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Forward and backward fuzzy rule base interpolation using fuzzy geometry

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Abstract

Fuzzy rule interpolation (FRI) predicts an accountable outcome of a possible course of action in sparse fuzzy rule base system (FRBS). The geometry based linear fuzzy rule interpolation (GLFRI) is extended for multi-dimensional fuzzy rule base interpolation. Expansion/contraction (EC) of triangular, trapezoidal and complex polygonal fuzzy sets has been also proposed which enables the proposed FRI method to incorporate with fuzzy rules which include triangular, trapezoidal, hexagonal or complex fuzzy sets. The study further extends to introduce the process of backward rule base interpolation. It has been shown that the scale and move transformation-based FRI method can yield a non-convex fuzzy consequent which can be avoided by using the proposed method. The proposed method performs better without any risk of obtaining non-convex fuzzy consequent. The efficiency of proposed forward and backward FRI methods is projected with several numerical examples. A detailed comparison of EC transformation with scale and move transformation is also presented here.

Keywords: Inverse rule base interpolation, scale and move transformation, transformation of fuzzy point, translation of fuzzy number, multi-dimensional rule base interpolation.

1 Introduction

Fuzzy rule-base interpolation (FRI) is becoming relevant more than ever with the growing demand of explainability in artificial intelligent. A recent study [34] discuss how FRI can play a crucial role on the explainability of artificial intelligent.

Fuzzy rule base consists fuzzy rules $\widetilde{R}_i : \widetilde{A}_i \to \widetilde{B}_i$ which are implication of known antecedents or inputs $(\widetilde{A}_i)$ to consequent or outputs. FRI finds the consequent for an antecedent, commonly known as observation which is not known from FRBS.

Bellman and Zadeh [5] introduced the fundamental concepts of fuzzy reasoning in 1970 and Zadeh [41] used it in decision making process of a complex system in 1973. Later on, Mamdani and Assilian [33], Tsukamoto [40] and Sugeno and Takagi [39] made further contribution on fuzzy reasoning in decision making process.

All those above mentioned methods determine the desired consequent or outcome by rule matching scale which fails to produce any outcome if the upcoming observation has negligible or zero match with the given FRBS. This type of FRBS is called sparse. Kóczy an Hirota [25] introduced the first FRI technique for sparse FRBS in 1993. In the subsequent years, the KH method has been improved and generalized by several researchers in [26, 27, 28]. This methods generally obtain the desired outcome in one step by using the fundamental fuzzy interpolation rule: $d(\widetilde{A}^*, \widetilde{A}_1) : d(\widetilde{A}^*, \widetilde{A}_2) = d(\widetilde{B}^*, \widetilde{B}_1) : d(\widetilde{B}^*, \widetilde{B}_2)$ given in [25]. Here, B^* is the desired outcome for an observation $\widetilde{A}^*$. This type of FRI methods are categorized as first group of methods. Other methods which can be put into this first groups are [4, 13, 14].

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The FRI methodologies of second group obtain the desired consequent conclusion in two steps. In the first step, an auxiliary rule is obtained from the given FRBS and the conclusion is drawn using that auxiliary rule. Solid cutting method [3, 2], fixed point law, fixed value law [35, 38], least square method [18] and polar cut method [19] are among few of the methodologies which belong to second group.

In real life, some practical situations demand a desired consequent from a given FRBS. Obtaining the suitable course of action or antecedents in order to achieve the desired consequent is known as inverse fuzzy rule interpolation (IFRI) or backward fuzzy rule interpolation (BFRI). Baranyi et al. [1] introduced the first inverse interpolation process which is two step process. In the first step, an inverse rule base (IRB) is constructed from the given rule base. In the second step, required missing antecedent is obtained by using the driven IRB. Backward fuzzy rule interpolation (BFRI) has been proposed by Jin et al. [15] based on scale and move transformation. An α-cut based BFRI were introduced with the help of scale and move (SM) transformation by Jin et al. [16] in 2014. A comprehensive study on forward and backward interpolation based on SM transformation can be found in [17].

The SM transformation based FRI method was introduced by Huang and Shen [22]. This method belong to the second group of FRI technique [23, 24]. In the second step of SM, the desired outcome is driven from the auxiliary rule using scale and move transformation [8, 23, 24, 42]. The scale transformation changes the support length of fuzzy number while keeping the representative value same. The move transformation changes the starting and final position of fuzzy sets while keeping its support length same. The SM transformation is defined on complex polygonal fuzzy sets which includes all triangular, trapezoidal, hexagonal and Gaussian membership function.

Over the years FRI and BFRI have been used extensively in multi-class classification [43], approximate reasoning [32], inference mechanism [31], prediction of terrorist bombing threats [15], mammographic mass classification [30] and many other fields [6, 20, 21, 29, 36, 37]. SM transformation based FRI technique is mainly use in these recent studies [15, 30, 31, 43]. In this study, we have shown that there is a possibility that SM transformation based FRI method fails to provide a convex and interpretable fuzzy set as output. A new EC based approach is proposed as an alternative which does not have the prior limitations. The geometry based multi-dimensional fuzzy rule base interpolation (GMFRI) [11] provided a complete visualization of FRI in geometrical plane. Expansion/contraction (EC) of fuzzy numbers and points are used in GMFRI method. The EC transformation of fuzzy numbers alters the support lengths without changing its core [7, 9, 10, 11, 12]. In other words, in EC transformation core of fuzzy number were used as its representative value. The GMFRI method were introduced with rules having single dimensional antecedent and consequent. The rules involved in GMFRI have triangular fuzzy numbers (TFN) or Gaussian fuzzy number in its antecedent and consequent [11]. In the present study, the EC transformation has been re-defined for fuzzy numbers having polygonal membership function which covers TFN, trapezoidal fuzzy number (TrFN), hexagonal fuzzy number (HFN). Using the redefined EC transformation, a multi-dimensional FRI method has been developed which is compatible with rules having multi-dimensional antecedents. Also, a method for inverse or backward FRI has been proposed. A detail comparison between SM and EC transformation is also illustrated in the following. The overall of contribution of the paper can be summarized as follows:

- The GMFRI has been extended for multi-dimensional fuzzy rule base interpolation which can incorporate fuzzy rules including triangular, trapezoidal, hexagonal or complex fuzzy sets.
- Based on the same geometrical principle a backward rule base interpolation is also proposed.
- A detailed comparison between SM transformation and EC transformation is presented.
- It has been discussed that SM transformation based method inherits chances for failure to provide any interpretable fuzzy output whereas proposed method is robust and free from such volatility.

In the next section, an overview on existing forward and backward FRI is given. Section 3 provides a complete study on transformation of fuzzy numbers and fuzzy points. The comparison of expansion/contraction with scale and move transformation are also presented in Section 3. The forward FRI method for multiple antecedents and single output is proposed in Section 4. In Section 5, an inverse FRI technique is presented. Both of Sections 4 and 5 present workout examples to illustrate of the proposed methods. Finally, Section 6 concludes the work.

2 Overview of fuzzy rule base interpolation

The simplest case of FRI consist two rules with single TFN antecedent and consequent is considered to illustrate the fundamental idea of FRI. Suppose, two rules $\widetilde{R}_1 :\ X = \widetilde{A}_1 \Rightarrow Y = \widetilde{B}_1$ and $\widetilde{R}_2 :\ X = \widetilde{A}_2 \Rightarrow Y = \widetilde{B}_2$ are known and a conclusion or consequent has to be determined for an observation $\widetilde{O}^* :\ X = \widetilde{A}^* \Rightarrow Y = ?$. It is also assumed that $\widetilde{R}_1$ and

$\widetilde{R}_2$ are two adjacent rules of the observation $\widetilde{A}^*$, i.e. $Rep(\widetilde{A}_1) \le Rep(\widetilde{A}) \le Rep(\widetilde{A}_2)$ or $Rep(\widetilde{A}_2) \le Rep(\widetilde{A}) \le Rep(\widetilde{A}_1)$ where $Rep(\widetilde{A})$ is representative value of $\widetilde{A}$. The study of Huang and Shen [23] used the center of gravity as representative value, i.e. $Rep(\widetilde{A}) = \frac{a_0+a_1+a_2}{3}$ whereas Das et. al [11] used core of $\widetilde{A}$ as $Rep(\widetilde{A}) = a_1$. In literature, such as [22, 23, 15] use of other representative values, such as $Rep(\widetilde{A}) = \frac{a_0+a_1+a_2}{3}$ or $Rep(\widetilde{A}) = \frac{a_0+2*a_1+a_2}{3}$ may be found. In this study, we have used core of fuzzy number as the representative value $Rep(\widetilde{A}) = a_1$.

The auxiliary rule $\widetilde{R}$ is constructed by the formula $\widetilde{R} = (1-\lambda)\widetilde{R}_1 + \lambda\widetilde{R}_2 = (1-\lambda)\widetilde{A}_1 + \lambda\widetilde{A}_2 \Rightarrow (1-\lambda)\widetilde{B}_1 + \lambda\widetilde{B}_2$ where $\lambda = \frac{d(\widetilde{A}_1,\widetilde{A}^*)}{d(\widetilde{A}_1,\widetilde{A}_2)}$. The distance $d(\widetilde{A}_1, \widetilde{A}^*)$ between two fuzzy sets is defined as $d(\widetilde{A}_1, \widetilde{A}_2) = |Rep(\widetilde{A}_2) - Rep(\widetilde{A}_1)|$.

In the next step, the desired conclusion B^* (say) is determined from the auxiliary rule $\widetilde{R}: \widetilde{A} \Rightarrow \widetilde{B}$ where $\widetilde{A} = (1-\lambda)\widetilde{A}_1 + \lambda\widetilde{A}_2$ and $\widetilde{B} = (1-\lambda)\widetilde{B}_1 + \lambda\widetilde{B}_2$.

The transformation of $\widetilde{B}$ to B^* depends on the required transformation needed to match the auxiliary observation $\widetilde{A}$ with the given observation $\widetilde{A}^*$. Huang and Shen [23] used the scale and move (SM) transformation in this regard whereas Das et al. [11] used expansion/contraction (EC) operation. Both of SM and EC transformation are illustrated in the following.

The scale and move transformations are used to transform a fuzzy set $\widetilde{A}$ in another fuzzy set $\widetilde{B}$ through two intermediate fuzzy sets $\widetilde{A}^s$ and $\widetilde{A}^m$ (say) respectively.

2.1 Scale and move transformation of fuzzy number

The SM transformation has been defined for fuzzy numbers in the shape of triangular, trapezoidal, hexagonal, polygonal. Here, the SM transformation is briefly described for triangular fuzzy numbers (TFN) and compared with proposed expansion/contraction (EC) transformation. The transformation of $\widetilde{A}$ into $\widetilde{B}$ followed by two successive operations which are scale and move transformations respectively.

Scale Transformation: In the scale transformation, the fuzzy set $\widetilde{A}$ transformed into another fuzzy set $\widetilde{B}$ (say) in such a way that support length of $\widetilde{A}^s$ matches with the support length of $\widetilde{B}$. Other two conditions of this process are $\widetilde{A}$ and $\widetilde{A}^s$ have the same representative values and same slope ratios.

Suppose, $\widetilde{A}$ and $\widetilde{B}$ are two triangular fuzzy numbers $\widetilde{A}(a_0, a_1, a_2)$, $\widetilde{B}(b_0, b_1, b_2)$. Then, the TFN $\widetilde{A}^s(a_0^s, a_1^s, a_2^s)$ after scale transformation of TFN $\widetilde{A}$ is obtained by solving these three conditions *i)* $a_1^s = a_1$; *ii)* $a_2^s - a_0^s = S(a_2 - a_0)$ and *iii)* $\frac{a_1^s - a_0^s}{a_2^s - a_1^s} = \frac{a_1 - a_0}{a_2 - a_1}$. The scale parameter S is obtained from the formula: $S = \frac{b_2 - b_0}{a_2 - a_0}$.

So, after scale transformation, a new fuzzy number $\widetilde{A}^s$ is obtained which have same support length and representative value with $\widetilde{B}$. But, it might happen that $\widetilde{A}^s$ and $\widetilde{B}$ do not have same start and finish point, i.e. $a_0^s \ne b_0$ and $a_2^s \ne b_2$. So, this desired result is obtained through move transformation of $\widetilde{A}^s$.

Move Transformation: In move transformation, $\widetilde{A}^s$ is transformed into another fuzzy number $\widetilde{A}^m$ (say) in such a way that $\widetilde{A}^m$ and $\widetilde{B}$ have same the initial points. The other two criteria which are preserved in move transformation are, $\widetilde{A}^m$ and $\widetilde{A}^s$ that must have the same support length and the same representative value.

Suppose, $\widetilde{A}$ and $\widetilde{B}$ are two triangular fuzzy numbers $\widetilde{A}(a_0, a_1, a_2)$, $\widetilde{B}(b_0, b_1, b_2)$ respectively and $\widetilde{A}^s(a_0^s, a_1^s, a_2^s)$ is obtained from scale transformation. Then, $\widetilde{A}^m(a_0^m, a_1^m, a_2^m)$ after scale transformation of $\widetilde{A}^s$ is obtained by solving these three conditions *i)* $a_1^m = a_1^s$; *ii)* $a_2^m - a_0^m = a_2^s - a_0^s$ and *iii)* $a_0^m = a_0^s + m$ where the move parameter m between $\widetilde{A}^s$ and $\widetilde{B}$ is calculated as $m = b_0 - a_0^s$.

One more feature of move transformation of TFN needs to be addressed. It might happen that the $\widetilde{A}^m$ obtained through move transformation is non-convex when the center of gravity, i.e. $Rep(\widetilde{A}) = \frac{a_0+a_1+a_2}{3}$ is consider as representative value (see section-III [23]). But, if we consider $Rep(\widetilde{A}) = a_1$ then the $\widetilde{A}^m$ is always convex since $a_0^m = a_0^s + m = b_0 \le b_1 = a_1^m$. The obtained TFN $\widetilde{A}^m$ is expected to be identical with $\widetilde{B}$. The SM transformation is illustrated further in the following example.

Example 2.1. *Suppose, a TFN $\widetilde{A}(2,4,5)$ has to be transformed into another TFN $\widetilde{B}(3,4,7)$ (see figure 1).*

Scale Transformation: *In this step, the TFN $\widetilde{A}(2,4,5)$ is transformed into another TFN $\widetilde{A}^s(a_0^s, a_1^s, a_2^s)$ such that*

- *The cores of $\widetilde{A}$ and $\widetilde{A}^s$ remain the same, i.e. $a_1^s = a_1 = 4$.*
- *The ratios of the left support and right support remain the same i.e., $\frac{a_1^s - a_0^s}{a_2^s - a_1^s} = \frac{a_1 - a_0}{a_2 - a_1} = \frac{4-2}{5-4}$.*
- *The support length changes to new length, i.e., $a_2^s - a_0^s = S(a_2 - a_0) = \frac{4}{3} \times 3$.*

Applying above conditions one may obtain the three equations: i) $a_1^s = a_1 = 4$ ii) $2a_2^s + a_0^s = 12$ and iii) $a_2^s - a_0^s = 4$. Solving the above equations, the TFN $\widetilde{A}^s(a_0^s, a_1^s, a_2^s)$ is obtained as $\widetilde{A}^s(\frac{4}{3}, 4, \frac{16}{3})$ (see Figure 2).

Here, the fuzzy numbers $\widetilde{A}^s(\frac{4}{3}, 4, \frac{16}{3})$ and $\widetilde{B}(3,4,7)$ have the same representative value $a_1^s = a_1 = 4$ and the same support length $a_2^s - a_0^s = b_2 - b_0 = 4$. But, $\widetilde{A}^s$ and $\widetilde{B}$ are not identical since they do not have the same initial and final points. This is achieved in the following move transformation.

Move Transformation: *In this step, the TFN $\widetilde{A}^s(\frac{4}{3}, 4, \frac{16}{3})$ is transformed into $\widetilde{A}^m(a_0^m, a_1^m, a_2^m)$ in such way that $\widetilde{A}^m$ and $\widetilde{B}(3,4,7)$ become identical. The following steps are followed in move transformation:*

- *The cores of $\widetilde{A}^s$ and $\widetilde{A}^m$ remain the same, i.e. $a_1^m = a_1^s = 4$.*
- *The starting point of $\widetilde{A}^m$ and $\widetilde{B}$ should match, i.e., $a_0^m = b_0 = 3$.*
- *The support lengths of $\widetilde{A}^m$ and $\widetilde{A}^s$ should remain the same, i.e., $a_2^m - a_0^m = a_2^s - a_0^s = 4$.*

Applying above three conditions, one may obtain the equations i) $a_1^m = a_1^s = 4$ ii) $a_0^m = b_0 = 3$ and iii) $a_2^m - a_0^m = 4$. Thus, the TFN $\widetilde{A}^m(3,4,7)$ is obtained by solving these equations as $\widetilde{A}^m(3,4,7)$. It can be observed that $\widetilde{A}^m$ and $\widetilde{B}$ are identical (see figure 3).

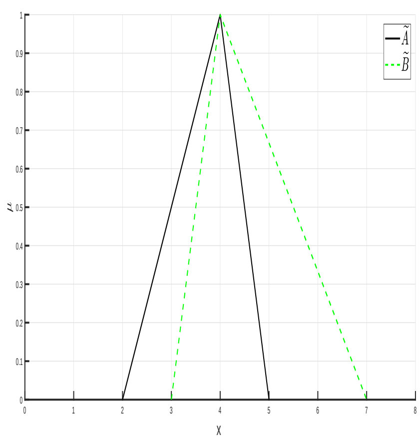

Figure 1: Fuzzy numbers with same core

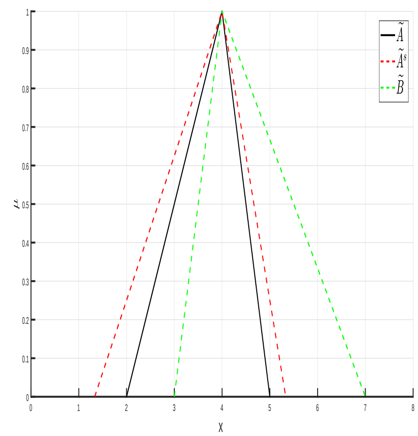

Figure 2: Scale transformation

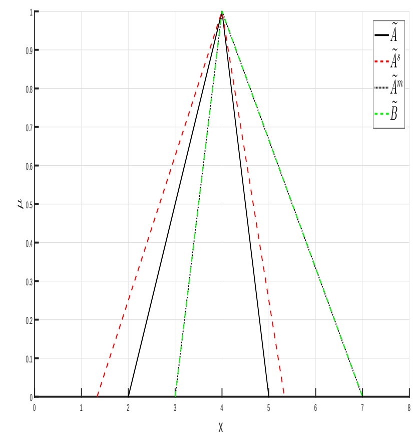

Figure 3: Move transformation

So, going back to the second step of FRI process, the scale and move transformation parameters S and m are calculated from auxiliary antecedent $\widetilde{A}$ and $\widetilde{A}^*$. Then the auxiliary consequent $\widetilde{B}$ is transformed with the same parameters S and m to obtain the final conclusion $\widetilde{B}^*$.

In the following, the expansion/contraction (EC) of TFN are defined. The process of obtaining final conclusion $\widetilde{B}^*$ from the auxiliary consequent $\widetilde{B}$ using EC transformation is illustrated.

2.2 Expansion/Contraction of fuzzy number

Let us consider that the membership function of TFN $\widetilde{A}(a_0, a_1, a_2)$ is defined as follows:

$$\mu(x \mid \widetilde{A}) = \begin{cases} f(x - a_1), & \text{if } a_0 \leq x \leq a_1 \\ g(x - a_1), & \text{if } a_1 \leq x \leq a_2 \\ 1, & \text{if } x = a_1 \\ 0, & elsewhere. \end{cases} \tag{1}$$

The point $x = a_1$ is called core the TFN $\widetilde{A}$ and the intervals $[a_0, a_1]$ and $[a_1, a_2]$ are known as the left and the right spreads respectively. In the following definition, the length of the left and the right spreads are altered while keeping the core at a_1 which produces transformed fuzzy number $\widetilde{A}^t$ (say). In this scenario the core is treated as representative value of the fuzzy numbers $\widetilde{A}$ and $\widetilde{A}^t$, i.e. $Rep(\widetilde{A}) = a_1$ and $Rep(\widetilde{A})^t = a_1$.

Definition 2.2. ***(Expansion/contraction of TFN)*[11]:** *Expansion/contraction (EC) of a triangular fuzzy number*

$\widetilde{A}(a_0, a_1, a_2)$ *by the parameters* $\gamma_l, \gamma_r \geq 0$ *is defined by its membership function as follows:*

$$\mu(x \mid \widetilde{A}^t) = \begin{cases} f(\frac{x-a}{\gamma}), & \text{if } \gamma_l(a_0 - a_1) + a_1 \leq x \leq a_1 \text{ and } \gamma_l > 0 \\ g(\frac{x-a}{\delta}), & \text{if } a_1 \leq x \leq a_1 + \delta(a_2 - a_1) \text{ and } \gamma_r > 0 \\ 1, & \text{if } x = a_1 \text{ and } \gamma_l = 0 \text{ and/or } \gamma_r = 0 \\ 0, & elsewhere \end{cases} \tag{2}$$

If $0 \leq \gamma_l \leq 1$ *and/or* $0 \leq \gamma_r \leq 1$ *then equation (2) gives contraction of* $\widetilde{A}$ *on the respective spreads. Otherwise, for* $\gamma_l > 1$ *and/or* $\gamma_r > 1$ *then equation (2) gives expansion of* $\widetilde{A}$ *on the respective spreads.*

Note 1. *After the EC transformation, the obtained fuzzy set* $\widetilde{A}^t(a_0^t, a_1^t, a_2^t)$ *is always convex. Since* $\gamma_l, \gamma_r \geq 0$ *and* $a_0^t = \gamma_l(a_0 - a_1) + a_1 \leq a_1^t$ *and* $a_2^t = a_1 + \delta(a_2 - a_1) \geq a_1^t$.

Example 2.3. *Let us consider a TFN* $\widetilde{A}(2, 3, 4)$ *whose membership function is given in the following.*

$$\mu(x \mid \widetilde{A}) = \begin{cases} x - 2, & 2 \leq x \leq 3 \\ 4 - x, & 3 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$$

It is given that the TFN has to be contracted on the left spread and expanded right spread with parameters $\gamma_l = 0.5$ *and* $\gamma_r = 2$ *respectively. Then the transformed TFN* $\widetilde{A}^t$ *with* $Rep(\widetilde{A}^t) = 3$ *can be obtained by its membership function as follows:*

$$\mu(x \mid \widetilde{A}^t) = \begin{cases} \frac{x-3}{0.5} + 1, & 2 + (2-3).5 = 2.5 \leq x \leq 3 \\ 1 - \frac{x-3}{2}, & 3 \leq x \leq 3 + (4-3)2 = 5 \\ 0, & elsewhere \end{cases}$$

From the Figure 4, it can be said that $\widetilde{A}$ *and* $\widetilde{A}^t$ *have the same representative value.*

The fuzziness of expanded (or contracted) fuzzy number increases (or decreases) as γ_l and γ_r are changing. In particular, contraction of a fuzzy number $\widetilde{A}(a_0, a_1, a_2)$ by $\gamma_l = \gamma_r = 0$ converts fuzzy number $\widetilde{A}$ into a crisp number $\widetilde{A} = a_1$.

Note 2. *If the scenario demands to transform the TFN* $\widetilde{A}(2, 3, 4)$ *into another TFN* $\widetilde{A}^t(2.5, 3, 5)$ *having the same representative value* $Rep(\widetilde{A}) = Rep(\widetilde{A})^t = 3$, *then the transformation parameters* γ_l *and* γ_r *are required. The parameters* γ_l *and* γ_r *are calculated as* $\gamma_l = \frac{a_1^t - a_0^t}{a_1 - a_0} = \frac{3-2.5}{3-2} = 0.5$ *and* $\gamma_r = \frac{a_2^t - a_1^t}{a_2 - a_1} = \frac{5-3}{4-3} = 2$ *respectively. Finally, the transformation of TFN* $\widetilde{A}$ *into* $\widetilde{A}^t$ *can be done by using the parameters* $\gamma_l = 0.5$ *and* $\gamma_r = 2$ *in the equation (2).*

Up to this point, the existing FRI methodologies are briefed using SM and EC transformation of triangular fuzzy numbers. In the following, the EC of fuzzy numbers are refined for trapezoidal fuzzy number (TrFN) and general polygonal shaped fuzzy sets. EC of fuzzy points are also defined in the next section. A relative comparison between SM and EC transformations is also presented in the next section.

3 Transformation of fuzzy numbers and fuzzy points

Let use consider the trapezoidal fuzzy number (TrFN) $\widetilde{A}(a_0, a_1, a_2, a_3)$ whose membership function is defined as follows:

$$\mu(x \mid \widetilde{A}) = \begin{cases} f(x - a_1), & \text{if } a_0 \leq x \leq a_1 \\ 1, & \text{if } a_1 \leq x \leq a_2 \\ g(x - a_2), & \text{if } a_2 \leq x \leq a_3 \\ 0, & elsewhere \end{cases} \tag{3}$$

The representative value $Rep(\widetilde{A})$ of the TrFN $\widetilde{A}$ is defined as $Rep(\widetilde{A}) = \frac{a_1+a_2}{2}$, i.e. the mid-point of the core $[a_1, a_2]$. In addition to left and right spreads, trapezoidal fuzzy numbers also have non-zero core spread. So, three parameters γ_l, γ_c and γ_r are required to define the expansion/contraction transformation on a TrFN.

Definition 3.1. ***(Expansion/Contraction of TrFN):*** *Expansion/Contraction of a trapezoidal fuzzy number* $\widetilde{A}(a_0, a_1, a_2, a_3)$ *by the parameters* γ_l, γ_c *and* γ_r *can be defined by its membership function as follows:*

$$\mu(x \mid \widetilde{A}^t) = \begin{cases} f(\frac{x-b_1}{\gamma}), & \text{if } b_0 \le x \le b_1 \text{ and } \gamma_l > 0 \\ 1, & \text{if } b_1 \le x \le b_2 \text{ and } \gamma_l = 0 \text{ and/or } \gamma_r = 0 \\ g(\frac{x-b_2}{\gamma_r}), & \text{if } b_2 \le x \le b_2 \text{ and } \gamma_r > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{4}$$

where b_0, b_1, b_2 *and* b_3 *are given as* $b_1 = \frac{a_1+a_2}{2} + \frac{a_1-a_2}{2}\gamma_c$, $b_2 = \frac{a_1+a_2}{2} + \frac{a_2-a_1}{2}\gamma_c$, $b_0 = b_1 + (a_0 - a_1)\gamma_l$ *and* $b_3 = b_2 + (a_3 - a_2)\gamma_r$.

If $0 \le \gamma_l \le 1$ *and/or* $0 \le \gamma_r \le 1$ *then equation (4) gives contraction of* $\widetilde{A}$ *on the respective spreads. Otherwise, for* $\gamma_l > 1$ *and/or* $\gamma_r > 1$ *then equation (2) gives expansion of* $\widetilde{A}$ *on the respective spreads.*

Example 3.2. *Let us consider a TrFN* $\widetilde{A}(3, 4, 5, 7)$ *defined by its membership function as follows:*

$$\mu(x \mid \widetilde{A}) = \begin{cases} x-3, & 3 \le x \le 4 \\ 1, & 4 \le x \le 5 \\ \frac{7-x}{2}, & 5 \le x \le 7 \\ 0, & \text{elsewhere} \end{cases}$$

Let the TrFN $\widetilde{A}(3, 4, 5, 7)$ *has to be expanded on the left spread and contract on the right spread with parameters* $\gamma_l = 1.5$ *and* $\gamma_r = 0.5$ *respectively. Also, the core* $[4, 5]$ *of* $\widetilde{A}$ *has to be contracted with parameter* $\gamma_c = 0.5$. *Then, the transformed TrFN* $\widetilde{A}^t$ *can be obtained by using equation (4) as follows*

$$\mu(x \mid \widetilde{A}^t) = \begin{cases} x-3, & 4.25 + (3-4)1.5 = 2.75 \le x \le 4.25 \\ 1, & \frac{4+5}{2} + \frac{4-5}{2}.5 = 4.25 \le x \le \frac{4+5}{2} + \frac{5-4}{2}.5 = 4.75 \\ \frac{7-x}{2}, & 4.75 \le x \le 4.75 + (7-5).5 = 5.75 \\ 0, & \text{elsewhere} \end{cases}$$

From the figure 5, it can be observed that the transformed TrFN $\widetilde{A}^t(2.75, 4.25, 4.75, 5.75)$ has the same representative value $Rep(\widetilde{A}^t) = \frac{4.25+4.75}{2} = 4.5 = \frac{4+5}{2} = Rep(\widetilde{A})$.

Note 3. *If the TrFN* $\widetilde{A}(3, 4, 5, 7)$ *had to transfer into another TrFN* $\widetilde{A}^t(2.75, 4.25, 4.75, 5.75)$ *having the same representative value* $Rep(\widetilde{A}) = Rep(\widetilde{A})^t = 4.5$, *then the transformation parameters* γ_l, γ_c *and* γ_r *were required. The parameters* γ_l, γ_c *and* γ_r *are calculated as* $\gamma_l = \frac{a_1^t - a_0^t}{a_1 - a_0} = \frac{4.25-2.75}{4-3} = 1.5$, $\gamma_c = \frac{a_2^t - a_1^t}{a_2 - a_1} = \frac{4.75-4.25}{5-4} = 0.5$ *and* $\gamma_r = \frac{a_3^t - a_2^t}{a_3 - a_2} = \frac{5.75-4.75}{7-5} = 0.5$ *respectively. Finally, the transformation of TrFN* $\widetilde{A}$ *into TrFN* $\widetilde{A}^t$ *can be done by using the parameters* $\gamma_l = 1.5, \gamma_c = 0.5$ *and* $\gamma_r = 0.5$ *in the equation (4).*

In the similar fashion the EC transformation can be defined on polygonal fuzzy sets. Let us consider a polygonal fuzzy set $\widetilde{A}(a_0, a_1, \ldots a_m, a_{m+1}, \ldots a_{n-1}, a_n)$ (see figure 6) which is defined by its membership function as follows:

$$\mu(x \mid \widetilde{A}) = \begin{cases} f_1(x-a_1), & \text{if } a_0 \le x \le a_1 \\ f_2(x-a_2), & \text{if } a_1 \le x \le a_2 \\ \vdots & \\ 1, & \text{if } a_m \le x \le a_{m+1} \\ g_1(x-a_{m+1}), & \text{if } a_{m+1} \le x \le a_{m+2} \\ \vdots & \\ g_{n-m-2}(x-a_{n-1}), & \text{if } a_{n-1} \le x \le a_n \\ 0, & \text{elsewhere.} \end{cases} \tag{5}$$

The EC transformation of the above defined polygonal fuzzy set $\widetilde{A}(a_0, a_1, \ldots a_m, a_{m+1}, \ldots a_{n-1}, a_n)$ by a set of $n-1$

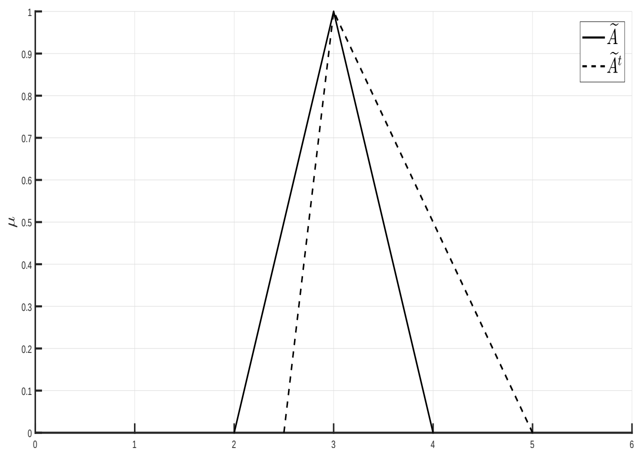

Figure 4: EC transformation of triangular fuzzy number

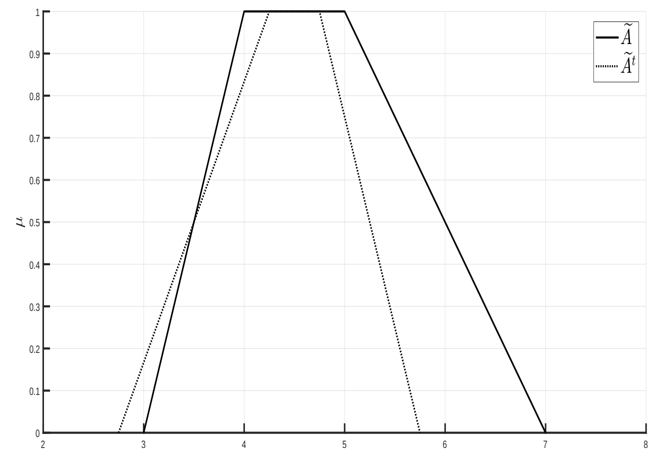

Figure 5: EC transformation of trapezoidal fuzzy number

parameters $\{\gamma_{l_1}, \gamma_{l_2}, \dots \gamma_{l_m}, \gamma_c, \gamma_{r_1}, \gamma_{r_2}, \dots \gamma_{r_{n-m-2}}\}$ can be defined by the following membership function

$$\mu(x \mid \widetilde{A}^t) = \begin{cases} f_1(\frac{x-b_1}{\gamma_1}), & \text{if } b_0 \le x \le b_1 \\ f_2(\frac{x-b_2}{\gamma_2}), & \text{if } b_1 \le x \le b_2 \\ \vdots & \\ 1, & \text{if } b_m \le x \le b_{m+1} \\ g_1(\frac{x-b_{m+1}}{\gamma_{r_1}}), & \text{if } b_{m+1} \le x \le b_{m+2} \\ \vdots & \\ g_{n-m-2}(\frac{x-b_{n-1}}{\gamma_{r_{n-m-2}}}), & \text{if } b_{n-1} \le x \le b_n \\ 0, & elsewhere. \end{cases} \tag{6}$$

where the points $b_0, b_1, \dots b_n$ are defined as $b_m = \frac{a_m + a_{m+1}}{2} + \frac{a_{m+1} - a_m}{2}\gamma_c$, $b_{m+1} = \frac{a_m + a_{m+1}}{2} + \frac{a_{m+1} - a_m}{2}\gamma_c$, $b_{m-1} = a_m + (a_m - a_{m-1})\gamma_{l_m}$, $b_{m-2} = b_{m-1} + (a_{m-1} - a_{m-2})\gamma_{l_{m-1}}, \dots b_0 = b_1 + (a_0 - a_1)\gamma_1$; $b_{m+2} = b_{m+1} + (a_{m+2} - a_{m+1})\gamma_{r_1}$, $b_{m+3} = b_{m+2} + (a_{m+3} - a_{m+2})\gamma_{r_2}, \dots b_n = b_{n-1} + (a_n - a_{n-1})\gamma_{r_{n-m-2}}$.

Note 4. $\gamma = \{\gamma_{l_1}, \gamma_{l_2}, \dots \gamma_{l_m}, \gamma_c, \gamma_{r_1}, \gamma_{r_2}, \dots \gamma_{r_{n-m-2}}\}$ *is a set of non-negative parameters. So, from the above definition it can be observed that the numbers* $b_0, b_1, \dots b_n$ *forms a monotonically increasing sequence. This, proves that the transformed fuzzy set* $\widetilde{A}^t$ *is always convex.*

The above definition of expansion/contraction of a polygonal fuzzy set is elaborated in the following example where transformation of a hexagonal fuzzy set is described.

Example 3.3. *Let us consider a hexagonal fuzzy set* $\widetilde{A}(2, 4, 5, 6, 8, 9 | 0, .5, 1, 1, .25, 0)$ *which is defined by its membership function as follows:*

$$\mu(x \mid \widetilde{A}) = \begin{cases} \frac{x-4}{4} + \frac{1}{2}, & \text{if } 2 \le x \le 4 \\ \frac{x-5}{2} + 1, & \text{if } 4 \le x \le 5 \\ 1, & \text{if } 5 \le x \le 6 \\ 1 - \frac{3(x-6)}{8}, & \text{if } 6 \le x \le 8 \\ \frac{1}{4} - \frac{x-8}{4}, & \text{if } 8 \le x \le 9 \\ 0, & elsewhere. \end{cases} \tag{7}$$

Suppose, the fuzzy set $\widetilde{A}(2, 4, 5, 6, 8, 9 | 0, .5, 1, 1, .25, 0)$ *has to be expanded/contracted by a set of parameters* $\{\gamma_{l_1}, \gamma_{l_2}, \gamma_c, \gamma_{r_1}, \gamma_{r_2}\}$

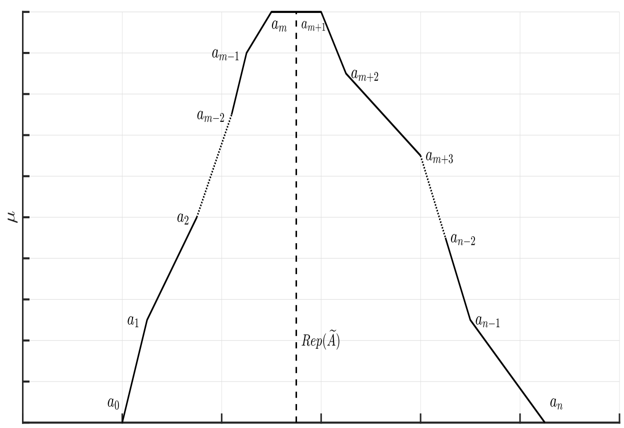

Figure 6: Polygonal fuzzy set

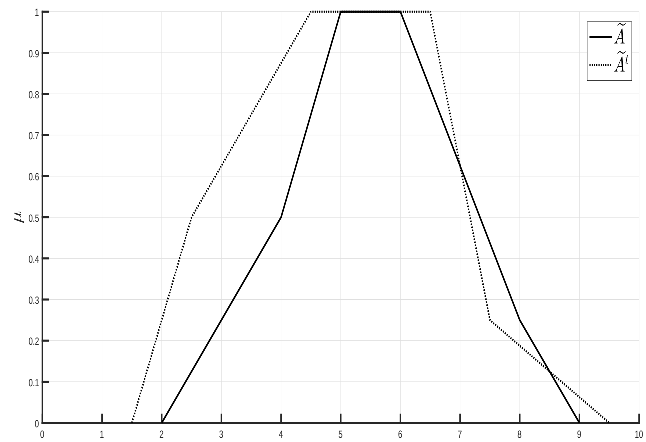

Figure 7: Transformation of Hexagonal fuzzy set

$= \{.5, 2, 2, .5, 2\}$. *The desired transformation the fuzzy set* $\widetilde{A}^t$ *can be obtained by using equation (6) as follows:*

$$\mu(x \mid \widetilde{A}) = \begin{cases} \frac{\frac{x-2.5}{.5}}{4} + \frac{1}{2}, & \textit{if } 2.5 + (2-4).5 = 1.5 \leq x \leq 2.5 \\ \frac{\frac{x-4.5}{2}}{2} + 1, & \textit{if } 4.5 + (4-5)2 = 2.5 \leq x \leq 4.5 \\ 1, & \textit{if } \frac{5+6}{2} + \frac{5-6}{2}2 = 4.5 \leq x \leq \frac{5+6}{2} + \frac{6-5}{2}2 = 6.5 \\ 1 - \frac{3\frac{(x-6.5)}{.5}}{8}, & \textit{if } 6.5 \leq x \leq 6.5 + (8-6).5 = 7.5 \\ \frac{1}{4} - \frac{\frac{x-7.5}{2}}{4}, & \textit{if } 7.5 \leq x \leq 7.5 + (9-8)2 = 9.5 \\ 0, & \textit{elsewhere.} \end{cases} \tag{8}$$

Thus, the transformed hexagonal fuzzy set $\widetilde{A}^t(1.5, 2.5, 4.5, 6.5, 7.5, 9.5|0, .5, 1, 1, .25, 0)$ *has been obtained. From figure 7, it can be observed that* $\widetilde{A}$ *and* $\widetilde{A}^t$ *have the same representative point* $\frac{4.5+6.5}{2} = 5.5$ *but different support lengths.*

Another important transformation, say translation of fuzzy number is defined here. Translation of fuzzy set changes the position of the fuzzy set without changing its shape or size, i.e. without changing support length and geometrical shape.

Definition 3.4. ***(Translation of Fuzzy Set):*** *Translation of a fuzzy number* $\widetilde{A}(a_0, a_1, \ldots a_m, a_{m+1}, \ldots a_{n-1}, a_n)$ *by a parameter t (say) can be defined by its membership function as follows:*

$$\mu(x \mid \widetilde{A}^{Tr}) = \begin{cases} f_1(x-b_1), & \textit{if } b_0 \leq x \leq b_1 \\ f_2(x-b_2), & \textit{if } b_1 \leq x \leq b_2 \\ \vdots & \\ 1, & \textit{if } b_m \leq x \leq b_{m+1} \\ g_1(x-b_{m+1}), & \textit{if } b_{m+1} \leq x \leq b_{m+2} \\ \vdots & \\ g_{n-m-2}(x-b_{n-1}), & \textit{if } b_{n-1} \leq x \leq b_n \\ 0, & \textit{elsewhere.} \end{cases} \tag{9}$$

where $(b_0, b_1, \ldots, b_{n-1}, b_n) = (a_0 + t, a_1 + t, \ldots a_{n-1} + t, a_n + t)$. It can be noticed that, translation of a fuzzy number $\widetilde{A}$ by a parameter is the same as the scalar addition of $\widetilde{A}$ and l, i.e. $\widetilde{A} + l$.

Note 5. *Commonly we use triangular and trapezoidal fuzzy numbers. But, it is very oversimplified assumption that we can represent every fuzzy set as linear fuzzy number (e.g, TFN or TrFN). Complex scenarios can take non-linear fuzzy sets and the non-linear fuzzy set might be approximated by hexagonal or polygonal fuzzy number in more accurate way. This study provides the technique to work with such scenarios as well.*

3.1 Expansion/Contraction of fuzzy point

Let $\widetilde{Q}(a,b)$ be a fuzzy point with membership function $\mu((x,y) \mid \widetilde{Q})$ and $\widetilde{Q}(0)$ be the 0-level set of $\widetilde{Q}$. We will consider the following representation of membership function: $\mu((x,y) \mid \widetilde{Q}) = f(x-a, y-b)$ to define expansion or contraction of fuzzy point.

Definition 3.5. ***(Expansion/Contraction of FP)*[11]:** *Expansion/Contraction of a fuzzy point $\widetilde{Q}(a,b)$ with membership function $\mu((x,y) \mid \widetilde{Q}) = f(x-a, y-b)$ by a set parameters $t = \{t_1, t_2 ... t_m\}$ and $s = \{s_1, s_2 ... s_m\}$, where $t_i, s_i \geq 0$ and $i = 1, 2...m$, in m different regions $D = \{D_1, D_2 ... D_m\}$ can be defined by the following membership function:*

$$\mu((x,y) \mid \widetilde{Q}') = \begin{cases} f(\frac{x-a}{t_i}, \frac{y-b}{s_i}), & \text{if } (x,y) \in D_i \text{ and } t_i, s_i > 0,\ i=1,2...m \\ 1, & \text{if } (x,y) = (a,b) \in D_i, \text{ and } t_i = 0 \text{ and/or } s_i = 0 \\ 0, & \text{if } (x,y) \neq (a,b),\ (x,y) \in D_i,\ t_i = 0 \text{ and/or } s_i = 0 \\ f(x-a, y-b), & \text{elsewhere} \end{cases} \tag{10}$$

Now if $t_i = s_i$ for region D_i, then equation (10) gives uniform expansion/contraction (according to $t_i \geq 1$ or $t_i \leq 1$) on the region D_i. In particular, if $t_i = s_i$ (for all $i = 1, 2, ...m$) then the expansion/contraction of any fuzzy point given by the definition 3.5 is uniform expansion or contraction of $\widetilde{Q}$.

But if $t_i \neq s_i$ (for any $i = 1, 2, \cdots m$) then the expansion/contraction of fuzzy point $\widetilde{Q}(a,b)$ on region D_i is not uniform.

The fuzziness of expanded (or contracted) fuzzy point increases (or decreases) depending on the values t and s. The definition 3.5 gives different possible types of expansions or contractions depending on the values of $t = \{t_1, t_2, \cdots t_m\}$ and $s = \{s_1, s_2, \cdots s_m\}$. In particular, contraction of a fuzzy point $\widetilde{Q}(a,b)$ by $t_i = s_i = 0$ (for all $i = 1, 2, \cdots m$) converts fuzzy point $\widetilde{Q}(a,b)$ into a crisp point $\widetilde{Q} = (a,b)$.

Example 3.6. *Let us consider the fuzzy point $\widetilde{Q}(5,5)$ with membership function defined as follows:*

$$\mu((x,y) \mid \widetilde{Q}) = \begin{cases} 1 - \sqrt{(\frac{x-5}{2})^2 + (\frac{y-5}{2})^2}, & \text{if } (\frac{x-5}{2})^2 + (\frac{y-5}{2})^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Then expansion/contraction of a fuzzy point $\widetilde{Q}(5,5)$ in regions $D = \{D_1, D_2, D_3, D_4\}$ with set of parameters $t = \{t_1, t_2, t_3, t_4\}$ and $s = \{s_1, s_2, s_3, s_4\}$, where $D_1 = \{(x,y) \mid x \geq 5, y \geq 5\}$, $D_2 = \{(x,y) \mid x \leq 5, y \geq 5\}$, $D_3 = \{(x,y) \mid x \leq 5, y \leq 5\}$ and $D_4 = \{(x,y) \mid x \geq 5, y \leq 5\}$, is defined by its membership function as follows:

$$\mu((x,y) \mid \widetilde{Q}') = \begin{cases} 1 - \sqrt{(\frac{x-5}{2\times t_1})^2 + (\frac{y-5}{2\times s_1})^2}, & \text{if } (\frac{x-5}{2\times t_1})^2 + (\frac{y-5}{2\times s_1})^2 \leq 1 \text{ and } (x,y) \in D_1 \\ 1 - \sqrt{(\frac{x-5}{2\times t_2})^2 + (\frac{y-5}{2\times s_2})^2}, & \text{if } (\frac{x-5}{2\times t_2})^2 + (\frac{y-5}{2\times s_2})^2 \leq 1 \text{ and } (x,y) \in D_2 \\ 1 - \sqrt{(\frac{x-5}{2\times t_3})^2 + (\frac{y-5}{2\times s_3})^2}, & \text{if } (\frac{x-5}{2\times t_3})^2 + (\frac{y-5}{2\times s_3})^2 \leq 1 \text{ and } (x,y) \in D_3 \\ 1 - \sqrt{(\frac{x-5}{2\times t_4})^2 + (\frac{y-5}{2\times s_4})^2}, & \text{if } (\frac{x-5}{2\times t_4})^2 + (\frac{y-5}{2\times s_4})^2 \leq 1 \text{ and } (x,y) \in D_4 \\ 0 & \text{elsewhere} \end{cases}$$

Different types of expansion/contraction of $\widetilde{Q}(5,5)$ with different choices of t and s may be obtained.

3.2 Comparison between scale and move transformation with expansion/contraction

There are two major differences between the SM and EC transformation. One is the risk of obtaining non-convex fuzzy sets and another is computational difficulties. In the following both of the issues are discussed.

3.2.1 Possibility of non-convex fuzzy sets:

In the sections 2.1, 2.2 and note 1 it has been discussed that there are no possibilities of obtaining non-convex fuzzy sets in both of the SM and EC transformation when core of TFN chosen as its representative value. But, representative value of a TFN $\widetilde{A}(a_0, a_1, a_2)$ is chosen as its centre of gravity $\frac{a_0+a_1+a_2}{3}$, then there is chance of obtaining non-convex fuzzy sets through move transformation which has been discussed in the study of Huang and Shen [23].

From equation (6) and note 4, it can be concluded that EC transformation has no risk of obtaining non-convex fuzzy sets for generalized polygonal fuzzy sets which include TFN, TrFN and HFN. Also, it is possible to find positive parameters γ between $\widetilde{A}$ and $\widetilde{A}^t$ and give the EC transformation to transfer $\widetilde{B}$ to $\widetilde{B}^t$ by parameter γ. But, this is not possible SM transformation.

For example, suppose a particular scenario demands to transfer the TrFN $\widetilde{B}(3.75, 5.38, 6.38, 7.38)$ into another TrFN $\widetilde{B}^t$ in a similar way the TrFN $\widetilde{A}(4.12, 7, 8, 9)$ and $\widetilde{A}^t(6, 6, 9, 10)$ are related to. That is, the transformation parameters between $\widetilde{B}$ and $\widetilde{B}^t$ are the same with which $\widetilde{A}$ would be transformed into $\widetilde{A}^t$.

If the EC transformation is chosen to the task, the EC parameters between $\widetilde{A}$ and $\widetilde{A}^t$ are calculated as $\gamma_l = \frac{6-6}{7-4.12} = 0, \gamma_c = \frac{9-6}{8-7} = 3$ and $\gamma_r = \frac{10-9}{9-8} = 1$. And , the required $\widetilde{B}^t$ is obtained as $b_1^t = \frac{5.38+6.38}{2} - 3 \times (\frac{5.38+6.38}{2} - 5.38) = 4.38, b_2^t = \frac{5.38+6.38}{2} + 3 \times (6.38 - \frac{5.38+6.38}{2}) = 7.38, b_0^t = 4.38 - 0 \times (5.38 - 3.75) = 4.38$ and $b_0^t = 7.38 + 1 \times (7.38 - 6.38) = 8.38$, i.e. $\widetilde{B}^t(4.38, 4.38, 7.38, 8.38)$.

Now, let us try the SM transformation for the same task. The scale parameters between $\widetilde{A}(4.12, 7, 8, 9)$ and $\widetilde{A}^t(6, 6, 9, 10)$ are calculated as $S_t = \frac{9-6}{8-7} = 3$ and $S_b = \frac{10-6}{9-4.12} = 0.82$. If the parameters $S_t = 3$ and $S_b = 0.82$ are used transform $\widetilde{B}(3.75, 5.38, 6.38, 7.38)$ then $\widetilde{B}^s\mathbf{(4.39,4.38,7.38,7.36)}$ is obtained after scale transformation. So, it can be observed that scale transformation might obtain non-convex fuzzy sets like $\widetilde{B}^s(4.39, 4.38, 7.38, 7.36)$. To avoid this non-convex out come, the scale parameters of SM transformation for TrFN are redefined [23] as

$$S_b = \frac{a_3^t - a_0^t}{a_3 - a_0} \quad \text{and} \quad S_t = \begin{cases} \dfrac{\frac{a_2^t - a_1^t}{a_3^t - a_0^t} - \frac{a_2 - a_1}{a_3 - a_0}}{1 - \frac{a_2 - a_1}{a_3 - a_0}} & \text{if } \frac{a_2^t - a_1^t}{a_2 - a_1} \geq \frac{a_3^t - a_0^t}{a_3 - a_0} \geq 0 \\ \dfrac{\frac{a_2^t - a_1^t}{a_3^t - a_0^t} - \frac{a_2 - a_1}{a_3 - a_0}}{\frac{a_2 - a_1}{a_3 - a_0}} & \text{if } \frac{a_3^t - a_0^t}{a_3 - a_0} \geq \frac{a_2^t - a_1^t}{a_2 - a_1} \geq 0 \end{cases} \tag{11}$$

Using the redefined $S_b = 0.82$ and $S_t = 0.69$ then we obtain $\widetilde{B}^s(4.39, 4.38, 7.38, 9.26)$ and $\widetilde{A}^s(4.7, 7.16, 7.84, 8.7)$. The final transformed $\widetilde{B}^t(5.41, 5.53, 6.22, 8.39)$ is obtained after move transformation of $\widetilde{B}^s$ with move parameter $m = a_0^t - a_0^s = 6 - 4.7 = 1.3$.

One major point which has to noticed here is that the transformation of $\widetilde{A}(4.12, 7, 8, 9)$ with $S_b = 0.82$, $S_t = 0.69$ and $m = 1.3$ yield $\widetilde{A}^m(6, 7.16, 7.84, 10)$ which is not identical with desired $\widetilde{A}^t(6, 6, 9, 10)$. So, it could be concluded that the required transformation could not be performed through SM transformation.

3.2.2 Complexity:

EC transformation of TFN $\widetilde{A}$ to $\widetilde{B}$ require five operations, i.e. $\gamma_l = \frac{b_1 - b_0}{a_1 - a_0}$, $\gamma_r = \frac{b_2 - b_1}{a_2 - a_1}$; $a_1^t = a_1$, $a_0^t = a_1 + \gamma_l(a_0 - a_1)$ and $a_2^t = a_1 + \gamma_r(a_2 - a_1)$. Whereas, in the SM transformation the scale parameter is calculated $S = \frac{b_2 - b_1}{a_2 - a_1}$ and then $a_1^s = a_1$, $a_0^s = a_1 + S(a_0 - a_1)$ and $a_2^s = a_1 + S(a_2 - a_1)$ are calculated. After that, the move parameter is calculated $m = b_0 - a_0^s$. Then, the TFN $\widetilde{A}^m$ is obtained from $a_1^m = a_1^s$, $a_0^m = a_0^s + m$ and $a_2^m = a_2^s + m$ are calculated. So, a total eight number operations are needed.

In general, to transform a polygonal fuzzy set $\widetilde{A}(a_0, a_1, \dots a_m, a_{m+1}, \dots a_{n-1}, a_n)$ by SM transformation total $\lceil \frac{n}{2} \rceil - 2 + n + \lfloor \frac{n}{2} \rfloor - 2 + n = 3n - 3$ number of operation needed [22]. Whereas, in EC transformation required number of operation is $n - 1 + n = 2n - 1$.

Also, the EC transformation is being defined for more generalized polygonal fuzzy sets. Meaning of which, the type of PFN were chosen in the study of Huang and Shen [22] had symmetric odd points. That is, if a PFN $\widetilde{A}$ has an odd point at $\alpha = 0.5$ level in the left spread, it must have an odd point at $\alpha = 0.5$ level in the right spread as well. But, in this study no such assumptions were made to define the PFN in equation (5).

The beauty of EC transformation also lies within its extension on higher dimension. Fuzzy point, a higher dimensional notion of fuzzy numbers, can be expanded/contracted which has been discussed in section 3.1.

4 Proposed forward FRI technique

Suppose the given knowledge base is presented by a collection of fuzzy rules $\mathbf{R} = \bigcup\{\widetilde{R}_i\}$. The fuzzy rules consist of M number of antecedents and single consequent of the following form:

$$\widetilde{R}_i : \ if \ x_1 = \widetilde{A}_{i1}, \ x_2 = \widetilde{A}_{i2}, \dots x_M = \widetilde{A}_{iM} \ then \ y = \widetilde{B}_i \tag{12}$$

Based on the given rule base $\mathbf{R}$, decision maker has to determine the conclusion $\widetilde{B}^*$ corresponding to an observation $\widetilde{O} : \ if \ x_1 = \widetilde{A}_1^*, \ x_2 = \widetilde{A}_2^*, \dots x_M = \widetilde{A}_M^* \ then \ y = ?$.

4.1 Finding the nearest rules

It might happen that the decision maker does not want to include all the rules of the rule base $\mathbf{R}$, rather choose to use only n number of closest or nearest rules from the observation $\widetilde{O}$. The distance between the rule $\widetilde{R}_i$ and the observation $\widetilde{O}$ is defined as:

$$d_i = \sqrt{\sum_{j=1}^{M} d^2(\widetilde{A}_{ij}, \widetilde{A}_j^*)} \qquad \text{where } \; d(\widetilde{A}_{ij}, \widetilde{A}_j^*) = \frac{|Rep(\widetilde{A}_{ij}) - Rep(\widetilde{A}_j^*)|}{range_j}. \tag{13}$$

Difference of domain ranges of different antecedents are normalized by dividing $range_j = \underset{i}{max}\{Rep(\widetilde{A}_{ij})\} - \underset{i}{min}\{Rep(\widetilde{A}_{ij})\}$. Based on the distances d_i associated with every rule $\widetilde{R}_i$, the decision maker can select n number of rules and remake the rule base as $\mathbf{R} = \bigcup_{i=1}^{n}\{\widetilde{R}_i\}$.

4.2 Obtaining of the auxiliary rule

Based on the n nearest rules $\{\widetilde{R}_i, \widetilde{R}_2, \ldots \widetilde{R}_n\}$ and observation $\widetilde{O}$ and auxiliary rule $\widetilde{R}: \; x_1 = A_1, \; x_2 = \widetilde{A}_2 \ldots x_m = \widetilde{A}_m \Rightarrow y = \widetilde{B}$ is obtained. Auxiliary rule $\widetilde{R}$ is being constructed as a weighted summation of the rules $\{\widetilde{R}_i, \widetilde{R}_2, \ldots \widetilde{R}_n\}$, i.e. $\widetilde{R} = \sum_{i=1}^{n} \omega_i \widetilde{R}_i$. Logically, the weights ω_i has been calculated in such way that the far the $\widetilde{R}_i$ from the observation $\widetilde{O}$ the less amount weight ω_i it gets. Mathematically which implies ω_i has to inversely proportional to the distance d_i between the rules $\widetilde{R}_i$ and the observation $\widetilde{O}$, i.e. $\omega_i \propto \frac{1}{d(\widetilde{R}_i, \widetilde{O})}$. In practice, the distance d_i is calculated in the antecedent space between antecedent $\widetilde{A}_i$ and observation $\widetilde{O}_i$.

Here, the M components $\omega_{i1}, \omega_{i2} \ldots \omega_{iM}$ of ω_i are calculated as:

$$\omega_{ik} = \frac{\omega'_{ik}}{\sum\limits_k \omega'_{ik}}; \qquad \text{where } \omega'_{ik} = \frac{1}{d(\widetilde{A}_{ik}, \widetilde{A}_k^*)}. \tag{14}$$

The distance component $d(\widetilde{A}_{ik}, \widetilde{A}_k^*)$ is defined in equation (13). The consequent component of the weight ω_{iB} (say) is being calculated as an average of $\omega_{i1}, \omega_{i2} \ldots \omega_{iM}$, i.e. $\omega_{iB} = \frac{\sum_{k=1}^{M} \omega_{ik}}{M}$. So, the weight ω_i associated with the rule $\widetilde{R}_i$ is obtained as $\omega_i = (\omega_{i1}, \omega_{i2} \ldots \omega_{iM}, \omega_{iB})$.

Thus, the auxiliary rule $\widetilde{R}$ is obtained as:

$$\widetilde{R} = \sum_{i=1}^{n} \omega_i \widetilde{R}_i = \Big(\sum_{i=1}^{n} \omega_{i1} \widetilde{A}_{i1}, \sum_{i=1}^{n} \omega_{i2} \widetilde{A}_{i2}, \cdots \sum_{i=1}^{n} \omega_{iM} \widetilde{A}_{iM}, \sum_{i=1}^{n} \omega_{iB} \widetilde{B}_i\Big). \tag{15}$$

4.3 Translation

It might happen that the representative value $Rep(\widetilde{R})$ of the auxiliary rule $\widetilde{R}$ does not coincides with the representative value $Rep(\widetilde{O})$ of the observation $\widetilde{O}$. But, it is necessary to derive the conclusion from a rule (calculated) whose representative value coincides with the of observation. So, it is needed to give a transformation to the obtained rule $\widetilde{R}$ into a new position with out changing its spread length. The required transformation is termed and defined as translation (see definition 3.4). So, every antecedent $\widetilde{A}_k$ is translated to $\widetilde{A}'_k$ by the parameter $l_k = Rep(\widetilde{A}_k^*) - Rep(\widetilde{A}_k)$. The translation parameter for the consequent is calculated as average of l_k, i.e. $l_B = \frac{\sum_{k=1}^{M} l_k}{M}$.

Thus, from this step we obtain a rule $\widetilde{R}' : \; x_1 = \widetilde{A}'_1, \; x_2 = \widetilde{A}'_2, \ldots x_m = \widetilde{A}'_m \Rightarrow y = \widetilde{B}'$ whose antecedent part has same representative value as the observation $\widetilde{O}$.

4.4 Expansion/Contraction

In this step, the difference between the amount of uncertainty of the antecedents $(\widetilde{A}'_1, \widetilde{A}'_2, \ldots \widetilde{A}'_m)$ and $(\widetilde{A}^*_1, \widetilde{A}^*_2, \ldots \widetilde{A}^*_m)$ are incorporated. The parameters γ of EC between the antecedents $\widetilde{A}'_k$ and $\widetilde{A}^*_k$ are calculated. If $\widetilde{A}'_k$ and $\widetilde{A}^*_k$ are TrFN then γ_{kl}, γ_{kc} and γ_{kr} are calculated from the formulas $\gamma_{kl} = \frac{a^*_{k1} - a^*_{k0}}{a'_{k1} - a'_{k0}}$, $\gamma_{kc} = \frac{a^*_{k2} - a^*_{k1}}{a'_{k2} - a'_{k2}}$ and $\gamma_{kr} = \frac{a^*_{k3} - a^*_{k2}}{a'_{k3} - a'_{k2}}$.

The desired conclusion $\widetilde{B}^*$ is obtained from $\widetilde{B}'$ through EC transformation. The parameters γ_B of the EC transformation are obtained as average of γ_k, i.e. $\gamma_B = \frac{\sum_{k=1}^{M} \gamma_k}{M}$. For the case of TrFN, the parameters γ_{Bl}, γ_{Bc} and γ_{Br} are computed as $\gamma_{Bl} = \frac{\sum_{k=1}^{M} \gamma_{kl}}{M}, \gamma_{Bc} = \frac{\sum_{k=1}^{M} \gamma_{kc}}{M}$ and $\gamma_{Br} = \frac{\sum_{k=1}^{M} \gamma_{kr}}{M}$.

Algorithm 1 Algorithm of FFRI

Require: Given rule base $\mathbf{R}$ with N rules and observation $\widetilde{O}^*(\widetilde{A}_1^*, \widetilde{A}_2^*, \ldots \widetilde{A}_M^*)$.
Find the distances d_i for each rule $\widetilde{R}_i \in \mathbf{R}$ using equation (13).
Find least n distances $d_1, d_2, \ldots d_n$ and reconstruct $\mathbf{R} = \{\widetilde{R}_1, \widetilde{R_2}, \ldots \widetilde{R}_n\}$.
for $i = 1$ to n **do**
  for $k = 1$ to M **do**
    Calculate the weights ω_{ik} using equation (18).
    Calculate the antecedents $\widetilde{A}_k = \sum_{i=1}^{n} \omega_{ik} \widetilde{A}_{ik}$.
  end for
  Calculate $\omega_{iB} = \frac{\sum_{k=1}^{M} \omega_{ik}}{M}$.
  Calculate the consequent $\widetilde{B} = \sum_{i=1}^{n} \omega_{iB} \widetilde{B}_i$.
end for
for $k = 1$ to M **do**
  Calculate the translation parameters $l_k = Rep(\widetilde{A}_k^*) - Rep(\widetilde{A}_k)$.
  Obtain $\widetilde{A}_k'$ by translating $\widetilde{A}_k$ by parameter l_k using definition (3.4).
end for
Calculate $l_B = \frac{\sum_{k=1}^{M} l_k}{M}$.
Obtain $\widetilde{B}'$ by translating $\widetilde{B}$ by parameter l_B using definition (3.4).
for $k = 1$ to M **do**
  Find the parameters γ_k between $\widetilde{A}_k'$ and $\widetilde{A}_k^*$.
end for
Obtain $\gamma_B = \frac{\sum_{k=1}^{M} \gamma_k}{M}$.
Obtain final conclusion $\widetilde{B}^*$ by EC transformation of $\widetilde{B}'$ with parameters γ_B.

The proposed method is furthermore explained with the following algorithm 1 and examples. The following Examples 4.1, 4.3,4.2, 4.4 and 4.5 are taken from the study of Huang and Shen [23] to illustrate the applicability of proposed method in various situation. The results obtained from proposed methods are compared with the results obtained from SM transformation based fuzzy interpolation method.

Example 4.1. *The following rule base* $\boldsymbol{R} = \{\widetilde{R}_1, \widetilde{R}_2\}$ *and observation* $\widetilde{O}^*$ *given in Table 1 have triangular fuzzy numbers in their antecedent and consequent part.*

In the first step, the auxiliary rule $\widetilde{R} = (1-\lambda)\widetilde{R}_1 + \lambda\widetilde{R}_2$ *is being obtained as* $\widetilde{R}: \ \widetilde{A} = (4.12, 8, 9) \Rightarrow \widetilde{B} = (3.75, 5.38, 7.38)$. *Here,* λ *is being calculated as* $\lambda = \frac{Rep(\widetilde{A}^*) - Rep(\widetilde{A}_1)}{Rep(\widetilde{A}_2) - Rep(\widetilde{A}_1)} = \frac{8-5}{13-5} = \frac{3}{8}$. *The auxiliary rule* $\widetilde{R}$ *obtained by using proposed method and SM transformation based method (say, SM method) [23] are identical if the representative value* $Rep(\widetilde{A})$ *is chosen as core of* $\widetilde{A}$, *i.e.* $Rep(\widetilde{A}(a_0, a_1, a_2)) = a_1$.

The auxiliary antecedent $\widetilde{A} = (4.12, 8, 9)$ *and observation* $\widetilde{O}^* = (7, 8, 9)$ *have the same core and the same right spread. The EC parameters* γ_l *and* γ_r *calculated as* $\gamma_l = \frac{8-7}{8-4.12} = .26$ *and* $\gamma_r = \frac{9-8}{9-8} = 1$. *Thus, the desired conclusion* $\widetilde{B}^*(b_0, b_1, b_2) =$ ***(4.96,5.38,7.38)*** *is obtained by giving EC transformation to the auxiliary consequent* $\widetilde{B} = (3.75, 5.38, 7.38)$ *by parameters* $\gamma_l = .26$ *and* $\gamma_r = 1$, *i.e.* $b_1^* = 5.38$; $b_0^* = 5.38 - 0.26(5.38 - 3.75) = 4.96$ *and* $b_2^* = 5.38 + 1 * (7.38 - 5.38) = 7.38$.

Now, the scale parameter S in SM method is obtained as $S = \frac{a_2^* - a_0^*}{a_2 - a_0} = \frac{9-7}{9-4.12} = 0.41$. *That is, the support length of* $\widetilde{A}$ *and* $\widetilde{B}$ *will be reduced* 0.41 *times to produce* $\widetilde{A}^s$ *and* $\widetilde{B}^s$ *while keeping the same core. Applying scale transformation, the fuzzy numbers* $\widetilde{A}^s$ *and* $\widetilde{B}^s$ *are obtained as* $\widetilde{A}^s(6.41, 8, 8.41)$ *and* $\widetilde{B}^s(4.71, 5.38, 6.2)$. *It can be observed that* $\widetilde{A}^s$ *and* $\widetilde{O}^*$ *have the same support length and core but their starting and final points are not the same. Thus, the move transformation is given to both* $\widetilde{A}^s(6.41, 8, 8.41)$ *and* $\widetilde{B}^s(4.71, 5.38, 6.2)$ *by move parameter* $m = a_0^* - a_0^s = 7 - 6.41 = 0.59$. *After the move transformation* $\widetilde{A}^m(7, 8, 9)$ *and* $\widetilde{B}^m(5.3, 5.38, 6.79)$ *are obtained. Since,* $\widetilde{A}^m(7, 8, 9)$ *and the observation* $\widetilde{O}^* = (7, 8, 9)$ *are identical, the desired conclusion is* $\widetilde{B}_{SM}^*$***(5.3,5.38,6.79)***.

In a closer look it can be observe that $\widetilde{A} = (4.12, 8, 9)$ *and observation* $\widetilde{O}^* = (7, 8, 9)$ *have the same core and the same right spread.* $\widetilde{A}$ *and* $\widetilde{O}^*$ *have only difference in left spread lengths. In proposed EC method, only the left spread length of* $\widetilde{B} = (3.75, 5.38, 7.38)$ *is changed to obtain* $\widetilde{B}^* =$ ***(4.96,5.38,7.38)***. *Whereas, in SM method both of left and right spread lengths of* $\widetilde{B} = (3.75, 5.38, 7.38)$ *is being changed to obtain* $\widetilde{B}_{SM}^*$***(5.3,5.38,6.79)***.

In the Table 2, KH refers to method proposed by Kóczy and Hirota [25]. HScg and HSc represents the SM method

Rules	Antecedent	Consequent
$\widetilde{R}_1$	(0,5,6)	(0,2,4)
$\widetilde{R}_2$	(11, 13, 14)	(10,11,13)
$\widetilde{O}^*$	(7,8,9)	?

Table 1: Rules for Example 4.1

Results	
Methods	B^*
KH	(6.36,5.38,7.38)
$HScg$	(5.83,6.26,7.38)
HSc	(5.3, 5.38, 6.79)
DCK	(4.96,5.38,7.38)

Table 2: Results of Example 4.1

proposed by Huang and Shen [23] using center of gravity and core as the representative value of fuzzy number respectively. DCK refers to the proposed method.

Example 4.2. *The following rule base* $\boldsymbol{R} = \{\widetilde{R}_1, \widetilde{R}_2\}$ *and observation* O^* *given in Table 3 consist trapezoidal fuzzy numbers in antecedent and consequent part.*

In the first step, the auxiliary rule $\widetilde{R} = (1-\lambda)\widetilde{R}_1 + \lambda\widetilde{R}_2$ *is being obtained as* $\widetilde{R}: \ \widetilde{A} = (4.12, 7, 8, 9) \Rightarrow \widetilde{B} = (3.75, 5.38, 6.38, 7.38)$. *Here,* λ *is being calculated as* $\lambda = \frac{Rep(\widetilde{A}^*)-Rep(\widetilde{A}_1)}{Rep(\widetilde{A}_2)-Rep(\widetilde{A}_1)} = \frac{\frac{6+9}{2}-\frac{4+5}{2}}{\frac{12+13}{2}-\frac{4+5}{2}} = \frac{3}{8}$. *The auxiliary rule* $\widetilde{R}$ *obtained by using proposed method and SM method [23] are identical if the representative value* $Rep(\widetilde{A})$ *of TrFN* $\widetilde{A}$ *is chosen as average of the core extremities, i.e.* $Rep\big(\widetilde{A}(a_0, a_1, a_2, a_3)\big) = \frac{a_1+a_2}{2}$.

Rules	Antecedent	Consequent
R_1	(0,4,5,6)	(0,2,3,4)
R_2	(11,12,13,14)	(10,11,12,13)
O^*	(6,6,9,10)	?

Table 3: Rules for Example 4.2

Results	
Methods	B^*
KH	(5.45,4.25,7.5, 8.5)
$HScg$	(5.23,5.23,7.61,8.32)
HSc	(5.41,5.53,6.22,8.39)
DCK	(4.38,4.38,7.38,8.38)

Table 4: Results of Example 4.2

EC transformation: *The auxiliary antecedent* $\widetilde{A} = (4.12, 7, 8, 9)$ *and observation* $\widetilde{O}^* = (6, 6, 9, 10)$ *have the same representative values* $Rep(\widetilde{A}) = \frac{7+8}{2} = Rep(\widetilde{O}^*) = \frac{6+9}{2}$ *but different left, right and core spread lengths. The EC parameters* γ_l, γ_c *and* γ_r *are calculated as* $\gamma_l = \frac{6-6}{7-4.12} = 0$; $\gamma_c = \frac{9-6}{8-7} = 3$ *and* $\gamma_r = \frac{10-9}{9-8} = 1$.

Thus, the desired conclusion $\widetilde{B}^*(b_0, b_1, b_2, b_3) =$ ***(4.38,4.38,7.38,8.38)*** *is obtained by giving EC transformation to the auxiliary consequent* $\widetilde{B} = (3.75, 5.38, 6.38, 7.38)$ *by parameters* $\gamma_l = 0$; $\gamma_c = 3$ *and* $\gamma_r = 1$, *i.e.* $b_1^* = \frac{5.38+6.38}{2} - 3*(\frac{5.38+6.38}{2} - 5.68) = 4.38; b_2^* = \frac{5.38+6.38}{2} + 3*(6.38 - \frac{5.38+6.38}{2}) = 7.38$; $b_0^* = 4.38 - 0 \times (5.38 - 3.75) = 4.38$ *and* $b_3^* = 7.38 + 1 \times (7.38 - 6.38) = 8.38$.

SM transformation: *The scale parameters* S_b *and* S_t *in SM method are calculated as obtained as* $S_b = \frac{a_3^*-a_0^*}{a_3-a_0} = \frac{10-6}{9-4.12} = 0.82$ *and* $S_t = \frac{\frac{a_2^*-a_1^*}{a_3^*-a_0^*} - \frac{a_2-a_1}{a_3-a_0}}{1-\frac{a_2-a_1}{a_3-a_0}} = 0.68$ *since* $\frac{a_2^*-a_1^*}{a_2-a_1} \geq \frac{a_3^*-a_0^*}{a_3-a_0}$ *(see equation (11)).*

Applying scale transformation, the fuzzy numbers $\widetilde{A}^s$ *and* $\widetilde{B}^s$ *are obtained as* $\widetilde{A}^s(4.7, 7.16, 7.84, 8.7)$ *and* $\widetilde{B}^s(4.11, 5.53, 6.22, 7.09)$. *It can be observed that* $\widetilde{A}^s$ *and* $\widetilde{O}^*$ *have the same support length of* 4 *units but they do not have the same core length. The starting and final positions of* $\widetilde{A}^s$ *and* $\widetilde{O}^*$ *are also not the same. Thus, the move transformation is given to both* $\widetilde{A}^s(4.7, 7.16, 7.84, 8.7)$ *and* $\widetilde{B}^s(4.11, 5.53, 6.22, 7.09)$ *by move parameter* $m = a_0^* - a_0^s = 6 - 4.7 = 1.3$. *After the move transformation* $\widetilde{A}^m(6, 7.16, 7.84, 10)$ *and* $\widetilde{B}^m(5.41, 5.53, 6.22, 8.39)$ *are obtained. So, the desired conclusion is obtained by SM method is* $\widetilde{B}^*_{SM}$ ***(5.41,5.53,6.22,8.39)***.

In a closer look it can be observe that $\widetilde{A}^m(6, 7.16, 7.84, 10)$ *and observation* $\widetilde{O}^* = (6, 6, 9, 10)$ *do not have the same core though they have the same representative value. The desired conclusion is obtained corresponding to the antecedent* $\widetilde{A}^m(6, 7.16, 7.84, 10)$. *So, acceptance of this result as desired conclusion is questionable.*

A relative comparison between the results obtained from KH, SMcg, SMc and DCK methods is listed in Table 4.

Example 4.3. *This is beautiful example that shows the competency of producing the same result from proposed and SM methods.*

Both the methods obtain the same auxiliary rule $\widetilde{R}: \ \widetilde{A} = (4.12, 8, 9) \Rightarrow \widetilde{B} = (3.75, 5.38, 7.38)$. *It can be observed that* γ_l *and* γ_r *are obtained as* $\gamma_l = 0$ *and* $\gamma_r = 0$ *between the auxiliary antecedent* $\widetilde{A}$ *and observation* $\widetilde{O}^* = (8, 8, 8)$. *So,*

Rules	Antecedent	Consequent
$\widetilde{R}_1$	(0, 5, 6)	(0,2,4)
$\widetilde{R}_2$	(11, 13, 14)	(10,11,13)
$\widetilde{O}^*$	(8,8,8)	?

Table 5: Rules for Example 4.3

Results	
Methods	B^*
KH	(7.27, 5.34, 6.25)
$HScg$	(6.49,6.49,6.49)
HSc	(5.38,5.38,5.38)
DCK	(5.38,5.38,5.38)

Table 6: Results of Example 4.3

*the desired conclusion $\widetilde{B}^*$ is obtained as $\widetilde{B}^*$(**5.38,5.38,5.38**) by giving EC transformation to $\widetilde{B} = (3.75, 5.38, 7.38)$ by parameters $\gamma_l = 0$ and $\gamma_r = 0$.*

The scale parameter S in SM method is obtained as $S = \frac{8-8}{9-4.12} = 0$. After scale transformation, the antecedent $\widetilde{A}^s$ and consequent $\widetilde{B}^s$ are obtained as $\widetilde{A}^s(8,8,8)$ and $\widetilde{B}^s(5.38, 5.38, 5.38)$. Since the antecedent $\widetilde{A}^s(8,8,8)$ coincides with the observation $\widetilde{O}^* = (8,8,8)$, the desired conclusion is being obtained as $\widetilde{B}^* = (5.38, 5.38, 5.38)$.

It can be observed that conclusions obtained from HSc and proposed DCK method are identical. The crisp nature of the observation $\widetilde{O}^* = (8,8,8)$ results this coincidence.

Example 4.4. *The following rule base $\boldsymbol{R} = \{\widetilde{R}_1, \widetilde{R}_2\}$ and observation O^* given in Table 7 have hexagonal fuzzy sets in antecedent and consequent part. The ability to perform the proposed method with polygonal fuzzy sets is depicted in this example.*

Rules	Antecedent	Consequent
$\widetilde{R}_1$	(0,1,3,4,5,5.5)	(0,.5,1,3,4,4.5)
$\widetilde{R}_2$	(11,11.5,12, 13,13.5,14)	(10.5,11,12,13,13.5,14)
$\widetilde{O}^*$	(6, 6.5, 7, 9, 10, 10.5)	?

Table 7: Rules for Example 4.4

Results	
Methods	$\widetilde{B}^*$
KH	(5.73,6.00,5.89,8.56,9.59,10.09)
$HScg$	(5.64,5.98,6.29,8.63,9.46,9.93)
HSc	(5.47,5.79,6.08,8.42,9.23,9.70)
DCK	(5.12,5.45,5.75,8.75,9.75,10.75)

Table 8: Results of Example 4.4

The auxiliary rule is obtained as $\widetilde{R}: \ \widetilde{A} = (5.5, 6.25, 7.5, 8.5, 9.25, 9.75) \Rightarrow \widetilde{B} = (5.25, 5.75, 6.5, 8, 8.75, 9.75)$. The EC parameters γ between antecedent $\widetilde{A}(5.5, 6.25, 7.5, 8.5, 9.25, 9.75)$ and observation $\widetilde{O}^*(6, 6.5, 7, 9, 10, 10.5)$ are calculated as $\gamma_{l_1} = \frac{6.5-6}{6.25-5.5} = 0.67$; $\gamma_{l_2} = \frac{7-6.5}{7.5-6.25} = 0.4$; $\gamma_c = \frac{9-7}{8.5-7.5} = 2$; $\gamma_{r_1} = \frac{10-9}{9.25-8.5} = 1.33$ and $\gamma_{r_2} = \frac{10.5-10}{9.75-9.25} = 1$.

*Thus, the desired conclusion $\widetilde{B}^*$ **(5.12,5.45,5.75,8.75,9.75,10.75)** is obtained as $b_2^* = \frac{8+6.5}{2} - 2(\frac{8+6.5}{2} - 6.5) = 5.75$; $b_3^* = \frac{8+6.5}{2} + 2(8 - \frac{8+6.5}{2}) = 8.75$; $b_1^* = 5.75 - 0.4(6.5 - 5.75) = 5.45$; $b_0^* = 5.45 - 0.67(5.75 - 5.25) = 5.12$; $b_4^* = 8.75 - 1.33(8.75 - 8) = 9.75$ and $b_5^* = 9.75 - (9.75 - 8.75) = 10.75$.*

The results obtained from KH, HScg, HSc and proposed DCK are presented in the Table 8.

Example 4.5. *In this example, the rules $\{\widetilde{R}_1, \widetilde{R}_2\}$ and observation O^* have two dimensional antecedents of trapezoidal fuzzy numbers. The desired conclusion is obtained by the following Algorithm 1.*

Rules	Antecedent	Consequent
$\widetilde{R}_1$	A_{11} =(0,4,5,6) ; A_{12} =(12,14,15,16)	B_1 =(0,2,3,4)
$\widetilde{R}_2$	A_{21} =(11,12, 13,14); A_{22} =(1,2,3,4)	B_2 =(10,11,12,13)
$\widetilde{O}^*$	A_1^* =(6, 7, 9, 11); A_2^* =(6, 8, 10, 12)	?

Table 9: Rules for Example 4.5

Results	
Methods	$\widetilde{B}^*$
KH	(5.45,5.94,7.13,8.31)
$HScg$	(4.37,5.55,7.48,9.33)
HSc	(5.78, 6.79, 8.79, 10.74)
DCK	(2.45,5.55,7.55,9.55)

Table 10: Results of Example 4.5

The auxiliary rule $\widetilde{R} = \widetilde{A}_1(4.84, 7.52, 8.52, 9.52) \wedge \widetilde{A}_2(6.94, 8.48, 9.48, 10.48) \Rightarrow \widetilde{B}(4.5, 6.05, 7.05, 8.05)$ is obtained by following steps 3 to 9 of Algorithm 1. Since the representative values of $\widetilde{A}_1$ and $\widetilde{A}_2$ matches with representative values of $\widetilde{A}_1^*$ and $\widetilde{A}_2^*$, translation of the rule $\widetilde{R}$ is not required.

*In the next step, the EC parameters γ_{k_l}, γ_{k_c} γ_{k_r} ($k = 1, 2$) are being calculated as $\gamma_1 = (0.37, 2, 2)$ and $\gamma_2 = (1.3, 2, 2)$. Thus, γ_B is obtained as $\gamma_B = (\frac{0.37+1.3}{2}, \frac{2+2}{2}, \frac{2+2}{2}) = (0.84, 2, 2)$. The desired conclusion $\widetilde{B}^*$ **(4.25,5.55,7.55,9.55)*** *is obtained from EC transformation of* $\widetilde{B}(4.5, 6.05, 7.05, 8.05)$ *with the parameters* $\gamma_B(0.84, 2, 2)$ *as* $b_1^* = \frac{6.05+7.05}{2} - 2(\frac{6.05+7.05}{2} - 6.05) = 5.55$; $b_2^* = \frac{6.05+7.05}{2} + 2(7.05 - \frac{6.05+7.05}{2}) = 7.55$; $b_0^* = 5.55 - 0.84(6.05 - 4.5) = 4.25$ *and* $b_3^* = 7.55 + 2(8.05 - 7.05) = 9.55$.

Results obtained from different methods are compared in Table 10.

In the above Example 4.1 the proposed method is described step by step and the obtained results is compared with the result obtained by SM method. Example 4.2 shows the possibility of compromise final conclusion to avoid non-convex conclusion through SM method whereas the proposed EC transformation based method provides desirable convex conclusion. The coincidence of SM and proposed EC methods are discussed in 4.3. The applicability of proposed method for rules involving hexagonal fuzzy numbers are shown in 4.4. Finally, a multi-dimensional rule base is used in example 4.5 to further illustrate the proposed method.

5 Proposed inverse or backward FRI technique

The objective of inverse or backward interpolation is to find an appropriate course of action $\widetilde{A}_t^*$ (say) in order to achieve a desired outcome $\widetilde{B}^*$ when the other inputs of the system, i.e. $\widetilde{A}_1^*, \widetilde{A}_2^* \dots \widetilde{A}_{t-1}^*, \widetilde{A}_{t+1}^*, \dots \widetilde{A}_m^*$ are given.

So, for a given knowledge base $\mathbf{R} = \bigcup\{\widetilde{R}_i\}$ which consists fuzzy rules of the form equation (12), the observation can be presented as follows:

$$\widetilde{O}: \; if \; x_1 = \widetilde{A}_1^*, \; x_2 = \widetilde{A}_2^*, \dots x_{t-1} = \widetilde{A}_{t-1}^*, x_t = ?, x_{t+1} = \widetilde{A}_{t+1}^* \dots x_M = \widetilde{A}_M^* \; then \; y = \widetilde{B}^*. \tag{16}$$

Based on the given rule base $\mathbf{R}$, decision maker has to determine the appropriate antecedent $\widetilde{A}_t^*$ which will yield $\widetilde{B}^*$ when $\widetilde{A}_1^*, \widetilde{A}_2^* \dots \widetilde{A}_{t-1}^*, \widetilde{A}_{t+1}^*, \dots \widetilde{A}_m^*$ are given.

5.1 Obtaining nearest rules

The nearest rules of inverse interpolation are obtained in a similar fashion with forward interpolation process. The distance between the rule $\widetilde{R}_i$ and the observation $\widetilde{O}$ is defined as:

$$d_i = \sqrt{\sum_{j \in I} d^2(\widetilde{A}_{ij}, \widetilde{A}_j^*) + M \cdot d^2(\widetilde{B}_i, \widetilde{B}^*)} \qquad \text{where} \;\; d(\widetilde{A}_{ij}, \widetilde{A}_j^*) = \frac{|Rep(\widetilde{A}_{ij}) - Rep(\widetilde{A}_j^*)|}{range_j}. \tag{17}$$

The set of indices I is defined as $I = \{1, 2, \dots t-1, t+1, \dots M\}$. Based on the distances d_i associated with every rule $\widetilde{R}_i$, the decision maker may sorted n number of rules and remake the rule base as $\mathbf{R} = \bigcup_{i=1}^n \{\widetilde{R}_i\}$.

5.2 Obtaining of the auxiliary rule

Based on the n nearest rules $\{\widetilde{R}_1, \widetilde{R}_2, \dots \widetilde{R}_n\}$ and observation $\widetilde{O}$, the auxiliary rule $\widetilde{R}: \; x_1 = \widetilde{A}_1, \; x_2 = \widetilde{A}_2, \dots x_m = \widetilde{A}_m \Rightarrow y = \widetilde{B}$ is obtained. Auxiliary rule $\widetilde{R}$ is being constructed as an weighted summation of the rules $\{\widetilde{R}_i, \widetilde{R}_2, \dots \widetilde{R}_n\}$, i.e. $\widetilde{R} = \sum_{i=1}^n \omega_i \widetilde{R}_i$. Here, the M components $\omega_{i1}, \omega_{i2} \dots \omega_{it-1}, \omega_{it+1} \dots \omega_{iM}, \omega_B$ of ω_i are calculated as:

$$\omega_{ik} = \frac{\omega'_{ik}}{\sum\limits_{k \in I} \omega'_{ik} + \omega_B} \text{ and } \omega_B = \frac{\omega'_B}{\sum\limits_{k \in I} \omega'_{ik} + \omega_B} \text{ where } \omega'_{ik} = \frac{1}{d(\widetilde{A}_{ik}, \widetilde{A}_k^*)} \text{ and } \omega'_B = \frac{1}{d(\widetilde{B}_i, \widetilde{B}^*)}. \tag{18}$$

The distance components $d(\widetilde{A}_{ik}, \widetilde{A}_k^*)$ is defined in equation (13). The weight component ω_{it} (say) of the missing antecedent is being calculated from the formulation of ω_B in the forward FRI mentioned in section 4.2 as: $\omega_{iB} = \frac{\sum_{k=1}^M \omega_{ik}}{M} \Rightarrow \omega_{it} = M\omega_{iB} - (\sum_{k \in I} \omega_{ik})$. So, the weight $\omega_i = (\omega_{i1}, \omega_{i2} \dots \omega_{iM}, \omega_{iB})$ associated with the rule $\widetilde{R}_i$ is obtained.

Thus, the auxiliary rule $\widetilde{R}$ is obtained as:

$$\widetilde{R} = \sum_{i=1}^n \omega_i \widetilde{R}_i = \Big(\sum_{i=1}^n \omega_{i1} \widetilde{A}_{i1}, \sum_{i=1}^n \omega_{i2} \widetilde{A}_{i2}, \cdots \sum_{i=1}^n \omega_{iM} \widetilde{A}_{iM}, \sum_{i=1}^n \omega_{iB} \widetilde{B}_i \Big). \tag{19}$$

5.3 Translation

The translation of $\widetilde{R}$ is needed to match the representative value $Rep(\widetilde{R})$ with the representative value $Rep(\widetilde{O})$. The translation of antecedents or consequent do not change the amount of uncertainty or spread lengths. Every given antecedent $\widetilde{A}_k$; $k \in I$ and consequent $\widetilde{B}$ are translated to $\widetilde{A}'_k$ and consequent $\widetilde{B}'$ by the parameters $l_k = Rep(\widetilde{A}^*_k) - Rep(\widetilde{A}_k)$ and $l_B = Rep(\widetilde{B}^*) - Rep(\widetilde{B})$ respectively. Then, translation parameter of the missing antecedent $\widetilde{A}_t$ is calculated from the formulation of l_B (see section 4.3) as $l_B = \frac{\sum_{k=1}^{M} l_k}{M} \Rightarrow l_t = Ml_B - (\sum_{k\in I} l_k)$.

Thus, from this step we obtain rule $\widetilde{R}' : x_1 = \widetilde{A}'_1, x_2 = \widetilde{A}'_2, \ldots x_m = \widetilde{A}'_m \Rightarrow y = \widetilde{B}'$ whose antecedent and consequent part have the same representative values as the observation $\widetilde{O}$. But, the amount of uncertainty of the rule $\widetilde{R}'$ and the observation $\widetilde{O}$ may not be same which is being addressed in the following.

5.4 Expansion/Contraction

In this step, the difference between the amount of uncertainty of the rule $\widetilde{R}'$ and the observation $\widetilde{O}$ are incorporated. The EC parameters γ_k between the antecedents $\widetilde{A}'_k$ and $\widetilde{A}^*_k$ ($k \in I$) are calculated. Also, the parameter γ_B between $\widetilde{B}'$ and $\widetilde{B}^*$ is calculated. Finally, the EC parameter γ_t for the missing antecedent $\widetilde{A}_t$ is calculated from the formulation of γ_B (see section 4.4) as $\gamma_B = \frac{\sum_{k=1}^{M} \gamma_k}{M} \Rightarrow \gamma_t = M\gamma_B - (\sum_{k\in I} \gamma_k)$. Finally, the missing antecedent $\widetilde{A}^*_t$ is obtained through EC transformation of $\widetilde{A}'_t$ by the parameter γ_t.

The proposed method is furthermore explained by the following Algorithm 2 and examples 5.1 and 5.2.

Algorithm 2 Algorithm of IFRI

Require: Given rule base $\mathbf{R}$ with N rules and observation $\widetilde{O}^*(\widetilde{A}^*_1, \widetilde{A}^*_2, \ldots \widetilde{A}^*_{t-1}, ?, \widetilde{A}^*_{t+1}, \ldots \widetilde{A}^*_M, \widetilde{B}^*)$.

Find the distances d_i for each rule $\widetilde{R}_i \in \mathbf{R}$ using equation (17).
Find least n distances $d_1, d_2, \ldots d_n$ and reconstruct $\mathbf{R} = \{\widetilde{R}_1, \widetilde{R_2}, \ldots \widetilde{R}_n\}$.
for $i = 1$ to n **do**
    for $k \in I$; $I = \{1, 2, \ldots, t-1, t+1, \dot{M}\}$ **do**
        Calculate the weights ω_{ik} using equation (18).
        Calculate the antecedents $\widetilde{A}_k = \sum_{i\in I} \omega_{ik} \widetilde{A}_{ik}$.
    end for
    Calculate the weights ω_{iB} using equation (18).
    Calculate the consequent $\widetilde{B} = \sum_{i=1}^{n} \omega_{iB} \widetilde{B}_i$.
    Calculate $\omega_{it} = m\omega_{iB} - \sum_{k\in I} \omega_{ik}$.
    Calculate the antecedent $\widetilde{A}_t = \sum_{i=1}^{n} \omega_{it} \widetilde{A}_{it}$.
end for
for $k \in I$ **do**
    Calculate the translation parameters $l_k = Rep(\widetilde{A}^*_k) - Rep(\widetilde{A}_k)$.
    Obtain $\widetilde{A}'_k$ by translating $\widetilde{A}_k$ by parameter l_k using definition (3.4).
end for
Calculate the translation parameter $l_B = Rep(\widetilde{B}^*) - Rep(\widetilde{B})$.
Obtain $\widetilde{B}'$ by translating $\widetilde{B}$ by parameter l_B using definition (3.4).
Calculate $l_t = Ml_B - \sum_{k\in I} l_k$.
Obtain $\widetilde{A}'_t$ by translating $\widetilde{A}_t$ by parameter l_t using definition (3.4).
for $k \in I$ **do**
    Find the parameters γ_k between $\widetilde{A}'_k$ and $\widetilde{A}^*_k$.
end for
Find the parameters γ_B between $\widetilde{B}'$ and $\widetilde{B}^*$.
Obtain $\gamma_t = M\gamma_B - \sum_{k\in I} \gamma_k$.
Obtain required antecedent $\widetilde{A}^*_t$ by EC transformation of $\widetilde{A}'_t$ with parameters γ_t.

The following Examples 5.1 and 5.2 are taken from [15] on BFRI to illustrated the applicabity of proposed method.

Example 5.1. *The rules $\widetilde{R}_i$ used in this example have three dimensional antecedents and TFN in antecedent and consequent parts. The required course of action or the missing antecedent $\widetilde{A}^*_3$ is obtained by following the steps mentioned in the above Algorithm 2.*

Rules	Antecedent	Consequent
R_1	(2,2,2) ; (3,3,3); (4,4,4)	(7,7,7)
R_2	(7,9,10); (8,9,10);(9,10,11)	(15,17,19)
O^*	(4,5,6); (5,6,7); *Missing*	(10,11,13)

Table 11: Rules for Example 5.1

Results	
Methods	$\widetilde{A}_3^*$
$HScg$	(5.65,5.80,6.85)
DCK	(6,6.66,7.66)

Table 12: Results of Example 5.1

The auxiliary rule $\widetilde{R} := \widetilde{A}_1(4.14, 5, 5.43) \wedge \widetilde{A}_2(5.5, 6, 6.5) \wedge \widetilde{A}_3(5.36, 5.63, 5.9) \Rightarrow \widetilde{B}(10.2, 11, 11.8)$ *is obtained by following steps 2 to 12 of Algorithm 2. Since the core of* $\widetilde{A}_1, \widetilde{A}_2$ *and* $\widetilde{B}$ *are the same as the core of* $\widetilde{A}_1^*, \widetilde{A}_2^*$ *and* $\widetilde{B}^*$*, translation of* $\widetilde{R}$ *is not needed.*

The parameters of EC γ_1, γ_2 *and* γ_B *are calculated between* $\widetilde{A}_1, \widetilde{A}_2, \widetilde{B}$ *and* $\widetilde{A}_1^*, \widetilde{A}_2^*, \widetilde{B}^*$ *respectively. The parameters are obtained as* $\gamma_{1l} = 1.16, \gamma_{1r} = 2.33$*;* $\gamma_{2l} = 2, \gamma_{2r} = 2$ *and* $\gamma_{1l} = 1.25, \gamma_{1r} = 2.5$*. So, the required parameter* γ *for the missing antecedent* $\widetilde{A}_3$ *is obtained as* $\gamma_{3l} = 3 \times 1.25 - (1.16 + 2) = 0.58$ *and* $\gamma_{3r} = 3 \times 2.5 - (2.33 + 2) = 3.17$.

The missing antecedent $\widetilde{A}_3^*$ ***(5.47,5.63,6.49)*** *is obtained through EC transformation of* $\widetilde{A}_3$ *by parameters* $\gamma_{3l} = 0.58$ *and* $\gamma_{3r} = 3.17$ *as* $a_1^* = 5.63; a_0^* = 5.63 - 0.58(5.63 - 5.47) = 5.47$ *and* $a_3^* = 5.63 + 3.17(5.9 - 5.63) = 6.49$.

Validity: *The validity of the proposed methods can be assured by repeating the proposed forward FRI on the same rule base given in Table 11 and observation* $\widetilde{O}_1^* :\ (4, 5, 6); (5, 6, 7); (5.47, 5.63, 6.49) \Rightarrow ?$*. Using Algorithm 1 of forward FRI, the consequent* $\widetilde{B}^*$ *is obtained as* $\widetilde{B}^*(10, 11, 13)$ *which is the same as given consequent in observation* $\widetilde{O}^*$.

Example 5.2. *In this example, the rule base consists four rules* $\boldsymbol{R} = \{\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, \widetilde{R}_4\}$*. The rules* $\widetilde{R}_i$ *and observation* O^* *have TrFN in antecedent and consequent parts.*

Rules	Antecedent	Consequent
$\widetilde{R}_1$	(0.2,1.1,2.2,2.7) ;(1.5,2,2.5,3); (0.4,1.5,2,2.5); (1.1,1.5,2.1,2.5)	(0.2,2,2.5,3)
$\widetilde{R}_2$	(2.0,2.3,2.5,3.4); (3.1,3.2,3.5,4.3); (2.5,3.5,4.2,4.5); (6.1,7.0,8.0,8.6)	(4.0,4.8,5.3,6.0)
$\widetilde{R}_3$	(8.2,9.510.5,11); (7.5,9,10.2,11.3); (7.3,9.2,10.5,11.1); (3.8,4.1,4.3,5.0)	(9.5,10.0,11.3,12.5)
$\widetilde{R}_4$	(10.5,11.5, 12.5,13.1); (10.0,11.2,12.3,13); (10.2,11.0,11.5,13.2); (10.1,12.0,12.5,14.3)	(12,13,13.5,14.2)
$\widetilde{O}^*$	(3.5,4.0,5.0,7.0); (5.0,5.5,6.0,7.5); *Missing* ; (4.5,5.2,6.5,7.5)	(5.5,6.5,7.0,8.7)

Table 13: Rules for Example 5.2

Results	
Methods	$\widetilde{A}_3^*$
$HScg$	(4.01,5.46, 5.98,6.50)
HSc	(5.78, 6.79, 8.79, 10.74)
DCK	(4.15,5.21, 6.30,7.89)

Table 14: Results of Example 5.1

Proceeding by Algorithm 2, the auxiliary rule is obtained as $\widetilde{R} := \widetilde{A}_1(3.43, 4.15, 4.85, 5.53) \wedge \widetilde{A}_2(4.69, 5.36, 6.03, 6.8) \wedge \widetilde{A}_3(3.82, 4.87, 5.58, 6.1) \wedge \widetilde{A}_4(4.86, 5.55, 6.03, 6.30) \Rightarrow \widetilde{B}(5.42, 6.37, 7.03, 7.8)$*. The representative values* $Rep(\widetilde{A}_1) = 4.5, Rep(\widetilde{A}_2) = 5.69, Rep(\widetilde{A}_4) = 5.79$ *and* $Rep(\widetilde{B}) = 6.7$ *do not matches with the representative values* $Rep(\widetilde{A}_1^*) = 4.5, Rep(\widetilde{A}_2^*) = 5.75, Rep(\widetilde{A}_4) = 5.85$ *and* $Rep(\widetilde{B}) = 6.75$*. So, translation of* $\widetilde{R}$ *is needed. The parameters of translation* $l_1 = 0, l_2 = 0.05, l_4 = 0.06, l_B = 0.05$ *and* $l_3 = 0.05 \times 4 - (0 + 0.05 + .06) = 0.08$ *is obtained by using lines 17 and 18 of Algorithm 2.*

After translation, the rule $\widetilde{R}'$ *is obtained as* $\widetilde{R}' := \widetilde{A}_1'(3.43, 4.15, 4.85, 5.53) \wedge \widetilde{A}_2'(4.74, 5.41, 6.08, 6.85) \wedge \widetilde{A}_3'(3.90, 4.95, 5.66, 6.18) \wedge \widetilde{A}_4'(4.92, 5.61, 6.09, 6.36) \Rightarrow \widetilde{B}'(5.47, 6.42, 7.08, 7.85)$.

The EC parameters $\gamma_1, \gamma_2, \gamma_4$ *and* γ_B *between* $\widetilde{A}_1', \widetilde{A}_2', \widetilde{A}_4', \widetilde{B}'$ *and* $\widetilde{A}_1^*, \widetilde{A}_2^*, \widetilde{A}_4^*, \widetilde{B}^*$ *are calculated as* $\gamma_1(0.69, 1.43, 2.95)$, $\gamma_2(0.74, 0.75, 1.94)$, $\gamma_4(1.01, 1.68, 1.67)$ *and* $\gamma_B(1.05, 0.75, 2.22)$*. The* γ_3 *for the missing antecedent* $\widetilde{A}_3$ *is calculated as* $\gamma_3 = (0.86, 1.76, 2.32)$.

The missing antecedent $\widetilde{A}_3^*$ ***(3.17,5,5.61,6.82)*** *is obtained through EC transformation of* $\widetilde{A}_3'(4.16, 5.37, 6.14, 6.73)$ *by parameters* $\gamma_{3l} = 0.86, \gamma_{3c} = 1.76$ *and* $\gamma_{3r} = 2.32$.

Validity: *Here also, the validation of the proposed forward and backward FRI methods are checked by repeating the proposed forward FRI on the same rule base given in Table 13 and observation* $\widetilde{O}_1^* :\ (3.5, 4.0, 5.0, 7.0) \wedge (5.0, 5.5, 6.0, 7.5) \wedge\ (3.17, 5, 5.61, 6.82); (4.5, 5.2, 6.5, 7.5) \Rightarrow ?$*. Using Algorithm 1 of forward FRI, the consequent* $\widetilde{B}^*$ *is obtained as* $\widetilde{B}^*(5.49, 6.5, 7.0, 8.68)$ *which is almost the same as given consequent in observation* $\widetilde{O}^*$.

6 Conclusion

Geometry based expansion/contraction of different type of fuzzy sets are defined here. A detailed comparison of expansion/contraction with scale and move transformation is presented. It has been observed that when representative value of fuzzy number is treated its core then expansion/contraction of fuzzy numbers is more preferable over scale and move transformation.

Forward and backward rule base interpolation methods are proposed based on expansion/contraction transformation. Both of the proposed forward and backward interpolation methods are summarized in algorithms. Fundamental differences of proposed method with scale and move transformation based method are discussed by worked out examples. It has been established that the proposed FRI method is more preferable when the representative value of fuzzy sets are treated as their core. The proposed method can be further extended for extrapolation in future. The exactness of the proposed method for real life scenarios remain to check. Behaviour of the proposed methods can be analyzed for other choices representative values.

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