ at time step t, and $\mu > 0$. The oracle (2) is useful as it can correctly estimate the partial gradient of the Gaussian-smoothed cost function $\nabla_{x_k^i} f_{j,\mu}^i(\mathbf{x}_t)$.

We define a Gaussian-smoothed game associated with the N-cluster game Γ, denoted by $\Gamma_\mu(\mathcal{N}, \{f_\mu^i\}, \{\mathbb{R}^{n_i}\})$, having the same set of clusters and action sets as game Γ, but the cost function is given by

$$f_\mu^i(\mathbf{x}^i, \mathbf{x}^{-i}) \triangleq \frac{1}{n_i} \sum_{j=1}^{n_i} f_{j,\mu}^i(\mathbf{x}^i, \mathbf{x}^{-i}), \quad \forall i \in \mathcal{N},$$

where $f_{j,\mu}^i$ is a Gaussian-smoothed function of f_j^i defined in (1). Similar to the game mapping of Γ, we define the game mapping of Γ_μ by

$$\Phi_\mu(\mathbf{x}) \triangleq [\nabla_{\mathbf{x}^1} f_\mu^1(\mathbf{x})^\top, \ldots, \nabla_{\mathbf{x}^N} f_\mu^N(\mathbf{x})^\top]^\top.$$

The following lemma shows the strong monotonicity condition of $\Phi_\mu(\mathbf{x})$, and quantifies the distance between the NE of the smoothed game Γ_μ and the NE of the original game Γ in terms of the smoothing parameter μ.

Lemma 1: (see [17, Lemma 1]) Under Assumptions 2 and 3, for $\forall \mu \geq 0$, the smoothed game $\Gamma_\mu(\mathcal{N}, \{f_\mu^i\}, \{\mathbb{R}^{n_i}\})$ holds that

1) The game mapping $\Phi_\mu(\mathbf{x})$ is χ-strongly monotone.
2) The smoothed game Γ_μ admits a unique NE (denoted by $\mathbf{x}_\mu^*$) satisfying that

$$\|\mathbf{x}_\mu^* - \mathbf{x}^*\| \leq \frac{n(n+3)^{\frac{3}{2}} L\gamma}{2(1 - \sqrt{1 - \gamma\chi})} \mu,$$

where $\mathbf{x}^*$ is the unique NE of the original game Γ, and $\gamma \in (0, \frac{\chi}{n^2 L^2}]$ is a constant.

It follows from Lemma 1 that $\mathbf{x}_\mu^*$ is the unique NE of the smoothed game $\Gamma_\mu(\mathcal{N}, \{f_\mu^i\}, \{\mathbb{R}^{n_i}\})$, and hence holds that $\Phi_\mu(\mathbf{x}_\mu^*) = \mathbf{0}_n$. We define $G \triangleq \max_{j \in \mathcal{V}^i, i \in \mathcal{N}} \|\nabla f_{j,\mu}^i(\mathbf{x}_\mu^*)\|$.

III. NE Seeking Algorithm for N-Cluster Games

A. Algorithm

In this part, we present an NE seeking strategy for the N-cluster game. Specifically, each player $j \in \mathcal{V}^i$, $i \in \mathcal{N}$ needs to maintain its own action variable x_j^i, an auxiliary variable y_{jk}^i and gradient tracker variables φ_{jk}^i for $\forall k \in \mathcal{V}^i$. Let $x_{j,t}^i, y_{jk,t}^i, \varphi_{jk,t}^i$ denote the values of these variables at time-step t. The update laws for each player $j \in \mathcal{V}^i$, $i \in \mathcal{N}$ are designed as

$$y_{jk,t+1}^i = \sum_{l=1}^{n_i} a_{jl}^i y_{lk,t}^i - \alpha^i \varphi_{jk,t}^i, \tag{3a}$$

$$x_{j,t+1}^i = y_{jj,t+1}^i, \tag{3b}$$

$$\varphi_{jk,t+1}^i = \sum_{l=1}^{n_i} a_{jl}^i \varphi_{lk,t}^i + g_{jk}^i(\mathbf{x}_{t+1}) - g_{jk}^i(\mathbf{x}_t), \tag{3c}$$

with arbitrary $x_{j,0}^i, y_{jk,0}^i \in \mathbb{R}$ and $\varphi_{jk,0}^i = g_{jk}^i(\mathbf{x}_0)$, where $g_{jk}^i(\mathbf{x}_t)$ is the gradient estimator given by (2), and $\alpha^i > 0$ is a constant step-size sequence adopted by all players in cluster $i \in \mathcal{N}$. Denote the largest step-size by $\alpha_{\max} \triangleq \max_{i \in \mathcal{N}} \alpha^i$ and the average of all step-sizes by $\bar{\alpha} \triangleq \frac{1}{n} \sum_{i \in \mathcal{N}} n_i \alpha^i$. Define the heterogeneity of the step-size as the following ratio, $\epsilon_\alpha \triangleq \|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}\|/\|\bar{\boldsymbol{\alpha}}\|$, where $\boldsymbol{\alpha} \triangleq [\alpha^1 \mathbf{1}_{n_1}^\top, \ldots, \alpha^N \mathbf{1}_{n_N}^\top]^\top$ and $\bar{\boldsymbol{\alpha}} \triangleq \bar{\alpha} \mathbf{1}_n$.

B. Main Results

This part presents the main results of the proposed algorithm, as stated in the following theorem. Detailed proof is given in Sec. IV-B.

Theorem 1: Suppose Assumptions 1, 2 and 3 hold. Generate the auxiliary variables $\{y_{jk,t}^i\}_{t \geq 0}$, the player's action $\{x_{j,t}^i\}_{t \geq 0}$ and gradient tracker $\{\varphi_{jk,t}^i\}_{t \geq 0}$ by (3) with the uncoordinated constant step-size α^i satisfying $\epsilon_\alpha < \frac{\chi}{2\sqrt{n}L}$ and

$$0 < \alpha_{\max} < \min\left\{\alpha_1, \alpha_2, \alpha_3, \frac{1}{\chi - 2\sqrt{n}L\epsilon_\alpha}, 1\right\},$$

where α_1, α_2 and α_3 are defined in Sec. IV-B. Then, all players' decisions $\mathbf{x}_t$ linearly converges to a neighborhood of the unique NE $\mathbf{x}^*$, and

$$\limsup_{t \to \infty} \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] \leq \mathcal{O}(\alpha_{\max}) + \mathcal{O}(\mu).$$

Remark 2: Theorem 1 shows that the players' actions converge to a neighborhood of the NE linearly with the error bounded by two terms: one is proportional to the largest step-size, and the other is proportional to the smoothing parameter due to the gradient estimation.

IV. Convergence Analysis

Let $\mathcal{H}_t$ denote the σ-field generated by the entire history of the random variables from time-step 0 to $t-1$. We introduce the following notations for easy presentation. $n_s \triangleq \sum_{i=1}^N n_i^2$ and $n_c \triangleq \sum_{i=1}^N n_i^3$. For $\forall k \in \mathcal{V}^i, i \in \mathcal{N}$, $\mathbf{y}_{k,t}^i \triangleq [y_{1k,t}^i, \ldots, y_{n_ik,t}^i]^\top \in \mathbb{R}^{n_i}, \bar{y}_{k,t}^i \triangleq \frac{1}{n_i}\mathbf{1}_{n_i}^\top \mathbf{y}_{k,t}^i \in \mathbb{R}, \bar{\mathbf{y}}_t^i \triangleq [\bar{y}_{1,t}^i, \ldots, \bar{y}_{n_i,t}^i]^\top \in \mathbb{R}^{n_i}, \bar{\mathbf{y}}_t \triangleq [\bar{\mathbf{y}}_t^{1\top}, \ldots, \bar{\mathbf{y}}_t^{N\top}]^\top \in$

$\mathbb{R}^n, \varphi_{k,t}^i \triangleq [\varphi_{1k,t}^i, \ldots, \varphi_{n_ik,t}^i]^\top \in \mathbb{R}^{n_i}, \bar{\varphi}_{k,t}^i \triangleq \frac{1}{n_i}\mathbf{1}_{n_i}^\top \varphi_{k,t}^i \in \mathbb{R}$.

The convergence analysis of the proposed algorithm will be conducted by: 1) constructing a linear system of three terms $\sum_{i=1}^N \sum_{k=1}^{n_i} \|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2$, $\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2$ and $\sum_{i=1}^N \sum_{k=1}^{n_i} \|\varphi_{k,t}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t}^i\|^2$ in terms of their past iterations and some constants, 2) analyzing the convergence of the established linear system.

A. Auxiliary Analysis

In this part, present the results on $\sum_{i=1}^N \sum_{k=1}^{n_i} \|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2$, $\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2$ and $\sum_{i=1}^N \sum_{k=1}^{n_i} \|\varphi_{k,t}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t}^i\|^2$ in the following three lemmas, respectively. Due to limited space, their proofs are extended to the online version [21].

Lemma 2: Under Assumptions 1, 2 and 3, the total consensus error of the auxiliary variables $\sum_{i=1}^N \sum_{k=1}^{n_i} \|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2$ satisfies that

$$\sum_{i=1}^N \sum_{k=1}^{n_i} \mathbb{E}[\|\mathbf{y}_{k,t+1}^i - \mathbf{1}_{n_i}\bar{y}_{k,t+1}^i\|^2|\mathcal{H}_t] \leq (1 + m_2\alpha_{\max}^2$$
$$- m_1) \sum_{i=1}^N \sum_{k=1}^{n_i} \|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2 + m_2\alpha_{\max}^2\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2$$
$$+ m_3\alpha_{\max}^2 \sum_{i=1}^N \sum_{k=1}^{n_i} \mathbb{E}[\|\varphi_{k,t}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t}^i\|^2|\mathcal{H}_t] + m_{12}\alpha_{\max}^2,$$

where $m_1 \triangleq \frac{1-\bar{\sigma}^2}{2}$, $m_2 \triangleq 24(n+4)n_c\varsigma L^2$, $m_3 \triangleq 2\varsigma$, and $m_{12} \triangleq 24(n+4)n_c\varsigma G^2 + 6(n+4)^3 n_c\varsigma B^2$.

Lemma 3: Under Assumptions 1, 2 and 3, the gap between the stacked averaged auxiliary variable and the NE of game Γ_μ, $\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2$ holds that

$$\mathbb{E}[\|\bar{\mathbf{y}}_{t+1} - \mathbf{x}^*\|^2|\mathcal{H}_t] \leq (1 - m_6\bar{\alpha} + m_5\alpha_{\max}^2)\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2$$
$$+ (m_5\alpha_{\max}^2 + m_4\alpha_{\max}) \sum_{i=1}^N \sum_{k=1}^{n_i} \|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2$$
$$+ m_{13}\alpha_{\max}^2,$$

where $m_4 \triangleq n^2 L^2/\chi$, $m_5 \triangleq 12n(n+4)L^2$, $m_6 \triangleq \chi - 2\sqrt{n}L\epsilon_\alpha$, and $m_{13} \triangleq 12n(n+4)G^2 + 3n(n+4)^3 B^2$.

Lemma 4: Under Assumptions 1 and 2, the total gradient tracking error $\sum_{i=1}^N \sum_{k=1}^{n_i} \|\varphi_{k,t}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t}^i\|^2$ satisfies

$$\sum_{i=1}^N \sum_{k=1}^{n_i} \mathbb{E}[\|\varphi_{k,t+1}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t+1}^i\|^2|\mathcal{H}_t] \leq (1 - m_1$$
$$+ m_{11}\alpha_{\max}^2) \sum_{i=1}^N \sum_{k=1}^{n_i} \mathbb{E}[\|\varphi_{k,t}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t}^i\|^2|\mathcal{H}_t]$$
$$+ (m_7 + m_8\alpha_{\max} + m_9\alpha_{\max}^2) \sum_{i=1}^N \sum_{k=1}^{n_i} \|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2$$
$$+ (m_9\alpha_{\max}^2 + m_{10})\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2 + m_{14} + m_{15}\alpha_{\max}^2,$$

where $m_7 \triangleq 12(n+4)n_s\varsigma L^2(3 + \bar{\sigma}^2)$, $m_8 \triangleq 24(n+4)n_s\varsigma L^2 m_4$, $m_9 \triangleq 24(n+4)n_s\varsigma L^2(m_2 + m_5)$, $m_{10} \triangleq 48(n+4)n_s\varsigma L^2$, $m_{11} \triangleq \varsigma m_{10}$, $m_{14} \triangleq 48(n+4)n_s\varsigma G^2 + 12(n+4)^3 n_s\varsigma B^2$ and $m_{15} \triangleq 24(n+4)n_s\varsigma L^2(m_{12} + m_{13})$.

B. Proof of Theorem 1

Now, we proceed to the proof of Theorem 1. Based on the results in Lemmas 2, 3 and 4, we can construct a linear system by taking the total expectation on the corresponding relations.

$$\Psi_{t+1} \leq \mathbf{M}\Psi_t + \Upsilon, \tag{4}$$

where

$$\Psi_t \triangleq \begin{bmatrix} \sum_{i=1}^{N}\sum_{k=1}^{n_i} \mathbb{E}[\|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i}\bar{y}_{k,t}^i\|^2] \\ \mathbb{E}[\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2] \\ \sum_{i=1}^{N}\sum_{k=1}^{n_i} \mathbb{E}[\|\varphi_{k,t}^i - \mathbf{1}_{n_i}\bar{\varphi}_{k,t}^i\|^2] \end{bmatrix},$$

$$\mathbf{M} \triangleq \begin{bmatrix} 1-m_1+m_2\alpha_{\max}^2 & m_2\alpha_{\max}^2 & m_3\alpha_{\max}^2 \\ m_4\alpha_{\max}+m_5\alpha_{\max}^2 & 1-m_6\bar{\alpha}+m_5\alpha_{\max}^2 & 0 \\ m_7+m_8\alpha_{\max}+m_9\alpha_{\max}^2 & m_{10}+m_9\alpha_{\max}^2 & 1-m_1+m_{11}\alpha_{\max}^2 \end{bmatrix},$$

$$\Upsilon \triangleq \begin{bmatrix} m_{12}\alpha_{\max}^2 \\ m_{13}\alpha_{\max}^2 \\ m_{14}+m_{15}\alpha_{\max}^2 \end{bmatrix}.$$

For the linear system (4), we aim to prove $\rho(\mathbf{M}) < 1$ such that each component of Ψ_t can linearly converge to a neighborhood of 0 [22].

We adopt the following result to guarantee $\rho(\mathbf{M}) < 1$:

Lemma 5: (see [22, Cor. 8.1.29]) Let $A \in \mathbb{R}^{m \times m}$ be a matrix with non-negative entries and $\boldsymbol{\nu} \in \mathbb{R}^m$ be a vector with positive entries. If there exists a constant $\lambda \geq 0$ such that $A\boldsymbol{\nu} < \lambda\boldsymbol{\nu}$, then $\rho(A) < \lambda$.

To apply Lemma 5, each element of $\mathbf{M}$ should be non-negative. Hence, we may set $m_6 > 0$ and $\alpha_{\max} < \frac{1}{m_6}$, *i.e.*,

$$\alpha_{\max} < \frac{1}{m_6}, \epsilon_\alpha < \frac{\chi}{2\sqrt{n}L}.$$

Next, based on Lemma 5, it suffices to find a vector $\boldsymbol{\nu} \triangleq [\nu_1, \nu_2, \nu_3]^\top$ with $\nu_1, \nu_2, \nu_3 > 0$ such that $\mathbf{M}_\alpha \boldsymbol{\nu} < \boldsymbol{\nu}$, *i.e.*,

$$(1 - m_1 + m_2\alpha_{\max}^2)\nu_1 + (m_2\alpha_{\max}^2)\nu_2 + (m_3\alpha_{\max}^2)\nu_3 < \nu_1,$$
$$(m_4\alpha_{\max} + m_5\alpha_{\max}^2)\nu_1 + (1 - m_6\bar{\alpha} + m_5\alpha_{\max}^2)\nu_2 < \nu_2,$$
$$(m_7 + m_8\alpha_{\max} + m_9\alpha_{\max}^2)\nu_1 + (m_{10} + m_9\alpha_{\max}^2)\nu_2$$
$$+ (1 - m_1 + m_{11}\alpha_{\max}^2)\nu_3 < \nu_3.$$

Without loss of generality, we may set $\nu_3 = 1$. It remains to find ν_1 and ν_2 such that the following inequalities hold

$$(m_2\nu_1 + m_2\nu_2 + m_3)\alpha_{\max}^2 < m_1\nu_1, \tag{5a}$$
$$(m_5\nu_1 + m_5\nu_2)\alpha_{\max} < \frac{m_6\nu_2}{n} - m_4\nu_1, \tag{5b}$$
$$(m_9\nu_1 + m_9\nu_2 + m_{11})\alpha_{\max}^2 < m_1$$
$$- (m_7 + m_8)\nu_1 - m_{10}\nu_2, \tag{5c}$$

where we have applied $\frac{\bar{\alpha}}{\alpha_{\max}} > \frac{1}{n}$ in (5b), and forced $\alpha_{\max} < 1$ in (5c).

To ensure the existence of $\alpha_{\max}$, the RHS of (5) has to be positive. Hence, we may set

$$\nu_1 = \frac{m_1 m_6}{4nm_4 m_{10} + 2m_6 m_7 + 2m_6 m_8},$$
$$\nu_2 = \frac{nm_1 m_4}{2nm_4 m_{10} + m_6 m_7 + m_6 m_8}.$$

Then, the three inequalities in (5) can be solved, which gives

$$\alpha_{\max} < \alpha_1, \alpha_{\max} < \alpha_2, \alpha_{\max} < \alpha_3,$$

where

$$\alpha_1 \triangleq \sqrt{\frac{m_1^2 m_6}{\begin{array}{c} m_1 m_2 m_6 + 2nm_1 m_2 m_4 \\ + 4nm_3 m_4 m_{10} + 2m_3 m_6 m_7 + 2m_3 m_6 m_8 \end{array}}},$$

$$\alpha_2 \triangleq \frac{m_1 m_4 m_6}{m_1 m_5 m_6 + 2nm_1 m_4 m_5},$$

$$\alpha_3 \triangleq \sqrt{\frac{m_1(2nm_4 m_{10} + m_6 m_7 + m_6 m_8)}{\begin{array}{c} m_1 m_6 m_9 + 2nm_1 m_4 m_9 \\ + m_{11}(4nm_4 m_{10} + 2m_6 m_7 + 2m_6 m_8) \end{array}}}.$$

Therefore, the range of the step-size is given by

$$0 < \alpha_{\max} < \min\left\{\alpha_1, \alpha_2, \alpha_3, \frac{1}{m_6}, 1\right\}, \epsilon_\alpha < \frac{\chi}{2\sqrt{n}L}.$$

Furthermore, taking the limsup on both sides of (4),

$$\limsup_{t \to \infty} \Psi_t \leq \mathbf{M} \limsup_{t \to \infty} \Psi_t + \Upsilon,$$

which gives

$$(I_3 - \mathbf{M}) \limsup_{t \to \infty} \Psi_t \leq \Upsilon.$$

where $I_3 - \mathbf{M}$ is given by

$$\begin{bmatrix} m_1 - m_2\alpha_{\max}^2 & -m_2\alpha_{\max}^2 & -m_3\alpha_{\max}^2 \\ -m_4\alpha_{\max}-m_5\alpha_{\max}^2 & m_6\bar{\alpha}-m_5\alpha_{\max}^2 & 0 \\ -m_7-m_8\alpha_{\max}-m_9\alpha_{\max}^2 & -m_{10}-m_9\alpha_{\max}^2 & m_1-m_{11}\alpha_{\max}^2 \end{bmatrix}.$$

It can be obtained that

$$det(I_3 - \mathbf{M}) \triangleq (m_1 - m_{11}\alpha_{\max}^2)[(m_1 - m_2\alpha_{\max}^2)$$
$$\cdot (m_6\bar{\alpha} - m_5\alpha_{\max}^2) - m_2\alpha_{\max}^2(m_4\alpha_{\max} + m_5\alpha_{\max}^2)]$$
$$> (m_1 - m_{11}\alpha_{\max}^2)[(m_1 - m_2\alpha_{\max}^2)(\tfrac{m_6\alpha_{\max}}{n}$$
$$- m_5\alpha_{\max}^2) - m_2\alpha_{\max}^2(m_4\alpha_{\max} + m_5\alpha_{\max}^2)]$$
$$= \alpha_{\max}(m_1 - m_2\alpha_{\max}^2)[\tfrac{m_1 m_6}{n} - m_1 m_5\alpha_{\max}$$
$$- m_2(m_4 + \tfrac{m_6}{n})\alpha_{\max}^2],$$
$$adj(I_3 - \mathbf{M})_{11} \triangleq (m_1 - m_{11}\alpha_{\max}^2)(m_6\bar{\alpha} - m_5\alpha_{\max}^2)$$
$$\leq \alpha_{\max}(m_1 - m_{11}\alpha_{\max}^2)(m_6 - m_5\alpha_{\max}),$$
$$adj(I_3 - \mathbf{M})_{12} \triangleq \alpha_{\max}^2[m_2(m_1 - m_{11}\alpha_{\max}^2)$$
$$+ m_3(m_{10} + m_9\alpha_{\max}^2)],$$
$$adj(I_3 - \mathbf{M})_{13} \triangleq m_3\alpha_{\max}^2(m_6\bar{\alpha} - m_5\alpha_{\max}^2)$$
$$\leq m_3\alpha_{\max}^3(m_6 - m_5\alpha_{\max}),$$
$$adj(I_3 - \mathbf{M})_{21} \triangleq \alpha_{\max}(m_1 - m_{11}\alpha_{\max}^2)(m_4 + m_5\alpha_{\max}),$$
$$adj(I_3 - \mathbf{M})_{22} \triangleq (m_1 - m_2\alpha_{\max}^2)(m_1 - m_{11}\alpha_{\max}^2)$$
$$- m_3\alpha_{\max}^2(m_7 + m_8\alpha_{\max} + m_9\alpha_{\max}^2),$$
$$adj(I_3 - \mathbf{M})_{23} \triangleq m_3\alpha_{\max}^3(m_4 + m_5\alpha_{\max}).$$

Then, we have

$$\limsup_{t\to\infty} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \mathbb{E}[\|\mathbf{y}_{k,t}^i - \mathbf{1}_{n_i} \bar{y}_{k,t}^i\|^2] \le [(I_3 - \mathbf{M})^{-1}\Upsilon]_1$$

$$= \frac{adj(I_3 - \mathbf{M})_{11}[\Upsilon]_1}{det(I_3 - \mathbf{M})} + \frac{adj(I_3 - \mathbf{M})_{12}[\Upsilon]_2}{det(I_3 - \mathbf{M})}$$

$$+ \frac{adj(I_3 - \mathbf{M})_{13}[\Upsilon]_3}{det(I_3 - \mathbf{M})} = \mathcal{O}(\alpha_{\max}^2),$$

and

$$\limsup_{t\to\infty} \mathbb{E}[\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2] \le [(I_3 - \mathbf{M})^{-1}\Upsilon]_2$$

$$= \frac{adj(I_3 - \mathbf{M})_{21}[\Upsilon]_1}{det(I_3 - \mathbf{M})} + \frac{adj(I_3 - \mathbf{M})_{22}[\Upsilon]_2}{det(I_3 - \mathbf{M})}$$

$$+ \frac{adj(I_3 - \mathbf{M})_{23}[\Upsilon]_3}{det(I_3 - \mathbf{M})} = \mathcal{O}(\alpha_{\max}).$$

Thus,

$$\limsup_{t\to\infty} \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_\mu^*\|^2]$$

$$\le 2\limsup_{t\to\infty} \mathbb{E}[\|\mathbf{x}_t - \bar{\mathbf{y}}_t\|^2] + 2\limsup_{t\to\infty} \mathbb{E}[\|\bar{\mathbf{y}}_t - \mathbf{x}_\mu^*\|^2]$$

$$\le \mathcal{O}(\alpha_{\max}^2) + \mathcal{O}(\alpha_{\max}) = \mathcal{O}(\alpha_{\max}).$$

Invoking Lemma 1 yields

$$\limsup_{t\to\infty} \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] \le 2\limsup_{t\to\infty} \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_\mu^*\|^2]$$

$$+ 2\|\mathbf{x}_\mu^* - \mathbf{x}^*\|^2 = \mathcal{O}(\alpha_{\max}) + \mathcal{O}(\mu),$$

which completes the proof.

V. NUMERICAL SIMULATIONS

We illustrate the proposed NE seeking strategy on a connectivity control game [23], played among a number of sensor networks. Specifically, there are N sensor networks, where each sensor network contains n_i sensors. Let $x_j^i = [x_{j,1}^i, x_{j,2}^i]^\top \in \mathbb{R}^2$ denote the position of sensor j (referred to as an player) from a sensor network i (referred to as a cluster). Then, this sensor aims to seek a tradeoff between a local cost, $l_j^i(\mathbf{x}^i)$ (e.g., source seeking and positioning) and the global cost, $h_j^i(\mathbf{x})$ (e.g., connectivity preservation with other sensor networks). Hence, the cost function to be minimized by this sensor is given by

$$f_j^i(\mathbf{x}) = l_j^i(\mathbf{x}^i) + h_j^i(\mathbf{x}),$$

where

$$l_j^i(\mathbf{x}^i) = \mathbf{x}^{i\top} a_j^i \mathbf{x}^i + b_j^{i\top} \mathbf{x}^i + c_j^i,$$
$$h_j^i(\mathbf{x}) = \sum_{k\in\mathcal{N}_i} d_j^i \|e_{jk}^i x_j^i - \mathbf{x}^k\|^2,$$

and $a_j^i, b_j^i, c_j^i, d_j^i, e_{jk}^i$ are constant matrices or vectors of appropriate dimensions, and $\mathcal{N}_i$ stands for the set of neighbors of sensor network i in a connected graph characterizing their position dependence. Specifically, if $k \in \mathcal{N}_i$, then the corresponding term $\|e_{jk}^i x_j^i - \mathbf{x}^k\|^2$ represents the intention of sensor j from a sensor network i to preserve the connectivity with the sensors from sensor network k.

In this simulation, we consider $N = 3$ and $n_i = 4$. The local and global costs are set as $l_j^i = i[\|x_j^i\|^2 + \mathbf{1}_2^\top x_j^i + j]$ for $j = 1, \ldots, 4$, $i = 1, 2, 3$ and $h_j^1 = \|x_j^1 - x_j^2\|^2$, $h_j^2 = \|x_j^2 - x_j^3\|^2$, $h_j^3 = \|x_j^3 - x_j^1\|^2$ for $j = 1, \ldots, 4$. Then, it is readily verified that Assumptions 2 and 3 hold. The directed communication graph for each sensor network i is as shown in Fig. 1. For the algorithm parameters, we let the smoothing parameter be $\mu = 10^{-4}$, and the constant step-sizes for sensors of network i be $\alpha^i = 0.1, 0.08, 0.06$, respectively. Thus, $\epsilon_\alpha = 0.2041$. We initialize the algorithm with arbitrary $x_{j,0}^i$, $y_{jk,0}^i$ and $\varphi_{jk,0}^i = g_{jk}^i(\mathbf{x}_0)$. The trajectories of the sensors' positions for the three sensor networks are plotted in Fig. 2. It can be seen that the positions of all sensors can almost converge to the NE. Also, more 'zigzags' can be observed for the case of a larger step-size, since the update is more aggressive.

Next, we illustrate the convergence rate results. First, we set the constant step-sizes for sensors of network i be $\alpha_j^i = 0.1a, 0.08a$ and $0.06a$, respectively, and let $a = 1.2$, 1 and 0.6, respectively. Hence, we fix the heterogeneity of the step-size $\epsilon_\alpha = 0.2041$, and set the largest step-size to $\alpha_{\max} = 0.12, 0.1$ and 0.06, respectively. The trajectories of the error gap $\|\mathbf{x}_t - \mathbf{x}^*\|$ with these settings are plotted in Fig. 3a. Then, we fix the largest step-size to $\alpha_{\max} = 0.1$ and the averaged step-size $\bar{\alpha} = 0.06$, and set the heterogeneity of the step-size $\epsilon_\alpha = 0.2041, 0.4714, 0.4907$ and 0.5443, respectively. The trajectories of the error gap $\|\mathbf{x}_t - \mathbf{x}^*\|$ with these settings are plotted in Fig. 3b. As can be seen from both figures, the error gap descends linearly for all cases. Moreover, the convergence speed is faster with larger step-sizes and smaller heterogeneity, which verifies the derived results in Theorem 1.

VI. CONCLUSIONS

In this work, we have studied an N-cluster non-cooperative game problem, where the players' cost functions are possibly non-smooth and the explicit expressions are unknown. By integrating the Gaussian smoothing techniques with the gradient tracking, a gradient-free NE seeking algorithm has been developed, in which the players across different clusters are allowed to select their own preferred constant step-sizes. We have shown that, when the largest step-size is sufficiently small, the players' actions approximately converge to the unique NE under a strongly monotone game mapping condition, and the error gap is proportional to the largest step-size and the smoothing parameter. Finally, the derived results have been verified by numerical simulations on a connectivity control game.
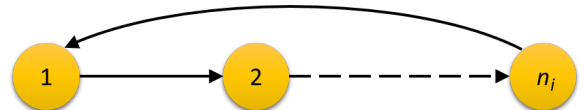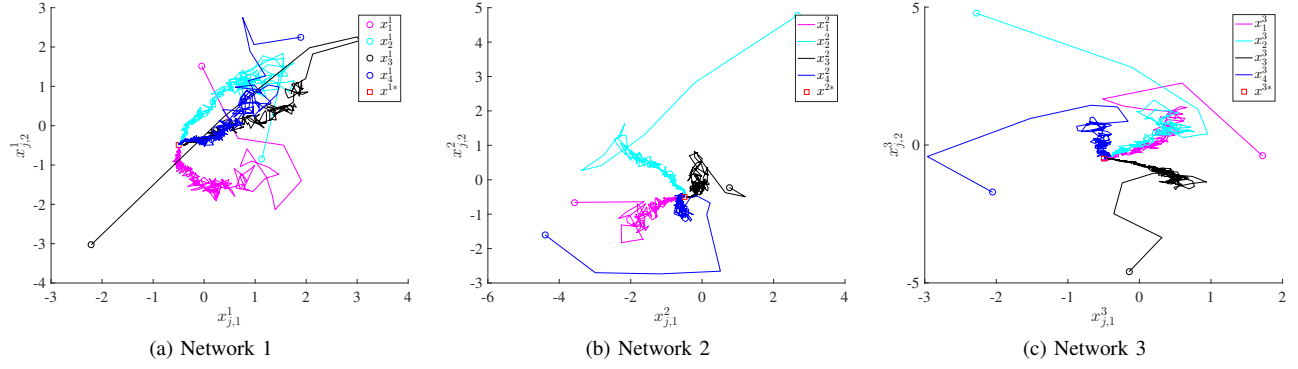

Fig. 1.   Communication network.

(a) Network 1   (b) Network 2   (c) Network 3

Fig. 2. Trajectories of sensors' positions in different networks.

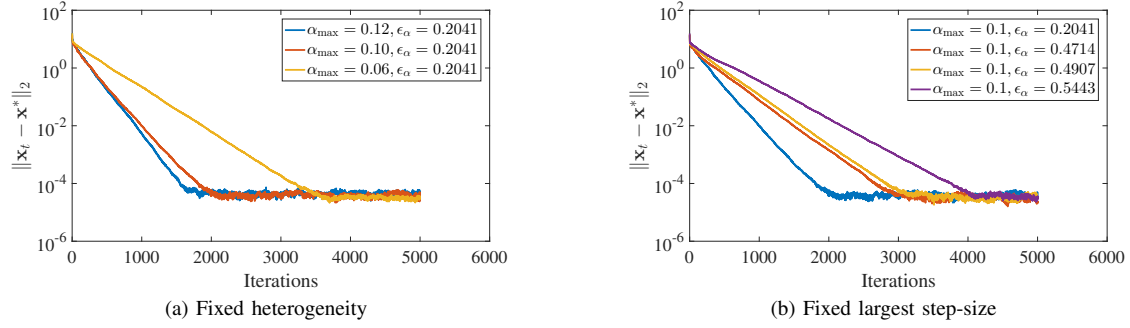(a) Fixed heterogeneity   (b) Fixed largest step-size

Fig. 3. Trajectories of the error gap $\|\mathbf{x}_t - \mathbf{x}^*\|$.

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