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One-Shot Yield-Cost Relations in General Quantum Resource Theories

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Although it is well known that the amount of resources that can be asymptotically distilled from a quantum state or channel does not exceed the resource cost needed to produce it, the corresponding relation in the nonasymptotic regime hitherto has not been well understood. Here, we establish a quantitative relation between the one-shot distillable resource yield and dilution cost in terms of transformation errors involved in these processes. Notably, our bound is applicable to quantum state and channel manipulation with respect to any type of quantum resource and any class of free transformations thereof, encompassing broad types of settings, including entanglement, quantum thermodynamics, and quantum communication. We also show that our techniques provide strong converse bounds relating the distillable resource and the resource dilution cost in the asymptotic regime. Moreover, we introduce a class of channels that generalize the twirling maps encountered in many resource theories, and by directly connecting it with resource quantification, we compute analytically several smoothed resource measures and improve our one-shot yield-cost bound in relevant theories. We use these operational insights to exactly evaluate important measures for various resource states in the resource theory of magic states.

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I. INTRODUCTION

Resource distillation and dilution are two fundamental resource manipulation tasks concerned, respectively, with the purification of noisy quantum resources and with the synthesis of general resources from pure ones. There have been many efforts to investigate the optimal rates of these tasks, which are known, respectively, as the *distillable resource yield* and the *resource dilution cost*, under various physical settings in the asymptotic regime with vanishing error [1–6] and in the nonasymptotic regime with nonzero error [7–12]. In the asymptotic regime, an operational argument paralleling the second law of thermodynamics implies that the distillable resource is bounded from above by the dilution cost, because otherwise one could produce an unbounded amount of resources by repeating the distillation-dilution cycle *ad infinitum* [2,5,13–19]. (For related discussions of second-order asymptotics in the context of entanglement theory, see also Refs. [20–22]). However, in the nonasymptotic regime, the relation between the distillable resource and the resource cost is more subtle, especially when errors incurred in the transformations are also taken into account.

The recent work of Ref. [23] has established a quantitative relation in the nonasymptotic regime for entanglement transformations that use local operations and classical communication and it has recovered the well-known asymptotic relation as a limit of the one-shot bound. A natural question following these findings is to what extent we can strengthen the relations obtained there and establish the corresponding bounds in more general settings beyond the distillation and dilution of entangled states.

One possibility is to extend the result for entanglement manipulation to the manipulation of other types of quantum resources, such as quantum superposition [9,10,24,25] and thermal nonequilibrium [3,26]. The ultimate form of this extension is to consider *general resource theories* [27], which encompass diverse types of physical resources in a single framework. Results obtained for general resource theories are not only readily applicable to practical settings of interest but also provide foundational insights into what properties are universally shared by wide classes of quantum resources, which might seem very different from

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each other on the surface. Despite the apparent difficulty of getting meaningful results out of this abstract setup, recent years have seen significant development in the understanding of operational aspects of general resource theories, including the settings of resource erasure [28,29], discrimination tasks [30–35], resource quantification [27,36,37], and resource transformations [4,12,16,19,31,38–42].

Another possible extension is to the setting of quantum channel manipulation. Although research in resource theories has largely focused on quantum states, there are various situations in which quantum dynamics, generally represented by quantum channels, play the role of the main objects of study. As a framework to accommodate these settings, *resource theories of quantum channels* have recently been under rapid development [11,31,43–69]. These dynamical resources include resource theories of states as special cases, because quantum states can be equivalently represented as quantum channels that prepare them. Thus, channel theories are strictly more general than state theories and results that hold for arbitrary quantum channel resources are automatically carried over to state theories. Importantly, we can impose additional structures to channels under study so that the resulting resource theory can describe various settings involving quantum channels with predetermined structures, such as the scenario of Bell nonlocality represented by no-signaling boxes [70–72], and measurement incompatibility [73], where a channel describes a set of positive operator-valued measures (POVMs) [63]. Although the performance of channel distillation and one-shot channel manipulation in general resource theories has recently been studied [58,59, 63,67,68], the relation between the one-shot distillable resource and the resource cost under arbitrary free-channel transformations has still been elusive.

In this work, we establish a fundamental quantitative relation between the one-shot distillable resource and the dilution cost that is applicable to general resource theories of quantum channels, accomplishing the two extensions in a single framework. We further extend the one-shot bound to the asymptotic regime and obtain strong converse bounds between transformation rates, also revealing new relations in the asymptotic manipulation of quantum states. Our framework not only accommodates well-known quantum resources that have been previously studied using related approaches—such as quantum entanglement and coherence—but also other settings that involve dynamical resources as central objects of study, e.g., quantum communication. Notably, a major part of our main results does not even assume the convexity or closedness of the set of free channels, unlike most of the works dealing with general resource theories. Moreover, our bounds hold for all chosen sets of free operations satisfying only minimal requirements, thus accommodating all physically motivated choices of free operations, including ones that may be equipped with complex structures.

An important technical aspect of our result is that, for the bounds to be most effective, they require the collapse of a class of resource measures based on Rényi divergences. Focusing on resource theories of states, we provide an operational interpretation for this and related conditions in terms of a special class of free operations that reflects the essential features of the "twirling" operation commonly encountered in various theories [74,75]. We use this class of operations to obtain analytical expressions for a number of smoothed resource measures, leading also to an improved yield-cost bound for the case of state manipulation. Finally, to demonstrate the applicability of our results, we use these operational insights in the resource theory of magic [54,76,77] and compute resource measures for several special states of interest.

We focus on discussing the main results and their implications in the main text, while deferring all proofs and technical details to the appendices.

II. RESOURCE THEORIES OF STATES

We begin by introducing the framework of resource theories of quantum states, in which quantum states are the central objects under study [78]. Quantum information processing necessarily involves the manipulation of quantum states but is commonly performed under some physical restrictions, which limits the accessible set of quantum states and operations. A primary example is a scenario in which two parties, Alice and Bob, are physically separated. Unless they are connected by a quantum communication channel, they can only apply local quantum operations and exchange classical messages. This restriction limits the accessible set of operations to the class known as local operations and classical communication (LOCC). Importantly, LOCC channels cannot create entanglement for free but can only prepare states that are separable [79].

The framework of resource theories allows us to deal formally with such restrictions [27]. It provides a recipe to quantify the amount of resource that cannot be created due to the restriction and characterize feasible resource manipulation using only accessible operations. Each resource theory comes with a set $\mathbb{F}$ of *free states* and a set $\mathbb{O}$ of *free operations*. They are subsets of quantum states and quantum channels (completely positive trace-preserving maps), respectively, and are considered to be accessible for free under a given setting. The minimal requirement for free operations is that they should not create resourceful states out of free states, i.e., $\Lambda \in \mathbb{O} \Rightarrow \Lambda(\sigma) \in \mathbb{F}, \forall \sigma \in \mathbb{F}$. Physical considerations usually impose more constraints but it is useful to consider the set of free operations consisting of all the operations satisfying this minimal requirement; we call this the maximal set of free operations and write it as $\mathbb{O}_{\max}$. Consequently, any valid set of free operations is a subset of $\mathbb{O}_{\max}$.

A major strength of resource theories is that they allow us to evaluate the resourcefulness of a given state quantitatively. This is made possible by introducing *resource measures*, which are functions from quantum states to real numbers that represent the amount of resource contained in the state. The requirement for resource measures is that they output the same value for free states and that they do not increase under the application of free operations.

Here, we consider two resource measures that can be defined for an arbitrary set $\mathbb{F}$ of free states and an arbitrary set $\mathbb{O}$ of free operations. The first one is known as the *generalized robustness* or, alternatively, the *max-relative entropy measure* [80–83], defined as

$$\begin{aligned} D_{\max,\mathbb{F}}(\rho) &:= \inf\left\{\log(1+s)\ \middle|\ \frac{\rho+s\tau}{1+s}\in\mathbb{F},\tau\in\mathbb{D}\right\} \\ &= \inf\left\{\lambda\ \middle|\ \rho\le 2^{\lambda}\sigma,\sigma\in\mathbb{F}\right\}, \end{aligned} \tag{1}$$

where $\mathbb{D}$ is the set of all quantum states. The other type of resource measure relevant to this work is the *min-relative entropy measure* [82], defined as

$$D_{\min,\mathbb{F}}(\rho) := \inf_{\sigma\in\mathbb{F}} D_{\min}(\rho\|\sigma), \tag{2}$$

where $D_{\min}(\rho\|\sigma) := -\log\mathrm{Tr}[\Pi_\rho\sigma]$, with Π_ρ denoting the projector onto the support of ρ.

These quantities lay out a platform for establishing relations between resource manipulation and resource quantification. Resource distillation and dilution in particular stand out as important subclasses of resource manipulation tasks. Resource distillation is a protocol to transform a given state to a state in the family $\mathbb{T}$ of *reference states* using free operations, while resource dilution is the opposite task, in which a reference state is to be transformed to the desired state. The optimal performance of these tasks is characterized by the one-shot *distillable resource* and the *dilution cost*, defined, respectively, as

$$\begin{aligned} d^{\epsilon}_{\mathbb{O}}(\rho) &:= \sup\left\{\mathfrak{R}_{\mathbb{F}}(\Phi)\ \middle|\ F(\Lambda(\rho),\Phi)\ge 1-\epsilon,\right. \\ &\qquad\qquad \left.\Phi\in\mathbb{T},\Lambda\in\mathbb{O}\right\}, \\ c^{\epsilon}_{\mathbb{O}}(\rho) &:= \inf\left\{\mathfrak{R}_{\mathbb{F}}(\Phi)\ \middle|\ F(\rho,\Lambda(\Phi))\ge 1-\epsilon,\right. \\ &\qquad\qquad \left.\Phi\in\mathbb{T},\Lambda\in\mathbb{O}\right\}, \end{aligned} \tag{3}$$

where $F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ is the fidelity [84] and $\mathfrak{R}_{\mathbb{F}}$ refers to an arbitrary resource monotone. For simplicity, we take $\mathfrak{R}_{\mathbb{F}} = D_{\min,\mathbb{F}}$ in the above definition throughout this paper.

To provide more insight into these definitions and connect them with notions of distillation and dilution familiar from commonly encountered resource theories, let us consider the case of quantum entanglement. Here, $\mathbb{F}$ denotes all separable quantum states and the family $\mathbb{T}$ of reference states can be taken simply as copies of the maximally entangled qubit Bell state:

$$\mathbb{T} = \{\Phi_2^{\otimes n} | n\in\mathbb{N}\}, \tag{4}$$

where $|\Phi_2\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and we use the shorthand notation $\Phi_2 = |\Phi_2\rangle\langle\Phi_2|$. Combined with the fact that [85]

$$D_{\min,\mathbb{F}}(\Phi_2^{\otimes n}) = n, \tag{5}$$

the quantities $d^{\epsilon}_{\mathbb{O}}$ and $c^{\epsilon}_{\mathbb{O}}$ can be alternatively understood as asking, respectively, how many copies of $|\Phi_2\rangle$ can be obtained from a given state, or how many copies of $|\Phi_2\rangle$ are necessary to produce a given state. We use more general definitions of distillation yield and resource cost to accommodate potentially more complex families of reference states and resource theories where a simple choice of reference states $\mathbb{T}$ might not be available.

III. RESOURCE THEORIES OF CHANNELS

We now introduce resource theories of quantum channels, extending the framework described in the previous section. To take into account the various structures that may be imposed on channels, we employ the approach of Ref. [63] and consider a subset $\mathbb{O}_{\rm all}$ of quantum channels as the set of channels of interest. Each situation designates a set $\mathbb{O}\subset\mathbb{O}_{\rm all}$ of *free channels* that does not contain any resource considered precious under the given setting. We also need to introduce the set of free operations that are accessible to manipulate channels. General channel transformations are described by quantum *superchannels* [86,87]. Superchannels are linear maps that map quantum channels to quantum channels and they are physically realizable by sandwiching an input channel with a preprocessing channel and a postprocessing channel, both of which can also be connected by a quantum memory. Letting $\mathbb{O}'_{\rm all}$ denote the set of output channels of interest, the set of relevant superchannels is specified as

$$\mathbb{S}_{\rm all} := \left\{\Xi\ \middle|\ \Xi(\mathcal{E})\in\mathbb{O}'_{\rm all},\ \forall\mathcal{E}\in\mathbb{O}_{\rm all}\right\}. \tag{6}$$

Then, one can consider a subset $\mathbb{S}\subset\mathbb{S}_{\rm all}$ of superchannels as *free superchannels*, serving as free operations. The minimal requirement for free operations forces $\Theta\in\mathbb{S}\Rightarrow\Theta(\mathcal{M})\in\mathbb{O}',\forall\mathcal{M}\in\mathbb{O}$. Analogous to the case of state theories, we also define the maximal set of free superchannels as the set of superchannels satisfying the above minimal condition and call it $\mathbb{S}_{\max}$.

We can analogously introduce resource quantifiers that do not increase under any set of free superchannels. The relevant channel resource measures that we need are the max-relative entropy measure [31,46,56,57], defined for every $\mathcal{E}\in\mathbb{O}_{\rm all}$ as

$$D_{\max,\mathbb{O}}(\mathcal{E}) := \inf_{\mathcal{M}\in\mathbb{O}} \sup_{\psi} D_{\max}(\mathrm{id}\otimes\mathcal{E}(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi)), \tag{7}$$

and the min-relative entropy measure [56,57], defined as

$$D_{\min,\mathbb{O}}(\mathcal{E}) := \inf_{\mathcal{M}\in\mathbb{O}} \sup_{\psi} D_{\min}(\mathrm{id}\otimes\mathcal{E}(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi)), \tag{8}$$

where the optimization is restricted to every pure input state ψ, without loss of generality [88].

Then, we can formalize the tasks of resource distillation and dilution. Resource distillation is a task that transforms a given channel $\mathcal{E}\in\mathbb{O}_{\mathrm{all}}$ to a channel in the set $\mathbb{T}\subset\mathbb{O}'_{\mathrm{all}}$ of reference channels using free superchannels $\mathbb{S}$, while resource dilution transforms a reference channel to the desired channel $\mathcal{E}$. The optimal performance of these tasks is characterized by the one-shot distillable resource and dilution cost, defined, respectively, as

$$\begin{aligned} d_{\mathbb{S}}^{\epsilon}(\mathcal{E}) &:= \sup\left\{\mathfrak{R}_{\mathbb{O}'}(\mathcal{T}) \;\middle|\; F(\Theta(\mathcal{E}),\mathcal{T})\geq 1-\epsilon,\right.\\ &\qquad\qquad \left.\mathcal{T}\in\mathbb{T}, \Theta\in\mathbb{S}\right\},\\ c_{\mathbb{S}}^{\epsilon}(\mathcal{E}) &:= \inf\left\{\mathfrak{R}_{\mathbb{O}'}(\mathcal{T}) \;\middle|\; F(\mathcal{E},\Theta(\mathcal{T}))\geq 1-\epsilon,\right.\\ &\qquad\qquad \left.\mathcal{T}\in\mathbb{T}, \Theta\in\mathbb{S}\right\}, \end{aligned} \tag{9}$$

where

$$F(\mathcal{E}_1,\mathcal{E}_2) := \min_{\psi} F(\mathrm{id}\otimes\mathcal{E}_1(\psi), \mathrm{id}\otimes\mathcal{E}_2(\psi)) \tag{10}$$

is the worst-case fidelity [88] (also called the channel fidelity) and $\mathfrak{R}_{\mathbb{O}}$ refers to an arbitrary resource monotone, which we take as $\mathfrak{R}_{\mathbb{O}} = D_{\min,\mathbb{O}}$ in the above definition for simplicity.

In our discussion below, we make the natural assumption that the reference channel $\mathcal{T}$ is a pure target channel, in the sense that $\mathrm{id}\otimes\mathcal{T}(\psi)$ is pure for every pure state ψ. This covers general types of distillation processes encountered in physical resource theories, where $\mathcal{T}$ plays the role of a noiseless resource. If the input and output systems are identical, then the condition implies that $\mathcal{T}$ is unitary, while if the input and output systems have different dimensions, then it implies that $\mathcal{T}$ is an isometry. When the input space is trivial—which is the case, e.g., in the special case of resource theories of states—then $\mathcal{T}$ reduces to a pure-state-preparation channel.

IV. YIELD-COST RELATIONS

Our first main result is a relation between the one-shot distillable resource and dilution cost, which we dub a yield-cost relation, applicable to (1) dynamical resource theories of channels, (2) any set of free channels, and (3) any set of free superchannels defined for the given free channels.

Theorem 1: *Suppose that for every channel $\mathcal{T}$ in the chosen reference set $\mathbb{T}$, $\mathrm{id}\otimes\mathcal{T}(\psi)$ is pure for every pure state ψ and $\mathcal{T}$ satisfies $D_{\min,\mathbb{O}}(\mathcal{T}) = D_{\max,\mathbb{O}}(\mathcal{T})$. Then, for all $\epsilon_1\in[0,1)$ and $\epsilon_2\in[0,1-\epsilon_1]$, for every quantum channel $\mathcal{E}$, and for every set $\mathbb{S}\subseteq\mathbb{S}_{\max}$ of free superchannels, the following inequality holds:*

$$d_{\mathbb{S}}^{\epsilon_1}(\mathcal{E}) \leq c_{\mathbb{S}}^{\epsilon_2}(\mathcal{E}) + \log f(\epsilon_1,\epsilon_2), \tag{11}$$

where $f(\epsilon_1,\epsilon_2)$ is a function defined as

$$f(\epsilon_1,\epsilon_2) := \min\left\{(1-\epsilon_1-\sqrt{\epsilon_2})^{-1}, (\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1})^{-2}\right\} \tag{12}$$

for $\epsilon_1+\sqrt{\epsilon_2}<1$ and

$$f(\epsilon_1,\epsilon_2) := (\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1})^{-2} \tag{13}$$

otherwise.

For $\epsilon_1+\sqrt{\epsilon_2}<1$, each quantity can be tighter than the other for certain error regions. Indeed, direct calculation reveals that $(1-\epsilon_1-\sqrt{\epsilon_2})^{-1} \leq (\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1})^{-2}$ if and only if

$$\begin{aligned} &\frac{1}{2}(1-\sqrt{1-\epsilon_2})(1-\sqrt{\epsilon_2})\\ &\qquad \leq \epsilon_1 \leq \frac{1}{2}(1+\sqrt{1-\epsilon_2})(1-\sqrt{\epsilon_2}). \end{aligned} \tag{14}$$

This gives a fundamental relation between distillable resource and dilution cost in the one-shot regime under any chosen set of free superchannels. In particular, when transformation errors are taken into account, the one-shot cost could be smaller than the one-shot yield. Our result establishes a quantitative trade-off relation between a potential yield-cost gap and the transformation inaccuracy.

The only assumption required for the result to hold is the choice of reference channels that satisfy $D_{\min,\mathbb{O}}(\mathcal{T}) = D_{\max,\mathbb{O}}(\mathcal{T})$. The collapse of the two measures to the same value is a common property of maximally resourceful states or channels [12,41,63], which are precisely the most appropriate references employed in distillation and dilution protocols in practice. We review examples of reference states and channels in several physical settings in Appendix B. We also return to this condition in Sec. V to provide more intuition behind the assumption and give an operationally motivated understanding of it.

Theorem 1 is a direct consequence of the following lemma.

Lemma 2: *Let $\mathcal{E}$ be an arbitrary quantum channel and let ϵ_1,ϵ_2 be arbitrary real numbers such that $\epsilon_1\in[0,1)$, $\epsilon_2\in[0,1-\epsilon_1]$. Also, let $\mathcal{T}_1$ be a channel for which $\mathrm{id}\otimes\mathcal{T}_1(\psi)$*

is pure for every pure state ψ and there exists $\Theta_1 \in \mathbb{S}$ such that $F(\Theta_1(\mathcal{E}), \mathcal{T}_1) \geq 1-\epsilon_1$ and let $\mathcal{T}_2$ be an arbitrary channel for which there exists $\Theta_2 \in \mathbb{S}$ such that $F(\Theta_2(\mathcal{T}_2), \mathcal{E}) \geq 1-\epsilon_2$. Then,

$$D_{\min,\mathbb{O}}(\mathcal{T}_1) \leq D_{\max,\mathbb{O}}(\mathcal{T}_2) + \log f(\epsilon_1, \epsilon_2), \tag{15}$$

where $f(\cdot,\cdot)$ is the function introduced in Theorem 1.

A natural question following these results is whether they smoothly connect to an asymptotic yield-cost relation. In asymptotic distillation (and analogously for dilution), the goal of the tasks is commonly to obtain as many copies of a fixed reference channel as possible from multiple copies of the given channel, in such a way that the transformation error vanishes in the limit of infinite copies. The figure of merit is then the ratio of the number of obtained copies of the reference channel to the number of used copies of the given channel. The following result provides a relation between the optimal rates for the asymptotic distillation and dilution in a more general setting, in which the errors do not necessarily approach zero in the asymptotic limit.

Theorem 3: *Let $\mathcal{E}$ be an arbitrary input channel and let $\mathcal{T}$ be some target reference channel for which* $\mathrm{id} \otimes \mathcal{T}(\psi)$ *is pure for every pure state ψ. Let d be any rate of distillation such that there exists a sequence $\{\Theta_n\}_n$ of free superchannels with*

$$1 - F\left(\Theta_n(\mathcal{E}^{\otimes n}), \mathcal{T}^{\otimes \lfloor dn \rfloor}\right) =: \delta_n. \tag{16}$$

Also, let c be any rate of dilution such that there exists a sequence $\{\Theta_n\}_n$ of free superchannels with

$$1 - F\left(\Theta_n(\mathcal{T}^{\otimes \lceil cn \rceil}), \mathcal{E}^{\otimes n}\right) =: \epsilon_n. \tag{17}$$

Suppose that the following conditions are satisfied:

(i) It holds that $\liminf_{n\to\infty} \epsilon_n + \delta_n < 1$.
(ii) The resource theory is closed under tensor products; that is, $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{O} \Rightarrow \mathcal{M}_1 \otimes \mathcal{M}_2 \in \mathbb{O}$.

Then the following inequality holds:

$$d \cdot D^{\infty}_{\min,\mathbb{O}}(\mathcal{T}) \leq c\, D^{\infty}_{\max,\mathbb{O}}(\mathcal{T}), \tag{18}$$

where

$$\begin{aligned} D^{\infty}_{\min,\mathbb{O}}(\mathcal{T}) &:= \lim_{n\to\infty} \frac{1}{n} D_{\min}\left(\mathcal{T}^{\otimes n}\right), \\ D^{\infty}_{\max,\mathbb{O}}(\mathcal{T}) &:= \lim_{n\to\infty} \frac{1}{n} D_{\max}\left(\mathcal{T}^{\otimes n}\right). \end{aligned} \tag{19}$$

Let us briefly discuss the assumptions of Theorem 3. Condition (i) simply means that the errors of distillation and dilution do not get too large at the same time; it is satisfied, for instance, when we take $\lim_{n\to\infty} \epsilon_n = 0$ and $\liminf_{n\to\infty} \delta_n < 1$ or vice versa. Condition (ii) is a basic property obeyed by virtually all theories encountered in practice.

Although the result of Theorem 3 is appealing since it follows straightforwardly from our one-shot relation in Theorem 1, one can ask whether it is possible to derive asymptotic relations that have a better dependence on quantities such as $D^{\infty}_{\max,\mathbb{O}}(\mathcal{T})$ and $D^{\infty}_{\min,\mathbb{O}}(\mathcal{T})$, as well as being free from the constraint on the target channel $\mathcal{T}$; that is, the constraint that $\mathrm{id} \otimes \mathcal{T}(\psi)$ is pure for every pure state ψ. We discuss several variations on this idea in Appendix C, where we present other bounds that are potentially tighter and do not impose any condition on $\mathcal{T}$ but come with an additional condition on the sequences $\{\epsilon_n\}_n$ and $\{\delta_n\}_n$ of achievable errors or require the computation of more complicated regularized quantities. For our purposes, it is sufficient to use the condition of Theorem 3 and, indeed, we see that the reliance on the max-/min-relative entropies is not a problem in most theories of practical interest.

The result of Theorem 3 implies in particular the strong converse bounds given in Corollary 4 below, for the rates of resource manipulation tasks. Before stating this result, let us recall the definitions of the basic asymptotic quantities involved. We define the asymptotic distillable resource $d^{\infty}_{\mathbb{S}}$ as the largest achievable rate at which copies of $\mathcal{T}$ can be extracted from a given channel and, analogously, the asymptotic resource cost $c^{\infty}_{\mathbb{S}}$ as the least rate at which copies of $\mathcal{T}$ are needed to produce a given channel. Imposing that the error in such transformations vanishes in the asymptotic limit, we have

$$\begin{aligned} d^{\infty}_{\mathbb{S}}(\mathcal{E}, \mathcal{T}) &:= \sup\left\{ d \,\middle|\, \lim_{n\to\infty} \sup_{\Theta_n \in \mathbb{S}} F\left(\Theta_n(\mathcal{E}^{\otimes n}), \mathcal{T}^{\otimes \lfloor dn \rfloor}\right) = 1 \right\}, \\ c^{\infty}_{\mathbb{S}}(\mathcal{E}, \mathcal{T}) &:= \inf\left\{ c \,\middle|\, \lim_{n\to\infty} \sup_{\Theta_n \in \mathbb{S}} F\left(\Theta_n(\mathcal{T}^{\otimes \lceil cn \rceil}), \mathcal{E}^{\otimes n}\right) = 1 \right\}. \end{aligned} \tag{20}$$

As counterparts to the above asymptotic quantities, we also define the corresponding strong converse quantities as [17,89]

$$\begin{aligned} \tilde{d}^{\infty}_{\mathbb{S}}(\mathcal{E}, \mathcal{T}) &:= \inf\left\{ d \,\middle|\, \lim_{n\to\infty} \sup_{\Theta_n \in \mathbb{S}} F\left(\Theta_n(\mathcal{E}^{\otimes n}), \mathcal{T}^{\otimes \lfloor dn \rfloor}\right) = 0 \right\}, \\ \tilde{c}^{\infty}_{\mathbb{S}}(\mathcal{E}, \mathcal{T}) &:= \sup\left\{ c \,\middle|\, \lim_{n\to\infty} \sup_{\Theta_n \in \mathbb{S}} F\left(\Theta_n(\mathcal{T}^{\otimes \lceil cn \rceil}), \mathcal{E}^{\otimes n}\right) = 0 \right\}, \end{aligned} \tag{21}$$

with the interpretation of the strong converse distillable resource as the smallest rate at which the error is guaranteed to converge to one in the asymptotic limit and that for the strong converse dilution cost as the largest rate at which the error is guaranteed to converge to one. The following equivalent expressions for the strong converse quantities hold:

$$\tilde{d}_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) = \sup\left\{ d \;\middle|\; \limsup_{n\to\infty} \sup_{\Theta_n\in\mathbb{S}} F\left(\Theta_n(\mathcal{E}^{\otimes n}), \mathcal{T}^{\otimes\lfloor dn\rfloor}\right) > 0 \right\},$$
$$\tilde{c}_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) = \inf\left\{ c \;\middle|\; \limsup_{n\to\infty} \sup_{\Theta_n\in\mathbb{S}} F\left(\Theta_n(\mathcal{T}^{\otimes\lceil cn\rceil}), \mathcal{E}^{\otimes n}\right) > 0 \right\}, \tag{22}$$

which can alternatively be understood as weakening the requirements imposed on the transformation error in Eq. (20). Here, the interpretation is as follows: unlike the quantities in Eq. (20), the strong converse quantities no longer ask that the error converges to zero but represent the best rates at which the error does not converge to one—they therefore provide a general threshold on achievable asymptotic transformations even when perfect conversion is not required. Note that $d_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) \leq \tilde{d}_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T})$ and $c_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) \geq \tilde{c}_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T})$ by definition.

Theorem 3 implies the following strong converse bounds.

Corollary 4: *Let $\mathcal{E}$ be an arbitrary input channel and let $\mathcal{T}$ be some target reference channel for which* $\mathrm{id}\otimes\mathcal{T}(\psi)$ *is pure for every pure state ψ. Suppose that the following conditions are satisfied:*

(i) The resource theory is closed under tensor products; that is, $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{O} \Rightarrow \mathcal{M}_1 \otimes \mathcal{M}_2 \in \mathbb{O}$.
(ii) The target channel satisfies $D_{\min,\mathbb{O}}(\mathcal{T}) = D_{\max,\mathbb{O}}(\mathcal{T})$ and $D_{\min,\mathbb{O}}(\mathcal{T}^{\otimes n}) = nD_{\min,\mathbb{O}}(\mathcal{T})$ for every n.

Then the following inequalities hold:

$$d_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) \leq \tilde{c}_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}), \quad \tilde{d}_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) \leq c_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}). \tag{23}$$

We remark here that the definitions of strong converse quantities and Corollary 4 imply the following fundamental inequality:

$$d_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}) \leq c_{\mathbb{S}}^{\infty}(\mathcal{E},\mathcal{T}). \tag{24}$$

Here, condition (i) is generally satisfied in physical theories, but condition (ii) imposes nontrivial requirements on the choice of a suitable target channel $\mathcal{T}$. In particular, in addition to the collapse of the max- and min-relative entropy measures to the same value, it requires that the min-relative entropy measure is additive, i.e., $D_{\min,\mathbb{O}}(\mathcal{T}^{\otimes n}) = nD_{\min,\mathbb{O}}(\mathcal{T})$. Although this constitutes a seemingly restrictive condition, the existence of channels or states satisfying this requirement is actually a common property of important resource theories, obeyed, e.g., by quantum entanglement, quantum thermodynamics, magic, and quantum communication (cf. Ref. [58, Table 1]). Intuitively, the condition can be thought of as characterizing how well the given reference $\mathcal{T}$ serves as an intermediary channel in distillation and dilution protocols.

We stress that all the results in this section automatically provide bounds for state theories as special cases; when one is interested in a resource theory of quantum states with free states $\mathbb{F}$ and free operations $\mathbb{O}$, the same results hold by replacing free superchannels $\mathbb{S}$ with free operations $\mathbb{O}$ and free channels $\mathbb{O}$ with free states $\mathbb{F}$. Furthermore, in the case of state transformations, the generalized quantum Stein's lemma [15,90] implies that the asymptotic distillable resource coincides with its strong converse rate when the set of free states and the reference state satisfy some mild assumptions—for details, see Appendix D. When the generalized quantum Stein's lemma holds, for the asymptotic distillation and dilution for an arbitrary state ρ with respect to a reference state Φ under free operations $\mathbb{O}$, we can improve the bound in Corollary 4 to

$$\tilde{d}_{\mathbb{O}}^{\infty}(\rho,\Phi) \leq \tilde{c}_{\mathbb{O}}^{\infty}(\rho,\Phi). \tag{25}$$

This applies in particular to the resource theory of quantum entanglement, which solves an open problem posed in Ref. [23]. The implication of Eq. (25) is that the yield-cost relation still holds even when neither of the errors is required to vanish in the asymptotic limit, which gives a significant strengthening of the inequalities in Eq. (23).

We remark that another approach to asymptotic transformation rates of states and the consequent relations between asymptotic resource yield and cost in general resource theories has been considered in Ref. [19].

V. OPERATIONAL ACCOUNT FOR REFERENCE STATES

We saw that the relation between $D_{\min,\mathbb{O}}$ and $D_{\max,\mathbb{O}}$ plays a major role in establishing the yield-cost relation in general resource theories. Here, we give operational insights into the properties of these quantifiers and in particular the condition $D_{\min,\mathbb{O}}(\mathcal{T}) = D_{\max,\mathbb{O}}(\mathcal{T})$ that we have encountered previously. By studying the interplay between these and other related measures, we are able to quantify a number of smoothed resource monotones for target reference states in general resource theories. In this section, we restrict our attention to theories of quantum states and we use $\mathbb{F}$ and $\mathbb{O}$ to specify the set of free states and free operations, respectively. In what follows, we also assume that $\mathbb{F}$ is a convex and closed set.

In addition to $D_{\min,\mathbb{F}}$ and $D_{\max,\mathbb{F}}$, we also consider another type of robustness measure known as the *standard* *robustness* [80], defined as

$$D_{s,\mathbb{F}}(\rho) := \inf\left\{\log(1+s) \;\middle|\; \frac{\rho+s\tau}{1+s} \in \mathbb{F}, \tau \in \mathbb{F}\right\}, \quad (26)$$

We also define two smooth robustness quantities as [82,90]

$$D^{\epsilon}_{s,\mathbb{F}}(\rho) := \inf\left\{D_{s,\mathbb{F}}(\rho') \;\middle|\; F(\rho',\rho) \geq 1-\epsilon\right\}, \quad (27)$$

$$D^{\epsilon}_{\max,\mathbb{F}}(\rho) := \inf\left\{D_{\max,\mathbb{F}}(\rho') \;\middle|\; F(\rho',\rho) \geq 1-\epsilon\right\} \quad (28)$$

and two types of hypothesis-testing relative entropy measures [10,41,90]:

$$\begin{aligned} D^{\epsilon}_{H,\mathbb{F}}(\rho) &:= \inf_{\sigma\in\mathbb{F}} D^{\epsilon}_{H}(\rho\|\sigma), \\ D^{\epsilon}_{H,\mathrm{aff}(\mathbb{F})}(\rho) &:= \inf_{\sigma\in\mathrm{aff}(\mathbb{F})} D^{\epsilon}_{H}(\rho\|\sigma), \end{aligned} \quad (29)$$

where

$$D^{\epsilon}_{H}(\rho\|\sigma) := \sup_{\substack{0\leq P\leq \mathbb{1}\\ \mathrm{Tr}(P\rho)\geq 1-\epsilon}} \log \mathrm{Tr}(P\sigma)^{-1} \quad (30)$$

is the hypothesis-testing relative entropy [7,91,92]. We use $\mathrm{aff}(\mathbb{F})$ to denote the affine hull of $\mathbb{F}$, which is the smallest affine subspace that contains $\mathbb{F}$. The min-relative entropy is obtained as a special case of the hypothesis-testing relative entropy measure as $D_{\min,\mathbb{F}}(\rho) = D^{\epsilon=0}_{H,\mathbb{F}}(\rho)$. We use this correspondence to define the affine min-relative entropy measure [41,63] as

$$\begin{aligned} D_{\min,\mathrm{aff}(\mathbb{F})}(\rho) &:= D^{\epsilon=0}_{H,\mathrm{aff}(\mathbb{F})}(\rho) \\ &= \inf_{\sigma\in\mathrm{aff}(\mathbb{F})} D^{\epsilon=0}_{H}(\rho\|\sigma). \end{aligned} \quad (31)$$

Then, the following ordering holds for an arbitrary state ρ:

$$D_{\min,\mathrm{aff}(\mathbb{F})}(\rho) \leq D_{\min,\mathbb{F}}(\rho) \leq D_{\max,\mathbb{F}}(\rho) \leq D_{s,\mathbb{F}}(\rho), \quad (32)$$

where the first inequality follows because $\mathbb{F} \subseteq \mathrm{aff}(\mathbb{F})$, the second inequality because $D_{\min}(\rho\|\sigma) \leq D_{\max}(\rho\|\sigma)$ for all states ρ and σ, and the third inequality from the definitions of the generalized and standard robustness measures. Depending on the structure of $\mathbb{F}$, some of these measures may exhibit drastic behavior. $D_{\min,\mathbb{F}}$ and $D_{\max,\mathbb{F}}$ take finite values for every state ρ as long as $\mathbb{F}$ contains at least one full-rank state, which is satisfied by most of the relevant theories. However, $D_{s,\mathbb{F}}$ may diverge if $\mathbb{F}$ is *reduced dimensional* [41], meaning that $\mathbb{F}$ has zero volume in the set of all states and $\mathrm{span}(\mathbb{F})$ is not the whole space of self-adjoint operators. Examples of reduced-dimensional theories include the theories of coherence and thermal nonequilibrium. On the other hand, for *full-dimensional* theories, which are not reduced dimensional and include the theories of entanglement and magic as examples, $D_{s,\mathbb{F}}$ remains finite but $D_{\min,\mathrm{aff}(\mathbb{F})}$ takes the value zero for every state ρ. Therefore, $D_{s,\mathbb{F}}$ is usually a relevant resource quantifier in full-dimensional theories, while $D_{\min,\mathrm{aff}(\mathbb{F})}$ is relevant in reduced-dimensional theories [41,63].

Although these measures are introduced in a rather abstract manner, they play crucial roles in the quantitative characterization of distillation and dilution—under certain assumptions on the target state Φ, the value of the one-shot resource yield $d^{\epsilon}_{\mathbb{O}}(\rho)$ under the maximal set of free operations is directly related to $D^{\epsilon}_{H,\mathbb{F}}(\rho)$ or $D^{\epsilon}_{H,\mathrm{aff}(\mathbb{F})}(\rho)$, while the value of the resource cost $c^{\epsilon}_{\mathbb{O}}(\rho)$ corresponds to the value of $D^{\epsilon}_{s,\mathbb{F}}(\rho)$ or $D^{\epsilon}_{\max,\mathbb{F}}(\rho)$ [12,41]. A necessary requirement for a precise description of distillation or dilution to be possible is that, when evaluated on the reference state Φ, the resource measures all collapse to the same value; specifically, in full-dimensional theories one needs that

$$D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi), \quad (33)$$

while in reduced-dimensional theories one instead requires

$$D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi). \quad (34)$$

In some theories, such as entanglement or coherence, the existence of such states is natural: the maximally entangled or coherent states always satisfy the requirement. It is rather remarkable that maximally resourceful states satisfy $D_{\min,\mathbb{F}}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$ in *any* convex resource theory [41] but this is still not sufficient to guarantee that Eqs. (33) and (34) hold in general—we expect this to be a resource-dependent property that needs to be verified explicitly in each specific setting.

Here, we find that Eqs. (33) and (34) can have operational implications, which then help to evaluate other resource measures introduced in this section. In particular, we provide an understanding of the conditions given in Eqs. (33) and (34) in terms of a free operation that parallels twirling.

Lemma 5: *If a state Φ satisfies $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi)$ or $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$, there exists a free operation $\Lambda \in \mathbb{O}_{\max}$ defined by an operator $0 \leq P^\star \leq \mathbb{1}$ with $\mathrm{Tr}[P^\star\Phi] = 1$ of the form*

$$\Lambda(\cdot) = \mathrm{Tr}[P^\star \cdot]\Phi + \mathrm{Tr}[(\mathbb{1}-P^\star)\cdot]\sigma^\star, \quad (35)$$

such that $\sigma^\star$ is a state orthogonal to Φ, satisfying $\mathrm{Tr}[P\sigma^\star] = \mathrm{Tr}[\Phi\sigma^\star] = 0$. When $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi)$, one can further take $\sigma^\star$ to be a free state.

This free operation possesses a property similar to well-known group twirling operations such as the isotropic twirling $\int U \otimes U^*(\cdot)U^\dagger \otimes U^{*\dagger}\, dU$ in entanglement theory [75], in that it maps states to the given reference state or its complement, while stabilizing a specific state Φ.

Such twirling operations have been found to be useful to evaluate several entanglement measures for states invariant under them [93]. In particular, the existence of such an operation allows us to obtain exact expressions for smoothed resource measures, which *a priori* require a nontrivial optimization over all states within an error ϵ.

Proposition 6: *If a state Φ satisfies $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi) =: r$, then for every $\epsilon \in [0,1)$, it holds that*

$$D^{\epsilon}_{H,\mathbb{F}}(\Phi) = r + \log \frac{1}{1-\epsilon} \tag{36}$$

and

$$D^{\epsilon}_{\max,\mathbb{F}}(\Phi) = D^{\epsilon}_{s,\mathbb{F}}(\Phi) = \max\left\{ r - \log \frac{1}{1-\epsilon}, 0 \right\}. \tag{37}$$

Similarly, if a state Φ satisfies $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$ $=: r$, then

$$D^{\epsilon}_{H,\mathrm{aff}(\mathbb{F})}(\Phi) = D^{\epsilon}_{H,\mathbb{F}}(\Phi) = r + \log \frac{1}{1-\epsilon} \tag{38}$$

and

$$D^{\epsilon}_{\max,\mathbb{F}}(\Phi) = \max\left\{ r - \log \frac{1}{1-\epsilon}, 0 \right\}. \tag{39}$$

A crucial property that makes this result possible is that optimal states in the optimization of the smoothed resource measures can always be taken of the form $\Phi_\kappa = \kappa\Phi + (1-\kappa)\sigma^\star$, as obtained through the operation in Lemma 5. Such states share many useful properties with isotropic states of entanglement theory and, indeed, several quantitative insights into the description of isotropic states in works such as Ref. [93] can be extended to general resource theories. In particular, the entropic resource quantifiers can be computed exactly for such states.

Proposition 7: *Suppose a state Φ satisfies $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi) =: r$, and let $\sigma^\star$ be the state in Eq. (35). Then,*

$$D_{\min,\mathbb{F}}(\Phi_\kappa) = \begin{cases} 0, & 0 \le \kappa < 1, \\ r, & \kappa = 1, \end{cases} \tag{40}$$

and

$$\begin{aligned} D_{\max,\mathbb{F}}(\Phi_\kappa) &= D_{s,\mathbb{F}}(\Phi_\kappa) \\ &= \max\left\{ r - \log\frac{1}{\kappa}, 0 \right\} \end{aligned} \tag{41}$$

for every $\Phi_\kappa = \kappa\Phi + (1-\kappa)\sigma^\star$ with $\kappa \in [0,1]$. Similarly, if a state Φ satisfies $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi) =: r$, then

$$\begin{aligned} D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi_\kappa) &= D_{\min,\mathbb{F}}(\Phi_\kappa) \\ &= \begin{cases} 0, & 0 < \kappa < 1, \\ \log \dfrac{1}{1-2^{-r}}, & \kappa = 0, \\ r, & \kappa = 1, \end{cases} \end{aligned} \tag{42}$$

and

$$D_{\max,\mathbb{F}}(\Phi_\kappa) = \max\left\{ r - \log\frac{1}{\kappa}, \log\frac{1-\kappa}{1-2^{-r}} \right\} \tag{43}$$

for every $\Phi_\kappa = \kappa\Phi + (1-\kappa)\sigma^\star$ with $\kappa \in [0,1]$.

We present a more complete discussion of the quantitative properties of the isotropiclike states Φ_κ in Appendix E, where we also show how the smoothed entropic measures can be computed for this class of states.

These exact evaluations of the resource measures allow us to employ the argument in Ref. [23] to obtain an alternative bound to that given in Theorem 1.

Theorem 8: *If the chosen reference set obeys $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi)\ \forall \Phi \in \mathbb{T}$ or $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi)\ \forall \Phi \in \mathbb{T}$, then for every set $\mathbb{O} \subseteq \mathbb{O}_{\max}$ of free operations and all $\epsilon_1, \epsilon_2 \ge 0$ satisfying $\epsilon_1 + \epsilon_2 < 1$, the following inequality holds:*

$$d^{\epsilon_1}_{\mathbb{O}}(\rho) \le c^{\epsilon_2}_{\mathbb{O}}(\rho) + \log\frac{1}{1-\epsilon'}, \tag{44}$$

where $\epsilon' := \left[\sqrt{\epsilon_1(1-\epsilon_2)} + \sqrt{\epsilon_2(1-\epsilon_1)}\right]^2$.

This improves the bound in Theorem 1, as it gives a tighter upper bound for all regions of (ϵ_1, ϵ_2) with $\epsilon_1 + \epsilon_2 < 1$, as we prove in Appendix G. We also remark that this bound becomes essentially tight when $\mathbb{O} = \mathbb{O}_{\max}$, $\rho \in \mathbb{T}$ and either ϵ_1 or ϵ_2 is 0. This can be shown by using $d^{\epsilon_1}_{\mathbb{O}_{\max}}(\rho) = D^{\epsilon_1}_{H,\mathbb{F}}(\rho)$ and $c^{\epsilon_2}_{\mathbb{O}_{\max}}(\rho) = D^{\epsilon_2}_{s,\mathbb{F}}(\rho)$ when $D_{\min,\mathbb{F}}(\rho) = D_{s,\mathbb{F}}(\rho)$ and $d^{\epsilon_1}_{\mathbb{O}_{\max}}(\rho) = D^{\epsilon_1}_{H,\mathrm{aff}(\mathbb{F})}(\rho)$ and $c^{\epsilon_2}_{\mathbb{O}_{\max}}(\rho) = D^{\epsilon_2}_{\max,\mathbb{F}}(\rho)$ when $D_{\min,\mathrm{aff}(\mathbb{F})}(\rho) = D_{\max,\mathbb{F}}(\rho)$ (up to some floor or ceiling due to a discrete structure of $\mathbb{T}$) [63], as well as explicitly evaluating these measures by Proposition 6.

So far, we have employed the property of several resource measures evaluated for a reference state to show the existence of a twirlinglike operation as given in Eq. (35) and presented its applications. We now also investigate the opposite direction, asking whether the existence of a certain type of free operation provides insights into the relation between different resource measures. The following result addresses this question.

Lemma 9: *Let $\mathcal{S}(\Phi)$ be a set of channels stabilizing Φ, defined as*

$$\mathcal{S}(\Phi) := \left\{ \mathrm{Tr}[P\cdot]\Phi + \tilde{\Lambda}(\cdot) \;\middle|\; 0 \le P \le \mathbb{1}, \right.$$
$$\left. \mathrm{Tr}[\Phi P] = 1, \tilde{\Lambda} \in CP \right\}, \tag{45}$$

where CP *is the set of completely positive maps. Then, if there exists a free operation $\Lambda \in \mathbb{O}_{\max} \cap \mathcal{S}(\Phi)$, the equality*

$$D_{\min,\mathbb{F}}(\Phi) = D_{\max,\mathbb{F}}(\Phi) \tag{46}$$

holds. In addition, if Λ is completely free, i.e., $\mathrm{id} \otimes \Lambda \in \mathbb{O}_{\max}$*, where* id *is the identity map on an arbitrary ancillary system, then*

$$D_{\min,\mathbb{F}}(\Phi^{\otimes n}) = D_{\max,\mathbb{F}}(\Phi^{\otimes n}), \quad \forall n. \tag{47}$$

Moreover, if $\tilde{\Lambda}$ can be taken as a free subchannel in Eq. (45), i.e., $\tilde{\Lambda}(\sigma) \propto \mathrm{cone}(\mathbb{F})$, $\forall \sigma \in \mathbb{F}$, then

$$D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi), \tag{48}$$

where $\mathrm{cone}(X) := \left\{ \lambda x \;\middle|\; \lambda \ge 0, x \in X \right\}$.

Together with Lemma 5, we conclude a general operational correspondence between the collapse of resource measures and the existence of twirlinglike free operations: we have that $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi)$ if and only if there exists a map of the form given in Eq. (35) with a free state $\sigma^\star$.

However, it is not always easy to find a channel $\Lambda \in \mathbb{O}_{\max} \cap \mathcal{S}(\Phi)$ to apply Lemma 9. The following result helps to find such a channel by formally relating the group-theoretic twirling operations inspired by the original LOCC twirlings [74,75] to the form of Eq. (45) necessary to apply Lemma 9.

Proposition 10: *Let Φ be a pure state. Suppose that there exists a convex set $\mathbb{O} \subseteq \mathbb{O}_{\max}$ of free operations and a finite or a compact Lie group G with a unitary representation $\{U(g)\}_{g\in G}$ that satisfies $U(g) \cdot U(g)^\dagger \in \mathbb{O}$ and $U(g)\left|\Phi\right\rangle = e^{i\phi_g}\left|\Phi\right\rangle$ $\forall g \in G$ for some set of eigenvalues $\{e^{i\phi_g}\}_g$. If $\left|\Phi\right\rangle$ is the unique simultaneous eigenvector of all $U(g)$'s with eigenvalues $\{e^{i\phi_g}\}_g$, then $D_{\min,\mathbb{F}}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$ holds. Moreover, if $\mathbb{O}$ is completely free, then $D_{\min,\mathbb{F}}(\Phi^{\otimes n}) = D_{\max,\mathbb{F}}(\Phi^{\otimes n})$ holds for every positive integer n.*

This result allows us to find an appropriate free operation to ensure the condition required in Theorems 1 and 8. We indeed find that Lemma 9 and Proposition 10 are helpful to obtain new insights into specific resource theories, which we now demonstrate.

VI. APPLICATION: THEORY OF MAGIC

As we have emphasized earlier, our results immediately hold in general types of quantum resources of both states and channels and we direct the interested reader to, e.g., the recent Refs. [12,58,59,63,67], as well as to Appendix B, for a discussion of how such general methods can be applied in specific theories such as quantum communication. Here, we discuss a nontrivial example in which the quantification of resource measures for many important states is still not understood—the theory of magic (nonstabilizerness) [76,77]. Applying the results of our work, we show how they can be used to provide new quantitative insights and reveal broad and useful relations for this resource.

To realize scalable fault-tolerant quantum computation, it is essential to encode the whole computation in a higher-dimensional space using an error-correcting code. Clifford gates in particular stand out as a subset of quantum gates that admits efficient fault-tolerant encoding on many error-correcting codes. However, to form a universal gate set, we also need to implement a non-Clifford gate. This is usually accomplished by the gate-teleportation technique [94,95], which simulates the action of a non-Clifford gate by combining a Clifford operation and a *magic state*, which cannot be created solely by Clifford operations. Since magic states are hard to produce in a fault-tolerant manner, they are precious resources under this setting, motivating us to consider the quantification and manipulation of them using a resource-theoretic formalism [76,77].

In particular, the optimal performance of magic state distillation and dilution has been a central question in the field, as it is the most resource-demanding part to realize fault-tolerant universal quantum computation. Our results establish a relation between the optimal performance of these two, under the assumption that the reference resource states satisfy the aforementioned conditions on resource measures. Here, we investigate these conditions for some well-known states that can be good candidates for reference states in distillation and dilution protocols. We find that our approach, particularly Lemma 9 and Proposition 10, provides new operational insights into the evaluation of resource measures, which may be of independent interest.

Stabilizer states are those that can be represented by a probabilistic mixture of eigenstates of Pauli operators. In the resource theory of magic, stabilizer states form the set $\mathbb{F}_{\mathrm{stab}}$ of free states. Although it is often assumed that quantum computation is carried out in multiqubit systems, higher-dimensional qu*d*it systems also stand as potential candidates for a large-scale quantum computing architecture [96,97]. Resource theories have been developed for both settings [76,77,98] and we consider several standard magic states defined in these scenarios.

Let us start with multiqubit systems. For single-qubit states, the farthest states from the set of stabilizer states are positioned at $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$ in Bloch coordinates, which are connected to each other by Clifford unitaries. We call one of them the *face state*, which is written as

$$|F\rangle\langle F| = \frac{\mathbb{1} + (X+Y+Z)/\sqrt{3}}{2}, \tag{49}$$

where X, Y, Z are qubit Pauli operators. Then, Lemma 9 and Proposition 10 allow us to obtain the following relation:

Proposition 11: *The qubit face state satisfies*

$$D_{\min,\mathbb{F}_{\rm stab}}(F^{\otimes n}) = D_{\max,\mathbb{F}_{\rm stab}}(F^{\otimes n}). \tag{50}$$

for every integer $n \geq 1$.

Since the face state is a maximizer of $D_{\min,\mathbb{F}_{\rm stab}}$, called a golden state [12,41], the case of $n=1$ recovers the result in Ref. [12]. Also, noting that the stabilizer extent introduced in Ref. [99] is identical to $D_{\max,\mathbb{F}_{\rm stab}}$ [36], Eq. (50) can be shown by using the additivity of $D_{\min,\mathbb{F}_{\rm stab}}(F^{\otimes n})$ and $D_{\max,\mathbb{F}_{\rm stab}}(F^{\otimes n})$, as well as the equivalence of $D_{\min,\mathbb{F}_{\rm stab}}(F)$ and $D_{\max,\mathbb{F}_{\rm stab}}(F)$ shown in Ref. [99]. Our result provides a different operational approach to this relation based on a strategy with a broad potential applicability.

We also obtain an analogous result for a special class of magic states that includes the T state and the Toffoli state, which are usually used for gate-teleportation protocols.

Proposition 12: *Let V be a unitary in the third level of the Clifford hierarchy and let $|\phi\rangle$ be a stabilizer state defined on a multiqudit system with local dimension d, where $d = 2$ or d is an odd prime. Then, the state $|\psi\rangle = V|\phi\rangle$ satisfies $D_{\min,\mathbb{F}_{\rm stab}}(\psi^{\otimes n}) = D_{\max,\mathbb{F}_{\rm stab}}(\psi^{\otimes n})$ for every integer $n \geq 1$.*

This provides an alternative proof of the result in Ref. [99] and extends it to qudit systems, including the qudit generalization of the T state [97]. Note that $D_{\max,\mathbb{F}_{\rm stab}}(\psi^{\otimes n}) = nD_{\max,\mathbb{F}_{\rm stab}}(\psi)$ holds for every ψ of up to three qubits [99] and, indeed, also for single-qubit mixed states [100].

Although $D_{\min,\mathbb{F}_{\rm stab}}$ and $D_{\max,\mathbb{F}_{\rm stab}}$ coincide for the above cases, one can actually show that for every single-qubit state, $D_{\max,\mathbb{F}_{\rm stab}}$ is strictly less than $D_{s,\mathbb{F}_{\rm stab}}$. The natural question is whether there exists a state for which the three quantities collapse to the same value, allowing us to use both Theorems 1 and 8. No such state has previously been known in the theory of multiqubit magic, which has prevented previously known bounds for one-shot distillation and dilution from being tight [12,41]. To show that a suitable choice does indeed exist, let us consider the three-qubit fiducial Hoggar state [77,101–104], defined as

$$|\text{Hog}\rangle = \frac{1}{\sqrt{6}}(1+i, 0, -1, 1, -i, -1, 0, 0)^T. \tag{51}$$

The Hoggar state serves as one of the fiducial states from which group-covariant symmetric informationally complete POVMs (SIC POVMs) can be generated [104]. We show that the three measures collapse for the Hoggar state.

Proposition 13: *The three-qubit Hoggar state satisfies*

$$D_{\min,\mathbb{F}_{\rm stab}}(\text{Hog}) = D_{\max,\mathbb{F}_{\rm stab}}(\text{Hog}) = D_{s,\mathbb{F}_{\rm stab}}(\text{Hog}) = \log\frac{12}{5}, \tag{52}$$

and

$$D_{\min,\mathbb{F}_{\rm stab}}(\text{Hog}^{\otimes n}) = D_{\max,\mathbb{F}_{\rm stab}}(\text{Hog}^{\otimes n}) \tag{53}$$

for every integer $n \geq 1$.

We note that the value of $D_{s,\mathbb{F}_{\rm stab}}(\text{Hog})$ has been reported in Ref. [77].

Let us now turn our attention to qutrit states. In the qutrit magic theory, two classes of states are identified to have the maximum sum negativity of the discrete Wigner function [76]. The first class is represented by the Strange state [76,98,105,106], defined as

$$|S\rangle := \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle). \tag{54}$$

We can use Lemma 9 to show the collapse of the three measures for this state.

Proposition 14: *The qutrit Strange state satisfies*

$$D_{\min,\mathbb{F}_{\rm stab}}(S) = D_{\max,\mathbb{F}_{\rm stab}}(S) = D_{s,\mathbb{F}_{\rm stab}}(S) = 1 \tag{55}$$

and

$$D_{\min,\mathbb{F}_{\rm stab}}(S^{\otimes n}) = D_{\max,\mathbb{F}_{\rm stab}}(S^{\otimes n}) \tag{56}$$

for every integer $n \geq 1$.

Our results complement the findings in Ref. [98], which has considered $D_{\min,\mathbb{F}'}$ and $D_{\max,\mathbb{F}'}$ with respect to a larger set of free operators (not necessarily normalized quantum states) based on the negativity of the discrete Wigner function and has found that $D_{\min,\mathbb{F}'}(S) = D_{\max,\mathbb{F}'}(S) = \log(5/3)$.

Another maximizer of the sum negativity is the Norrell state [76], defined as

$$|N\rangle := \frac{1}{\sqrt{6}}(-|0\rangle + 2|1\rangle - |2\rangle). \tag{57}$$

We can obtain a similar collapse for the measures but to a different value.

Proposition 15: *The qutrit Norrell state satisfies*

$$D_{\min,\mathbb{F}_{\text{stab}}}(N) = D_{\max,\mathbb{F}_{\text{stab}}}(N) = D_{s,\mathbb{F}_{\text{stab}}}(N) = \log\frac{3}{2}. \tag{58}$$

We note that $D_{\min,\mathbb{F}_{\text{stab}}}(N) = D_{\max,\mathbb{F}_{\text{stab}}}(N) = \log\frac{3}{2}$ has originally been reported in Ref. [98].

These results may make one wonder whether there is a general characterization for when the three measures take the same value. Although we still do not have a definitive answer for this question, we can make the following observation. For a given state ψ, Lemma 5 implies that if $D_{\min,\mathbb{F}_{\text{stab}}}(\psi) = D_{\max,\mathbb{F}_{\text{stab}}}(\psi)$, we must have $D_{\max,\mathbb{F}_{\text{stab}}}(\psi) < D_{s,\mathbb{F}_{\text{stab}}}(\psi)$ unless there exists a free state acting on $\operatorname{supp}[(\mathbb{1}-\psi)/d-1]$. This is equivalent to the condition that the *weight resource measure* [58,107], defined for the set $\mathbb{F}$ of free states as

$$W_{\mathbb{F}}(\rho) := \sup\left\{ w \;\middle|\; \rho = w\sigma + (1-w)\tau,\ \sigma \in \mathbb{F}\right\}, \tag{59}$$

satisfies

$$W_{\mathbb{F}_{\text{stab}}}\left(\frac{\mathbb{1}-\psi}{d-1}\right) > 0. \tag{60}$$

This condition can be explicitly verified to hold for the Strange state and Norrell state, for which we have seen that the three measures coincide. On the other hand, for the qutrit T state [97], defined as

$$|T\rangle := \frac{1}{\sqrt{3}}(e^{2\pi i/9}|0\rangle + |1\rangle + e^{-2\pi i/9}|2\rangle), \tag{61}$$

one can check that $W_{\mathbb{F}_{\text{stab}}}((\mathbb{1}-T)/2) = 0$. Combining it with Proposition 12, we recover the fact that

$$D_{\min,\mathbb{F}_{\text{stab}}}(T) = D_{\max,\mathbb{F}_{\text{stab}}}(T) < D_{s,\mathbb{F}_{\text{stab}}}(T). \tag{62}$$

Interestingly, numerical investigations suggest that, for most qutrit states, the weight measure for the complement is equal to zero, indicating that the nonzero gap between $D_{\max,\mathbb{F}_{\text{stab}}}$ and $D_{s,\mathbb{F}_{\text{stab}}}$ is a generic feature shared by the majority of qutrit states. We leave a thorough investigation on the relationship between the weight resource measure and the robustness measures for future work.

VII. CONCLUSIONS

We establish a quantitative relation between the one-shot distillable resource and the resource cost for general quantum resource theories, including both state-based resources as well as dynamical resources of quantum channels. We also show the corresponding bounds in the asymptotic regime and recover the familiar relation between the distillable resource and the resource cost in the form of strong converse bounds. We investigate the conditions that appear in the yield-cost relation and relate it to a class of free operations that have properties similar to those of twirling operations. We employ such operations to obtain analytical expressions for several smoothed resource measures for general resource theories of states and tighten the yield-cost relation. We then apply our operational technique to evaluate resource measures for several standard resource states in the resource theory of magic, recovering previous results with different techniques and presenting new evaluations of measures for some magic states.

Outstanding questions include the extension of the results that are only shown for state theories in this work to channel theories, such as the two-sided strong converse bound given in Eq. (25) and the relation to the twirlinglike operation. The difficulty in characterizing the manipulation of quantum channels, and in particular their asymptotic properties [60], makes such questions nontrivial to answer. Another interesting direction is to use the operational techniques introduced here to shed light on settings other than magic theory. On the other hand, it will also be beneficial to gain a deeper understanding of the structure of magic theory, for which additional operational insights might be helpful.

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APPENDIX A: PROOFS OF THEOREM 1, LEMMA 2, AND THEOREM 3

Let us first define the smoothed channel divergences [46,109]:

$$D_H^\epsilon(\mathcal{E}\|\mathcal{M}) := \sup_\psi D_H^\epsilon(\mathrm{id}\otimes\mathcal{E}(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi)),$$
$$D_{\max}^\epsilon(\mathcal{E}\|\mathcal{M}) := \inf_{F(\mathcal{E}',\mathcal{E})\geq 1-\epsilon} D_{\max}(\mathcal{E}'\|\mathcal{M}), \tag{A1}$$

which construct the following resource monotones [46,56,57]:

$$D_{H,\mathbb{O}}^\epsilon(\mathcal{E}) := \inf_{\mathcal{M}\in\mathbb{O}} D_H^\epsilon(\mathcal{E}\|\mathcal{M}),$$
$$D_{\max,\mathbb{O}}^\epsilon(\mathcal{E}) := \inf_{\mathcal{M}\in\mathbb{O}} D_{\max}^\epsilon(\mathcal{E}\|\mathcal{M}). \tag{A2}$$

We begin by showing some useful lemmas. The first two are channel extensions of the relations shown in Refs. [110, 111] between the hypothesis-testing relative entropy and the max-relative entropy of quantum states. They have an interpretation in the resource theory of asymmetric distinguishability as a one-shot yield-cost relation for bits of asymmetric distinguishability [55].

Lemma 16: *For all $\epsilon_1,\epsilon_2\geq 0$ with $\epsilon_1+\sqrt{\epsilon_2}<1$, two arbitrary channels $\mathcal{E},\mathcal{M}$ satisfy*

$$D_H^{\epsilon_1}(\mathcal{E}\|\mathcal{M}) \leq D_{\max}^{\epsilon_2}(\mathcal{E}\|\mathcal{M}) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}}. \tag{A3}$$

Proof. Reference [110] has shown that

$$D_H^{\epsilon_1}(\rho\|\sigma) \leq D_{\max}^{\epsilon_2}(\rho\|\sigma) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}} \tag{A4}$$

for all $\epsilon_1,\epsilon_2\geq 0$, $\epsilon_1+\sqrt{\epsilon_2}<1$ and all states ρ,σ. Then, we obtain

$$\begin{aligned}
D_H^{\epsilon_1}(\mathcal{E}\|\mathcal{M}) &= \sup_\psi D_H^{\epsilon_1}(\mathrm{id}\otimes\mathcal{E}(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi))\\
&\overset{(1)}{\leq} \sup_\psi D_{\max}^{\epsilon_2}(\mathrm{id}\otimes\mathcal{E}(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi)) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}}\\
&= \sup_\psi \inf_{F(\phi,\mathrm{id}\otimes\mathcal{E}(\psi))\geq 1-\epsilon_2} D_{\max}(\phi\|\mathrm{id}\otimes\mathcal{M}(\psi)) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}}\\
&\overset{(2)}{\leq} \sup_\psi \inf_{F(\mathcal{E}',\mathcal{E})\geq 1-\epsilon_2} D_{\max}(\mathrm{id}\otimes\mathcal{E}'(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi)) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}}\\
&\overset{(3)}{\leq} \inf_{F(\mathcal{E}',\mathcal{E})\geq 1-\epsilon_2} \sup_\psi D_{\max}(\mathrm{id}\otimes\mathcal{E}'(\psi)\|\mathrm{id}\otimes\mathcal{M}(\psi)) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}}\\
&= D_{\max}^{\epsilon_2}(\mathcal{E}\|\mathcal{M}) + \log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}},
\end{aligned} \tag{A5}$$

where we use (1) Eq. (A4), (2) the restriction of the optimization over ϕ to the form $\phi=\mathrm{id}\otimes\mathcal{E}'(\psi)$ with $F(\mathcal{E}',\mathcal{E})\geq 1-\epsilon_2$, which is justified by $F(\mathrm{id}\otimes\mathcal{E}'(\psi),\mathrm{id}\otimes\mathcal{E}(\psi))\geq F(\mathcal{E}',\mathcal{E})\geq 1-\epsilon_2$ for every ψ, and (3) the max-min inequality. ■

We note that alternative bounds with trace-distance and diamond-distance smoothing have been shown in Refs. [55,112].

We also have the following alternative bound.

Lemma 17: *For all $\epsilon_1,\epsilon_2\geq 0$ with $\epsilon_1+\epsilon_2<1$, two arbitrary channels $\mathcal{E},\mathcal{M}$ satisfy*

$$D_H^{\epsilon_1}(\mathcal{E}\|\mathcal{M}) \leq D_{\max}^{\epsilon_2}(\mathcal{E}\|\mathcal{M}) + \log\frac{1}{\left(\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1}\right)^2}. \tag{A6}$$

Proof. The proof of Theorem 4 of Ref. [111] establishes that

$$D_H^{\epsilon_1}(\rho\|\sigma) \leq D_{\max}^{\epsilon_2}(\rho\|\sigma) + \log\frac{1}{\left(\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1}\right)^2} \tag{A7}$$

for all $\epsilon_1,\epsilon_2\geq 0$, $\epsilon_1+\epsilon_2<1$ and all states ρ,σ. The statement for the channels follows by employing the same argument in the proof of Lemma 16. ■

We also recall monotonicity properties under one-shot channel transformations involving smooth measures.

Lemma 18 (Theorems 1 and 3 in Ref. [63])**:** *If there exists a free superchannel $\Theta\in\mathbb{S}_{\max}$ such that $F(\Theta(\mathcal{E}),\mathcal{N})\geq 1-\epsilon$, then for every resource* *monotone* $\mathfrak{R}_{\mathbb{O}}$*,*

$$\mathfrak{R}_{\mathbb{O}}(\mathcal{E}) \geq \mathfrak{R}^{\epsilon}_{\mathbb{O}}(\mathcal{N}), \tag{A8}$$

where

$$\mathfrak{R}^{\epsilon}_{\mathbb{O}}(\mathcal{E}) := \inf_{F(\mathcal{E}',\mathcal{E})\geq 1-\epsilon} \mathfrak{R}_{\mathbb{O}}(\mathcal{E}'). \tag{A9}$$

In particular,

$$D_{\max,\mathbb{O}}(\mathcal{E}) \geq D^{\epsilon}_{\max,\mathbb{O}}(\mathcal{N}). \tag{A10}$$

Now let $\mathcal{N}$ *be a channel such that* $\mathrm{id}\otimes\mathcal{N}(\psi)$ *is pure for every pure state* ψ*. If there exists a free superchannel* Θ *such that* $F(\Theta(\mathcal{E}),\mathcal{N}) \geq 1-\delta$*, then*

$$D^{\delta}_{H,\mathbb{O}}(\mathcal{E}) \geq D_{\min,\mathbb{O}}(\mathcal{N}). \tag{A11}$$

We are now in a position to prove Theorem 1 and Lemma 2.

Proof of Theorem 1 and Lemma 2. Theorem 1 is obtained from Lemma 2 by maximizing the left-hand side over $\mathcal{T}_1 \in \mathbb{T}$ and minimizing the right-hand side over $\mathcal{T}_2 \in \mathbb{T}$ for fixed errors ϵ_1, ϵ_2. Thus, it suffices to show Lemma 2. Let $\mathcal{T}_1$ be an arbitrary channel for which there exists $\Theta_1 \in \mathbb{S}$ such that $F(\Theta_1(\mathcal{E}),\mathcal{T}_1) \geq 1-\epsilon_1$ and let $\mathcal{T}_2$ be an arbitrary channel for which there exists $\Theta_2 \in \mathbb{S}$ such that $F(\Theta_2(\mathcal{T}_2),\mathcal{E}) \geq 1-\epsilon_2$. Then,

$$\begin{aligned} D_{\min,\mathbb{O}}(\mathcal{T}_1) &\leq D^{\epsilon_1}_{H,\mathbb{O}}(\mathcal{E}) \\ &\leq D^{\epsilon_2}_{\max,\mathbb{O}}(\mathcal{E}) + \log f(\epsilon_1,\epsilon_2) \\ &\leq D_{\max,\mathbb{O}}(\mathcal{T}_2) + \log f(\epsilon_1,\epsilon_2), \end{aligned} \tag{A12}$$

where in the first and the third inequalities we use Lemma 18 and in the second inequality we use Lemmas 16 and 17. ■

Theorem 3 can be shown similarly.

Proof of Theorem 3. Let us first observe that the regularized quantities $D^{\infty}_{\min,\mathbb{O}}$ and $D^{\infty}_{\max,\mathbb{O}}$ are both well defined. This follows since $D_{\max,\mathbb{O}}$ and $D_{\min,\mathbb{O}}$ are subadditive under assumption (ii)—as can be seen from their definitions using the additivity of $D_{\max}$ and $D_{\min}$ on tensor-product arguments—and hence the limit in the definition of the regularized quantities exists by Fekete's lemma. Now, by assumption (i), there exists a subsequence $\{n_k\}_k$ of indices such that $\lim_{k\to\infty} \epsilon_{n_k} + \delta_{n_k} < 1$. Then,

$$\begin{aligned} d\, D^{\infty}_{\min,\mathbb{O}}(\mathcal{T}) &\overset{(1)}{=} \lim_{k\to\infty} \frac{\lfloor dn_k\rfloor}{n_k}\frac{1}{\lfloor dn_k\rfloor} D_{\min,\mathbb{O}}(\mathcal{T}^{\otimes\lfloor dn_k\rfloor}) \\ &\overset{(2)}{\leq} \lim_{k\to\infty}\frac{1}{n_k} D^{\delta_{n_k}}_{H,\mathbb{O}}(\mathcal{E}^{\otimes n_k}) \\ &\overset{(3)}{\leq} \lim_{k\to\infty}\frac{1}{n_k}\left[D^{\epsilon_{n_k}}_{\max,\mathbb{O}}(\mathcal{E}^{\otimes n_k}) + \log f(\epsilon_{n_k},\delta_{n_k})\right] \\ &\overset{(4)}{\leq} \lim_{k\to\infty}\frac{1}{n_k}\left[D_{\max,\mathbb{O}}(\mathcal{T}^{\otimes\lceil cn_k\rceil}) + \log f(\epsilon_{n_k},\delta_{n_k})\right] \\ &\overset{(5)}{=} \lim_{k\to\infty}\frac{\lceil cn_k\rceil}{n_k}\frac{1}{\lceil cn_k\rceil} D_{\max,\mathbb{O}}(\mathcal{T}^{\otimes\lceil cn_k\rceil}) \\ &\overset{(6)}{=} c\, D^{\infty}_{\max,\mathbb{O}}(\mathcal{T}), \end{aligned} \tag{A13}$$

where we use: (1) the fact that $\lim_{k\to\infty}\lfloor dn_k\rfloor/n_k = d$ and that

$$\lim_{k\to\infty}\frac{1}{\lfloor dn_k\rfloor} D_{\min,\mathbb{O}}(\mathcal{T}^{\otimes\lfloor dn_k\rfloor}) = D^{\infty}_{\min,\mathbb{O}}(\mathcal{T}) \tag{A14}$$

by definition; (2) Lemma 18; (3) Lemmas 16 and 17 and that $\lim_{k\to\infty}\epsilon_{n_k}+\delta_{n_k} < 1$; (4) Lemma 18; (5) $\lim_{k\to\infty}(1/n_k) f(\epsilon_{n_k},\delta_{n_k}) = 0$; and (6) the same argument as in (1). ■

We remark that the subadditivity of $D_{\max,\mathbb{O}}$ under assumption (ii) implies that $D^{\infty}_{\max,\mathbb{O}}(\mathcal{T}) \leq D_{\max,\mathbb{O}}(\mathcal{T})$, also giving the general bound

$$d\, D^{\infty}_{\min,\mathbb{O}}(\mathcal{T}) \leq c\, D_{\max,\mathbb{O}}(\mathcal{T}). \tag{A15}$$

APPENDIX B: APPLICABILITY TO SPECIFIC SETTINGS

We briefly review several examples of physical settings, as well as a reference channel $\mathcal{T}$, that satisfy the conditions of Theorem 1. The examples discussed here are either state-preparation channels with one-dimensional input or unitaries with input and output of the same dimension, ensuring that $\mathrm{id}\otimes\mathcal{T}(\psi)$ is pure for every pure state ψ. They further satisfy $D_{\min,\mathbb{O}}(\mathcal{T}) = D_{\max,\mathbb{O}}(\mathcal{T})$ and thus serve as appropriate references for distillation and dilution protocols that meet the conditions in Theorem 1.

Let us first consider state theories, where $\mathbb{O}$ is a set of channels preparing free states (denoted by $\mathbb{F}$ in Sec. II). The theory of bipartite entanglement [79], in which separable states serve as free states, takes a maximally entangled state $|\Phi^+_m\rangle = \sum_{i=1}^m m^{-1/2}|ii\rangle$ as a reference state and obeys $D_{\min,\mathbb{O}}(\Phi^+_m) = D_{\max,\mathbb{O}}(\Phi^+_m) = \log m$. In the case of the theory of coherence [25], where the free states are the diagonal states with respect to a given preferred basis, a maximally coherent state, i.e., a uniform superposition of the basis states $|\phi^+_m\rangle = \sum_{i=1}^m m^{-1/2}|i\rangle$, meets the condition $D_{\min,\mathbb{O}}(\phi^+_m) = D_{\max,\mathbb{O}}(\phi^+_m) = \log m$. The theory of thermal nonequilibrium [3] is defined by fixed temperature and the Hamiltonian. The "work bit" represented by the eigenstate of the Hamiltonian with the highest energy is considered as a standard unit resource and it also satisfies the conditions of Theorem 1. In fact, the above observation can be generalized to an arbitrary state theory; every theory with an arbitrary convex set $\mathbb{F}$ of free states is equipped with a *golden state* $\Phi_{\rm gold}$ [12] such that $D_{\min,\mathbb{F}}(\Phi_{\rm gold}) =$

$D_{\max,\mathbb{F}}(\Phi_{\text{gold}})$ [41]. See also Sec. VI in the main text for further discussion on the theory of magic states.

Many important dynamical resource theories are relevant to the setting of quantum communication. A central purpose of quantum communication is to transmit a quantum state from one party to another. In such a scenario, the identity channel connecting the two parties is considered as the most useful channel. It is then natural to take the m-dimensional identity channel id_m as a reference in distillation and dilution protocols and, indeed, the identity channel satisfies the conditions in Theorem 1 under several coding schemes that are specified by different choices of $\mathbb{O}$ [58,63]. For instance, in the theory of no-signaling–assisted communication, we obtain $D_{\min,\mathbb{O}}(\text{id}_m) = D_{\max,\mathbb{O}}(\text{id}_m) = 2\log m$, while in quantum communication assisted by LOCC, separable, or positive partial transpose codes we obtain $D_{\min,\mathbb{O}}(\text{id}_m) = D_{\max,\mathbb{O}}(\text{id}_m) = \log m$. A related setting is a framework that quantifies how much quantum memory a given channel can preserve [47]. This theory takes the set of entanglement-breaking channels as free channels and the identity channel again serves as an appropriate reference channel [53]. In relation to Sec. VI, dynamical resource theories of magic [54,113] admit several unitary gates as reference channels satisfying the conditions in Theorem 1 [58].

APPENDIX C: ALTERNATIVE ASYMPTOTIC BOUNDS

Besides Theorem 3, we can also show alternative asymptotic bounds, which are tighter but have additional conditions on the achievable errors. Another advantage of our alternative bounds is that they do not require the reference channel $\mathcal{T}$ to be pure, i.e., $\text{id}\otimes\mathcal{T}(\psi)$ does not need to be pure for every pure state ψ.

Let us begin by presenting some useful lemmas. We first introduce the smoothed hypothesis-testing relative entropy measure as

$$D_{H,\mathbb{O}}^{\epsilon,\delta}(\mathcal{E}) := \inf_{F(\mathcal{E}',\mathcal{E})\geq 1-\delta} D_{H,\mathbb{O}}^{\epsilon}(\mathcal{E}'). \tag{C1}$$

Then, we can relate this smoothed measure and the standard hypothesis-testing measure as follows.

Lemma 19: *For an arbitrary channel $\mathcal{E}$ and all $\epsilon,\delta\geq 0$ satisfying $\epsilon+\sqrt{\delta}\leq 1$,*

$$D_{H,\mathbb{O}}^{\epsilon}(\mathcal{E}) \leq D_{H,\mathbb{O}}^{\epsilon+\sqrt{\delta},\delta}(\mathcal{E}). \tag{C2}$$

Proof. The hypothesis-testing measure can be written explicitly by

$$D_{H,\mathbb{O}}^{\epsilon}(\mathcal{E}) = -\log \max_{\mathcal{M}\in\mathbb{O}} \min_{\psi} \min_{\substack{0\leq P\leq \mathbb{1}\\ \text{Tr}[P\,\text{id}\otimes\mathcal{E}(\psi)]\geq 1-\epsilon}} \text{Tr}[P\,\text{id}\otimes\mathcal{M}(\psi)], \tag{C3}$$

where we define $\log 0 := -\infty$ here and throughout the paper.

Note that for an arbitrary channel $\mathcal{E}'$ satisfying $F(\mathcal{E}',\mathcal{E})\geq 1-\delta$, an arbitrary positive semidefinite operator P satisfying $0\leq P\leq\mathbb{1}$, and an arbitrary state ψ,

$$\begin{aligned}&\left|\text{Tr}[P\,\text{id}\otimes\mathcal{E}(\psi)] - \text{Tr}[P\,\text{id}\otimes\mathcal{E}'(\psi)]\right|\\ &\quad\leq \frac{1}{2}\|\text{id}\otimes\mathcal{E}(\psi) - \text{id}\otimes\mathcal{E}'(\psi)\|_1\\ &\quad\leq \sqrt{1-F(\mathcal{E}',\mathcal{E})},\end{aligned} \tag{C4}$$

where the first inequality is because of the optimization form of the trace distance $\frac{1}{2}\|\rho-\sigma\|_1 = \max_{0\leq P\leq\mathbb{1}} \text{Tr}[P(\rho-\sigma)]$ satisfied for an arbitrary two states ρ and σ [114] and the second inequality is because

$$\begin{aligned}&\frac{1}{2}\|\text{id}\otimes\mathcal{E}(\psi) - \text{id}\otimes\mathcal{E}'(\psi)\|_1\\ &\quad\leq \sqrt{1-F(\text{id}\otimes\mathcal{E}(\psi),\text{id}\otimes\mathcal{E}'(\psi))}\\ &\quad\leq \sqrt{1-F(\mathcal{E},\mathcal{E}')},\end{aligned} \tag{C5}$$

where we use the relation between trace distance and fidelity [115] in the first inequality and the definition of channel fidelity (10) in the second inequality. This implies that as long as $F(\mathcal{E}',\mathcal{E})\geq 1-\delta$, every P with $0\leq P\leq \mathbb{1}$ and state ψ satisfying $\text{Tr}[P\ \text{id}\otimes\mathcal{E}(\psi)]\geq 1-\epsilon$ also satisfy $\text{Tr}[P\,\text{id}\otimes\mathcal{E}'(\psi)]\geq 1-\epsilon-\sqrt{\delta}$. This gives

$$\begin{aligned}&\max_{\mathcal{M}\in\mathbb{O}}\min_{\psi} \min_{\substack{0\leq P\leq\mathbb{1}\\ \text{Tr}[P\ \text{id}\otimes\mathcal{E}(\psi)]\geq 1-\epsilon}} \text{Tr}[P\,\text{id}\otimes\mathcal{M}(\psi)]\\ &\geq \max_{\mathcal{M}\in\mathbb{O}}\min_{\psi}\max_{F(\mathcal{E}',\mathcal{E})\geq 1-\delta}\\ &\qquad \min_{\substack{0\leq P\leq\mathbb{1}\\ \text{Tr}[P\ \text{id}\otimes\mathcal{E}'(\psi)]\geq 1-\epsilon-\sqrt{\delta}}} \text{Tr}[P\,\text{id}\otimes\mathcal{M}(\psi)]\\ &\geq \max_{F(\mathcal{E}',\mathcal{E})\geq 1-\delta}\max_{\mathcal{M}\in\mathbb{O}}\min_{\psi}\\ &\qquad \min_{\substack{0\leq P\leq\mathbb{1}\\ \text{Tr}[P\,\text{id}\otimes\mathcal{E}'(\psi)]\geq 1-\epsilon-\sqrt{\delta}}} \text{Tr}[P\,\text{id}\otimes\mathcal{M}(\psi)],\end{aligned} \tag{C6}$$

where in the last line, we use the max-min inequality. Taking $-\log$ on both sides concludes the proof. ■

The following result, which is a variant of Theorem 1 in Ref. [63], is also useful.

Lemma 20: *If there exists a free superchannel* $\Theta \in \mathbb{S}_{\max}$ *such that* $F(\Theta(\mathcal{E}),\mathcal{N}) \geq 1-\epsilon$*, then for an arbitrary resource monotone* $\mathfrak{R}_{\mathbb{O}}$*,*

$$\mathfrak{R}^{\delta}_{\mathbb{O}}(\mathcal{E}) \geq \mathfrak{R}^{(\sqrt{\delta}+\sqrt{\epsilon})^2}_{\mathbb{O}}(\mathcal{N}) \tag{C7}$$

for every $0 \leq \delta \leq 1$*, where*

$$\mathfrak{R}^{\delta}_{\mathbb{O}}(\mathcal{E}) := \inf_{F(\mathcal{E}',\mathcal{E})\geq 1-\delta} \mathfrak{R}_{\mathbb{O}}(\mathcal{E}'). \tag{C8}$$

Proof. Let $P(\mathcal{E},\mathcal{N}) := \sqrt{1-F(\mathcal{E},\mathcal{N})}$ be the sine distance of quantum channels (also known as the purified distance), which satisfies the triangle inequality

$$P(\mathcal{E},\mathcal{T}) \leq P(\mathcal{E},\mathcal{N}) + P(\mathcal{N},\mathcal{T}) \tag{C9}$$

for all channels $\mathcal{E},\mathcal{N},\mathcal{T}$ [88]. Noting that $\mathfrak{R}^{\delta}_{\mathbb{O}}$ is a resource monotone for all δ, we can use Lemma 18 to obtain

$$\begin{aligned}\mathfrak{R}^{\delta}_{\mathbb{O}}(\mathcal{E}) &\geq \inf_{P(\mathcal{N}',\mathcal{N})\leq\sqrt{\epsilon}} \mathfrak{R}^{\delta}_{\mathbb{O}}(\mathcal{N}') \\ &= \inf_{P(\mathcal{N}',\mathcal{N})\leq\sqrt{\epsilon}} \inf_{P(\mathcal{N}'',\mathcal{N}')\leq\sqrt{\delta}} \mathfrak{R}_{\mathbb{O}}(\mathcal{N}''). \end{aligned}\tag{C10}$$

Using the triangle inequality of the purified distance, we obtain $P(\mathcal{N}'',\mathcal{N}) \leq \sqrt{\epsilon}+\sqrt{\delta}$ for all channels $\mathcal{N}$, $\mathcal{N}'$, $\mathcal{N}''$ that satisfy $P(\mathcal{N}',\mathcal{N}) \leq \sqrt{\epsilon}$ and $P(\mathcal{N}'',\mathcal{N}') \leq \sqrt{\delta}$. Noting also that $P(\mathcal{N}'',\mathcal{N}) \leq \sqrt{\epsilon}+\sqrt{\delta}$ is equivalent to $F(\mathcal{N}'',\mathcal{N}) \geq 1-(\sqrt{\epsilon}+\sqrt{\delta})^2$, we obtain

$$\begin{aligned}\mathfrak{R}^{\delta}_{\mathbb{O}}(\mathcal{E}) &\geq \inf_{F(\mathcal{N}'',\mathcal{N})\geq 1-(\sqrt{\epsilon}+\sqrt{\delta})^2} \mathfrak{R}_{\mathbb{O}}(\mathcal{N}'') \\ &= \mathfrak{R}^{(\sqrt{\delta}+\sqrt{\epsilon})^2}_{\mathbb{O}}(\mathcal{N}). \end{aligned}\tag{C11}$$

■

Remark: We can improve the triangle inequality given in Eq. (C9). Define the following "best-case" fidelity:

$$\begin{aligned}F_{\max}(\mathcal{E},\mathcal{N}) &:= \sup_{\rho_{RA}} F(\mathrm{id}_R\otimes\mathcal{E}(\rho_{RA}), \mathrm{id}_R\otimes\mathcal{N}(\rho_{RA})) \\ &= \max_{\rho} F(\mathcal{E}(\rho),\mathcal{N}(\rho)), \end{aligned}\tag{C12}$$

where the supremum in the first line is taken over every possible ancillary system R and the second equality is due to the data-processing inequality, ensuring that tracing out the ancillary system does not decrease the fidelity. The last form in particular allows for an efficient computation via the following semidefinite program, which follows from Eq. (C12) and the known semidefinite program for fidelity [116]:

$$\begin{aligned}\sqrt{F_{\max}(\mathcal{E},\mathcal{N})} &= \max_{\substack{\rho\geq 0,\\ X\in \mathrm{L}(\mathcal{H})}} \left\{ \mathrm{Re}[\mathrm{Tr}[X]] \,\middle|\, \begin{bmatrix} \mathcal{E}(\rho) & X^\dagger \\ X & \mathcal{N}(\rho)\end{bmatrix} \geq 0, \right. \\ &\qquad \left. \mathrm{Tr}[\rho]=1 \right\}, \end{aligned}\tag{C13}$$

where $\mathrm{L}(\mathcal{H})$ denotes the set of linear operators acting on the output Hilbert space $\mathcal{H}$ for the channels $\mathcal{E}$ and $\mathcal{N}$. Then, we have the following inequality.

Lemma 21: *Every set of channels* $\mathcal{E}$*,* $\mathcal{N}$*, and* $\mathcal{T}$ *with* $P(\mathcal{E},\mathcal{N})^2 + P(\mathcal{N},\mathcal{T})^2 \leq 1$ *satisfies*

$$\begin{aligned}P(\mathcal{E},\mathcal{T}) &\leq P(\mathcal{E},\mathcal{N})\sqrt{F_{\max}(\mathcal{N},\mathcal{T})} \\ &\quad + P(\mathcal{N},\mathcal{T})\sqrt{F_{\max}(\mathcal{E},\mathcal{N})}. \end{aligned}\tag{C14}$$

Proof. Recall that the purified distance $P(\rho,\sigma) := \sqrt{1-F(\rho,\sigma)}$ defined for an arbitrary two states ρ and σ satisfies the following triangle inequality [108]:

$$P(\rho,\tau) \leq P(\rho,\sigma)\sqrt{F(\sigma,\tau)} + P(\sigma,\tau)\sqrt{F(\rho,\sigma)} \tag{C15}$$

for every set of three states ρ,σ,τ such that $P(\rho,\sigma)^2 + P(\sigma,\tau)^2 \leq 1$. We can employ this to obtain

$$\begin{aligned}P(\mathcal{E},\mathcal{T}) &= \max_{\psi} P(\mathrm{id}\otimes\mathcal{E}(\psi), \mathrm{id}\otimes\mathcal{T}(\psi)) \\ &\leq \max_{\psi} P(\mathrm{id}\otimes\mathcal{E}(\psi), \mathrm{id}\otimes\mathcal{N}(\psi))\sqrt{F_{\max}(\mathcal{N},\mathcal{T})} \\ &\quad + \max_{\psi} P(\mathrm{id}\otimes\mathcal{N}(\psi), \mathrm{id}\otimes\mathcal{T}(\psi))\sqrt{F_{\max}(\mathcal{E},\mathcal{N})} \\ &= P(\mathcal{E},\mathcal{N})\sqrt{F_{\max}(\mathcal{N},\mathcal{T})} \\ &\quad + P(\mathcal{N},\mathcal{T})\sqrt{F_{\max}(\mathcal{E},\mathcal{N})}. \end{aligned}\tag{C16}$$

■

Equation (C14) is tighter than (C9) in general because $F_{\max}(\mathcal{E},\mathcal{N}) \leq 1$ for every $\mathcal{E}$ and $\mathcal{N}$. Although (C9) and Lemma 20 suffice to establish the forthcoming results, Lemma 21 may find use in other different settings.

We are ready to show the first alternative asymptotic bound, which is tighter than Theorem 3 and does not assume anything on $\mathcal{T}$, while having less flexibility in the achievable errors.

Proposition 22: *Let* $\mathcal{E}$ *be an arbitrary input channel and let* $\mathcal{T}$ *be some target reference channel. Let* d *be any rate* *of distillation such that there exists a sequence* $\{\Theta_n\}_n$ *of free superchannels with*

$$1 - F\left(\Theta_n(\mathcal{E}^{\otimes n}), \mathcal{T}^{\otimes \lfloor dn \rfloor}\right) =: \delta_n. \tag{C17}$$

Also, let c be any rate of dilution such that there exists a sequence $\{\Theta_n\}_n$ *of free superchannels with*

$$1 - F\left(\Theta_n(\mathcal{T}^{\otimes \lceil cn \rceil}), \mathcal{E}^{\otimes n}\right) =: \epsilon_n. \tag{C18}$$

Suppose that

$$\lim_{n\to\infty} \delta_n = 0, \quad \liminf_{n\to\infty} \epsilon_n < 1. \tag{C19}$$

Then, the following inequality holds:

$$d \cdot \tilde{D}^{\infty}_{H,\mathbb{O}}(\mathcal{T}) \le c \cdot \tilde{D}^{\infty}_{\max,\mathbb{O}}(\mathcal{T}), \tag{C20}$$

where

$$\begin{aligned} \tilde{D}^{\infty}_{H,\mathbb{O}}(\mathcal{T}) &:= \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} D^{\epsilon}_{H,\mathbb{O}}(\mathcal{T}^{\otimes n}), \\ \tilde{D}^{\infty}_{\max,\mathbb{O}}(\mathcal{E}) &:= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} D^{\epsilon}_{\max,\mathbb{O}}(\mathcal{E}^{\otimes n}). \end{aligned} \tag{C21}$$

Proof. Let $\{n_k\}_k$ be a subsequence of indices such that $\epsilon_\infty := \lim_{k\to\infty} \epsilon_{n_k} < 1$. Then, for every ξ with $0 < \xi < 1 - \epsilon_\infty$, we obtain

$$\begin{aligned} c\tilde{D}^{\infty}_{\max,\mathbb{O}}(\mathcal{T}) &\ge \lim_{\eta\to 0} \liminf_{k\to\infty} \frac{\lceil cn_k \rceil}{n_k} \frac{1}{\lceil cn_k \rceil} D^{\eta}_{\max,\mathbb{O}}(\mathcal{T}^{\otimes \lceil cn_k \rceil}) \\ &\ge \lim_{\eta\to 0} \liminf_{k\to\infty} \frac{1}{n_k} D^{(\sqrt{\eta}+\sqrt{\epsilon_{n_k}})^2}_{\max,\mathbb{O}}(\mathcal{E}^{\otimes n_k}) \\ &\ge \lim_{\eta\to 0} \liminf_{k\to\infty} \frac{1}{n_k} \left\{ D^{\xi}_{H,\mathbb{O}}(\mathcal{E}^{\otimes n_k}) - \log f\left(\xi, (\sqrt{\eta} + \sqrt{\epsilon_{n_k}})^2\right) \right\} \\ &\ge \lim_{\eta\to 0} \liminf_{k\to\infty} \frac{1}{n_k} \left\{ D^{\xi,\delta_{n_k}}_{H,\mathbb{O}}(\mathcal{T}^{\otimes \lfloor dn_k \rfloor}) - \log f\left(\xi, (\sqrt{\eta} + \sqrt{\epsilon_{n_k}})^2\right) \right\} \\ &\ge \lim_{\eta\to 0} \liminf_{k\to\infty} \frac{1}{n_k} \left\{ D^{\xi-\sqrt{\delta_{n_k}}}_{H,\mathbb{O}}(\mathcal{T}^{\otimes \lfloor dn_k \rfloor}) - \log f\left(\xi, (\sqrt{\eta} + \sqrt{\epsilon_{n_k}})^2\right) \right\}. \end{aligned} \tag{C22}$$

The second inequality follows from Lemma 20, the third inequality from Lemmas 16 and 17 and the fact that $\xi + (\sqrt{\eta} + \sqrt{\epsilon_{n_k}})^2 < 1$ holds for η sufficiently close to 0 and for sufficiently large k, the fourth inequality from Lemma 18, noting that $D^{\xi}_{H,\mathbb{O}}$ is a resource monotone for a fixed ξ, and the fifth inequality from Lemma 19 together with that $\xi - \sqrt{\delta_{n_k}} \in [0,1]$ for sufficiently large k.

The second term of the last line, $\log f\left(\xi, (\sqrt{\eta} + \sqrt{\epsilon_{n_k}})^2\right)/n_k$, vanishes at the limit of $k \to \infty$, which also removes the η dependence. Also, for every $\delta' > 0$, we have $\sqrt{\delta_k} < \delta'$ for sufficiently large k. Noting that $D^{\epsilon}_{H,\mathbb{O}}$ is nondecreasing with respect to ϵ, we can bound the last line as

$$\begin{aligned} &\ge \lim_{\delta'\to 0} \liminf_{k\to\infty} \frac{1}{n_k} D^{\xi-\delta'}_{H,\mathbb{O}}(\mathcal{T}^{\otimes \lfloor dn_k \rfloor}) \\ &= \lim_{\delta'\to 0} \liminf_{k\to\infty} \frac{\lfloor dn_k \rfloor}{n_k} \frac{1}{\lfloor dn_k \rfloor} D^{\xi-\delta'}_{H,\mathbb{O}}(\mathcal{T}^{\otimes \lfloor dn_k \rfloor}). \end{aligned} \tag{C23}$$

Since this holds for every ξ with $0 < \xi < 1 - \epsilon_\infty$, we can further take $\lim_{\xi\to 0}$ and use

$$\lim_{\xi\to 0} \lim_{\delta'\to 0} \liminf_{k\to\infty} \frac{1}{\lfloor dn_k \rfloor} D^{\xi-\delta'}_{H,\mathbb{O}}(\mathcal{T}^{\otimes \lfloor dn_k \rfloor}) \ge \tilde{D}^{\infty}_{H,\mathbb{O}}(\mathcal{T}) \tag{C24}$$

to obtain

$$c\tilde{D}^{\infty}_{\max,\mathbb{O}}(\mathcal{T}) \ge d\tilde{D}^{\infty}_{H,\mathbb{O}}(\mathcal{T}). \tag{C25}$$

■

In particular, this gives a strong converse inequality

$$\tilde{c}^{\infty}_{\mathbb{S}}(\mathcal{E},\mathcal{T})\tilde{D}^{\infty}_{\max,\mathbb{O}}(\mathcal{T}) \ge d^{\infty}_{\mathbb{S}}(\mathcal{E},\mathcal{T})\tilde{D}^{\infty}_{H,\mathbb{O}}(\mathcal{T}). \tag{C26}$$

In the case of state transformations with mild assumptions (cf. Appendix D), the asymptotic equipartition property $\tilde{D}^{\infty}_{\max,\mathbb{F}}(\Phi) = \tilde{D}^{\infty}_{H,\mathbb{F}}(\Phi)$ holds [90] and consequently we obtain

$$d^{\infty}_{\mathbb{O}}(\mathcal{E},\Phi) \le \tilde{c}^{\infty}_{\mathbb{O}}(\mathcal{E},\Phi). \tag{C27}$$

In addition, if Φ satisfies $D_{\min,\mathbb{F}}(\Phi^{\otimes m}) = D_{s,\mathbb{F}}(\Phi^{\otimes m}) = mD_{\min,\mathbb{F}}(\Phi)\ \forall m$, this leads to the double-sided strong converse inequality

$$\tilde{d}_{\mathbb{O}}^{\infty}(\rho,\Phi) \le \tilde{c}_{\mathbb{O}}^{\infty}(\rho,\Phi), \tag{C28}$$

as we discuss in Appendix D.

Finally, if we further impose a stronger condition to the achievable errors, we can obtain an even tighter bound.

Proposition 23: *Let us consider the setting in which we instead have the condition* $\lim_{k\to\infty}\delta_{n_k} = \lim_{k\to\infty}\epsilon_{n_k} = 0$ *for some subsequence* $\{n_k\}_k$*. For every resource monotone* $\mathfrak{R}_{\mathbb{O}}$*, define*

$$\begin{aligned}\overline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{E}) &:= \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\mathfrak{R}_{\mathbb{O}}^{\epsilon}(\mathcal{E}^{\otimes n}),\\ \underline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{E}) &:= \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\mathfrak{R}_{\mathbb{O}}^{\epsilon}(\mathcal{E}^{\otimes n}).\end{aligned} \tag{C29}$$

Then,

$$d\cdot\underline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{T}) \le c\cdot\overline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{T}). \tag{C30}$$

Proof.

$$\begin{aligned}c\overline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{T}) &\ge \lim_{\eta\to 0}\liminf_{k\to\infty}\frac{\lceil cn_k\rceil}{n_k}\frac{1}{\lceil cn_k\rceil}\mathfrak{R}_{\mathbb{O}}^{\eta}(\mathcal{T}^{\otimes\lceil cn_k\rceil})\\ &\ge \lim_{\eta\to 0}\liminf_{k\to\infty}\frac{1}{n_k}\mathfrak{R}_{\mathbb{O}}^{(\sqrt{\eta}+\sqrt{\epsilon_{n_k}})^2}(\mathcal{E}^{\otimes n_k})\\ &\ge \lim_{\eta\to 0}\liminf_{k\to\infty}\frac{1}{n_k}\mathfrak{R}_{\mathbb{O}}^{(\sqrt{\eta}+\sqrt{\epsilon_{n_k}}+\sqrt{\delta_{n_k}})^2}(\mathcal{T}^{\otimes\lfloor dn_k\rfloor}),\end{aligned} \tag{C31}$$

where the second and third lines follow from Lemma 20.

Since $\lim_{k\to\infty}\epsilon_{n_k} = \lim_{k\to\infty}\delta_{n_k} = 0$, for arbitrary constant $\xi > 0$, it is ensured that $\epsilon_{n_k} < \xi$ and $\delta_{n_k} < \xi$ for sufficiently large k. Since $\mathfrak{R}_{\mathbb{O}}^{\epsilon}$ is nonincreasing with respect to ϵ, we can bound the last expression as

$$\ge \lim_{\eta\to 0}\liminf_{k\to\infty}\frac{1}{n_k}\mathfrak{R}_{\mathbb{O}}^{(\sqrt{\eta}+2\sqrt{\xi})^2}(\mathcal{T}^{\otimes\lfloor dn_k\rfloor}). \tag{C32}$$

Since this holds for every $\xi > 0$, we can bound the expression in Eq. (C31) by taking the limit $\xi\to 0$ as

$$\begin{aligned}&\lim_{\eta\to 0}\lim_{\xi\to 0}\liminf_{k\to\infty}\frac{1}{n_k}\mathfrak{R}_{\mathbb{O}}^{(\sqrt{\eta}+\sqrt{\epsilon_{n_k}}+\sqrt{\delta_{n_k}})^2}(\mathcal{T}^{\otimes\lfloor dn_k\rfloor})\\ &\ge \lim_{\eta\to 0}\lim_{\xi\to 0}\liminf_{n\to\infty}\frac{1}{n}\mathfrak{R}_{\mathbb{O}}^{(\sqrt{\eta}+2\sqrt{\xi})^2}(\mathcal{T}^{\otimes\lfloor dn\rfloor})\\ &= \lim_{\eta\to 0}\lim_{\xi\to 0}\liminf_{n\to\infty}\frac{\lfloor dn\rfloor}{n}\frac{1}{\lfloor dn\rfloor}\mathfrak{R}_{\mathbb{O}}^{(\sqrt{\eta}+2\sqrt{\xi})^2}(\mathcal{T}^{\otimes\lfloor dn\rfloor})\\ &\ge d\underline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{T}),\end{aligned} \tag{C33}$$

resulting in the desired inequality in Eq. (C30). ■

This extends and complements similar relations known for state transformations in settings such as entanglement [14,18] and a general class of resources [16,19]. In particular, this reduces to the intuitive bound $c \ge d$ when $\overline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{T}) = \underline{\mathfrak{R}}_{\mathbb{O}}^{\infty}(\mathcal{T}) > 0$. Note, however, that the regularized resource measure may take 0 for all channels in some cases [117].

APPENDIX D: STRONG CONVERSE PROPERTY OF DISTILLABLE RESOURCE

Reference [90] has discussed a generalization of quantum Stein's lemma for resource theories satisfying mild assumptions. Using this, Ref. [15] has characterized the asymptotic distillable entanglement under the set of nonentangling operations with the regularized relative entropy of entanglement. In fact, their argument shows more than that—the regularized relative entropy of entanglement also serves as a strong converse distillation rate.

Here, we extend this strong converse property to general resource theories by combining the results in Refs. [15] and [41] in the case when the generalized quantum Stein's lemma holds. This can then turn the one-sided strong converse inequality in Corollary 4 to a double-sided inequality, making both quantities—yield and cost—strong converse rates for each other.

Before stating the result, we recall the following characterization of the fidelity of distillation.

Lemma 24 ([41])**:** *For an arbitrary convex and closed set* $\mathbb{F}$*, if* $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi) = r$*, then*

$$\sup_{\mathcal{E}\in\mathbb{O}} F(\mathcal{E}(\rho),\Phi) = G_{\mathbb{F}}(\rho;2^r), \tag{D1}$$

where

$$G_{\mathbb{F}}(\rho;K) := \sup\left\{\mathrm{Tr}[W\rho]\;\middle|\;0\le W\le\mathbb{1},\ \mathrm{Tr}[W\sigma]\le\frac{1}{K}\ \forall\sigma\in\mathbb{F}\right\}. \tag{D2}$$

Then, we obtain the following relations.

Proposition 25: *Let* $\mathbb{F}$ *be a set of free states that:*

(1) is convex and closed
(2) contains a full-rank state
(3) is closed under partial trace and composition of free states

Also, let Φ *be a state that satisfies* $D_{\min,\mathbb{F}}(\Phi^{\otimes m}) = D_{s,\mathbb{F}}(\Phi^{\otimes m}) = mD_{\min,\mathbb{F}}(\Phi)\ \forall m$*. Then,*

$$\tilde{d}_{\mathbb{O}_{\max}}^{\infty}(\rho,\Phi) = d_{\mathbb{O}_{\max}}^{\infty}(\rho,\Phi). \tag{D3}$$

In particular, for every $\mathbb{O} \subseteq \mathbb{O}_{\max}$, we have that

$$\tilde{d}^{\infty}_{\mathbb{O}}(\rho, \Phi) \le \tilde{c}^{\infty}_{\mathbb{O}}(\rho, \Phi). \tag{D4}$$

Proof. Let $F_{\mathbb{O}}(\rho \to \phi)$ be the fidelity of distillation from ρ to ϕ under free operations $\mathbb{O}$, defined as

$$F_{\mathbb{O}}(\rho \to \phi) := \sup_{\mathcal{E}\in\mathbb{O}} F(\mathcal{E}(\rho), \phi). \tag{D5}$$

Lemma 24 and the assumed additivity of $D_{\min,\mathbb{F}}$ give

$$F_{\mathbb{O}_{\max}}(\rho^{\otimes n} \to \Phi^{\otimes ny}) = G_{\mathbb{F}}(\rho^{\otimes n}; 2^{rny}), \tag{D6}$$

where $r = D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi)$.

Note that $G_{\mathbb{F}}(\rho; K)$ is a convex-optimization program and one can obtain its dual program by following standard techniques in convex-optimization theory [118] (cf. [90]). For operators $W \ge 0$, $Y \ge 0$, $Z \in \operatorname{cone}(\mathbb{F})$, consider the Lagrangian

$$\begin{aligned}\mathcal{L}(\rho, W; Y, Z) &:= \operatorname{Tr}[W\rho] + \operatorname{Tr}[(\mathbb{1} - W)Y] + \operatorname{Tr}[(\mathbb{1} - KW)Z] \\ &= \operatorname{Tr}[Y] + \operatorname{Tr}[Z] + \operatorname{Tr}[W(\rho - Y - KZ)]. \end{aligned}\tag{D7}$$

This form leads to a dual program:

$$\begin{aligned}&\inf\left\{ \operatorname{Tr}[Y] + \operatorname{Tr}[Z] \;\middle|\; Y \ge 0, Y \ge \rho - KZ, Z \in \operatorname{cone}(\mathbb{F})\right\} \\ &\quad = \inf\left\{ \operatorname{Tr}[\rho - KZ]_+ + \operatorname{Tr}[Z] \;\middle|\; Z \in \operatorname{cone}(\mathbb{F})\right\}, \end{aligned}\tag{D8}$$

where we define $\operatorname{Tr}[A]_+$ to be the trace over the positive part of the operator A. Since Slater's condition [118, Sec. 5.2.3] is satisfied, which can be confirmed by taking $W = \mathbb{1}/(K+1)$ in Eq. (D2), we obtain

$$G_{\mathbb{F}}(\rho; K) = \inf_{\tilde{\sigma}\in\operatorname{cone}(\mathbb{F})} \left\{ \operatorname{Tr}[\rho - \tilde{\sigma}]_+ + \frac{1}{K}\operatorname{Tr}[\tilde{\sigma}] \right\}. \tag{D9}$$

Taking $\tilde{\sigma} = 2^{rnb}\sigma$ for some $b \in \mathbb{R}$ and $\sigma \in \mathbb{F}$ allows us to write

$$G_{\mathbb{F}}(\rho^{\otimes n}; 2^{rny}) = \inf_{\sigma\in\mathbb{F}, b\in\mathbb{R}} \left\{ \operatorname{Tr}[\rho - 2^{rnb}\sigma]_+ + 2^{-r(y-b)n} \right\} \tag{D10}$$

for all n and y. Let $R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)$ be the regularized relative entropy resource measure [15,16], defined as

$$R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} \inf_{\sigma\in\mathbb{F}} D(\rho^{\otimes n}\|\sigma), \tag{D11}$$

where $D(\rho\|\sigma)$ is the relative entropy defined for an arbitrary two states ρ and σ taking $\operatorname{Tr}[\rho\log\rho] - \operatorname{Tr}[\rho\log\sigma]$ if $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$ and $+\infty$ otherwise. Let us take $y = (1/r)R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho) + \epsilon$ for some $\epsilon > 0$. Then, for each n, we can take $b = (1/r)R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho) + (\epsilon/2)$ to obtain

$$\begin{aligned}&G_{\mathbb{F}}(\rho^{\otimes n}; 2^{rn(R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)+\epsilon)}) \\ &\quad \le \inf_{\sigma\in\mathbb{F}} \left\{ \operatorname{Tr}[\rho^{\otimes n} - 2^{rn[(1/r)R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)+(\epsilon/2)]}\sigma]_+ \right\} + 2^{-(rn\epsilon/2)} \\ &\quad = \inf_{\sigma\in\mathbb{F}} \left\{ \operatorname{Tr}[\rho^{\otimes n} - 2^{n[R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)+(r\epsilon/2)]}\sigma]_+ \right\} + 2^{-(rn\epsilon/2)}. \end{aligned}\tag{D12}$$

Reference [90, Prop. III.1] has shown that the first term approaches 0 in the limit of $n \to \infty$ for every $\epsilon > 0$. Therefore, combining it with Eq. (D6), we obtain that for all $\epsilon > 0$,

$$\lim_{n\to\infty} F_{\mathbb{O}_{\max}}\left(\rho^{\otimes n} \to \Phi^{\otimes n(R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)/r+\epsilon)}\right) = 0. \tag{D13}$$

On the other hand, if we take $y = (1/r)R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho) - \epsilon$, the optimum b for each n needs to satisfy $b < y$ because otherwise $G_{\mathbb{F}}(\rho^{\otimes n}, r^{ny})$ would diverge as $n \to \infty$. Therefore,

$$G_{\mathbb{F}}(\rho^{\otimes n}; r^{ny}) \ge \inf_{\sigma\in\mathbb{F}} \operatorname{Tr}[\rho - 2^{n(R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)-\epsilon r)}\sigma]_+. \tag{D14}$$

The right-hand side approaches 1 for every $\epsilon > 0$ [90] and thus leads to

$$\lim_{n\to\infty} F_{\mathbb{O}_{\max}}\left(\rho^{\otimes n} \to \Phi^{\otimes n(R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)/r-\epsilon)}\right) = 1 \tag{D15}$$

for every $\epsilon > 0$. Equations (D13) and (D15) imply

$$d^{\infty}_{\mathbb{O}_{\max}}(\rho, \Phi) = \tilde{d}^{\infty}_{\mathbb{O}_{\max}}(\rho, \Phi) = \frac{R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)}{r} = \frac{R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)}{D_{\min,\mathbb{F}}(\Phi)}. \tag{D16}$$

Finally, combining this with Corollary 4, we obtain

$$\tilde{d}_{\mathbb{O}_{\max}}(\rho, \Phi) \le \tilde{c}_{\mathbb{O}_{\max}}(\rho, \Phi). \tag{D17}$$

Noting that

$$\tilde{d}_{\mathbb{O}}(\rho, \Phi) \le \tilde{d}_{\mathbb{O}_{\max}}(\rho, \Phi), \quad \tilde{c}_{\mathbb{O}_{\max}}(\rho, \Phi) \le \tilde{c}_{\mathbb{O}}(\rho, \Phi) \tag{D18}$$

for every $\mathbb{O} \subseteq \mathbb{O}_{\max}$ immediately leads to

$$\tilde{d}_{\mathbb{O}}(\rho, \Phi) \le \tilde{c}_{\mathbb{O}}(\rho, \Phi), \tag{D19}$$

which concludes the proof. ■

We note that the fact that $(1/r)\tilde{D}^{\infty}_{\max,\mathbb{O}}(\mathcal{E})$ is a strong converse rate for distillation in general resource theories of channels has previously been shown in Ref. [58]. However, the relation

$$\tilde{D}^{\infty}_{\max,\mathbb{F}}(\rho) = \tilde{D}^{\infty}_{H,\mathbb{F}}(\rho) = R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho) \tag{D20}$$

and, in particular, the fact that $(1/r)R^{\infty}_{\mathrm{rel},\mathbb{F}}(\rho)$ constitutes an *achievable* rate of distillation under $\mathbb{O}_{\max}$ is a very nontrivial result established in Ref. [90], applicable only to resources of quantum states. It is an open question whether an extension of this result to channel theories can be obtained (cf. Ref. [60]). It would also be interesting to understand whether the double-sided strong converse bound described in this section can be shown without relying on the generalized quantum Stein's lemma of Ref. [90].

APPENDIX E: PROOF OF LEMMA 5, PROPOSITION 6, AND PROPOSITION 7

We first show Lemma 5.

Proof of Lemma 5. Suppose that $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi)$. By definition of the standard robustness and the closedness of the set $\mathbb{F}$, there exist states $\tau, \sigma^\star \in \mathbb{F}$ such that

$$\tau = \frac{\Phi + [2^{D_{s,\mathbb{F}}(\Phi)} - 1]\sigma^\star}{2^{D_{s,\mathbb{F}}(\Phi)}}. \tag{E1}$$

Also, by definition of $D_{\min,\mathbb{F}}(\Phi)$ and noting that $D_{\min,\mathbb{F}}(\Phi) = D^{\epsilon=0}_{H,\mathbb{F}}(\Phi)$, one can confirm that an operator $P^\star = \Pi_\Phi$ satisfies $0 \le P^\star \le \mathbb{1}$, $\mathrm{Tr}[P^\star\Phi] = 1$ and $\mathrm{Tr}[P^\star\eta] \le 2^{-D_{\min,\mathbb{F}}(\Phi)}, \forall\eta \in \mathbb{F}$. Using this operator, define the following channel:

$$\Lambda(\cdot) := \mathrm{Tr}[P^\star\cdot]\Phi + \mathrm{Tr}[(\mathbb{1} - P^\star)\cdot]\sigma^\star. \tag{E2}$$

One can check that $\Lambda \in \mathbb{O}_{\max}$ as follows. For every $\eta \in \mathbb{F}$, we obtain

$$\Lambda(\eta) = \mathrm{Tr}[P^\star\eta]\Phi + \mathrm{Tr}[(\mathbb{1} - P^\star)\eta]\sigma^\star, \tag{E3}$$

with

$$\mathrm{Tr}[P^\star\eta] \le 2^{-D_{\min,\mathbb{F}}(\Phi)} = 2^{-D_{s,\mathbb{F}}(\Phi)}. \tag{E4}$$

The convexity of $\mathbb{F}$ and the form of Eq. (E1) imply that all states of the form

$$\alpha\Phi + (1-\alpha)\sigma^\star, \quad 0 \le \alpha \le 2^{-D_{s,\mathbb{F}}(\Phi)} \tag{E5}$$

are free states. Therefore, Eq. (E4) ensures $\Lambda(\eta) \in \mathbb{F}$, $\forall\eta \in \mathbb{F}$, implying $\Lambda \in \mathbb{O}_{\max}$.

Next, we show that $\mathrm{Tr}[P^\star\sigma^\star] = \mathrm{Tr}[\Phi\sigma^\star] = 0$. If we apply Λ to τ in Eq. (E1), we obtain

$$\begin{aligned}\Lambda(\tau) &= \frac{\Lambda(\Phi) + [2^{D_{s,\mathbb{F}}(\Phi)} - 1]\Lambda(\sigma^\star)}{2^{D_{s,\mathbb{F}}(\Phi)}}\\ &= \frac{\Phi + [2^{D_{s,\mathbb{F}}(\Phi)} - 1]\{\mathrm{Tr}[P^\star\sigma^\star]\Phi + \mathrm{Tr}[(\mathbb{1} - P^\star)\sigma^\star]\sigma^\star\}}{2^{D_{s,\mathbb{F}}(\Phi)}}\\ &= \left\{2^{-D_{s,\mathbb{F}}(\Phi)} + \mathrm{Tr}[P^\star\sigma^\star]\left[1 - 2^{-D_{s,\mathbb{F}}(\Phi)}\right]\right\}\Phi + \left[1 - 2^{-D_{s,\mathbb{F}}(\Phi)}\right]\left(1 - \mathrm{Tr}[P^\star\sigma^\star]\right)\sigma^\star.\end{aligned} \tag{E6}$$

Since $\tau \in \mathbb{F}$ and $\Lambda \in \mathbb{O}_{\max}$, we have $\Lambda(\tau) \in \mathbb{F}$. Therefore, the definition of $D_{s,\mathbb{F}}(\Phi)$ [or, in other words, Eq. (E5)] forces $\mathrm{Tr}[P^\star\sigma^\star]\left[1 - 2^{-D_{s,\mathbb{F}}(\Phi)}\right] \le 0$. Since $2^{-D_{s,\mathbb{F}}(\Phi)} \le 1$, we must have that $\mathrm{Tr}[P^\star\sigma^\star] = 0$. Combining the fact that $P^\star \ge \Phi$ because of $\mathrm{Tr}[P^\star\Phi] = 1$, we also have $0 = \mathrm{Tr}[P^\star\sigma^\star] \ge \mathrm{Tr}[\Phi\sigma^\star] \ge 0$, leading to $\mathrm{Tr}[\Phi\sigma^\star] = 0$.

The proof for the case of $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$ goes analogously. The only difference is that the inequality in Eq. (E4) becomes an equality and $P^\star$ does not necessarily coincide with Π_Φ. By definition of the generalized robustness, there exist a free state $\tau \in \mathbb{F}$ and some state $\sigma^\star \in \mathbb{D}$ such that

$$\tau = \frac{\Phi + [2^{D_{\max,\mathbb{F}}(\Phi)} - 1]\sigma^\star}{2^{D_{\max,\mathbb{F}}(\Phi)}}. \tag{E7}$$

Also, by definition of $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) := D^{\epsilon=0}_{H,\mathrm{aff}(\mathbb{F})}$, there exists an operator $P^\star$ that satisfies $0 \le P^\star \le \mathbb{1}$, $\mathrm{Tr}[P^\star\Phi] = 1$, and $\mathrm{Tr}[P^\star\eta] \le 2^{-D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi)}, \forall\eta \in \mathrm{aff}(\mathbb{F})$. In fact, these conditions impose a strong constraint

$$\mathrm{Tr}[P^\star\eta] = 2^{-D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi)}, \quad \forall\eta \in \mathbb{F}. \tag{E8}$$

To see this, observe first that $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) \ge 0$ because in Eq. (30), the choice of $P = \mathbb{1}$ ensures $D^\epsilon_H(\rho\|\sigma) \ge 0$ for all $\sigma \in \mathrm{aff}(\mathbb{F})$ and all ϵ, resulting in $D^\epsilon_{H,\mathrm{aff}(\mathbb{F})}(\rho) \ge 0$ for an arbitrary state ρ. This particularly ensures that

$$\mathrm{Tr}[P^\star\eta] \le 1, \quad \forall\eta \in \mathrm{aff}(\mathbb{F}). \tag{E9}$$

Then, suppose that there exist two free states $\eta_1, \eta_2 \in \mathbb{F}$ such that $\mathrm{Tr}[P^\star\eta_1] \ne \mathrm{Tr}[P^\star\eta_2]$, where we assume $\mathrm{Tr}[P^\star\eta_1] - \mathrm{Tr}[P^\star\eta_2] =: \Delta > 0$ without loss of generality. Define an affine combination $\eta(c) := c\eta_1 - (1-c)\eta_2 \in \mathrm{aff}(\mathbb{F})$ for an arbitrary real number c. This operator realizes $\mathrm{Tr}[P^\star\eta(c)] = c\Delta - \mathrm{Tr}[P^\star\eta_2]$. However, since $\Delta > 0$, one could violate Eq. (E9) by taking sufficiently large c, which is a contradiction. Thus, we must have $\mathrm{Tr}[P^\star\eta] = \mathrm{const}$, $\forall\eta \in \mathbb{F}$. Combining this with the definition of $D_{\min,\mathrm{aff}(\mathbb{F})}$ leads to the condition given in Eq. (E8).

Using this operator, define the following channel:

$$\Lambda(\cdot) := \mathrm{Tr}[P^\star \cdot]\Phi + \mathrm{Tr}[(\mathbb{1} - P^\star)\cdot]\sigma^\star. \tag{E10}$$

It is now easy to check that $\Lambda \in \mathbb{O}_{\max}$ using Eqs. (E8) and (E7).

Next, we show that $\mathrm{Tr}[P^\star\sigma^\star] = \mathrm{Tr}[\Phi\sigma^\star] = 0$. If we apply Λ to τ in Eq. (E1), we obtain

$$\begin{aligned}\Lambda(\tau) &= \frac{\Lambda(\Phi) + [2^{D_{\max,\mathbb{F}}(\Phi)} - 1]\Lambda(\sigma^\star)}{2^{D_{\max,\mathbb{F}}(\Phi)}} \\ &= \frac{\Phi + [2^{D_{\max,\mathbb{F}}(\Phi)} - 1]\left(\mathrm{Tr}[P^\star\sigma^\star]\Phi + \mathrm{Tr}[(\mathbb{1} - P^\star)\sigma^\star]\sigma^\star\right)}{2^{D_{\max,\mathbb{F}}(\Phi)}} \\ &= \left\{2^{-D_{\max,\mathbb{F}}(\Phi)} + \mathrm{Tr}[P^\star\sigma^\star]\left[1 - 2^{-D_{\max,\mathbb{F}}(\Phi)}\right]\right\}\Phi + \left[1 - 2^{-D_{\max,\mathbb{F}}(\Phi)}\right]\left(1 - \mathrm{Tr}[P^\star\sigma^\star]\right)\sigma^\star.\end{aligned} \tag{E11}$$

Since $\tau \in \mathbb{F}$ and $\Lambda \in \mathbb{O}_{\max}$, we have $\Lambda(\tau) \in \mathbb{F}$. The definition of $D_{\max,\mathbb{F}}(\Phi)$ states that $2^{-D_{\max,\mathbb{F}}(\Phi)}$ is the maximum coefficient in front of Φ such that a mixture with another state becomes a free state. This forces $\mathrm{Tr}[P^\star\sigma^\star]\left[1 - 2^{-D_{\max,\mathbb{F}}(\Phi)}\right] \leq 0$. Since $2^{-D_{\max,\mathbb{F}}(\Phi)} \leq 1$, we must have that $\mathrm{Tr}[P^\star\sigma^\star] = 0$. Combining the fact that $P^\star \geq \Phi$ because of $\mathrm{Tr}[P^\star\Phi] = 1$, we also have $0 = \mathrm{Tr}[P^\star\sigma^\star] \geq \mathrm{Tr}[\Phi\sigma^\star] \geq 0$, leading to $\mathrm{Tr}[\Phi\sigma^\star] = 0$. ■

Using Lemma 5, we obtain the following simplification of the evaluation of resource measures:

Lemma 26: *Let $\mathfrak{R}$ be a resource measure defined by*

$$\mathfrak{R}(\rho) = \inf_{\sigma\in\mathbb{F}} D(\rho,\sigma), \tag{E12}$$

where D is a contractive measure under free operations, i.e., $D(\rho,\sigma) \geq D(\Lambda(\rho),\Lambda(\sigma))$, $\forall \rho,\sigma$ for an arbitrary free channel $\Lambda \in \mathbb{O}_{\max}$. Suppose that $D_{s,\mathbb{F}}(\Phi) = D_{\min,\mathbb{F}}(\Phi) =: r$ for some state Φ and let $\sigma^\star$ be the state that appears in Eq. (35). Also, let $\tilde{\mathbb{D}}$ and $\tilde{\mathbb{F}}$ be the sets of states defined as

$$\begin{aligned}\tilde{\mathbb{D}} &:= \left\{\kappa\Phi + (1-\kappa)\sigma^\star \;\middle|\; 0 \leq \kappa \leq 1\right\}, \\ \tilde{\mathbb{F}} &:= \left\{\alpha\Phi + (1-\alpha)\sigma^\star \;\middle|\; 0 \leq \alpha \leq 2^{-r}\right\}.\end{aligned} \tag{E13}$$

Then, for every $\rho \in \tilde{\mathbb{D}}$, we can restrict the optimization in Eq. (E12) as

$$\mathfrak{R}(\rho) = \inf_{\sigma\in\tilde{\mathbb{F}}} D(\rho,\sigma). \tag{E14}$$

On the other hand, if $D_{\max,\mathbb{F}}(\Phi) = D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) =: r$, then every $\rho \in \tilde{\mathbb{D}}$ satisfies

$$\mathfrak{R}(\rho) = D(\rho,\tilde{\sigma}), \quad \tilde{\sigma} := 2^{-r}\Phi + (1-2^{-r})\sigma^\star. \tag{E15}$$

Proof. When $D_{s,\mathbb{F}}(\Phi) = D_{\min,\mathbb{F}}(\Phi) = r$, Lemma 5 ensures the existence of a channel $\Lambda \in \mathbb{O}_{\max}$ of the form given in Eq. (35). Crucially, all states in $\tilde{\mathbb{D}}$ are invariant under Λ. Thus, every $\rho \in \tilde{\mathbb{D}}$ satisfies

$$\begin{aligned}\mathfrak{R}(\rho) &= \inf_{\sigma\in\mathbb{F}} D(\rho,\sigma) \\ &\geq \inf_{\sigma\in\mathbb{F}} D(\Lambda(\rho),\Lambda(\sigma)) \\ &= \inf_{\sigma\in\mathbb{F}} D(\rho,\Lambda(\sigma)) \\ &\geq \inf_{\sigma\in\tilde{\mathbb{F}}} D(\rho,\sigma) \\ &\geq \inf_{\sigma\in\mathbb{F}} D(\rho,\sigma) \\ &= \mathfrak{R}(\rho),\end{aligned} \tag{E16}$$

where we use the contractivity of D in the second line, the fact that $\Lambda(\rho) = \rho$ in the third line, $\Lambda(\sigma) \in \tilde{\mathbb{F}}\;\forall\sigma \in \mathbb{F}$ due to Lemma 5 and the definition of the robustness measures in the fourth line, and $\tilde{\mathbb{F}} \subseteq \mathbb{F}$ in the fifth line, leading to Eq. (E14).

When $D_{\max,\mathbb{F}}(\rho) = D_{\min,\mathrm{aff}(\mathbb{F})}(\rho) = r$, the operator $P^\star$ in Eq. (35) satisfies $\mathrm{Tr}[P^\star\sigma] = 2^{-r}, \forall\sigma \in \mathbb{F}$ as in Eq. (E8). Thus, Eq. (E15) is obtained by replacing $\tilde{\mathbb{F}}$ in Eq. (E16) with $\{\tilde{\sigma}\}$. ■

We are now ready to prove Propositions 6 and 7. In fact, we show a more general result, which immediately implies both of the propositions and allows for the computation of the smoothed entropic measures for all isotropiclike states Φ_κ.

Proposition 27: *Suppose that a state Φ satisfies $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi) =: r$ and let $\sigma^\star$ be the state in Eq. (35). Then, every state $\Phi_\kappa = \kappa\Phi + (1-\kappa)\sigma^\star$ with $0 \leq \kappa \leq 1$ satisfies*

$$D^{\epsilon}_{H,\mathbb{F}}(\Phi_\kappa) = \begin{cases} 0, & \epsilon = 0,\ 0 \le \kappa < 1, \\ r + \log \dfrac{1}{1-\epsilon}, & 0 \le \epsilon < 1,\ \kappa = 1. \end{cases} \tag{E17}$$

Also, let $\eta^{\epsilon}_{\min}$ *and* $\eta^{\epsilon}_{\max}$ *be the minimum and maximum* η *that satisfy* $F_{cl}((\eta, 1-\eta), (\kappa, 1-\kappa)) \ge 1-\epsilon$*, where* $F_{cl}(p,q) := \left(\sum_i \sqrt{p_i q_i}\right)^2$ *is the fidelity for two classical distributions. Then,*

$$D^{\epsilon}_{\max,\mathbb{F}}(\Phi_\kappa) = D^{\epsilon}_{s,\mathbb{F}}(\Phi_\kappa) = \max\left\{ r - \log \frac{1}{\eta^{\epsilon}_{\min}}, 0 \right\} \tag{E18}$$

for all $\epsilon \in [0,1)$.

Similarly, if a state Φ *satisfies* $D_{\min,\mathrm{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi) =: r$*, then*

$$\begin{aligned} D^{\epsilon}_{H,\mathrm{aff}(\mathbb{F})}(\Phi_\kappa) &= D^{\epsilon}_{H,\mathbb{F}}(\Phi_\kappa) \\ &= \begin{cases} 0, & \epsilon = 0,\ 0 < \kappa < 1, \\ \dfrac{1}{1-2^{-r}}, & \epsilon = 0,\ \kappa = 0, \\ r + \log \dfrac{1}{1-\epsilon}, & 0 \le \epsilon < 1,\ \kappa = 1, \end{cases} \end{aligned} \tag{E19}$$

and

$$D^{\epsilon}_{\max,\mathbb{F}}(\Phi_\kappa) = \begin{cases} r - \log \dfrac{1}{\eta^{\epsilon}_{\min}}, & \eta^{\epsilon}_{\min} \ge 2^{-r}, \\ \log \dfrac{1-\eta^{\epsilon}_{\max}}{1-2^{-r}}, & \eta^{\epsilon}_{\max} \le 2^{-r}, \\ 0, & \eta^{\epsilon}_{\min} \le 2^{-r} \le \eta^{\epsilon}_{\max}, \end{cases} \tag{E20}$$

for all $\epsilon \in [0,1)$.

We remark that one can obtain an analogous result for smoothing with different distance measures. In particular, the trace-distance smoothing leads to simple expressions with $\eta^{\epsilon}_{\min} = \kappa - \epsilon$ and $\eta^{\epsilon}_{\max} = \kappa + \epsilon$.

Proof. Let us first consider the case when $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi) = r$. Since $D^{\epsilon}_{H}(\cdot\|\cdot)$ satisfies the data-processing inequality, we can use Lemma 26 to obtain

$$\begin{aligned} D^{\epsilon}_{H,\mathbb{F}}(\Phi_\kappa) &= \log \min_{\tau\in\mathbb{F}} \max_{\substack{0\le Q\le \mathbb{1} \\ \mathrm{Tr}[Q\Phi_\kappa]\ge 1-\epsilon}} \mathrm{Tr}[Q\tau]^{-1} \\ &= \log \min_{\tau\in\tilde{\mathbb{F}}} \max_{\substack{0\le Q\le \mathbb{1} \\ \mathrm{Tr}[Q\Phi_\kappa]\ge 1-\epsilon}} \mathrm{Tr}[Q\tau]^{-1}, \end{aligned} \tag{E21}$$

where $\tilde{\mathbb{F}}$ is the set of free states defined in Eq. (E13). Let Λ and $P^\star$ be the channel and operator in Eq. (35). Then, since $\Lambda(\Phi_\kappa) = \Phi_\kappa$ and $\Lambda(\tau) = \tau$, $\forall \tau \in \tilde{\mathbb{F}}$, if Q is a feasible solution in Eq. (E21), $\Lambda^\dagger(Q) = \mathrm{Tr}[Q\Phi]P^\star + \mathrm{Tr}[Q\sigma](\mathbb{1} - P^\star)$ is also a feasible solution giving the same objective function, i.e., $\mathrm{Tr}[\Lambda^\dagger(Q)\tau]^{-1} = \mathrm{Tr}[Q\tau]^{-1}$. Thus, it suffices to take the optimization over operators of the form $Q = \eta P^\star + \lambda(\mathbb{1} - P^\star)$. The conditions $0 \le Q \le \mathbb{1}$, $\mathrm{Tr}[Q\Phi_\kappa] \ge 1-\epsilon$ are equivalent to

$$0 \le \lambda \le 1, \quad 0 \le \eta \le 1, \quad \eta\kappa + \lambda(1-\kappa) \ge 1-\epsilon. \tag{E22}$$

Let $\mathcal{A}$ be the set of (λ, η) that satisfies Eq. (E22). Then, we can compute $D^{\epsilon}_{H,\mathbb{F}}(\Phi_\kappa)$ as

$$\begin{aligned} &\log \min_{0\le\alpha\le 2^{-r}} \max_{(\lambda,\eta)\in\mathcal{A}} \mathrm{Tr}[\{\eta P^\star + \lambda(\mathbb{1} - P^\star)\} \\ &\quad \times \{\alpha\Phi + (1-\alpha)\sigma^\star\}]^{-1} \\ &\quad = -\log \max_{0\le\alpha\le 2^{-r}} \min_{(\lambda,\eta)\in\mathcal{A}} [\eta\alpha + \lambda(1-\alpha)], \end{aligned} \tag{E23}$$

where we use $\mathrm{Tr}[P^\star\Phi] = 1$ and $\mathrm{Tr}[P^\star\sigma^\star] = 0$.

Let us first consider the case $\kappa = 1$. The condition for $\mathcal{A}$ in this case turns to

$$0 \le \lambda \le 1, \quad 1-\epsilon \le \eta \le 1. \tag{E24}$$

Then, Eq. (E23) can be further computed as

$$-\log \max_{0\le\alpha\le 2^{-r}} (1-\epsilon)\alpha = r + \log \frac{1}{1-\epsilon}. \tag{E25}$$

On the other hand, suppose that $\epsilon = 0$ and $0 \le \kappa \le 1$. Then, the condition for $\mathcal{A}$ takes the form

$$0 \le \lambda \le 1, \quad 0 \le \eta \le 1, \quad \eta\kappa + \lambda(1-\kappa) = 1. \tag{E26}$$

From this, it is clear that for the case $0 < \kappa < 1$, we must have $\lambda = \eta = 1$, which makes the quantity in Eq. (E23) equal to 0. In the case of $\kappa = 0$, which forces $\lambda = 1$, $0 \le \eta \le 1$, the optimum in Eq. (E23) is achieved at $\alpha = \eta = 0$, $\lambda = 1$, also resulting in 0. The case of $\kappa = 1$ is included in Eq. (E25); we obtain r by setting $\epsilon = 0$.

To summarize, we show that

$$D^{\epsilon}_{H,\mathbb{F}}(\Phi_\kappa) = \begin{cases} 0, & \epsilon = 0,\ 0 \le \kappa < 1, \\ r + \log \dfrac{1}{1-\epsilon}, & 0 \le \epsilon < 1,\ \kappa = 1. \end{cases} \tag{E27}$$

To show the expression for $D^{\epsilon}_{\max,\mathbb{F}}$, note that

$$\begin{aligned} &D^{\epsilon}_{\max,\mathbb{F}}(\Phi_\kappa) \\ &\quad = \inf_{\sigma\in\mathbb{F}} D^{\epsilon}_{\max}(\Phi_\kappa\|\sigma) \\ &\quad = \inf\left\{ \log s \;\middle|\; \rho \le s\sigma,\ \sigma \in \tilde{\mathbb{F}}, F(\rho, \Phi_\kappa) \ge 1-\epsilon \right\}, \end{aligned} \tag{E28}$$

where in the second line we use Lemma 26 because $D^{\epsilon}_{\max}(\cdot\|\cdot)$ satisfies the data-processing inequality for all quantum channels. Consider again the map Λ in Eq. (35), which maps an arbitrary state to a state in $\tilde{\mathbb{D}}$ [cf. Eq. (E13)], while stabilizing every state $\Phi_\kappa \in \tilde{\mathbb{D}}$ and $\sigma \in \tilde{\mathbb{F}}$ as $\Lambda(\Phi_\kappa) = \Phi_\kappa$ and $\Lambda(\sigma) = \sigma$. Then, $\rho \leq s\sigma$ implies that $\Lambda(\rho) \leq s\Lambda(\sigma) = s\sigma, \forall \sigma \in \tilde{\mathbb{F}}$. We also have

$$F(\Lambda(\rho), \Phi_\kappa) = F(\Lambda(\rho), \Lambda(\Phi_\kappa)) \geq F(\rho, \Phi_\kappa) \geq 1 - \epsilon. \tag{E29}$$

Thus, the optimization in Eq. (E28) is achieved by the states of the form $\rho = \eta\Phi + (1-\eta)\sigma^\star$, $\sigma = \alpha\Phi + (1-\alpha)\sigma^\star$ with constraints $0 \leq \eta \leq 1$, $F(\rho, \Phi_\kappa) \geq 1 - \epsilon$ and $0 \leq \alpha \leq 2^{-r}$. Noting that $\mathrm{Tr}[\Phi\sigma^\star] = 0$ and thus $\rho = \eta\Phi \oplus (1-\eta)\sigma^\star$, the condition on η can equivalently be written as a condition for two classical distributions $(\eta, 1-\eta)$ and $(\kappa, 1-\kappa)$:

$$F_{\mathrm{cl}}((\eta, 1-\eta), (\kappa, 1-\kappa)) \geq 1 - \epsilon. \tag{E30}$$

Let $\eta_{\min}^\epsilon$ and $\eta_{\max}^\epsilon$ be the minimum and maximum η that satisfy Eq. (E30). Then, we can compute $D_{\max,\mathbb{F}}^\epsilon(\Phi_\kappa)$ as

$$\begin{aligned}
&\inf_{\substack{\alpha\in[0,2^{-r}]\\ \eta\in[\eta_{\min}^\epsilon,\eta_{\max}^\epsilon]}} \left\{\log s \;\middle|\; \eta\Phi + (1-\eta)\sigma^\star \leq s[\alpha\Phi + (1-\alpha)\sigma^\star]\right\} \\
&= \inf_{\substack{\alpha\in[0,2^{-r}]\\ \eta\in[\eta_{\min}^\epsilon,\eta_{\max}^\epsilon]}} \left\{\log s \;\middle|\; s\alpha - \eta \geq 0,\ s(1-\alpha) - (1-\eta) \geq 0\right\} \\
&= \log \inf_{\substack{\alpha\in[0,2^{-r}]\\ \eta\in[\eta_{\min}^\epsilon,\eta_{\max}^\epsilon]}} \max\left\{\frac{\eta}{\alpha}, \frac{1-\eta}{1-\alpha}\right\}.
\end{aligned} \tag{E31}$$

Note that $\max\{\eta/\alpha, (1-\eta)/(1-\alpha)\}$ is clearly lower bounded by 1 and it is achieved when $\eta = \alpha$. Thus, when $\eta_{\min}^\epsilon \leq 2^{-r}$, we immediately obtain $D_{\max,\mathbb{F}}^\epsilon(\Phi_\kappa) = 0$. On the other hand, when $\eta_{\min}^\epsilon \geq 2^{-r}$, we always have $\max\{\eta/\alpha, (1-\eta)/(1-\alpha)\} = \eta/\alpha$, and the minimization over α and η gives $D_{\max,\mathbb{F}}^\epsilon(\Phi_\kappa) = r - \log(1/\eta_{\min}^\epsilon)$. These can concisely be written as

$$D_{\max,\mathbb{F}}^\epsilon(\Phi_\kappa) = \max\left\{r - \log\frac{1}{\eta_{\min}^\epsilon}, 0\right\}, \tag{E32}$$

which concludes the proof for the expression of $D_{\max,\mathbb{F}}^\epsilon$.

The smooth standard robustness can be computed similarly. Note that $D_{s,\mathbb{F}}^\epsilon(\Phi) = \inf_{\sigma\in\mathbb{F}} D_{s,\mathbb{F}}^\epsilon(\Phi\|\sigma)$, where

$$D_{s,\mathbb{F}}^\epsilon(\rho_1\|\rho_2) := \inf\left\{\log(1+s) \;\middle|\; \frac{\rho_1' + s\tau}{1+s} = \rho_2,\ \tau \in \mathbb{F},\ F(\rho_1', \rho_1) \geq 1 - \epsilon\right\}, \tag{E33}$$

and $D_{s,\mathbb{F}}^\epsilon$ is contractive under every free operation $\mathcal{E} \in \mathbb{O}_{\max}$, as for all states ρ and σ,

$$\begin{aligned}
D_{s,\mathbb{F}}^\epsilon(\rho\|\sigma) &= D_{s,\mathbb{F}}(\tilde{\rho}\|\sigma) \\
&\geq D_{s,\mathbb{F}}(\mathcal{E}(\tilde{\rho})\|\mathcal{E}(\sigma)) \\
&\geq D_{s,\mathbb{F}}^\epsilon(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)),
\end{aligned} \tag{E34}$$

where we set $\tilde{\rho}$ as the optimal state realizing the standard robustness and in the third line we use that $F(\mathcal{E}(\tilde{\rho}), \mathcal{E}(\rho)) \geq F(\tilde{\rho}, \rho) \geq 1 - \epsilon$. Thus, we can use Lemma 26 to compute $D_{s,\mathbb{F}}^\epsilon(\Phi_\kappa)$ as

$$\inf_{\substack{\alpha\in[0,2^{-r}]\\ \eta\in[\eta^\epsilon_{\min},\eta^\epsilon_{\max}]}} \left\{\log s \;\middle|\; \eta\Phi+(1-\eta)\sigma^\star \le_{\mathbb{F}} s[\alpha\Phi+(1-\alpha)\sigma^\star]\right\}$$
$$= \inf_{\substack{\alpha\in[0,2^{-r}]\\ \eta\in[\eta^\epsilon_{\min},\eta^\epsilon_{\max}]}} \left\{\log s \;\middle|\; s\alpha-\eta\ge 0,\; s(1-\alpha)-(1-\eta)\ge 0,\; \frac{s(1-\alpha)-(1-\eta)}{s\alpha-\eta}\ge 2^r-1\right\}$$
$$= \log \inf_{\substack{\alpha\in[0,2^{-r}]\\ \eta\in[\eta^\epsilon_{\min},\eta^\epsilon_{\max}]}} \max\left\{\frac{\eta}{\alpha},\frac{1-\eta}{1-\alpha},\frac{1-2^r\eta}{1-2^r\alpha}\right\}, \tag{E35}$$

where in the first line, we use the notation $A\le_{\mathbb{F}} B \iff B-A\in\text{cone}(\mathbb{F})$. When $\eta^\epsilon_{\min}\le 2^{-r}$, we immediately obtain $D^\epsilon_{s,\mathbb{F}}(\Phi_\kappa)=0$, which is achieved at $\alpha=\eta$. On the other hand, when $\eta^\epsilon_{\min}\ge 2^{-r}$, we have $\eta/\alpha>(1-\eta)/(1-\alpha)$ and $\eta/\alpha>(1-2^r\eta)(1-2^r\alpha)$, in which the minimization over α and η results in $D^\epsilon_{s,\mathbb{F}}(\Phi_\kappa)=r-\log(1/\eta_{\min})$. These two can be combined as

$$D^\epsilon_{s,\mathbb{F}}(\Phi_\kappa)=\max\left\{r-\log\frac{1}{\eta^\epsilon_{\min}},0\right\}. \tag{E36}$$

The proof for the case when $D_{\min,\text{aff}(\mathbb{F})}(\Phi)=D_{\max,\mathbb{F}}(\Phi)$ goes analogously, where we basically change the region of optimization over free states from $\tilde{\mathbb{F}}$ to $\{\tilde\sigma\}$ defined in Eq. (E15). Since $D^\epsilon_H(\rho\|\sigma)$ satisfies the data-processing inequality even if σ is not a positive operator [63], we can employ the same argument as that in Lemma 26 to obtain

$$D^\epsilon_{H,\text{aff}(\mathbb{F})}(\Phi_\kappa)=\log \sup_{\substack{0\le Q\le \mathbb{1}\\ \text{Tr}[Q\Phi_\kappa]\ge 1-\epsilon}} \text{Tr}[Q\tilde\sigma]^{-1}, \tag{E37}$$

where we use that the map Λ in Eq. (35) transforms all $\sigma\in\text{aff}(\mathbb{F})$ to $\tilde\sigma$. We can then restrict the optimization to $Q=\eta P^\star+\lambda(\mathbb{1}-P^\star)$ with the condition given in Eq. (E22). Recall that $\mathcal{A}$ is the set of (λ,η) that satisfies Eq. (E22). Then, we can compute $D^\epsilon_{H,\text{aff}(\mathbb{F})}(\Phi_\kappa)$ as

$$\log\sup_{(\lambda,\eta)\in\mathcal{A}} \text{Tr}[\{\eta P^\star+\lambda(\mathbb{1}-P^\star)\}\{2^{-r}\Phi+(1-2^{-r})\sigma^\star\}]^{-1}$$
$$=-\log\inf_{(\lambda,\eta)\in\mathcal{A}}[\eta 2^{-r}+\lambda(1-2^{-r})]. \tag{E38}$$

When $\kappa=1$, the condition on $\mathcal{A}$ becomes Eq. (E24). Then, we evaluate this as

$$-\log\left[(1-\epsilon)2^{-r}\right]=r+\log\frac{1}{1-\epsilon}. \tag{E39}$$

On the other hand, when $\epsilon=0$ and $0<\kappa<1$, we are constrained to $\lambda=\eta=1$, leading to value 0. When $\kappa=0$, we have $\lambda=1$, $0\le\eta\le 1$, giving $\log[(1/(1-2^{-r})]$. When $\kappa=1$, we have $0\le\lambda\le 1$, $\eta=1$, giving r.

Noting that the above argument gives the same conclusion for $D^\epsilon_{H,\mathbb{F}}$ as well, we reorganize the above form to reach

$$D^\epsilon_{H,\text{aff}(\mathbb{F})}(\Phi_\kappa)=D^\epsilon_{H,\mathbb{F}}(\Phi_\kappa)$$
$$=\begin{cases}0, & \epsilon=0,\ 0<\kappa<1,\\ \log\dfrac{1}{1-2^{-r}}, & \epsilon=0,\ \kappa=0,\\ r+\log\dfrac{1}{1-\epsilon}, & 0\le\epsilon<1,\ \kappa=1.\end{cases} \tag{E40}$$

As for $D^\epsilon_{\max,\mathbb{F}}$, we follow the same argument up to Eq. (E31) to obtain

$$D^\epsilon_{\max,\mathbb{F}}(\Phi_\kappa)=\log\inf_{\eta\in[\eta^\epsilon_{\min},\eta^\epsilon_{\max}]}\max\left\{\frac{\eta}{2^{-r}},\frac{1-\eta}{1-2^{-r}}\right\}. \tag{E41}$$

If $\eta^\epsilon_{\min}\ge 2^{-r}$, we always have $\eta/2^{-r}\ge(1-\eta)/(1-2^{-r})$ and thus $D^\epsilon_{\max,\mathbb{F}}(\Phi_\kappa)=\log(\eta^\epsilon_{\min}/2^{-r})$. If $\eta^\epsilon_{\max}\le 2^{-r}$, we always have $\eta/2^{-r}\le(1-\eta)/(1-2^{-r})$ and thus $D^\epsilon_{\max,\mathbb{F}}(\Phi_\kappa)=\log[(1-\eta^\epsilon_{\max})/(1-2^{-r})]$. If $\eta^\epsilon_{\min}\le 2^{-r}\le\eta^\epsilon_{\max}$, then $D^\epsilon_{\max,\mathbb{F}}(\Phi_\kappa)=0$, which is achieved at $\eta=2^{-r}$.

Summarizing, we obtain

$$D^\epsilon_{\max,\mathbb{F}}(\Phi_\kappa)=\begin{cases} r-\log\dfrac{1}{\eta^\epsilon_{\min}}, & \eta^\epsilon_{\min}\ge 2^{-r},\\ \log\dfrac{1-\eta^\epsilon_{\max}}{1-2^{-r}}, & \eta^\epsilon_{\max}\le 2^{-r},\\ 0, & \eta^\epsilon_{\min}\le 2^{-r}\le\eta^\epsilon_{\max},\end{cases} \tag{E42}$$

concluding the proof. ■

The generality of Lemma 26 has wide applicability beyond the above measures. As an example, it allows us to provide an exact evaluation for a measure based on the trace distance.

Proposition 28: *Define the trace-distance measure* $R_{\text{tr},\mathbb{F}}(\rho):=\min_{\sigma\in\mathbb{F}}\frac{1}{2}\|\rho-\sigma\|_1$. *If* $D_{\min,\mathbb{F}}(\Phi)=D_{s,\mathbb{F}}(\Phi)=:$

r or $D_{\min,\text{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi) =: r$*, then*

$$R_{\text{tr},\mathbb{F}}(\Phi) = 1 - 2^{-r}. \tag{E43}$$

Proof. For the case when $D_{\min,\mathbb{F}}(\Phi) = D_{s,\mathbb{F}}(\Phi) =: r$,

$$\begin{aligned} R_{\text{tr},\mathbb{F}}(\Phi) &= \min_{\sigma\in\tilde{\mathbb{F}}} \frac{1}{2}\|\Phi - \sigma\|_1 \\ &= \min_{0\le\alpha\le 2^{-r}} \frac{1}{2}\|\Phi - \left[\alpha\Phi + (1-\alpha)\sigma^\star\right]\|_1 \\ &= \min_{0\le\alpha\le 2^{-r}} (1-\alpha) \\ &= 1 - 2^{-r}, \end{aligned} \tag{E44}$$

where in the first line we use Lemma 26 and in the third line we use that $\text{Tr}[\Phi\sigma] = 0$. The case for $D_{\min,\text{aff}(\mathbb{F})}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$ can be shown analogously. ■

APPENDIX F: PROOF OF THEOREM 8

Proof. The proof combines Proposition 6 and the argument in Ref. [23]. For two states $\Phi_1, \Phi_2 \in \mathbb{T}$, consider the following sequence of transformations:

$$\Phi_2 \xrightarrow[\mathbb{O}]{\epsilon_2} \rho \xrightarrow[\mathbb{O}]{\epsilon_1} \Phi_1. \tag{F1}$$

Let Λ_2 and Λ_1 be free transformations corresponding to the first and the second transformations above. Define the purified distance $P(\rho,\sigma) := \sqrt{1 - F(\rho,\sigma)}$, which satisfies the following triangle inequality [108]:

$$P(\rho,\tau) \le P(\rho,\sigma)\sqrt{F(\sigma,\tau)} + P(\sigma,\tau)\sqrt{F(\rho,\sigma)} \tag{F2}$$

for every set of three states ρ,σ,τ such that $P(\rho,\sigma)^2 + P(\sigma,\tau)^2 \le 1$. Applying this to our setting, we obtain that $\Lambda_1 \circ \Lambda_2 \in \mathbb{O}$ achieves the transformation $\Phi_2 \xrightarrow[\mathbb{O}]{\epsilon'} \Phi_1$ with

$$\epsilon' := \left[\sqrt{\epsilon_2(1-\epsilon_1)} + \sqrt{\epsilon_1(1-\epsilon_2)}\right]^2, \tag{F3}$$

whenever $\epsilon_1 + \epsilon_2 \le 1$ is satisfied. Moreover, $\epsilon_1 + \epsilon_2 < 1$ ensures $\epsilon' < 1$. This can be shown as follows. Direct calculation gives

$$\epsilon' = \epsilon_1 + \epsilon_2 + 2\sqrt{\epsilon_1\epsilon_2}\left[\sqrt{(1-\epsilon_1)(1-\epsilon_2)} - \sqrt{\epsilon_1\epsilon_2}\right]. \tag{F4}$$

Let us change parameters as $\Delta := 1 - (\epsilon_1 + \epsilon_2)$ and $p := \epsilon_1\epsilon_2$, which behave as independent variables under the condition that $(1-\Delta)^2 - 4p \ge 0$, $0 < \Delta \le 1$, $p \ge 0$, noting that ϵ_1, ϵ_2 are solutions of the quadratic equation $x^2 - (1 - \Delta)x + p = 0$ while satisfying $\epsilon_1, \epsilon_2 \ge 0$ and $\epsilon_1 + \epsilon_2 < 1$. Then, we obtain

$$1 - \epsilon' = \Delta - 2\sqrt{p}\left(\sqrt{\Delta + p} - \sqrt{p}\right). \tag{F5}$$

This gives

$$\frac{\partial}{\partial\Delta}(1 - \epsilon') = 1 - \frac{\sqrt{p}}{\sqrt{\Delta + p}} > 0, \tag{F6}$$

implying that $1 - \epsilon'$ is a strictly increasing function of Δ for an arbitrary fixed p. Since $(1-\epsilon')|_{\Delta=0} = 0$ for every p, we obtain that $1 - \epsilon' > 0$ for all ϵ_1 and ϵ_2 satisfying $\epsilon_1 + \epsilon_2 < 1$.

Therefore, for every dilution operation Λ_2 with error ϵ_2 and distillation operation Λ_1 with error ϵ_1, we obtain

$$\begin{aligned} D_{\min,\mathbb{F}}(\Phi_1) &\le D_{H,\mathbb{F}}^{\epsilon'}(\Phi_2) \\ &= D_{\min,\mathbb{F}}(\Phi_2) + \log\frac{1}{1-\epsilon'}, \end{aligned} \tag{F7}$$

where the first line follows from Lemma 18 and the second line from Proposition 6. Optimizing over all feasible Λ_1 and Λ_2 and using the definition of distillable resource and resource cost gives the desired relation:

$$d_{\mathbb{O}}^{\epsilon_1}(\rho) \le c_{\mathbb{O}}^{\epsilon_2}(\rho) + \log\frac{1}{1-\epsilon'}. \tag{F8}$$

■

APPENDIX G: COMPARING THE BOUNDS IN THEOREMS 1 AND 8

Here, we show that the bound in Theorem 8 is always tighter than the bound in Theorem 1.

Let us first compare the expression in Theorem 1 valid for $\epsilon_1 + \epsilon_2 < 1$ and the bound in Theorem 8. Comparing the denominators of the expressions inside the logarithm, we obtain

$$\begin{aligned} &(1-\epsilon') - \left(\sqrt{1-\epsilon_2} - \sqrt{\epsilon_1}\right)^2 \\ &\quad = -2\epsilon_1 + 2\epsilon_1\epsilon_2 - 2\sqrt{\epsilon_1\epsilon_2(1-\epsilon_2)(1-\epsilon_1)} \\ &\qquad + 2\sqrt{\epsilon_1(1-\epsilon_2)} \\ &\quad = 2\sqrt{\epsilon_1(1-\epsilon_2)}\left[1 - \sqrt{\epsilon_2(1-\epsilon_1)}\right] - 2\epsilon_1(1-\epsilon_2) \\ &\quad = 2\sqrt{\epsilon_1(1-\epsilon_2)}\left[1 - \sqrt{\epsilon_2(1-\epsilon_1)} - \sqrt{\epsilon_1(1-\epsilon_2)}\right] \\ &\quad = 2\sqrt{\epsilon_1(1-\epsilon_2)}\left(1 - \sqrt{\epsilon'}\right), \end{aligned} \tag{G1}$$

where ϵ' is the one introduced in Eq. (F3). As shown in Appendix F, we have $\epsilon' < 1$ (and thus $1 - \sqrt{\epsilon'} > 0$)

for all ϵ_1 and ϵ_2 satisfying $\epsilon_1+\epsilon_2<1$. This implies that $\left(1-\epsilon'\right)-\left(\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1}\right)^2>0$ and, in particular,

$$\log\frac{1}{1-\epsilon'}<\log\left(\sqrt{1-\epsilon_2}-\sqrt{\epsilon_1}\right)^{-2}, \tag{G2}$$

for $\epsilon_1+\epsilon_2<1$.

We next compare the expression in Theorem 1 to the bound in Theorem 8 for $\epsilon_1+\sqrt{\epsilon_2}<1$. Direct calculation gives

$$\begin{aligned}&\left(1-\epsilon'\right)-\left(1-\epsilon_1-\sqrt{\epsilon_2}\right)\\&\quad=\epsilon_1+\sqrt{\epsilon_2}-\left[\sqrt{\epsilon_1(1-\epsilon_2)}+\sqrt{\epsilon_2(1-\epsilon_1)}\right]^2\\&\quad=\sqrt{\epsilon_2}-\epsilon_2+2\epsilon_1\epsilon_2-2\sqrt{\epsilon_1\epsilon_2(1-\epsilon_1)(1-\epsilon_2)}.\end{aligned} \tag{G3}$$

Let us define $g(\epsilon_1,\epsilon_2):=\left(1-\epsilon'\right)-\left(1-\epsilon_1-\sqrt{\epsilon_2}\right)$ and consider varying ϵ_1 for a fixed ϵ_2. Although we are interested in the region $0\le\epsilon_1<1-\sqrt{\epsilon_2}$, let us consider an extended region $0\le\epsilon_1<1-\epsilon_2$ as a domain of $g(\epsilon_1,\epsilon_2)$. Since $1-\sqrt{\epsilon_2}\le 1-\epsilon_2$, if we can show that $g(\epsilon_1,\epsilon_2)\ge 0$ for $0\le\epsilon_1\le 1-\epsilon_2$, then $g(\epsilon_1,\epsilon_2)\ge 0$ for $0\le\epsilon_1\le 1-\sqrt{\epsilon_2}$ automatically follows.

Since $g(0,\epsilon_2)=g(1-\epsilon_2,\epsilon_2)=\sqrt{\epsilon_2}-\epsilon_2\ge 0$, it suffices to show that $g(\cdot,\epsilon_2)\ge 0$ at local minima. Since

$$\begin{aligned}&\frac{\partial}{\partial\epsilon_1}g(\epsilon_1,\epsilon_2)\\&\quad=2\epsilon_2-\frac{\epsilon_2(1-\epsilon_2)(1-2\epsilon_1)}{\sqrt{\epsilon_1\epsilon_2(1-\epsilon_1)(1-\epsilon_2)}}\\&\quad\propto 2\sqrt{\epsilon_1\epsilon_2(1-\epsilon_1)(1-\epsilon_2)}-(1-\epsilon_2)(1-2\epsilon_1),\end{aligned} \tag{G4}$$

the local minima occur at $\epsilon_1^\star$ such that

$$2\sqrt{\epsilon_1^\star\epsilon_2(1-\epsilon_1^\star)(1-\epsilon_2)}-(1-\epsilon_2)(1-2\epsilon_1^\star)=0, \tag{G5}$$

for which we obtain a simplified form of $g(\epsilon_1^\star,\epsilon_2)$ as

$$\begin{aligned}g(\epsilon_1^\star,\epsilon_2)&=\sqrt{\epsilon_2}-\epsilon_2+2\epsilon_1^\star\epsilon_2-(1-\epsilon_2)(1-2\epsilon_1^\star)\\&=\sqrt{\epsilon_2}-1+2\epsilon_1^\star.\end{aligned} \tag{G6}$$

Equation (G5) can alternatively be written as

$$-1+\epsilon_1^\star+\left[\sqrt{\epsilon_1^\star(1-\epsilon_2)}+\sqrt{\epsilon_2(1-\epsilon_1^\star)}\right]^2=0, \tag{G7}$$

leading to

$$\sqrt{\epsilon_1^\star(1-\epsilon_2)}=(1-\sqrt{\epsilon_2})\sqrt{1-\epsilon_1^\star}, \tag{G8}$$

for the region $\epsilon_1^\star\le 1$. This gives the expression of the local minimum as

$$\epsilon_1^\star=\frac{1-\sqrt{\epsilon_2}}{2}. \tag{G9}$$

Plugging this into Eq. (G6) gives

$$g(\epsilon_1^\star,\epsilon_2)=0. \tag{G10}$$

This concludes the proof that $\left(1-\epsilon'\right)-\left(1-\epsilon_1-\sqrt{\epsilon_2}\right)\ge 0$ for all ϵ_1 and ϵ_2 satisfying $\epsilon_1+\sqrt{\epsilon_2}<1$ and, in particular,

$$\log\frac{1}{1-\epsilon'}\le\log\frac{1}{1-\epsilon_1-\sqrt{\epsilon_2}}. \tag{G11}$$

APPENDIX H: PROOF OF LEMMA 9

Proof. Let us write $\Lambda\in\mathbb{O}_{\max}\cap\mathcal{S}(\Phi)$ as

$$\Lambda(\cdot)=\mathrm{Tr}[P\cdot]\Phi+\tilde{\Lambda}(\cdot), \tag{H1}$$

where $\mathrm{Tr}[P\Phi]=1$ and $\tilde{\Lambda}\in\mathrm{CP}$. Let $\sigma\in\mathbb{F}$ be an arbitrary free state. Since $\Lambda\in\mathbb{O}_{\max}$, we obtain

$$\mathrm{Tr}[P\sigma]\Phi+(1-\mathrm{Tr}[P\sigma])\,\tau\in\mathbb{F}, \tag{H2}$$

where we define a state $\tau:=\tilde{\Lambda}(\sigma)/\mathrm{Tr}[\tilde{\Lambda}(\sigma)]$ and use that $\mathrm{Tr}[\tilde{\Lambda}(\sigma)]=1-\mathrm{Tr}[P\sigma]$ because Λ is trace preserving. Since this holds for every free state σ, the first expression of $D_{\max,\mathbb{F}}$ in Eq. (1) implies that

$$\mathrm{Tr}[P\sigma]^{-1}\ge 2^{D_{\max,\mathbb{F}}(\Phi)},\quad\forall\sigma\in\mathbb{F}. \tag{H3}$$

Note that

$$\begin{aligned}2^{D_{\min,\mathbb{F}}(\Phi)}&=\min_{\sigma\in\mathbb{F}}\max_{\substack{0\le Q\le\mathbb{1}\\ \mathrm{Tr}[Q\Phi]=1}}\mathrm{Tr}[Q\sigma]^{-1}\\&\ge\max_{\substack{0\le Q\le\mathbb{1}\\ \mathrm{Tr}[Q\Phi]=1}}\min_{\sigma\in\mathbb{F}}\mathrm{Tr}[Q\sigma]^{-1},\end{aligned} \tag{H4}$$

where the inequality is due to the max-min inequality. (This can be made into an equality by further using the convexity of $\mathbb{F}$, the linearity of the trace, and Sion's minimax theorem—which, however, we do not need here.) These give

$$2^{D_{\min,\mathbb{F}}(\Phi)} \geq \max_{\substack{0\leq Q\leq \mathbb{1}\\ \mathrm{Tr}[Q\Phi]=1}} \min_{\sigma\in\mathbb{F}} \mathrm{Tr}[Q\sigma]^{-1} \geq \min_{\sigma\in\mathbb{F}} \mathrm{Tr}[P\sigma]^{-1} \geq 2^{D_{\max,\mathbb{F}}(\Phi)}, \tag{H5}$$

implying that $D_{\min,\mathbb{F}}(\Phi) \geq D_{\max,\mathbb{F}}(\Phi)$. On the other hand, note that $D_{\min,\mathbb{F}}(\Phi) \leq D_{\max,\mathbb{F}}(\Phi)$ for every Φ because

$$\begin{aligned} D_{\min,\mathbb{F}}(\Phi) &= \inf_{\sigma\in\mathbb{F}} D_{\min}(\Phi\|\sigma), \\ D_{\max,\mathbb{F}}(\Phi) &= \inf_{\sigma\in\mathbb{F}} D_{\max}(\Phi\|\sigma), \\ D_{\min}(\Phi\|\sigma) &\leq D_{\max}(\Phi\|\sigma) \quad \forall\Phi,\sigma. \end{aligned} \tag{H6}$$

These result in $D_{\min,\mathbb{F}}(\Phi) = D_{\max,\mathbb{F}}(\Phi)$. The proof for $D_{s,\mathbb{F}}$ goes analogously.

Furthermore, $\Lambda^{\otimes n}$ takes the form

$$\Lambda^{\otimes n} = \mathrm{Tr}[P^{\otimes n}\cdot]\Phi^{\otimes n} + \Lambda'(\cdot), \tag{H7}$$

where Λ' is another completely positive map. Noting that $0 \leq P^{\otimes n} \leq \mathbb{1}$, $\mathrm{Tr}[P^{\otimes n}\Phi^{\otimes n}] = 1$, and

$$\begin{aligned} \Lambda^{\otimes n} = &(\Lambda\otimes \mathrm{id}\otimes\cdots\otimes \mathrm{id})\circ(\mathrm{id}\otimes\Lambda\otimes\cdots\otimes\mathrm{id}) \\ &\circ\cdots\circ(\mathrm{id}\cdots\otimes\mathrm{id}\otimes\Lambda), \end{aligned} \tag{H8}$$

is in $\mathbb{O}_{\max}$ because Λ is completely free, we obtain $\Lambda^{\otimes n} \in \mathbb{O}_{\max} \cap \mathcal{S}(\Phi^{\otimes n})$. Thus, by using the same argument, we obtain $D_{\min,\mathbb{F}}(\Phi^{\otimes n}) = D_{\max,\mathbb{F}}(\Phi^{\otimes n})$ for every positive integer n. ■

APPENDIX I: PROOF OF PROPOSITION 10

Proof. Let us define the group twirling operation

$$\Xi(\cdot) := \int_G dg\; U(g)\cdot U(g)^\dagger, \tag{I1}$$

where the integral is taken over the Haar measure of the group G. We first show a general form of how the twirling operation acts on an arbitrary state ρ [in particular, Eq. (I11)]. Although Eq. (I11) has already been presented in Ref. [119, Sec. 2.6], here we provide a self-contained explanation for completeness.

An arbitrary unitary representation $\{U(g)\}_{g\in G}$ of a finite or a compact Lie group G can be decomposed into a direct sum of irreducible representations [120]. Accordingly, the Hilbert space on which each $U(g)$ acts admits the following decomposition:

$$\mathcal{H} = \bigoplus_\mu \mathcal{H}_\mu^{(r)} \otimes \mathcal{H}_\mu^{(m)}, \tag{I2}$$

where μ labels the irreducible representations, $\mathcal{H}_\mu^{(r)}$ denotes the subspace on which each irreducible representation acts nontrivially, and $\mathcal{H}_\mu^{(m)}$ denotes the multiplicity subspace. We write the dimensions of $\mathcal{H}_\mu^{(r)}$ and $\mathcal{H}_\mu^{(m)}$ as $d_\mu^{(r)}$ and $d_\mu^{(m)}$, respectively.

Each $U(g)$ can be written as

$$U(g) = \bigoplus_\mu U_\mu^{(r)}(g)\otimes \mathbb{1}_\mu^{(m)}. \tag{I3}$$

Also, Schur's lemma [120] imposes a structure to every *symmetric state* σ satisfying $U(g)\sigma U(g)^\dagger = \sigma\ \forall g$ as

$$\sigma = \bigoplus_\mu q_\mu \frac{\mathbb{1}_\mu^{(r)}}{d_\mu^{(r)}} \otimes \eta_\mu^{(m)}, \tag{I4}$$

where $\{q_\mu\}_\mu$ is a probability distribution, $\eta_\mu^{(m)}$ is a quantum state, and $\mathbb{1}_\mu^{(r)}$ is the projector onto $\mathcal{H}_\mu^{(r)}$.

It is easy to see that every output of the group twirling operation is a symmetric state because, for every $g\in G$,

$$\begin{aligned} U_g\Xi(\rho)U_g^\dagger &= \int dg'\, U(g)U(g')\rho U(g')^\dagger U(g)^\dagger \\ &= \int_G dg'\, U(gg')\rho U(gg')^\dagger \\ &= \int_G dg'\, U(g')\rho U(g')^\dagger \\ &= \Xi(\rho), \end{aligned} \tag{I5}$$

where in the third line we use the left and right invariances of the Haar measure.

Let us also define the projection onto invariant subspaces as

$$\mathcal{P}(\cdot) := \sum_\mu \mathbb{1}_\mu^{(r)}\otimes \mathbb{1}_\mu^{(m)} \cdot \sum_\mu \mathbb{1}_\mu^{(r)}\otimes \mathbb{1}_\mu^{(m)}. \tag{I6}$$

The projection turns every state ρ into the form

$$\mathcal{P}(\rho) = \bigoplus_\mu p_\mu \sigma_\mu^{(r,m)}, \tag{I7}$$

where $\{p_\mu\}_\mu$ is probability distribution and $\sigma_\mu^{(r,m)}$ is a quantum state acting on $\mathcal{H}_\mu^{(r)}\otimes\mathcal{H}_\mu^{(m)}$. Also, the forms of Eqs. (I3) and (I6) imply that $U_g\mathcal{P}(\rho)U_g^\dagger = \mathcal{P}(U_g\rho U_g^\dagger)\ \forall\rho, \forall g\in G$ and hence

$$\begin{aligned}\Xi\circ\mathcal{P}(\rho) &= \mathcal{P}\circ\Xi(\rho)\\ &= \Xi(\rho)\end{aligned}\tag{I8}$$

for every state ρ, where in the second line, we use the fact that Ξ maps every state to a symmetric state, which has the form given in Eq. (I4). Then,

$$\begin{aligned}\Xi(\rho) &= \Xi(\mathcal{P}(\rho))\\ &= \bigoplus_\mu \int_G dg\, p_\mu \left[U_\mu^{(r)}(g)\otimes \mathbb{1}_\mu^{(m)}\right]\sigma_\mu^{(r,m)}\\ &\quad\times\left[U_\mu^{(r)}(g)^\dagger\otimes\mathbb{1}_\mu^{(m)}\right].\end{aligned}\tag{I9}$$

Since $\Xi(\rho)$ is a symmetric state having a structure of Eq. (I4) and $U_\mu^{(r)}$ in Eq. (I9) acts only on $\mathcal{H}_\mu^{(r)}$ nontrivially, we must have

$$\begin{aligned}&\int_G dg\, p_\mu\left[U_\mu^{(r)}(g)\otimes\mathbb{1}_\mu^{(m)}\right]\sigma_\mu^{(r,m)}\left[U_\mu^{(r)}(g)^\dagger\otimes\mathbb{1}_\mu^{(m)}\right]\\ &= p_\mu\frac{\mathbb{1}_\mu^{(r)}}{d_\mu^{(r)}}\otimes\sigma_\mu^{(m)},\quad\forall\mu,\end{aligned}\tag{I10}$$

where $\sigma_\mu^{(m)} := \mathrm{Tr}_r[\sigma_\mu^{(r,m)}]$ and Tr_r denotes the partial trace over the system with $\mathcal{H}_\mu^{(r)}$. Therefore, we obtain

$$\begin{aligned}\Xi(\rho) &= \sum_\mu p_\mu\frac{\mathbb{1}_\mu^{(r)}}{d_\mu^{(r)}}\otimes\sigma_\mu^{(m)}\\ &= \sum_\mu\frac{\mathbb{1}_\mu^{(r)}}{d_\mu^{(r)}}\otimes\mathrm{Tr}_r\left[\mathcal{P}(\rho)\mathbb{1}_\mu^{(r)}\otimes\mathbb{1}_\mu^{(m)}\right]\\ &= \sum_\mu\frac{\mathbb{1}_\mu^{(r)}}{d_\mu^{(r)}}\otimes\mathrm{Tr}_r\left[\rho\mathbb{1}_\mu^{(r)}\otimes\mathbb{1}_\mu^{(m)}\right].\end{aligned}\tag{I11}$$

We now show that this group twirling operation satisfies the condition for the operation that appears in Lemma 9. By assumption, $|\Phi\rangle$ is an eigenvector of U_g with eigenvalue $e^{i\phi_g}$ for every $g\in G$. This implies that $|\Phi\rangle$ is in the invariant subspace of $U(g)$ corresponding to the one-dimensional irreducible representation $\{e^{i\phi_g}\}_{g\in G}$. We assign the label $\mu^\star$ for this representation. Then, every $U(g)$ has the form

$$U(g) = \begin{pmatrix} e^{i\phi_g}\mathbb{1}_{\mu^\star}^{(m)} & \\ & \bigoplus_{\mu\neq\mu^\star}U_\mu^{(r)}(g)\otimes\mathbb{1}_\mu^{(m)}\end{pmatrix},\tag{I12}$$

implying that there exist $d_{\mu^\star}^{(m)}$ simultaneous eigenvectors of all $U(g)$'s with eigenvalues $\{e^{i\phi_g}\}_g$. Using the assumption that $|\Phi\rangle$ is the unique simultaneous eigenvector with eigenvalues $\{e^{i\phi_g}\}_g$, we identify $d_{\mu^\star}^{(m)}=1$. Noting $d_{\mu^\star}^{(r)}=1$ and $\mathbb{1}_{\mu^\star}^{(r)}=\Phi$, we obtain from Eq. (I11) that

$$\Xi(\rho) = \mathrm{Tr}[\rho\Phi]\Phi + \sum_{\mu\neq\mu^\star}\frac{\mathbb{1}_\mu^{(r)}}{d_\mu^{(r)}}\otimes\mathrm{Tr}_r\left[\rho\mathbb{1}_\mu^{(r)}\otimes\mathbb{1}_\mu^{(m)}\right].\tag{I13}$$

The second term is a completely positive map. Also, the convexity of $\mathbb{O}$ and the assumption that $U_g\cdot U_g^\dagger\in\mathbb{O}$ imply that $\Xi\in\mathbb{O}\subseteq\mathbb{O}_{\max}$. Thus, we can apply Lemma 9 to conclude the proof. ■

APPENDIX J: PROOFS OF PROPOSITIONS 11–15

We first show Proposition 11.

Proof of Proposition 11. The face state is the $+1$ eigenstate of the Clifford unitary $K := SH$, which cycles Pauli operators as $X\to Z\to Y$. This implies that one can construct the dephasing map Λ with respect to $|F\rangle$ and the -1 eigenstate $|\overline{F}\rangle$ of K of the form

$$\Lambda(\rho) = \frac{1}{2}\rho+\frac{1}{2}K\rho K^\dagger = \mathrm{Tr}[F\rho]F + \mathrm{Tr}[(\mathbb{1}-F)\rho]\overline{F}.\tag{J1}$$

Since this is realized by a probabilistic application of the Clifford gate K, it is a stabilizer operation and thus $\Lambda\in\mathbb{O}_{\rm all}$. We obtain the statement by applying Lemma 9 or, alternatively, Proposition 10. ■

This idea can be employed to prove Proposition 12.

Proof of Proposition 12. Consider a t-qudit system with local dimension d. Since ϕ is a stabilizer state, there exists a Clifford unitary U such that $|\phi\rangle = U|0\rangle^{\otimes t}$, where $|0\rangle$ is the $+1$ eigenstate of $\{Z^j\}_{j=0}^{d-1}$ with $Z := \sum_{j=0}^{d-1}e^{i2\pi j/d}|j\rangle\langle j|$. Since U is Clifford, ϕ is the $+1$ eigenvalue of the $t(d-1)$ generalized Pauli operators $\left\{U^\dagger Z_k^j U\right\}_{k,j}$, where Z_k is the Z operator that only acts on the kth qudit and acts trivially on the other qudits. Since V is in the third level of the Clifford hierarchy, ψ is the unique $+1$ eigenstate of $t(d-1)$ Clifford unitaries $\left\{V^\dagger U^\dagger Z_k^j UV\right\}_{k,j}$. Let $W_{jk} := V^\dagger U^\dagger Z_k^j UV$ and $|\psi_{\vec z}\rangle := VU|\vec z\rangle$ with $\vec z\in\{0,\ldots,d-1\}^t$ be another eigenstate of W_{jk} parametrized by $\vec z$, satisfying $W_{jk}|\psi_{\vec z}\rangle = e^{i2\pi j z_k/d}|\psi_{\vec z}\rangle$. The uniformly random application of Clifford unitaries $\left\{\prod_{k=1}^t W_{j_k k}\right\}_{j_1\ldots j_t}$ works as a dephasing among

$\{\psi_{\vec{z}}\}_{\vec{z}}$ because

$$\begin{aligned}
&\frac{1}{d^t}\sum_{j_1=0}^{d-1}\cdots\sum_{j_t=0}^{d-1}\left(\prod_{k=1}^{t} W_{j_k k}\right)|\psi_{\vec{z}}\rangle\langle\psi_{\vec{z}'}|\left(\prod_{k=1}^{t} W_{j_k k}^{\dagger}\right)\\
&= |\psi_{\vec{z}}\rangle\langle\psi_{\vec{z}'}|\frac{1}{d^t}\sum_{j_1,\ldots,j_t}\prod_{k=1}^{t} e^{i2\pi j_k(z_k - z_k')/d}\\
&= |\psi_{\vec{z}}\rangle\langle\psi_{\vec{z}'}|\prod_{k=1}^{t}\left(\frac{1}{d}\sum_{j=0}^{d-1} e^{i2\pi j(z_k - z_k')/d}\right)\\
&= |\psi_{\vec{z}}\rangle\langle\psi_{\vec{z}'}|\,\delta_{\vec{z}\vec{z}'}. \qquad (J2)
\end{aligned}$$

Thus, it acts as a projector onto $\{\psi_{\vec{z}}\}_{\vec{z}}$ as

$$\frac{1}{d^t}\sum_{j_1\ldots j_t}^{d-1}\left(\prod_{k=1}^{t} W_{j_k k}\right)\cdot\left(\prod_{k=1}^{t} W_{j_k k}^{\dagger}\right) = \sum_{\vec{z}} |\psi_{\vec{z}}\rangle\langle\psi_{\vec{z}}|\cdot|\psi_{\vec{z}}\rangle\langle\psi_{\vec{z}}|. \qquad (J3)$$

Importantly, this is a free operation because any $\prod_{k=1}^{t} W_{j_k k}$ is a Clifford unitary. On the other hand, this is clearly in $\mathcal{S}(\psi)$ and thus we can apply Lemma 9 or, alternatively, Proposition 10. ■

Next, we prove Proposition 13.

Proof of Proposition 13. The crucial property of the Hoggar state is that it has a completely flat representation in the Pauli basis, in the sense that

$$|\langle \mathrm{Hog}|P|\mathrm{Hog}\rangle| = \frac{1}{3}, \qquad (J4)$$

for every nontrivial Pauli operator $P \neq \mathbb{1}$.

On the one hand, we know that Hog is a maximizer of $D_{s,\mathbb{F}_{\mathrm{stab}}}$ with $D_{s,\mathbb{F}_{\mathrm{stab}}}(\mathrm{Hog}) = \log\frac{12}{5}$ [77]. On the other hand, we consider the stabilizer norm [121], defined for every n-qubit operator A as

$$\|A\|_{\mathrm{st}} := \frac{1}{2^n}\sum_{P\in\mathcal{P}} |\mathrm{Tr}(A\,P)|\,, \qquad (J5)$$

where $\mathcal{P}$ denotes all n-qubit Pauli operators. Crucially, since any stabilizer state satisfies $\|\sigma\|_{\mathrm{st}} \leq 1$, we can define the set

$$\mathbb{F}_{\mathcal{P}} := \left\{\sigma \;\middle|\; \sigma \geq 0,\ \mathrm{Tr}\,\sigma = 1,\ \|\sigma\|_{\mathrm{st}} \leq 1\right\} \supseteq \mathbb{F}_{\mathrm{stab}}. \qquad (J6)$$

Then $D_{\min,\mathbb{F}_{\mathrm{stab}}}(\rho) \geq D_{\min,\mathbb{F}_{\mathcal{P}}}(\rho)$ for every state ρ.

Then take the Hoggar state, which can be written as

$$|\mathrm{Hog}\rangle\langle\mathrm{Hog}| = \frac{1}{2^3}\left[\mathbb{1} + \sum_{P\in\mathcal{P}\setminus\{\mathbb{1}\}}\frac{1}{3}e^{i\phi_P}P\right], \qquad (J7)$$

and consider any state $\sigma \in \mathbb{F}_{\mathcal{P}}$, which can always be written as

$$\sigma = \frac{1}{2^3}\left[\mathbb{1} + \sum_{P\in\mathcal{P}\setminus\{\mathbb{1}\}} z_P P\right], \qquad (J8)$$

with $\sum_P |z_P| \leq 2^3 - 1$ since $\|\sigma\|_{\mathrm{st}} \leq 1$. Then,

$$\begin{aligned}
2^{-D_{\min,\mathbb{F}_{\mathcal{P}}}(\mathrm{Hog})} &\leq \langle\mathrm{Hog}|\sigma|\mathrm{Hog}\rangle\\
&\leq \frac{1}{2^3}\left[1 + \sum_{P\in\mathcal{P}\setminus\{\mathbb{1}\}}\frac{1}{3}e^{i\phi_P}z_P\right]\\
&\leq \frac{1}{2^3}\left[1 + \frac{1}{3}\left(2^3 - 1\right)\right]\\
&= \frac{1}{8} + \frac{7}{24}\\
&= \frac{5}{12}. \qquad (J9)
\end{aligned}$$

Using the fact that $D_{s,\mathbb{F}_{\mathrm{stab}}}(\mathrm{Hog}) \geq D_{\max,\mathbb{F}_{\mathrm{stab}}}(\mathrm{Hog}) \geq D_{\min,\mathbb{F}_{\mathrm{stab}}}(\mathrm{Hog}) \geq D_{\min,\mathbb{F}_{\mathcal{P}}}(\mathrm{Hog})$ concludes the proof of the first part of the result.

To show the second part of the result, note that the Hoggar state $|\mathrm{Hog}\rangle$ is the $+1$ eigenvector of all unitaries in the group generated by two unitaries with order 7 and order 12 [104], defined as

$$\tilde{U}_7 := \frac{\omega^5}{\sqrt{2}}\begin{pmatrix}
0 & 0 & 1 & 0 & -i & 0 & 0 & 0\\
0 & 0 & i & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & -i & 0 & -1 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & -i & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & -i & 0\\
-i & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & -i & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & i
\end{pmatrix}, \qquad (J10)$$

$$\tilde{U}_{12} := \frac{\omega^3}{\sqrt{2}}\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & i & 0 & 0\\
0 & 0 & 0 & 0 & -1 & i & 0 & 0\\
1 & -i & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & -i & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & i & 0 & 0 & 0 & 0\\
0 & 0 & -1 & i & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -i
\end{pmatrix}, \qquad (J11)$$

where $\omega = e^{2\pi i/8}$. Let us call this group G and consider the representation $\{U_g\}_{g\in G}$ that has the identical form to the element of the unitary group G. As one can explicitly check, $\tilde{U}_{12}$ has a one-dimensional $+1$-eigenspace with the unique $+1$-eigenvector $|\mathrm{Hog}\rangle$, meaning that the Hoggar state is the unique simultaneous $+1$-eigenvector of all unitaries $\{U_g\}_{g\in G}$.

Since $\tilde{U}_7$ and $\tilde{U}_{12}$ are Clifford unitaries, G is a subgroup of the three-qubit Clifford group and hence $U_g \in \mathbb{O}_{\rm stab}$ $\forall g \in G$. Noting also that $\mathbb{O}_{\rm stab}$ is completely free [113], we can apply Proposition 10 to conclude the proof. ■

We now focus on qutrit states. First, we show Proposition 14.

Proof of Proposition 14. The Strange state is a fiducial state for a SIC POVM for dimension 3. Recall that a set $\{|\psi_j\rangle\langle\psi_j|\}_{j=0}^{d^2-1}$ of projectors is called a SIC POVM if $(1/d)\sum_j |\psi_j\rangle\langle\psi_j| = \mathbb{1}$ and $|\langle\psi_i|\psi_j\rangle|^2 = (d\delta_{ij}+1)/d+1$. Moreover, a SIC POVM is called covariant with group G if any state $|\psi_i\rangle$ in the SIC POVM can be constructed by applying some unitary representation $U_{g_i}, g_i \in G$ to a fiducial state $|\psi_0\rangle$ as $U_{g_i}|\psi_0\rangle = |\psi_i\rangle$. The SIC POVM generated from the Strange state is covariant with respect to the Heisenberg-Weyl group and the Strange state is stabilized by a subset of the Clifford group that is isomorphic to the special linear group $\mathrm{SL}(2,\mathbb{Z}_3)$, which is the set of matrices with unit determinant the entries of which are over $\mathbb{Z}_3$ [105]. It has been shown that there is a one-to-one correspondence between any $F \in \mathrm{SL}(2,\mathbb{Z}_3)$ and a Clifford unitary U_F (up to global phase) acting as

$$U_F D_{\mathbf{k}} U_F^\dagger = D_{F\mathbf{k}}, \tag{J12}$$

where $D_{\mathbf{k}} = D_{k_1,k_2} = -e^{i\pi/d}X^{k_1}Z^{k_2}$ [105,106]. Noting that the projectors in the SIC POVM generated from the Strange state can be parametrized by $\mathbf{k}$ as $|\psi_{\mathbf{k}}\rangle\langle\psi_{\mathbf{k}}| = D_{\mathbf{k}}|S\rangle\langle S|D_{\mathbf{k}}^\dagger$ and

$$\begin{aligned}U_F|\psi_{\mathbf{k}}\rangle\langle\psi_{\mathbf{k}}|U_F &= U_F D_{\mathbf{k}} U_F^\dagger |S\rangle\langle S| U_F D_{\mathbf{k}}^\dagger U_F^\dagger\\ &= |\psi_{F\mathbf{k}}\rangle\langle\psi_{F\mathbf{k}}|,\end{aligned} \tag{J13}$$

where we use that $U_F|S\rangle = |S\rangle$, we obtain that, for $\mathbf{k} \neq \mathbf{0}$,

$$\begin{aligned}\Lambda(\psi_{\mathbf{k}}) &:= \frac{1}{|\mathrm{SL}(2,\mathbb{Z}_3)|}\sum_{F\in\mathrm{SL}(2,\mathbb{Z}_3)} U_F\psi_{\mathbf{k}}U_F^\dagger\\ &= \frac{1}{8}\sum_{\mathbf{k}'\neq\mathbf{0}}\psi_{\mathbf{k}'}\\ &= \frac{3\mathbb{1}-|S\rangle\langle S|}{8}.\end{aligned} \tag{J14}$$

In the third equality, we use the fact that $\{\frac{1}{3}\psi_{\mathbf{k}}\}_{\mathbf{k}}$ constitutes a POVM and thus $\sum_{\mathbf{k}}\psi_{\mathbf{k}} = 3\mathbb{1}$. The second equality follows from the following observation. The set of Clifford unitaries of the form (J12) is the collection of all possible mappings from a Pauli operator to another Pauli operator. This means that a certain given nonidentity Pauli operator $D_{\mathbf{k}}$ can be mapped to an arbitrary nonidentity Pauli operator $D_{\mathbf{k}'}$ by some U_F. Let $\{U^{(j)}_{\mathbf{k}\to\mathbf{k}'}\}_{j=1}^{N_{\mathbf{k}'}}$ be the set of Clifford unitaries that map $D_{\mathbf{k}}$ to $D_{\mathbf{k}'}$ as in Eq. (J12), where $N_{\mathbf{k}'}$ is the number of such Clifford unitaries, and set that $U^{(j_1)}_{\mathbf{k}\to\mathbf{k}'} \neq U^{(j_2)}_{\mathbf{k}\to\mathbf{k}'}$ for $j_1 \neq j_2$. The second equality of Eq. (J14) is equivalent to showing that $N_{\mathbf{k}'}$ takes the same value for all $\mathbf{k}' \neq \mathbf{0}$. Suppose, to the contrary, that there exists $\mathbf{k}'$ such that $N_{\mathbf{k}'} > N_{\mathbf{k}''}$ for all $\mathbf{k}'' \neq \mathbf{k}'$. Pick an arbitrary $\mathbf{k}''$ with $\mathbf{k}'' \neq \mathbf{k}'$ and let $\{U_{\mathbf{k}'\to\mathbf{k}''}\}_{\mathbf{k}''\neq\mathbf{0}}$ be a fixed set of unitaries that satisfies

$$U_{\mathbf{k}'\to\mathbf{k}''}D_{\mathbf{k}'}U^\dagger_{\mathbf{k}'\to\mathbf{k}''} = D_{\mathbf{k}''}. \tag{J15}$$

This implies that for every j, $U_{\mathbf{k}'\to\mathbf{k}''}U^{(j)}_{\mathbf{k}\to\mathbf{k}'}$ maps $D_{\mathbf{k}}$ to $D_{\mathbf{k}''}$ and thus is a member of the set $\{U^{(l)}_{\mathbf{k}\to\mathbf{k}''}\}_{l=1}^{N_{\mathbf{k}''}}$. Moreover, we have

$$U_{\mathbf{k}'\to\mathbf{k}''}U^{(j_1)}_{\mathbf{k}\to\mathbf{k}'} \neq U_{\mathbf{k}'\to\mathbf{k}''}U^{(j_2)}_{\mathbf{k}\to\mathbf{k}'}, \quad \forall j_1 \neq j_2, \tag{J16}$$

because otherwise it would result in $U^{(j_1)}_{\mathbf{k}\to\mathbf{k}'} = U^{(j_2)}_{\mathbf{k}\to\mathbf{k}'}$, violating the assumption. This implies that for every $U^{(j)}_{\mathbf{k}\to\mathbf{k}'}$, we can construct a corresponding element in $\{U^{(l)}_{\mathbf{k}\to\mathbf{k}''}\}_{l=1}^{N_{\mathbf{k}''}}$, which are distinct from each other for different values of j. This shows that we must at least have $N_{\mathbf{k}''} \geq N_{\mathbf{k}'}$ but this violates the assumption that $N_{\mathbf{k}'} > N_{\mathbf{k}''}$. This concludes the proof that $N_{\mathbf{k}'}$ takes the same value for all $\mathbf{k}' \neq \mathbf{0}$, leading to the second equality in Eq. (J14).

On the other hand, for $\mathbf{k} = \mathbf{0}$, we obtain $\Lambda(\psi_{\mathbf{0}}) = \Lambda(S) = S$. Since $\{\frac{1}{3}\psi_{\mathbf{k}}\}_{\mathbf{k}}$ is informationally complete, any state ρ can be expanded as

$$\rho = \sum_{\mathbf{k}} c_{\mathbf{k}}\psi_{\mathbf{k}}. \tag{J17}$$

This gives an action of Λ for an arbitrary state ρ as

$$\begin{aligned}\Lambda(\rho) &= c_{\mathbf{0}}S + (1-c_{\mathbf{0}})\frac{3\mathbb{1}-S}{8}\\ &= (1-\epsilon_\rho)S + \epsilon_\rho\frac{\mathbb{1}-S}{2},\end{aligned} \tag{J18}$$

where we set $\epsilon_\rho := [3(1-c_{\mathbf{0}})]/4$. In order for Λ to be linear in ρ, ϵ_ρ must have the form $\epsilon_\rho = \mathrm{Tr}[H\rho]$ for some Hermitian operator H. Combining this with the conditions $\Lambda(S) = S$ and $\Lambda((\mathbb{1}-S)/2) = (\mathbb{1}-S)/2$, we obtain

$$\Lambda(\rho) = \mathrm{Tr}[S\rho]S + \mathrm{Tr}[(\mathbb{1}-S)\rho]\frac{\mathbb{1}-S}{2}, \tag{J19}$$

showing that $\Lambda \in \mathcal{S}(S)$.

Moreover, $(\mathbb{1}-S)/2$ is a stabilizer state because $\Lambda(|0\rangle\langle 0|) = (\mathbb{1}-S)/2$, while $|0\rangle$ is a stabilizer state and Λ is a stabilizer operation. Indeed, we explicitly find

$$\frac{\mathbb{1}-S}{2} = \frac{1}{4}(|0\rangle\langle 0| + |x\rangle\langle x| + |xz\rangle\langle xz| + |xz^2\rangle\langle xz^2|), \tag{J20}$$

where $|x\rangle, |xz\rangle, |xz^2\rangle$ are the eigenstates of X with eigenvalue $+1$, XZ with eigenvalue $e^{i2\pi/3}$, and XZ^2 with eigenvalue $e^{-i2\pi/3}$, respectively, defined as

$$\begin{aligned} |x\rangle &:= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \\ |xz\rangle &:= \frac{1}{\sqrt{3}}(|0\rangle + e^{-i2\pi/3}|1\rangle + e^{-i2\pi/3}|2\rangle) \\ |xz^2\rangle &:= \frac{1}{\sqrt{3}}(|0\rangle + e^{i2\pi/3}|1\rangle + e^{i2\pi/3}|2\rangle). \end{aligned} \tag{J21}$$

These allow us to apply Lemma 9 to obtain the statement. The value for $D_{\min,\mathbb{F}_{\rm stab}}(S)$ can be explicitly calculated by computing the overlaps between the Strange state and all the pure stabilizer states and taking the logarithm of the inverse of the maximum overlap. ∎

Finally, we show Proposition 15.

Proof of Proposition 15. $D_{s,\mathbb{F}_{\rm stab}}(N) \le \log\frac{3}{2}$ is obtained by noting that

$$|N\rangle\langle N| + \frac{1}{2}|+\rangle\langle +| = \frac{1}{2}(|1\rangle\langle 1| + |xz'\rangle\langle xz'| + |xzz'\rangle\langle xzz'|), \tag{J22}$$

where

$$\begin{aligned} |+\rangle &:= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \\ |xz'\rangle &:= \frac{1}{\sqrt{3}}(|0\rangle + e^{i2\pi/3}|1\rangle + |2\rangle) \\ |xzz'\rangle &:= \frac{1}{\sqrt{3}}(|0\rangle + e^{-i2\pi/3}|1\rangle + |2\rangle) \end{aligned} \tag{J23}$$

are stabilizer states. On the other hand, the optimization in $D_{\min,\mathbb{F}_{\rm stab}}$ is achieved by

$$D_{\min,\mathbb{F}_{\rm stab}}(N) = \log\langle N|xzz'\rangle^{-2} = \log\frac{3}{2}. \tag{J24}$$

Use of the inequalities $D_{\min,\mathbb{F}_{\rm stab}}(N) \le D_{\max,\mathbb{F}_{\rm stab}}(N) \le D_{s,\mathbb{F}_{\rm stab}}(N)$ concludes the proof. ∎

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