Hybrid matrix method for stable analysis of wave propagation in multilayered complex media

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**HYBRID MATRIX METHOD FOR STABLE
ANALYSIS OF WAVE PROPAGATION IN
MULTILAYERED COMPLEX MEDIA**

HYBRID MATRIX METHOD FOR STABLE ANALYSIS OF WAVE
PROPAGATION IN MULTILAYERED COMPLEX MEDIA

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Hybrid Matrix Method for Stable Analysis of Wave Propagation in Multilayered Complex Media

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A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the degree of
Doctor of Philosophy

2008

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research done by me and has not been submitted for a higher degree to any other University or Institute.

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Summary

This thesis presents the analysis of electromagnetic and acoustic wave propagation in multilayered complex media.

First of all, a new hybrid matrix method is proposed for stable analysis of electromagnetic wave propagation in multilayered bianisotropic media. The method overcomes the numerical instability of transfer and impedance matrix methods known for the entire thickness and frequency range. To determine the hybrid matrix of each layer, a new and simple self-recursive asymptotic method along with thin-layer asymptotic approximation is introduced. It only requires elementary matrix operations and bypasses the intricacies of eigenvalue-eigenvector approach. The stack hybrid matrix for multilayered bianisotropic media is also obtained by employing matrix recursions. The method is then applied to the study of the dispersion relation of layered bianisotropic waveguides, and to calculating the reflection coefficient and shielding effectiveness of multilayered anisotropic materials for the design of radar absorbers and laminated shields.

Then, the propagation, reflection and transmission of light in one-dimensional photonic crystals are investigated. The Bloch-Floquet waves are determined by a new generalized eigenproblem of hybrid matrix method. It overcomes the numerical

instability in the standard eigenproblem of transfer matrix method. Using the imaginary part of the Bloch-Floquet wavenumbers, we demonstrate that it is convenient to determine (if any) the frequency range of omnidirectional reflection. The effects of chirality, loss and tunable anisotropy are also discussed along with the numerical results.

Finally, the hybrid matrix method is extended for the analysis of acoustic wave propagation in multilayered solids and fluids. The method can still provide robust results even when the thickness tends to infinity or zero. The matrix recursions for multilayered media with different solid and fluid phases are presented. To increase the computational speed, the surface hybrid matrix is introduced that can reduce the number of recursion equations required. The frequency spectra of transmission coefficient and the dispersion relation for multilayered solids and fluids structures can be investigated efficiently by this method.

List of Abbreviations and Symbols

Abbreviations

EM	electromagnetic
TE	transverse electric field
TM	transverse magnetic field
SE	shielding effectiveness
PEC	perfect electric conductor
MIC	microwave integrated circuit
ODTR	omnidirectional total reflection
DNG	double-negative
SAW	surface acoustic wave
BAW	bulk acoustic wave

Symbols

Matrices are represented in bold and non-italic and without overline; Vectors are represented in italic and with overline.

$\overline{D}$	electric flux density
$\overline{B}$	magnetic flux density
$\overline{E}$	electric field
$\overline{H}$	magnetic field
$\overline{E}_t$	transverse electric field components
$\overline{H}_t$	transverse magnetic field components
ϵ	permittivity
μ	permeability
ξ, ζ	magneto-electric tensors
ω	angular frequency
k	wavenumber
Ψ	eigenwave matrix
$\mathbf{T}$	transfer matrix
$\mathbf{Z}$	impedance matrix
$\mathbf{H}$	hybrid matrix
$\mathbf{T}_f, \mathbf{Z}_f, \mathbf{H}_f$	layer matrices for layer f
$\mathbf{T}^{(l,f)}, \mathbf{Z}^{(l,f)}, \mathbf{H}^{(l,f)}$	stack matrices from layer l to layer f

$\mathbf{T}_c, \mathbf{Z}_c, \mathbf{H}_c$	cell matrices for unit cell
$\mathbf{I}$	identity matrix (size varies accordingly)
$\mathbf{0}$	zero matrix (size varies accordingly)
r	reflection coefficient
t	transmission coefficient
η	intrinsic impedance of free space
$\bar{\mathbf{v}}$	particle velocity vector
$\bar{\boldsymbol{\tau}}$	normal stress vector
$\mathbf{C}$	stiffness constant
ρ	charge density, mass density

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Chapter 1

Introduction

1.1 Motivation

Although considerable efforts have been invested in the analysis of electromagnetic wave propagation in multilayered complex media, many challenging problems still remain to be solved. For instance, the transfer matrix method, well known in the analysis of multilayered problem, is applicable in principle, but its practical implementation may lead to numerical instability (also known as exponential dichotomy). In particular, the transfer matrix computations may become inaccurate or overflow when a layer is electrically thick enough. This is due to the mixture of exponentially growing and decaying terms that lead to loss of precision during computations.

Several methods have been developed for overcoming the numerical problem of the transfer matrix method. However, these methods often rely on the exact solutions to eigenvalue-eigenvector problem or nonlinear Riccati differential equation, which involve complicated manipulations and calculations. Recently, an impedance matrix

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method has been proposed. Although the method maintains its numerical stability when the layer thickness grows to infinity, it is inaccurate when a layer thickness decreases toward zero. Since the methods mentioned above are conditionally stable or comparatively complex, it is highly desirable to find a stable and simple method to analyze electromagnetic wave propagation in multilayered media.

Light propagation in photonic crystals, which are artificial periodic composites, has attracted much attention in recent years. These materials can manipulate light in a way like conventional semiconductors do to electrons. Photonic crystals are expected to serve as the platform for future integrated optical circuits, due to their ability to effectively control light. Generally, the Bloch-Floquet waves are determined by a standard eigenproblem of transfer matrix method. Once the transfer matrix for the whole unit cell has been determined through proper cascading, Bloch-Floquet waves are obtained simply as its eigenvectors with the associated eigenvalues being the exponentials of Bloch-Floquet wavenumbers. This method however, also suffers from the numerical instability problem when the unit cell is electrically thick as stated above. Hence, we aim to find a stable method to determine the light propagation in photonic crystals.

Omnidirectional total reflection (ODTR) has a wide variety of applications, such as in optical communication, electromagnetic energy transportation, and solar energy engineering. Light incident to ODTR will be totally reflected in all directions. Remarkably, this effect extends over finite spectral bands, rather than single frequencies. ODTR may occur when light propagates along one-dimensional photonic

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crystals. Therefore it is also interesting for this phenomenon to be investigated in this thesis.

By properly organizing the field variables and applying the appropriate matrix methods, one can analyze the acoustic wave propagation in a similar manner as electromagnetic wave propagation. However, as opposed to electromagnetics, the size of pertaining matrices in acoustics may change when the phases of media layers are different (solid and fluid). Therefore, the matrix methods to be dealt with need to be modified accordingly during the analysis. Furthermore, acoustic wave propagation in multilayered media may be expected to encounter the same numerical problem for certain matrix methods. This calls for alternative method that is robust and simple as mentioned previously.

1.2 Objectives

The objectives of this thesis are as follows:

- (i). To find a stable and simple method to analyze electromagnetic wave propagation in multilayered bianisotropic media — A new hybrid matrix method that overcomes the numerical instability of transfer and impedance matrix methods for the entire thickness and frequency range will be investigated. To bypass the intricacies of eigenvalue-eigenvector approach, a

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simple self-recursive asymptotic algorithm applying thin-layer asymptotic approximation will be presented. The stability and accuracy of the method will also be demonstrated.

- (ii). To analyze the reflection, transmission and propagation of light in one-dimensional photonic crystals — A new method based on a generalized eigenproblem of hybrid matrix will be introduced to determine the Bloch-Floquet waves. The effects of chirality, loss and tunable anisotropy will also be investigated.
- (iii). To extend the new matrix method for applications in acoustics — Acoustic wave propagation in multilayered solids and fluids will be investigated. The hybrid matrix method for multilayered media with different solid and fluid phases will be presented.
- (iv). To increase the computational speed of new matrix method whenever possible, e.g. when only the reflection and transmission coefficients or the dispersion relations are needed.

1.3 Contributions

The contributions of this thesis include the following:

- (i). A new *stable* matrix method, called hybrid matrix method, for analysis of electromagnetic wave propagation in multilayered bianisotropic media is proposed. It overcomes the numerical instability of transfer and impedance matrix methods for the entire thickness and frequency range. The stability and accuracy of hybrid matrix are demonstrated analytically and numerically.
- (ii). To determine the hybrid matrix of each layer, a new *simple* self-recursive asymptotic method along with thin-layer asymptotic approximation is introduced. It requires only elementary matrix operations and bypasses the intricacies of eigenvalue-eigenvector approach.
- (iii). For propagation of light in one-dimensional photonic crystals, a new *generalized eigenproblem* of hybrid matrix method is presented to determine the Bloch-Floquet waves. It overcomes the numerical instability in the standard eigenproblem of transfer matrix method. By using the imaginary part of the Bloch-Floquet wavenumbers, it is convenient to determine (if any) the frequency range of omnidirectional reflection. The effects of chirality, loss and tunable anisotropy are also investigated.

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- (iv). The hybrid matrix method is extended for analysis of acoustic wave propagation in multilayered media with different *solid and fluid phases*. The method can still provide stable results even when the thickness tends to infinity and zero.
- (v). To increase the computational speed, a new *surface hybrid matrix method* is introduced. It can reduce the number of recursion equations required when only the reflection and transmission coefficients or the dispersion relations are of concern.

1.4 Organization of the Thesis

The thesis is organized as follows

Chapter 2 provides a brief literature survey of electromagnetic and acoustic wave propagation in multilayered complex media. The multilayered electromagnetic media and its applications are introduced firstly. Then the constitutive relations of some common media are listed out, and a convenient method based on Euler angles is described to perform coordinate transformation. A brief review of the matrix methods to analyze electromagnetic wave propagation in multilayered media is given. In the same manner, the acoustic wave propagation is discussed accordingly for its

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multilayered media, constitutive relations, coordinate transformation and matrix methods.

Chapter 3 presents the hybrid matrix method for stable analysis of electromagnetic wave propagation in multilayered bianisotropic media. We formulate the 4×4 system matrix directly from Maxwell's equations and deduce the corresponding eigenwaves. Next, the transfer, impedance and hybrid matrices for each layer are introduced along with their relations, stability and accuracy analysis. It is shown that the hybrid matrix method overcomes the numerical instability of transfer matrix and impedance matrix methods, even when the thickness tends to infinity or zero. To bypass the intricacies of eigenvalue-eigenvector approach, a set of simple self-recursive asymptotic algorithms, along with thin-layer asymptotic approximations, are presented to calculate these matrices for each layer. The definitions of stack transfer, impedance and hybrid matrices, as well as their matrix recursions are also provided. The hybrid matrix method is applied to study the dispersion relation of layered bianisotropic waveguides. It is also used to calculate the reflection coefficient and shielding effectiveness of multilayered anisotropic materials, which are useful for design of radar absorbers and laminated shields.

Chapter 4 presents a method based on the solutions to a generalized eigenproblem of hybrid matrix for analysis of light propagation in one-dimensional photonic crystals. It enables Bloch-Floquet waves to be determined reliably and overcomes the numerical instability in the standard eigenproblem of transfer matrix. Using the imaginary part of Bloch-Floquet wavenumbers, we demonstrate that it is convenient to determine (if any)

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the frequency range of omnidirectional reflection. The effects of chirality, loss and tunable anisotropy are also discussed along with the numerical results.

Chapter 5 extends the hybrid matrix method for analysis of acoustic wave propagation in multilayered solids and fluids. The method overcomes the numerical instability of transfer matrix and impedance matrix methods even when the thickness tends to infinity and zero. The matrix recursions for multilayered media with different solid and fluid phases are presented. To increase the computational speed, the surface hybrid matrix is introduced which can reduce the number of recursion equations required. The frequency spectra of transmission coefficient and the dispersion relation for multilayered solids and fluids structures are demonstrated.

Chapter 6 provides the conclusions of this thesis and some suggestions for future research.

Chapter 2

A Brief Literature Survey of Wave Propagation in Multilayered Complex Media

Multilayered media have found wide applications in military and industrial applications involving electromagnetic and/or acoustic wave propagation. The media can be arbitrarily complex involving the most general bianisotropic materials for electromagnetics and anisotropic media for acoustics. This chapter provides a brief literature survey of electromagnetic and acoustic wave propagation in multilayered complex media. The multilayered electromagnetic media and their applications are introduced in Section 2.1.1. Then the constitutive relations of some common media are listed out in the section following. In Section 2.1.3, a convenient method based on Euler angles is described to perform coordinate transformation for (bi)anisotropic media. A brief review of the matrix methods to analyze electromagnetic wave propagation in multilayered media is given in Section 2.1.4. In the same manner, the acoustic wave propagation is discussed accordingly for its multilayered media, constitutive relations, coordinate transformation and matrix methods in Section 2.2.

2.1 Electromagnetic Wave Propagation

2.1.1 Multilayered Electromagnetic Media

Electromagnetic waves propagation in multilayered media is an extremely important problem in microwave theory, antenna theory, optics, and electromagnetics in general [1]-[9]. This problem has been studied for many applications in electromagnetics involving a wide variety of different material layers, e.g. dielectrics, magnetic materials, anisotropic dielectric and magnetic materials, and bianisotropic materials. One of the most important applications is that it can be applied as substrates of planar microwave components and printed circuit antennas. In fact, this type of media has been successfully employed in the fabrication of microwave filters and couplers [10], waveguides [11], high frequency interconnects [12], [13], microstrip patch antennas [14], [15], etc. Besides being low cost, light weight, and small volume, planar circuits and antenna elements are highly conformable and easy to integrate with microwave solid state devices. In some occasions, the materials used in the substrates of planar microwave circuits and printed antennas are anisotropic, i.e., they are characterized by a second rank permittivity tensor or/and a second rank permeability tensor. One such class of examples can be found in some crystalline substrate materials such as sapphire, alumina, quartz, etc. [16]. The sapphire has attracted considerable attention since it exhibits several very desirable properties in that it is optically transparent, compatible with high-resistivity silicon and its electrical properties are reproducible from batch to

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batch. Some plastic substrates impregnated with either ceramic particles or glass fibers (Epsilam-10, impregnated PTFE) also show dielectric anisotropy. The anisotropy effects have to be taken into account when planar transmission lines or printed antennas are fabricated on these anisotropic dielectric substrates. Dielectric anisotropy is not necessarily viewed as an inconvenience which complicates the design of circuit components and antennas. It can be utilized beneficially, for instance, in the fabrication of high directivity microstrip couplers [16].

By combining the desirable properties of different materials, it is possible that the performances of multilayered media be superior to those of homogeneous ones. More specifically, materials which are anisotropic and exhibit nonreciprocal behavior can be readily integrated with isotropic, reciprocal substrates in multilayered circuits. For example, magnetic materials such as ferrites can be integrated with semiconductor substrates such as GaAs or Si to produce nonreciprocal circuit or antenna components that are merged with other microwave integrated circuit (MIC) structures [17]-[19]. The nonreciprocal properties of ferrites result in phase differentiation for forward and backward waves, which have been widely applied in the design of the key elements in microwave devices such as phase shifters, isolators, and circulators [19], [20]. This nonreciprocal behavior has been shown to be present in planar transmission lines printed on magnetized ferrite substrates [17], [21], [22]. Understanding the effects of interfacing these nonreciprocal media with isotropic substrates is desirable for improving MIC performance. Further study has been revived nowadays because many materials used as substrates for integrated microwave/millimeter-wave circuits and

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printed-circuit antennas can exhibit some form of anisotropy that is intentionally introduced during manufacturing.

Most media mentioned above are particular cases of the general linear bianisotropic media. The concept of bianisotropic medium was coined by Cheng and Kong [23], [24] during which the problems involving moving media, e.g. plasma, ionosphere, took quite a central role in the world-wide research on electromagnetics. In fact, there were extensive studies on electromagnetics of different media, dispersive or nondispersive, in relativistic or nonrelativistic motion, in free or bounded space, in frequency or time domain, describing solutions for radiation fields, wave numbers, Green's functions, antenna impedances, etc. It was shown that almost any media in motion can be described as a stationary bianisotropic medium, thus made working with moving frames and Lorentz transformations obsolete [25]. The main theorems and principles governing the bianisotropic medium were subsequently worked out [26]. In due time, the content was transferred to the monograph by Kong [3], which covers almost all fundamental electromagnetic properties of bianisotropic media. With the emergence of the concept of bianisotropic medium, the separate branches of electromagnetics research on moving media, Tellegen media, optically active materials, magneto-electric crystals [27], etc., were all united under the same topic. In light of recent interest and attention being paid to composite media, more creative use of various novel materials for monolithic integrated circuits would be expected. For instance, very complex materials with a combination of birefringent, gyroelectric, gyromagnetic or other anisotropic and magneto-electric coupling properties may be

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utilized. The presence of these complex features necessitates the development of rigorous and accurate mathematical models to analyze the wave propagation characteristics in these media, as to be discussed later.

Apart from microwaves and millimeter-waves, multilayered media also play a very important role in many applications of modern optics. In recent years, much attention has been paid to photonic crystals, a new kind of optical material to control electromagnetic waves [28]-[31]. Photonic crystals are composite structures with a periodic arrangement of refractive index in one dimension, two dimensions, or three dimensions. Wave propagation in these media exhibits many interesting and potentially useful phenomena. These include Bragg reflection, holography, optical stop band, negative refraction, superprism etc. The simplest photonic crystal is that of one-dimensional which is easier to fabricate and lower cost. In essence, the one-dimensional photonic crystal is a special class of multilayered media in which layers of optical materials are stacked in a periodic fashion. They are often used to suppress or enhance reflectance, to alter the spectral or the polarization characteristics of light or as gratings that reflect optical waves incident at certain angles, or as filters that selectively reflect waves of certain frequencies, and many more applications are in the offing [31]-[40].

Optical waves interact closely with periodic media particularly when the scale of the periodicity is of the same order as that of the wavelength. For instance, spectral bands emerge where light waves cannot propagate through the medium without severe attenuation. With frequencies lying within these forbidden bands, called photonic

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bandgaps, the waves behave in a manner similar as total internal reflection, but applicable for all directions. The dissolution of the transmitted waves is a result of destructive interference among the waves scattered by elements of the periodic structure in the forward direction. Remarkably, this omnidirectional reflection extends over finite spectral bands, rather than for just single frequencies. This phenomenon is analogous to the electronic properties of crystalline solids such as semiconductors, where the periodic wave associated with an electron travels in a periodic crystal lattice, and energy bandgaps are commonly found. (Because of this analogy, the photonic periodic structures have come to be called photonic crystals!) Omnidirectional dielectric reflectors can have wide potential applications, as in microcavities, nanoantenna substrates, optical communication cables, and solar energy engineering.

2.1.2 Constitutive Relations

Let us start with the Maxwell's equations

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t} \bar{B} \quad (2.1)$$

$$\nabla \times \bar{H} = \frac{\partial}{\partial t} \bar{D} + \bar{J} \quad (2.2)$$

$$\nabla \cdot \bar{B} = 0 \quad (2.3)$$

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$$\nabla \cdot \bar{D} = \rho \quad (2.4)$$

The electric and magnetic fluxes are related to the electric and magnetic fields via the constitutive relations:

$$\begin{bmatrix} \bar{D} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{H} \end{bmatrix} \quad (2.5)$$

where $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ represent respectively the medium permittivity, permeability and magneto-electric dyadics. Below we list down some types of media commonly considered from the simplest to the most complex ones. It is assumed that the parameters are not equal unless otherwise stated.

- Isotropic:

$$\boldsymbol{\varepsilon} = \varepsilon \mathbf{I} \quad (2.6)$$

$$\boldsymbol{\mu} = \mu \mathbf{I} \quad (2.7)$$

$$\boldsymbol{\xi} = \boldsymbol{\zeta} = 0 \quad (2.8)$$

In recent years, metamaterial is one of the most interesting isotropic media where permittivity and/or permeability are assumed to have negative real values in the following manner [41]-[47]:

- Double-negative (DNG)

$$\varepsilon < 0 \quad \text{and} \quad \mu < 0 \quad (2.9)$$

- Epsilon-negative (ENG)

$$\varepsilon < 0 \quad \text{and} \quad \mu > 0 \quad (2.10)$$

- Mu-negative (MNG)

$$\varepsilon > 0 \quad \text{and} \quad \mu < 0 \quad (2.11)$$

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- Uniaxial (electrically) anisotropic:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{\perp} & 0 & 0 \\ 0 & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{//} \end{bmatrix} \quad (2.12)$$

$$\boldsymbol{\mu} = \mu \mathbf{I} \quad (2.13)$$

Sometimes one may encounter different representations for such medium, e.g. $\varepsilon_{//}$ may be in the first or second instead of third diagonal position. This is only a choice of coordinate systems which can be converted easily to one another through Euler rotations, as to be introduced shortly later.

- Uniaxial (magnetically) anisotropic:

$$\boldsymbol{\varepsilon} = \varepsilon \mathbf{I} \quad (2.14)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{\perp} & 0 & 0 \\ 0 & \mu_{\perp} & 0 \\ 0 & 0 & \mu_{//} \end{bmatrix} \quad (2.15)$$

- Biaxial (electrically) anisotropic:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \quad (2.16)$$

$$\boldsymbol{\mu} = \mu \mathbf{I} \quad (2.17)$$

- Biaxial (magnetically) anisotropic:

$$\boldsymbol{\varepsilon} = \varepsilon \mathbf{I} \quad (2.18)$$

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$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix} \quad (2.19)$$

- Electrically anisotropic:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad (2.20)$$

$$\boldsymbol{\mu} = \mu \mathbf{I} \quad (2.21)$$

- Magnetically anisotropic:

$$\boldsymbol{\varepsilon} = \varepsilon \mathbf{I} \quad (2.22)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \quad (2.23)$$

- General anisotropic:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad (2.24)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \quad (2.25)$$

- Bi-isotropic [6]:

$$\boldsymbol{\varepsilon} = \varepsilon \mathbf{I} \quad (2.26)$$

$$\boldsymbol{\mu} = \mu \mathbf{I} \quad (2.27)$$

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$$\xi = \xi \mathbf{I} \quad (2.28)$$

$$\zeta = \zeta \mathbf{I} \quad (2.29)$$

A reciprocal type of such medium that has attracted lots of attention is called chiral medium [6], [48], with

$$\xi = -\zeta = \xi \mathbf{I} \quad (2.30)$$

- Uniaxial bianisotropic:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{\perp} & 0 & 0 \\ 0 & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{//} \end{bmatrix} \quad (2.31)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{\perp} & 0 & 0 \\ 0 & \mu_{\perp} & 0 \\ 0 & 0 & \mu_{//} \end{bmatrix} \quad (2.32)$$

$$\xi = \begin{bmatrix} \xi_{\perp} & 0 & 0 \\ 0 & \xi_{\perp} & 0 \\ 0 & 0 & \xi_{//} \end{bmatrix} \quad (2.33)$$

$$\zeta = \begin{bmatrix} \zeta_{\perp} & 0 & 0 \\ 0 & \zeta_{\perp} & 0 \\ 0 & 0 & \zeta_{//} \end{bmatrix} \quad (2.34)$$

- Chiroplasma [49], [52]:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_t & \varepsilon_g & 0 \\ -\varepsilon_g & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_p \end{bmatrix} \quad (2.35)$$

$$\boldsymbol{\mu} = \mu \mathbf{I} \quad (2.36)$$

$$\xi = -\zeta = \xi \mathbf{I} \quad (2.37)$$

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- Gyrotropic bianisotropic [49], [52]:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_t & \varepsilon_g & 0 \\ -\varepsilon_g & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_p \end{bmatrix} \quad (2.38)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_t & \mu_g & 0 \\ -\mu_g & \mu_t & 0 \\ 0 & 0 & \mu_p \end{bmatrix} \quad (2.39)$$

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_t & \xi_g & 0 \\ -\xi_g & \xi_t & 0 \\ 0 & 0 & \xi_p \end{bmatrix} \quad (2.40)$$

$$\boldsymbol{\zeta} = \begin{bmatrix} \zeta_t & \zeta_g & 0 \\ -\zeta_g & \zeta_t & 0 \\ 0 & 0 & \zeta_p \end{bmatrix} \quad (2.41)$$

- General bianisotropic:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad (2.42)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \quad (2.43)$$

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_{xx} & \xi_{xy} & \xi_{xz} \\ \xi_{yx} & \xi_{yy} & \xi_{yz} \\ \xi_{zx} & \xi_{zy} & \xi_{zz} \end{bmatrix} \quad (2.44)$$

$$\boldsymbol{\zeta} = \begin{bmatrix} \zeta_{xx} & \zeta_{xy} & \zeta_{xz} \\ \zeta_{yx} & \zeta_{yy} & \zeta_{yz} \\ \zeta_{zx} & \zeta_{zy} & \zeta_{zz} \end{bmatrix} \quad (2.45)$$

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2.1.3 Coordinate Transformation

It is useful to have a convenient method to perform coordinate transformation for (bi)anisotropic media. Under arbitrary transformation of coordinates, it could make the constitutive parameters “look” very different. For instance, the components that are equal before the transformation may become unequal and some previously zero elements may become nonzero.

Euler angles are a means of representing the spatial orientation of any frame of the space as a composition of rotations from a reference frame [50], [51]. In the following, we denote the fixed system in lower case (x, y, z) and the rotated system in upper case (X, Y, Z) . Fig. 2.1 illustrates the Euler angles system. The intersection of the xy and XY coordinate planes is called the line of nodes (N). α is the angle between the x axis and the line of nodes. β is the angle between the z axis and the Z axis. γ is the angle between the line of nodes and the X axis. This definition is called $z-x-z$ convention and is the most common one in use. The relation between the original coordinate system vectors $\hat{x}, \hat{y}, \hat{z}$ and the coordinate system vectors $\hat{X}, \hat{Y}, \hat{Z}$ obtained by rotating $\hat{x}, \hat{y}, \hat{z}$ with the Euler angles α, β, γ is:

$$\begin{bmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{bmatrix} = \mathbf{R} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (2.46)$$

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Fig. 2.1 Illustration of Euler angles

$$\begin{aligned}
 \mathbf{R} &= \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & \sin \alpha \cos \gamma + \cos \alpha \cos \beta \sin \gamma & \sin \beta \sin \gamma \\ -\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \beta \cos \gamma & \sin \beta \cos \gamma \\ \sin \beta \sin \alpha & -\sin \beta \cos \alpha & \cos \beta \end{bmatrix} \quad (2.47)
 \end{aligned}$$

Then the matrix $\mathbf{m}'$ in (X, Y, Z) coordinate system can be transformed from $\mathbf{m}$ in (x, y, z) coordinate system by

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$$\mathbf{m}' = \mathbf{R} \mathbf{m} \mathbf{R}^{-1} \quad (2.48)$$

One can notice that

$$\begin{aligned} \mathbf{R}^{-1} &= \left(\begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix}^{-1} \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix}^T \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \\ &= \mathbf{R}^T \end{aligned} \quad (2.49)$$

where the superscript T stands for transpose operation. Thus (2.48) can be written as

$$\mathbf{m}' = \mathbf{R} \mathbf{m} \mathbf{R}^T \quad (2.50)$$

Here, $\mathbf{m}$ and $\mathbf{m}'$ may stand for the original and transformed constitutive tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$ or $\boldsymbol{\zeta}$ as mentioned in the previous section.

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2.1.4 Matrix Methods

Analysis of electromagnetic waves and their propagation, reflection, transmission, and scattering in multilayered media has been receiving wide attention for long time [1]-[9]. The problem of wave propagation in (bi)anisotropic media is considerably more difficult to analyze than the isotropic one. Since the (bi)anisotropic constitutive parameters couple the electromagnetic field components together, they constitute a much more complicated system than that arises in the isotropic case. In dealing with most isotropic problems, the typical approach is to decouple the electric and magnetic fields components from one another and then derive a second-order partial differential wave equation from which the solution can be obtained. For most (bi)anisotropic problems, this procedure is quite intractable.

Attempting this procedure for most (bi)anisotropic systems would lead to fourth or higher order partial differential equations that would be quite difficult to solve. For multilayered media manifested in the above-mentioned microwave/millimeter-wave integrated circuits and optical devices, a very popular and more direct technique utilizes the 4×4 transfer matrix method (or exponential matrix) [53]-[67]. The procedure involves Fourier transform for all electromagnetic field quantities with respect to the transverse coordinate(s). Then the reduced Fourier transformed field variable equations are manipulated into a standard state variable form. Eigensolutions of these first-order state variable equations yield the propagation constants and propagation modes of the system. In this procedure, the two longitudinal field

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components are expressed in terms of the four transverse field components and then substituted into Maxwell's equations to write in a 4×4 matrix form. This allows simple boundary matching of the tangential field components from one layer interface to another.

The transfer matrix method was implemented by Teitler and Henvis [53] and was further developed by Berreman [54]. Berreman has also studied several anisotropic material examples, including propagation in orthorhombic crystal and optically active material using transfer matrix method. Many authors subsequently have also used the transfer matrix method to study propagation of plane waves in multilayered (bi)anisotropic media [57]-[67]. Most of these references apply the eigenvalue-eigenvector approach to determine the transfer matrix. In spite of being straightforward, solving eigenvalue-eigenvector is usually very tedious, which requires complex root searching, degeneracy treatment and upward/downward eigenvector sorting or selection. Despite the extensive literature, one still finds that the system matrices given by most authors are often cumbersome to manipulate. Especially for general complex media, the matrix elements are expressed in very lengthy and exhaustive forms that are prone to making mistakes. This renders lucid interpretation of the results difficult and prohibits identification of key constituents in the solutions. An alternative approach that obviates the need to solve for eigenvalue-eigenvector is through the direct application of finite difference approximation proposed by Morgan et al. [68]. The transfer matrix is obtained by successive multiplications of an approximated transfer matrix for thin layers.

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A more serious problem is that although the transfer matrix method is applicable in principle, its practical implementation may lead to numerical instability (also known as exponential dichotomy) [69]-[71]. In particular, the transfer matrix computations may become inaccurate or overflow when a layer is electrically thick enough with the presence of inhomogeneous waves within the layer. The physical explanation is that from one layer interface to another, parts of the waves die out before reaching the interface. Therefore, the transfer matrix can no longer pass the complete information of fields over. Several methods have been developed which can overcome the numerical problem of the transfer matrix method such as recursive transformation, eigen- or Riccati-based admittance, impedance and reflection matrix methods [70]-[76]. Since these methods often rely on the exact solutions to eigenvalue-eigenvector problem or nonlinear Riccati differential equation, the simplicity of previous finite difference approximation is generally lost. A self-recursive asymptotic stiffness matrix method has been developed for analysis of acoustic waves in layered piezoelectric media [77]. The analysis has been adapted and extended for electromagnetic waves in bianisotropic media through self-recursive asymptotic impedance matrix method [78]. The use of impedance matrix overcomes the numerical instability problem associated with transfer matrix above and retains the simplicity without having to deal with eigenvalue-eigenvector or nonlinear Riccati equation.

2.2 Acoustic Wave Propagation

2.2.1 Multilayered Acoustic Media

For many years there has been considerable interest in the acoustic wave propagation in multilayered anisotropic media due to its many applications in composite materials, nondestructive evaluation, geophysics, surface acoustic wave (SAW) devices, bulk acoustic wave (BAW) devices etc. [1], [2], [79]-[85]. In recent years, continued efforts have been expanded upon modeling wave propagation interaction with layered anisotropic media. This interest has been prompted by the use of composite materials in a wide variety of applications. Typically, composite structural components are made up of a stack of layers called plies or lamina to form a laminate. The individual lamina in structural composites is composed of brittle, stiff fibers embedded in a matrix of more ductile material which bonds the fibers together and acts as a load-transfer medium. However, the morphology of these classes of materials can seriously complicate their mechanical response, as compared with that of homogeneous isotropic media. Furthermore, these classes of materials also differ from isotropic homogeneous materials in that they are both anisotropic and inhomogeneous. The degree of anisotropy depends upon the specific material under consideration, the interfacial conditions, and the scale lengths involved.

Fiber reinforced composites exhibit anisotropy on micro- as well as on macro-scales, referred respectively to as micro- and macro-anisotropy.

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Micro-anisotropy results if one or more of the composite's basic constituents are anisotropic. It is also produced as a consequence of microstructure combinations that are highly dependent upon their specific geometric ordering. On the other hand, macro-anisotropy arises from combinations of different composite laminates to make a new composite structure, such as a plate made up of several arbitrarily oriented layers of the same basic composite medium. Due to the anisotropy of the individual lamina, composite components constitute a more complicated class of layered media.

Problems relating to the interaction of acoustic waves with fluid-loaded solids have also been widely studied [82], [83]. Most fluid loading conditions are introduced for the sake of ultrasonic inspection of the solid components. For example, the inspection of composite components above can be based upon the ultrasonic wave interactions with it. Such ultrasonic wave interactions with general solid materials may use the common water immersed techniques. Inspection systems ranging from the relatively simple one of isotropic solid to the most general ones of multilayered anisotropic media have been considered. Undoubtedly, analytical solutions to the fluid-loaded systems require adaptations, modifications or extensions of previous solutions pertaining to the corresponding dry cases. Inspection applications may include finding anomalies, properties, quality of bonds and even the morphology of the solid. The role of the fluid is often to facilitate the experiments. There are situations in which the fluid can substantially alter the behavior of the solid.

To analyze acoustic wave propagation in multilayered anisotropic media, the elastic properties of the individual layers are needed through certain constitutive

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relations. For composite materials, these properties are not necessarily available directly and one needs to deduce them. The properties will depend on the intrinsic properties and volume fractions of the individual components comprising the composite. For layered and fibrous composites, certain anisotropy will result depending upon the geometric arrangement of the individual components.

2.2.2 Constitutive Relations

Let us start with the equation of motion

$$\nabla \cdot \mathbf{T} = -\omega^2 \rho \bar{\mathbf{u}} = -i\omega \rho \bar{\mathbf{v}} \quad (2.51)$$

$$\mathbf{S} = \frac{1}{2} [\nabla \bar{\mathbf{u}} + (\nabla \bar{\mathbf{u}})^T] \quad (2.52)$$

where $\mathbf{T}$ is the stress tensor of 2nd rank, $\mathbf{S}$ is the strain tensor of 2nd rank, ρ is the density of the material, $\bar{\mathbf{v}}$ and $\bar{\mathbf{u}}$ are velocity and displacement vectors, respectively.

The constitutive relation for $\mathbf{T}$ and $\mathbf{S}$ is in the form [85]

$$\mathbf{T} = \mathbf{C} : \mathbf{S} \quad (2.53)$$

Here, $\mathbf{C}$ is a stiffness tensor of 4th rank, and it is inconvenient to use full subscript notation C_{ijkl} to express the constitutive relations. In acoustics, it is common to introduce a system of abbreviated subscripts to simplify the strain components. As the

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strain tensor is symmetric, each component can be specified by one subscript rather than two. These are defined according to the scheme

$$\mathbf{S} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} = \begin{bmatrix} S_1 & \frac{1}{2}S_6 & \frac{1}{2}S_5 \\ \frac{1}{2}S_6 & S_2 & \frac{1}{2}S_4 \\ \frac{1}{2}S_5 & \frac{1}{2}S_4 & S_3 \end{bmatrix} \quad (2.54)$$

where the order of numbering in the abbreviated system follows the cyclic pattern shown. The convention of introducing factors 1/2 is standard practice in elasticity theory, the reason being that it simplifies some of the key equations. In this abbreviated subscript notation, the strain may be written as a six-element column vector rather than as a nine-element square matrix. That is

$$\bar{\mathbf{S}} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} \quad (2.55)$$

Since the stress matrix is also symmetric, the abbreviated subscript notation introduced for strain tensor can also be used to describe stress components. In this case the convention is to omit the factors 1/2 that appeared in (2.54),

$$\mathbf{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = \begin{bmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{bmatrix} \quad (2.56)$$

and the stress can now be written as a six-element column vector

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$$\bar{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} \quad (2.57)$$

By applying the symmetry condition of stress and strain, we will have

$$C_{ijkl} = C_{jikl} \quad (2.58)$$

$$C_{ijkl} = C_{ijlk} \quad (2.59)$$

With these constraints on the stiffness constants, the four subscripts may be reduced to two by using abbreviated subscript notation as for **S** and **T**, see Table 2.1

Table 2.1 Abbreviated subscript notation

<i>I</i>	<i>ij</i>
1	<i>xx</i>
2	<i>yy</i>
3	<i>zz</i>
4	<i>yz, zy</i>
5	<i>xz, zx</i>
6	<i>xy, yx</i>

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Therefore, the abbreviated stiffness constants for general anisotropic (triclinic) system is

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \quad (2.60)$$

The forms of $\mathbf{C}$ for other symmetric systems can be referred to Appendix A. With the abbreviated $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{C}$, the constitutive relation for solid can be written in a rather compact form

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} \quad (2.61)$$

As to be introduced in Chapter 5, we will also deal with the normal stress vector $\bar{\tau}$, specifically

$$\bar{\tau} = \hat{\mathbf{z}} \cdot \mathbf{T} = \begin{bmatrix} T_{zx} \\ T_{zy} \\ T_{zz} \end{bmatrix} = \begin{bmatrix} T_5 \\ T_4 \\ T_3 \end{bmatrix} \quad (2.62)$$

Meanwhile, the constitutive relation for fluid has much simpler form

$$\mathbf{T} = \rho c^2 \nabla \cdot \bar{\mathbf{u}} \mathbf{I} \quad (2.63)$$

where c is the speed of sound in the fluid.

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2.2.3 Coordinate Transformation

Unlike for electromagnetics whose permittivity, permeability and magneto-electric tensors are 3×3 matrices, the stiffness tensors in abbreviated form for acoustics are 6×6 matrices. The coordinate rotation relation with Euler angles earlier cannot be applied directly. A very concise matrix technique has been developed for this purpose by W. L. Bond [85]. In essence, it involves the construction of 6×6 matrices that may be used to transform stress or strain means of a single matrix multiplication.

The relation between the stress field $\bar{T}$ in the original coordinate system (x, y, z) and $\bar{T}'$ in the rotated coordinate system (X, Y, Z) is:

$$\bar{T}' = \mathbf{M} \bar{T} \quad (2.64)$$

$$\mathbf{M} = \begin{bmatrix} R_{xx}^2 & R_{xy}^2 & R_{xz}^2 & \vdots & 2R_{xy}R_{xz} & 2R_{xz}R_{xx} & 2R_{xx}R_{xy} \\ R_{yx}^2 & R_{yy}^2 & R_{yz}^2 & \vdots & 2R_{yy}R_{yz} & 2R_{yz}R_{yx} & 2R_{yx}R_{yy} \\ R_{zx}^2 & R_{zy}^2 & R_{zz}^2 & \vdots & 2R_{zy}R_{zz} & 2R_{zz}R_{zx} & 2R_{zx}R_{zy} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ R_{yx}R_{zx} & R_{yy}R_{zy} & R_{yz}R_{zz} & \vdots & R_{yy}R_{zz} + R_{yz}R_{zy} & R_{yx}R_{zz} + R_{yz}R_{zx} & R_{yy}R_{zx} + R_{yx}R_{zy} \\ R_{zx}R_{xx} & R_{zy}R_{xy} & R_{zz}R_{xz} & \vdots & R_{xy}R_{zz} + R_{xz}R_{zy} & R_{xz}R_{zx} + R_{xx}R_{zz} & R_{xx}R_{zy} + R_{xy}R_{zx} \\ R_{xx}R_{yx} & R_{xy}R_{yy} & R_{xz}R_{yz} & \vdots & R_{xy}R_{yz} + R_{xz}R_{yy} & R_{xz}R_{yx} + R_{xx}R_{yz} & R_{xx}R_{yy} + R_{xy}R_{yx} \end{bmatrix} \quad (2.65)$$

where R_{ij} is obtained from (2.47)

$$\mathbf{R} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \quad (2.66)$$

The relation between $\mathbf{C}$ in the original coordinate system (x, y, z) and $\mathbf{C}'$ in the rotated coordinate system (X, Y, Z) is:

$$\mathbf{C}' = \mathbf{M} \mathbf{C} \mathbf{M}^T \quad (2.67)$$

2.2.4 Matrix Methods

The first significant work on the study of acoustic wave interaction with multilayered media is attributed to Thomson [86] who introduced the transfer matrix method in order to facilitate his analysis. A small error in his work was subsequently corrected by Haskell [87]. According to this technique one can easily execute the transition of field variables from one layer to the next while satisfying the interfacial condition between them. In its original form, the specific steps taken can be summarized as follows: Formal solutions are first obtained for the individual layer in terms of its wave potential amplitudes. By specializing these solutions to the top and bottom surfaces of the layer, followed by eliminating its common wave amplitudes, one can directly relate the field variables (the stress and displacement/velocity components) of one of its surfaces to the other. This relation is conveniently written in a matrix form which defines the individual layer transfer matrix. Such a matrix relation can be used in conjunction with satisfying appropriate interface conditions across neighboring layers, to directly relate the stress and displacement/velocity at the top of individual layer to the bottom of its neighbor. If this procedure is carried out consecutively for all layers, a stack transfer matrix as the product of the individual transfer matrices will result. The stack transfer matrix relates the field variables at the top of the layered system to those at its bottom or visa-versa. It can then be used to present results for a wide variety of problems [88]-[90]. All we need to do is to invoke the appropriate values of the field variables at the external surfaces of the layered system. Because the method is used to

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transfer (propagate) boundary condition from one external surface of the layered system to the other via matrix multiplications, it is also known as “the propagator matrix method”.

On problems relating to multilayered anisotropic media, Nayfeh [81], [91], [92], used the transfer matrix method and presented solutions for horizontally and generally polarized waves in multilayered anisotropic media, respectively. Fryer and Frazer [93], [94] studied wave interaction with stratified geophysical media. Harmonic wave propagation in a variety of anisotropic layered systems has been investigated by many researchers, e.g. Nemat-Nasser et al. [95]-[97], Helbig [98], Braga and Herrmann [99] and Datta [100].

Although the transfer matrix method is applicable in principle, its direct implementation has been found to suffer from numerical instabilities. These instabilities often occur when the frequency is high and/or the thickness is large. The cause of this problem is the poor conditioning of the transfer matrix due to the combinations of both decaying and growing coefficients. Many researchers have proposed various techniques to overcome the numerical instability of transfer matrix method. An early reformulation based on the delta operator technique has been developed to alleviate the problem for isotropic media [101]. The technique has been extended to anisotropic media but it requires the computation of high-order delta matrices [102]. An alternative approach that is able to circumvent the numerical instability is the direct global matrix method [103]-[105]. The method involves a global banded matrix whose size grows with the number of layers. For many layers, it

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will lead to substantial memory storage and computation time. Another stable approach that involves a constant size matrix is the reflection matrix method (sometimes called invariant imbedding method) [79] [106]. This method utilizes a recursive scheme to build up the overall response of a multilayered medium from the scattering properties of each interface. The original implementation of the reflection matrix method has been made more efficient and concise in scattering matrix method [107]-[109]. Through the transition region reflection and transmission operators, the invariant imbedding method has also been applied to transversely inhomogeneous multilayers [110].

Recently, a different approach called the stiffness/impedance matrix method [111]-[114] has been proposed to resolve the numerical instability of transfer matrix method. The method operates with total stress and displacement via the stiffness matrix applied in a recursive algorithm. Using these total field variables makes it more convenient to incorporate imperfect interfaces that must be considered in certain ultrasonic problems [115], [116]. It also constitutes naturally the framework of self-recursive asymptotic method [77], [117], which is a simple asymptotic method that obviates the need to compute the exact wave propagation solution. When not all submatrices of the full matrix are required, e.g., for a layered half space, a partial algorithm of surface stiffness/impedance matrix method can be utilized [118], [119]. This method deals with surface matrices that are of smaller dimension at each recursion, thus giving rise to higher computational speed. Although the stiffness/impedance matrix method has been demonstrated to be computationally

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stable for large layer thickness, it becomes inaccurate and approaches being singular when the layer thickness reduces toward zero. The search thus continues for a more robust and stable matrix method that is able to accommodate all thicknesses including the extreme, very thick or very thin layers.

Chapter 3

Hybrid Matrix Method for Stable Analysis of Electromagnetic Wave Propagation in Multilayered Bianisotropic Media

3.1 Introduction

The increasing use of multilayered bianisotropic materials in military and industrial applications, such as radar absorbers and laminated shields [120]-[125] calls for intensive studies of their electromagnetic properties. However, the bianisotropic materials are complicated and their performance cannot be estimated simply by the equations derived previously for the dielectric (isotropic) media [125]-[127]. For analysis of such materials, a technique based on the transfer matrix has been introduced [54]. This method facilitates the transition of field variables from one layer to the next while satisfying the interfacial condition between them. Despite its convenience, the transfer matrix method suffers from inherent deficiency in numerical implementation. This problem often occurs when the frequency is high and/or the thickness is large. To overcome the numerical instability, other techniques have been

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proposed such as recursive transformation, eigen- or Riccati-based admittance, impedance and reflection matrix methods [70]-[76]. However, these methods often rely on the exact solutions to eigenvalue-eigenvector problem or nonlinear Riccati differential equation, which involve complicated manipulations and calculations.

Lately, an approach called the impedance matrix method [78] has been introduced. The method operates with total electric and magnetic fields via the impedance matrix applied in a recursive algorithm. It constitutes naturally the framework of self-recursive asymptotic method, which is a simple asymptotic method that obviates the need to compute the exact wave propagation solution using eigenwaves. Although the method maintains the numerical stability when the layer thickness grows to infinity, it is inaccurate when a layer thickness reduces toward zero (as required for thin-layer approximation).

In this chapter, we develop a hybrid matrix method for stable analysis of electromagnetic wave propagation in multilayered bianisotropic media for all thickness and frequency range. Similar to the impedance matrix method, the hybrid matrix method is able to eliminate the numerical instability of transfer matrix method. Moreover, contrary to the impedance matrix, the hybrid matrix remains to be well-conditioned and accurate even for zero or small thickness.

In Section 3.2, we formulate the 4×4 system matrix directly from Maxwell's equations and deduce the corresponding eigenwaves in spectral domain. Section 3.3 introduces the transfer, impedance and hybrid matrices for each layer along with their relations, stability and accuracy analysis. To bypass the intricacies of

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eigenvalue-eigenvector approach, a set of simple self-recursive asymptotic algorithms, along with thin-layer asymptotic approximations, are presented to calculate these matrices for each layer. The definitions of stack transfer, impedance and hybrid matrices, as well as their matrix recursions are provided in Section 3.4. A comparison of the CPU time for hybrid, transfer and impedance matrix methods is also discussed in this section. The expressions for reflection and transmission coefficients using hybrid matrix are deduced in Section 3.5. In Section 3.6, the numerical implementation and examples are described. We begin with a two layered dielectric structure to verify our method with transmission line theory. Then, we use Computer Systems Technology (CST) software and published work to verify our method for more complex structures. After that, more applications of our method are presented. Two situations of uncovered and covered layered bianisotropic waveguides and the frequency response of reflection coefficient for a tunable biased ferrite absorber are studied. The shielding effectiveness of a laminated graphite/epoxy (G/E) composite is also illustrated.

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3.2 Basic Formulation

3.2.1 4×4 System Matrix

The homogeneous bianisotropic medium of each layer is characterized by permittivity ϵ , permeability μ and magneto-electric tensors ξ and ζ through constitutive relations [3]

$$\begin{bmatrix} \overline{D} \\ \overline{B} \end{bmatrix} = \begin{bmatrix} \epsilon & \xi \\ \zeta & \mu \end{bmatrix} \begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} \quad (3.1)$$

Substituting these relations into source-incorporated Maxwell equations (including magnetic current source for completeness), we have

$$\begin{bmatrix} \nabla \times \overline{E} \\ \nabla \times \overline{H} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \left(-i\omega \begin{bmatrix} \epsilon & \xi \\ \zeta & \mu \end{bmatrix} \begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} + \begin{bmatrix} \overline{J} \\ \overline{M} \end{bmatrix} \right) \quad (3.2)$$

Let us describe our Cartesian coordinate system by unit vectors $\hat{t}_1$, $\hat{t}_2$ and $\hat{p}$, with $\hat{p}$ chosen as the preferred direction. To facilitate subsequent analysis, it is expedient to partition the matrices and vectors according to

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{tt}^{(2 \times 2)} & \mathbf{A}_{tp}^{(2 \times 1)} \\ \mathbf{A}_{pt}^{(1 \times 2)} & \mathbf{A}_{pp}^{(1 \times 1)} \end{bmatrix} \quad (3.3)$$

$$\overline{A} = \begin{bmatrix} \overline{A}_t^{(2 \times 1)} \\ \overline{A}_p^{(1 \times 1)} \end{bmatrix} \quad (3.4)$$

where subscripts t and p signify quantities transverse and parallel to $\hat{p}$, respectively. The size of each submatrix is as indicated in brackets. Henceforth, we will carry out analysis in the two-dimensional Fourier transform domain which

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assumes field and source dependence of $\exp(i\bar{k}_t \cdot \bar{r}_t)$ with

$$\bar{k}_t = k_{t1}\hat{t}_1 + k_{t2}\hat{t}_2 \quad (3.5)$$

$$\bar{r}_t = t_1\hat{t}_1 + t_2\hat{t}_2 \quad (3.6)$$

(More details of the Fourier transform and derivation can be referred to Appendix B.)

Applying the decomposition (3.3)-(3.4) into Maxwell equations (3.2) cast in Fourier domain, we obtain two linear algebraic equations and four coupled linear first-order ordinary differential equations as follows.

The algebraic equations relate the longitudinal ($\hat{p}$) components of electric and magnetic fields to their transverse components plus the longitudinal components of current sources:

$$\begin{bmatrix} \bar{E}_p \\ \bar{H}_p \end{bmatrix} = \begin{bmatrix} \alpha_{ee} & \alpha_{em} \\ \alpha_{me} & \alpha_{mm} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} + \frac{1}{i\omega} \begin{bmatrix} \epsilon_{pp} & \xi_{pp} \\ \zeta_{pp} & \mu_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \bar{J}_p \\ \bar{M}_p \end{bmatrix} \quad (3.7)$$

Here, the four α 's represent 1×2 matrices performing transverse-to-longitudinal transformations of field vectors:

$$\begin{bmatrix} \alpha_{ee} & \alpha_{em} \\ \alpha_{me} & \alpha_{mm} \end{bmatrix} = \begin{bmatrix} \epsilon_{pp} & \xi_{pp} \\ \zeta_{pp} & \mu_{pp} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{0} & -\mathbf{\kappa}_{pt} \\ \mathbf{\kappa}_{pt} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \epsilon_{pt} & \xi_{pt} \\ \zeta_{pt} & \mu_{pt} \end{bmatrix} \right) \quad (3.8)$$

with

$$\mathbf{\kappa} = \frac{\bar{k}_t}{\omega} \times \mathbf{I} = \frac{1}{\omega} \begin{bmatrix} 0 & 0 & k_{t2} \\ 0 & 0 & -k_{t1} \\ -k_{t2} & k_{t1} & 0 \end{bmatrix} \quad (3.9)$$

to be partitioned according to (3.3) as well. Notice that for convenience sake, the variables in (3.7) have been written using the same notations as those in (3.2), although they should be understood as (transverse) Fourier transformed quantities.

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Introducing an antisymmetric 2×2 matrix

$$\mathbf{\Gamma}_a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (3.10)$$

we have four differential equations given by

$$\begin{aligned} \frac{d}{dp} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} &= i\omega \begin{bmatrix} \Lambda_{ee} & \Lambda_{em} \\ \Lambda_{me} & \Lambda_{mm} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_a \\ -\mathbf{\Gamma}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{J}_t \\ \bar{M}_t \end{bmatrix} + \begin{bmatrix} \beta_{ee} & \beta_{em} \\ \beta_{me} & \beta_{mm} \end{bmatrix} \begin{bmatrix} \bar{J}_p \\ \bar{M}_p \end{bmatrix} \end{aligned} \quad (3.11)$$

Here, the four β 's complement those α 's above and represent 2×1 matrices performing longitudinal-to-transverse transformations of source vectors:

$$\begin{bmatrix} \beta_{ee} & \beta_{em} \\ \beta_{me} & \beta_{mm} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & -\kappa_{tp} \\ \kappa_{tp} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \epsilon_{tp} & \xi_{tp} \\ \zeta_{tp} & \mu_{tp} \end{bmatrix} \right) \begin{bmatrix} \epsilon_{pp} & \xi_{pp} \\ \zeta_{pp} & \mu_{pp} \end{bmatrix}^{-1} \quad (3.12)$$

Observe the highly symmetric form of expression in (3.12) compared to (3.8). Using these β 's and α 's, the four Λ 's can be expressed succinctly as

$$\begin{aligned} \begin{bmatrix} \Lambda_{ee} & \Lambda_{em} \\ \Lambda_{me} & \Lambda_{mm} \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_a \\ -\mathbf{\Gamma}_a & \mathbf{0} \end{bmatrix} \\ &\cdot \left(\begin{bmatrix} \beta_{ee} & \beta_{em} \\ \beta_{me} & \beta_{mm} \end{bmatrix} \begin{bmatrix} \epsilon_{pp} & \xi_{pp} \\ \zeta_{pp} & \mu_{pp} \end{bmatrix} \begin{bmatrix} \alpha_{ee} & \alpha_{em} \\ \alpha_{me} & \alpha_{mm} \end{bmatrix} - \begin{bmatrix} \epsilon_{tt} & \xi_{tt} \\ \zeta_{tt} & \mu_{tt} \end{bmatrix} \right) \end{aligned} \quad (3.13)$$

Equation (3.11) constitutes the basis of source-incorporated 4×4 matrix formalism [128]-[131]. As is evident from this equation, the key ingredients of various system matrices α , β , Λ have been clearly identified. Their concise appearance is mostly appreciated in view of the previous expressions of matrix elements via a number of auxiliary notations in rather elaborated forms [63], [73]. Note that the usual duality relationships which can be handy in the course of their derivation are immediately apparent from our presentation above [61], [65].

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At this point, it is appropriate to introduce some shorthand notations for (3.11), (3.7), (3.8), (3.12) and (3.13) as

$$\frac{d}{dp} \bar{f}_t = i\omega \Lambda \bar{f}_t + \Gamma_v (\bar{s}_t + \beta \bar{s}_p) \quad (3.14)$$

$$\bar{f}_p = \alpha \bar{f}_t + \frac{1}{i\omega} \mathbf{M}_{pp}^{-1} \bar{s}_p \quad (3.15)$$

$$\alpha = \mathbf{M}_{pp}^{-1} (\mathbf{K}_{pt} - \mathbf{M}_{pt}) \quad (3.16)$$

$$\beta = (\mathbf{K}_{tp} - \mathbf{M}_{tp}) \mathbf{M}_{pp}^{-1} \quad (3.17)$$

$$\Lambda = \Gamma_v (\beta \mathbf{M}_{pp} \alpha - \mathbf{M}_{tt}) \quad (3.18)$$

The definitions of various symbols follow readily from above, e.g.

$$\Gamma_v = \begin{bmatrix} \mathbf{0} & \Gamma_a \\ -\Gamma_a & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad (3.19)$$

$$\mathbf{K}_{pt} = \begin{bmatrix} \mathbf{0} & -\kappa_{pt} \\ \kappa_{pt} & \mathbf{0} \end{bmatrix} \quad (3.20)$$

$$\mathbf{K}_{tp} = \begin{bmatrix} \mathbf{0} & -\kappa_{tp} \\ \kappa_{tp} & \mathbf{0} \end{bmatrix} \quad (3.21)$$

$$\mathbf{M}_{tt} = \begin{bmatrix} \boldsymbol{\varepsilon}_{tt} & \boldsymbol{\xi}_{tt} \\ \boldsymbol{\zeta}_{tt} & \boldsymbol{\mu}_{tt} \end{bmatrix} \quad (3.22)$$

$$\mathbf{M}_{tp} = \begin{bmatrix} \boldsymbol{\varepsilon}_{tp} & \boldsymbol{\xi}_{tp} \\ \boldsymbol{\zeta}_{tp} & \boldsymbol{\mu}_{tp} \end{bmatrix} \quad (3.23)$$

$$\mathbf{M}_{pt} = \begin{bmatrix} \boldsymbol{\varepsilon}_{pt} & \boldsymbol{\xi}_{pt} \\ \boldsymbol{\zeta}_{pt} & \boldsymbol{\mu}_{pt} \end{bmatrix} \quad (3.24)$$

$$\mathbf{M}_{pp} = \begin{bmatrix} \boldsymbol{\varepsilon}_{pp} & \boldsymbol{\xi}_{pp} \\ \boldsymbol{\zeta}_{pp} & \boldsymbol{\mu}_{pp} \end{bmatrix} \quad (3.25)$$

$\bar{f}_t$ can be termed as transverse field vector. In the following, we shall omit the source terms and consider the eigenwaves being the homogeneous solutions of (3.14).

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3.2.2 Eigenwaves

Within a source-free homogeneous medium, Λ is a constant 4×4 matrix in the homogeneous differential equation

$$\frac{d}{dp} \bar{f}_t = i\omega \Lambda \bar{f}_t \quad (3.26)$$

This equation admits nontrivial solutions of $\exp(ik_p p)$ dependence subjected to the dispersion relation

$$\det(\omega \Lambda - k_p \mathbf{I}) = 0 \quad (3.27)$$

Since (3.27) yields a quartic equation for k_p , there are four eigenvalues and four eigenvectors associated with it. As a result, the general solution for the transverse field vector can be written as

$$\bar{f}_t(p) = \Psi \mathbf{P}(p) \bar{c} = \Psi \bar{w}(p) \quad (3.28)$$

where the eigenvalues k_{pj} , $j = 1, 2, 3, 4$ are included in phase matrix $\mathbf{P}$:

$$p^{(j)} = \exp(ik_{pj} p) \quad (3.29)$$

$$\mathbf{P}(p) = \text{diag}(p^{(j)}) \quad (3.30)$$

while their corresponding eigenvectors form the columns of eigenwave matrix Ψ :

$$\Psi = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 & \bar{e}_4 \\ \bar{h}_1 & \bar{h}_2 & \bar{h}_3 & \bar{h}_4 \end{bmatrix} \quad (3.31)$$

In accordance with the composition of $\bar{f}_t$, each eigenvector is seen to be made up of electric ($\bar{e}_j$) and magnetic ($\bar{h}_j$) eigenwaves which are individually two-component column vectors. Furthermore, $\bar{c}$ is a four-component coefficient vector containing the unknown constants (independent of p) to be specified by primary excitations or

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determined from boundary and/or radiation conditions. These unknowns can be lumped together with the exponential phase factors into another wave vector $\bar{w}$ whose four components are now dependent on p . For either unknown representation, equation (3.28) states that the total tangential field vector can be viewed as a weighted sum of four eigenwaves.

Note that in arriving at (3.28), it has been assumed that Λ is a diagonalizable matrix, so that there are four linearly independent eigenvectors and the inverse of Ψ exists. In addition, let us assume in the sequel that k_{p1} and k_{p2} have positive imaginary parts while k_{p3} and k_{p4} have negative imaginary parts. This follows because the imaginary parts normally occur in positive and negative pairs (in the limit of small loss for lossless case). Since these k_{pj} appear in the exponents as $\exp(ik_{pj}p)$, one can regard the eigenwaves $\bar{e}_1, \bar{h}_1$ (for k_{p1}) and $\bar{e}_2, \bar{h}_2$ (for k_{p2}) as upward-bounded waves which remain bounded as $p \rightarrow +\infty$. Likewise, $\bar{e}_3, \bar{h}_3$ (for k_{p3}) and $\bar{e}_4, \bar{h}_4$ (for k_{p4}) correspond to downward-bounded waves that are still bounded as $p \rightarrow -\infty$. Note that it is actually not very appropriate to term the waves as ‘upward-propagating’ and ‘downward-propagating’. This is because for general bianisotropic media, phase propagation direction (according to real parts of k_{pj}) may not coincide with energy flow direction [129]. In fact, downward-bounded waves may be upward-propagating at infinity. Therefore, the radiation condition should be in general based on bounded solutions which require all waves to be physically bounded. Under the above association, we can partition the phase matrix and eigenwave matrix into 2×2 submatrices as

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$$\mathbf{P} = \begin{bmatrix} \mathbf{P}^> & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^< \end{bmatrix} \quad (3.32)$$

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{e}^> & \mathbf{e}^< \\ \mathbf{h}^> & \mathbf{e}^< \end{bmatrix} \quad (3.33)$$

where “>” and “<” in the superscripts refer to upward-bounded and downward-bounded waves respectively. Likewise, the coefficient vector and wave vector can also be decomposed into 2×1 partitions

$$\bar{\mathbf{c}} = \begin{bmatrix} \bar{\mathbf{c}}^> \\ \bar{\mathbf{c}}^< \end{bmatrix} \quad (3.34)$$

$$\bar{\mathbf{w}} = \begin{bmatrix} \bar{\mathbf{w}}^> \\ \bar{\mathbf{w}}^< \end{bmatrix} \quad (3.35)$$

3.2.3 Notations

In the following, we let $\hat{p}$ stands for $\hat{z}$, $\hat{t}_1$ and $\hat{t}_2$ stand for $\hat{x}$ and $\hat{y}$, respectively. The differential equation for layer f will be written as

$$\frac{d}{dz} \begin{bmatrix} \bar{\mathbf{E}}_t \\ \bar{\mathbf{H}}_t \end{bmatrix} = \mathbf{A}_f \begin{bmatrix} \bar{\mathbf{E}}_t \\ \bar{\mathbf{H}}_t \end{bmatrix} \quad (3.36)$$

where

$$\mathbf{A}_f = i\omega \mathbf{\Lambda}_f = \begin{bmatrix} \mathbf{A}_{f,11} & \mathbf{A}_{f,12} \\ \mathbf{A}_{f,21} & \mathbf{A}_{f,22} \end{bmatrix} \quad (3.37)$$

The $\mathbf{K}$ and $\mathbf{M}$ can be specialized as

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$$\mathbf{K}_{zt} = \frac{1}{\omega} \begin{bmatrix} 0 & 0 & k_y & -k_x \\ -k_y & k_x & 0 & 0 \end{bmatrix} \quad (3.38)$$

$$\mathbf{K}_{tz} = \frac{1}{\omega} \begin{bmatrix} 0 & -k_y \\ 0 & k_x \\ k_y & 0 \\ -k_x & 0 \end{bmatrix} \quad (3.39)$$

$$\mathbf{M}_{tt} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \xi_{xx} & \xi_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} & \xi_{yx} & \xi_{yy} \\ \zeta_{xx} & \zeta_{xy} & \mu_{xx} & \mu_{xy} \\ \zeta_{yx} & \zeta_{yy} & \mu_{yx} & \mu_{yy} \end{bmatrix} \quad (3.40)$$

$$\mathbf{M}_{tz} = \begin{bmatrix} \varepsilon_{xz} & \xi_{xz} \\ \varepsilon_{yz} & \xi_{yz} \\ \zeta_{xz} & \mu_{xz} \\ \zeta_{yz} & \mu_{yz} \end{bmatrix} \quad (3.41)$$

$$\mathbf{M}_{zt} = \begin{bmatrix} \varepsilon_{zx} & \varepsilon_{zy} & \xi_{zx} & \xi_{zy} \\ \zeta_{zx} & \zeta_{zy} & \mu_{zx} & \mu_{zy} \end{bmatrix} \quad (3.42)$$

$$\mathbf{M}_{zz} = \begin{bmatrix} \varepsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{bmatrix} \quad (3.43)$$

3.3 Layer Matrices

3.3.1 Definitions

A planar multilayered structure comprising N homogeneous bianisotropic layers stratified in $\hat{z}$ direction is shown in Fig. 3.1.

Consider first a single layer f , (3.36) can be solved by using the layer transfer matrix, which provides a linear relationship between the fields at two surfaces of layer f as

$$\begin{bmatrix} \bar{E}_t(z_f^>) \\ \bar{H}_t(z_f^>) \end{bmatrix} = \mathbf{T}_f \begin{bmatrix} \bar{E}_t(z_f^<) \\ \bar{H}_t(z_f^<) \end{bmatrix} \quad (3.44)$$

$$\mathbf{T}_f = \exp(\mathbf{A}_f h_f) \quad (3.45)$$

For the convenience of following discussion, the transfer matrix will also be written in the submatrix form as

$$\mathbf{T}_f = \begin{bmatrix} \mathbf{T}_{f,11} & \mathbf{T}_{f,11} \\ \mathbf{T}_{f,11} & \mathbf{T}_{f,11} \end{bmatrix} \quad (3.46)$$

With the eigensolutions available, the transfer matrix can be given as

$$\begin{aligned} \mathbf{T}_f &= \boldsymbol{\Psi}_f \mathbf{P}_f(h_f) \boldsymbol{\Psi}_f^{-1} \\ &= \begin{bmatrix} \mathbf{e}_f^> \mathbf{P}_f^>(h_f) & \mathbf{e}_f^< \\ \mathbf{h}_f^> \mathbf{P}_f^>(h_f) & \mathbf{h}_f^< \end{bmatrix} \begin{bmatrix} \mathbf{e}_f^> & \mathbf{e}_f^< \mathbf{P}_f^<(-h_f) \\ \mathbf{h}_f^> & \mathbf{h}_f^< \mathbf{P}_f^<(-h_f) \end{bmatrix}^{-1} \end{aligned} \quad (3.47)$$

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Fig. 3.1 Cross section of a planar N -multilayered structure. The upper and lower bounding interfaces of each layer f are denoted by $Z_f^>$ and $Z_f^<$, respectively.

Although straightforward and elegant, it is well known [71], [112] that when the layer is lossy and thickness increases, the transfer matrix exhibits numerical instability.

To overcome such problem, a layer impedance matrix has been introduced as

$$\begin{bmatrix} \bar{E}_t(z_f^<) \\ \bar{E}_t(z_f^>) \end{bmatrix} = \mathbf{Z}_f \begin{bmatrix} \bar{H}_t(z_f^<) \\ \bar{H}_t(z_f^>) \end{bmatrix} \quad (3.48)$$

$$\mathbf{Z}_f = \begin{bmatrix} \mathbf{Z}_{f,11} & \mathbf{Z}_{f,12} \\ \mathbf{Z}_{f,21} & \mathbf{Z}_{f,22} \end{bmatrix} \quad (3.49)$$

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$$\mathbf{Z}_f = \begin{bmatrix} \mathbf{e}_f^{\rightarrow} & \mathbf{e}_f^{\leftarrow} \mathbf{P}_f^{\leftarrow}(-h_f) \\ \mathbf{e}_f^{\rightarrow} \mathbf{P}_f^{\rightarrow}(h_f) & \mathbf{e}_f^{\leftarrow} \end{bmatrix} \begin{bmatrix} \mathbf{h}_f^{\rightarrow} & \mathbf{h}_f^{\leftarrow} \mathbf{P}_f^{\leftarrow}(-h_f) \\ \mathbf{h}_f^{\rightarrow} \mathbf{P}_f^{\rightarrow}(h_f) & \mathbf{h}_f^{\leftarrow} \end{bmatrix}^{-1} \quad (3.50)$$

It can be demonstrated that the impedance matrix is free from numerical instability for large layer thickness. Unfortunately, it is ill-conditioned when the layer thickness becomes too small as to be shown later.

The problems mentioned above can be overcome altogether by resorting to layer hybrid matrix defined as

$$\begin{bmatrix} \overline{E}_t(z_f^{\leftarrow}) \\ \overline{H}_t(z_f^{\rightarrow}) \end{bmatrix} = \mathbf{H}_f \begin{bmatrix} \overline{H}_t(z_f^{\leftarrow}) \\ \overline{E}_t(z_f^{\rightarrow}) \end{bmatrix} \quad (3.51)$$

$$\mathbf{H}_f = \begin{bmatrix} \mathbf{H}_{f,11} & \mathbf{H}_{f,12} \\ \mathbf{H}_{f,21} & \mathbf{H}_{f,22} \end{bmatrix} \quad (3.52)$$

$\mathbf{H}_f$ is called hybrid matrix since its elements are a mixture of impedance, admittance and transfer quantities. The hybrid matrix can be expressed in terms of eigensolutions as

$$\mathbf{H}_f = \begin{bmatrix} \mathbf{e}_f^{\rightarrow} & \mathbf{e}_f^{\leftarrow} \mathbf{P}_f^{\leftarrow}(-h_f) \\ \mathbf{h}_f^{\rightarrow} \mathbf{P}_f^{\rightarrow}(h_f) & \mathbf{h}_f^{\leftarrow} \end{bmatrix} \begin{bmatrix} \mathbf{h}_f^{\rightarrow} & \mathbf{h}_f^{\leftarrow} \mathbf{P}_f^{\leftarrow}(-h_f) \\ \mathbf{e}_f^{\rightarrow} \mathbf{P}_f^{\rightarrow}(h_f) & \mathbf{e}_f^{\leftarrow} \end{bmatrix}^{-1} \quad (3.53)$$

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3.3.2 Relations

The layer transfer, impedance and hybrid matrices provide linear relationship between the fields at two surfaces of each layer. In turn, they also possess relations with each others. In particular, the hybrid matrix can be related to the transfer matrix by

$$\mathbf{H}_f = \begin{bmatrix} -(\mathbf{T}_{f,11})^{-1}\mathbf{T}_{f,12} & (\mathbf{T}_{f,11})^{-1} \\ \mathbf{T}_{f,12} - \mathbf{T}_{f,21}(\mathbf{T}_{f,11})^{-1}\mathbf{T}_{f,12} & \mathbf{T}_{f,21}(\mathbf{T}_{f,11})^{-1} \end{bmatrix}. \quad (3.54)$$

It can also be deduced from the impedance matrix as

$$\mathbf{H}_f = \begin{bmatrix} \mathbf{Z}_{f,11} - \mathbf{Z}_{f,12}(\mathbf{Z}_{f,22})^{-1}\mathbf{Z}_{f,21} & \mathbf{Z}_{f,12}(\mathbf{Z}_{f,22})^{-1} \\ -(\mathbf{Z}_{f,22})^{-1}\mathbf{Z}_{f,21} & (\mathbf{Z}_{f,22})^{-1} \end{bmatrix}. \quad (3.55)$$

Meanwhile, the transfer matrix can be written in terms of the hybrid and impedance matrices:

$$\mathbf{T}_f = \begin{bmatrix} (\mathbf{H}_{f,12})^{-1} & -(\mathbf{H}_{f,12})^{-1}\mathbf{H}_{f,11} \\ \mathbf{H}_{f,22}(\mathbf{H}_{f,12})^{-1} & \mathbf{H}_{f,21} - \mathbf{H}_{f,22}(\mathbf{H}_{f,12})^{-1}\mathbf{H}_{f,11} \end{bmatrix} \quad (3.56)$$

$$\mathbf{T}_f = \begin{bmatrix} \mathbf{Z}_{f,22}(\mathbf{Z}_{f,12})^{-1} & -\mathbf{Z}_{f,22}(\mathbf{Z}_{f,12})^{-1}\mathbf{Z}_{f,11} + \mathbf{Z}_{f,21} \\ (\mathbf{Z}_{f,12})^{-1} & -(\mathbf{Z}_{f,12})^{-1}\mathbf{Z}_{f,11} \end{bmatrix}. \quad (3.57)$$

For completeness, the impedance matrix can be given in terms of the hybrid and transfer matrices:

$$\mathbf{Z}_f = \begin{bmatrix} \mathbf{H}_{f,11} - \mathbf{H}_{f,12}(\mathbf{H}_{f,22})^{-1}\mathbf{H}_{f,21} & \mathbf{H}_{f,12}(\mathbf{H}_{f,22})^{-1} \\ -(\mathbf{H}_{f,22})^{-1}\mathbf{H}_{f,21} & (\mathbf{H}_{f,22})^{-1} \end{bmatrix} \quad (3.58)$$

$$\mathbf{Z}_f = \begin{bmatrix} -(\mathbf{T}_{f,21})^{-1}\mathbf{T}_{f,22} & (\mathbf{T}_{f,21})^{-1} \\ \mathbf{T}_{f,12} - \mathbf{T}_{f,11}(\mathbf{T}_{f,21})^{-1}\mathbf{T}_{f,22} & \mathbf{T}_{f,11}(\mathbf{T}_{f,21})^{-1} \end{bmatrix} \quad (3.59)$$

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3.3.3 Stability and Accuracy Analysis

As mentioned above, the transfer matrix method suffers from the numerical instability as the layer is lossy and thickness approaches infinite. This can be easily understood from the eigensolution expression (3.47) that when $h_f \rightarrow \infty$, $\mathbf{P}_f^<(-h_f)$ and $\mathbf{P}_f^>(h_f)$ tend to zero, and the layer transfer matrix becomes

$$\mathbf{T}_f|_{h_f \rightarrow \infty} = \begin{bmatrix} \mathbf{0} & \mathbf{e}_f^< \\ \mathbf{0} & \mathbf{h}_f^< \end{bmatrix} \begin{bmatrix} \mathbf{e}_f^> & \mathbf{0} \\ \mathbf{h}_f^> & \mathbf{0} \end{bmatrix}^{-1} \quad (3.60)$$

which is obviously not computable (overflow) numerically. In practice, numerical difficulty also occurs for transfer matrix at high frequency even before the extreme state of being overflow. This is due to the mixture of exponentially growing and decaying terms that lead to loss of precision during computations.

On the contrary, the impedance matrix method stays stable when the layer thickness becomes large. From its expression in terms of eigensolutions (3.50), one can obtain the limit

$$\mathbf{Z}_f|_{h_f \rightarrow \infty} = \begin{bmatrix} \mathbf{e}_f^>(\mathbf{h}_f^>)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_f^<(\mathbf{h}_f^<)^{-1} \end{bmatrix} \quad (3.61)$$

In a similar manner, the limit of hybrid matrix is reduced from (3.53) to

$$\mathbf{H}_f|_{h_f \rightarrow \infty} = \begin{bmatrix} \mathbf{e}_f^>(\mathbf{h}_f^>)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_f^<(\mathbf{e}_f^<)^{-1} \end{bmatrix} \quad (3.62)$$

Thus it can be seen that the hybrid matrix for thick layer asymptotically decomposes into impedance and admittance submatrices for the lower and upper surfaces. Moreover, such reduction to the asymptotic hybrid matrix is regular and

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computationally stable because only the decreasing exponential terms are involved and the matrices are well conditioned. Therefore, the hybrid matrix will be particularly useful to characterize the exterior semi-infinite regions.

On the other hand, when the layer thickness tends to zero, the transfer matrix do not encounter numerical difficulty. This is evident from its eigensolution expression, which when $h_f = 0$ becomes simply

$$\mathbf{T}_f|_{h_f=0} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (3.63)$$

noting that $\mathbf{P}_f^<(-h_f) = \mathbf{P}_f^>(h_f) = \mathbf{I}$.

On the contrary, the impedance matrix is not immune to numerical difficulty when the thickness reduces toward zero. This can be understood easily from

$$\mathbf{Z}_f|_{h_f=0} = \begin{bmatrix} \mathbf{e}_f^> & \mathbf{e}_f^< \\ \mathbf{e}_f^> & \mathbf{e}_f^< \end{bmatrix} \begin{bmatrix} \mathbf{h}_f^> & \mathbf{h}_f^< \\ \mathbf{h}_f^> & \mathbf{h}_f^< \end{bmatrix}^{-1} \quad (3.64)$$

The second matrix on the right is obviously not invertible due to its repeated rows. Therefore, computational accuracy would be affected when dealing with thin layer modeling using impedance matrices.

Unlike the impedance matrix, the hybrid matrix is still well-conditioned as is evident from

$$\mathbf{H}_f|_{h=0} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (3.65)$$

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Fig. 3.2 Condition numbers versus layer thickness for $\mathbf{T}_f$, $\mathbf{Z}_f$, $\mathbf{H}_f$ matrices of single layer dielectric with (a) $\epsilon_r = 5 + 18.8i$ (b) $\epsilon_r = 5 + 1.88i$. The condition numbers for three methods are averaged values over a range of frequencies $0.5\text{GHz} \leq f \leq 10\text{GHz}$

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In Fig. 3.2(a) and Fig. 3.2(b), we show the condition numbers versus layer thickness for transfer, impedance and hybrid matrices of single layer dielectric with $\varepsilon_r = 5 + 18.8i$ and $\varepsilon_r = 5 + 1.88i$, respectively. The condition numbers for three methods are averaged values over a range of frequencies $0.5GHz \leq f \leq 10GHz$. The condition number of matrix $\mathbf{A}$ is defined as

$$cond(\mathbf{A}) = \|\mathbf{A}^{-1}\| \cdot \|\mathbf{A}\| \quad (3.66)$$

In numerical analysis, the condition number associated with a problem is a measure of that problem's amenability to digital computation, that is, how numerically well-conditioned the problem is. From the figures, it is obvious that when the layer is thin, the impedance matrix is ill-conditioned. As the layer thickness increases, the transfer matrix becomes unstable rapidly. Being superior to both impedance and transfer matrices, the condition number of hybrid matrix stays at low level for all thickness range. Furthermore, comparing Fig. 3.2(a) and Fig. 3.2(b), one can find that the condition number of transfer matrix for medium with higher loss (Fig. 3.2 (a)) will become large earlier than the lower loss one (Fig. 3.2 (b)). This indicates that the transfer matrix will become more ill-conditioned for higher loss media.

From the above discussion, it is evident that the stable ranges of layer thickness for transfer and impedance matrices are not complete. On the contrary, the hybrid matrix is shown to preserve numerical stability and condition even when the thickness tends to infinity or zero.

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3.3.4 Self-Recursive Asymptotic Method

For each individual layer, one can deduce the layer matrix by implementing simple self-recursive asymptotic algorithm. This method will circumvent the need to solve eigenvalue-eigenvector problems and bypass their intricacies. To that end, we geometrically subdivide the layer into $n+1$ sublayers having thicknesses as $d_i = h_f / 2^i$ for $i = 1, 2, \dots, n$ and $d_{n+1} = h_f / 2^n$. Fig. 3.3 shows the cross section of the geometrical subdivision of layer f .

For the layer transfer matrix, the self-recursive asymptotic algorithm is to be initialized by thin-layer asymptotic approximation

$$\mathbf{T}_f^{(n+1)} \approx \left[\mathbf{I} - \frac{d_{n+1}}{2} \mathbf{A}_f \right]^{-1} \left[\mathbf{I} + \frac{d_{n+1}}{2} \mathbf{A}_f \right] \quad (3.67)$$

Then starting from $i=n$ and using the self-recursions

$$\mathbf{T}_f^{(i)} = [\mathbf{T}_f^{(i+1)}]^2 \quad (3.68)$$

The algorithm proceeds until $i=1$ and the layer transfer matrix can be obtained as

$$\mathbf{T}_f = \mathbf{T}_f^{(1)}.$$

The layer impedance matrix can also be deduced via self-recursions

$$\mathbf{Z}_{f,11}^{(i)} = \mathbf{Z}_{f,11}^{(i+1)} + \mathbf{Z}_{f,12}^{(i+1)} [\mathbf{Z}_{f,11}^{(i+1)} - \mathbf{Z}_{f,22}^{(i+1)}]^{-1} \mathbf{Z}_{f,21}^{(i+1)} \quad (3.69)$$

$$\mathbf{Z}_{f,12}^{(i)} = -\mathbf{Z}_{f,12}^{(i+1)} [\mathbf{Z}_{f,11}^{(i+1)} - \mathbf{Z}_{f,22}^{(i+1)}]^{-1} \mathbf{Z}_{f,12}^{(i+1)} \quad (3.70)$$

$$\mathbf{Z}_{f,21}^{(i)} = \mathbf{Z}_{f,21}^{(i+1)} [\mathbf{Z}_{f,11}^{(i+1)} - \mathbf{Z}_{f,22}^{(i+1)}]^{-1} \mathbf{Z}_{f,21}^{(i+1)} \quad (3.71)$$

$$\mathbf{Z}_{f,22}^{(i)} = \mathbf{Z}_{f,22}^{(i+1)} - \mathbf{Z}_{f,21}^{(i+1)} [\mathbf{Z}_{f,11}^{(i+1)} - \mathbf{Z}_{f,22}^{(i+1)}]^{-1} \mathbf{Z}_{f,12}^{(i+1)} \quad (3.72)$$

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Fig. 3.3 Cross section of the geometrical subdivision of layer f

The self-recursions are to be initialized with

$$\mathbf{Z}_{f,11}^{(n+1)} \approx \frac{d_{n+1}}{4} \mathbf{B} - \frac{1}{d_{n+1}} \mathbf{A}_{f,21}^{-1} + \frac{1}{2} \mathbf{A}_{f,11} \mathbf{A}_{f,21}^{-1} - \frac{1}{2} \mathbf{A}_{f,21}^{-1} \mathbf{A}_{f,22} \quad (3.73)$$

$$\mathbf{Z}_{f,12}^{(n+1)} \approx \frac{d_{n+1}}{4} \mathbf{B} + \frac{1}{d_{n+1}} \mathbf{A}_{f,21}^{-1} - \frac{1}{2} \mathbf{A}_{f,11} \mathbf{A}_{f,21}^{-1} - \frac{1}{2} \mathbf{A}_{f,21}^{-1} \mathbf{A}_{f,22} \quad (3.74)$$

$$\mathbf{Z}_{f,21}^{(n+1)} \approx -\frac{d_{n+1}}{4} \mathbf{B} - \frac{1}{d_{n+1}} \mathbf{A}_{f,21}^{-1} - \frac{1}{2} \mathbf{A}_{f,11} \mathbf{A}_{f,21}^{-1} - \frac{1}{2} \mathbf{A}_{f,21}^{-1} \mathbf{A}_{f,22} \quad (3.75)$$

$$\mathbf{Z}_{f,22}^{(n+1)} \approx -\frac{d_{n+1}}{4} \mathbf{B} + \frac{1}{d_{n+1}} \mathbf{A}_{f,21}^{-1} + \frac{1}{2} \mathbf{A}_{f,11} \mathbf{A}_{f,21}^{-1} - \frac{1}{2} \mathbf{A}_{f,21}^{-1} \mathbf{A}_{f,22} \quad (3.76)$$

where $\mathbf{B} = \mathbf{A}_{f,11} \mathbf{A}_{f,21}^{-1} \mathbf{A}_{f,22} - \mathbf{A}_{f,12}$.

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In a similar manner, the layer hybrid matrix can be obtained by using self-recursive asymptotic method, which calls for initialization with

$$\mathbf{H}_f^{(n+1)} = \begin{bmatrix} \mathbf{H}_{f,11}^{(n+1)} & \mathbf{H}_{f,12}^{(n+1)} \\ \mathbf{H}_{f,21}^{(n+1)} & \mathbf{H}_{f,22}^{(n+1)} \end{bmatrix} \approx \begin{bmatrix} \mathbf{I} + \frac{d_{n+1}}{2} \mathbf{A}_{f,11} & \frac{d_{n+1}}{2} \mathbf{A}_{f,12} \\ \frac{d_{n+1}}{2} \mathbf{A}_{f,21} & -\mathbf{I} + \frac{d_{n+1}}{2} \mathbf{A}_{f,22} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{d_{n+1}}{2} \mathbf{A}_{f,12} & \mathbf{I} - \frac{d_{n+1}}{2} \mathbf{A}_{f,11} \\ -\mathbf{I} - \frac{d_{n+1}}{2} \mathbf{A}_{f,22} & -\frac{d_{n+1}}{2} \mathbf{A}_{f,21} \end{bmatrix} \quad (3.77)$$

and self-recursions

$$\mathbf{H}_{f,11}^{(i)} = \mathbf{H}_{f,11}^{(i+1)} + \mathbf{H}_{f,12}^{(i+1)} \mathbf{H}_{f,11}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,22}^{(i+1)} \mathbf{H}_{f,11}^{(i+1)}]^{-1} \mathbf{H}_{f,21}^{(i+1)} \quad (3.78)$$

$$\mathbf{H}_{f,12}^{(i)} = \mathbf{H}_{f,12}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,11}^{(i+1)} \mathbf{H}_{f,22}^{(i+1)}]^{-1} \mathbf{H}_{f,12}^{(i+1)} \quad (3.79)$$

$$\mathbf{H}_{f,21}^{(i)} = \mathbf{H}_{f,21}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,22}^{(i+1)} \mathbf{H}_{f,11}^{(i+1)}]^{-1} \mathbf{H}_{f,21}^{(i+1)} \quad (3.80)$$

$$\mathbf{H}_{f,22}^{(i)} = \mathbf{H}_{f,22}^{(i+1)} + \mathbf{H}_{f,21}^{(i+1)} \mathbf{H}_{f,22}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,11}^{(i+1)} \mathbf{H}_{f,22}^{(i+1)}]^{-1} \mathbf{H}_{f,12}^{(i+1)} \quad (3.81)$$

With this method, the hybrid matrix can be calculated stably and accurately even for very thick or thin layer. Moreover, the calculation is free from all intricacies of solving the eigenvalues and eigenvectors, which require complex root searching, degeneracy treatment and upward/downward eigenvector sorting or selection.

To assess the accuracy of self-recursive asymptotic methods using impedance and hybrid matrices, we investigate the relative error changes with the number of subdivisions. Fig. 3.4 shows the average relative errors versus the number of recursive operations n for a single layer with $\varepsilon_r = 5 + 1.88i$. The relative error defined as

$$E(n) = \frac{\|\mathbf{X}_a - \mathbf{X}_e\|}{\|\mathbf{X}_e\|} \quad (3.82)$$

Here, $\mathbf{X}_a$ and $\mathbf{X}_e$ represent the impedance ($\mathbf{Z}$) or hybrid ($\mathbf{H}$) matrices obtained from the self-recursive asymptotic method and eigensolution method, respectively.

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The error is computed by taking the average over a range of incident angles $0 \leq \theta \leq \pi/2$ and frequencies $0.5\text{GHz} \leq f \leq 10\text{GHz}$. It is noted that the errors of both impedance and hybrid matrices decrease initially due to the smaller asymptotic approximation error for smaller initial sublayer thickness. After certain minimum point, the error of hybrid matrix increases slightly and then reaches a plateau. On the other hand, the error of impedance matrix increases much faster than that of hybrid matrix and it keeps growing as the recursion number increases. Therefore it is worthwhile adopting the hybrid matrix in conjunction with the self-recursive asymptotic method for its overall high accuracy and stability.

Fig. 3.4 Relative errors of $\mathbf{Z}$ and $\mathbf{H}$ matrices versus recursion number n for a single layer with $\epsilon_r = 5 + 1.88i$.

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3.4 Stack Matrices

3.4.1 Definitions

For solving multilayered problem, we also define the stack transfer matrix as

$$\begin{bmatrix} \bar{E}_t(z_f^>) \\ \bar{H}_t(z_f^>) \end{bmatrix} = \mathbf{T}^{(l,f)} \begin{bmatrix} \bar{E}_t(z_l^<) \\ \bar{H}_t(z_l^<) \end{bmatrix} \quad (3.83)$$

where $\mathbf{T}^{(l,f)}$ is the stack transfer matrix from layer l to layer f (usually we choose $l < f$). In a similar manner, the stack impedance and hybrid matrices can be defined as

$$\begin{bmatrix} \bar{E}_t(z_l^<) \\ \bar{E}_t(z_f^>) \end{bmatrix} = \mathbf{Z}^{(l,f)} \begin{bmatrix} \bar{H}_t(z_l^<) \\ \bar{H}_t(z_f^>) \end{bmatrix} \quad (3.84)$$

$$\begin{bmatrix} \bar{E}_t(z_l^<) \\ \bar{H}_t(z_f^>) \end{bmatrix} = \mathbf{H}^{(l,f)} \begin{bmatrix} \bar{H}_t(z_l^<) \\ \bar{E}_t(z_f^>) \end{bmatrix} \quad (3.85)$$

3.4.2 Matrix Recursions

We consider the matrix recursions starting from top layer N downward to lower layer f .

From the definitions of layer and stack transfer matrices in (3.44) and (3.83) while noting the continuity of the fields

$$\bar{E}_t(z_f^>) = \bar{E}_t(z_{f+1}^<) \quad (3.86)$$

$$\bar{H}_t(z_f^>) = \bar{H}_t(z_{f+1}^<) \quad (3.87)$$

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we can obtain the matrix recursions as

$$\mathbf{T}^{(f,N)} = \mathbf{T}^{(f+1,N)} \mathbf{T}_f \quad (3.88)$$

Thus, the total stack transfer matrix until bottom layer 1 can be obtained simply by successive multiplications of all intervening layer transfer matrices as

$$\mathbf{T}^{(1,N)} = \mathbf{T}_N \mathbf{T}_{N-1} \cdots \mathbf{T}_1 \quad (3.89)$$

From the definitions of layer and stack impedance matrices in (3.48) and (3.84), the impedance matrix recursions can be deduced from top layer N to lower layer f as

$$\mathbf{Z}_{11}^{(f,N)} = \mathbf{Z}_{f,11} + \mathbf{Z}_{f,12} [\mathbf{Z}_{11}^{(f+1,N)} - \mathbf{Z}_{f,22}]^{-1} \mathbf{Z}_{f,21} \quad (3.90)$$

$$\mathbf{Z}_{12}^{(f,N)} = -\mathbf{Z}_{f,12} [\mathbf{Z}_{11}^{(f+1,N)} - \mathbf{Z}_{f,22}]^{-1} \mathbf{Z}_{12}^{(f+1,N)} \quad (3.91)$$

$$\mathbf{Z}_{21}^{(f,N)} = \mathbf{Z}_{21}^{(f+1,N)} [\mathbf{Z}_{11}^{(f+1,N)} - \mathbf{Z}_{f,22}]^{-1} \mathbf{Z}_{f,21} \quad (3.92)$$

$$\mathbf{Z}_{22}^{(f,N)} = \mathbf{Z}_{22}^{(f+1,N)} - \mathbf{Z}_{21}^{(f+1,N)} [\mathbf{Z}_{11}^{(f+1,N)} - \mathbf{Z}_{f,22}]^{-1} \mathbf{Z}_{12}^{(f+1,N)} \quad (3.93)$$

Likewise, from the definitions of layer and stack hybrid matrices in (3.51) and (3.85), we can also obtain the hybrid matrix recursions from top layer N to lower layer f as

$$\mathbf{H}_{11}^{(f,N)} = \mathbf{H}_{f,11} + \mathbf{H}_{f,12} \mathbf{H}_{11}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (3.94)$$

$$\mathbf{H}_{12}^{(f,N)} = \mathbf{H}_{f,12} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (3.95)$$

$$\mathbf{H}_{21}^{(f,N)} = \mathbf{H}_{21}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (3.96)$$

$$\mathbf{H}_{22}^{(f,N)} = \mathbf{H}_{22}^{(f+1,N)} + \mathbf{H}_{21}^{(f+1,N)} \mathbf{H}_{f,22} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (3.97)$$

In the opposite direction, the hybrid matrix recursions starting from bottom layer 1 to upper layer f can be deduced as

$$\mathbf{H}_{11}^{(1,f)} = \mathbf{H}_{11}^{(1,f-1)} + \mathbf{H}_{12}^{(1,f-1)} \mathbf{H}_{f,11} [\mathbf{I} - \mathbf{H}_{22}^{(1,f-1)} \mathbf{H}_{f,11}]^{-1} \mathbf{H}_{21}^{(1,f-1)} \quad (3.98)$$

$$\mathbf{H}_{12}^{(1,f)} = \mathbf{H}_{12}^{(1,f-1)} [\mathbf{I} - \mathbf{H}_{f,11} \mathbf{H}_{22}^{(1,f-1)}]^{-1} \mathbf{H}_{f,12} \quad (3.99)$$

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$$\mathbf{H}_{21}^{(1,f)} = \mathbf{H}_{f,21} [\mathbf{I} - \mathbf{H}_{22}^{(1,f-1)} \mathbf{H}_{f,11}]^{-1} \mathbf{H}_{21}^{(1,f-1)} \quad (3.100)$$

$$\mathbf{H}_{22}^{(1,f)} = \mathbf{H}_{f,22} + \mathbf{H}_{f,21} \mathbf{H}_{22}^{(1,f-1)} [\mathbf{I} - \mathbf{H}_{f,11} \mathbf{H}_{22}^{(1,f-1)}]^{-1} \mathbf{H}_{f,12} \quad (3.101)$$

Comparing (3.94)-(3.97) with (3.98)-(3.101), one can find that they both have similar forms. Therefore the hybrid matrix recursions can be written in a general form

$$\mathbf{H}_{11} = \mathbf{H}_{11}^< + \mathbf{H}_{12}^< \mathbf{H}_{11}^> [\mathbf{I} - \mathbf{H}_{22}^< \mathbf{H}_{11}^>]^{-1} \mathbf{H}_{21}^< \quad (3.102)$$

$$\mathbf{H}_{12} = \mathbf{H}_{12}^< [\mathbf{I} - \mathbf{H}_{11}^> \mathbf{H}_{22}^<]^{-1} \mathbf{H}_{12}^> \quad (3.103)$$

$$\mathbf{H}_{21} = \mathbf{H}_{21}^> [\mathbf{I} - \mathbf{H}_{22}^< \mathbf{H}_{11}^>]^{-1} \mathbf{H}_{21}^< \quad (3.104)$$

$$\mathbf{H}_{22} = \mathbf{H}_{22}^> + \mathbf{H}_{21}^> \mathbf{H}_{22}^< [\mathbf{I} - \mathbf{H}_{11}^> \mathbf{H}_{22}^<]^{-1} \mathbf{H}_{12}^> \quad (3.105)$$

where $\mathbf{H}^>$ and $\mathbf{H}^<$ stand for the hybrid matrix for upper and lower layer/stack respectively.

3.4.3 Comparison of Algorithms

For comparison of the various matrix algorithms discussed above, we have acquired their respective CPU time. Table 3.1 lists the CPU time for obtaining total stack matrices of hybrid, transfer and impedance matrix methods. The CPU time is obtained by taking the average over 10^5 loops of each method for a two-layered structure. The methods are programmed by using Matlab 6.1 software and are run on laptop of Intel(R) Pentium(R) M processor 1.40 GHz, 504 MB of RAM, Microsoft

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Windows XP system. From Table 3.1, one can see that the efficiencies of all three methods are comparable. Furthermore, the stable and accurate range of layer thickness for transfer and impedance matrices is not complete – excluding very large and very small thicknesses. On the contrary, the hybrid matrix is unconditionally stable throughout.

Table 3.1 CPU time for each recursion of hybrid, transfer and impedance matrix methods

	Algorithm	CPU time (s)	Remarks
H	$\psi_f \xrightarrow{\text{Eq.(3.53)}} \mathbf{H}_f \xrightarrow{\text{Eq.(3.94)-(3.97)}} \mathbf{H}^{(f,N)}$	1.8×10^{-3}	Unconditionally stable
T	$\psi_f \xrightarrow{\text{Eq.(3.47)}} \mathbf{T}_f \xrightarrow{\text{Eq.(3.88)}} \mathbf{T}^{(f,N)}$	1.5×10^{-3}	Ill conditioned: $d_f \rightarrow \infty$
Z	$\psi_f \xrightarrow{\text{Eq.(3.50)}} \mathbf{Z}_f \xrightarrow{\text{Eq.(3.90)-(3.93)}} \mathbf{Z}^{(f,N)}$	1.7×10^{-3}	Ill conditioned: $d_f \rightarrow 0$

3.5 Reflection and Transmission Coefficients

With the total stack hybrid matrix $\mathbf{H}^{(1,N)}$ available for the whole stack from layer 1 to layer N , it is simple to compute the corresponding reflection and transmission coefficients. Let the electromagnetic waves be incident from layer 0, and the reflection

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$(\mathbf{r}_{0,1})$ and transmission $(\mathbf{t}_{0,N+1})$ coefficient matrices be defined as

$$\bar{\mathbf{w}}_0^<(Z_0^>) = \mathbf{r}_{0,1} \bar{\mathbf{w}}_0^>(Z_0^>) \quad (3.106)$$

$$\bar{\mathbf{w}}_{N+1}^>(Z_{N+1}^<)= \mathbf{t}_{0,N+1} \bar{\mathbf{w}}_0^>(Z_0^>) \quad (3.107)$$

In case the external layer $N+1$ is semi-infinite, the radiation condition reads

$$\bar{\mathbf{w}}_{N+1}^<(Z_{N+1}^<)=0 \quad (3.108)$$

Then the reflection and transmission coefficient matrices can be solved explicitly in terms of stack hybrid matrix

$$\mathbf{r}_{0,1} = [\mathbf{H}_s \mathbf{h}_0^< - \mathbf{e}_0^<]^{-1} [\mathbf{e}_0^> - \mathbf{H}_s \mathbf{h}_0^>] \quad (3.109)$$

$$\mathbf{t}_{0,N+1} = [\mathbf{h}_{N+1}^> - \mathbf{H}_{22}^{(1,N)} \mathbf{e}_{N+1}^>]^{-1} \mathbf{H}_{21}^{(1,N)} [\mathbf{h}_0^> + \mathbf{h}_0^< \mathbf{r}_{0,1}] \quad (3.110)$$

$$\mathbf{H}_s = \mathbf{H}_{11}^{(1,N)} + \mathbf{H}_{12}^{(1,N)} [\mathbf{h}_{N+1}^> (\mathbf{e}_{N+1}^>)^{-1} - \mathbf{H}_{22}^{(1,N)}]^{-1} \mathbf{H}_{21}^{(1,N)} \quad (3.111)$$

where $\mathbf{e}_0^<$, $\mathbf{h}_0^<$ and $\mathbf{e}_{N+1}^<$, $\mathbf{h}_{N+1}^<$ are the eigenwaves for the lower and upper external layers (0 and $N+1$). In most practical applications, these external layers are air for which the eigenwave matrix may take the form

$$\begin{aligned} \Psi_0 &= \begin{bmatrix} \mathbf{e}_0^> & \mathbf{e}_0^< \\ \mathbf{h}_0^> & \mathbf{h}_0^< \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & 0 & -\cos \theta & 0 \\ 0 & -1 & 0 & -1 \\ 0 & \cos \theta / \eta & 0 & -\cos \theta / \eta \\ 1 / \eta & 0 & 1 / \eta & 0 \end{bmatrix} \end{aligned} \quad (3.112)$$

where θ is the incident angle, η is the intrinsic impedance of free space $\eta = \sqrt{\mu_0 / \epsilon_0}$. Here we have chosen to compose the eigenwave matrix with the common TE and TM waves.

One should notice that both $\mathbf{r}_{0,1}$ and $\mathbf{t}_{0,N+1}$ are 2×2 matrices:

$$\mathbf{r}_{0,1} = \begin{bmatrix} (r_{0,1})_{11} & (r_{0,1})_{12} \\ (r_{0,1})_{21} & (r_{0,1})_{22} \end{bmatrix} \quad (3.113)$$

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$$\mathbf{t}_{0,N+1} = \begin{bmatrix} (t_{0,N+1})_{11} & (t_{0,N+1})_{12} \\ (t_{0,N+1})_{21} & (t_{0,N+1})_{22} \end{bmatrix} \quad (3.114)$$

Let r_{co} and t_{co} represent the reflection and transmission coefficients for the same polarization as the incident waves, while r_{cross} and t_{cross} correspond to the cross polarization. We have for TE incidence:

$$(r_{0,1})_{11} = r_{co}, \quad (r_{0,1})_{21} = r_{cross} \quad (3.115)$$

$$(t_{0,N+1})_{11} = t_{co}, \quad (t_{0,N+1})_{21} = t_{cross} \quad (3.116)$$

and for TM incidence:

$$(r_{0,1})_{22} = r_{co}, \quad (r_{0,1})_{12} = r_{cross} \quad (3.117)$$

$$(t_{0,N+1})_{22} = t_{co}, \quad (t_{0,N+1})_{12} = t_{cross} \quad (3.118)$$

The shielding effectiveness is defined as

$$SE(dB) = -10 \log P_t / P_i \quad (3.119)$$

Here P_i and P_t are the average powers associated with the incident and transmitted waves, respectively. It can be deduced simply as:

$$SE(dB) = -10 \log(|t_{co}|^2 + |t_{cross}|^2) \quad (3.120)$$

In the case of layer $N+1$ being perfect electric conductor (PEC), the boundary condition reads

$$\bar{E}_t(z_N^>) = \bar{E}_t(z_{N+1}^<) = 0 \quad (3.121)$$

Then the reflection coefficient can be determined as

$$\mathbf{r}_{0,1} = [\mathbf{H}_{11}^{(1,N)} \mathbf{h}_0^< - \mathbf{e}_0^<]^{-1} [\mathbf{e}_0^> - \mathbf{H}_{11}^{(1,N)} \mathbf{h}_0^>] \quad (3.122)$$

3.6 Numerical Results

The analytical expressions derived for the hybrid matrix method have been applied to calculate the electromagnetic wave propagation in a number of multilayered structures to be presented in the following. The algorithms are programmed by using Matlab 6.1 software.

First, we shall verify our method with some reduced or simplified cases. Let a plane TEM wave be incident on a multilayered dielectric sheet. The dielectric constants of the two layers are $\epsilon_1 = 4.2$ and $\epsilon_2 = 10.2$. Their thicknesses are $d_1 = 5\text{mm}$ and $d_2 = 10\text{mm}$. So the electrical lengths are $\phi_1 = \sqrt{\epsilon_1} k_0 d_1$ and $\phi_2 = \sqrt{\epsilon_2} k_0 d_2$. The structure and its equivalent circuit are illustrated in Fig. 3.5. The characteristic impedance of both dielectric layers are $Z_1 = Z_0 / \sqrt{\epsilon_1}$ and $Z_2 = Z_0 / \sqrt{\epsilon_2}$. The input impedance at the interface of dielectric 1 and dielectric 2 is given by [5]

$$Z_{in1} = \frac{Z_2(Z_0 + jZ_2 \tan \phi_2)}{Z_2 + jZ_0 \tan \phi_2} \quad (3.123)$$

The input impedance at the air-dielectric 1 interface is given by

$$Z_{in} = \frac{Z_1(Z_{in1} + jZ_1 \tan \phi_1)}{Z_1 + jZ_{in1} \tan \phi_1} \quad (3.124)$$

Then the reflection coefficient can be calculated as

$$R = \frac{Z_{in1} - Z_0}{Z_{in1} + Z_0} \quad (3.125)$$

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In Fig. 3.6, we show the reflection coefficients versus frequency of two layered dielectric obtained by transmission line theory and hybrid matrix method. From the figure, one can see that the results of the two methods fit very well.

Fig. 3.5. A two layered dielectric sheet and its equivalent circuit.

Fig. 3.6. Reflection coefficients versus frequency of two layered dielectric obtained by transmission line theory and hybrid matrix method.

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Next, we verify our method by comparing with Computer Systems Technology (CST) software. The example structure is composed of two liquid crystal (electrically biaxial anisotropic) layers with different optical axis orientations. The permittivities of the two layers are $\epsilon_{1xx} = 13.0$, $\epsilon_{1yy} = \epsilon_{1zz} = 2.25$ and $\epsilon_{2yy} = 13.0$, $\epsilon_{2xx} = \epsilon_{2zz} = 2.25$, respectively. Their thicknesses are $d_1 = 2d_2 = 2d_c/3$. Fig. 3.7 shows the reflection and transmission coefficients calculated by CST and hybrid matrix method. The programs for both methods are run on laptop mentioned in Section 3.4.3. The version of CST is 2006 and the hybrid matrix method is programmed on Matlab 6.1.

Fig. 3.7. The reflection and transmission coefficients for two layered anisotropic media calculated by CST and hybrid matrix method. The permittivities of the two layers are $\epsilon_{1xx} = 13.0$, $\epsilon_{1yy} = \epsilon_{1zz} = 2.25$ and $\epsilon_{2yy} = 13.0$, $\epsilon_{2xx} = \epsilon_{2zz} = 2.25$, respectively. Their thicknesses are $d_1 = 2d_2 = 2d_c/3$.

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The numerical results agree very well. But the efficiency of our method (samples: 1000, CPU time: ~4.8s) is much higher than CST (samples: 1000, CPU time: ~70s). Furthermore, CST software can only simulate the biaxial anisotropic materials while our method can deal with more complex materials.

Then, we consider a more complex practical example. Multilayered magneto-optic recording media are based on the fact that the optical interference of a multilayered medium enhances the reflectivity and, therefore, increases the signal-to-noise ratio [33]. The recording medium is composed of four layers on a substrate. The glass substrate with refractive index $n_{sub}=1.5$ is coated with a reflecting aluminum layer of thickness 500 nm and complex refractive index $n_{Al}=2.75+8.31i$. A quarter-wave layer of SiO_x (143.2nm thick), with $n_{\text{SiO}_x}=1.449$, separates the aluminum layer from the magnetic film, which is 20nm thick and has the following dielectric tensor: $\epsilon_{xx}=\epsilon_{yy}=\epsilon_{zz}=-4.8984+19.415i$ and $\epsilon_{xy}=-\epsilon_{yx}=0.4322+0.0058i$. The overcoat layer is another quarter-wave SiO_x and the light is incident from air with a wavelength of $\lambda_0=830\text{nm}$. The reflectivities obtained by our calculation and previous publication [33] are presented in Fig. 3.8. R_{pp} and R_{sp} are, respectively, the P and S components of the reflected beam when the incident polarization is P . Similarly, R_{ss} and R_{ps} correspond to S -polarized incident light. It is observed that R_{ps} and R_{sp} are identical. The results by our method are in full agreement with the publication.

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Fig. 3.8. The reflectivity coefficients as a function of the incident angle for four-layered magneto-optic recording medium obtained by our method and published paper.

Moving on, let us consider a planar stratified bianisotropic waveguide shown in Fig. 3.1. The homogeneous medium of each layer f is characterized by permittivity ϵ_f , permeability μ_f , and magneto-electric tensors ξ_f and ζ_f as

$$\epsilon_f = \epsilon_0 \begin{bmatrix} \epsilon_{ff} & \epsilon_{gf} & 0 \\ -\epsilon_{gf} & \epsilon_{ff} & 0 \\ 0 & 0 & \epsilon_{pf} \end{bmatrix} \quad (3.126)$$

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$\mu_f = \mu_0 \mathbf{I}$ and $\xi_f = -\varsigma_f = -j\kappa_f \sqrt{\mu_0 \varepsilon_0} \mathbf{I}$. Fig. 3.9 shows the dispersion relations for (a) covered and (b) uncovered grounded two-layered bianisotropic waveguides, with $h_1 = h_2 = 2\text{mm}$, $\varepsilon_{t1} = 10$, $\varepsilon_{p1} = 8$, $\varepsilon_{g1} = 2i$, $\kappa_1 = 0.27$, $\varepsilon_{t2} = 5$, $\varepsilon_{p2} = 14$, $\varepsilon_{g2} = 11i$, $\kappa_2 = 0.3$. The dispersion curves can be obtained by finding the solution of

$$\det(\mathbf{H}_{11}^{(l)}) = 0 \quad (3.127)$$

The results have been computed simply and reliably using the self-recursive asymptotic hybrid matrix method ($n=10\sim 22$) in conjunction with hybrid matrix recursions. We have also applied the corresponding impedance matrix method. Although it works well for thicker case, the impedance matrix with thin initial layer ($n\sim 22$) becomes inaccurate and results in many incorrect zeros (due to spurious sign changes in the root-searching function). This is illustrated in the inset of Fig. 3.9(a). Such incorrect zeros also appear in other parts of the figure but have been omitted for clarity. Note that no such problem occurs if one resorts to the hybrid matrix method. For the uncovered situation in Fig. 3.9(b), we take the air as the third layer. Actually this layer is not restricted to air but can be any bianisotropic material. To calculate the hybrid matrix of the semi-infinite layer, one can still apply the self-recursive asymptotic method. We perform the self-recursions until the results converge (e.g. 10-20 recursions for 10^{-6} accuracy). In this way we bypass all intricacies of eigensolution method and one can also trade-off the efficiency and accuracy in a convenient manner. Since the transfer matrix fails to converge for the semi-infinite layer, it cannot be utilized at all in this case.

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(a)

(b)

Fig. 3.9 Dispersion relations for (a) covered and (b) uncovered grounded two-layered bianisotropic waveguides, with $h_1 = h_2 = 2\text{mm}$, $\epsilon_{t1} = 10$, $\epsilon_{p1} = 8$, $\epsilon_{g1} = 2i$, $\kappa_1 = 0.27$, $\epsilon_{t2} = 5$, $\epsilon_{p2} = 14$, $\epsilon_{g2} = 11i$, $\kappa_2 = 0.3$ as defined in (3.126).

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Besides the application for waveguides, we can also apply the hybrid matrix method above to analyze the performance characteristics of tunable absorber and laminated composite shield. Consider first the tunable absorber made of ferrite, with the permeability dependent on the dc biasing magnetic field H_0 and its bias angles θ_H and φ_H as follows [133]:

$$\mu_{xx} = \mu_H + (\mu_0 - \mu_H) \sin^2 \theta_H \cos^2 \varphi_H \quad (3.128)$$

$$\mu_{xy} = (\mu_0 - \mu_H) \sin^2 \theta_H \sin \varphi_H \cos \varphi_H + i\mu_\kappa \cos \theta_H \quad (3.129)$$

$$\mu_{xz} = (\mu_0 - \mu_H) \sin \theta_H \cos \theta_H \cos \varphi_H - i\mu_\kappa \sin \theta_H \sin \varphi_H \quad (3.130)$$

$$\mu_{yx} = (\mu_0 - \mu_H) \sin^2 \theta_H \sin \varphi_H \cos \varphi_H - i\mu_\kappa \cos \theta_H \quad (3.131)$$

$$\mu_{yy} = \mu_H + (\mu_0 - \mu_H) \sin^2 \theta_H \sin^2 \varphi_H \quad (3.132)$$

$$\mu_{yz} = (\mu_0 - \mu_H) \sin \theta_H \cos \theta_H \sin \varphi_H + i\mu_\kappa \sin \theta_H \cos \varphi_H \quad (3.133)$$

$$\mu_{zx} = (\mu_0 - \mu_H) \sin \theta_H \cos \theta_H \cos \varphi_H + i\mu_\kappa \sin \theta_H \sin \varphi_H \quad (3.134)$$

$$\mu_{zy} = (\mu_0 - \mu_H) \sin \theta_H \cos \theta_H \sin \varphi_H - i\mu_\kappa \sin \theta_H \cos \varphi_H \quad (3.135)$$

$$\mu_{zz} = \mu_H - (\mu_0 - \mu_H) \sin^2 \theta_H \quad (3.136)$$

$$\omega_0 = |\gamma| H_0 - i|\gamma| \frac{\Delta H}{2} \quad (3.137)$$

$$\omega_m = |\gamma| M_s \quad (3.138)$$

$$\mu_H = \mu_0 \left(1 + \frac{\omega_0 \omega_m}{\omega_0^2 - \omega^2} \right) \quad (3.139)$$

$$\mu_\kappa = -\mu_0 \frac{\omega \omega_m}{\omega_0^2 - \omega^2} \quad (3.140)$$

$\gamma = 1.759 \times 10^{11} \text{ kg/C}$ is the gyromagnetic ratio, ΔH is 3dB line width and M_s is the saturation magnetization for the ferrite. Fig. 3.10(a) and (b) show the calculated frequency response of $|r_{co}|^2$ and $|r_{cross}|^2$ for the TE waves incident normally to a

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biased ferrite layer backed with PEC. The layer thickness is $h = 5\text{mm}$ and the parameters are $\varepsilon_r = 12.6$, $\tan \delta = 0.001$, $\mu_0 H_0 = 0.028\text{ T}$, $\mu_0 \Delta H = 0.046\text{ T}$, $\mu_0 M_s = 0.21\text{ T}$, $\varphi_H = 45^\circ$ and $\theta_H = 0^\circ, 30^\circ, 60^\circ, 90^\circ$. The results suggest that the absorbing performance is tunable easily by adjusting the angle of bias magnetic field. For instance, when $\varphi_H = 45^\circ$ and θ_H changes from 0° to 90° , the location of maximum reflection loss is shifted from 2 GHz to 4 GHz in Fig. 3.10(a).

(a)

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Fig. 3.10 Frequency response of $|r_{co}|^2$ and $|r_{cross}|^2$ for the TE waves incident normally to a biased ferrite layer backed with PEC: (a) $|r_{co}|^2$; (b) $|r_{cross}|^2$. The layer thickness is $h = 5\text{mm}$ and the parameters are $\epsilon_r = 12.6$, $\tan \delta = 0.001$, $\mu_0 H_0 = 0.028$ T, $\mu_0 \Delta H = 0.046$ T, $\mu_0 M_s = 0.21$ T, $\varphi_H = 45^\circ$ and $\theta_H = 0^\circ, 30^\circ, 60^\circ, 90^\circ$.

In Fig. 3.11 and Fig. 3.12 we show the shielding effectiveness of a laminated graphite/epoxy fiber composite. The permittivity tensor for the graphite/epoxy can be expressed as [124]

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_r \boldsymbol{\epsilon}_0 + i \boldsymbol{\sigma} / \omega \quad (3.141)$$

where the relative permittivity and conductivity have the form

$$\boldsymbol{\epsilon}_r = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \quad (3.142)$$

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$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \quad (3.143)$$

For different fiber orientation, $\boldsymbol{\sigma}$ is provided by

$$\sigma_{xx} = \sigma_{//} \cos^2 \varphi + \sigma_{\perp} \sin^2 \varphi \quad (3.144)$$

$$\sigma_{xy} = \sigma_{yx} = (\sigma_{//} - \sigma_{\perp}) \cos \varphi \sin \varphi \quad (3.145)$$

$$\sigma_{yy} = \sigma_{//} \sin^2 \varphi + \sigma_{\perp} \cos^2 \varphi \quad (3.146)$$

$$\sigma_{zz} = \sigma_{\perp} \quad (3.147)$$

where $\sigma_{//}$ and $\sigma_{\perp}$ are the conductivities parallel and perpendicular to the fiber direction respectively; φ is the fiber orientation angle. Similar expressions are used for the relative permittivities:

$$\varepsilon_{xx} = \varepsilon_{//} \cos^2 \varphi + \varepsilon_{\perp} \sin^2 \varphi \quad (3.148)$$

$$\varepsilon_{xy} = \varepsilon_{yx} = (\varepsilon_{//} - \varepsilon_{\perp}) \cos \varphi \sin \varphi \quad (3.149)$$

$$\varepsilon_{yy} = \varepsilon_{//} \sin^2 \varphi + \varepsilon_{\perp} \cos^2 \varphi \quad (3.150)$$

$$\varepsilon_{zz} = \varepsilon_{\perp} \quad (3.151)$$

The graphite/epoxy is nonmagnetic and has $\varepsilon_r = \varepsilon_{//} = \varepsilon_{\perp} = 5$, $\sigma_{//} = 4 \times 10^4$ S/m, and $\sigma_{\perp} = 50$ S/m. Fig. 3.11 illustrates the frequency response of the shielding effectiveness for graphite/epoxy composites having different number of layers. The fiber orientation pattern is $[0^\circ, 45^\circ, 90^\circ, -45^\circ]$ to be repeated within a cell, and the TE waves are incident normally to the composites. The thickness of each layer is $d = d_0 / N$ and the total thickness $d_0 = 4\text{mm}$ stays unchanged. It is demonstrated that with the increasing number of cells, the shielding effectiveness may improve significantly at high frequency assuming the total thickness stays unchanged.

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Fig. 3.11 Frequency response of the shielding effectiveness for graphite/epoxy composites having different number of layers. The graphite/epoxy is nonmagnetic and has $\varepsilon_r = \varepsilon_{//} = \varepsilon_{\perp} = 5$, $\sigma_{//} = 4 \times 10^4$ S/m, and $\sigma_{\perp} = 50$ S/m.

Fig. 3.12 Frequency response of the shielding effectiveness for graphite/epoxy composites having different orientation patterns. The material parameter is same as that in Fig. 3.11

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Fig. 3.12 illustrates the frequency response of the shielding effectiveness for (one cell) graphite/epoxy composites having different orientation patterns. The thickness of each layer is 0.127mm. Four orientation patterns have been considered: $[0^\circ, 45^\circ, 90^\circ, -45^\circ]$, $[0^\circ, 30^\circ, 60^\circ, 90^\circ]$, $[0^\circ, 60^\circ, -30^\circ, 30^\circ]$ and $[0^\circ, 90^\circ]$. It is found that with the orientation pattern being $[0^\circ, 60^\circ, -30^\circ, 30^\circ]$, the shielding effectiveness is highest at low frequency and grows at the fastest rate when the frequency increases. From Fig. 3.11 and Fig. 3.12 one can see that the number of layers/cells as well as their fiber orientations will strongly influence the shielding characteristics of the composites.

3.7 Conclusion

This chapter has presented a hybrid matrix method for stable analysis of electromagnetic waves in multilayered bianisotropic media. The hybrid matrix method eliminates the numerical instability of transfer and impedance matrix methods even when the thickness tends to infinity and zero. A self-recursive asymptotic method has been proposed which bypasses the intricacies of eigensolution approach while requiring only simple elementary matrix operations based on asymptotic thin-layer approximation. For calculating the reflection coefficient and shielding effectiveness, one requires only the eigenvectors of two external layers, which can often be found

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analytically. With its simplicity and robustness to accommodate the complete range of thickness, the hybrid matrix method will find usefulness in designing radar absorbers and laminated shields, etc.

Chapter 4

Generalized Eigenproblem of Hybrid Matrix Method for Bloch-Floquet Waves Propagation in One-Dimensional Photonic Crystals

4.1 Introduction

In previous chapter, we have introduced the hybrid matrix method for multilayered bianisotropic media. In this chapter, we will pay attention to photonic crystals, which are artificial periodic composites useful to control and manipulate light propagation [28]-[31]. The simplest photonic crystals are that of one-dimensional, which possess several advantages such as low cost, ease of fabrication and excellent features of omnidirectional reflections [31]-[40]. In essence, the one-dimensional photonic crystals represent a kind of periodic multilayered structure, which may be composed of various dielectric, uniaxial, biaxial, magneto-optic or generally anisotropic media. One of the celebrated techniques for analysis of such media is based on the combination of transfer matrix method and Bloch-Floquet wave theory [32]-[40], [54], [56]. This technique features the advantage of being convenient in implementation. Once the

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transfer matrix for the whole unit cell has been determined through proper cascading, Bloch-Floquet waves are found simply as its eigenvectors with the associated eigenvalues being the exponentials of Bloch-Floquet wavenumbers. For photonic crystals with many or infinite periodic repetitions of few elementary layers, these Bloch-Floquet waves can be employed to analyze the whole crystals very efficiently.

As mentioned in previous chapter, the transfer matrix method suffers from the inherent numerical instabilities, particularly when the layer thickness is large, the frequency is high and/or the material is dissipative/active [134]. There have been various techniques proposed to alleviate the numerical problem including impedance and transmittance or scattering matrix methods [135]-[138]. However, these techniques cannot be applied to find Bloch-Floquet waves directly. This is because the Bloch-Floquet waves are the standard eigenvectors of transfer matrix and not those of impedance or scattering matrices. If the unit cell is thick enough to cause the cell transfer matrix to exhibit numerical instability, the Bloch-Floquet solutions for such periodic media cannot be determined correctly and reliably.

The objective of this chapter is to devise a stable and reliable method to determine the Bloch-Floquet waves for one-dimensional photonic crystals composed of general bianisotropic media. The method is based on the solutions to a generalized eigenproblem of hybrid matrix, instead of the standard eigenproblem of transfer matrix. The hybrid matrix is found to be stable for arbitrary thickness, frequency and material as discussed in the previous chapter. The exploitation of such formulation using hybrid

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matrix overcomes the numerical instability of transfer matrix. It enables stable and accurate analysis of photonic crystals with wide range of thickness at high frequency.

4.2 Standard Eigenproblem

Fig. 4.1 Geometry of one-dimensional photonic crystals.

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Fig. 4.1 shows the geometry of one-dimensional photonic crystals. There are in general n (could be ∞) unit cells and the unit cell may comprise N homogeneous layers stratified in $\hat{z}$ direction. The tangential components of electric and magnetic fields satisfy a first-order differential system for each layer f , cf. Section 3.2:

$$\frac{d}{dz} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} = \mathbf{A}_f \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} \quad (4.1)$$

Equation (4.1) admits solutions in terms of the superposition of eigenwaves:

$$\begin{bmatrix} \bar{E}_t(z) \\ \bar{H}_t(z) \end{bmatrix} = \boldsymbol{\Psi}_f \mathbf{P}_f(z) \bar{c}_f = \boldsymbol{\Psi}_f \bar{w}_f(z) \quad (4.2)$$

Here, $\boldsymbol{\Psi}_f$ is a 4×4 eigenwave matrix comprising the eigenvectors; $\mathbf{P}_f(z)$ is a diagonal matrix whose elements are $p_f^{(j)}(z) = \exp(ik_{zf}^{(j)}z)$, $k_{zf}^{(j)}$ being the j th wavenumber; $\bar{c}_f$ is a 4×1 coefficient vector containing the unknown constants to be determined, and $\bar{w}_f(z)$ is a 4×1 weight vector which lumps the exponentials and coefficients together. Henceforth, the eigensolutions are decomposed into upward-bounded and downward-bounded waves corresponding to $\text{Im}(k_{zf}^{(j)}) > 0$ and $\text{Im}(k_{zf}^{(j)}) < 0$, which remain bounded in the upper and lower region respectively.

Accordingly, $\boldsymbol{\Psi}_f$, $\mathbf{P}_f$ and $\bar{w}_f$ can be decomposed into partitions as

$$\boldsymbol{\Psi}_f = \begin{bmatrix} \mathbf{e}_f^> & \mathbf{e}_f^< \\ \mathbf{h}_f^> & \mathbf{h}_f^< \end{bmatrix} \quad (4.3)$$

$$\mathbf{P}_f = \begin{bmatrix} \mathbf{P}_f^> & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_f^< \end{bmatrix} \quad (4.4)$$

$$\bar{w}_f = \begin{bmatrix} \bar{w}_f^> \\ \bar{w}_f^< \end{bmatrix} \quad (4.5)$$

where the superscripts “ $>$ ” and “ $<$ ” stand for “upward-bounded” and “downward-bounded” decomposition, respectively.

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As in Chapter 3, the fields at the top of layer f in the unit cell can be related to those at the bottom by the layer transfer matrix

$$\begin{bmatrix} \bar{E}_t(Z_f^>) \\ \bar{H}_t(Z_f^>) \end{bmatrix} = \mathbf{T}_f \begin{bmatrix} \bar{E}_t(Z_f^<) \\ \bar{H}_t(Z_f^<) \end{bmatrix} \quad (4.6)$$

$$\mathbf{T}_f = \boldsymbol{\Psi}_f \mathbf{P}_f(h_f) \boldsymbol{\Psi}_f^{-1} \quad (4.7)$$

By properly cascading the pertaining layer transfer matrices, the fields at the top of the unit cell can be inferred from those at the bottom directly via the cell transfer matrix

$$\mathbf{T}_c = \mathbf{T}_N \cdots \mathbf{T}_2 \mathbf{T}_1 \quad (4.8)$$

Using the cell transfer matrix, the problem of wave propagation in a periodic stack of unit cells that constitute the one-dimensional photonic crystals can be treated conveniently. In particular, Bloch-Floquet (F) waves are determined from the solutions to the standard eigenproblem

$$\mathbf{T}_c \boldsymbol{\psi}_F^{(j)} = p_F^{(j)} \boldsymbol{\psi}_F^{(j)} \quad (4.9)$$

In accordance with the boundedness association in (4.3)-(4.4), the eigensolutions are decomposed into upward-bounded ($|p_F^{(j)}| < 1$) and downward-bounded ($|p_F^{(j)}| > 1$) waves – $p_F^{(j)}$ can be assembled into $\mathbf{P}_F^>$ and $\mathbf{P}_F^<$ forming the matrix $\mathbf{P}_F$, while their associated eigenvectors can also be assembled as $\mathbf{e}_F^>$, $\mathbf{h}_F^>$ and $\mathbf{e}_F^<$, $\mathbf{h}_F^<$ forming the Bloch-Floquet eigenwave matrix $\boldsymbol{\Psi}_F$:

$$\mathbf{P}_F = \begin{bmatrix} \mathbf{P}_F^> & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_F^< \end{bmatrix} \quad (4.10)$$

$$\boldsymbol{\Psi}_F = \begin{bmatrix} \mathbf{e}_F^> & \mathbf{e}_F^< \\ \mathbf{h}_F^> & \mathbf{h}_F^< \end{bmatrix} \quad (4.11)$$

The transfer matrix can also be written as

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$$\mathbf{T}_c = \boldsymbol{\Psi}_F \mathbf{P}_F(h_F) \boldsymbol{\Psi}_F^{-1} \quad (4.12)$$

where h_F is total thickness of the unit cell.

Equation (4.9) gives rise to a fourth-degree characteristic polynomial with its coefficients expressible in terms of the cell transfer matrix elements:

$$\det(\mathbf{T}_c - p_F \mathbf{I}) = p_F^4 + a_3 p_F^3 + a_2 p_F^2 + a_1 p_F + a_0 = 0 \quad (4.13)$$

$$a_3 = -\text{tr}(\mathbf{T}_c) \quad (4.14)$$

$$a_2 = [\text{tr}^2(\mathbf{T}_c) - \text{tr}(\mathbf{T}_c^2)]/2 \quad (4.15)$$

$$a_1 = -\text{tr}^3(\mathbf{T}_c)/6 + \text{tr}(\mathbf{T}_c)\text{tr}(\mathbf{T}_c^2)/2 - \text{tr}(\mathbf{T}_c^3)/3 \quad (4.16)$$

$$a_0 = \det(\mathbf{T}_c) \quad (4.17)$$

When the unit cell is thick or the frequency is high, the calculation of cell transfer matrix leads to numerical difficulties with the result being erroneous or unstable. This renders the computation of the eigenvalues and eigenvectors inaccurate or wrong based on the incorrect $\mathbf{T}_c$. In Fig. 4.2 we show the characteristic polynomial coefficients of transfer matrix as functions of $k_0 h_F$. The cell is made up by stacked LiNbO3 and liquid crystal layers, both of which have permittivity tensors of the form

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 \begin{bmatrix} \varepsilon_{//} & 0 & 0 \\ 0 & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{bmatrix} \quad (4.18)$$

and $\boldsymbol{\mu} = \mu_0 \mathbf{I}$. The thickness of each layer is $h_1 = 2h_F/3$, $h_2 = h_F/3$ and the parameters are: $\varepsilon_{\perp 1} = 4.12 + 9.38i$, $\varepsilon_{//1} = 6.27 + 6.40i$, $\varepsilon_{\perp 2} = 2.25$, $\varepsilon_{//2} = 13.2$. It can be seen that as $k_0 h_F$ increases, the characteristic polynomial coefficients may become very large, which makes the calculation of their roots (the eigenvalues of transfer matrix) inaccurate.

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Fig. 4.2 Characteristic polynomial coefficients of transfer matrix versus $k_0 h_F$. The cell is made up by stacked LiNbO₃ and liquid crystal layers, both with permittivity tensors (4.18), $h_1 = 2h_F/3$, $h_2 = h_F/3$, $\varepsilon_{a1} = 4.12 + 9.38i$, $\varepsilon_{c1} = 6.27 + 6.40i$, $\varepsilon_{a2} = 2.25$, $\varepsilon_{c2} = 13.2$.

There has been attempt to overcome the numerical problem of transfer matrix using impedance matrix [136]:

$$\begin{bmatrix} \bar{E}_t(Z_1^<) \\ \bar{E}_t(Z_N^>) \end{bmatrix} = \mathbf{Z}_c \begin{bmatrix} \bar{H}_t(Z_1^<) \\ \bar{H}_t(Z_N^>) \end{bmatrix} \quad (4.19)$$

$$\mathbf{Z}_c = \begin{bmatrix} \mathbf{Z}_{c11} & \mathbf{Z}_{c12} \\ \mathbf{Z}_{c21} & \mathbf{Z}_{c22} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_F^> & \mathbf{e}_F^< \mathbf{P}_F^{<-1} \\ \mathbf{e}_F^> \mathbf{P}_F^> & \mathbf{e}_F^< \end{bmatrix} \begin{bmatrix} \mathbf{h}_F^> & \mathbf{h}_F^< \mathbf{P}_F^{<-1} \\ \mathbf{h}_F^> \mathbf{P}_F^> & \mathbf{h}_F^< \end{bmatrix}^{-1} \quad (4.20)$$

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It can be demonstrated that the impedance matrix is free from numerical instability for large cell thickness.

Although the characteristic polynomial can be written in the form of impedance matrix as [112]

$$\det(p_F \mathbf{Z}_{c12} - p_F^{-1} \mathbf{Z}_{c21} + \mathbf{Z}_{c11} - \mathbf{Z}_{c22}) = 0 \quad (4.21)$$

for the periodic media considered here, the impedance matrix method cannot be applied directly also. This is because the Bloch-Floquet waves are really the eigenvectors of transfer matrix and not those of impedance matrix. Moreover, when the cell thickness becomes large, similar to (3.61)

$$\mathbf{Z}_c |_{h_F=\infty} = \begin{bmatrix} \mathbf{e}_F^> (\mathbf{h}_F^>)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_F^< (\mathbf{h}_F^<)^{-1} \end{bmatrix} \quad (4.22)$$

One can see that both $\mathbf{Z}_{c21}$ and $\mathbf{Z}_{c12}$ are zero, thus (4.21) is no longer a function of p_F .

On the other hand, when the cell thickness becomes small, the impedance matrix is ill-conditioned similar to (3.64)

$$\mathbf{Z}_c |_{h_F=0} = \begin{bmatrix} \mathbf{e}_F^> & \mathbf{e}_F^< \\ \mathbf{e}_F^> & \mathbf{e}_F^< \end{bmatrix} \begin{bmatrix} \mathbf{h}_F^> & \mathbf{h}_F^< \\ \mathbf{h}_F^> & \mathbf{h}_F^< \end{bmatrix}^{-1} \quad (4.23)$$

Therefore, we see from above that the eigenproblems for both transfer and impedance matrix methods will have numerical difficulties at large and/or small thicknesses.

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4.3 Generalized Eigenproblem of Hybrid Matrix

We shall resort to an alternative method based on hybrid matrix of Chapter 3, which for convenience we rewrite its definition for layer hybrid matrix as

$$\begin{bmatrix} \bar{E}_t(Z_f^<) \\ \bar{H}_t(Z_f^>) \end{bmatrix} = \mathbf{H}_f \begin{bmatrix} \bar{H}_t(Z_f^<) \\ \bar{E}_t(Z_f^>) \end{bmatrix}, \quad (4.24)$$

$$\mathbf{H}_f = \begin{bmatrix} \mathbf{e}_f^> & \mathbf{e}_f^< \mathbf{P}_f^<(-h_f) \\ \mathbf{h}_f^> \mathbf{P}_f^>(h_f) & \mathbf{h}_f^< \end{bmatrix} \begin{bmatrix} \mathbf{h}_f^> & \mathbf{h}_f^< \mathbf{P}_f^<(-h_f) \\ \mathbf{e}_f^> \mathbf{P}_f^>(h_f) & \mathbf{e}_f^< \end{bmatrix}^{-1} \quad (4.25)$$

For solving multilayered problem, we also define the stack hybrid matrix

$$\begin{bmatrix} \bar{E}_t(Z_i^<) \\ \bar{H}_t(Z_j^>) \end{bmatrix} = \mathbf{H}^{(i,j)} \begin{bmatrix} \bar{H}_t(Z_i^<) \\ \bar{E}_t(Z_j^>) \end{bmatrix} \quad (4.26)$$

From the matrix recursions starting from top layer downward to bottom layer

$$\mathbf{H}_{11}^{(f,N)} = \mathbf{H}_{f,11} + \mathbf{H}_{f,12} \mathbf{H}_{11}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (4.27)$$

$$\mathbf{H}_{12}^{(f,N)} = \mathbf{H}_{f,12} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (4.28)$$

$$\mathbf{H}_{21}^{(f,N)} = \mathbf{H}_{21}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (4.29)$$

$$\mathbf{H}_{22}^{(f,N)} = \mathbf{H}_{22}^{(f+1,N)} + \mathbf{H}_{21}^{(f+1,N)} \mathbf{H}_{f,22} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (4.30)$$

the cell hybrid matrix is obtained as

$$\mathbf{H}_c = \mathbf{H}^{(1,N)} \quad (4.31)$$

Using the cell hybrid matrix, Bloch-Floquet waves can be determined via a generalized eigenproblem

$$\mathbf{H}_A \boldsymbol{\psi}_F^{(j)} \boldsymbol{\beta}^{(j)} = \mathbf{H}_B \boldsymbol{\psi}_F^{(j)} \boldsymbol{\alpha}^{(j)} \quad (4.32)$$

where

$$\mathbf{H}_A = \begin{bmatrix} -\mathbf{I} & \mathbf{H}_{c11} \\ \mathbf{0} & \mathbf{H}_{c21} \end{bmatrix} \quad (4.33)$$

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$$\mathbf{H}_B = \begin{bmatrix} -\mathbf{H}_{c12} & \mathbf{0} \\ -\mathbf{H}_{c22} & \mathbf{I} \end{bmatrix} \quad (4.34)$$

With the aid of stable QZ algorithms [139], the eigensolutions to (4.32) can be computed reliably through $\alpha^{(j)}$ and $\beta^{(j)}$ rather than $p_F^{(j)} = \alpha^{(j)} / \beta^{(j)}$ directly, where $\alpha^{(j)}$ and $\beta^{(j)}$ are the diagonal elements of the triangular matrices $\mathbf{QH}_A\mathbf{Z}$ and $\mathbf{QH}_B\mathbf{Z}$ obtained by QZ factorization. Note that for some j , it is possible that $\beta^{(j)} \rightarrow 0$ which could then make $|p_F^{(j)}| \rightarrow \infty$. In practice, such infinite eigenvalue can be avoided if one adopts the upward-bounded and downward-bounded associations corresponding to $\mathbf{P}_F^> = \text{diag}(\alpha^{(j)} / \beta^{(j)})$ for $|\alpha^{(j)}| < |\beta^{(j)}|$ and $\mathbf{P}_F^{<-1} = \text{diag}(\beta^{(j)} / \alpha^{(j)})$ for $|\beta^{(j)}| < |\alpha^{(j)}|$, respectively. Then the eigenvectors of (4.32) are assembled as $\mathbf{e}_F^>$, $\mathbf{h}_F^>$ and $\mathbf{e}_F^<$, $\mathbf{h}_F^<$ forming the Bloch-Floquet eigenwave matrix $\boldsymbol{\Psi}_F$ as before. Having found the Bloch-Floquet waves, one can apply various efficient techniques for analysis of light propagation in the corresponding photonic crystals.

To highlight the distinctions between the present method and the previous one, let us summarize the procedure for standard eigenproblem of transfer matrix as below:

- 1) Obtain the eigensolutions for each layer of the unit cell and form the layer transfer matrix (4.7).
- 2) Apply the cascading relations until the transfer matrix of the unit cell is available in (4.8).
- 3) Solve the standard eigenproblem of transfer matrix using (4.9).
- 4) Determine the Bloch-Floquet waves as (4.10)-(4.11), involving $\boldsymbol{\Psi}_F$, $\mathbf{P}_F^>$ and (potentially unbounded) $\mathbf{P}_F^<$.

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On the other hand, our procedure for generalized eigenproblem of hybrid matrix can be summarized as:

- 1') Obtain the eigensolutions for each layer of the unit cell, and form the layer hybrid matrix (4.25).
- 2') Apply the matrix recursions (4.27)-(4.30) until the hybrid matrix of the unit cell is available in (4.31).
- 3') Solve the generalized eigenproblem (4.32) and retain $\alpha^{(j)}$ and $\beta^{(j)}$ of the QZ algorithm.
- 4') Determine the Bloch-Floquet waves as Ψ_F , $\mathbf{P}_F^>$ and (bounded) $\mathbf{P}_F^{<-1}$.

From (4.32), one can also write the cell hybrid matrix in terms of Bloch-Floquet waves:

$$\mathbf{H}_c = \begin{bmatrix} \mathbf{e}_F^> & \mathbf{e}_F^<\mathbf{P}_F^{<-1} \\ \mathbf{h}_F^>\mathbf{P}_F^> & \mathbf{h}_F^< \end{bmatrix} \begin{bmatrix} \mathbf{h}_F^> & \mathbf{h}_F^<\mathbf{P}_F^{<-1} \\ \mathbf{e}_F^>\mathbf{P}_F^> & \mathbf{e}_F^< \end{bmatrix}^{-1} \quad (4.35)$$

The stability of cell hybrid matrix can be deduced from this expression. When $h_F \rightarrow \infty$, $\mathbf{P}_F^>$ and $\mathbf{P}_F^{<-1}$ tend to zero, and the cell hybrid matrix is reduced to

$$\mathbf{H}_c |_{h_F \rightarrow \infty} = \begin{bmatrix} \mathbf{e}_F^>(\mathbf{h}_F^>)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_F^<(\mathbf{e}_F^<)^{-1} \end{bmatrix} \quad (4.36)$$

On the other hand, the cell transfer matrix can be reduced from (4.12) to

$$\mathbf{T}_c |_{h_F \rightarrow \infty} = \begin{bmatrix} \mathbf{0} & \mathbf{e}_F^< \\ \mathbf{0} & \mathbf{h}_F^< \end{bmatrix} \begin{bmatrix} \mathbf{e}_F^> & \mathbf{0} \\ \mathbf{h}_F^> & \mathbf{0} \end{bmatrix}^{-1} \quad (4.37)$$

which is obviously not computable numerically.

When the cell thickness becomes small, the impedance matrix is ill-conditioned as shown before. Still, the hybrid matrix is well-conditioned as

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$$\mathbf{H}_c|_{h_F=0} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \quad (4.38)$$

Fig. 4.3 shows the condition numbers of cell transfer, impedance and hybrid matrices versus $k_0 h_F$ for the example in Fig. 4.2. It is obvious that when the cell is thin, the impedance matrix is ill-conditioned. As the cell thickness increases, the transfer matrix becomes unstable rapidly. Being superior to both impedance and transfer matrices, the condition number of hybrid matrix stays low over the entire thickness range.

Fig. 4.3 Condition numbers of $\mathbf{T}_c$, $\mathbf{Z}_c$ and $\mathbf{H}_c$ matrices versus $k_0 h_F$. The cell is same as the one in Fig. 4.2.

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Fig. 4.4 Dispersion relations of k_{zF} and $k_0 h_F$ as calculated by (a) standard eigenproblem of transfer matrix and (b) generalized eigenproblem of hybrid matrix. The cell is same as the one in Fig. 4.2.

In Fig. 4.4, we show the dispersion relations of Bloch-Floquet wavenumbers k_{zF} and $k_0 h_F$ as calculated by standard eigenproblem of transfer matrix and generalized eigenproblem of hybrid matrix. We again make use of the example in Fig. 4.2. k_{zF} can be deduced from (4.9) and (4.32) for both methods as

$$\begin{aligned}
 k_{zF}^{(j)} &= (2m\pi + \arg p_F^{(j)} - i \ln |p_F^{(j)}|) / h_F \\
 &= (2m\pi + \arg \alpha^{(j)} - \arg \beta^{(j)}) / h_F - i(\ln |\alpha^{(j)}| - \ln |\beta^{(j)}|) / h_F \quad (4.39) \\
 m &= 0, \pm 1, \pm 2, \dots
 \end{aligned}$$

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It is shown that the first method can give uncorrupted results only for small $k_0 h_F$, but when $k_0 h_F$ is larger than about 2.5, errors start to occur. This can be understood from the previous discussion of Fig. 4.3 that as $k_0 h_F$ increases, the transfer matrix is numerically unstable and inaccuracies may arise in calculating k_{zF} . On the other hand, the second method using generalized eigenproblem of hybrid matrix can alleviate the problem. We see that it returns uncorrupted k_{zF} for the whole range considered. (For extreme $k_0 h_F$, one may anticipate that some $|p_F^{(j)}| \rightarrow 0, \infty$ or $\alpha^{(j)}, \beta^{(j)} \rightarrow 0$, during which the eigenvectors are more relevant as follows.)

It is more delicate to identify the potential problem associated with eigenvectors, due to the lack of exact eigenvectors for verification, and also the need for proper sorting and normalization during comparison. To get the exact results for reference, one can let the unit cell to comprise merely a single material layer, say LiNbO₃. In this case, the exact eigenvectors can be obtained by directly solving (4.1)-(4.2), which are then used to verify those obtained by (4.9) and (4.32). When the unit cell comprises a multilayered structure, the exact eigenvectors cannot be solved using (4.1)-(4.2). Alternatively one may use the cell impedance matrix as reference, which can be obtained by the formulas listed in Section 3.4.2. This cell impedance matrix, provided it being well-conditioned (for $k_0 h_F > 10^{-1}$ as dictated by Fig. 4.3), can then be used to verify those of (4.20) using eigensolutions of (4.9) and (4.32). Fig. 4.5(a) shows the relative norm errors of eigenvectors calculated by both methods as functions of $k_0 h_F$ for single layer, while Fig. 4.5(b) shows those of impedance matrices for unit cell taken from the example of Fig. 4.3. It is clear that the standard eigenproblem of

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transfer matrix cannot provide accurate eigenvectors or impedance matrix for large $k_0 h_F$. In contrast, the generalized eigenproblem of hybrid matrix leads to accurate results throughout the whole range considered.

Fig. 4.5 Relative norm errors of (a) eigenvectors for single layer (LiNbO_3); and (b) impedance matrices for unit cell (taken from Fig. 4.2); as functions of $k_0 h_F$.

4.4 Reflection and Transmission for One-Dimensional Photonic Crystals

Consider the structure of one-dimensional photonic crystals with n cells (nN layers) as shown in Fig. 4.1. Let the light be incident from layer 0, and reflection ($\mathbf{r}_{0,1}$) and transmission ($\mathbf{t}_{0,nN+1}$) coefficient matrices be defined as

$$\bar{\mathbf{w}}_0^<(Z_0^>) = \mathbf{r}_{0,1} \bar{\mathbf{w}}_0^>(Z_0^>) \quad (4.40)$$

$$\bar{\mathbf{w}}_{nN+1}^>(Z_{nN+1}^<)= \mathbf{t}_{0,nN+1} \bar{\mathbf{w}}_0^>(Z_0^>) \quad (4.41)$$

The hybrid matrix can be easily obtained in terms of Bloch-Floquet waves as:

$$\mathbf{H}^{(1,nN)} = \begin{bmatrix} \mathbf{e}_F^> & \mathbf{e}_F^<(\mathbf{P}_F^{<-1})^n \\ \mathbf{h}_F^>(\mathbf{P}_F^>)^n & \mathbf{h}_F^< \end{bmatrix} \begin{bmatrix} \mathbf{h}_F^> & \mathbf{h}_F^<(\mathbf{P}_F^{<-1})^n \\ \mathbf{e}_F^>(\mathbf{P}_F^>)^n & \mathbf{e}_F^< \end{bmatrix}^{-1} \quad (4.42)$$

In case the external layer $nN+1$ is semi-infinite, the reflection and transmission coefficients matrices can be solved explicitly as

$$\mathbf{r}_{0,1} = [\mathbf{H}_S \mathbf{h}_0^< - \mathbf{e}_0^<]^{-1} [\mathbf{e}_0^> - \mathbf{H}_S \mathbf{h}_0^>] \quad (4.43)$$

$$\mathbf{t}_{0,nN+1} = [\mathbf{h}_{nN+1}^> - \mathbf{H}_{22}^{(1,nN+1)} \mathbf{e}_{nN+1}^>]^{-1} \mathbf{H}_{21}^{(1,nN+1)} [\mathbf{h}_0^> + \mathbf{h}_{0,1}^< \mathbf{r}_{0,1}] \quad (4.44)$$

$$\mathbf{H}_S = \mathbf{H}_{11}^{(1,nN+1)} + \mathbf{H}_{12}^{(1,nN+1)} [\mathbf{h}_{nN+1}^> (\mathbf{e}_{nN+1}^>)^{-1} - \mathbf{H}_{22}^{(1,nN+1)}]^{-1} \mathbf{H}_{21}^{(1,nN+1)} \quad (4.45)$$

where $\mathbf{e}_0^>$, $\mathbf{e}_0^<$, $\mathbf{h}_0^>$, $\mathbf{h}_0^<$ and $\mathbf{e}_{nN+1}^>$, $\mathbf{e}_{nN+1}^<$, $\mathbf{h}_{nN+1}^>$, $\mathbf{h}_{nN+1}^<$ are the eigenwaves for the lower and upper external layers (0 and $nN+1$). In most practical applications, these external layers are air for which the eigenwave matrix can be found analytically. When the structure has infinite number of cells ($n = \infty$), the reflection coefficient is given by

$$\mathbf{r}_{0,1} = [\mathbf{e}_F^>(\mathbf{h}_F^>)^{-1} \mathbf{h}_0^< - \mathbf{e}_0^<]^{-1} [\mathbf{e}_0^> - \mathbf{e}_F^>(\mathbf{h}_F^>)^{-1} \mathbf{h}_0^>] \quad (4.46)$$

Note that both $\mathbf{r}_{0,1}$ and $\mathbf{t}_{0,nN+1}$ are 2×2 matrices:

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$$\mathbf{r}_{0,1} = \begin{bmatrix} r_{pp} & r_{ps} \\ r_{sp} & r_{ss} \end{bmatrix}, \quad (4.47)$$

$$\mathbf{t}_{0,nN+1} = \begin{bmatrix} t_{pp} & t_{ps} \\ t_{sp} & t_{ss} \end{bmatrix} \quad (4.48)$$

The reflectivity is defined as $R_{ij} = |r_{ij}|^2$, i, j stand for p or s polarization and total reflectivity is $R_{tp} = |r_{pp}|^2 + |r_{sp}|^2$, $R_{ts} = |r_{ss}|^2 + |r_{ps}|^2$.

4.5 Numerical Results

We begin with a published example to verify our algorithm. Fig. 4.6 shows the band structure (left) and reflection coefficients R (middle and right) of a uniaxial 1D photonic crystal obtained by (a) published work (Fig. 4 of [36]) and (b) hybrid matrix method. The unit cell is composed of two layers of arbitrarily oriented liquid crystals with permittivity tensor

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 \begin{bmatrix} \varepsilon_c \cos^2 \phi + \varepsilon_a \sin^2 \phi & (\varepsilon_c - \varepsilon_a) \cos \phi \sin \phi & 0 \\ (\varepsilon_c - \varepsilon_a) \cos \phi \sin \phi & \varepsilon_a \cos^2 \phi + \varepsilon_c \sin^2 \phi & 0 \\ 0 & 0 & \varepsilon_a \end{bmatrix} \quad (4.49)$$

R_x ($R_{xx} = |r_{pp}|^2$: blue or $R_{yx} = |r_{sp}|^2$: green) and R_y ($R_{yy} = |r_{ss}|^2$: red or $R_{xy} = |r_{ps}|^2$: green) indicate the reflection coefficients for an incident electric field parallel to the x and y axes, respectively. The permittivities of the two layers are $\varepsilon_c = 13.00$, $\varepsilon_a = 2.25$, the optical axis are $\phi_1 = 0$ and $\phi_2 = 30^\circ$ and the thicknesses are $a_1 = a_2 = h_F / 2$. The multilayered structure is made with five periods.

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Fig. 4.6 Band structure (left) and reflection coefficients R (middle and right) of a uniaxial 1D photonic crystal obtained by (a) published work and (b) generalized eigenproblem of hybrid matrix method. The permittivities of the two layers are $\varepsilon_c = 13.00$, $\varepsilon_a = 2.25$, the optical axis are $\phi_1 = 0$ and $\phi_2 = 30^\circ$ and the thicknesses are $a_1 = a_2 = h_F / 2$. The multilayered structure is made with five periods.

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From the figure, one can see that two bandgaps exist for the structure within $0.17 \leq \omega a / 2\pi \leq 0.22$ and $0.57 \leq \omega a / 2\pi \leq 0.61$. The results obtained by generalized eigenproblem of hybrid matrix method fit very well with published work.

After that, we consider one unit cell of a periodic structure made of two stacked bianisotropic layers. The media are characterized by permittivity $\boldsymbol{\epsilon}_f$,

$$\boldsymbol{\epsilon}_f = \boldsymbol{\epsilon}_0 \begin{bmatrix} \epsilon_{\perp f} & \epsilon_{gf} & 0 \\ -\epsilon_{gf} & \epsilon_{\perp f} & 0 \\ 0 & 0 & \epsilon_{\parallel f} \end{bmatrix} \quad (4.50)$$

permeability $\boldsymbol{\mu}_f = \mu_0 \mathbf{I}$, and magneto-electric tensors $\boldsymbol{\xi}_f = -\boldsymbol{\varsigma}_f = -j\kappa_f \sqrt{\mu_0 \epsilon_0} \mathbf{I}$. The thickness of each layer is $h_1 = h_2 = 0.5h_F$, and the parameters are: $\epsilon_{\perp 1} = 40$, $\epsilon_{\parallel 1} = 1$, $\epsilon_{g1} = 80i$, $\kappa_1 = 0.3$, $\epsilon_{\perp 2} = 50$, $\epsilon_{\parallel 2} = 14$, $\epsilon_{g2} = 100i$, $\kappa_2 = 0.27$, $k_0 h_F = 6.28$. Fig. 4.7 shows the reflection coefficients versus incident angles for various numbers of cells: (a) one cell; (b) 4 cells; (c) semi-infinite. The results have been computed using standard eigenproblem of transfer matrix and generalized eigenproblem of hybrid matrix. It is found that numerical instabilities arise for the former (Fig. 4.7a) especially for the small incident angles. Such numerical breakdown has been demonstrated earlier in [71] for the normal incidence case with $\kappa = 0$ and $k_0 h_F > 5.3$. On the other hand, the latter determines the Floquet waves in a stable manner (Fig. 4.7a-c).

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(a)

(b)

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Fig. 4.7 Reflection coefficients versus incident angles for periodic multilayered bianisotropic media: (a) one cell; (b) 4 cells; (c) semi-infinite. The thickness of each layer is $h_1 = h_2 = 0.5h_F$, and the parameters are: $\varepsilon_{\perp 1} = 40$, $\varepsilon_{//1} = 1$, $\varepsilon_{g1} = 80i$, $\kappa_1 = 0.3$, $\varepsilon_{\perp 2} = 50$, $\varepsilon_{//2} = 14$, $\varepsilon_{g2} = 100i$, $\kappa_2 = 0.27$, $k_0 h_F = 4.71$.

In Fig. 4.8, we plot both real and imaginary parts of k_{zF} as functions of $k_0 h_F$ for a range of incident angles ($0^\circ - 90^\circ$, step 10°). The unit cell is composed of two layers of arbitrarily oriented liquid crystals. The parameters follow the example in Fig. 4.2, except with different orientation angles in the layers: $\phi_1 = 0^\circ$ and $\phi_2 = 90^\circ$. We find that it is especially convenient to determine (if any) the frequency range of

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omnidirectional reflection using such scan plot of the imaginary part of k_{zF} . Recall that if the imaginary parts of all k_{zF} are not zero, the light waves will decay rapidly along $\hat{z}$ direction. From the figure, one can clearly see that there is a bandgap at around $0.20 \leq k_0 h_F / 2\pi \leq 0.23$. To confirm this, we plot the corresponding reflection coefficient at $k_0 h_F / 2\pi = 0.20$ in Fig. 4.9 with $n=10$ cells. Notice that $R_{pp} = R_{ss} = R_{tp} = R_{ts} = 1$ and $R_{ps} = R_{sp} = 0$ for all incident angles and hence we achieve omnidirectional reflection. Since both R_{pp} and R_{ss} are unity, the polarization of the light does not change after reflection.

Fig. 4.8 Real and imaginary parts of k_{zF} as functions of $k_0 h_F$ for a range of incident angles. The unit cell is composed of two layers of arbitrarily oriented liquid crystals with permittivity tensors (4.49), $\varepsilon_a = 2.25$, $\varepsilon_c = 13.2$, $\phi_1 = 0^\circ$ and $\phi_2 = 90^\circ$.

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Fig. 4.9 Variation of reflectivity against incident angles at $k_0 h_F / 2\pi = 0.20$ when omnidirectional reflection occurs, cf. Fig. 4.8, with $n = 10$ cells.

Next let us investigate the effect of chirality [140], which is introduced into the above parameters via additional magneto-electric tensors $\xi_f = -\zeta_f = -j\kappa_f \sqrt{\mu_0 \epsilon_0} \mathbf{I}$, $\kappa = 0.3$. Following the previous approach using the imaginary part of k_{zf} , we determine that the frequency range of omnidirectional reflection shifts to $k_0 h_F / 2\pi = 0.43$ with chirality included. The corresponding variation of reflectivity against incident angles is shown in Fig. 4.10. Notice that the graphs of R_{pp} , R_{ss} , R_{ps} and R_{sp} are different from the previous ones without chirality. In particular, R_{pp} and R_{ss} can be smaller than unity, while R_{ps} and R_{sp} can be larger than zero. Still R_{tp} and R_{ts} are unity for all incident angles. Thus unpolarized light is totally reflected unpolarized while linearly polarized incident light becomes elliptically polarized after totally reflected.

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Fig. 4.10 Variation of reflectivity against incident angles with chirality included at $k_0 h_F / 2\pi = 0.43$. Chirality is introduced into the parameters of Fig. 4.8 via additional magneto-electric tensors, $\xi = -\varsigma = i\kappa\sqrt{\mu_0\epsilon_0}\mathbf{I}$, $\kappa = 0.3$.

When the materials are dissipative (as most materials do), one expects that the performance of one-dimensional photonic crystals will be affected [134]. Fig. 4.11 shows both real and imaginary parts of k_{zF} and $k_0 h_F$ at normal incidence for lossless and lossy materials. The material parameters are same as those in Fig. 4.8 except that for lossy case, the relative permittivity of liquid crystal has small imaginary part, i.e. $\epsilon_a = 2.25 + 0.1i$, $\epsilon_{c2} = 13.2 + 0.1i$. It is observed that when the materials are lossy, the imaginary parts of k_{zF} will depart from zero. Fig. 4.12 shows the variation of total reflectivity against incident angles at $k_0 h_F / 2\pi = 0.20$ for lossy

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materials with tunable anisotropy. The reflectivity performance of each polarization may be changed by tuning the orientation angles as in the figure part (a) and (b). Note that while we have considered only slight loss here, as soon as the loss is larger or $k_0 h_F$ is higher, numerical problem may quickly emerge for standard eigenproblem of transfer matrix as demonstrated before. The generalized eigenproblem of hybrid matrix herein provides a robust method for analysis of photonic crystals with wide range of thickness at high frequency.

Fig. 4.11 Real and imaginary parts of k_{zF} as functions of $k_0 h_F$ at normal incidence for lossless “•” and lossy “+” unit cells. The cell parameters are same as those in Fig. 4.8 except that for lossy case, $\varepsilon_a = 2.25 + 0.1i$, $\varepsilon_c = 13.2 + 0.1i$.

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Fig. 4.12 Variation of total reflectivity against incident angles at $k_0 h_F / 2\pi = 0.20$ for lossy materials (same as Fig. 4.11) with tunable anisotropy: (a) $\phi_1 = 0^\circ$, $\phi_2 = 90^\circ$; (b) $\phi_1 = 90^\circ$, $\phi_2 = 0^\circ$.

4.6 Conclusion

We have presented a stable method for analysis of light propagation in one-dimensional photonic crystals composed of general bianisotropic media. Our method is based on the solutions to a generalized eigenproblem of hybrid matrix. It enables Bloch-Floquet waves to be determined reliably and overcomes the numerical instability in the standard eigenproblem of transfer matrix. When the unit cell is thick or the frequency is high, the transfer matrix or its characteristic polynomial may become ill-conditioned, whereas the hybrid matrix is always well-conditioned. Using

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the imaginary part of Bloch-Floquet wavenumbers, we have demonstrated that it is especially convenient to determine (if any) the frequency range of omnidirectional reflection. Some numerical results have been illustrated to investigate the effects of chirality, loss and tunable anisotropy.

Chapter 5

Hybrid Matrix Method for Stable Analysis of Acoustic Wave Propagation in Multilayered Solids and Fluids

5.1 Introduction

In recently years, there has been great interest in the study of phononic bandgap structures, which are periodic composite materials with acoustic analogues of photonic crystals [141]-[143]. One of the celebrated techniques for analysis of such media is based on the transfer matrix method [86], [87]. This method facilitates the transition of field variables from one layer to the next while satisfying the interfacial condition between them. Although the transfer matrix method is applicable in principle, its direct implementation has been found to suffer from numerical instabilities when the layer thickness is large. To circumvent the problem, many techniques have been proposed [101]-[110]. Recently, an approach called the stiffness/impedance matrix method [111]-[114], [118], [119] has been introduced. The method operates with total stress and displacement/velocity via the impedance matrix applied in a recursive algorithm. Using these total field variables makes it more convenient to incorporate imperfect

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interfaces that must be considered in certain ultrasonic problems. It also constitutes naturally the framework of self-recursive asymptotic method [77], [117], which is a simple asymptotic method that obviates the need to compute the exact wave propagation solution using eigenvalues-eigenvectors [107]-[109]. Although the method maintains the numerical stability when the layer thickness grows to infinity, it is inaccurate when a layer thickness reduces toward zero (as required for thin-layer approximation).

In this chapter, we extend the hybrid matrix method of previous chapters for stable analysis of acoustic wave propagation in multilayered solids and fluids for all thickness range. Although the propagations of electromagnetic wave and acoustic wave obey different laws, by properly organizing the field variables and applying the appropriate matrix methods, one can analyze them in a similar manner. We also extend the hybrid matrix of solid layer [144] in a recursive algorithm to deduce the stack hybrid matrix for a multilayered structure, which may now comprise layers of various solid and/or fluid phases. Similar to the impedance matrix method, the hybrid matrix method is able to eliminate the numerical instability of transfer matrix method. Moreover, contrary to the impedance matrix, the hybrid matrix remains to be well-conditioned and accurate even for zero or small thickness.

Section 5.2 describes the layer hybrid matrix definition for solid and fluid along with its stability and accuracy analysis. The self-recursive asymptotic method is applied to obtain the layer hybrid matrix. Section 5.3 describes the stack hybrid matrix and surface matrix method. The matrix recursions for multilayered structure of various

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phases are shown. In Section 5.4, numerical implementation and examples are described. The frequency spectra of transmission coefficient and the dispersion relation for multilayered solids and fluids structures are demonstrated.

5.2 Layer Matrices

5.2.1 Definitions

A planar multilayered structure comprising N homogeneous layers stratified in $\hat{z}$ direction is shown in Fig. 5.1, where each layer can be anisotropic solid or fluid material.

Assuming a plane harmonic wave with $\exp(-i\omega t)$ time dependence, each field vector satisfies a first-order differential system for layer f as

$$\frac{d}{dz} \begin{bmatrix} \bar{v} \\ \bar{\tau} \end{bmatrix} = \mathbf{A}_f \begin{bmatrix} \bar{v} \\ \bar{\tau} \end{bmatrix} \quad (5.1)$$

Here, $\bar{v}$ and $\bar{\tau}$ are the particle velocity vector and normal stress vector respectively.

For solid, $\mathbf{A}_f$ takes the form as

$$\mathbf{A}_f = -i\omega \begin{bmatrix} s\Gamma_{f,33}^{-1}\Gamma_{f,31} & \Gamma_{f,33}^{-1} \\ \rho_f \mathbf{I} - s^2(\Gamma_{f,11} - \Gamma_{f,13}\Gamma_{f,33}^{-1}\Gamma_{f,31}) & s\Gamma_{f,13}\Gamma_{f,33}^{-1} \end{bmatrix} \quad (5.2)$$

Here, s is the transverse slowness along $\hat{x}$ direction, ρ_f is the mass density and

Γ_f 's can be constructed from the stiffness constants $C_{f,mn}$ using the abbreviated

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subscripts [85], cf. Section 2.2.2:

$$\mathbf{\Gamma}_{f,11} = \begin{bmatrix} C_{f,11} & C_{f,16} & C_{f,15} \\ C_{f,61} & C_{f,66} & C_{f,65} \\ C_{f,51} & C_{f,56} & C_{f,55} \end{bmatrix} \quad (5.3)$$

$$\mathbf{\Gamma}_{f,33} = \begin{bmatrix} C_{f,55} & C_{f,54} & C_{f,53} \\ C_{f,45} & C_{f,44} & C_{f,43} \\ C_{f,35} & C_{f,34} & C_{f,33} \end{bmatrix} \quad (5.4)$$

$$\mathbf{\Gamma}_{f,13} = \begin{bmatrix} C_{f,15} & C_{f,14} & C_{f,13} \\ C_{f,65} & C_{f,64} & C_{f,63} \\ C_{f,55} & C_{f,54} & C_{f,53} \end{bmatrix} \quad (5.5)$$

$$\mathbf{\Gamma}_{f,31} = \begin{bmatrix} C_{f,51} & C_{f,56} & C_{f,55} \\ C_{f,41} & C_{f,46} & C_{f,45} \\ C_{f,31} & C_{f,36} & C_{f,35} \end{bmatrix} \quad (5.6)$$

The symmetry characteristics of stiffness/compliance can be found in Appendix A.

Fig. 5.1. Cross section of a planar N -multilayered structure. Each layer can be solid or fluid.

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For fluid, $\bar{v}$ and $\bar{\tau}$ reduce to their $\hat{z}$ components only (henceforth assumed for fluid) and the corresponding $\mathbf{A}_f$ reads

$$\mathbf{A}_f = -i\omega \begin{bmatrix} 0 & \frac{1}{\rho_f}(\frac{1}{c_f^2} - s^2) \\ \rho_f & 0 \end{bmatrix} \quad (5.7)$$

Here, c_f is the speed of sound in the fluid.

Based on the solution to a standard eigenvalue problem of (5.1), the field vectors in layer f can be written in terms of the superposition of eigenwaves as

$$\begin{bmatrix} \bar{v}_f(z) \\ \bar{\tau}_f(z) \end{bmatrix} = \Psi_f \mathbf{w}_f(z) = \begin{bmatrix} \mathbf{v}_f^> & \mathbf{v}_f^< \\ \mathbf{\tau}_f^> & \mathbf{\tau}_f^< \end{bmatrix} \begin{bmatrix} \bar{w}_f^>(z) \\ \bar{w}_f^<(z) \end{bmatrix} \quad (5.8)$$

The eigenwave matrix Ψ_f has been designated by $\mathbf{v}_f^<>$ and $\mathbf{\tau}_f^<>$ accordingly.

Equation (5.1) can also be solved by using the layer transfer matrix which provides a linear relationship between the field vectors at two surfaces of the layer f as

$$\begin{bmatrix} \bar{v}(z_f^>) \\ \bar{\tau}(z_f^>) \end{bmatrix} = \mathbf{T}_f \begin{bmatrix} \bar{v}(z_f^<) \\ \bar{\tau}(z_f^<) \end{bmatrix}, \quad \mathbf{T}_f = e^{\mathbf{A}h_f} \quad (5.9)$$

Although simple and straightforward, it is well known that when the layer thickness increases, the transfer matrix exhibits numerical instability. To overcome such problem, an impedance matrix can be introduced as

$$\begin{bmatrix} \bar{\tau}(z_f^<) \\ \bar{\tau}(z_f^>) \end{bmatrix} = \mathbf{Z}_f \begin{bmatrix} \bar{v}(z_f^<) \\ \bar{v}(z_f^>) \end{bmatrix} \quad (5.10)$$

It can be demonstrated that the impedance matrix is free from numerical instability for large layer thickness. Unfortunately, it suffers from numerical instability when the layer thickness becomes too small. The problems mentioned above can be overcome altogether by resorting to the hybrid matrix defined as

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$$\begin{bmatrix} \bar{v}(z_f^<) \\ \bar{\tau}(z_f^>) \end{bmatrix} = \mathbf{H}_f \begin{bmatrix} \bar{\tau}(z_f^<) \\ \bar{v}(z_f^>) \end{bmatrix} \quad (5.11)$$

Equations (5.1) and (5.8)-(5.11) are for solid layer with $\mathbf{T}_f$, $\mathbf{Z}_f$, $\mathbf{H}_f$ being 6×6 matrices. These equations should be reduced to only $\hat{z}$ components for fluid layer in the same form. Then, $\mathbf{T}_f$, $\mathbf{Z}_f$, $\mathbf{H}_f$ will become 2×2 matrices instead.

5.2.2 Self-Recursive Asymptotic Hybrid Matrix Method

Following Chapter 3, we also implement the self-recursive asymptotic hybrid matrix method to obtain $\mathbf{H}_f$ for each layer. For convenience, we rewrite the main formula here. The algorithm is initialized with asymptotic thin-layer approximation

$$\begin{aligned} \mathbf{H}_f^{(n+1)} &= \begin{bmatrix} \mathbf{H}_{f,11}^{(n+1)} & \mathbf{H}_{f,12}^{(n+1)} \\ \mathbf{H}_{f,21}^{(n+1)} & \mathbf{H}_{f,22}^{(n+1)} \end{bmatrix} \\ &\approx \begin{bmatrix} \mathbf{I} + \frac{d_{n+1}}{2} \mathbf{A}_{f,11} & \frac{d_{n+1}}{2} \mathbf{A}_{f,12} \\ \frac{d_{n+1}}{2} \mathbf{A}_{f,21} & -\mathbf{I} + \frac{d_{n+1}}{2} \mathbf{A}_{f,22} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{d_{n+1}}{2} \mathbf{A}_{f,12} & \mathbf{I} - \frac{d_{n+1}}{2} \mathbf{A}_{f,11} \\ -\mathbf{I} - \frac{d_{n+1}}{2} \mathbf{A}_{f,22} & -\frac{d_{n+1}}{2} \mathbf{A}_{f,21} \end{bmatrix} \end{aligned} \quad (5.12)$$

Then starting from $i=n$ and using the self-recursions

$$\mathbf{H}_{f,11}^{(i)} = \mathbf{H}_{f,11}^{(i+1)} + \mathbf{H}_{f,12}^{(i+1)} \mathbf{H}_{f,11}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,22}^{(i+1)} \mathbf{H}_{f,11}^{(i+1)}]^{-1} \mathbf{H}_{f,21}^{(i+1)} \quad (5.13)$$

$$\mathbf{H}_{f,12}^{(i)} = \mathbf{H}_{f,12}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,11}^{(i+1)} \mathbf{H}_{f,22}^{(i+1)}]^{-1} \mathbf{H}_{f,12}^{(i+1)} \quad (5.14)$$

$$\mathbf{H}_{f,21}^{(i)} = \mathbf{H}_{f,21}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,22}^{(i+1)} \mathbf{H}_{f,11}^{(i+1)}]^{-1} \mathbf{H}_{f,21}^{(i+1)} \quad (5.15)$$

$$\mathbf{H}_{f,22}^{(i)} = \mathbf{H}_{f,22}^{(i+1)} + \mathbf{H}_{f,21}^{(i+1)} \mathbf{H}_{f,22}^{(i+1)} [\mathbf{I} - \mathbf{H}_{f,11}^{(i+1)} \mathbf{H}_{f,22}^{(i+1)}]^{-1} \mathbf{H}_{f,12}^{(i+1)} \quad (5.16)$$

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the algorithm proceeds until $i=1$ and the layer hybrid matrix can be obtained as $\mathbf{H}_f = \mathbf{H}_f^{(1)}$. With this method, the hybrid matrix can be calculated stably and accurately even for very thick and thin layer. Moreover, the hybrid matrix is determined simply by self-recursive asymptotic method which is free from all intricacies of solving the eigenvalues and eigenvectors. Therefore the method circumvents the need for complex root searching, degeneracy treatment and upward/downward eigenvector sorting or selection.

5.2.3 Stability and Accuracy Analysis

It can be analytically demonstrated that the layer hybrid matrix is numerically stable when the layer thickness tends to infinity or zero, using its expression in terms of eigensolutions

$$\mathbf{H}_f = \begin{bmatrix} \mathbf{v}_f^{\rightarrow} & \mathbf{v}_f^{\leftarrow} \mathbf{P}_f^{\leftarrow}(-h_f) \\ \boldsymbol{\tau}_f^{\rightarrow} \mathbf{P}_f^{\rightarrow}(h_f) & \boldsymbol{\tau}_f^{\leftarrow} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_f^{\rightarrow} & \boldsymbol{\tau}_f^{\leftarrow} \mathbf{P}_f^{\leftarrow}(-h_f) \\ \mathbf{v}_f^{\rightarrow} \mathbf{P}_f^{\rightarrow}(h_f) & \mathbf{v}_f^{\leftarrow} \end{bmatrix}^{-1} \quad (5.17)$$

Indeed, when the layer thickness tends to zero, the limit can be seen

$$\mathbf{H}_f |_{h_f=0} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (5.18)$$

On the contrary, the impedance matrix is not immune to numerical deficiency when the thickness reduces to zero. This can be understood from its eigensolution expression

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$$\mathbf{Z}_f = \begin{bmatrix} \boldsymbol{\tau}_f^> & \boldsymbol{\tau}_f^< \mathbf{P}_f^<(-h_f) \\ \boldsymbol{\tau}_f^> \mathbf{P}_f^>(h_f) & \boldsymbol{\tau}_f^< \end{bmatrix} \begin{bmatrix} \mathbf{v}_f^> & \mathbf{v}_f^< \mathbf{P}_f^<(-h_f) \\ \mathbf{v}_f^> \mathbf{P}_f^>(h_f) & \mathbf{v}_f^< \end{bmatrix}^{-1} \quad (5.19)$$

With $h_f \rightarrow 0$, $\mathbf{P}_f^<(-h_f) = \mathbf{P}_f^>(h_f) = \mathbf{I}$, the impedance matrix reduces to

$$\mathbf{Z}_f = \begin{bmatrix} \boldsymbol{\tau}_f^> & \boldsymbol{\tau}_f^< \\ \boldsymbol{\tau}_f^> & \boldsymbol{\tau}_f^< \end{bmatrix} \begin{bmatrix} \mathbf{v}_f^> & \mathbf{v}_f^< \\ \mathbf{v}_f^> & \mathbf{v}_f^< \end{bmatrix}^{-1} \quad (5.20)$$

The matrix on the right is obviously not invertible due to its repeated rows and columns. Therefore computational accuracy would be affected when dealing with thin layer modeling using impedance matrices. On the other hand, when the layer thickness increases to infinity, i.e. $h_f \rightarrow \infty$, $\mathbf{P}_f^<(-h_f)$ and $\mathbf{P}_f^>(h_f)$ tend to zero, the hybrid matrix is reduced to

$$\mathbf{H}_f |_{h_f \rightarrow \infty} = \begin{bmatrix} \mathbf{v}_f^> (\boldsymbol{\tau}_f^>)^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\tau}_f^< (\mathbf{v}_f^<)^{-1} \end{bmatrix} \quad (5.21)$$

while the impedance matrix becomes

$$\mathbf{Z}_f |_{h_f \rightarrow \infty} = \begin{bmatrix} \boldsymbol{\tau}_f^> (\mathbf{v}_f^>)^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\tau}_f^< (\mathbf{v}_f^<)^{-1} \end{bmatrix} \quad (5.22)$$

Therefore it is shown that both hybrid and impedance matrix are stable for large thickness.

To assess the accuracy of self-recursive asymptotic methods using hybrid and impedance matrices, we investigate the relative error changes with number of subdivisions. Fig. 5.2 shows the average relative errors versus the number of recursive operations n for aluminum (solid) and water (fluid) whose parameters can be read from Table 5.1. The relative error is defined as

$$E(n) = \|\mathbf{X}_a - \mathbf{X}_e\| / \|\mathbf{X}_e\| \quad (5.23)$$

Here, $\mathbf{X}_a$ and $\mathbf{X}_e$ represent the impedance ($\mathbf{Z}$) or hybrid matrix ($\mathbf{H}$) obtained from

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the self-recursive asymptotic method and eigensolution method, respectively. The error is computed by taking the average over a range of transverse slownesses and frequencies. It is noted that the errors of both impedance and hybrid matrices decrease initially due to the smaller truncation error for smaller sublayer thickness. After certain minimum point, the error of hybrid matrix increases slightly and then reaches a plateau. On the other hand, the error of impedance matrix increases much faster than that of hybrid matrix and it keeps growing as the recursion number increases. For fluid, there is also a plateau at very high error level for the impedance matrix at large n .

From the above discussions, it is evident that the stable and accurate range of layer thickness for impedance matrix is not complete. On the contrary, the hybrid matrix is shown to preserve the numerical stability and accuracy even when the thickness tends to infinity or zero.

Table 5.1 Parameters for periodic unit cell

	Density (kg/m ³)	C_{11} (10 ¹⁰ N/m ²)	C_{12} (10 ¹⁰ N/m ²)	C_{44} (10 ¹⁰ N/m ²)	C_{13} (10 ¹⁰ N/m ²)	C_{33} (10 ¹⁰ N/m ²)
Al	2695	10.8	6.13	2.85	--	--
Ti	4506	16.2	9.2	4.67	6.9	18.1

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Fig. 5.2. Relative errors of $\mathbf{Z}$ and $\mathbf{H}$ versus recursion number n for aluminum (solid) and water (fluid).

5.3 Stack Hybrid Matrix

5.3.1 Definitions and Matrix Recursions

For solving multilayered solids and fluids problem, we define the stack hybrid matrix as

$$\begin{bmatrix} \bar{\mathbf{v}}(z_l^<) \\ \bar{\mathbf{t}}(z_f^>) \end{bmatrix} = \mathbf{H}^{(l,f)} \begin{bmatrix} \bar{\mathbf{t}}(z_l^<) \\ \bar{\mathbf{v}}(z_f^>) \end{bmatrix} \quad (5.24)$$

where $\mathbf{H}^{(l,f)}$ is the total stack hybrid matrix from layer l to layer f (usually we choose

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$l < f$). In the above equation one should note that $\bar{v}$ and $\bar{\tau}$ are 3×1 vectors for solid and 1×1 vectors for fluid. The stack hybrid matrix can be partitioned for several scenarios as follows:

$$[\mathbf{H}^{(l,f)}]_{6 \times 6} = \begin{bmatrix} [\mathbf{H}_{11}^{(l,f)}]_{3 \times 3} & [\mathbf{H}_{12}^{(l,f)}]_{3 \times 3} \\ [\mathbf{H}_{21}^{(l,f)}]_{3 \times 3} & [\mathbf{H}_{22}^{(l,f)}]_{3 \times 3} \end{bmatrix}, \text{ layer } l \text{ is solid \& layer } f \text{ is solid} \quad (5.25)$$

$$[\mathbf{H}^{(l,f)}]_{2 \times 2} = \begin{bmatrix} [\mathbf{H}_{11}^{(l,f)}]_{1 \times 1} & [\mathbf{H}_{12}^{(l,f)}]_{1 \times 1} \\ [\mathbf{H}_{21}^{(l,f)}]_{1 \times 1} & [\mathbf{H}_{22}^{(l,f)}]_{1 \times 1} \end{bmatrix}, \text{ layer } l \text{ is fluid \& layer } f \text{ is fluid} \quad (5.26)$$

$$[\mathbf{H}^{(l,f)}]_{4 \times 4} = \begin{bmatrix} [\mathbf{H}_{11}^{(l,f)}]_{3 \times 3} & [\mathbf{H}_{12}^{(l,f)}]_{3 \times 1} \\ [\mathbf{H}_{21}^{(l,f)}]_{1 \times 3} & [\mathbf{H}_{22}^{(l,f)}]_{1 \times 1} \end{bmatrix}, \text{ layer } l \text{ is solid \& layer } f \text{ is fluid} \quad (5.27)$$

$$[\mathbf{H}^{(l,f)}]_{4 \times 4} = \begin{bmatrix} [\mathbf{H}_{11}^{(l,f)}]_{1 \times 1} & [\mathbf{H}_{12}^{(l,f)}]_{1 \times 3} \\ [\mathbf{H}_{21}^{(l,f)}]_{3 \times 1} & [\mathbf{H}_{22}^{(l,f)}]_{3 \times 3} \end{bmatrix}, \text{ layer } l \text{ is fluid \& layer } f \text{ is solid} \quad (5.28)$$

The size of partition is indicated as the subscript outside the bracket. Then we consider the hybrid matrix recursions starting from top layer N . Let us first discuss the two adjacent layers of the same phase, i.e. solid-solid or fluid-fluid. From the definitions of layer and stack hybrid matrices in (5.11) and (5.24) while noting that

$$\bar{v}(z_f^>) = \bar{v}(z_{f+1}^<) \quad (5.29)$$

$$\bar{\tau}(z_f^>) = \bar{\tau}(z_{f+1}^<) \quad (5.30)$$

we can obtain the matrix recursions starting from the top layer N downward to the lower layer f as

$$\mathbf{H}_{11}^{(f,N)} = \mathbf{H}_{f,11} + \mathbf{H}_{f,12} \mathbf{H}_{11}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (5.31)$$

$$\mathbf{H}_{12}^{(f,N)} = \mathbf{H}_{f,12} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (5.32)$$

$$\mathbf{H}_{21}^{(f,N)} = \mathbf{H}_{21}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (5.33)$$

$$\mathbf{H}_{22}^{(f,N)} = \mathbf{H}_{22}^{(f+1,N)} + \mathbf{H}_{21}^{(f+1,N)} \mathbf{H}_{f,22} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (5.34)$$

For the two adjacent layers being of different phases, say layer f is fluid and layer $f+1$

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is solid, the stress and particle velocity in the normal direction are continuous at the interface, while the transverse stress is null:

$$\bar{\tau}_t(z_{f+1}^<) = 0 \quad (5.35)$$

$$\bar{\tau}_z(z_f^>) = \bar{\tau}_z(z_{f+1}^<) \quad (5.36)$$

$$\bar{v}_z(z_f^>) = \bar{v}_z(z_{f+1}^<) \quad (5.37)$$

Without loss of generality, let us suppose that layer N is solid. We partition the ij -submatrix of stack matrix as (i and j stand for 1 and/or 2)

$$\begin{aligned} [\mathbf{H}_{ij}^{(f+1,N)}]_{3 \times 3} &= [(\mathbf{H}_{ijIt}^{(f+1,N)})_{3 \times 2} \quad (\mathbf{H}_{ijIz}^{(f+1,N)})_{3 \times 1}] \\ &= \begin{bmatrix} (\mathbf{H}_{ijIt}^{(f+1,N)})_{2 \times 3} \\ (\mathbf{H}_{ijIz}^{(f+1,N)})_{1 \times 3} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{H}_{ijtt}^{(f+1,N)})_{2 \times 2} & (\mathbf{H}_{ijtz}^{(f+1,N)})_{2 \times 1} \\ (\mathbf{H}_{ijzt}^{(f+1,N)})_{1 \times 2} & (\mathbf{H}_{ijzz}^{(f+1,N)})_{1 \times 1} \end{bmatrix} \end{aligned} \quad (5.38)$$

By applying the continuity condition, we can obtain the matrix recursions for the stack matrix as

$$\mathbf{H}_{11}^{(f,N)} = \mathbf{H}_{f,11} + \mathbf{H}_{f,12} \mathbf{H}_{11zz}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11zz}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (5.39)$$

$$\mathbf{H}_{12}^{(f,N)} = \mathbf{H}_{f,12} [\mathbf{I} - \mathbf{H}_{11zz}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12zt}^{(f+1,N)} \quad (5.40)$$

$$\mathbf{H}_{21}^{(f,N)} = \mathbf{H}_{21Iz}^{(f+1,N)} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11zz}^{(f+1,N)}]^{-1} \mathbf{H}_{f,21} \quad (5.41)$$

$$\mathbf{H}_{22}^{(f,N)} = \mathbf{H}_{22}^{(f+1,N)} + \mathbf{H}_{21Iz}^{(f+1,N)} \mathbf{H}_{f,22} [\mathbf{I} - \mathbf{H}_{11zz}^{(f+1,N)} \mathbf{H}_{f,22}]^{-1} \mathbf{H}_{12zt}^{(f+1,N)} \quad (5.42)$$

Likewise, if layer $f+1$ is fluid and layer f is solid, its matrix recursions can be obtained as

$$\mathbf{H}_{11}^{(f,N)} = \mathbf{h}_{f,11} + \mathbf{h}_{f,12} \mathbf{H}_{11}^{(f+1,N)} [\mathbf{I} - \mathbf{h}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{h}_{f,21} \quad (5.43)$$

$$\mathbf{H}_{12}^{(f,N)} = \mathbf{h}_{f,12} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{h}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (5.44)$$

$$\mathbf{H}_{21}^{(f,N)} = \mathbf{H}_{21}^{(f+1,N)} [\mathbf{I} - \mathbf{h}_{f,22} \mathbf{H}_{11}^{(f+1,N)}]^{-1} \mathbf{h}_{f,21} \quad (5.45)$$

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$$\mathbf{H}_{22}^{(f,N)} = \mathbf{H}_{22}^{(f+1,N)} + \mathbf{H}_{21}^{(f+1,N)} \mathbf{h}_{f,22} [\mathbf{I} - \mathbf{H}_{11}^{(f+1,N)} \mathbf{h}_{f,22}]^{-1} \mathbf{H}_{12}^{(f+1,N)} \quad (5.46)$$

where

$$\mathbf{h}_{f,11} = \mathbf{H}_{f,11} + \mathbf{H}_{f,12t} (\mathbf{H}_{f,22tt})^{-1} \mathbf{H}_{f,21t} \quad (5.47)$$

$$\mathbf{h}_{f,12} = \mathbf{H}_{f,12Iz} + \mathbf{H}_{f,12It} (\mathbf{H}_{f,22tt})^{-1} \mathbf{H}_{f,22tz} \quad (5.48)$$

$$\mathbf{h}_{f,21} = \mathbf{H}_{f,21zI} + \mathbf{H}_{f,22zt} (\mathbf{H}_{f,22zz})^{-1} \mathbf{H}_{f,21t} \quad (5.49)$$

$$\mathbf{h}_{f,22} = \mathbf{H}_{f,22zz} + \mathbf{H}_{f,22zt} (\mathbf{H}_{f,22tt})^{-1} \mathbf{H}_{f,22tz} \quad (5.50)$$

From the above discussion, we can see that both sets of (5.39)-(5.42) and (5.43)-(5.46) have very similar form as (5.31)-(5.34). The layer hybrid sub-matrices of layer f in (5.31)-(5.34) are replaced by their corresponding interface sub-matrices $\mathbf{h}$ in (5.43)-(5.46) due to the interface condition.

In a similar manner, the matrix recursions starting from the bottom layer 1 to the upper layer f can be obtained for two adjacent layers of the same phase as

$$\mathbf{H}_{11}^{(1,f)} = \mathbf{H}_{11}^{(1,f-1)} + \mathbf{H}_{12}^{(1,f-1)} \mathbf{H}_{f,11} [\mathbf{I} - \mathbf{H}_{22}^{(1,f-1)} \mathbf{H}_{f,11}]^{-1} \mathbf{H}_{21}^{(1,f-1)} \quad (5.51)$$

$$\mathbf{H}_{12}^{(1,f)} = \mathbf{H}_{12}^{(1,f-1)} [\mathbf{I} - \mathbf{H}_{f,11} \mathbf{H}_{22}^{(1,f-1)}]^{-1} \mathbf{H}_{f,12} \quad (5.52)$$

$$\mathbf{H}_{21}^{(1,f)} = \mathbf{H}_{f,21} [\mathbf{I} - \mathbf{H}_{22}^{(1,f-1)} \mathbf{H}_{f,11}]^{-1} \mathbf{H}_{21}^{(1,f-1)} \quad (5.53)$$

$$\mathbf{H}_{22}^{(1,f)} = \mathbf{H}_{f,22} + \mathbf{H}_{f,21} \mathbf{H}_{22}^{(1,f-1)} [\mathbf{I} - \mathbf{H}_{f,11} \mathbf{H}_{22}^{(1,f-1)}]^{-1} \mathbf{H}_{f,12} \quad (5.54)$$

Other matrix recursions starting from the bottom can also be obtained for multilayered solids and fluids structures.

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5.3.2 Surface Hybrid Matrix Method

For the computation of reflection and transmission coefficients, let the acoustic waves be incident from layer 0, and the reflection ($\mathbf{r}_{0,1}$) and transmission ($\mathbf{t}_{0,N+1}$) coefficients be defined as

$$\bar{\mathbf{w}}_0^<(Z_0^>) = \mathbf{r}_{0,1} \bar{\mathbf{w}}_0^>(Z_0^>) \quad (5.55)$$

$$\bar{\mathbf{w}}_{N+1}^>(Z_{N+1}^<.) = \mathbf{t}_{0,N+1} \bar{\mathbf{w}}_0^>(Z_0^>) \quad (5.56)$$

Without loss of generality, we assume that the two external layers of the multilayered structure are composed by the materials of the same phase. For cases with different material phases, the corresponding formulas can be deduced in the similar form as described in Section 5.3.1. In the total hybrid matrix method, we first obtain the full stack matrix $\mathbf{H}^{(1,N)}$ via its hybrid matrix recursions. Then by applying the boundary condition $\bar{\mathbf{w}}_{N+1}^<(Z_{N+1}^<.) = 0$, the reflection and transmission coefficients matrices can be solved explicitly in terms of stack hybrid matrix as

$$\mathbf{r}_{0,1} = [\mathbf{H}_S \boldsymbol{\tau}_0^< - \mathbf{v}_0^<]^{-1} [\mathbf{v}_0^> - \mathbf{H}_S \boldsymbol{\tau}_0^>] \quad (5.57)$$

$$\mathbf{t}_{0,N+1} = [\boldsymbol{\tau}_{N+1}^> - \mathbf{H}_{22}^{(1,N)} \mathbf{v}_{N+1}^>]^{-1} \mathbf{H}_{21}^{(1,N)} [\boldsymbol{\tau}_0^> + \boldsymbol{\tau}_0^< \mathbf{r}_{0,1}] \quad (5.58)$$

$$\mathbf{H}_S = \mathbf{H}_{11}^{(1,N)} + \mathbf{H}_{12}^{(1,N)} [\boldsymbol{\tau}_{N+1}^> (\mathbf{v}_{N+1}^>)^{-1} - \mathbf{H}_{22}^{(1,N)}]^{-1} \mathbf{H}_{21}^{(1,N)} \quad (5.59)$$

To increase the computational speed, one may resort to the surface hybrid matrix defined as

$$\bar{\mathbf{v}}(z_f^<) = \mathbf{H}_{11}^{\{f,N+1\}} \bar{\boldsymbol{\tau}}(z_f^<) \quad (5.60)$$

$$\bar{\boldsymbol{\tau}}(z_{N+1}^<) = \mathbf{H}_{21}^{\{f,N+1\}} \bar{\boldsymbol{\tau}}(z_f^<) \quad (5.61)$$

Unlike the total hybrid matrix method, we first impose the radiation condition in the

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external layer $N+1$ at the outset given by

$$\mathbf{H}_{11}^{\{N+1,N+1\}} = \mathbf{v}_{N+1}^> (\boldsymbol{\tau}_{N+1}^>)^{-1} \quad (5.62)$$

$$\mathbf{H}_{21}^{\{N+1,N+1\}} = \mathbf{I} \quad (5.63)$$

The matrix recursions are then applied starting from $f = N$ as

$$\mathbf{H}_{11}^{\{f,N+1\}} = \mathbf{H}_{f,11} + \mathbf{H}_{f,12} \mathbf{H}_{11}^{\{f+1,N+1\}} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{\{f+1,N+1\}}]^{-1} \mathbf{H}_{f,21} \quad (5.64)$$

$$\mathbf{H}_{21}^{\{f,N+1\}} = \mathbf{H}_{21}^{\{f+1,N+1\}} [\mathbf{I} - \mathbf{H}_{f,22} \mathbf{H}_{11}^{\{f+1,N+1\}}]^{-1} \mathbf{H}_{f,21} \quad (5.65)$$

These steps are repeated until $\mathbf{H}_{11}^{\{1,N+1\}}$ and $\mathbf{H}_{21}^{\{1,N+1\}}$ are made available. Owing to the fact that only two equations are needed instead of four, the matrix recursions can be performed more efficiently. The reflection and transmission coefficients are determined using surface matrix method as

$$\mathbf{r}_{0,1} = [\mathbf{H}_{11}^{\{1,N+1\}} \boldsymbol{\tau}_0^< - \mathbf{v}_0^<]^{-1} [\mathbf{v}_0^> - \mathbf{H}_{11}^{\{1,N+1\}} \boldsymbol{\tau}_0^>] \quad (5.66)$$

$$\mathbf{t}_{0,N+1} = (\boldsymbol{\tau}_{N+1}^>)^{-1} \mathbf{H}_{21}^{\{1,N+1\}} (\boldsymbol{\tau}_0^> + \boldsymbol{\tau}_0^< \mathbf{r}_{0,1}) \quad (5.67)$$

For the cases that the two external layers of the multilayered structure are composed by the materials of different phases, the discussion above is still applicable except that the partition of $\mathbf{H}^{\{1,N+1\}}$ is different. When layer 1 is solid and layer N is fluid,

$$[\mathbf{H}^{\{1,N\}}]_{4 \times 4} = \begin{bmatrix} [\mathbf{H}_{11}^{\{f+1,N\}}]_{1 \times 1} & [\mathbf{H}_{11}^{\{f+1,N\}}]_{1 \times 3} \\ [\mathbf{H}_{11}^{\{f+1,N\}}]_{3 \times 1} & [\mathbf{H}_{11}^{\{f+1,N\}}]_{3 \times 3} \end{bmatrix}. \quad (5.68)$$

When layer 1 is fluid and layer N is solid,

$$[\mathbf{H}^{\{1,N\}}]_{4 \times 4} = \begin{bmatrix} [\mathbf{H}_{11}^{\{f+1,N\}}]_{3 \times 3} & [\mathbf{H}_{11}^{\{f+1,N\}}]_{3 \times 1} \\ [\mathbf{H}_{11}^{\{f+1,N\}}]_{1 \times 3} & [\mathbf{H}_{11}^{\{f+1,N\}}]_{1 \times 1} \end{bmatrix}. \quad (5.69)$$

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5.4 Numerical Results:

In Fig. 5.3, we show the calculated frequency spectra of the transmission coefficient for acoustic waves incident to the multilayered solids and fluids structure at an incident angle of $\theta = 45^\circ$. The layers are made of two materials with different thicknesses. The two materials are aluminum with density of 2695 kg m^{-3} , elastic constants $C_{11} = 10.8 \times 10^{10} (\text{kg m}^{-1} \text{s}^{-2})$, $C_{12} = 6.13 \times 10^{10} (\text{kg m}^{-1} \text{s}^{-2})$, $C_{44} = 2.85 \times 10^{10} (\text{kg m}^{-1} \text{s}^{-2})$ and water with density of 1000 kg m^{-3} and phase velocity of 1480 m s^{-1} . The four layers in one unit cell are $h_1 = 1.23 \text{ mm}$ (aluminum), $h_2 = 1.01 \text{ mm}$ (water), $h_3 = 2.23 \text{ mm}$ (aluminum) and $h_4 = 1.33 \text{ mm}$ (water). The structure is placed in the water environment.

By applying the matrix recursions, it is very simple to obtain multi-cell results from single cell one. We only need to perform self-recursions of the hybrid matrix for single cell to obtain the hybrid matrices for multi-cell structures. Fig. 5.3(a), (b) and (c) show the results for one, two and four unit cells respectively. From the results, we can note that the structure has three narrow passbands at 0.5-2.0 MHz frequency range. It also shows that two unit cells appear to be sufficient to provide well defined bandgaps, cf. Fig. 5.3 (b). Therefore the total thickness of the structure would not be too large in applications.

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Fig. 5.3. Transmission coefficient as function of frequency for periodic solids (aluminum, parameters can be read from Table 5.1.) and fluids (water) structures containing four layers in the unit cell. (a) One unit cell; (b) Two unit cells; (c) Four unit cells.

Fig. 5.4. Dispersion relation for titanium (parameters can be read from Table 5.1.) /water/titanium structure.

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Fig. 5.4 shows the dispersion relation for acoustic waves propagating in the multilayered structure which consists of two titanium (parameters can be read from Table 5.1.) layers (5 mm thick) and a water layer (1 mm thick). The structure is submerged in water, i.e. layers 0, 2, 4 are water; layers 1 and 3 are titanium. For testing of surface hybrid matrix method, we consider the fluid-solid interface between layer 2 and 3. We calculate the surface hybrid matrix from two sides and apply the continuity condition to obtain the dispersion relation. The result is very similar to those in [83], [118]. The lowest-order symmetric mode, which is called the “fluid” mode, is found. It exists in the case of two identical plates with a thin fluid layer between them.

5.5 Conclusion

This chapter has extended the hybrid matrix method for stable analysis of acoustic wave propagation in multilayered solids and fluids. The hybrid matrix method eliminates the numerical instability of transfer and impedance matrix methods over a wide range of thickness. The self-recursive asymptotic method requires only elementary matrix operations along with thin-layer asymptotic approximation and bypasses the intricacies of eigensolution approach. The stability and accuracy of hybrid matrix has been demonstrated analytically and numerically. With its simplicity and robustness to accommodate the complete range of thickness, the method finds

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usefulness in many situations. It has been applied to calculate the frequency spectra of transmission coefficient for analyzing the bandgap of the periodic multilayered solids and fluids. Dispersion relation of multilayered solids and fluids can also be obtained efficiently by the proposed surface hybrid matrix method.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this thesis, we have studied the electromagnetic and acoustic wave propagation in multilayered complex media. A new hybrid matrix method has been proposed for stable analysis of electromagnetic waves propagation in multilayered bianisotropic media. The method eliminates the numerical instability of transfer and impedance matrices for the entire thickness and frequency range. To determine the hybrid matrix of each layer, a new simple self-recursive asymptotic method along with thin-layer asymptotic approximation has been introduced. It requires only elementary matrix operations and bypasses the intricacies of eigenvalue-eigenvector approach. The stability and accuracy of hybrid matrix has been demonstrated analytically and numerically. By employing matrix recursions, the stack hybrid matrix for multilayered bianisotropic media can also be obtained. With its simplicity and robustness to accommodate the complete range of thickness, the method finds usefulness in many applications, e.g. to study the dispersion relation of layered bianisotropic waveguides,

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and to calculate the reflection coefficient and shielding effectiveness of multilayered anisotropic materials for design of radar absorbers and laminated shields.

The propagation, reflection and transmission of light in one-dimensional photonic crystals have been investigated. The Bloch-Floquet waves have been determined by a new generalized eigenproblem of hybrid matrix method. It overcomes the numerical instability in the standard eigenproblem of transfer matrix method. Using the imaginary part of the Bloch-Floquet wavenumbers, we have demonstrated that it is convenient to determine (if any) the frequency range of omnidirectional reflection. The effects of chirality, loss and tunable anisotropy have also been discussed along with the numerical results.

Finally, the hybrid matrix method has been extended for analysis of acoustic wave propagation in multilayered solids and fluids. The method can still provide stable results even when the thickness tends to infinity and zero. The matrix recursions for multilayered media with different solid and fluid phases have been presented. To increase the computational speed, the surface hybrid matrix has been introduced which can reduce the number of recursion equations required. The frequency spectra of transmission coefficient and the dispersion relation for multilayered solids and fluids structures can be investigated efficiently by this method.

6.2 Suggestions for Future Research

Although an intensive study on wave propagation in multilayered complex media has been carried out in this thesis, there still exist some interesting issues to be investigated. In this section, we present a number of possible extensions to our current work.

- (i). The photonic crystals considered in this thesis are periodic in one dimension.

In practice, more complex photonic crystals might be employed to further manipulate the wave propagation pattern in the way that we want. For analysis of the 2D or 3D photonic crystals, 1D or 2D Fourier transformations are required. While the Fourier transformation is employed, it is common that the high frequency modes have to be taken into account. Thus, the stability of hybrid matrix in the entire frequency range makes it extremely suitable for those situations.

- (ii). Phononic crystals are similar to photonic crystals except for their peculiarities of acoustic as compared to optical waves. They are receiving increasingly much attention as they enable the realization of perfect acoustic mirrors, the confinement of acoustic energy in defect modes, and the fabrication of very efficient acoustic waveguides. We can carry out study of phononic crystals in a similar manner as that for photonic crystals.

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- (iii). Piezoelectric solid is a kind of materials that can generate an electric potential in response to an applied mechanical stress. They find useful applications such as in the production and detection of sound, generation of high voltages, electronic frequency generation and SAW devices that are widely employed for high-frequency ultrasonics. Since this thesis has discussed both electromagnetic and acoustic waves, it is natural to extend the investigation into this kind of materials, which exhibit close coupling effects between the two waves.

- (iv). As discussed in Chapter 4, we have used generalized eigenproblem of hybrid matrix method to find the Bloch-Floquet wave number k_{zF} . Although it can alleviate the instability problem caused by standard eigenproblem of transfer matrix, for extreme $k_0 h_F$, one may anticipate that some $|p_F^{(j)}| \rightarrow 0, \infty$ or $\alpha^{(j)}, \beta^{(j)} \rightarrow 0$, then the wave number k_{zF} cannot be obtained correctly. It is therefore desirable to find a robust method to determine k_{zF} stably over the entire range of $k_0 h_F$.

Author's Publications

Journal Papers:

1. E. L. Tan, and J. Ning, "Alternative Implementation of Combined-Field Integral Equation Using Rao-Wilton-Glisson Basis Functions for Conducting Scatterers," *Microw. Opt. Technol. Lett.*, vol. 48, no. 4, pp. 753-756, Apr. 2006.
2. J. Ning and E. L. Tan, "Hybrid Matrix Method for Stable Analysis of Electromagnetic Waves in Stratified Bianisotropic Media," accepted by *IEEE Microw. Wireless Compon. Lett.*
3. J. Ning and E. L. Tan, "Simple and stable analysis of multilayered anisotropic materials for design of absorbers and shields," accepted by *Mater Design*.
4. J. Ning and E. L. Tan, "Generalized eigenproblem of hybrid matrix for Bloch-Floquet waves in one-dimensional photonic crystals," accepted by *J. Opt. Soc. Am. B*.
5. J. Ning and E. L. Tan, "Generalized eigenproblem of hybrid matrix for Floquet wave propagation in one-dimensional phononic crystals with solids and fluids," submitted to *Ultrasonics*.

Conference Papers

6. J. Ning and E. L. Tan, "Generalized eigenproblem of hybrid matrix method for stable analysis of periodic multilayered bianisotropic media," accepted by *Asia-Pacific Microwave Conference (APMC2008)*, Hong Kong, China, 2008.
7. E. L. Tan and J. Ning, "Geometrical Stability Criteria for Two-Port Networks in Invariant Immittance Parameters Representation," accepted by *Asia-Pacific Microwave Conference (APMC2008)*, Hong Kong, China, 2008.

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Appendix A

Appendix A

Symmetry characteristics of Stiffness and Compliance

The matrix elements are referred to coordinate axes x, y, z that coincide with the crystal axes. Except where noted, the compliance matrices have the same form as the stiffness matrices [85].

Triclinic System

21 constants

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \quad (\text{A } 1)$$

Monoclinic System

13 constants

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix} \quad (\text{A } 2)$$

Appendix A

Orthorhombic System

$$\begin{array}{c} 9 \text{ constants} \end{array} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad (\text{A } 3)$$

Tetragonal System (Classes $4, \bar{4}, 4/m$)

$$\begin{array}{c} 7 \text{ constants} \end{array} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{16} & -C_{16} & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad (\text{A } 4)$$

Tetragonal System (Classes $4mm, 422; \bar{4}2m, 4/mmm$)

$$\begin{array}{c} 6 \text{ constants} \end{array} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad (\text{A } 5)$$

Appendix A

Trigonal System (Classes $3, \bar{3}$)

$$7 \text{ constants} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{25} & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & C_{25} & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & C_{25} \\ -C_{25} & C_{25} & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 2C_{25} & 2C_{14} & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \quad (\text{A } 6)$$

$$s_{46} = 2s_{25} \quad (\text{A } 7)$$

$$s_{56} = 2s_{14} \quad (\text{A } 8)$$

$$s_{66} = 2(s_{11} - s_{12}) \quad (\text{A } 9)$$

Trigonal System ($32, 3m, \bar{3}m$)

$$6 \text{ constants} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 0 & 2C_{14} & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \quad (\text{A } 10)$$

$$s_{56} = 2s_{14} \quad (\text{A } 11)$$

$$s_{66} = 2(s_{11} - s_{12}) \quad (\text{A } 12)$$

Appendix A

Hexagonal System

$$\begin{array}{c} 5 \text{ constants} \end{array} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \quad (\text{A } 13)$$

$$s_{66} = 2(s_{11} - s_{12}) \quad (\text{A } 14)$$

Cubic System

$$\begin{array}{c} 3 \text{ constants} \end{array} \quad \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \quad (\text{A } 15)$$

Isotropic

$$\begin{array}{c} 2 \text{ constants} \end{array} \quad \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \quad (\text{A } 16)$$

$$c_{12} = c_{11} - 2c_{44} \quad (\text{A } 17)$$

$$s_{12} = s_{11} - \frac{1}{2}s_{44} \quad (\text{A } 18)$$

Appendix B

Concise spectral formalism of Maxwell equations

For convenience, we rewrite (3.2) here:

$$\begin{bmatrix} \nabla \times \bar{E} \\ \nabla \times \bar{H} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \left(-i\omega \begin{bmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{H} \end{bmatrix} + \begin{bmatrix} \bar{J} \\ \bar{M} \end{bmatrix} \right) \quad (\text{B } 1)$$

Fourier transform and inverse transform are defined as

$$F(k) = \int_{-\infty}^{\infty} f(r) e^{-ik \cdot r} dr \quad (\text{B } 2)$$

$$f(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ik \cdot r} dk \quad (\text{B } 3)$$

In Fourier (spectral) domain, the operator “ ∇ ” takes the form

$$\nabla \sim \begin{bmatrix} ik_{t1} \\ ik_{t2} \\ \frac{d}{dp} \end{bmatrix} \quad (\text{B } 4)$$

$$\nabla \times \bar{E} \sim \begin{bmatrix} ik_{t1} \\ ik_{t2} \\ \frac{d}{dp} \end{bmatrix} \times \mathbf{I} \bar{E} = \begin{bmatrix} 0 & -\frac{d}{dp} & ik_{t2} \\ \frac{d}{dp} & 0 & -ik_{t1} \\ -ik_{t2} & ik_{t1} & 0 \end{bmatrix} \bar{E} \quad (\text{B } 5)$$

One should note here that $\bar{E}$ represents the spectral domain electric field at the right side of “ $\sim$ ”. The two equations for $\hat{p}$ direction can be written as

Appendix B

$$\begin{bmatrix} ik_{t2}\bar{H}_{t1} - ik_{t1}\bar{H}_{t2} \\ ik_{t1}\bar{E}_{t2} - ik_{t2}\bar{E}_{t1} \end{bmatrix} = i\omega \left(\begin{bmatrix} \boldsymbol{\varepsilon}_{pp} & \boldsymbol{\xi}_{pp} \\ \boldsymbol{\zeta}_{pp} & \boldsymbol{\mu}_{pp} \end{bmatrix} \begin{bmatrix} \bar{E}_p \\ \bar{H}_p \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_{pt} & \boldsymbol{\xi}_{pt} \\ \boldsymbol{\zeta}_{pt} & \boldsymbol{\mu}_{pt} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} \right) - \begin{bmatrix} \bar{J}_p \\ \bar{M}_p \end{bmatrix} \quad (\text{B } 6)$$

Then one can obtain

$$\begin{bmatrix} \bar{E}_p \\ \bar{H}_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_{ee} & \boldsymbol{\alpha}_{em} \\ \boldsymbol{\alpha}_{me} & \boldsymbol{\alpha}_{mm} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} + \frac{1}{i\omega} \begin{bmatrix} \boldsymbol{\varepsilon}_{pp} & \boldsymbol{\xi}_{pp} \\ \boldsymbol{\zeta}_{pp} & \boldsymbol{\mu}_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \bar{J}_p \\ \bar{M}_p \end{bmatrix} \quad (\text{B } 7)$$

Here, the four $\boldsymbol{\alpha}$'s represent 1×2 matrices performing transverse-to-longitudinal transformations of field vectors:

$$\begin{bmatrix} \boldsymbol{\alpha}_{ee} & \boldsymbol{\alpha}_{em} \\ \boldsymbol{\alpha}_{me} & \boldsymbol{\alpha}_{mm} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{pp} & \boldsymbol{\xi}_{pp} \\ \boldsymbol{\zeta}_{pp} & \boldsymbol{\mu}_{pp} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{0} & -\boldsymbol{\kappa}_{pt} \\ \boldsymbol{\kappa}_{pt} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\varepsilon}_{pt} & \boldsymbol{\xi}_{pt} \\ \boldsymbol{\zeta}_{pt} & \boldsymbol{\mu}_{pt} \end{bmatrix} \right) \quad (\text{B } 8)$$

with

$$\boldsymbol{\kappa} = \frac{\bar{k}_t}{\omega} \times \mathbf{I} = \frac{1}{\omega} \begin{bmatrix} 0 & 0 & k_{t2} \\ 0 & 0 & -k_{t1} \\ -k_{t2} & k_{t1} & 0 \end{bmatrix} \quad (\text{B } 9)$$

to be partitioned according to (3.3) as well.

Introducing an antisymmetric 2×2 matrix

$$\boldsymbol{\Gamma}_a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (\text{B } 10)$$

The four equations for transverse direction can be written as

Appendix B

$$\begin{bmatrix} \frac{d}{dp} \bar{H}_{t2} - ik_{t2} \bar{H}_p \\ ik_{t1} \bar{H}_p - \frac{d}{dp} \bar{H}_{t1} \\ ik_{t2} \bar{E}_p - \frac{d}{dp} \bar{E}_{t2} \\ \frac{d}{dp} \bar{E}_{t1} - ik_{t1} \bar{E}_p \end{bmatrix} = i\omega \left(\begin{bmatrix} \boldsymbol{\varepsilon}_{tp} & \boldsymbol{\xi}_{tp} \\ \boldsymbol{\zeta}_{tp} & \boldsymbol{\mu}_{tp} \end{bmatrix} \begin{bmatrix} \bar{E}_p \\ \bar{H}_p \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_{tt} & \boldsymbol{\xi}_{tt} \\ \boldsymbol{\zeta}_{tt} & \boldsymbol{\mu}_{tt} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} \right) - \begin{bmatrix} \bar{J}_t \\ \bar{M}_t \end{bmatrix} \quad (\text{B } 11)$$

$$= \begin{bmatrix} \mathbf{0} & -\boldsymbol{\Gamma}_a \\ \boldsymbol{\Gamma}_a & \mathbf{0} \end{bmatrix} \frac{d}{dp} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{0} & -\boldsymbol{\kappa}_{tp} \\ \boldsymbol{\kappa}_{tp} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{E}_p \\ \bar{H}_p \end{bmatrix}$$

Applying (B 7), we have

$$\begin{aligned} \frac{d}{dp} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & \boldsymbol{\Gamma}_a \\ -\boldsymbol{\Gamma}_a & \mathbf{0} \end{bmatrix}^{-1} \cdot \\ &\left(-i\omega \begin{bmatrix} \boldsymbol{\varepsilon}_{tt} & \boldsymbol{\xi}_{tt} \\ \boldsymbol{\zeta}_{tt} & \boldsymbol{\mu}_{tt} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} + i\omega \left(\begin{bmatrix} \mathbf{0} & -\boldsymbol{\kappa}_{tp} \\ \boldsymbol{\kappa}_{tp} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\varepsilon}_{tp} & \boldsymbol{\xi}_{tp} \\ \boldsymbol{\zeta}_{tp} & \boldsymbol{\mu}_{tp} \end{bmatrix} \right) \begin{bmatrix} \bar{E}_p \\ \bar{H}_p \end{bmatrix} + \begin{bmatrix} \bar{J}_t \\ \bar{M}_t \end{bmatrix} \right) \\ &= \begin{bmatrix} \mathbf{0} & \boldsymbol{\Gamma}_a \\ -\boldsymbol{\Gamma}_a & \mathbf{0} \end{bmatrix} \cdot \quad (\text{B } 12) \\ &\left[i\omega \left(\left(\begin{bmatrix} \mathbf{0} & -\boldsymbol{\kappa}_{tp} \\ \boldsymbol{\kappa}_{tp} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\varepsilon}_{tp} & \boldsymbol{\xi}_{tp} \\ \boldsymbol{\zeta}_{tp} & \boldsymbol{\mu}_{tp} \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\alpha}_{ee} & \boldsymbol{\alpha}_{em} \\ \boldsymbol{\alpha}_{me} & \boldsymbol{\alpha}_{mm} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\varepsilon}_{tt} & \boldsymbol{\xi}_{tt} \\ \boldsymbol{\zeta}_{tt} & \boldsymbol{\mu}_{tt} \end{bmatrix} \right) \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} + \right. \\ &\left. \left(\begin{bmatrix} \mathbf{0} & -\boldsymbol{\kappa}_{tp} \\ \boldsymbol{\kappa}_{tp} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\varepsilon}_{tp} & \boldsymbol{\xi}_{tp} \\ \boldsymbol{\zeta}_{tp} & \boldsymbol{\mu}_{tp} \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\varepsilon}_{pp} & \boldsymbol{\xi}_{pp} \\ \boldsymbol{\zeta}_{pp} & \boldsymbol{\mu}_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \bar{J}_p \\ \bar{M}_p \end{bmatrix} + \begin{bmatrix} \bar{J}_t \\ \bar{M}_t \end{bmatrix} \right] \end{aligned}$$

One can write the equation in a concise form

$$\begin{aligned} \frac{d}{dp} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} &= i\omega \begin{bmatrix} \boldsymbol{\Lambda}_{ee} & \boldsymbol{\Lambda}_{em} \\ \boldsymbol{\Lambda}_{me} & \boldsymbol{\Lambda}_{mm} \end{bmatrix} \begin{bmatrix} \bar{E}_t \\ \bar{H}_t \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0} & \boldsymbol{\Gamma}_a \\ -\boldsymbol{\Gamma}_a & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \bar{J}_t \\ \bar{M}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{\beta}_{ee} & \boldsymbol{\beta}_{em} \\ \boldsymbol{\beta}_{me} & \boldsymbol{\beta}_{mm} \end{bmatrix} \begin{bmatrix} \bar{J}_p \\ \bar{M}_p \end{bmatrix} \right) \quad (\text{B } 13) \end{aligned}$$

Here, the four $\boldsymbol{\beta}$'s complement those $\boldsymbol{\alpha}$'s above and represent 2×1 matrices performing longitudinal-to-transverse transformations of source vectors:

Appendix B

$$\begin{bmatrix} \beta_{ee} & \beta_{em} \\ \beta_{me} & \beta_{mm} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & -\kappa_{tp} \\ \kappa_{tp} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \varepsilon_{tp} & \xi_{tp} \\ \zeta_{tp} & \mu_{tp} \end{bmatrix} \right) \begin{bmatrix} \varepsilon_{pp} & \xi_{pp} \\ \zeta_{pp} & \mu_{pp} \end{bmatrix}^{-1} \quad (\text{B } 14)$$

The four Λ 's can be expressed succinctly as

$$\begin{bmatrix} \Lambda_{ee} & \Lambda_{em} \\ \Lambda_{me} & \Lambda_{mm} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \Gamma_a \\ -\Gamma_a & \mathbf{0} \end{bmatrix} \cdot \left(\begin{bmatrix} \beta_{ee} & \beta_{em} \\ \beta_{me} & \beta_{mm} \end{bmatrix} \begin{bmatrix} \varepsilon_{pp} & \xi_{pp} \\ \zeta_{pp} & \mu_{pp} \end{bmatrix} \begin{bmatrix} \alpha_{ee} & \alpha_{em} \\ \alpha_{me} & \alpha_{mm} \end{bmatrix} - \begin{bmatrix} \varepsilon_{tt} & \xi_{tt} \\ \zeta_{tt} & \mu_{tt} \end{bmatrix} \right) \quad (\text{B } 15)$$