Study of continuous variable entanglement in coupled oscillator systems by means of the dynamical two-step approach

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**STUDY OF CONTINUOUS VARIABLE
ENTANGLEMENT IN COUPLED OSCILLATOR
SYSTEMS BY MEANS OF THE DYNAMICAL
TWO-STEP APPROACH**

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**A THESIS SUBMITTED
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Summary

In this thesis, we have extended the applicability of the two-step approach to higher-dimensional problem, and have further developed it to yield the quantum stationary states and to evaluate the quantum dynamics of coupled anharmonic oscillator system with fast convergence. This new approach has enabled us to investigate the main topic of this thesis: continuous variable entanglement of a general class of coupled oscillator systems, with good numerical efficiency. We have validated this technique against several coupled oscillator models. We have also applied it to study the relation between entanglement and quantum squeezing as well as quantum uncertainties within the ground state of a general class of coupled oscillator system. Interestingly, while the relation between quadrature squeezing and entanglement depends sensitively on the nonlinear potential, the influence of nonlinear perturbation on the entropy-uncertainty relation is minimal, especially when the entanglement between the two oscillators is large. We then explored into the dynamics of quantum entanglement where we found in three of our sample models that the generation of the dynamical pattern of entanglement depends completely on the global classical dynamical regime without being influenced by the local classical behavior. In other words, these entanglement dynamical patterns are independent of the initial coherent product states. Nonetheless, we found through our numerical experiments on two of these models that the entanglement dynamics can depend on the choice of the initial number product states.

Chapter 1

Background and motivation

1.1 Introduction to quantum entanglement

1.1.1 Quantum entanglement from the EPR paradox

In 1935, Einstein, Podolsky, and Rosen (EPR) introduced a quantum-mechanical “gedanken” experiment which highlights a phenomenon that is counter-intuitive. In this thought experiment, two objects are first prepared in an “entangled state”. After which, a measurement made on one of them can affect its partner instantaneously even though they are very far apart. Taking a pair of quantum dice as example, this results in the curious feature that although each throw of a dice gives a random number between 1 to 6, the outcome of the throw of the second dice is exactly the same, without being constrained by the distance that separates them. Einstein described this phenomenon as “spooky action at a distance”. Indeed, the agenda of EPR was to exploit the intriguing outcome of this thought experiment to reveal a flaw in the quantum theory. This had caused much philosophically oriented discussions and debates concerning quantum theory until John Bell finally came out with an inequality to test the completeness of the quantum formalism. Eventually, experiments carried out by Clause *et al.* and Aspect *et al.* showed the violation of the Bell’s inequality and thus demonstrated the completeness of quantum mechanics. Although EPR’s attempt to show the incompleteness of the quantum interpretation of physical reality

fails at the end, it brings to the forefront the aforementioned phenomenon implied by quantum mechanics. This phenomenon is known as quantum entanglement.

1.1.2 Definition, examples and practical application of quantum entanglement

Quantum entanglement is a quantum-mechanical effect which links the quantum states of two or more objects together, such that the correlation between the two entangled partners continue to exist even if they are far away from each other. Interestingly, the joint description of the entangled parties is found to be greater than the sum of their individual description. A good example of such entangled states is the spin singlet state:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2) . \quad (1.1)$$

In the spin singlet state, it is impossible to associate either electron with a state of definite spin. If measurement were made on the first electron and the outcome is found to be $\uparrow$ (or $\downarrow$), then the spin of the second electron collapses immediately to $\downarrow$ (or $\uparrow$). This nonlocal effect, which is a consequence of entanglement, has now been observed in diverse systems from condensed matter physics to elementary particle physics. In fact, physicists have managed to create entangled chains of up to five photons [1], six beryllium ions [2], eight calcium ions [3], and three diamond nuclei [4], experimentally.

Although entanglement is fascinating in its own right, what attract the scientists most are the practical applications that it promises. By treating entanglement as a resource, some of the classically impossible tasks have now become possible. For instance, one expects a qubit to carry at most one bit of classical information, but quantum dense coding introduced by Bennett and Weisner [5] has enabled the transmission of two bits of classical information by sending only one *a priori* entangled qubit. Also, in 1993, Charles H. Bennett *et al.* [6] suggested the possibility of quantum teleportation by means of entanglement – the quantum state of a particle can be transferred instantaneously to a remote location through a shared entangled resource.

This is shown to be possible if one does not gain any information from the state during the process of transfer. Subsequently, Zeilinger *et al.* [7] succeeded in the first experimental verification of quantum teleportation between photon pairs four years after its theoretical prediction. In addition to these novel ideas, quantum entanglement has also revealed its importance in the field of quantum information in areas such as quantum cryptography and quantum errors correction [8].

1.2 Detection and quantification of entanglement

The theoretical studies and development of entanglement have been intensive in numerous directions other than the field of quantum information. More fundamental investigations in this respect include the detection, characterization and quantification of entanglement, as well as its optimal detection and manipulation in the laboratory.

Suppose we are given N pairs of particles. A basic problem in quantum entanglement theory is to determine whether the particles in each pair are entangled or not. Unfortunately, this problem has a simple solution only in a few cases. The simplest case is the pure bipartite state with $N = 1$. Consider two noninteracting systems A and B with respective Hilbert spaces H_A and H_B . The Hilbert space of the composite system is the tensor product $H_A \otimes H_B$. If the first system is in the pure state $|\psi\rangle$ and the second in the pure state $|\phi\rangle$, then the state of the composite system can be written as a tensor product of the two pure states,

$$|\Psi_{AB}\rangle = |\psi\rangle \otimes |\phi\rangle. \quad (1.2)$$

States of composite system which can be represented in this form are called separable. Not all states can be written in this form. If a state is not separable, it is called an entangled state.

More generally, if one has a large collection of identical copies of a quantum system of interest, then the state of this ensemble can be described by a density matrix ρ_{AB} . From a mathematical point of view, an ensemble is separable if and only if it can be

expressed in the form of a convex sum as follows:

$$\rho_{AB} = \sum p_i \rho_i^A \otimes \rho_i^B, \quad (1.3)$$

where ρ_i^A and ρ_i^B are defined on the local Hilbert space H_A and H_B respectively. Note that p_i is the probability of occurrence of the i th subensemble such that $\sum_i p_i = 1$. For a pure ensemble, the state ρ_{AB} is separable if and only if

$$\rho_{AB} = \rho^A \otimes \rho^B. \quad (1.4)$$

In this case, ρ^A and ρ^B are equal to the reduced density matrices obtained by tracing out the influence of the other subsystem:

$$\begin{aligned} \rho^A &= \text{Tr}_B(\rho_{AB}), \\ \rho^B &= \text{Tr}_A(\rho_{AB}). \end{aligned} \quad (1.5)$$

It is very difficult, however, to check whether a given mixed ensemble ρ_{AB} is entangled based on the separable definition given in Eq. (1.4). In particular, one needs to construct the separable decomposition of state explicitly, which is a hard and lengthy task. Hence, researchers have created different separable criteria to detect entanglement. One of which is the positive partial transpose criterion developed by A. Peres [9] in 1996. The criterion says that if a state ρ_{AB} is separable, then the partial transpose of ρ_{AB} defined with respect to some fixed product basis as follows:

$$\langle m, \mu | \rho_{AB}^{T_B} | n, \nu \rangle = \langle m, \nu | \rho_{AB} | n, \mu \rangle, \quad (1.6)$$

has non-negative eigenvalues. Note that although the transposition above is performed on subsystem B , the criterion continues to apply if the partial transposition is performed on subsystem A . Indeed, this condition is found to be sufficient for systems with a dimension of 2×2 and 2×3 . In addition, researchers are also using observables that completely characterize separable states to detect entanglement physically. These observables are called entanglement witnesses [10]. An example of entanglement witness is the Hermitian swap operator [11]. Another approach to detect entanglement employs the uncertainty relations of arbitrary sets of physical

properties [12, 13, 14, 15]. This method of entanglement detection has been implemented experimentally through the quadrature phase measurements on the two output beams of a nondegenerate parametric amplifier, which has also been used to demonstrate the Einstein-Podolsky-Rosen paradox [16]. Moreover, the method [17] has been used to detect entanglement among cold atoms.

To further characterize entanglement, it is necessary to develop a measure on the amount of entanglement. Such entanglement measure should quantify the degree of quantum correlations contained in a given quantum state of a composite system, and in consequence, able to describe the changes and dynamical evolution of these correlations. Formally, entanglement measures are functions that map the state space of the quantum system onto the set of non-negative real numbers. The values of these functions cannot be increased under any local operation (i.e., operations that act separately on each subsystem) and classical communication (LOCC). The measures are expected to be zero for separable states. But for entangled states, its value is non-zero and is larger for a state that is more entangled. Although the theory of entanglement measure originates from research in quantum information and communication, it has now gained importance in many other research areas. In particular, it has been applied to the study of entanglement distribution in energy eigenstates and the dynamical production of entanglement.

For pure bipartite states, a good measure of quantum entanglement is the von Neumann entropy (S_{vN}) [18] of the reduced density matrix (ρ^A):

$$S_{vN} = -\text{Tr} \{ \rho^A \ln \rho^A \} . \quad (1.7)$$

Another widely accepted measure of entanglement is the linear entropy

$$S_L = 1 - \text{Tr} (\rho^A)^2 . \quad (1.8)$$

In the case of mixed states, the von Neumann entropy or linear entropy is no longer an adequate measure of entanglement. This is because the process of mixing can lead to a nonzero subsystem entropy even though the two subsystems are not entangled. As a result, an appropriate measure of entanglement for mixed states requires the

consideration of the average entanglement of the pure state $|\psi_i\rangle$. Furthermore, given a mixed state with the density matrix:

$$\rho = \sum p_i |\psi_i\rangle \langle \psi_i|, \quad (1.9)$$

there is no unique decomposition. Hence, the entanglement measure has to be minimized over all its possible decompositions. This defines the so-called entanglement of formation [19] which, for bipartite systems, come in the form of concurrence [20] and relative entropy [21]. Finally, with regards to multipartite states, only recently are there attempts to find suitable entanglement measures.

1.3 Entanglement in continuous variable systems

Continuous variable (CV) systems comprise subsystems which are associated with infinite dimensional Hilbert spaces. Examples of CV system include the field modes of light, harmonic oscillators, and the state of cold atomic gases [22]. As most of the research concerning entanglement has been carried out in low-dimensional quantum systems like qubits and quantum two-level systems, only recently has considerable amount of effort been put into the research of CV entanglement. In particular, scientists are now seeking for a fundamental understanding of CV entanglement and its possible application to the theory of quantum information and communication. The most notable current contribution to this field is the theory of Gaussian quantum states which have properties that make them more mathematically tractable. On the other hand, not much is known of the non-Gaussian states to date. Nevertheless, they are gaining increasing interest and attention due to their enhanced effect on entanglement [23].

The process of parametric down-conversion gave the first demonstration of the practical generation of CV entangled state through the creation of an entangled photon pair [24]. During the down-conversion, an ultraviolet photon splits spontaneously into two lower-energy infrared photons, whose polarizations are intimately related. Currently, the main focus of experimental research in CV entanglement is on the generation of macroscopic light entanglement [25].

Many properties of quantum entanglement become modified as the system passes from finite to infinite dimension. In particular, the entropy of the entanglement can become infinite. As a result, one has to introduce a restriction on the mean energy to bound the state of the system from above. Only then, the state may be truncated with respect to a finite dimensional subspace such that the evaluated degree of entanglement becomes continuous. For pure bipartite states, the von Neumann entropy remains a good measure of entanglement. In the more general case of mixed states and multipartite system, a complete characterization of entanglement is still an open problem.

1.4 Motivations and outline of the thesis

Seventy years after its inception, the theory of quantum entanglement is still undergoing an endless evolution in mathematical structures, conceptual foundations, and intellectual implications. In particular, the field of CV entanglement is to a large extent unexplored. Many of the well addressed issues in low-dimensional system still lack analogs in the CV system. For instance, the investigation on the impact of classical chaos on quantum entanglement has been confined mostly in finite dimensional systems like the interacting spin systems, the coupled kicked tops and the $4D$ standard maps. Only recently has a CV model of coupled oscillators been used by Zhang and Jie [26] to study the consequence of chaos on quantum entanglement. Nevertheless, they have considered the correspondence only within the regular and the mixed regimes, leaving the purely chaotic regime unexplored. One of the aim of this thesis is to address the issues of quantum chaos with regards to CV entanglement in this purely chaotic regime. In addition, we shall also explore the link between quantum entanglement and two quantum aspects: squeezing and the uncertainty product.

Currently, there are diverse physical systems being studied to gain a better understanding on CV entanglement despite scant attention on them in the earlier days. Indeed, CV entanglement has been investigated through the entangled macroscopic quantum states in two superconducting qubits [27], the interacting two-mode Bose-

Einstein condensate, the coupled cavity scheme with a Kerr medium [28], and the two-mode magnon in an antiferromagnet [29]. On the other hand, several recent experiments [30, 31, 32] have revealed the possibility of employing atomic ensembles for quantum teleportation and quantum memory by means of spin-squeezed states. For example, an unknown quantum state can be teleported from one location to another by entangling light with an atomic ensembles via a Kerr-like interaction. It is interesting to note that the theoretical modelling of all these systems can be formulated in terms of two-coupled-oscillator systems. For the example on atomic ensembles, the collective internal state of the atoms can be described by the continuous variables x_1 , p_1 , while their interaction with the optical field (described by the Stokes parameters x_2 , p_2), is given by a bilinear term of the form x_1x_2 [33]. The optimal use of this interaction to create entanglement and squeezing has been shown by Kraus *et al.* [34]. Furthermore, the effects of the nonlinear Kerr-like medium [28, 35] can be modelled by adding higher order interaction and anharmonic potential to the system's Hamiltonian.

Much of the theoretical work on CV entanglement has been resorted to a few solvable models [28, 29, 36, 37]. For models that lack analytical solutions, CV entanglement has been studied with the help of numerical methods that involve direct diagonalization. This leads to another main motivation behind this thesis, which is to develop an approximation method that will be applicable to a general class of coupled oscillator systems for the study of CV entanglement and its related topics. From the literature, we found that much research had been carried out to determine the stationary and dynamical quantum states of coupled oscillator systems. Traditionally, these problems have been investigated by the perturbation theory [38, 39], which applied only to a small class of coupled oscillator systems with very small anharmonic potential. Hioe *et al.* [40] have introduced several convergent algorithms which are able to calculate the eigenvalues of anharmonic oscillators depending on the magnitude of the coupling constant. Sharma and Fiase [41] have worked out an analytical expression for the energy eigenvalues of oscillators with an even-order of anharmonicities, while Datta and Rampal [42] have studied analytically the wave

function and excited energy levels of doubly anharmonic oscillators in the asymptotic regime. Moreover, there is a technique known as the Lanczos iteration method [43, 44] which yields fast converging results for both the eigenvalues and eigenvectors, but requires a pre-allocation of an appropriate trial wave function.

Many of the approaches above depend on the specific system being investigated without any standard procedure that is applicable to the general class of coupled oscillator systems. In consequence, we hope to develop a simple and direct approximation method that will enable us to investigate the topics of CV entanglement for a large class of coupled oscillator systems with greater computational efficiency. This has led us to further develop the two-step approach. Similar to the approach of direct diagonalization, the two-step approach involves (i) expressing the Hamiltonian and the quantum state in terms of a matrix and a vector respectively with respect to a chosen basis, and (ii) diagonalizing the matrix to extract the eigenvalues. The basis choosing procedure of the two-step approach involves a Bogoliubov transformation and the application of the variational principle. In order to study entanglement from composite systems, the applicability of the approach has to be extended to higher dimension. In addition, the method has to be further developed so that quantum dynamical problems can be addressed. Such a development of the extended dynamical two-step approach, which has been published in [45] and [46], shall constitute the main content of Chap. 2. In Chap. 3, we shall apply the extended two-step approach to compute the stationary states of a general class of coupled oscillators systems so as to investigate the relationship between quantum squeezing and quantum uncertainty product with CV entanglement. In the computation, we adopt the natural units for the harmonic oscillator, i.e. $\hbar\omega$ for energy, $\sqrt{\hbar/m\omega}$ for length, $\sqrt{\hbar m\omega}$ for momentum and ω^{-1} for time, unless otherwise specified. In chapter 4, entanglement dynamics and its classical correspondence [47] will be investigated by means of both analytical and numerical calculation. In the numerical calculation, we used the dynamical two-step approach that has been developed previously in Chap. 2. The direct diagonalization method will also be used as a check of accuracy. Finally in Chap. 5, we conclude the thesis.

Chapter 2

Development of mathematical tools

2.1 Introduction

As an important model in physics, quantum oscillator has been studied extensively in the past [39, 40, 41, 42, 48, 49, 50, 51, 52]. Motivated by its application to molecular physics and quantum field theory, researchers begin to investigate the consequences and effects of interaction within a system of coupled oscillators. This leads to studies into pairs of interacting harmonic oscillators [36, 53] which serve as useful starting points to understand the many interesting properties typical of quantum systems with many degrees of freedom. For example, Bosco *et al.* first look into a system of two coupled harmonic oscillators to arrive at the correct sequence of variable transformations, before generalizing their treatment to the N coupled oscillator systems [53].

However, to examine more realistic and complex physical systems, it is necessary to incorporate the anharmonic character of the oscillations. For example, a system of two coupled quantum anharmonic oscillators is interesting as a realistic model of physical systems. In fact, they have been employed to model molecular systems such as the five-membered ring compounds including 2,5-dihydrofuran, cyclopentanone, cyclopentene, silacyclopentane, and germacyclopentane; and have enabled the calculation of the bend/twist eigenvalues and energy transitions of these systems. Furthermore, they are able to describe the nonlinear optical activity of a dimer-like chiral

molecule [58], the semiclassical theory of bound state [56], the interaction of quantum system with classical environment [59], and structures that are generally accepted to be good candidates for the realization of quantum bits – the double-quantum-dots [60] and the Josephson junction [61]. In addition, the investigation of continuous variable entanglement and its related topics is also being actively pursued within these two-coupled-oscillator systems [26, 62, 63, 64, 65]. On the other hand, the more complicated systems of N -coupled-anharmonic-oscillator are used to model realistic quantum systems with many degrees of freedom. Recently, researchers are able to model the hole transfer problem in B-DNA double strands [67] by modelling clusters of nucleobases as a chain of coupled quantum anharmonic oscillators. They have been able to explore a larger size system with this model.

The improved modelling comes at a price of greater mathematical intractability as the resulting systems have become nonintegrable with perturbation analysis typically ending up with series that diverges. These difficulties have motivated the development of approximation methods, such as the mixed diagonalization [57], semiclassical calculation [56], Bargmann representation [54], and operator method [66]. In this section, we explore another solution to this problem by extending a two-step diagonalization approach in which a Bogoliubov transformation is used with a variational approach to define a basis that would give fast converging energy eigenvalues. The two-step approach pioneered by Hsue *et al.* [51] is applicable to single oscillator system with quartic and cubic anharmonic potential. It is capable of producing fast converging and precise result for the energy eigenvalues, and was later re-examined by Jafarpour *et al.* [49] who extended it to single oscillator systems with polynomial potential. In order to apply the two-step approach to examine entanglement properties within composite systems, we first extend its applicability to higher dimensional problems. We aim to use the extended two-step approach to investigate not only the energy eigenvalues but also quantum entanglement and other related quantum aspects within the general class of coupled-anharmonic-oscillator systems.

The original two-step approach has been applied only to quantum static problem to yield the energy eigenvalues. Since quantum systems evolve and need not exist in

their eigenstates most of the time, it is important to develop a numerical approach that enables the efficient study of quantum dynamics. In fact, many interesting subject areas, such as quantum tunnelling [68], quantum decoherence [69], quantum synchronization [70] and quantum chaos, involve detailed analysis of quantum dynamical evolution in nonlinear oscillator systems. For instance, the recent study of the influence of chaos on entanglement production requires a close examination of the quantum dynamics of a pair of oscillators coupled with biquadratic potential [26]. This has motivated us to further extend the applicability of the two-step approach to the computation of quantum dynamical problems. We term the new technique the dynamical two-step approach.

2.2 Stationary two-step approach

2.2.1 Generalized two-step approach to systems of N coupled quantum anharmonic oscillators

As shown in our work published in [45], the two-step approach [51] can be extended to a system of N coupled quantum anharmonic oscillators with Hamiltonian of the following general form:

$$H = \sum_{j=1}^N \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right) + \lambda \sum_{\mu_1=0}^{\kappa} \sum_{\mu_2=0}^{\kappa} \cdots \sum_{\mu_N=0}^{\kappa} c_{\mu_1 \mu_2 \cdots \mu_N} x_1^{\mu_1} x_2^{\mu_2} \cdots x_N^{\mu_N}, \quad (2.1)$$

where p_j are the kinetic momenta, x_j are the oscillators' positions for $j = 1, 2, \dots, N$, λ is the perturbative constant and $c_{\mu_1 \mu_2 \cdots \mu_N}$ is the coupling coefficients. In the first step, a Bogoliubov transformation is used to define a new set of creation and annihilation operators $b_j^\dagger$ and b_j from the original set $a_j^\dagger$ and a_j , as follows:

$$b_j = \frac{a_j - t_j a_j^\dagger}{\sqrt{1 - t_j^2}}, \quad (2.2)$$

$$b_j^\dagger = \frac{a_j^\dagger - t_j a_j}{\sqrt{1 - t_j^2}}, \quad (2.3)$$

where t_j is real and satisfies $|t_j| < 1$. Since the transformation is canonical, these new bosonic operators satisfy the commutation relation $[b_i, b_j^\dagger] = \delta_{ij}$, where δ_{ij} is the

Kronecker delta. Note that $i, j = 1, 2, \dots, N$. Furthermore, the corresponding ground state $|\phi_j, t_j\rangle_b$, which satisfies $b_j|\phi_j, t_j\rangle_b = 0$, is a squeezed vacuum state. It is defined through the action of the squeeze operator $S(t_j)$ on the original ground state $|0_j\rangle_a$:

$$\begin{aligned}
|\phi_j, t_j\rangle_b &= S(t_j)|0_j\rangle_a \\
&= \exp\left(\frac{t_j}{2}a_j^{\dagger 2}\right)|0_j\rangle_a \\
&= \sum_{n_j=0}^{\infty} \frac{1}{n_j!} \left(\frac{t_j}{2}a_j^{\dagger 2}\right)^{n_j} |0_j\rangle_a \\
&= \sum_{n_j=0}^{\infty} \frac{t_j^{n_j} \sqrt{(2n_j)!}}{2^{n_j} n_j!} |2n_j\rangle_a.
\end{aligned} \tag{2.4}$$

Note that n_j is the index to the original number state $|n_j\rangle_a$. The squeeze operator $S(t_j)$ is, however, different from the general one [71] by a factor of $\exp\left(-\frac{t_j}{2}a_j^2\right)$. Incidentally, the squeezed vacuum state is also the eigenstate $|\lambda, +1\rangle$ of the “two-photon annihilation operator” (TAO) $a_j^{\dagger-1}a_j$ [71] with eigenvalue t_j :

$$a_j^{\dagger-1}a_j|\phi_j, t_j\rangle_b = t_j|\phi_j, t_j\rangle_b, \tag{2.5}$$

where $a_j^{\dagger-1}$ is the inverse creation operator. In [71], the ground state is normalized by including a factor of $\eta_j = (1 - |t_j|^2)^{1/4}$.

Next, let us take the tensor product of the individual oscillator states as follows:

$$\begin{aligned}
|\phi_1, t_1\rangle_b \otimes |\phi_2, t_2\rangle_b \otimes \dots \otimes |\phi_N, t_N\rangle_b &= |\phi_1, \phi_2, \dots, \phi_N, t_1, t_2, \dots, t_N\rangle_b \\
&= \exp\left(\sum_{j=1}^N \frac{t_j}{2}a_j^{\dagger 2}\right) |0_1, 0_2, \dots, 0_N\rangle_a.
\end{aligned} \tag{2.6}$$

The new bosonic operators now operate on these states in the following way:

$$b_j|\phi_1, \phi_2, \dots, \phi_N, t_1, t_2, \dots, t_N\rangle_b = 0 \quad \text{for } j = 1, 2, \dots, N. \tag{2.7}$$

Thus, we observe that the new set of bosonic operators is associated with the new squeezed vacuum product state $|\phi_1, \phi_2, \dots, \phi_N, t_1, t_2, \dots, t_N\rangle_b$, and they are parameterized by N independent parameters $t_1, t_2, \dots, t_N$. By defining

$$\Omega_j = \frac{1 - t_j}{1 + t_j}, \tag{2.8}$$

which amounts to a rescaling of the form $x_j \rightarrow \sqrt{\Omega_j}x_j$, we select the optimal set of bosonic operators and squeezed vacuum product state by minimizing the ground state energy:

$$E(\Omega_1, \Omega_2, \dots, \Omega_N) = \frac{{}_b\langle\phi_1, \dots, \phi_N, \Omega_1, \dots, \Omega_N|H|\phi_1, \dots, \phi_N, \Omega_1, \dots, \Omega_N\rangle_b}{{}_b\langle\phi_1, \dots, \phi_N, \Omega_1, \dots, \Omega_N|\phi_1, \dots, \phi_N, \Omega_1, \dots, \Omega_N\rangle_b}, \quad (2.9)$$

with respect to the set of Ω_j . In other words, we solve for the $\bar{\Omega}_j$ which gives

$$\frac{\partial E(\Omega_1, \Omega_2, \dots, \Omega_N)}{\partial \Omega_j} = 0. \quad (2.10)$$

Then, the optimal generalized number product basis $\{|\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}}\}$ can be generated from the optimal squeezed vacuum product state

$$|\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_N\rangle_{\bar{b}} = |\phi_1, \phi_2, \dots, \phi_N, \bar{\Omega}_1, \bar{\Omega}_2, \dots, \bar{\Omega}_N\rangle_{\bar{b}} \quad (2.11)$$

in the following way:

$$|\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}} = \frac{\bar{b}_1^{\dagger \bar{n}_1} \bar{b}_2^{\dagger \bar{n}_2} \dots \bar{b}_N^{\dagger \bar{n}_N}}{\sqrt{\bar{n}_1! \bar{n}_2! \dots \bar{n}_N!}} |\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_N\rangle_{\bar{b}}. \quad (2.12)$$

Note that the Hamiltonian with respect to the new operator $\bar{b}_j$ (which results from $\bar{t}_j$) can be related to the original Hamiltonian according to the following transformation:

$$\begin{aligned} H(a_1, a_1^\dagger, a_2, a_2^\dagger, \dots, a_N, a_N^\dagger) \\ = S(\bar{t}_1)^\dagger S(\bar{t}_2)^\dagger \dots S(\bar{t}_N)^\dagger H(\bar{b}_1, \bar{b}_1^\dagger, \bar{b}_2, \bar{b}_2^\dagger, \dots, \bar{b}_N, \bar{b}_N^\dagger) S(\bar{t}_1) S(\bar{t}_2) \dots S(\bar{t}_N). \end{aligned} \quad (2.13)$$

In the second step, we use the optimal generalized number product basis to obtain the Heisenberg matrix through truncating the basis by limiting $\bar{n}_j < M$. The Heisenberg matrix is a multi-dimensional matrix. We convert it to a $M^N \times M^N$ matrix by means of the following transformation:

$$C(r, s) = {}_{\bar{b}}\langle \bar{n}'_1, \bar{n}'_2, \dots, \bar{n}'_N | H | \bar{n}_1, \bar{n}_2, \dots, \bar{n}_N \rangle_{\bar{b}}, \quad (2.14)$$

where

$$r = \sum_{k=1}^N \bar{n}'_k M^{N-k} + 1, \quad (2.15)$$

$$s = \sum_{k=1}^N \bar{n}_k M^{N-k} + 1, \quad (2.16)$$

with $\bar{n}'_k, \bar{n}_k = 0, 1, 2, \dots, M - 1$. The basis has been truncated at size M due to finite computational resources. Nonetheless, we have assumed M to be sufficiently large for the calculations to have converged. By diagonalizing $C(r, s)$, the eigen-energies of the system of N coupled quantum anharmonic oscillators can be determined.

In summary, the combination of Bogoliubov transformation and the variational method has rescaled the Hamiltonian such that in terms of the optimized operators and basis, it now describes N oscillators with much reduced coupling and anharmonic potential. More precisely, the transformation performed above has rescaled the Hamiltonian such that the frequencies of the normal modes are now given by $\omega_j \bar{\Omega}_j$, while the magnitude of the perturbation has diminished by a factor of the form $1/\sqrt{\bar{\Omega}_j^{\mu_j}}$, with the integer μ_j smaller than the maximum degree κ (see Eq. (2.1)). This leads to smaller off-diagonal elements in the Heisenberg matrix formed using the optimized number product states as basis. As a result, fast converging eigenvalues can be obtained.

2.2.2 Application to systems of two coupled quantum anharmonic oscillators

Let us apply the extended two-step approach to a pair of coupled quantum anharmonic oscillators, which is described by a Hamiltonian of the following general form:

$$H = \sum_{j=1}^2 \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right) + \lambda \sum_{\nu=0}^{\kappa} \sum_{\mu=0}^{\kappa-\nu} c_{\mu\nu} x_1^{\mu} x_2^{\nu}. \quad (2.17)$$

Note that the polynomial of the perturbing anharmonic potential is up to degree κ . The Hamiltonian can also be written in the second-quantized form based on the creation and annihilation operators of each oscillator as follows:

$$H = \sum_{j=1}^2 \hbar \omega_j (a_j^{\dagger} a_j + \frac{1}{2}) + \lambda \sum_{\nu=0}^{\kappa} \sum_{\mu=0}^{\kappa-\nu} c_{\mu\nu} \frac{\beta_1^{\mu} \beta_2^{\nu}}{2^{(\mu+\nu)/2}} (a_1^{\dagger} + a_1)^{\mu} (a_2^{\dagger} + a_2)^{\nu}, \quad (2.18)$$

where $x_j = \beta_j (a_j^{\dagger} + a_j)/\sqrt{2}$, $p_j = i\hbar (a_j^{\dagger} - a_j)/\sqrt{2}\beta_j$ and $\beta_j = \sqrt{\hbar/m_j \omega_j}$.

In applying our extended approach, we first employ Eqs. (2.2), (2.3) and (2.8) to

re-express the Hamiltonian in terms of $b_j^\dagger$, b_j and Ω_j in the following manner:

$$\begin{aligned}
H = & \sum_{j=1}^2 \hbar \omega_j \left(\frac{1 + \Omega_j^2}{4\Omega_j} + \frac{1 + \Omega_j^2}{2\Omega_j} b_j^\dagger b_j + \frac{1 - \Omega_j^2}{4\Omega_j} (b_j^{\dagger 2} + b_j^2) \right) \\
& + \lambda \sum_{\nu=0}^{\kappa} \sum_{\mu=0}^{\kappa-\nu} c_{\mu\nu} \frac{\beta_1^\mu \beta_2^\nu}{2^{(\mu+\nu)/2} \Omega_1^{\mu/2} \Omega_2^{\nu/2}} \sum_{p=0}^{[\mu/2]} \sum_{q=0}^{\mu-2p} \frac{\mu! b_1^{\dagger(\mu-2p-q)} b_1^q}{2^p p! q! (\mu - 2p - q)!} \\
& \times \sum_{r=0}^{[\nu/2]} \sum_{s=0}^{\nu-2r} \frac{\nu! b_2^{\dagger(\nu-2r-s)} b_2^s}{2^r r! s! (\nu - 2r - s)!},
\end{aligned} \tag{2.19}$$

where the Gauss' symbol $[x]$ means rounding the value of x to the nearest lower integer. Note that we have expressed the Hamiltonian in the normal-ordered form for the convenience of constructing the Heisenberg matrix later. By applying the variational principle through Eqs (2.9) and (2.10) with $N = 2$, the following set of simultaneous equations are obtained:

$$\hbar \omega_1 \frac{\bar{\Omega}_1^2 - 1}{\bar{\Omega}_1} - \lambda \sum_{\nu=0,2}^{\kappa} \sum_{\mu=0,2}^{\kappa-\nu} \frac{c_{\mu\nu} \beta_1^\mu \beta_2^\nu \mu! \nu! (\mu/2)}{2^{(\mu+\nu-2)} (\mu/2)! (\nu/2)! \bar{\Omega}_1^{\mu/2} \bar{\Omega}_2^{\nu/2}} = 0, \tag{2.20}$$

$$\hbar \omega_2 \frac{\bar{\Omega}_2^2 - 1}{\bar{\Omega}_2} - \lambda \sum_{\nu=0,2}^{\kappa} \sum_{\mu=0,2}^{\kappa-\nu} \frac{c_{\mu\nu} \beta_1^\mu \beta_2^\nu \mu! \nu! (\nu/2)}{2^{(\mu+\nu-2)} (\mu/2)! (\nu/2)! \bar{\Omega}_1^{\mu/2} \bar{\Omega}_2^{\nu/2}} = 0. \tag{2.21}$$

Equations (2.20) and (2.21) are then solved numerically to yield the optimal values of Ω_1 and Ω_2 . It is important to notice that Eqs. (2.20) and (2.21) also guarantee that the coefficients of b^2 and $b^{\dagger 2}$ in H are zero. As pointed out by Hsue and Chern [51], this is the condition for the new ladder operators to correspond to the normal modes, and is the general requirement for the Hartree approximation.

In terms of the optimized number product states basis

$$|\bar{n}_1, \bar{n}_2\rangle_{\bar{b}} = \frac{\bar{b}_1^{\dagger \bar{n}_1} \bar{b}_2^{\dagger \bar{n}_2}}{\sqrt{\bar{n}_1! \bar{n}_2!}} |\bar{\phi}_1, \bar{\phi}_2\rangle_{\bar{b}}, \tag{2.22}$$

the multi-dimension matrix representation of H for the system of two coupled oscillators is obtained as follows:

$$\begin{aligned}
& H(\bar{n}'_1 + 1, \bar{n}_1 + 1, \bar{n}'_2 + 1, \bar{n}_2 + 1) \\
& = {}_{\bar{b}} \langle \bar{n}'_1, \bar{n}'_2 | H | \bar{n}_1, \bar{n}_2 \rangle_{\bar{b}}
\end{aligned}$$

$$\begin{aligned}
&= \left(\hbar\omega_1 \frac{1 + \bar{\Omega}_1^2}{4\bar{\Omega}_1} + \hbar\omega_1 \bar{n}_1 \frac{1 + \bar{\Omega}_1^2}{2\bar{\Omega}_1} + \hbar\omega_2 \frac{1 + \bar{\Omega}_2^2}{4\bar{\Omega}_2} + \hbar\omega_2 \bar{n}_2 \frac{1 + \bar{\Omega}_2^2}{2\bar{\Omega}_2} \right) \delta_{\bar{n}'_1 \bar{n}_1} \delta_{\bar{n}'_2 \bar{n}_2} \\
&+ \hbar\omega_1 \frac{1 - \bar{\Omega}_1^2}{4\bar{\Omega}_1} \left(\sqrt{(\bar{n}_1(\bar{n}_1 - 1))} \delta_{\bar{n}'_1, \bar{n}_1-2} + \sqrt{(\bar{n}_1 + 1)(\bar{n}_1 + 2)} \delta_{\bar{n}'_1, \bar{n}_1+2} \right) \delta_{\bar{n}'_2 \bar{n}_2} \\
&+ \hbar\omega_2 \frac{1 - \bar{\Omega}_2^2}{4\bar{\Omega}_2} \left(\sqrt{(\bar{n}_2(\bar{n}_2 - 1))} \delta_{\bar{n}'_2, \bar{n}_2-2} + \sqrt{(\bar{n}_2 + 1)(\bar{n}_2 + 2)} \delta_{\bar{n}'_2, \bar{n}_2+2} \right) \delta_{\bar{n}'_1 \bar{n}_1} \\
&+ \lambda \sum_{\nu=0}^{\kappa} \sum_{\mu=0}^{\kappa-\nu} \frac{c_{\mu\nu} \beta_1^\mu \beta_2^\nu}{2^{(\mu+\nu)/2} \bar{\Omega}_1^{\mu/2} \bar{\Omega}_2^{\nu/2}} \sum_{p=0}^{[\mu/2]} \sum_{q=0}^{\mu-2p} \frac{\mu! \sqrt{\bar{n}_1!} (\mu - 2p - 2q + \bar{n}_1)!}{2^p p! q! (\bar{n}_1 - q)! (\mu - 2p - q)!} \\
&\times \sum_{r=0}^{[\nu/2]} \sum_{s=0}^{\nu-2r} \frac{\nu! \sqrt{\bar{n}_2!} (\nu - 2r - 2s + \bar{n}_2)!}{2^r r! s! (\bar{n}_2 - s)! (\nu - 2r - s)!} \delta_{\bar{n}'_1, \bar{n}_1 + \mu - 2p - 2q} \delta_{\bar{n}'_2, \bar{n}_2 + \nu - 2r - 2s},
\end{aligned} \tag{2.23}$$

where $\bar{n}'_1, \bar{n}'_2, \bar{n}_1, \bar{n}_2 = 0, 1, 2, \dots, M - 1$. This is a $M \times M \times M \times M$ 4-D matrix. Next, we convert this multi-dimension matrix into a $M^2 \times M^2$ matrix through the following transformation:

$$C(M(\bar{n}'_1) + \bar{n}'_2 + 1, M(\bar{n}_1) + \bar{n}_2 + 1) = H(\bar{n}'_1 + 1, \bar{n}_1 + 1, \bar{n}'_2 + 1, \bar{n}_2 + 1). \tag{2.24}$$

By applying the standard diagonalization scheme on the matrix $C(r, s)$, we obtain the eigen-energies of the coupled system from the eigenvalues of matrix C .

Finally, it is interesting to note that our approach for the systems of two coupled anharmonic oscillators is a generalization of that for a single oscillator with a polynomial potential [49]. This is because the latter system can be treated by simply setting the parameter of the second oscillator in the two-coupled-oscillator system to zero.

2.2.3 Energy levels of a pair of coupled anharmonic oscillators

Coupled Quartic Anharmonic Oscillators

We first apply our method to a pair of coupled quartic anharmonic oscillators with the Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2} m_1 \omega_1^2 x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 x_2^2 + \lambda (c_{40} x_1^4 + c_{22} x_1^2 x_2^2 + c_{04} x_2^4), \tag{2.25}$$

which models the dynamics of the five-membered ring compounds [55], and compare our results to those in [54].

We truncate the Hamiltonian matrix at $M = 5, 9, 15$ and 19 . Table 2.1 shows that the low-lying energy eigenstates of the symmetrically coupled quartic anharmonic oscillators converge throughout the whole range of λ . We then choose to truncate the basis at $M = 9$ and compare our results to those in [54] (see Table 2.2). Note that E_j is the result of our calculation, while E_j^H is that from Hioe *et al.* The results for unsymmetrically coupling, i.e. $c_{40} \neq c_{04}$, is shown in Table 2.3.

Coupled Cubic Anharmonic Oscillators

Next, we apply our method to a system of harmonic oscillators coupled by cubic potential

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda(c_{12}x_1x_2^2 + c_{30}x_1^3), \quad (2.26)$$

which has been employed to approximate the normal mode of an anharmonic triatomic oscillator. This system has in fact been used in the study of the semiclassical theory of bound states by Eastes *et al.* [56].

Low-lying energy levels are computed by truncating the Hamiltonian matrix at $M = 9$. Our results $E_{j_1j_2}$ are found to compare favorably to the numerically computed exact results $E_{j_1j_2}^W$ of [56] (see Table 2.4).

Coupled Oscillators With Cubic And Quartic Anharmonicity

We have also explored the applicability of our approach to a more complicated system of harmonic oscillators coupled by mixed cubic and quartic potential. The Hamiltonian of this system is given below:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda \left(c_{40}x_1^4 + c_{30}x_1^3 + c_{04}x_2^4 + c_{03}x_2^3 + c_{21}x_1^2x_2 + c_{12}x_1x_2^2 + c_{22}x_1^2x_2^2 \right). \quad (2.27)$$

Energy eigenvalues are computed by truncating the Hamiltonian matrix at $M = 11$. The energy levels of the ground state and the 50th excited state are shown in Fig.

Table 2.1: Convergence of the first three states of the system of coupled quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = c_{40} = c_{04} = 1, c_{22} = 2$.

λ	Size of determinant	E_0	E_1	E_2
0.1	$5^2 \times 5^2$	1.150238	2.417194	3.775260
	$9^2 \times 9^2$	1.150188	2.414344	3.772339
	$15^2 \times 15^2$	1.150188	2.414340	3.772322
	$19^2 \times 19^2$	1.150188	2.414340	3.772322
0.5	$5^2 \times 5^2$	1.476477	3.241042	5.203985
	$9^2 \times 9^2$	1.476036	3.231518	5.195597
	$15^2 \times 15^2$	1.476025	3.231453	5.195314
	$19^2 \times 19^2$	1.476025	3.231453	5.195313
1.0	$5^2 \times 5^2$	1.724994	3.844320	6.226393
	$9^2 \times 9^2$	1.724206	3.830464	6.214372
	$15^2 \times 15^2$	1.724184	3.830325	6.213817
	$19^2 \times 19^2$	1.724184	3.830323	6.213815
10.0	$5^2 \times 5^2$	3.304048	7.564173	12.430486
	$9^2 \times 9^2$	3.301300	7.527662	12.398972
	$15^2 \times 15^2$	3.301210	7.527050	12.396828
	$19^2 \times 19^2$	3.301210	7.527043	12.396815
100.0	$5^2 \times 5^2$	6.918703	15.952793	26.312626
	$9^2 \times 9^2$	6.912122	15.870528	26.241507
	$15^2 \times 15^2$	6.911900	15.868991	26.236273
	$19^2 \times 19^2$	6.911899	15.868972	26.236240
5000.0	$5^2 \times 5^2$	25.299783	58.446221	96.495492
	$9^2 \times 9^2$	25.274871	58.139655	96.230311
	$15^2 \times 15^2$	25.274025	58.133767	96.210410
	$19^2 \times 19^2$	25.274022	58.133692	96.210282

Table 2.2: Energy eigenvalues for the first three states of the system of coupled quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = c_{40} = c_{04} = 1, c_{22} = 2$.

λ	E_0	E_0^H	E_1	E_1^H	E_2	E_2^H
0.05	1.0843	1.0843	2.2388	2.2388	3.4542	3.4542
0.10	1.1502	1.1502	2.4143	2.4143	3.7723	3.7723
0.50	1.4760	1.4760	3.2315	3.2315	5.1956	5.1953
1.00	1.7242	1.7242	3.8305	3.8304	6.2144	6.2140
10.0	3.3013	3.3012	7.5277	7.5271	12.399	12.397
100	6.9121	6.9119	15.871	15.869	26.242	26.237
5000	25.275	25.274	58.140	58.134	96.230	96.213

Table 2.3: Energy eigenvalues for the first three states of the system of coupled quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = 1, c_{40} = 0.4, c_{04} = 1, c_{22} = 2$.

λ	E_0	E_0^H	E_1	E_1^H	E_2	E_2^H
0.05	1.0669	1.0669	2.1580	-	2.2231	-
0.10	1.1206	1.1206	2.2818	-	2.3889	-
0.50	1.3965	1.3965	2.8989	-	3.1691	-
1.00	1.6123	1.6123	3.3725	-	3.7451	-
10.0	3.0176	3.0175	6.4059	-	7.3205	-
100	6.2773	6.2772	13.3821	-	15.4119	-
5000	22.913	22.913	48.903	-	56.438	-

Table 2.4: Energy eigenvalues of the system of coupled cubic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = 1, c_{12} = 1, c_{30} = 0.1$.

λ	ω_1^2	ω_2^2	E_{00}	E_{00}^W	E_{10}	E_{10}^W	E_{20}	E_{20}^W
-0.10	0.36	1.96	0.9939	0.9939	1.5809	1.5809	2.1613	2.1612
-0.10	0.49	1.69	0.9955	0.9955	1.6870	1.6870	2.3750	2.3750
-0.08	0.81	1.21	0.9980	0.9980	1.8944	1.8944	2.7899	2.7899

2.1. The results correspond very well to those computed by Hernandez [57].

Coupled Quantum Pendulums

Finally, we have also applied our approach to a system of two pendulums coupled through the parameter ε . The Hamiltonian of this system is given by [72]:

$$\begin{aligned}
H &= H_1 + H_2 + H_{int} \\
&= \sum_{j=1}^2 \left(\frac{p_j^2}{2} + \alpha(1 - \cos x_j) \right) + \varepsilon(1 - \cos(x_1 - x_2)) \\
&= \sum_{j=1}^2 \left(\frac{p_j^2}{2} + \frac{\alpha x_j^2}{2} \right) + \alpha \sum_{k=4,6}^{\infty} (-1)^{k/2-1} \frac{x_j^k}{k!} + \varepsilon \sum_{k=2,4}^{\infty} \sum_{r=0}^k (-1)^{k/2-1} \frac{x_1^{k-r} x_2^r}{(k-r)!r!}
\end{aligned} \tag{2.28}$$

Notice that we have expressed the cosine potential as an infinite polynomial series about $x_j = 0$. In order to apply our approach, we have to approximate the cosine potential by truncating the series at $k = 8$. With $M = 7$, we have determined the low-lying energy eigenvalues (less than the energy of the separatix $E_s = 2\alpha$) from the Heisenberg matrix with $\alpha = 50$. The results are displayed in Table 2.5.

Figure 2.1: Energy eigenvalues of the coupled cubic-quartic anharmonic oscillators for the case of $\hbar = m_1 = m_2 = 1$, $\omega_1 = 1000 \text{ cm}^{-1}$, $\omega_2 = 800 \text{ cm}^{-1}$, $c_{30} = \alpha/6$, $c_{40} = \alpha/24$, $c_{03} = c_{04} = 0$, $c_{12} = c_{21} = \beta/2$, $c_{22} = \beta/4$, $\lambda = 1$.

Table 2.5: Energy eigenvalues of the coupled pendulum for the case of $\hbar = 1, \alpha = 50$.

ε	E_{00}	E_{10}, E_{01}	E_{02}, E_{20}	E_{11}
0.0	7.0080	13.9517	20.7651	20.8955
	-	13.9517	20.7651	-
0.5	7.0426	13.9870	20.8395	21.0225
	-	14.0536	20.8688	-
1.0	7.0768	14.0218	20.8867	21.1750
	-	14.1546	20.9716	-

Table 2.6: Convergence of the energy eigenvalues of the system of three coupled sextic anharmonic oscillators.

Size of determinant	E_0	E_1	E_2
$5^3 \times 5^3$	2.986296	5.339869	5.905519
$9^3 \times 9^3$	2.978875	5.298264	5.867776
$17^3 \times 17^3$	2.978305	5.296000	5.865828
Braun	2.978302	5.295992	5.865822

2.2.4 Energy levels of systems of three and four coupled anharmonic oscillators

By extending the analysis of Sec. 2.2.2, we have determined in this section the eigen-energies of systems of three, and four coupled anharmonic oscillators. This has enabled us to further verify the validity of the extended two-step approach to problems of higher dimension.

System of Three Coupled Sextic Oscillators

Applying the extended two-step approach to a system of three coupled sextic anharmonic oscillators with the following Hamiltonian:

$$\begin{aligned}
 H = & \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + 2x_1^4 + \frac{1}{2}x_1^6 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + 2x_2^4 + \frac{1}{2}x_2^6 + \frac{p_3^2}{2} + \frac{1}{2}x_3^2 + 2x_3^4 + \frac{1}{2}x_3^6 \\
 & + x_1x_2 + x_1x_3 + x_2x_3,
 \end{aligned} \tag{2.29}$$

we found close convergence of the energy eigenvalues to the results of [44] as shown in Table 2.6, where we have truncated the Hamiltonian matrix at $M = 5, 9$, and 17.

Table 2.7: Convergence of the energy eigenvalues of the system of four coupled sextic anharmonic oscillators.

Size of determinant	E_0	E_1	E_2
$3^4 \times 3^4$	4.047835	6.523707	7.230376
$5^4 \times 5^4$	3.970225	6.328065	7.057805
$9^4 \times 9^4$	3.960086	6.283305	7.017863
Kaluža	3.959304	-	-

System of Four Coupled Sextic Oscillators

Next, we apply the approach to a system of four coupled sextic anharmonic oscillators with the following Hamiltonian:

$$\begin{aligned}
H = & \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + 2x_1^4 + \frac{1}{2}x_1^6 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + 2x_2^4 + \frac{1}{2}x_2^6 + \frac{p_3^2}{2} + \frac{1}{2}x_3^2 + 2x_3^4 + \frac{1}{2}x_3^6 \\
& + \frac{p_4^2}{2} + \frac{1}{2}x_4^2 + 2x_4^4 + \frac{1}{2}x_4^6 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4. \quad (2.30)
\end{aligned}$$

We have computed the low-lying energy levels by truncating the Hamiltonian matrix at $M = 3, 5$, and 9 . Our results are found to compare favorably to those of [43] (see Table 2.7).

2.2.5 A perturbative study on the squeezing properties of the vacuum product state

We have observed that the transformation to the new set of bosonic operators has scaled the angular frequencies of the harmonic potential and diminished the magnitude of the perturbing anharmonic potential. This implies that one can analyze the rescaled system by means of perturbation theory about the squeezed vacuum product state. In this subsection, we shall investigate into the convergence of the perturbation series through the squeezing properties of coupled anharmonic oscillators systems.

By using the fact that the squeeze vacuum product state $|\bar{\phi}_1, \bar{\phi}_2\rangle_{\bar{b}}$ is an eigenstate

of the TAO, we can simply determine the following expectation values:

$$\langle x_j \rangle = \langle p_j \rangle = 0, \quad (2.31)$$

$$\langle a_j^\dagger a_j \rangle = \frac{\bar{t}_j^2}{1 - \bar{t}_j^2}, \quad (2.32)$$

$$\langle a_j^{\dagger 2} \rangle = \frac{\bar{t}_j}{1 - \bar{t}_j^2}, \quad (2.33)$$

$$\langle a_j^2 \rangle = \frac{\bar{t}_j}{1 - \bar{t}_j^2}, \quad (2.34)$$

where $j = 1, 2$. From Eqs. (2.31-2.34), we readily obtain the uncertainties of x_j and p_j :

$$\sigma_{x_j}^2 = \langle x_j^2 \rangle - \langle x_j \rangle^2 = \beta_j^2 \left(\frac{1}{2} + \frac{\bar{t}_j}{1 - \bar{t}_j} \right) = \frac{\beta_j^2}{2\bar{\Omega}_j}, \quad (2.35)$$

$$\sigma_{p_j}^2 = \langle p_j^2 \rangle - \langle p_j \rangle^2 = \frac{\hbar^2}{\beta_j^2} \left(\frac{1}{2} - \frac{\bar{t}_j}{1 + \bar{t}_j} \right) = \frac{\hbar^2 \bar{\Omega}_j}{2\beta_j^2}. \quad (2.36)$$

This gives the zeroth-order estimation of the coordinate and momentum uncertainties. Equation (2.35) shows that coordinate squeezing occurs when $-1 < t_j < 0$, while Eq. (2.36) shows that momentum squeezing occurs when $0 < t_j < 1$. Since the product of the uncertainties σ_{x_j} and σ_{p_j} is always equal to $\hbar/2$, the squeezed vacuum product state is a coherent state. The same results can also be obtained by operating the position and momentum operators on the squeezed vacuum state with the new bosonic operators $\bar{b}_j^\dagger$ and $\bar{b}_j$ as follows:

$$\bar{b} \langle \bar{\phi}_j | x_j | \bar{\phi} \rangle_{\bar{b}} = \frac{\beta_j}{\sqrt{2\bar{\Omega}_j}} \bar{b} \langle \bar{\phi}_j | \bar{b}_j^\dagger + \bar{b}_j | \bar{\phi}_j \rangle_{\bar{b}} \quad (2.37)$$

$$\bar{b} \langle \bar{\phi}_j | p_j | \bar{\phi}_j \rangle_{\bar{b}} = \frac{i\hbar\sqrt{\bar{\Omega}_j}}{\beta_j\sqrt{2}} \bar{b} \langle \bar{\phi}_j | \bar{b}_j^\dagger - \bar{b}_j | \bar{\phi}_j \rangle_{\bar{b}} \quad (2.38)$$

Since the squeezed vacuum product state is not the energy eigenstate of the Hamiltonian of the coupled anharmonic oscillators, it cannot adequately describe the ground state of these oscillators. Hence, a first-order perturbation analysis is required to determine their coordinate and momentum uncertainties.

Coupled Harmonic Oscillators

The Hamiltonian for a system of coupled harmonic oscillators can be written as follows:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda(c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2). \quad (2.39)$$

By transforming to the new harmonic oscillator states characterized by the bosonic operators $\bar{b}_j^\dagger$ and $\bar{b}_j$, the Hamiltonian can be formulated into an unperturbed H_0 and a small perturbation H_1 quantified by the parameter λ' :

$$H = H_0 + \lambda' H_1 \quad \text{where} \quad \lambda' = -\frac{\lambda \beta_1 \beta_2}{2 \bar{\Omega}_1 \bar{\Omega}_2}, \quad (2.40)$$

$$H_0 = \sum_{j=1}^2 \hbar \omega_j \bar{\Omega}_j \left(\frac{1}{2} + \bar{b}_j^\dagger \bar{b}_j \right), \quad (2.41)$$

$$H_1 = -(\bar{b}_2^\dagger \bar{b}_1^\dagger + \bar{b}_2^\dagger \bar{b}_1 + \bar{b}_2 \bar{b}_1^\dagger + \bar{b}_2 \bar{b}_1). \quad (2.42)$$

Next, we expand the vacuum state $|\psi_0\rangle$ of the coupled harmonic oscillators about the optimal squeezed vacuum state $|\psi_0^{(0)}\rangle = |\bar{\phi}_1, \bar{\phi}_2\rangle_{\bar{b}}$ perturbatively as follows:

$$|\psi_0\rangle = |\psi_0^{(0)}\rangle + \lambda' |\psi_0^{(1)}\rangle + \lambda'^2 |\psi_0^{(2)}\rangle + \dots. \quad (2.43)$$

This allows us to calculate the the position and momentum uncertainties for the first oscillator perturbatively,

$$\sigma_{x_1} = \sqrt{\langle \psi_0 | x_1^2 | \psi_0 \rangle - \langle \psi_0 | x_1 | \psi_0 \rangle^2}, \quad (2.44)$$

$$\sigma_{p_1} = \sqrt{\langle \psi_0 | p_1^2 | \psi_0 \rangle - \langle \psi_0 | p_1 | \psi_0 \rangle^2}, \quad (2.45)$$

up to first-order. Our calculated results for σ_{x_1} and σ_{p_1} are shown in Table 2.8. Compared to those calculated by MacDermott and Redmount [36] (σ^R), our first-order approximations are not as accurate for larger λ , but are sufficiently good for us to investigate the squeezing of the state. To get more accurate results, one has to perform higher-order perturbative calculations. From Table 2.8, we observe that our results lead to the same conclusion as MacDermott: the coupling squeezes the uncertainty in x , while it widens the uncertainty in p . The overall effect is an increase in the product of the uncertainties.

Table 2.8: Position and momentum uncertainties of the coupled harmonic oscillators for the case of $\hbar = m_1 = m_2 = \omega_1 = \omega_2 = 1, c_{20} = 0.5, c_{02} = 0.5, c_{11} = -1$.

λ	$\sigma_{x_1} (\sqrt{\hbar/2})$	$\sigma_{x_1}^R (\sqrt{\hbar/2})$	$\sigma_{p_1} (\sqrt{\hbar/2})$	$\sigma_{p_1}^R (\sqrt{\hbar/2})$	$\sigma_{x_1} \sigma_{p_1} (\hbar/2)$	$\sigma_{x_1}^R \sigma_{p_1}^R (\hbar/2)$
0.05	0.98809	0.98830	1.01249	1.01213	1.00043	1.00029
0.10	0.97721	0.97798	1.02491	1.02385	1.00155	1.00130
0.50	0.91297	0.92388	1.11815	1.09868	1.02083	1.01505
1.00	0.86038	0.88807	1.21676	1.16877	1.04688	1.03795

Coupled Quartic Anharmonic Oscillators

Let us consider again the pair of coupled quartic anharmonic oscillators with Hamiltonian given by Eq. (2.25):

$$H = \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + \lambda(x_1^4 + 2x_1^2x_2^2 + x_2^4). \quad (2.46)$$

As before, we formulate the Hamiltonian into the unperturbed harmonic oscillator part H_0 and a small perturbation H_1 quantified by λ' ,

$$H = H_0 + \lambda' H_1 \quad \text{where} \quad \lambda' = \lambda/4 \quad (2.47)$$

$$H_0 = E_0 + \sum_{j=1}^2 \bar{\Omega}_j \bar{b}_j^\dagger \bar{b}_j \quad \text{where} \quad E_0 = \sum_{j=1}^2 \frac{3\bar{\Omega}_j^2 + 1}{8\bar{\Omega}_j} \quad (2.48)$$

$$\begin{aligned} H_1 = & \sum_{j=1}^2 : (\bar{b}_j^\dagger + \bar{b}_j)^4 : / \bar{\Omega}_j^2 \\ & + \frac{2}{\bar{\Omega}_1 \bar{\Omega}_2} (\bar{b}_1^{\dagger 2} \bar{b}_2^{\dagger 2} + \bar{b}_1^{\dagger 2} \bar{b}_2^2 + \bar{b}_1^2 \bar{b}_2^{\dagger 2} + \bar{b}_1^2 \bar{b}_2^2 \\ & + 2\bar{b}_1^{\dagger 2} \bar{b}_2^\dagger \bar{b}_2 + 2\bar{b}_1^2 \bar{b}_2^\dagger \bar{b}_2 + 2\bar{b}_2^{\dagger 2} \bar{b}_1^\dagger \bar{b}_1 + 2\bar{b}_2^2 \bar{b}_1^\dagger \bar{b}_1 + 4\bar{b}_1^\dagger \bar{b}_1 \bar{b}_2^\dagger \bar{b}_2), \end{aligned} \quad (2.49)$$

where the symbol $::$ denotes normal ordering. We again expand the ground state perturbatively about the squeezed vacuum state. We then calculate the position and momentum uncertainties of the first oscillator for $\lambda = 1$ based on the standard first-order perturbative analysis. In Table 2.9, we compare the results from coupled harmonic oscillators (CHO), coupled anharmonic oscillators (CAHO) and single anharmonic oscillators (SAHO). The perturbation series is convergent, which follows the results (energy eigenvalues) of Table 2.1. In other words, it is often possible to

Table 2.9: Comparing the position and momentum uncertainties of the coupled harmonic oscillators (CHO), the coupled anharmonic oscillators (CAHO) and the single anharmonic oscillator (SAHO).

Oscillator	$\sigma_{x_1} (\sqrt{\hbar/2})$	$\sigma_{p_1} (\sqrt{\hbar/2})$	$\sigma_{x_1} \sigma_{p_1} (\hbar/2)$
CHO	0.8604	1.2168	1.0469
CAHO	0.6815	1.4763	1.0060
SAHO	0.7089	1.4178	1.0051

construct a converging perturbation series using knowledge of the variational principle [73, 74]. In order to study the effect of anharmonicity and coupling on the squeezing separately, we set the respective parameter to zero. The results are shown in Figs. 2.2 and 2.3 respectively.

Compared to the uncertainties of a single anharmonic oscillator, the coupling in the coupled anharmonic oscillators has the effect of decreasing the position uncertainty and increasing the momentum uncertainty of the individual oscillator. In addition, it also has the effect of increasing the uncertainty product without violating the Heisenberg uncertainty principle. By comparing with results obtained from the coupled harmonic oscillators system, the effect of nonlinearity in the anharmonic potential of the coupled anharmonic oscillator is to squeeze and antisqueeze the position and momentum uncertainties more without any significant increase in the uncertainty product.

It is important to note that although the observations given above show that the effect of coupling and anharmonicity leads to a squeezing in x and an anti-squeezing in p , this is not always the case. A perturbative calculation for the coupled quantum pendulum given in Eq. (2.28) shows that it can indeed happen the other way round, with an anti-squeezing in x ($\sigma_{x_1} = 1.00904 \sqrt{\hbar/2}$) and a squeezing in p ($\sigma_{p_1} = 0.99104 \sqrt{\hbar/2}$).

Figure 2.2: Position Uncertainty σ_{x_1} (dashed line), Momentum Uncertainty σ_{p_1} (dash-dotted line) and Uncertainty Product $\sigma_{x_1}\sigma_{p_1}$ (solid line) dependence on the anharmonicity c_{40} of coupled quartic anharmonic oscillators with $c_{22} = 0$ and $\lambda = 1$.

Figure 2.3: Position Uncertainty σ_{x_1} (dashed line), Momentum Uncertainty σ_{p_1} (dash-dotted line) and Uncertainty Product $\sigma_{x_1}\sigma_{p_1}$ (solid line) dependence on the coupling c_{22} of coupled quartic anharmonic oscillators with $c_{40} = c_{04} = 0$ and $\lambda = 1$.

2.3 Dynamical two-step approach

2.3.1 Generalized two-step approach to solve the dynamics of systems of N coupled quantum anharmonic oscillators

The dynamics of a system of N coupled quantum oscillators evolves according to the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle. \quad (2.50)$$

For the class of N coupled nonlinear oscillators system, it is convenient to select the product states $|n_1\rangle_a \otimes |n_2\rangle_a \otimes \cdots \otimes |n_N\rangle_a$ as the basis vectors. Note that $|n_j\rangle_a$ is the eigenvector of the j -th harmonic oscillator. With this basis, the matrix element

$${}_a \langle n'_1, n'_2, \dots, n'_N | H | n_1, n_2, \dots, n_N \rangle_a \quad (2.51)$$

can be easily computed. In addition, the Schrödinger equation becomes:

$$\begin{aligned} & i\hbar \frac{d}{dt} {}_a \langle n'_1, n'_2, \dots, n'_N | \psi(t) \rangle \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} {}_a \langle n'_1, n'_2, \dots, n'_N | H | n_1, n_2, \dots, n_N \rangle_a {}_a \langle n_1, n_2, \dots, n_N | \psi(t) \rangle, \end{aligned} \quad (2.52)$$

or more compactly

$$i\hbar \frac{d}{dt} \Psi_t = \mathcal{H} \Psi_t. \quad (2.53)$$

Assuming that $\mathcal{H}$ is diagonalizable such that $\mathcal{H} = v D v^{-1}$, with v being the matrix of eigenvectors and D the diagonal matrix of energy eigenvalues, Eq. (2.53) can be solved to obtain:

$$\begin{aligned} \Psi_t &= \exp(-i\mathcal{H}t/\hbar) \Psi_0 \\ &= v \exp(-iDt/\hbar) v^{-1} \Psi_0, \end{aligned} \quad (2.54)$$

where $\exp(-i\mathcal{H}t/\hbar)$ is the evolution operator and Ψ_0 is the initial quantum state with respect to the chosen basis. However, the evaluation of Eq. (2.54) up to a certain level of precision can be computationally demanding especially when anharmonicity

is present in the model. A way to overcome this is to employ the two-step approach, which we have shown to be able to determine D with fast convergence through the diagonalization of $C(r, s)$ (see Eq. (2.14)). However, we have yet to exploit the eigenvectors obtained through the diagonalization process. Indeed, the elements of the j -th eigenvector of $C(r, s)$, which is represented by $\mathcal{E}_{sj}$, are the coefficients of expansion of the j -th energy eigenstates of the coupled anharmonic oscillators $|\psi_j\rangle$ with respect to the basis $\{|\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}}\}$. More precisely, if $C = \mathcal{E}D\mathcal{E}^{-1}$, then

$$|\psi_j\rangle = \sum_{\bar{n}_1=0}^{M-1} \sum_{\bar{n}_2=0}^{M-1} \cdots \sum_{\bar{n}_N=0}^{M-1} \mathcal{E}_{sj} |\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}}, \quad (2.55)$$

whose corresponding eigenvalue is D_{jj} . Note that the value of the index s in $\mathcal{E}_{sj}$ is given by Eq. (2.16). We surmise that after the transformation, the orthonormal number state is closer to the energy eigenstate. In other words, the number of the tensor product state $|\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}}$ required to approximate the energy eigenstates $|\psi_j\rangle$ of the coupled anharmonic oscillators within the same error is much smaller than that with $\{|n_1, n_2, \dots, n_N\rangle_a\}$ as the product basis.

In terms of the new number basis, Eq. (2.54) becomes

$$\Psi'_t = \mathcal{E} \exp(-iDt/\hbar) \mathcal{E}^{-1} \Psi'_0. \quad (2.56)$$

In this sense, the eigenvectors of $C(r, s)$ constitute an essential part in the dynamical two-step approach.

From Eq. (2.56), we observe that in order to obtain Ψ'_t , it is also crucial to determine Ψ'_0 . This requires a non-trivial transformation of the representation of the initial state $|\psi(0)\rangle$ from the original basis $\{|n_1, n_2, \dots, n_N\rangle_a\}$ to the new basis $\{|\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}}\}$. The detail of this transformation is shown below.

An initial state can be expressed with respect to the original number basis in the following manner:

$$|\psi(0)\rangle = \sum_{n_1=0}^{M-1} \sum_{n_2=0}^{M-1} \cdots \sum_{n_N=0}^{M-1} A_{n_1, n_2, \dots, n_N} |n_1, n_2, \dots, n_N\rangle_a. \quad (2.57)$$

By substituting the completeness relation of the optimized basis

$$\sum_{\bar{n}_1=0}^{M-1} \sum_{\bar{n}_2=0}^{M-1} \cdots \sum_{\bar{n}_N=0}^{M-1} |\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}} \langle \bar{n}_1, \bar{n}_2, \dots, \bar{n}_N| = 1 \quad (2.58)$$

into Eq.(2.57), the initial state in terms of the new basis is

$$|\psi(0)\rangle = \sum_{\bar{n}_1=0}^{M-1} \sum_{\bar{n}_2=0}^{M-1} \cdots \sum_{\bar{n}_N=0}^{M-1} B_{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N} |\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N\rangle_{\bar{b}}, \quad (2.59)$$

where

$$B_{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N} = \sum_{n_1=0}^{M-1} \sum_{n_2=0}^{M-1} \cdots \sum_{n_N=0}^{M-1} A_{n_1, n_2, \dots, n_N} {}_{\bar{b}}\langle \bar{n}_1, \bar{n}_2, \dots, \bar{n}_N | n_1, n_2, \dots, n_N \rangle_a. \quad (2.60)$$

After substituting Eqs. (2.12), (2.4) and (2.2) into Eq. (2.60), we obtain

$$B_{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N} = \sum_{n_1=0}^{M-1} \sum_{n_2=0}^{M-1} \cdots \sum_{n_N=0}^{M-1} A_{n_1, n_2, \dots, n_N} \prod_{j=1}^N \left(\eta_j \sum_{l_j=0}^{\infty} \frac{\bar{t}_j^{l_j} \sqrt{(2l_j)!}}{2^{l_j} l_j! \sqrt{\bar{n}_j!} (1 - \bar{t}_j^2)^{\bar{n}_j}} \right) {}_a\langle 2l_1, 2l_2, \dots, 2l_N | (a_j - \bar{t}_j a_j^\dagger)^{\bar{n}_j} | n_1, n_2, \dots, n_N \rangle_a. \quad (2.61)$$

Note that the operator $(-\bar{t}_j a_j^\dagger + a_j)^{\bar{n}_j}$ has to be expressed in normal order form before we can determine the quantum expectation value

$${}_a\langle 2l_1, 2l_2, \dots, 2l_N | (a_j - \bar{t}_j a_j^\dagger)^{\bar{n}_j} | n_1, n_2, \dots, n_N \rangle_a. \quad (2.62)$$

With the additional factor of $-\bar{t}_j$ on the creation operator $a_j^\dagger$, the normal order form of $(-\bar{t}_j a_j^\dagger + a_j)^{\bar{n}_j}$ cannot be obtained directly from the expression given in [50]. However, by realizing that

$$: (-\bar{t}_j a^\dagger + a)^{\bar{n}_j} : (-\bar{t}_j a^\dagger + a) = : (-\bar{t}_j a^\dagger + a)^{\bar{n}_j+1} : -\bar{n}_j \bar{t}_j : (-\bar{t}_j a^\dagger + a)^{\bar{n}_j-1} :, \quad (2.63)$$

we are able to prove by mathematical induction (see appendix B) that the term $(-\bar{t}_j a^\dagger + a)^{\bar{n}_j}$ in Eq.(2.61) can be expressed in the normal order form as follows:

$$(-\bar{t}_j a^\dagger + a)^{\bar{n}_j} = \sum_{q=0,2,\dots}^{[\bar{n}_j]_e} \epsilon_q {}^{\bar{n}_j} C_q (-\bar{t}_j)^{q/2} : (-\bar{t}_j a^\dagger + a)^{\bar{n}_j-q} :, \quad (2.64)$$

with

$$\epsilon_q = \frac{(q-1)!}{2^{q/2-1}(q/2-1)!} \quad \text{for } q \geq 4, \quad (2.65)$$

and

$$\epsilon_0 = \epsilon_2 = 1. \quad (2.66)$$

Note that the symbol $[\cdot]_e$ means taking the value of the nearest smaller even integer.

In consequence, Eq. (2.61) becomes

$$B_{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N} = \sum_{n_1=0}^{M-1} \sum_{n_2=0}^{M-1} \dots \sum_{n_N=0}^{M-1} A_{n_1, n_2, \dots, n_N} \prod_{j=1}^N \left(\sum_{l_j=0}^{\infty} \sum_{q_j=0,2,\dots}^{[\bar{n}_j]_e} \sum_{k_j=0}^{\bar{n}_j-q_j} \frac{\eta_j \bar{t}_j^{l_j}}{2^{l_j} l_j! \sqrt{(1-\bar{t}_j^2)^{\bar{n}_j}}} \right. \\ \left. (-\bar{t}_j)^{\bar{n}_j-q_j/2-k_j} \epsilon_{q_j} \frac{\sqrt{\bar{n}_j!}}{q_j!} \frac{\sqrt{n_j!}}{k_j!} {}^{2l_j}C_{n_j-k_j} \delta_{2l_j, n_j-2k_j+\bar{n}_j-q_j} \right), \quad (2.67)$$

where $\delta_{i,j}$ is the Kronecker delta. The vector representation of the initial state is then given by

$$\Psi'_0(s) = B_{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_N}, \quad (2.68)$$

with the value of s determined through Eq. (2.16).

2.3.2 Application of the two-step approach to the evaluation of quantum squeezing parameter and the entropy of the eigenstate

To begin, let us validate the dynamical two-step approach for the simplest case when the initial state is an energy eigenstate. For example, Ψ'_0 is set equal to $(\mathcal{E}_{1j}, \mathcal{E}_{2j}, \dots, \mathcal{E}_{M^N j})^T$, with the $\mathcal{E}_{sj}$ being coefficients of the j -th eigenstate with respect to the optimal basis. Then, the resulting time evolved state is a stationary quantum state, i.e., $\Psi'_t = e^{-iE_j t/\hbar} (\mathcal{E}_{1j}, \mathcal{E}_{2j}, \dots, \mathcal{E}_{M^N j})^T$. We shall determine the efficacy of our approach by comparing the quantum expectation values determined from such a stationary state with that obtained through analytical derivation. The results obtained are found to improve over our earlier approach based on perturbation in Sec. 2.2.5. The system of interest is the two coupled harmonic oscillators (CHO) with the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2 x_2^2 + \lambda(x_2 - x_1)^2. \quad (2.69)$$

The coupling potential between the two oscillators is assumed to depend quadratically on the distance between the oscillators, and is proportional to the constant λ . Physical systems modeled by such a Hamiltonian include vibrating molecules and modes of the

electromagnetic field, and the entanglement properties of these systems are currently under active investigation [62, 63, 64, 65]. Note that this Hamiltonian can be obtained from Eq. (2.39) by setting $c_{20} = 1, c_{02} = 1, c_{11} = -2, m_1 = m_2 = m, \omega_1 = \omega_2 = \omega_0$ and hereafter, it will be denoted by H_c .

By second quantizing Eq. (2.69) in terms of b_j and $b_j^\dagger$ through Eqs. (2.2) and (2.3), as well as by means of the variational approach:

$$\Omega_j^2 - 1 - 2\lambda = 0, \quad (2.70)$$

which yields the optimized value $\bar{\Omega}_j = (1 - \bar{t}_j)/(1 + \bar{t}_j)$ for $j = 1, 2$, the Hamiltonian is found to adopt the following form:

$$H_c = \frac{1}{2\bar{\Omega}_1} + \frac{1}{2\bar{\Omega}_2} + \bar{\Omega}_1 b_1^\dagger b_1 + \bar{\Omega}_2 b_2^\dagger b_2 - \frac{\lambda}{\sqrt{\bar{\Omega}_1 \bar{\Omega}_2}} (b_1^\dagger + b_1) (b_2^\dagger + b_2). \quad (2.71)$$

Note that we have set the values of $\hbar, m$, and ω_0 to 1. By forming $C(r, s)$ according to Eq. (2.14) and diagonalizing it, the matrix $\mathcal{E}(r, s)$ is obtained. The representation of the j -th energy eigenstate of the coupled system with respect to the optimized basis is given by the j -th column of the $\mathcal{E}(r, s)$ matrix.

In particular, the determination of the quantum squeezing parameter of the j -th energy eigenstate of the coupled system first involves the following calculation:

$$\sigma_{j, x_1} = \sum_{r=1}^{M^N} \sum_{s=1}^{M^N} \mathcal{E}_{rj}^\dagger \hat{\sigma}_{rs} \mathcal{E}_{sj}, \quad (2.72)$$

where $\hat{\sigma}_{rs}$ is a $M^N \times M^N$ matrix which is formed by the position uncertainty operator σ_{x_1} (see Eq. 2.44) expressed in terms of b and $b^\dagger$. The dimensionless squeezing parameters is then defined by:

$$S_x = -\ln \frac{\sigma_{j, x_1}}{\sigma_{x_1}^{(0)}}. \quad (2.73)$$

The quantity $\sigma_{x_1}^{(0)}$ is the uncertainty with respect to the unperturbed system. It serves as normalization constants in the definition of the squeezing parameters. For symmetrical systems, there is no difference in the consideration of the squeezing parameter with respect to the first or to the the second oscillator. In consequence, we have defined the squeezing parameters in terms of the first oscillator.

In the calculation of the entanglement properties of the coupled harmonic oscillators (and coupled anharmonic oscillators in general), we have set $N = 2$ since we are concerned with a bipartite system. The density matrix of energy eigenstate $\rho = |\psi_j\rangle\langle\psi_j|$ is given by the $M^2 \times M^2$ matrix:

$$\rho(r, s) = \mathcal{E}_{rj} \mathcal{E}_{sj}^*. \quad (2.74)$$

The reduced density matrix,

$$\rho^{(1)}(r, s) = \sum_{k=0}^{M-1} \mathcal{E}_{(r-1)M+k+1, j} \mathcal{E}_{(s-1)M+k+1, j}^*, \quad (2.75)$$

which is of size $M \times M$, is determined by performing a trace on ρ over one of the subsystems. The entanglement entropy of the coupled system is then obtained by evaluating the eigenvalues ζ_j of $\rho^{(1)}$:

$$S_{vN} = -\text{Tr}[\rho^{(1)} \ln \rho^{(1)}] = -\sum_{j=1}^M \zeta_j \ln \zeta_j, \quad (2.76)$$

where Tr is the trace operation.

A comparison on the convergence rate between the two-step approach and direct diagonalization in the computation of the squeezing properties of the ground states of the coupled harmonic oscillators is illustrated in Fig. 2.4. The computation is performed by truncating the basis at $M = 5, 10, 15$ and 25 . The figure shows that the two-step approach gives calculations that converge faster to the exact values [36],

$$S_x = -\ln \sqrt{\frac{1+\gamma}{2}}, \quad (2.77)$$

where $\gamma = 1/\sqrt{1+4\lambda}$, throughout the whole range of λ . In addition, computation based on the two-step approach and direct diagonalization on the entanglement entropy shows convergence towards the exact analytical results [37]:

$$S_{vN} = 2 \left\{ \cosh^2 \left(\frac{\ln \gamma}{4} \right) \ln \left[\cosh \left(\frac{\ln \gamma}{4} \right) \right] - \sinh^2 \left(\frac{\ln \gamma}{4} \right) \ln \left[\sinh \left(\frac{\ln \gamma}{4} \right) \right] \right\}. \quad (2.78)$$

Here, we have employed the basis size of $M = 20, 25$ and 35 . As illustrated in Fig. 2.5, the rate of convergence of the calculations obtained by the two-step approach is again faster than that from direct diagonalization.

Figure 2.4: The position quadrature squeezing S_x of the ground state of H_c calculated by direct diagonalization (thick dashed line) and the extended two-step approach (dashed line). The basis size are: $M = 5$, 10 , and 15 compared to the exact result (solid line). Note that the unit of λ is $m\omega^2$.

Figure 2.5: The entanglement entropy S_{vN} of the ground state of H_c calculated by direct diagonalization (thick dashed line), and the extended two-step approach (dashed line). The basis size are: $M = 20$, 25 and 35 compared to the exact result (solid line). Note that the unit of λ is $m\omega^2$.

2.3.3 Application of the dynamical two-step approach to arbitrary initial states

In this subsection, we will apply the dynamical two-step approach to evaluate the quantum dynamics of momentum uncertainty, position uncertainty, and entanglement of several nonlinear oscillator systems with arbitrary initial conditions. The approach will be applied to systems with known result as a check of accuracy as well as to illustrate the applicability of the approach to systems that can be expressed in the general form given by Eq. (2.1).

Single anharmonic oscillator

Let us first consider a single quartic anharmonic oscillator with the Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^4. \quad (2.79)$$

By means of the Bogoliubov transformation and the variational approach, which yields:

$$\Omega^3 - \Omega - 6\lambda = 0, \quad (2.80)$$

whose solution gives $\bar{\Omega}$, the Hamiltonian can be re-expressed in terms of b , $b^\dagger$ and $\bar{\Omega}$ as follow:

$$H = \frac{1 + 3\bar{\Omega}^2}{8\bar{\Omega}} + \bar{\Omega}b^\dagger b + \frac{\lambda}{4\bar{\Omega}^2} : (b^\dagger + b)^4 :. \quad (2.81)$$

Note that $m = \omega = \hbar = 1$. The initial state $|\psi(0)\rangle$ is chosen to be the coherent state $|\alpha\rangle$, with $\alpha = (x + ip)/\sqrt{2}$. It can be expressed in terms of the optimized basis in the following manner:

$$|\alpha\rangle = \sum_{\bar{n}=0}^{M-1} \sum_{n=0}^{M-1} \sum_{l=0}^{M-1} \frac{e^{-|\alpha|^2/2} \alpha^n \bar{\eta}^l}{2^l l! \sqrt{(1 - \bar{t}^2)^{\bar{n}}}} \sum_{q=0,2,\dots}^{[\bar{n}]_e} \sum_{k=0}^{\bar{n}-q} (-\bar{t})^{\bar{n}-q/2-k} \epsilon_q \frac{\sqrt{\bar{n}}!}{q!k!} {}^{2l}C_{n-k} \delta_{2l,n-2k+\bar{n}-q} |\bar{n}\rangle_{\bar{b}}. \quad (2.82)$$

When $\lambda = 10$, $x = 0.01$ and $p = 0.01$, we have calculated the variance of the momentum quadrature $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ for the time evolved state with respect to both the original and the optimized basis. The results are shown in Fig. 2.6. Compared to the traditional direct diagonalization approach, the dynamical two-step method is found to give converging result for a much smaller basis size.

Harmonic oscillators with biquadratic coupling

The dynamical two-step approach is next applied to a system of coupled oscillators with the following Hamiltonian:

$$H_3 = \frac{p_1^2}{2} + \frac{1}{2}x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}x_2^2 + \lambda x_1^2 x_2^2. \quad (2.83)$$

Note that this model has been employed to investigate the influence of chaos on the dynamical production of entanglement [26].

We first express the Hamiltonian in terms of b_j , $b_j^\dagger$ and Ω as follows:

$$H_3 = \frac{1 + 3\Omega^2}{4\Omega} + \Omega b_1^\dagger b_1 + \Omega b_2^\dagger b_2 + \frac{\lambda}{4\Omega^2} : (b_1^\dagger + b_1)^2 : : (b_2^\dagger + b_2)^2 : . \quad (2.84)$$

The optimal Ω is then obtained by solving the equation

$$\Omega^3 - \Omega - \lambda = 0. \quad (2.85)$$

Next, the initial coherent state is evaluated and found to take the following form:

$$\begin{aligned} |\alpha_1, \alpha_2\rangle &= \sum_{\bar{n}_1=0}^{M-1} \sum_{\bar{n}_2=0}^{M-1} \sum_{n_1=0}^{M-1} \sum_{n_2=0}^{M-1} \sum_{l_1=0}^{M-1} \sum_{l_2=0}^{M-1} \frac{e^{-|\alpha_1|^2/2} \alpha_1^{n_1} \eta_1 \bar{t}_1^{l_1}}{2^{l_1} l_1! \sqrt{(1 - \bar{t}_1^2)^{\bar{n}_1}}} \frac{e^{-|\alpha_2|^2/2} \alpha_2^{n_2} \eta_2 \bar{t}_2^{l_2}}{2^{l_2} l_2! \sqrt{(1 - \bar{t}_2^2)^{\bar{n}_2}}} \\ &\quad \sum_{q_1=0,2,\dots}^{[\bar{n}_1]_e} \sum_{q_2=0,2,\dots}^{[\bar{n}_2]_e} \sum_{k_1=0}^{\bar{n}_1-q_1} \sum_{k_2=0}^{\bar{n}_2-q_2} (-\bar{t}_1)^{\bar{n}_1-q_1/2-k_1} \epsilon_{q_1} (-\bar{t}_2)^{\bar{n}_2-q_2/2-k_2} \epsilon_{q_2} \\ &\quad \frac{\sqrt{\bar{n}_1!}}{q_1! k_1!} \frac{\sqrt{\bar{n}_2!}}{q_2! k_2!} {}^{2l_1}C_{n_1-k_1} {}^{2l_2}C_{n_2-k_2} \delta_{2l_1, n_1-2k_1+\bar{n}_1-q_1} \delta_{2l_2, n_2-2k_2+\bar{n}_2-q_2} |\bar{n}_1, \bar{n}_2\rangle_{\bar{b}}. \end{aligned} \quad (2.86)$$

By using the same parameter value of $\lambda = 0.0075$ with $M = 85$, we are able to reproduce the results of the first 7 tori in [26] as illustrated in Fig. 2.7. This again validates the accuracy of the dynamical two-step approach.

Two-coupled Morse oscillator system

Finally, we demonstrate the application of the dynamical two-step approach to a system of two coupled Morse oscillators with the following Hamiltonian:

$$H = \frac{p_1^2}{2} + D_e(1 - e^{-x_1})^2 + \frac{p_2^2}{2} + D_e(1 - e^{-x_2})^2 + \lambda(x_2 - x_1)^2. \quad (2.87)$$

This system is well known to be a realistic model of the stretching modes of SO_2 and H_2O [75, 76]. It is also used in the Peyrard-Bishop model to simulate the H-bond between the DNA base pair [77]. In order to apply our approach, we have approximated the exponential potential by its Taylor series up to $O(x^{12})$:

$$\begin{aligned}
H \approx & \frac{p_1^2}{2} + \frac{p_2^2}{2} + \lambda(x_2 - x_1)^2 \\
& + D_e \sum_{k=1}^2 \left(x_k^2 - x_k^3 + \frac{7}{12}x_k^4 - \frac{1}{4}x_k^5 + \frac{31}{360}x_k^6 - \frac{1}{40}x_k^7 + \frac{127}{20160}x_k^8 \right. \\
& \left. - \frac{17}{12096}x_k^9 + \frac{73}{259200}x_k^{10} - \frac{31}{604800}x_k^{11} + \frac{2047}{239500800}x_k^{12} \right).
\end{aligned}$$

For $D_e = 10$ and $\lambda = 0.1$, the dynamics of the position uncertainty and the entropy of the first oscillator of the two-coupled Morse oscillator system are shown in Fig. 2.8 for the initial states $|0, 1\rangle_a$, and Fig. 2.9 for the coherent state $|\alpha_1, \alpha_2\rangle$.

At $t = 0$, the first oscillator has a “minimum” uncertainty value of $\sigma_{x_1} = \sqrt{1/2}$ because at that moment it is in the ground state $|0\rangle_a$ or the coherent state $|\alpha_1\rangle$. As time progresses, a sharp position uncertainty squeezing is observed to occur for a very short time (see Figs. 2.8(a) and 2.9(a)) before the variance acquires its oscillatory nature above $\sqrt{1/2}$. On the other hand, Figs. 2.8(b) and 2.9(b) show that interaction between the two oscillators which are initially separable, has led to a dynamical production of entanglement. The maximum degree of the generated entanglement depends largely on the number of accessible quantum state, and since state $|0, 1\rangle_a$ has a larger mean energy relative to $|\alpha_1 = i/10\sqrt{2}, \alpha_2 = \sqrt{2}/10\rangle$, it is expected to generate a larger entanglement because its time evolved states are able to access more quantum states dynamically.

Figure 2.6: Time series for the momentum variance (σ_p) of the system of single anharmonic oscillator in Eq.(2.79) with $\lambda = 10$, calculated through (a) direct diagonalization and (b) dynamics two-step approach. The basis size used is 5, 15, 65, 115 and 165 from top to bottom.

Figure 2.7: Time series of the entanglement entropy S_{vN} of the system of coupled anharmonic oscillator in Eq.(2.83) with $\lambda = 0.0075$ for different initial states (see [26]). Note that the von Neumann entropy of the reduced density matrix shown here is multiplied by a factor of $1/\ln 2$ in order to match the definition of entanglement entropy in [26].

Figure 2.8: Time series of (a) the position variance σ_{x_1} and (b) the entanglement entropy S_{vN} for the system of two coupled Morse oscillators with $\lambda = 0.1$ and $D_e = 10$ for the initial state $|0, 1\rangle_a$.

Figure 2.9: Time series of (a) the position variance σ_{x_1} and (b) the entanglement entropy S_{vN} for the system of two coupled Morse oscillators with $\lambda = 0.1$ and $D_e = 10$ for the initial state ($\alpha_1 = i/10\sqrt{2}, \alpha_2 = \sqrt{2}/10$).

Chapter 3

The relation of entanglement, squeezing and uncertainty product in two-coupled-oscillator system

3.1 Introduction

Squeezing, an intrinsically quantum mechanical property which cannot be explained through a classical theory, has attracted much attention recently as a potential resource for entanglement. This results from the realization that a reduction in uncertainty together with passive operations can generate non-classical correlations between Gaussian states, implying an inherent connection between squeezing and entanglement [78]. Indeed, it is found that spin squeezing is closely related to, and in fact implies, quantum entanglement [79]. In consequence, it is possible to characterize entanglement in terms of spin squeezing, which has been experimentally realized through the creation and manipulation of correlated collective atomic states [17]. Furthermore, the introduction of quantum correlations between the atomic spins in a Bose-Einstein condensate can also cause the collective spin state of the vapor to be squeezed [17, 80]. Such entanglement-induced squeezing has the important outcome of producing measuring instruments that go beyond the precision of current models.

Close association between squeezing and entanglement also exists in quantum open systems, where entanglement is found to be controllable by means of squeezing in the Jaynes-Cummings model, with stronger entanglement attained through a proper adjustment of the initial squeezed photon states [81]. All these results imply a subtle relationship between squeezing and entanglement.

On the other hand, uncertainties of arbitrary sets of physical properties have often been used for the derivation of separability criteria. Criteria for detecting entanglement have been developed in composite systems [12], continuous variable systems [13], finite dimensional systems [14], and N -level systems [15]. In contrast, there is less investigation on the fundamental connection between entanglement and the conventional uncertainty product. Kim and Mahler [82] started this a few years ago in the ‘which-path’ detection model. The model was first introduced by Einstein in the famous Bohr-Einstein debates, and later used to demonstrate the concept of complementarity. While the standard uncertainty principle has been widely recognized as the origin of complementarity, Scully, Englert and Walther [83] introduce “complementarity without uncertainty relation” by exploiting quantum entanglement between the atom and the detector. This idea is then experimentally realized by Rempe *et al.* [84], Kwiat *et al.* [85] and Schneider *et al.* [86]. In the new experiment, entanglement leads to a way of distinguishing paths and destroying interference. Kim and Mahler later readdressed the equivalence between the uncertainty relation and complementarity from a different perspective by pointing out the dependence of uncertainty relation on entanglement. Busch and Shilladay [87] hence conclude the close link between uncertainty and entanglement, and consider both as possible mechanisms that enforce complementarity. This can be discerned through their statement: “whenever an explanation of the loss of interference can be given in terms of entanglement, this can be accompanied with an explanation in terms of a form of uncertainty relation”. But as suggested by Knight [88], a great deal of work must still be done to quantify this.

In this chapter, we shall investigate into the relationship between entanglement and the two quantum aspects introduced above: quadrature squeezing and quantum

uncertainty product. In particular, these relations will first be established with respect to the energy ground state of a pair of coupled harmonic oscillator (see Eq. (2.69)). Note that the lowest energy eigenstate is an interesting quantum state that has attracted much investigation [37, 34]. For instance, the degree of entanglement of the ground state has been used to investigate quantum phase transition [89, 90, 91].

The nonlinear perturbative potentials considered in this chapter consist of two types: higher-order interaction and anharmonic potential. The Hamiltonian of the former type is given below:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda(x_2 - x_1)^\nu, \quad (3.1)$$

for $\nu \geq 2$. On the other hand, the Hamiltonian for the latter type is of the following form:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda(x_2 - x_1)^2 + \sum_{\mu=4,6,\dots}^k \varepsilon_\mu(x_1^\mu + x_2^\mu), \quad (3.2)$$

where a finite number k of anharmonic terms parameterized by the coefficients ε_μ have been added to the system. Interestingly, such anharmonic terms appear in the Hamiltonian of vibrating molecules or electromagnetic modes coupled to nonlinear Kerr-like medium [28, 35].

3.2 Quantum squeezing and entanglement

Let us first observe on the close relationship between squeezing and entanglement through the ground state of the two coupled harmonic oscillators with Hamiltonian H_c (see Eq.(2.69)). From Figs. 2.4 and 2.5, we see that as the coupling constant λ increases, both the squeezing and the entanglement become larger, which implies that greater interaction between the oscillators has led to a more entangled state.

Next, we study the effects of nonlinear perturbative potentials on squeezing and entanglement through the Hamiltonian given in Eq. (3.1). In particular, we shall focus on higher-order coupling potentials with $\nu = 4$ and $\nu = 6$. We found that both the squeezing parameter and the entanglement entropy increase as we increase

λ for these Hamiltonian systems (see Fig. 3.1). Interestingly, the observed increase is faster at small λ , but slower at large λ , compared to H_c for these oscillators with higher-order coupling.

The addition of anharmonic terms to H_c as given in Eq. (3.2) again leads to an increase in the squeezing of the position uncertainty quadrature relative to the coupled harmonic oscillator. However, these anharmonic terms have the effect of reducing the degree of entanglement for all λ . These results are obtained through the following coupled anharmonic oscillator systems by setting $k = 12$: two coupled quartic oscillators system (only $\varepsilon_4 \neq 0$); two-coupled-oscillator system with order 10th anharmonicity (only $\varepsilon_{10} \neq 0$); and two-coupled-oscillator system with quartic, sextic, octic, order 10th and 12th anharmonicity (in this case, $\varepsilon_\mu \neq 0$ and has the same value for $\mu = 4, 6, 8, 10, 12$).

Figure 3.2 shows that the amount of squeezing increases with both an increase in the coupling constant λ and the strength of the anharmonic potential ε_μ , for all coupled oscillator systems. On the other hand, the entanglement entropy is found to increase with an increase in λ , but decrease with an increase in ε_μ . Thus, λ and ε_μ have a co-operative effect on the squeezing parameter, but an opposing influence on the entanglement entropy. These results can be understood qualitatively through Eqs. (A.7) and (A.8) of Appendix A. In the presence of anharmonicity, the squeezing parameter changes from $\sqrt{(1 + \gamma)/2}$ to $\sqrt{(1 + \gamma')/2\bar{\Omega}}$. Since $\bar{\Omega} > 1$, this implies an increase in the squeezing parameter. Furthermore, with $\gamma' > \gamma$, Eq. (A.8) indicates that the introduction of anharmonicities indeed reduces the entanglement entropy. We have plotted Eqs. (A.7) and (A.8) as dotted curves in Fig. 3.2 for the two coupled quartic oscillators ($\varepsilon_\mu = 0.05$). They are noticed to follow closely the corresponding curves obtained by numerical simulation, especially when λ is small. In addition, we found that the amount of squeezing and the degree of entanglement depend sensitively on the order of the anharmonic potential. The higher the order of the anharmonicity, the greater is the increment in squeezing and the greater is the decrement in entanglement with respect to H_c .

In the case of coupled harmonic oscillator with Hamiltonian H_c , the squeezing

Figure 3.1: The effect of higher order coupling on (a) the squeezing parameter S_x and (b) the entanglement entropy S_{vN} of the ground state of the two-coupled oscillator systems against the coupling constant λ : H_c (solid line, analytical), $H_{\nu 4}$ (dashed line, numerical), and $H_{\nu 6}$ (dashed-dotted line, numerical). $H_{\nu 4}$ and $H_{\nu 6}$ are Hamiltonian given by Eq. (3.1) with $\nu = 4$ and $\nu = 6$ respectively. Note that $m = \omega_1 = \omega_2 = \hbar = 1$.

parameter S_x and the entanglement entropy S_{vN} of the ground state is related analytically by Eq. (A.9) with $\bar{\Omega}$ set to one in Eq. (A.10). This analytical result leads to an entropy-squeezing curve (see Figs. 3.3 and 3.4 for H_c) which serves to characterize the degree of entanglement in a two coupled quantum harmonic oscillators system. It is interesting to note that the variance in position quadrature measurement of the coupled harmonic oscillator reduces as the state becomes more correlated, with S_x saturating at the limiting value of $\ln \sqrt{2}$.

There will be modifications to the entropy and squeezing relation if we were to consider the two classes of coupled oscillator systems given by Eqs. (3.1) and (3.2) against H_c . For the system given by Eq. (3.1), the change is observed to be a very slight shift to the left as the order of the coupling increases (see Fig. 3.3). On the other hand, for the system given by Eq. (3.2), the change is observed to be a right-shift with the amount of shift determine by the strength of the added anharmonic potential (see Fig. 3.4). If the anharmonic potential is a slight perturbation, the right-shift can be predicted by the zeroth-order entropy-squeezing relation given by

Figure 3.2: The effect of strength ε_u and order of anharmonicities on (a) the squeezing parameter S_x and (b) the entanglement entropy S_{vN} of the ground state of coupled oscillators system. The quantum aspects are plotted against the coupling constant λ . From bottom to top, we show: analytical result for H_c (red), numerical result for coupled quartic oscillators with $\varepsilon_4 = 0.05$ (magenta), analytical approximation for coupled quartic oscillators with $\varepsilon_4 = 0.05$ (magenta star), numerical result for coupled quartic oscillators with $\varepsilon_4 = 1$ (black), numerical result for coupled anharmonic oscillators with $\varepsilon_{10} = 1$ (green) and numerical result for coupled anharmonic oscillators with $\varepsilon_4 = \varepsilon_6 = \varepsilon_8 = \varepsilon_{10} = \varepsilon_{12} = 1$ (blue) in figure (a). Results of the entanglement entropy is shown in the reversing order in figure (b). Note that $m = \omega_1 = \omega_2 = \hbar = 1$.

Figure 3.3: The ground state entropy-squeezing curves for the following two-coupled oscillator systems (from bottom to top): H_c (red line, analytical), $H_{\nu 4}$ (green line, numerical) and $H_{\nu 6}$ (blue line, numerical). Note that $m = \omega_1 = \omega_2 = \hbar = 1$.

Eq. (A.9). For $\varepsilon_4 = 0.05$, this is given by the dotted curve shown in Fig. 3.4. Notice that this curve follows rather closely the corresponding curve obtained via numerical simulation. Similar to the coupled harmonic oscillators system, there exist a maximum attainable squeezing parameter which is greater due to the anharmonic effects. In particular, for the sufficiently small perturbing anharmonic potential of $\varepsilon_4 = 0.05$, this limit occurs approximately at $S_x = \ln \sqrt{2\Omega} = 0.4056$ for the quartic oscillator system.

In conclusion, the relation between squeezing and entanglement can be altered by the perturbation potential. Nevertheless, by selecting from the set of entropy-squeezing curves according to the corresponding anharmonic potential or higher-order coupling of the coupled oscillator system, one is still able to characterize the degree of entanglement through a measurement of the squeezing parameter of the particular system.

Figure 3.4: The ground state entropy-squeezing curves for the following two-coupled oscillator systems (from left to right): CHO (red, analytical), coupled quartic oscillators (magenta, $\varepsilon_4 = 0.05$, numerical), coupled quartic oscillators (magenta star, $\varepsilon_4 = 0.05$, analytical approximation), coupled quartic oscillators (black, $\varepsilon_4 = 1$, numerical), coupled anharmonic oscillators (green, $\varepsilon_{10} = 1$, numerical), and coupled anharmonic oscillators (blue, $\varepsilon_4 = \varepsilon_6 = \varepsilon_8 = \varepsilon_{10} = \varepsilon_{12} = 1$, numerical). Note that $m = \omega_1 = \omega_2 = \hbar = 1$.

3.3 Quantum uncertainty product and entanglement

The relation between entanglement and uncertainty product was first investigated in the ground state of the CHO model. As shown by McDermott and Redmount [36], the net effect of bilinear coupling on the ground state of the CHO system is a decrease in position uncertainty

$$\sigma_{x_1} = \sqrt{\frac{\hbar}{2m\omega_0} \frac{1+\gamma}{2}} \quad (3.3)$$

and an increase in momentum uncertainty

$$\sigma_{p_1} = \sqrt{\frac{\hbar m\omega_0}{2} \frac{1+\gamma}{2\gamma}} \quad (3.4)$$

from their respective unperturbed values:

$$\sigma_{x_1}^{(0)} = \sqrt{\frac{\hbar}{2m\omega_0}} \quad (3.5)$$

and

$$\sigma_{p_1}^{(0)} = \sqrt{\frac{\hbar m\omega_0}{2}}. \quad (3.6)$$

The corresponding uncertainty product and its unperturbed version are given respectively by

$$U = \sigma_{x_1} \sigma_{p_1} = \frac{\hbar}{2} \frac{1+\gamma}{2\sqrt{\gamma}} \quad (3.7)$$

and

$$U^{(0)} = \frac{\hbar}{2}. \quad (3.8)$$

As the ground state's position quadrature is being squeezed, the momentum quadrature will be anti-squeezed, resulting in an increase in the uncertainty product without violating the Heisenberg uncertainty principle. Remarkably, this uncertainty product possesses a unique relationship with the entanglement entropy of the system's ground state as follows:

$$S_{vN} = \frac{1}{2}(\bar{U} + 1) \ln \left(\frac{1}{2}(\bar{U} + 1) \right) - \frac{1}{2}(\bar{U} - 1) \ln \left(\frac{1}{2}(\bar{U} - 1) \right), \quad (3.9)$$

where $\bar{U}$ is the normalized product of the uncertainties, i.e.,

$$\bar{U} = \frac{\sigma_{x_1} \sigma_{p_1}}{\sigma_{x_1}^{(0)} \sigma_{p_1}^{(0)}} \quad (3.10)$$

with $\hbar$ set to 1. Note that this relation does not depend on the coupling strength λ explicitly. Thus, entanglement between the two oscillators is a monotonically increasing function of the position-momentum uncertainty product of the subsystem. This is shown in Fig. 3.5, which gives the entropy-uncertainty curve of Eq. (3.9). Interestingly, similar relation is obtained by Vidal *et. al.* [89] in their investigation on quantum phase transition, where the entanglement entropy is found to be related instead to the elements of the Green function matrices.

Having established the quantitative link between entanglement and uncertainty product in the CHO system is encouraging but the question is: how robust is this relation against the additional perturbation by anharmonic potential? Besides, how will this relation respond to the higher-order interaction between the two oscillators? The answer to these questions may be important if one intends to exploit the uncertainty product as a means to characterize entanglement within physical systems which by nature are typically nonlinear. In order to answer these questions, let us first look at the effects of these perturbations in the zeroth-order limit. In the appendix, we have shown that if the nonlinear potential is small enough, each two coupled anharmonic oscillator system can be represented by a two coupled harmonic oscillator system after a Bogoliubov transformation, with the frequency being rescaled in the new representation. Nonetheless, as shown by Eq.(A.13), the relation between the entanglement entropy and the uncertainty product is not affected by such a frequency change. Hence, the entropy-uncertainty relation is not altered by the anharmonic perturbation in the zeroth-order limit, with the result that entanglement and uncertainty product continue to relate “harmonically”. This implies a universal relationship between these two quantities for a subclass of coupled oscillator system in the limit of small perturbations.

In order to have more precise results especially when the perturbation potential become large, we adopt the extended two-step approach introduced in Chapter 2 to

perform numerical calculation of the entanglement entropy and uncertainty product. This approach gives fast numerical convergence. The numerical results are shown in Fig. 3.6 for the Hamiltonian given by Eq. (3.1) with $\nu = 4, 6$, and 12. It is interesting that the entropy-uncertainty curves for these systems match the curve given by Eq.(3.9) for the CHO system. In other words, the relation between entanglement and the product of uncertainties is insensitive to the order of the interaction between the two oscillators.

Next, we explore numerically the two coupled anharmonic oscillator system given by Eq. (3.2) at the lowest energy eigenstate. We investigate the relation between entanglement entropy and uncertainty product by allowing both λ and ε_μ to vary between 0 and 50. The entropy-uncertainty relation (represented by the red squares) for coupled quartic oscillators:

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2x_1^2 + \varepsilon_4x_1^4 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2x_2^2 + \varepsilon_4x_2^4 + \lambda(x_2 - x_1)^2, \quad (3.11)$$

is shown in Fig. 3.7. As λ and ε_μ vary, almost all the data points lie on the CHO's entropy-uncertainty curve except when S_{vN} is small, where we observe a very small deviation from the CHO's curve. When $\lambda = 0$, the entropy is zero which indicates that the two oscillators are disentangled, while the presence of the quartic potential leads to quadratures squeezing which increases the uncertainty product, and produces the observed small deviation. Similar entropy-uncertainty curves (represented by triangles in magenta, circles in green and diamonds in black respectively) have been obtained for the coupled Morse oscillators in Eq. (2.87), coupled sextic oscillators:

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2x_1^2 + \varepsilon_6x_1^6 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2x_2^2 + \varepsilon_6x_2^6 + \lambda(x_2 - x_1)^2, \quad (3.12)$$

and the coupled oscillators with anharmonicity of order 12:

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2x_1^2 + \varepsilon_{12}x_1^{12} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2x_2^2 + \varepsilon_{12}x_2^{12} + \lambda(x_2 - x_1)^2, \quad (3.13)$$

respectively. As the perturbation order becomes larger, greater deviation of the entropy-uncertainty curves from the CHO's curve is observed at small entropy, but convergence towards the CHO's curve is found at large entropy. This is especially clear for the coupled oscillators with order 12th anharmonicity.

From the previous section, we learn that for system given by Eq. (3.2), the degree of entanglement increases with an increase in λ , but decreases with an increase in ε . In order to get a highly entangled state, the coupling constant λ should be as large as possible while the anharmonic potential ε as small as possible. This is equivalent to a coupled harmonic oscillator system with a very small anharmonic perturbation H_{ANH} . In other words, as the entanglement entropy increases, we envisage that the ratio between the anharmonic perturbation H_{ANH} to the coupled harmonic potential H_{CHO} reduces. Thus, the anharmonic Hamiltonian approaches the Hamiltonian of the coupled harmonic oscillator system, which explains the convergence of the coupled nonlinear oscillators' entropy-uncertainty curve to the coupled linear oscillators' curve at large entanglement. In contrast, small entanglement entropy results from a large relative ratio between H_{ANH} and H_{CHO} such that the zeroth-order limit can no longer approximate the solution adequately. In conclusion, while the relation between quadrature squeezing and entanglement depends sensitively on the nonlinear potential, the influence of nonlinear perturbation on the entropy-uncertainty product relation is minimal especially when the entanglement between the two oscillators is large.

3.4 Summary

We have found that the relation between squeezing and entanglement in the ground state of a class of two coupled harmonic oscillator systems can be altered by the addition of nonlinear perturbations. Hence, in order to characterize entanglement based on squeezing, a set of entropy-squeezing curves that corresponds to the form of the perturbations, i.e., (i) higher-order coupling (Fig. 3.3), or (ii) anharmonic potential (Fig. 3.4) will be required. On the other hand, we found that the relation between entanglement and uncertainty product in the ground state of the class of two coupled harmonic oscillator systems is unique. Furthermore, our analytical and numerical results show that this relation is robust against the influence of nonlinear perturbations especially when the entanglement between the two oscillators is large, indicating the

Figure 3.5: CHO system ground state's uncertainty-entropy relation – simulation results (star) and analytical results given by Eq. (3.9) (solid line). Note that $m = \omega_0 = \hbar = 1$.

Figure 3.6: Ground state's entropy-uncertainty relation for $H_{\nu 4}$ (red squares), $H_{\nu 6}$ (green circles) and $H_{\nu 12}$ (black diamond), compared to CHO system (blue solid curve). $H_{\nu 4}$, $H_{\nu 6}$ and $H_{\nu 12}$ are Hamiltonians given by Eq. (3.1) with $\nu = 4$, $\nu = 6$ and $\nu = 12$ respectively. Note that $m = \omega_0 = \hbar = 1$.

Figure 3.7: Ground state's entropy-uncertainty relation for coupled quartic oscillators (red squares), coupled Morse oscillators (magenta triangles), coupled sextic oscillators (green circles) and coupled oscillators with order 12th anharmonicity (black diamond), compared to CHO system (blue solid curve). Note that $m = \omega_0 = \hbar = 1$.

possible presence of a universal relation. This implies that the uncertainty product can serve as a useful candidate for the characterization of entanglement at the regime of large entropy.

Chapter 4

Dependence of entanglement dynamics on the global classical dynamical regime

4.1 Introduction

Generally, interaction between two particles leads to dynamical production of entanglement, especially when several quantum states are accessible to both particles. The study of such dynamical entanglement is particularly interesting for systems which have a classical counterpart. In these systems, classical effects can come into play in the quantum systems, whose influences are still not clear. In this chapter, we are interested to study these influences on the generation of CV entanglement by means of the two-coupled-oscillator systems.

This chapter is organized as follows. In Sec. 4.2, we study the correspondence between the classical-quantum regimes on the dynamics of CV entanglement by having the initial states of the two oscillators to be in the product of two coherent states. This choice results from the fact that coherent states gives good classical correspondence, which enables a clear relationship between the classical phase space picture to the quantum description. Then, in Sec. 4.3, by evolving an initial product state of two

number state (or a superposition of them) through Hamiltonians that have classical counterparts with globally different behavior, the dependence of the dynamics of CV entanglement on the initial state is investigated.

4.2 Classical-quantum correspondence

Extensive research has revealed the correspondence between quantum and classical mechanics. Notable examples include the manifestation of chaos in the energy level distribution of atomic systems, as well as the wave patterns associated with those levels. Interestingly, such correspondence can also exist within the entanglement of coupled quantum systems, whose research is especially fascinating at the regime where transition to chaos occurs. The investigation typically involves an examination into the maximal, mean-value and production rate of entanglement based on nonlinear models such as the 4D standard map [92], coupled kicked tops [93, 94, 95, 96, 97, 98, 99], interacting spins system [100, 101] and the Jaynes-Cummings model [102] which are typical examples of finite dimensional systems. In these studies, classical chaos is found to induce a stronger entanglement [92, 102], with the entanglement production rate linked to the Lyapunov exponent [95]. In addition, they have shown that increased chaos leads to an eventual saturation of entanglement production rate [96]. In contrast, less work has been done on entanglement in infinite dimensional systems (also known as continuous variable entanglement) and their correspondence to classical chaos. Recently, Zhang and Jie [26] have investigated the production of continuous variable entanglement through the quantization of classical tori and the mixed trajectories. They have found good quantum-classical correspondence. This has stimulated us to explore into the quantum-classical correspondence in a similar context, with an additional situation of a purely chaotic regime which has yet to be explored within infinite dimensional systems. Furthermore, we believe that interesting quantum-classical correspondence arises not only in the regime that transits to chaos, it can also occur within the regime of regular systems. In fact, detailed studies on the dynamics of entanglement when the corresponding classical

system is regular have been lacking. For example, there is as yet no investigation on the quantum-classical correspondence for system that displays periodic entanglement. Although quasi-periodic classical dynamics has been shown to correspond to entanglement dynamics with either quasi-periodicity or saturation, the underlying classical dynamics for a system that exhibits quasi-periodic entanglement dynamics is still not clear. Indeed, an interesting question that we hope to answer is: can a periodic classical trajectory give rise to a quasi-periodic entanglement dynamics upon quantization?

In this section, the initial states $|\psi(0)\rangle$ are chosen to be the coherent states $|\alpha_1\rangle \otimes |\alpha_2\rangle$. This serves as the connection between the quantum and classical domain [26]. The center of the coherent state is located precisely on a point (x_1, p_1, x_2, p_2) in the classical phase space. Note that $\alpha_l = (x_l + ip_l)/\sqrt{2}$. Entanglement of the time-evolved state is quantified by the von Neumann entropy of the reduced density matrix which one can compute by means of the dynamical two-step approach for faster converging results. In addition, we shall introduce the quantity ξ , which is closely related to the entanglement. According to the criterion developed by Duan *et al.* [13], as long as the parameter

$$\xi = (\Delta u)^2 + (\Delta v)^2 - 2 < 0, \quad (4.1)$$

the quantum state is entangled. Note that $u = x_1 + x_2$ and $v = p_1 - p_2$ are the two EPR-type operators, while Δu and Δv are the corresponding quantum fluctuations. Analytical solutions of the quantum fluctuation in the linear models will be worked out by means of the Heisenberg equation of motion:

$$\frac{d}{dt}a_j = \frac{1}{i\hbar}[a_j, H], \quad (4.2)$$

with a_j being the annihilation operators.

In this section, our investigation involves a study on three conservative Hamiltonian systems and is organized as follows. In subsection 4.2.1, we show the transition of a classical fixed point from a center to a saddle point in a two-mode magnon system. The response of the quantized system to this transition is then investigated by means of the dynamical generation of quantum entanglement and fluctuation. In section

4.2.2, we proceed to explore into the correspondence of periodic and quasi-periodic classical trajectories in a two-coupled harmonic oscillator system with the entanglement dynamics of their quantized counterparts. In section 4.2.3, we investigate into the change in entanglement dynamics as the corresponding classical dynamics transits from the regular to the chaotic regime through a coupled quartic oscillator model. For the models discussed in these sections, we aim to seek for the correspondence, if any, between the classical trajectories within different global dynamical regime and the continuous variable entanglement dynamics of the quantized system. Note that our main focus is on the dynamics rather than the production rate of quantum entanglement.

4.2.1 Two-mode magnon system

The Hamiltonian of our first model to study the quantum-classical correspondence is given by

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2x_2^2 + \lambda \left(m\omega_0x_1x_2 - \frac{p_1p_2}{m\omega_0} \right). \quad (4.3)$$

We have assumed that $m = \omega_0 = 1$, and $\lambda \geq 0$. In the classical regime, $\lambda = 1$ is the transition point where the fixed point at the origin transits from a center to a saddle – i.e., trajectories cycle with a frequency of $\omega = \sqrt{1 - \lambda^2}$ around the origin when $\lambda < 1$ (see appendix C), but evolve either towards or away from the origin when $\lambda > 1$.

The second-quantization of Eq. (4.3) gives the following Hamiltonian:

$$H = \hbar\omega_0 \left(a_1^\dagger a_1 + \frac{1}{2} \right) + \hbar\omega_0 \left(a_2^\dagger a_2 + \frac{1}{2} \right) + \lambda\hbar (a_1^\dagger a_2^\dagger + a_1 a_2), \quad (4.4)$$

which models the two-mode magnons excited by an external field [29]. For the quantized system, we determine the entanglement dynamics numerically from several initial coherent states for $\lambda = 0.1, 0.5$ and 0.8 with a basis size of $M = 50$. The resultant entanglement dynamics is found to depend only on the coupling strength λ , but not on the initial coherent state. Figures 4.1 (a), (b) and (c) show that the entanglement dynamics is periodic when $\lambda < 1$. Interestingly, the periods of entanglement dynamics are exactly half the periods of the corresponding classical motions. This

frequency-doubling phenomenon in the quantization of the classical periodic orbit has been reported earlier by Satchell and Sarben [103] for quantum moment in a dissipative system. Our result here and in the next section shows that this phenomenon remains true in a conservative system.

By increasing λ , we observe an increase in the maximum attainable entanglement. This means that the stronger the interaction between the two modes, the more they become entangled. Figure 4.1 (d) shows the entropy dynamics at different basis size in the saddle regime, where we simulate the entanglement dynamics with $\lambda = 1.1$. As time goes on, we observe that the number of basis required to approximate the time-evolved state increases rapidly. It seems that the strong interaction between the two mode has induced a rapid growth in the entropy as the quantized system crosses the $\lambda = 1$ transition point.

Next, we determine the time dependent equation of the ladder operators $a_j(t)$ and $a_j^\dagger(t)$ for $j = 1, 2$ by means of the Heisenberg formulation of quantum mechanics:

$$\frac{d}{dt} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} -i & -i\lambda \\ i\lambda & i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix}, \quad (4.5)$$

whose solutions are given by

$$a_1(t) = \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] a_1(0) - \frac{i\lambda}{\kappa} \sinh(\kappa t) a_2^\dagger(0), \quad (4.6)$$

$$a_2^\dagger(t) = \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right] a_2^\dagger(0) + \frac{i\lambda}{\kappa} \sinh(\kappa t) a_1(0), \quad (4.7)$$

where $\kappa = \sqrt{\lambda^2 - 1}$. The time dependence of ξ is given by

$$\begin{aligned} \xi(t) = & 2(\langle a_1^\dagger a_1 \rangle - \langle a_1^\dagger \rangle \langle a_1 \rangle + \langle a_2^\dagger a_2 \rangle - \langle a_2^\dagger \rangle \langle a_2 \rangle + \\ & \langle a_1^\dagger a_2^\dagger \rangle - \langle a_1^\dagger \rangle \langle a_2^\dagger \rangle + \langle a_1 a_2 \rangle - \langle a_1 \rangle \langle a_2 \rangle), \end{aligned} \quad (4.8)$$

where $\langle O \rangle$ in Eq. (4.8) is the time dependent expectation value taken over the coherent state, i.e. $\langle \alpha_1, \alpha_2 | O(t) | \alpha_1, \alpha_2 \rangle$. Substituting Eqs. (D.9), (D.10) and their conjugates into Eq. (4.8), we obtain

$$\xi(t) = \frac{4\lambda}{|\lambda^2 - 1|} (\lambda - 1) |\sinh(t\sqrt{\lambda^2 - 1})|^2. \quad (4.9)$$

See appendix D for details. For $\lambda < 1$, we have

$$\xi(t) = \frac{2\lambda}{\lambda + 1} \left\{ \cos(2t\sqrt{1 - \lambda^2}) - 1 \right\}, \quad (4.10)$$

which is always less than the lower bound – zero. According to Duan’s criterion, violation of the lower bound serves as a sufficient condition for entanglement. This analysis through the Duan’s criterion corresponds to our numerical quantification based on the entanglement entropy. First, the sum of quantum fluctuations is independent of the chosen initial coherent state ($|\alpha_1\rangle \otimes |\alpha_2\rangle$), just like our conjecture on the entropy dynamics. This can be observed from the value of $\xi(t)$, which depends only on λ . Thus, the entanglement dynamics of the two-mode magnon system is independent of the local classical behavior. Next, the periodic $\xi(t)$ has a frequency of $2\sqrt{1 - \lambda^2}$, which is exactly twice the corresponding classical frequency. It is important to note that $\xi(t)$ is always greater than zero when $\lambda > 1$. As a sufficient but not necessary criterion, Duan’s criterion can no longer determine whether the state is entangled or separable in the $\lambda > 1$ regime. Nonetheless, the analytical result for $\xi(t)$ does show a transition point at $\lambda = 1$. Indeed, $\xi(t)$ diverges to infinity for $\lambda \geq 1$, in direct correspondence to the fact that the entanglement between the two oscillators grows rapidly in time. Finally, these results illustrate the good quantum-classical correspondence in the bifurcation behavior of the two-mode magnon system, which exemplifies the manner in which entanglement dynamics can depend on the global classical dynamical regime.

4.2.2 Coupled harmonic oscillators

The Hamiltonian of our second model involves two harmonic oscillators coupled in a bilinear fashion:

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2 x_2^2 + \lambda x_1 x_2. \quad (4.11)$$

Physical realizations of such systems include vibrating molecules and modes of the electromagnetic field [62, 104]. Again, we have set $m = \omega_0 = 1$. Furthermore, we have restricted $\lambda < 1$. In general, the classical trajectories yield two-frequency

dynamics – periodic for the former and quasi-periodic for the latter within a time of 40π .

We next solve the Heisenberg equation of motion for this two-coupled harmonic oscillator system:

$$\frac{d}{dt} \begin{pmatrix} a_1 \\ a_1^\dagger \\ a_2 \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} -i & 0 & -i\frac{\lambda}{2} & -i\frac{\lambda}{2} \\ 0 & i & i\frac{\lambda}{2} & i\frac{\lambda}{2} \\ -i\frac{\lambda}{2} & -i\frac{\lambda}{2} & -i & 0 \\ i\frac{\lambda}{2} & i\frac{\lambda}{2} & 0 & i \end{pmatrix} \begin{pmatrix} a_1 \\ a_1^\dagger \\ a_2 \\ a_2^\dagger \end{pmatrix}, \quad (4.13)$$

and obtain

$$\begin{aligned} a_1(t) &= (-A_1 + B_1 - A_2 + B_2) a_1(0) - (C_1 + C_2) a_1^\dagger(0) \\ &+ (-A_1 + B_1 + A_2 - B_2) a_2(0) - (C_1 - C_2) a_2^\dagger(0), \end{aligned} \quad (4.14)$$

$$\begin{aligned} a_2^\dagger(t) &= -(C_1^* - C_2^*) a_1(0) + (B_1^* - A_1^* - B_2^* + A_2^*) a_1^\dagger(0) \\ &- (C_1^* + C_2^*) a_2(0) + (B_1^* - A_1^* + B_2^* - A_2^*) a_2^\dagger(0), \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} A_{1,2} &= \frac{(\omega_{1,2} - 1)^2}{8\omega_{1,2}} e^{i\omega_{1,2}t}, \\ B_{1,2} &= \frac{(\omega_{1,2} + 1)^2}{8\omega_{1,2}} e^{-i\omega_{1,2}t}, \\ C_{1,2} &= \frac{2i(\omega_{1,2} - 1)(\omega_{1,2} + 1)}{8\omega_{1,2}} \sin(\omega_{1,2}t). \end{aligned}$$

This solution enables us to determine the dynamics of the sum of quantum fluctuations which gives (see appendix D)

$$\xi(t) = -\frac{1}{2} \frac{\omega_1^2 - 1}{\omega_1^2} [1 - \cos(2\omega_1 t)] + \frac{1}{2} (\omega_2^2 - 1) [1 - \cos(2\omega_2 t)]. \quad (4.16)$$

Interestingly, the analytical result is in accordance with our numerical result – there is also a two-frequency dynamics in the sum of quantum fluctuations. The two frequencies, $\varpi_1 = 2\sqrt{1 + \lambda}$ and $\varpi_2 = 2\sqrt{1 - \lambda}$, are twice the corresponding classical

frequencies ω_1 and ω_2 respectively, which is again the frequency-doubling scenario that we have obtained in our first model. Furthermore, the dynamics, which is either periodic or quasi-periodic, depends on the ratio of the two frequencies ϖ_1/ϖ_2 , instead of the local dynamical behavior of the underlying classical trajectories. This is because both the entanglement and the sum of quantum fluctuations depend only on the coupling strength, and not on the chosen initial coherent state. Thus, the entanglement dynamics of the coupled harmonic oscillators is dependent solely on the global classical behavior. The most dramatic effect of this global dependence is observed for the quasi-periodic entanglement dynamics of Fig. 4.2 (d), which corresponds to the classical periodic orbit indicated by a cross in Fig. 4.2 (c). This indicates that a periodic classical trajectory can give rise to a quasi-periodic entanglement dynamics upon quantization, answering the question posed in section 4.2. Indeed, similar counterintuitive influence of the global classical dynamical regime applies to the one-frequency periodic orbit (indicated by a cross) in Fig. 4.2 (a), which corresponds to a two-frequency periodic entanglement dynamics as illustrated in Fig. 4.2 (b).

4.2.3 Coupled quartic oscillators

The classical dynamics of the coupled quartic oscillator system,

$$H = \frac{1}{2}(p_1^2 + p_2^2) + 3x_1^4 + x_2^4 - \lambda x_1^2 x_2^2, \quad (4.17)$$

has been found to range from regular behavior to completely chaotic behavior through the variation of the coupling constant λ . It is observed that the system becomes more chaotic as the coupling becomes larger. For instance, the phase space is a mixture of classical tori and chaotic sea when $\lambda = 0.8$, but becomes purely chaotic when $\lambda = 2.7$. Figures 4.3, 4.4 and 4.5 show the respective phase portraits of classically regular, mixed and chaotic dynamics for this system. Note that good quantum-classical correspondence has been found in the study of the quantum energy level distribution of this system [105].

In order to treat the quantum and classical regime with care, we adopt the parameter $R = \hbar/(|\alpha_1|^2 + |\alpha_2|^2)$ to quantify the ‘quantumness’ of the system [28]. By letting

Figure 4.2: Poincaré surface of section and entanglement dynamics of the coupled harmonic oscillator system: $\lambda = 11/61$ for (a) and (b), while $\lambda = 0.19$ for (c) and (d). Insets in (b) and (d) show the corresponding dynamics of $\xi(t)$.

$R \gg 1$, we simulate the entropy dynamics of the quantized system of Eq. (4.17) at $\lambda = 0.4, 0.8$ and 2.7 in the quantum regime. The results are shown in Fig. 4.6. On the other hand, we consider the case of $R < 0.1$ for the semiclassical treatment, for which we again simulate the entropy dynamics of the quantized system of Eq. (4.17) at $\lambda = 0.4, 0.8$ and 2.7 . The results are shown in Fig. 4.7. Compared to the quantum treatment, the entropy of entanglement is much larger in the semiclassical regime. Nevertheless, similar phenomena are observed to occur in the two regimes. First, in both treatments, the entanglement production rate is the highest for the case of pure chaos, lower for the mixed case and the lowest for the regular case, in correspondence to the finite dimensional systems [102, 92]. Second, the frequency of the oscillation becomes higher as λ becomes larger. Finally, identical results are obtained in the mixed case even though the entanglement dynamics is simulated numerically from different initial coherent states. This indicates that the entanglement dynamics is insensitive towards the choice of regular or chaotic initial conditions in the mixed case for a fixed energy. In other words, the behavior of the entanglement dynamics in this model depends entirely on the global classical dynamical regime and not on the local classical behavior. This is quite a surprising result which is different from those obtained by Zhang *et al.* [26] for an infinite dimensional system and Novaes [101] for a finite dimensional system. They have both observed a local influence of the classical regular and chaotic dynamics on the corresponding entanglement dynamical pattern. See Fig. 2.5 for the obvious difference between Zhang *et al.* results and ours.

4.3 Dependence of entanglement dynamics on the type of initial state

In this section, we study the dynamics of entanglement with number product states as the initial quantum states.

Figure 4.3: Classical phase portrait for the coupled quartic oscillator system with coupling constant $\lambda = 0.4$.

Figure 4.4: Classical phase portrait for the coupled quartic oscillator system with coupling constant $\lambda = 0.8$. Note that the marked points indicate the coherent states employed as initial conditions in the dynamical evolution.

Figure 4.5: Classical phase portrait for the coupled quartic oscillator system with coupling constant $\lambda = 2.7$.

Figure 4.6: Entanglement dynamics for the coupled quartic oscillator system with coupling constant (a) $\lambda = 0.4$, (b) $\lambda = 0.8$ and (c) $\lambda = 2.7$ in the quantum regime.

Figure 4.7: Entanglement dynamics for the coupled quartic oscillator system with coupling constant (a) $\lambda = 0.4$, (b) $\lambda = 0.8$ and (c) $\lambda = 2.7$. The upper curves (red) show results from the semiclassical treatment whereas the lower curves (blue) show results in the quantum regime.

4.3.1 Coupled harmonic oscillators

Our investigation is first carried out on the coupled harmonic oscillators with Hamiltonian given by Eq. (4.11) with initial states that are separable. From the numerical results illustrated in Fig. 4.8 with $\lambda = 11/61$, we notice that the *period* of the entanglement dynamics is independent of the choice of the initial state. However, the degree of entanglement is not independent of this choice. Since the maximum degree of the generated entanglement is dependent on the number of accessible quantum states, states with a larger mean energy is found to possess a greater degree of entanglement.

Our result in Fig. 4.8 (a) shows that when both the initial states of the oscillators are in the ground state, the resultant entanglement dynamics is similar to that when the initial states are coherent product state. It is periodic with two commensurate frequencies. By raising the initial state of one of the oscillator to the first excited state and then the second excited state, while the other remains in the ground state, one of the frequency is observed to disappear as illustrated in Fig. 4.8 (b). Now, if both of the oscillators are initially in the same excited states, entanglement dynamical pattern of a new form emerges. As shown in Figs. 4.8 (c), humps begin to appear in the pattern with the number of humps increases as the mean energy of the states increases. Next, let us consider initial state that is entangled. As illustrated in Fig. 4.8 (d), the entanglement between the two oscillators is found to oscillate. Essentially, the entanglement increases beyond its initial value implying that the initial state is not maximally entangled although it has the appearance of the maximally entangled state of a coupled qubit system. The inset of Fig. 4.8 (d) shows the oscillatory behavior of the entanglement dynamics of another initially entangled state. The amplitude of oscillation of states which are initially entangled is found to possess a much smaller value. Figure 4.9 shows similar results for the quasi-periodic entanglement dynamics when $\lambda = 0.19$.

In conclusion, the entanglement dynamics again depends on the underlying global classical dynamical regime when the initial state is the number product state. This

Figure 4.8: Entanglement generation from different initial states through the Hamiltonian of the coupled harmonic oscillators for $\lambda = 11/61$.

is observed for both the quasi-periodic and periodic entanglement dynamics of Figs. 4.8 and 4.9 which corresponds to the tori and periodic orbits in the classical domain respectively. However, unlike the case when the initial state is the coherent product state, the entanglement dynamics is dependent on the initial number product state. This is because the number product state does not correspond well to the classical state.

4.3.2 Coupled quartic oscillators

Next, let us investigate the generation of entanglement in the coupled quartic oscillators with Hamiltonian given by Eq. (4.17). For various initial number product states, we simulate the entropy dynamics of the system at $\lambda = 0.4, 0.8$ and 2.7 . The results are shown in Figs. 4.10, 4.11 and 4.12 respectively. The maximum degree of the generated entanglement is again found to depend on the mean energy. In contrast to the CHO, the maximum value of the entropy increases gradually as the coupling constant λ increases. Finally, we observe that the entanglement production rate is

Figure 4.9: Entanglement generation from different initial states through the Hamiltonian of the coupled harmonic oscillators for $\lambda = 0.19$.

much higher when the initial states are separable instead of entangled (refer to parts (a), (b), (c) and (d) of Figs. 4.10, 4.11 and 4.12 versus parts (e) and (f) of these figures).

Figure 4.10: Entanglement generation through the Hamiltonian of the coupled quartic oscillators at $\lambda = 0.4$ from the initial states: (a) $|0,0\rangle_a$, (b) $|0,1\rangle_a$, (c) $|1,1\rangle_a$, (d) $|2,3\rangle_a$, (e) $(|0,1\rangle_a + |1,0\rangle_a)/\sqrt{2}$ and (f) $(|0,0\rangle_a + |1,1\rangle_a + |2,2\rangle_a)/\sqrt{3}$.

Figure 4.11: Entanglement generation through the Hamiltonian of the coupled quartic oscillators at $\lambda = 0.8$ from the initial states: (a) $|0,0\rangle_a$, (b) $|0,1\rangle_a$, (c) $|1,1\rangle_a$, (d) $|2,3\rangle_a$, (e) $(|0,1\rangle_a + |1,0\rangle_a)/\sqrt{2}$ and (f) $(|0,0\rangle_a + |1,1\rangle_a + |2,2\rangle_a)/\sqrt{3}$.

Figure 4.12: Entanglement generation through the Hamiltonian of the coupled quartic oscillators at $\lambda = 2.7$ from the initial states: (a) $|0,0\rangle_a$, (b) $|0,1\rangle_a$, (c) $|1,1\rangle_a$, (d) $|2,3\rangle_a$, (e) $(|0,1\rangle_a + |1,0\rangle_a)/\sqrt{2}$ and (f) $(|0,0\rangle_a + |1,1\rangle_a + |2,2\rangle_a)/\sqrt{3}$.

Chapter 5

Conclusion

At the end of this thesis, we summarize what we have achieved and take a general view of the results presented in the last three chapters with regards to continuous variable entanglement. In Chapter 2, we have presented our development of a new approach which has enabled us to perform efficient numerical studies on the property of quantum entanglement for a general class of coupled oscillator systems. We have termed this method “the dynamical two-step approach”, which involves a Bogoliubov transformation with an application of the variational principle, followed by matrix diagonalization. The eigenvalues and eigenstates obtained from the diagonalization are then exploited, together with the transformation of the initial states, to yield the time-evolved states of high-dimensional quantum systems. By extending the original approach which is capable of determining precise energy eigenvalues within system of single anharmonic oscillator system with general polynomial potentials, the new procedure is able to determine both the quantum stationary states and quantum dynamics of coupled anharmonic oscillator systems with fast converging results. We have validated this technique against several coupled oscillator models.

In addition, we have employed our extended approach to study the relation between entanglement and quantum squeezing as well as quantum uncertainties within the ground state of a general class of coupled oscillator system. Interestingly, while the relation between quadrature squeezing and entanglement depends very much on the nonlinear potential of the quantum system, the influence of nonlinear perturba-

tion on the uncertainty-entropy relation is minimal especially when the entanglement between the two oscillators is large. Our results hence open up a new era for entanglement characterization in the ground states of quantum systems. We will continue to look into the relation between entanglement and entropy within the excited states of quantum systems in our future work.

Meanwhile, we have explored the link between classical trajectories and continuous variable entanglement dynamics in three pairs of two-coupled oscillator system. In the first model, quantization of periodic orbits yields periodic entanglement, whereas the quantization of saddle produces a rapid growth of entropy, with the transition occurs at $\lambda = 1$ for both the classical and quantum regimes. For the second model, the quantization of periodic orbits in the case of commensurate frequencies has led to periodic entanglement dynamics, while the quantization of both the periodic orbit and quasi-periodic tori in the case of incommensurate frequencies has led to quasi-periodic entanglement dynamics. Interestingly, the dynamical feature of the entanglement dynamics is solely parameterized by the two independent frequencies of the classical system. In the third model, quantization of either the regular, mixed or chaotic regimes of a coupled quartic oscillators yield entanglement dynamics that are again independent of the classical initial conditions, just like the first two models. This is unexpected as entanglement is generally known to depend on local dynamical behavior [101] and structure of the classical phase space [106], as well as non-classical features like the type of initial states [97] and their spectral width [107], for nonlinear systems. In conclusion, we have found that entanglement dynamics can be dependent on the global classical dynamical regime but independent of the local classical behavior in both integrable and non-integrable models. With entanglement being an important resource in quantum communication and computation, we would expect these “non-local” models to generate an encoding subspace [108] that is stable against any errors in the preparation of the initial separable coherent states. Such a feature will be physically significant in the design of robust quantum information processing protocols.

The results above on the dependence of entanglement dynamics on the global

classical dynamical regime have been obtained from initial coherent product states. In the last part of our work, the dependence of entanglement production on the chosen initial number product states is investigated. The maximum degree of the generated entanglement is found to depend on the mean energy in this case. On the other hand, this maximum is found to have a slight dependence on the coupling constant λ . In addition, states that are initially separable are observed to have a higher production rate of entanglement compared to states that are initially entangled.

After the investigation on coherent state and number product state, our next target is the squeezed product state. The squeezed product state has very interesting property and has been shown to be able to enhance the dynamical generation of entanglement in the finite dimensional systems. This will be part of our future work.

Appendix A

Zeroth-order approximation of a class of coupled quantum anharmonic oscillator systems

In this appendix, we provide a specific application of the Bogoliubov transformation and the variational approach to a class of two-coupled harmonic oscillator systems with its Hamiltonian perturb by a small anharmonic term:

$$H = \sum_{j=1}^2 \left(\frac{p_j^2}{2m} + \frac{1}{2} m \omega_0^2 x_j^2 \right) + \lambda (x_2 - x_1)^2 + \varepsilon \sum_{\nu=0}^{\kappa} \sum_{\mu=0}^{\kappa-\nu} c_{\mu\nu} x_1^{\mu} x_2^{\nu}, \quad (\text{A.1})$$

where ε is the perturbative constant and $c_{\mu\nu}$ are constant coefficients. Note that the polynomial of the anharmonic potential is up to degree κ . Furthermore, the unperturbed position uncertainty of this coupled anharmonic oscillator system is given by $\sigma_{x_1}^{(0)} = \sqrt{\hbar/2m\omega_0}$. Note that Eqs. (3.1) and (3.2) are special cases of Eq. (A.1).

After applying the Bogoliubov transformation to the perturbed potential, the Hamiltonian is re-expressed in terms of the new ladder operators $b_j^{\dagger}$ and b_j , as well as Ω , as follows:

$$\begin{aligned} H = & \sum_{j=1}^2 \hbar \omega_0 \left(\frac{1 + \Omega_j^2}{4\Omega_j} + \frac{1 + \Omega_j^2}{2\Omega_j} b_j^{\dagger} b_j + \frac{1 - \Omega_j^2}{4\Omega_j} (b_j^{\dagger 2} + b_j^2) \right) \\ & + \frac{\lambda}{2} \left\{ \frac{\beta_0}{\Omega_2} (b_2^{\dagger} + b_2) - \frac{\beta_0}{\Omega_1} (b_1^{\dagger} + b_1) \right\}^2 \end{aligned}$$

$$\begin{aligned}
& + \varepsilon \sum_{\nu=0}^{\kappa} \sum_{\mu=0}^{\kappa-\nu} c_{\mu\nu} \frac{\beta_0^{\mu+\nu}}{2^{(\mu+\nu)/2} \Omega_1^{\mu/2} \Omega_2^{\nu/2}} \sum_{p=0}^{[\mu/2]} \sum_{q=0}^{\mu-2p} \frac{\mu! b_1^{\dagger(\mu-2p-q)} b_1^q}{2^p p! q! (\mu-2p-q)!} \\
& \times \sum_{r=0}^{[\nu/2]} \sum_{s=0}^{\nu-2r} \frac{\nu b_2^{\dagger(\nu-2r-s)} b_2^s}{2^r r! s! (\nu-2r-s)!}, \tag{A.2}
\end{aligned}$$

where the symbol $[x]$ means rounding the value of x to the nearest lower integer.

Then, by applying the variational principle, we obtain the following equations:

$$\hbar\omega_0 \frac{\bar{\Omega}_1^2 - 1}{\bar{\Omega}_1} - \varepsilon \sum_{\nu=0,2}^{\kappa} \sum_{\mu=0,2}^{\kappa-\nu} \frac{c_{\mu\nu} \beta_0^{\mu} \beta_0^{\nu} \mu! \nu! (\mu/2)}{2^{(\mu+\nu-2)} (\mu/2)! (\nu/2)! \bar{\Omega}_1^{\mu/2} \bar{\Omega}_2^{\nu/2}} = 0, \tag{A.3}$$

$$\hbar\omega_0 \frac{\bar{\Omega}_2^2 - 1}{\bar{\Omega}_2} - \varepsilon \sum_{\nu=0,2}^{\kappa} \sum_{\mu=0,2}^{\kappa-\nu} \frac{c_{\mu\nu} \beta_0^{\mu} \beta_0^{\nu} \mu! \nu! (\nu/2)}{2^{(\mu+\nu-2)} (\mu/2)! (\nu/2)! \bar{\Omega}_1^{\mu/2} \bar{\Omega}_2^{\nu/2}} = 0. \tag{A.4}$$

The solutions of Eqs. (A.3) and (A.4) yield the optimal values $\bar{\Omega}_1$ and $\bar{\Omega}_2$. As we consider $c_{\mu\nu} = c_{\nu\mu}$ for the Hamiltonian in this paper, we have $\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}$. Furthermore, these equations guarantee that the coefficients of b_j^2 and $b_j^{\dagger 2}$ in $H - \lambda(x_2 - x_1)^2$ to be zero, while the coefficients of $b_j^{\dagger} b_j$ to be $\hbar\omega_0 \bar{\Omega}$, leading to the following form for the Hamiltonian equation:

$$\begin{aligned}
H &= \sum_{j=1}^2 \hbar\omega_0 \bar{\Omega} \left(b_j^{\dagger} b_j + \frac{1}{2} \right) + \frac{\lambda \beta_0}{2\bar{\Omega}} \left\{ (b_2^{\dagger} + b_2) - (b_1^{\dagger} + b_1) \right\}^2 + O^3(b_j^{\dagger}, b_j) \\
&= \frac{p_1^2}{2m} + \frac{1}{2} m (\omega_0 \bar{\Omega})^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2} m (\omega_0 \bar{\Omega})^2 x_2^2 + \lambda (x_2 - x_1)^2 + O^3(x_j, p_j). \tag{A.5}
\end{aligned}$$

If we were to neglect the perturbative potential (which is the last term of Eq. (A.5)) whose effects have become much smaller after the Bogoliubov transformation, we obtain the zeroth-order approximation of the Hamiltonian equation. In this zeroth-order limit, the ground state position uncertainty σ_{x_1} of the original pair of coupled anharmonic oscillators is equal to that of a pair of linearly coupled harmonic oscillators with a frequency of $\omega_0 \bar{\Omega}$, i.e.,

$$\sigma_{x_1} = \sqrt{\frac{\hbar}{2m\omega_0 \bar{\Omega}} \frac{1 + \gamma'}{2}} \tag{A.6}$$

with $\gamma' = 1/\sqrt{1 + 4\lambda/m(\omega_0 \bar{\Omega})^2}$. Dividing this uncertainty by $\sigma_{x_1}^{(0)}$, we obtain the zeroth-order approximation of the squeezing parameter of the coupled anharmonic

oscillator systems:

$$S_x = -\ln \sqrt{\frac{1 + \gamma'}{2\bar{\Omega}}} . \quad (\text{A.7})$$

The zeroth-order limit of the entanglement entropy is then given by

$$S_{vN} = 2 \left\{ \cosh^2 \left(\frac{\ln \gamma'}{4} \right) \ln \left[\cosh \left(\frac{\ln \gamma'}{4} \right) \right] - \sinh^2 \left(\frac{\ln \gamma'}{4} \right) \ln \left[\sinh \left(\frac{\ln \gamma'}{4} \right) \right] \right\} . \quad (\text{A.8})$$

By eliminating γ' in Eqs. (A.7) and (A.8), we arrive at an analytical expression between S_{vN} and S_x :

$$S_{vN} = \frac{(\xi' + 1)^2}{4\xi'} \ln \left(\frac{(\xi' + 1)^2}{4\xi'} \right) - \frac{(\xi' - 1)^2}{4\xi'} \ln \left(\frac{(\xi' - 1)^2}{4\xi'} \right) , \quad (\text{A.9})$$

where

$$\xi' = \left(2\bar{\Omega} e^{-2S_x} - 1 \right)^{1/2} . \quad (\text{A.10})$$

The presence of $\bar{\Omega}$ in Eq. (A.10) indicates that the entropy-squeezing relation is non-unique and depends on the anharmonicities of the two-coupled anharmonic oscillator systems.

Similarly, the zeroth-order solution of the squeezing parameter in the momentum quadrature can be obtained:

$$S_p = -\ln \sqrt{\frac{\bar{\Omega}(1 + \gamma')}{2\sqrt{\gamma'}}} . \quad (\text{A.11})$$

This leads to the normalized product of position and momentum uncertainties as follows:

$$\bar{U} = \frac{1 + \gamma'}{2\sqrt{\gamma'}} . \quad (\text{A.12})$$

By eliminating γ' in Eqs. (A.12) and (A.8), we arrive at the following relation

$$S_{vN} = \frac{1}{2}(\bar{U} + 1) \ln \left(\frac{1}{2}(\bar{U} + 1) \right) - \frac{1}{2}(\bar{U} - 1) \ln \left(\frac{1}{2}(\bar{U} - 1) \right) . \quad (\text{A.13})$$

Notice that this relation does not depend on $\bar{\Omega}$ explicitly. Hence, anharmonicities have no influence on the entropy-uncertainty relation in the zeroth-order limit.

Appendix B

Normal ordered form

The ordering of the operators plays a crucial role in quantum mechanics. For any operator function $f(a, a^\dagger)$ of the usual bosonic annihilation and creation operators a and $a^\dagger$, which satisfy the commutation relation

$$[a, a^\dagger] = 1, \quad (\text{B.1})$$

one can write $f(a, a^\dagger)$ in such a way that all powers of $a^\dagger$ always appear to the left of all powers of a . Then one will obtain the normal ordered form of $f(a, a^\dagger)$. In this appendix, we want to write $(sa^\dagger + a)^n$ in the normal ordered form for integral values of n . We shall proceed by using the following lemma:

Lemma 1

$$: (sa^\dagger + a)^n : (sa^\dagger + a) = : (sa^\dagger + a)^{n+1} : + ns : (sa^\dagger + a)^{n-1} : . \quad (\text{B.2})$$

Note that $: (sa^\dagger + a)^n :$ is a binomial expansion in which powers of the $a^\dagger$ are always kept to the left of the powers of a .

Proof:

$$\begin{aligned}
& : (sa^\dagger + a)^n : (sa^\dagger + a) \\
&= \{s^n a^{\dagger n} + {}^nC_1 s^{n-1} a^{\dagger n-1} a + \dots + {}^nC_r s^{n-r} a^{\dagger n-r} a^r + \dots + a^n\} \{sa^\dagger + a\} \\
&= \{s^{n+1} a^{\dagger n+1} + {}^nC_1 s^n a^{\dagger n-1} (a^\dagger a + 1) + \dots + {}^nC_r s^{n-r+1} a^{\dagger n-r} (a^\dagger a^r + r a^{r-1}) + \dots \\
&\quad + s(a^\dagger a^n + n a^{n-1})\} + \{s^n a^{\dagger n} a + {}^nC_1 s^{n-1} a^{\dagger n-1} a^2 + \dots + {}^nC_r s^{n-r} a^{\dagger n-r} a^{r+1} \\
&\quad + \dots + a^{n+1}\} \\
&= \{s^{n+1} a^{\dagger n+1} + ({}^nC_1 + {}^nC_0) s^n a^{\dagger n} a + \dots + ({}^nC_r + {}^nC_{r-1}) s^{n-r+1} a^{\dagger n-r+1} a^r + \dots \\
&\quad + a^{n+1}\} + \{{}^nC_1 s^n a^{\dagger n-1} + \dots + r {}^nC_r s^{n-r+1} a^{\dagger n-r} a^{r-1} + \dots + s n a^{n-1}\} \\
&= \{s^{n+1} a^{\dagger n+1} + {}^{n+1}C_1 s^n a^{\dagger n} a + \dots + {}^{n+1}C_r s^{n-r+1} a^{\dagger n-r+1} a^r + \dots + a^{n+1}\} \\
&\quad + \{n s^n a^{\dagger n-1} + \dots + n {}^{n-1}C_{r-1} s^{n-r+1} a^{\dagger n-r} a^{r-1} + \dots + s n a^{n-1}\} \\
&=: (sa^\dagger + a)^{n+1} : + n s : (sa^\dagger + a)^{n-1} : .
\end{aligned}$$

Note that we have used the following identities in the above:

$$(a^r a^\dagger) = a^\dagger a^r + r a^{r-1} \quad (\text{B.3})$$

$$r {}^nC_r = n {}^{n-1}C_{r-1} \quad (\text{B.4})$$

$${}^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1} . \quad (\text{B.5})$$

Using lemma 1, we can write

$$\begin{aligned}
(sa^\dagger + a) &= : (sa^\dagger + a) : \\
(sa^\dagger + a)^2 &= : (sa^\dagger + a)^2 : + s \\
(sa^\dagger + a)^3 &= : (sa^\dagger + a)^3 : + 3s : (sa^\dagger + a) : \\
(sa^\dagger + a)^4 &= : (sa^\dagger + a)^4 : + 6s : (sa^\dagger + a)^2 : + 3s^2 . \quad (\text{B.6})
\end{aligned}$$

From Eq.(B.6), we can identify the general normal ordered form of $(sa^\dagger + a)^n$ as

$$(s a^\dagger + a)^n = \sum_{r=0,2,\dots}^{[n]_e} \epsilon_r {}^nC_r s^{r/2} : (s a^\dagger + a)^{n-r} : , \quad (\text{B.7})$$

with

$$\epsilon_r = \frac{(r-1)!}{2^{r/2-1}(r/2-1)!} \quad \text{for } r \geq 4 \quad (\text{B.8})$$

and

$$\epsilon_0 = \epsilon_2 = 1. \quad (\text{B.9})$$

Note that the symbol $[\cdot]_e$ means taking the value of the nearest smaller even integer. It can be easily verified that Eq.(B.7) gives all the expansion of Eq.(B.6) such that Eq.(B.7) is true for $n = 1, 2, 3$ and 4. Now, we prove the validity of Eq.(B.7) by means of mathematical induction.

Suppose

$$(s a^\dagger + a)^m = \sum_{r=0,2,\dots}^{[m]_e} \epsilon_r {}^m C_r s^{r/2} : (s a^\dagger + a)^{m-r} :, \quad (\text{B.10})$$

is true for some integers m , we shall show that it remains true for the integer $m + 1$:

$$\begin{aligned} & (s a^\dagger + a)^{m+1} \\ &= (s a^\dagger + a)^m (s a^\dagger + a) \\ &= \sum_{r=0,2,\dots}^{[m]_e} \epsilon_r {}^m C_r s^{r/2} : (s a^\dagger + a)^{m-r} : (s a^\dagger + a) \\ &= \sum_{r=0,2,\dots}^{[m]_e} \epsilon_r {}^m C_r s^{r/2} \{ : (s a^\dagger + a)^{m-r+1} : + (m-r)s : (s a^\dagger + a)^{m-1-r} : \} \\ &= \sum_{r=0,2,\dots}^{[m]_e} \epsilon_r {}^m C_r s^{r/2} : (s a^\dagger + a)^{m-r+1} : \\ &+ \sum_{q=0,2,\dots}^{[m]_e} \epsilon_q {}^m C_q s^{q/2+1} (m-q) : (s a^\dagger + a)^{m-1-q} : \\ &= \epsilon_0 {}^m C_0 : (a^\dagger + a)^{m+1} : + \sum_{r=2,\dots}^{[m]_e} \epsilon_r {}^m C_r s^{r/2} : (s a^\dagger + a)^{m-r+1} : \\ &+ \sum_{q=0,2,\dots}^{[m]_e-2} \epsilon_q {}^m C_q s^{q/2+1} (m-q) : (s a^\dagger + a)^{m-1-q} : \\ &+ \epsilon_{[m]_e} {}^m C_{[m]_e} s^{[m]_e/2+1} (m - [m]_e) : (s a^\dagger + a)^{m-1-[m]_e} : \end{aligned}$$

Note that the coefficients of the terms $: (s a^\dagger + a)^{m+1-r} :$ are given by:

$$\begin{aligned} & \epsilon_r {}^m C_r s^{r/2} + \epsilon_{r-2} {}^m C_{r-2} s^{r/2-1} (m-r+2) s \\ &= \frac{(r-1)!}{2^{r/2-1}(r/2-1)!} \frac{m!}{(m-r)! r!} s^{r/2} + \frac{(r-3)!}{2^{r/2-2}(r/2-2)!} \frac{m!(m-r+2)}{(m-r+2)!(r-2)!} s^{r/2} \\ &= \frac{(r-1)!}{2^{r/2-1}(r/2-1)!} \frac{m! s^{r/2}}{(m-r+1)! r!} \left\{ (m-r+1) + \frac{2(r/2-1)r(r-1)}{(r-2)(r-1)} \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{(r-1)!}{2^{r/2-1}(r/2-1)!} \frac{m!(m+1)s^{r/2}}{(m-r+1)!r!} \\
&= \epsilon_r^{m+1} C_r s^{r/2}.
\end{aligned} \tag{B.11}$$

A comprehensive treatment requires the consideration when m is even or odd. If m is even,

$$(s a^\dagger + a)^{m+1} \tag{B.12}$$

$$\begin{aligned}
&=: (a^\dagger + a)^{m+1} : + \sum_{r=2, \dots}^m \epsilon_r^{m+1} C_r s^{r/2} : (s a^\dagger + a)^{m+1-r} : \\
&= \sum_{r=0, 2, \dots}^{[m+1]_e} \epsilon_r^{m+1} C_r s^{r/2} : (s a^\dagger + a)^{m+1-r} :,
\end{aligned} \tag{B.13}$$

and if m is odd,

$$(s a^\dagger + a)^{m+1} \tag{B.14}$$

$$\begin{aligned}
&=: (a^\dagger + a)^{m+1} : + \sum_{r=2, \dots}^{m-1} \epsilon_r^{m+1} C_r s^{r/2} : (s a^\dagger + a)^{m+1-r} : + \epsilon_{m-1}^m C_{m-1} \\
&=: (a^\dagger + a)^{m+1} : + \sum_{r=2, \dots}^{m-1} \epsilon_r^{m+1} C_r s^{r/2} : (s a^\dagger + a)^{m+1-r} : + \epsilon_{m+1}^{m+1} C_{m+1} \\
&= \sum_{r=0, 2, \dots}^{[m+1]_e} \epsilon_r^{m+1} C_r s^{r/2} : (s a^\dagger + a)^{m+1-r} :.
\end{aligned} \tag{B.15}$$

Hence, proving Eq. (B.7).

Appendix C

Classical dynamics

C.1 Two-mode magnon system

The 4-dimensional linear dynamical system of the two-mode magnon system can be written in the following form:

$$\begin{aligned}\dot{\mathbf{y}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{m} & 0 & -\frac{\lambda}{m\omega_0} \\ -m\omega_0^2 & 0 & -\lambda m\omega_0 & 0 \\ 0 & -\frac{\lambda}{m\omega_0} & 0 & \frac{1}{m} \\ -\lambda m\omega_0 & 0 & -m\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix} \\ &= A\mathbf{y}.\end{aligned}\tag{C.1}$$

Given the initial condition $\mathbf{y}(t_0)$, the general solution is given by

$$\mathbf{y}(t) = e^{At}\mathbf{y}(t_0) = Ee^{\Lambda t}E^{-1}\mathbf{y}(t_0),\tag{C.2}$$

where

$$e^{\Lambda t} = \begin{pmatrix} e^{\zeta_1} & 0 & 0 & 0 \\ 0 & e^{\zeta_2} & 0 & 0 \\ 0 & 0 & e^{\zeta_3} & 0 \\ 0 & 0 & 0 & e^{\zeta_4} \end{pmatrix}, \quad (\text{C.3})$$

and

$$E = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{pmatrix}. \quad (\text{C.4})$$

Note that ζ_j and e_j ($j=1,2,3,4$) are the eigenvalues and eigenvectors of the matrix A .

The two distinct eigenvalues are found to be

$$\zeta_{1,2} = \pm \sqrt{\lambda^2 - \omega_0^2}. \quad (\text{C.5})$$

For $0 < \lambda^2 < \omega_0^2$, the eigenvalues are purely imaginary. Hence, the resultant classical motion is periodic with frequency of $|\zeta_1|$.

C.2 Coupled harmonic oscillators

The 4-dimensional linear dynamical system of the coupled harmonic oscillators system can be written in the following form:

$$\begin{aligned} \dot{\mathbf{y}} &= \begin{pmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ -m\omega_0^2 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{m} \\ -\lambda & 0 & -m\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix} \\ &= A\mathbf{y}. \end{aligned} \quad (\text{C.6})$$

Eigenvalues of matrix A are $\zeta = \pm i\sqrt{\omega_0^2 \pm \lambda}$, which are pure imaginary numbers for $-\omega_0^2 < \lambda < \omega_0^2$, the classical motions are thus two-frequency tori.

Appendix D

Time evolution of quantum fluctuation

D.1 Two-mode magnon system

By Heisenberg equation of motion

$$\begin{aligned}\frac{da_1}{dt} &= \frac{1}{i}[a_1, H] \\ &= -i[a_1, a_1^\dagger a_1 + a_2^\dagger a_2 + \lambda a_1^\dagger a_2^\dagger + \lambda a_1 a_2] \\ &= -i\{a_1^\dagger[a_1, a_1] + [a_1, a_1^\dagger]a_1 + a_2^\dagger[a_1, a_2] + [a_1, a_2^\dagger]a_2 \\ &\quad + \lambda[a_1, a_1^\dagger]a_2^\dagger + \lambda a_1^\dagger[a_1, a_2^\dagger] + \lambda a_1[a_1, a_2] + \lambda[a_1, a_1]a_2\} \\ &= -i\{a_1 + \lambda a_2^\dagger\},\end{aligned}\tag{D.1}$$

and

$$\begin{aligned}\frac{da_2^\dagger}{dt} &= \frac{1}{i}[a_2^\dagger, H] \\ &= -i[a_2^\dagger, a_1^\dagger a_1 + a_2^\dagger a_2 + \lambda a_1^\dagger a_2^\dagger + \lambda a_1 a_2] \\ &= -i\{a_1^\dagger[a_2^\dagger, a_1] + [a_2^\dagger, a_1^\dagger]a_1 + a_2^\dagger[a_2^\dagger, a_2] + [a_2^\dagger, a_2^\dagger]a_2 \\ &\quad + \lambda[a_2^\dagger, a_1^\dagger]a_2^\dagger + \lambda a_1^\dagger[a_2^\dagger, a_2^\dagger] + \lambda a_1[a_2^\dagger, a_2] + \lambda[a_2^\dagger, a_1]a_2\} \\ &= i\{a_2^\dagger + \lambda a_1\}.\end{aligned}\tag{D.2}$$

Hence,

$$\frac{d}{dt} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} -i & -i\lambda \\ i\lambda & i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix}. \quad (\text{D.3})$$

Let

$$A = \begin{pmatrix} -i & -i\lambda \\ i\lambda & i \end{pmatrix}, \quad (\text{D.4})$$

we then obtain the eigenvalues: $\zeta_{1,2} = \pm\sqrt{\lambda^2 - 1} = \pm\kappa$. The corresponding eigenvectors are

$$\mathbf{e}_{1,2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{1}{\lambda} \pm \sqrt{\frac{1}{\lambda^2} - 1} \end{pmatrix}. \quad (\text{D.5})$$

Given the initial conditions $a_1(0)$ and $a_2^\dagger(0)$, the general solution is given by

$$\begin{pmatrix} a_1(t) \\ a_2^\dagger(t) \end{pmatrix} = e^{At} \begin{pmatrix} a_1(0) \\ a_2^\dagger(0) \end{pmatrix} = E e^{\Lambda t} E^{-1} \begin{pmatrix} a_1(0) \\ a_2^\dagger(0) \end{pmatrix}, \quad (\text{D.6})$$

where

$$e^{\Lambda t} = \begin{pmatrix} e^{\zeta_1} & 0 \\ 0 & e^{\zeta_2} \end{pmatrix}, \quad (\text{D.7})$$

and

$$E = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}. \quad (\text{D.8})$$

This gives

$$a_1(t) = \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] a_1(0) - \frac{i\lambda}{\kappa} \sinh(\kappa t) a_2^\dagger(0), \quad (\text{D.9})$$

$$a_2^\dagger(t) = \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right] a_2^\dagger(0) + \frac{i\lambda}{\kappa} \sinh(\kappa t) a_1(0). \quad (\text{D.10})$$

Note that

$$\langle a_1^\dagger(0) a_1(0) \rangle - \langle a_1^\dagger(0) \rangle \langle a_1(0) \rangle = 0, \quad (\text{D.11})$$

$$\langle a_2^\dagger(0) a_2(0) \rangle - \langle a_2^\dagger(0) \rangle \langle a_2(0) \rangle = 0, \quad (\text{D.12})$$

$$\langle a_1^\dagger(0) a_2^\dagger(0) \rangle - \langle a_1^\dagger(0) \rangle \langle a_2^\dagger(0) \rangle = 0, \quad (\text{D.13})$$

$$\langle a_1(0) a_2(0) \rangle - \langle a_1(0) \rangle \langle a_2(0) \rangle = 0. \quad (\text{D.14})$$

Now,

$$\begin{aligned}
a_1^\dagger(t) a_1(t) &= \left[\cosh^*(\kappa t) + \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] a_1^\dagger(0) a_1(0) \\
&\quad - \frac{i\lambda}{\kappa} \sinh(\kappa t) \left[\cosh^*(\kappa t) + \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_1^\dagger(0) a_2^\dagger(0) \\
&\quad + \frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] a_2(0) a_1(0) \\
&\quad + \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2 a_2(0) a_2^\dagger(0)
\end{aligned} \tag{D.15}$$

$$\begin{aligned}
&= \left[\cosh^*(\kappa t) + \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] a_1^\dagger(0) a_1(0) \\
&\quad - \frac{i\lambda}{\kappa} \sinh(\kappa t) \left[\cosh^*(\kappa t) + \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_1^\dagger(0) a_2^\dagger(0) \\
&\quad + \frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] a_2(0) a_1(0) \\
&\quad + \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2 (a_2^\dagger(0) a_2(0) + 1).
\end{aligned} \tag{D.16}$$

Hence,

$$\langle a_1^\dagger(t) a_1(t) \rangle - \langle a_1^\dagger(t) \rangle \langle a_1(t) \rangle = \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2. \tag{D.17}$$

Similarly,

$$\begin{aligned}
a_2^\dagger(t) a_2(t) &= \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right] \left[\cosh^*(\kappa t) - \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_2^\dagger(0) a_2(0) \\
&\quad - \frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right] a_2^\dagger(0) a_1^\dagger(0) \\
&\quad + \frac{i\lambda}{\kappa} \sinh(\kappa t) \left[\cosh^*(\kappa t) - \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_1(0) a_2(0) \\
&\quad + \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2 (a_1^\dagger(0) a_1(0) + 1)
\end{aligned} \tag{D.18}$$

leads to

$$\langle a_2^\dagger(t) a_2(t) \rangle - \langle a_2^\dagger(t) \rangle \langle a_2(t) \rangle = \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2. \tag{D.19}$$

Furthermore,

$$\begin{aligned}
a_1^\dagger(t) a_2^\dagger(t) &= \left[\cosh^*(\kappa t) + \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right] a_1^\dagger(0) a_2^\dagger(0) \\
&\quad + \frac{i\lambda}{\kappa} \sinh(\kappa t) \left[\cosh^*(\kappa t) + \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_1^\dagger(0) a_1(0) \\
&\quad + \frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right] (a_2^\dagger(0) a_2(0) + 1) \\
&\quad - \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2 a_2(0) a_1(0)
\end{aligned} \tag{D.20}$$

results in

$$\langle a_1^\dagger(t) a_2^\dagger(t) \rangle - \langle a_1^\dagger(t) \rangle \langle a_2^\dagger(t) \rangle = \frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) + \frac{i}{\kappa} \sinh(\kappa t) \right], \quad (\text{D.21})$$

while

$$\begin{aligned} a_1(t) a_2(t) &= \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] \left[\cosh^*(\kappa t) - \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_1(0) a_2(0) \\ &\quad - \frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right] (a_1^\dagger(0) a_1(0) + 1) \\ &\quad - \frac{i\lambda}{\kappa} \sinh(\kappa t) \left[\cosh^*(\kappa t) - \frac{i}{\kappa^*} \sinh^*(\kappa t) \right] a_2^\dagger(0) a_2(0) \\ &\quad - \frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2 a_2^\dagger(0) a_1^\dagger(0) \end{aligned} \quad (\text{D.22})$$

leads to

$$\langle a_1(t) a_2(t) \rangle - \langle a_1(t) \rangle \langle a_2(t) \rangle = -\frac{i\lambda}{\kappa^*} \sinh^*(\kappa t) \left[\cosh(\kappa t) - \frac{i}{\kappa} \sinh(\kappa t) \right]. \quad (\text{D.23})$$

Therefore,

$$\begin{aligned} \xi(t) &= 2(\langle a_1^\dagger a_1 \rangle - \langle a_1^\dagger \rangle \langle a_1 \rangle + \langle a_2^\dagger a_2 \rangle - \langle a_2^\dagger \rangle \langle a_2 \rangle + \\ &\quad \langle a_1^\dagger a_2^\dagger \rangle - \langle a_1^\dagger \rangle \langle a_2^\dagger \rangle + \langle a_1 a_2 \rangle - \langle a_1 \rangle \langle a_2 \rangle) \\ &= 2\{2\frac{\lambda^2}{|\kappa|^2} |\sinh(\kappa t)|^2 - 2\frac{\lambda}{|\kappa|^2} |\sinh(\kappa t)|^2\} \\ &= \frac{4\lambda}{|\kappa|^2} (\lambda - 1) |\sinh(t\sqrt{\lambda^2 - 1})|^2. \end{aligned} \quad (\text{D.24})$$

D.2 Coupled harmonic oscillators

By Heisenberg equation of motion

$$\begin{aligned} \frac{da_1}{dt} &= \frac{1}{i} [a_1, H] \\ &= -i \left[a_1, \left(a_1^\dagger a_1 + a_2^\dagger a_2 + \frac{\lambda}{2} (a_1^\dagger a_2^\dagger + a_1^\dagger a_2 + a_1 a_2^\dagger + a_1 a_2) \right) \right] \\ &= -i \left\{ a_1 + \frac{\lambda}{2} a_2^\dagger + \frac{\lambda}{2} a_2 \right\}, \end{aligned} \quad (\text{D.25})$$

$$\begin{aligned} \frac{da_1^\dagger}{dt} &= \frac{1}{i} [a_1^\dagger, H] \\ &= -i \left[a_1^\dagger, \left(a_1^\dagger a_1 + a_2^\dagger a_2 + \frac{\lambda}{2} (a_1^\dagger a_2^\dagger + a_1^\dagger a_2 + a_1 a_2^\dagger + a_1 a_2) \right) \right] \\ &= i \left\{ a_1^\dagger + \frac{\lambda}{2} a_2^\dagger + \frac{\lambda}{2} a_2 \right\}, \end{aligned} \quad (\text{D.26})$$

$$\begin{aligned}
\frac{da_2}{dt} &= \frac{1}{i}[a_2, H] \\
&= -i \left[a_2, \left(a_1^\dagger a_1 + a_2^\dagger a_2 + \frac{\lambda}{2}(a_1^\dagger a_2^\dagger + a_1^\dagger a_2 + a_1 a_2^\dagger + a_1 a_2) \right) \right] \\
&= -i \left\{ a_2 + \frac{\lambda}{2} a_1^\dagger + \frac{\lambda}{2} a_1 \right\},
\end{aligned} \tag{D.27}$$

and

$$\begin{aligned}
\frac{da_2^\dagger}{dt} &= \frac{1}{i}[a_2^\dagger, H] \\
&= -i \left[a_2^\dagger, \left(a_1^\dagger a_1 + a_2^\dagger a_2 + \frac{\lambda}{2}(a_1^\dagger a_2^\dagger + a_1^\dagger a_2 + a_1 a_2^\dagger + a_1 a_2) \right) \right] \\
&= i \left\{ a_2^\dagger + \frac{\lambda}{2} a_1^\dagger + \frac{\lambda}{2} a_1 \right\}.
\end{aligned} \tag{D.28}$$

Thus,

$$\frac{d}{dt} \begin{pmatrix} a_1 \\ a_1^\dagger \\ a_2 \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} -i & 0 & -i\frac{\lambda}{2} & -i\frac{\lambda}{2} \\ 0 & i & i\frac{\lambda}{2} & i\frac{\lambda}{2} \\ -i\frac{\lambda}{2} & -i\frac{\lambda}{2} & -i & 0 \\ i\frac{\lambda}{2} & i\frac{\lambda}{2} & 0 & i \end{pmatrix} \begin{pmatrix} a_1 \\ a_1^\dagger \\ a_2 \\ a_2^\dagger \end{pmatrix}. \tag{D.29}$$

Similar to the previous section, we let

$$A = \begin{pmatrix} -i & 0 & -i\frac{\lambda}{2} & -i\frac{\lambda}{2} \\ 0 & i & i\frac{\lambda}{2} & i\frac{\lambda}{2} \\ -i\frac{\lambda}{2} & -i\frac{\lambda}{2} & -i & 0 \\ i\frac{\lambda}{2} & i\frac{\lambda}{2} & 0 & i \end{pmatrix}, \tag{D.30}$$

and obtain the eigenvalues: $\zeta_{1,2} = \pm\sqrt{1+\lambda} = \pm\omega_1$ and $\zeta_{3,4} = \pm\sqrt{1-\lambda} = \pm\omega_2$. The corresponding eigenvectors are

$$\mathbf{e}_{1,2} = \begin{pmatrix} -\frac{\omega_1 \mp 1}{\omega_1 \pm 1} \\ 1 \\ -\frac{\omega_1 \mp 1}{\omega_1 \pm 1} \\ 1 \end{pmatrix}, \tag{D.31}$$

and

$$\mathbf{e}_{3,4} = \begin{pmatrix} -\frac{\omega_2 \mp 1}{\omega_2 \pm 1} \\ 1 \\ -\frac{\omega_2 \mp 1}{\omega_2 \pm 1} \\ -1 \end{pmatrix}. \quad (\text{D.32})$$

Given the initial conditions $a_1(0)$, $a_2(0)$, $a_1^\dagger(0)$ and $a_2^\dagger(0)$, the general solution is given by

$$\begin{pmatrix} a_1(t) \\ a_1^\dagger(t) \\ a_2(t) \\ a_2^\dagger(t) \end{pmatrix} = e^{At} \begin{pmatrix} a_1(0) \\ a_1^\dagger(0) \\ a_2(0) \\ a_2^\dagger(0) \end{pmatrix} = E e^{\Lambda t} E^{-1} \begin{pmatrix} a_1(0) \\ a_1^\dagger(0) \\ a_2(0) \\ a_2^\dagger(0) \end{pmatrix}, \quad (\text{D.33})$$

where

$$e^{\Lambda t} = \begin{pmatrix} e^{\zeta_1} & 0 & 0 & 0 \\ 0 & e^{\zeta_2} & 0 & 0 \\ 0 & 0 & e^{\zeta_3} & 0 \\ 0 & 0 & 0 & e^{\zeta_4} \end{pmatrix}, \quad (\text{D.34})$$

and

$$E = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{pmatrix}. \quad (\text{D.35})$$

This gives

$$\begin{aligned} a_1(t) &= (-A_1 + B_1 - A_2 + B_2) a_1(0) - (C_1 + C_2) a_1^\dagger(0) \\ &+ (-A_1 + B_1 + A_2 - B_2) a_2(0) - (C_1 - C_2) a_2^\dagger(0), \end{aligned} \quad (\text{D.36})$$

$$\begin{aligned} a_1^\dagger(t) &= (-A_1^* + B_1^* - A_2^* + B_2^*) a_1^\dagger(0) - (C_1^* + C_2^*) a_1(0) \\ &+ (-A_1^* + B_1^* + A_2^* - B_2^*) a_2^\dagger(0) - (C_1^* - C_2^*) a_2(0), \end{aligned} \quad (\text{D.37})$$

$$\begin{aligned} a_2(t) &= -(C_1 - C_2) a_1^\dagger(0) + (B_1 - A_1 - B_2 + A_2) a_1(0) \\ &- (C_1 + C_2) a_2^\dagger(0) + (B_1 - A_1 + B_2 - A_2) a_2(0), \end{aligned} \quad (\text{D.38})$$

$$a_2^\dagger(t) = -(C_1^* - C_2^*) a_1(0) + (B_1^* - A_1^* - B_2^* + A_2^*) a_1^\dagger(0)$$

$$- (C_1^* + C_2^*) a_2(0) + (B_1^* - A_1^* + B_2^* - A_2^*) a_2^\dagger(0), \quad (\text{D.39})$$

where

$$\begin{aligned} A_{1,2} &= \frac{(\omega_{1,2} - 1)^2}{8\omega_{1,2}} e^{i\omega_{1,2}t}, \\ B_{1,2} &= \frac{(\omega_{1,2} + 1)^2}{8\omega_{1,2}} e^{-i\omega_{1,2}t}, \\ C_{1,2} &= \frac{2i(\omega_{1,2} - 1)(\omega_{1,2} + 1)}{8\omega_{1,2}} \sin(\omega_{1,2}t). \end{aligned}$$

By using Eqs. (D.11), (D.12), (D.13) and (D.14), we obtain the following results:

$$\begin{aligned} &\langle a_1^\dagger(t) a_1(t) \rangle - \langle a_1^\dagger(t) \rangle \langle a_1(t) \rangle \\ &= (C_1^* + C_2^*)(C_1 + C_2) + (C_1^* - C_2^*)(C_1 - C_2) \\ &= 2(C_1^* C_1 + C_2^* C_2), \end{aligned} \quad (\text{D.40})$$

$$\begin{aligned} &\langle a_2^\dagger(t) a_2(t) \rangle - \langle a_2^\dagger(t) \rangle \langle a_2(t) \rangle \\ &= (C_1^* - C_2^*)(C_1 - C_2) + (C_1^* + C_2^*)(C_1 + C_2) \\ &= 2(C_1^* C_1 + C_2^* C_2), \end{aligned} \quad (\text{D.41})$$

$$\begin{aligned} &\langle a_1^\dagger(t) a_2^\dagger(t) \rangle - \langle a_1^\dagger(t) \rangle \langle a_2^\dagger(t) \rangle \\ &= -(C_1^* + C_2^*)(B_1^* - A_1^* - B_2^* + A_2^*) - (C_1^* - C_2^*)(B_1^* - A_1^* + B_2^* - A_2^*) \\ &= 2C_1^*(A_1^* - B_1^*) - 2C_2^*(A_2^* - B_2^*), \end{aligned} \quad (\text{D.42})$$

$$\begin{aligned} &\langle a_1(t) a_2(t) \rangle - \langle a_1(t) \rangle \langle a_2(t) \rangle \\ &= -(C_1 - C_2)(-A_1 + B_1 - A_2 + B_2) - (C_1 + C_2)(-A_1 + B_1 + A_2 - B_2) \\ &= 2C_1(A_1 - B_1) - 2C_2(A_2 - B_2). \end{aligned} \quad (\text{D.43})$$

Hence,

$$\begin{aligned}
\xi(t) &= 2(\langle a_1^\dagger a_1 \rangle - \langle a_1^\dagger \rangle \langle a_1 \rangle + \langle a_2^\dagger a_2 \rangle - \langle a_2^\dagger \rangle \langle a_2 \rangle + \langle a_1^\dagger a_2^\dagger \rangle - \langle a_1^\dagger \rangle \langle a_2^\dagger \rangle + \langle a_1 a_2 \rangle \\
&\quad - \langle a_1 \rangle \langle a_2 \rangle) \\
&= 8(C_1^* C_1 + C_2^* C_2) + 4C_1^*(A_1^* - B_1^*) - 4C_2^*(A_2^* - B_2^*) \\
&\quad + 4C_1(A_1 - B_1) - 4C_2(A_2 - B_2)\} \\
&= \frac{(\omega_1 - 1)^2(\omega_1 + 1)^2}{2\omega_1^2} \sin^2(\omega_1 t) + \frac{(\omega_2 - 1)^2(\omega_2 + 1)^2}{2\omega_2^2} \sin^2(\omega_2 t) \\
&\quad - \frac{8i(\omega_1 - 1)(\omega_1 + 1)}{8\omega_1} \sin(\omega_1 t) \left(\frac{(\omega_1 - 1)^2}{8\omega_1} e^{-i\omega_1 t} - \frac{(\omega_1 + 1)^2}{8\omega_1} e^{i\omega_1 t} \right) \\
&\quad + \frac{8i(\omega_2 - 1)(\omega_2 + 1)}{8\omega_2} \sin(\omega_2 t) \left(\frac{(\omega_2 - 1)^2}{8\omega_2} e^{-i\omega_2 t} - \frac{(\omega_2 + 1)^2}{8\omega_2} e^{i\omega_2 t} \right) \\
&\quad + \frac{8i(\omega_1 - 1)(\omega_1 + 1)}{8\omega_1} \sin(\omega_1 t) \left(\frac{(\omega_1 - 1)^2}{8\omega_1} e^{i\omega_1 t} - \frac{(\omega_1 + 1)^2}{8\omega_1} e^{-i\omega_1 t} \right) \\
&\quad - \frac{8i(\omega_2 - 1)(\omega_2 + 1)}{8\omega_2} \sin(\omega_2 t) \left(\frac{(\omega_2 - 1)^2}{8\omega_2} e^{i\omega_2 t} - \frac{(\omega_2 + 1)^2}{8\omega_2} e^{-i\omega_2 t} \right) \\
&= \frac{(\omega_1 - 1)^2(\omega_1 + 1)^2}{2\omega_1^2} \sin^2(\omega_1 t) + \frac{(\omega_2 - 1)^2(\omega_2 + 1)^2}{2\omega_2^2} \sin^2(\omega_2 t) \\
&\quad + \frac{i(\omega_1 - 1)(\omega_1 + 1)}{\omega_1} \sin(\omega_1 t) \left(\frac{(\omega_1 - 1)^2}{8\omega_1} (e^{i\omega_1 t} - e^{-i\omega_1 t}) \right. \\
&\quad \left. + \frac{(\omega_1 + 1)^2}{8\omega_1} (e^{i\omega_1 t} - e^{-i\omega_1 t}) \right) - \frac{i(\omega_2 - 1)(\omega_2 + 1)}{\omega_2} \sin(\omega_2 t) \\
&\quad \times \left(\frac{(\omega_2 - 1)^2}{8\omega_2} (e^{i\omega_2 t} - e^{-i\omega_2 t}) + \frac{(\omega_2 + 1)^2}{8\omega_2} (e^{i\omega_2 t} - e^{-i\omega_2 t}) \right) \\
&= \frac{(\omega_1 - 1)^2(\omega_1 + 1)^2}{2\omega_1^2} \sin^2(\omega_1 t) + \frac{(\omega_2 - 1)^2(\omega_2 + 1)^2}{2\omega_2^2} \sin^2(\omega_2 t) \\
&\quad - \frac{(\omega_1 - 1)(\omega_1 + 1)}{4\omega_1^2} \sin^2(\omega_1 t) ((\omega_1 - 1)^2 + (\omega_1 + 1)^2) \\
&\quad + \frac{(\omega_2 - 1)(\omega_2 + 1)}{4\omega_2^2} \sin^2(\omega_2 t) ((\omega_2 - 1)^2 + (\omega_2 + 1)^2) \\
&= \frac{(\omega_1^2 - 1)(\omega_1^2 - 1)}{2\omega_1^2} \sin^2(\omega_1 t) + \frac{(\omega_2^2 - 1)(\omega_2^2 - 1)}{2\omega_2^2} \sin^2(\omega_2 t) \\
&\quad - \frac{(\omega_1^2 - 1)}{2\omega_1^2} \sin^2(\omega_1 t) (\omega_1^2 + 1) + \frac{(\omega_2^2 - 1)}{2\omega_2^2} \sin^2(\omega_2 t) (\omega_2^2 + 1) \\
&= \frac{(\omega_1^2 - 1)}{2\omega_1^2} \sin^2(\omega_1 t) (\omega_1^2 - 1 - \omega_1^2 - 1) \\
&\quad + \frac{(\omega_2^2 - 1)}{2\omega_2^2} \sin^2(\omega_2 t) (\omega_2^2 - 1 + \omega_2^2 + 1)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{(\omega_1^2 - 1)}{\omega_1^2} \sin^2(\omega_1 t) + (\omega_2^2 - 1) \sin^2(\omega_2 t) \\
&= -\frac{1}{2} \frac{\omega_1^2 - 1}{\omega_1^2} [1 - \cos(2\omega_1 t)] + \frac{1}{2} (\omega_2^2 - 1) [1 - \cos(2\omega_2 t)] .
\end{aligned}$$

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