Phase-lagged formulation of quantum mechanics : application to ultra-fast processes of energy and information transport

Abouzar, Kaboudian

2009

Abouzar, K. (2009). Phase-lagged formulation of quantum mechanics : application to ultra-fast processes of energy and information transport. Master' s thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/20651>

<https://doi.org/10.32657/10356/20651>

**PHASE-LAGGED FORMULATION OF QUANTUM
MECHANICS:
APPLICATION TO ULTRA-FAST PROCESSES OF
ENERGY AND INFORMATION TRANSPORT**

ABOUZAR KABOUDIAN

School of Mechanical and Aerospace Engineering

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Master of Engineering

2009

Acknowledgments

I would like to express my appreciation to my research advisor, Associate Professor Vladimir V. Kulish for his guidance and encouragement through this research project.

The candidate is also grateful to Nanyang Technological University for the use of facilities and A-Star for financial support.

Table of Contents

Acknowledgments 1

Table of Contents..... 2

Table of Figures 6

Abstract..... 7

Chapter 1 Introduction and Objectives..... 8

 1.1 Introduction..... 8

 1.2 Main Objectives..... 9

Chapter 2 Literature Review 10

 2.1 A Brief History of Quantum Mechanics..... 10

 2.2 Quantum Field Theory..... 16

 2.3 The Schrödinger Equation 20

 2.3.1 The Paradox of Instantaneous Propagation in the Schrödinger Equation .. 21

 2.4 Hidden Variables theory and the Bell Inequality 22

 2.4.1 Experimental Tests of Bell’s Theorem 24

 2.5 Relativistic Quantum Mechanics 29

 2.5.1 The Notation 31

 2.5.2 The Klein Gordon Equation..... 35

 2.5.3 A wave equation for Spin $-\frac{1}{2}$ Particles: The Dirac Equation..... 39

 2.6 Derivation of the Schrodinger equation from Newtonian Mechanics 43

 2.6.1 Stochastic mechanics 45

 2.6.2 The Ornstein-Uhlenbeck Theory 46

 2.6.3 Kinematics of the Markov Processes..... 48

2.6.4 The Hypothesis of Universal Brownian Motion.....53

2.6.5 The Real Time Independent Schrödinger Equation.....55

2.6.6 The Time-Dependent Schrödinger Equation57

2.7 Summary and Conclusion.....60

Chapter 3 Theoretical Background.....61

3.1 Introduction.....61

3.2 Heuristic derivation of the Schrödinger equation.....62

3.3 Derivation of the Phase-Lagged Schrödinger Equation from Conservation of Energy65

3.4 Constructive Derivation of the Classical Schrödinger Equation from Newtonian Mechanics69

3.4.1 Notes on the Constructive derivation of the Schrödinger equation from Newtonian Mechanics74

3.5 Derivation of the Coupled Phase-lagged Schrödinger Equation74

3.6 Derivation of the classical Schrödinger Equation from an Electromagnetic Wave Point of View.....77

3.7 Derivation of the Phase-lagged Schrödinger Equation from an Electromagnetic Wave Point of View.....78

3.8 Finiteness of the Time-Lag.....80

3.9 Recursive Solution of the Phase-Lagged Schrödinger Equation.....81

3.10 Summary and Conclusion.....83

Chapter 4 Macroscopic Phenomena and Phase-Lagged Formulation of Quantum Mechanics85

4.1 The Classical Schrödinger Equation and Diffusion Equation85

4.2 The Phase-Lagged Schrödinger Equation and The Diffusion Equation.....88

4.2.1 Transformation of the boundary and initial conditions for the classical Schrödinger equation for the diffusion equation91

4.2.1.1 Transformation of the Initial Condition91

4.2.1.2 Transformation of the Dirichlet Boundary Condition92

4.2.1.3 Transformation of the Neumann Boundary Condition.....93

4.3 The relationship between the fluid velocity and de Broglie’s wave function ..96

4.3.1 Wave Function.....97

4.3.2 Group Velocity98

4.3.3 Schrödinger’s Equation and the Continuity Equation102

4.3.4 Wave Packet Assumption105

4.3.5 Review of the Assumptions and Conditions for the Presented Derivations...
.....107

4.4 Superfluid Decay109

4.4.1 Relative Mean Free Time116

4.4.2 The Remainder Factor117

4.4.2.1 Correct Direction:.....118

4.4.2.2 Probability of Having Sufficient Speed.....119

4.4.2.3 Probability of hitting a target particle at distance r in a time less than τ 124

4.4.2.4 Collision of a Confined Projectile Particle with the Target Particle .125

4.4.2.5 Probability of total no collision with the target particle.....127

4.4.3 Superfluid Decay and Phase-Lagged Formulations of Quantum Mechanics.
.....132

4.5 Summary and Conclusion.....133

Chapter 5 Summary, Conclusion, and Future Work135

5.1 Summary and Conclusion.....135

5.2 Future Work.....138

Table of Contents

Appendix A. *Proof of Bell's Theorem* 139

References..... 151

Table of Figures

Fig 1 Domain of the solution of equation (4.18), where Ω is the domain and Γ is boundary of the domain91

Fig 2 Effective collision area 111

Fig 3 Effective swept collision volume 112

Fig 4 The particles have relative velocity..... 113

Fig 5 Correct direction probability 118

Fig 6 Distance between the target and projectile..... 120

Fig 7 Comparison of the linear approximation and exact values of the dimensionless mean distance..... 123

Fig 8 cross section of the spherical differential volume inside the confining sphere..... 125

Fig 9 Probability of Total No Collision with Target, in less than $\tau_{n/n}$ time, against $r_c / l_{n/n}$ 129

Fig 10 Dimensionless superfluid decay curve 131

Fig A-1 Photon pairs are released from the source and enter the Analyzers..... 146

Abstract

In this report, a brief history of quantum mechanics and the major attempts and developments in the early days of quantum mechanics is presented. It is addressed that the Schrödinger equation, the corner stone in quantum mechanics, gives rise to the paradox of instantaneous propagation of energy. It is underlined that the Schrödinger equation is not a relativistic formula. Major attempts to overcome the paradox and to make the formulation relativistic are addressed. It is discussed that, despite of all the attempts, *no* unique and general formulation which removes the paradox and gives relativistic results in all cases is ever proposed.

Three possible phased-lagged derivations of the Schrödinger equation have been derived, based on the assumption that there exists a *finite* time lag between the onset of the gradients and the corresponding fluxes. The extended version of the Schrödinger equation, therefore, has been proposed to reduce the paradox of instantaneous propagation, intrinsic to the classical Schrödinger equation. The method of steps is proposed to solve the phase-lagged Schrödinger equation.

Furthermore, the relationships between the phase-lagged Schrödinger equation and macroscopic equations like the diffusion equation are discussed.

Then, the relationship between the fluid velocity and the de Broglie wave is investigated. Consequently, the link between superfluidity and wave packet assumption is discussed and the superfluid decay is studied in detail. Finally, the relationship between the superfluid decay and the phase-lagged Schrödinger equation is argued.

Chapter 1 Introduction and Objectives

1.1 Introduction

The Schrödinger equation is one of the most important equations in quantum mechanics. It is the key equation, which is supposed to be valid for all time and space scales. However, the classical Schrödinger equation allows for the instantaneous energy propagation and, hence, it seems no longer valid for description of ultra-fast transport processes. Instantaneous energy propagation means that the speed of energy propagation must be infinite which is known to be wrong. Heuristic treatment of the Schrödinger equation has put this fact in shade.

In order to remove the paradox of instantaneous propagation, a phase-lagged Schrödinger equation has been derived (Kulish and Chan 2008). This means that a finite time lag exists between the onset of energy gradient and the corresponding energy flux.

It has been suggested that the phase-lagged formulation of the Schrödinger equation might be equivalent to the relativistic formulations (Kulish and Chan 2008). In addition, some apparent source terms appear in the phase-lagged Schrödinger equation. The physical meaning of these source terms has to be studied. It is believed that these source terms represent gravitational fields. The phase-lagged Schrödinger equation is a nonlinear equation. Hence, this nonlinearity can lead to *nonlinear* gravitational fields. Explanation of nonlinear gravitational fields by means of quantum mechanics has been impossible so far.

Nonlinearity in the phase-lagged Schrödinger equation, which was mentioned above, makes this equation hard to solve. In fact, techniques for solving phase-lagged equations are very rarely proposed. Hence, developing of a technique for solving the phase-lagged Schrödinger equation is crucial.

It has been shown that, the form of the Schrödinger equation is related to the form of the Navier-Stokes and diffusion equations (Strauss 1992). The links between these equations has been studied from different viewpoints (Kulish and Lage 2002; Christianto and Smarandache 2008).

1.2 Main Objectives

The main goals of the proposed project are to be achieved by:

- a) Derivation of the phase-lagged Schrödinger equation;
- b) Establishing the physical meaning of the apparent source terms in the phase-lagged equation: in particular, whether these terms describe gravitational fields;
- c) Proposing a technique of solving the phase-lagged Schrödinger equation;
- d) Establishing links between the phase-lagged Schrödinger equation and “macroscopic” transport equations (e.g., Navier-Stokes, diffusion/heat, etc).

Chapter 2 Literature Review

2.1 A Brief History of Quantum Mechanics

In 1896, the cathode rays from a cathode within an evacuated tube were reported by Jean-Baptiste Perrin based on experimental results (Perrin 1896). Hertz had proposed that the cathode rays were a form of light. However, Perrin's experiments contradicted with Hertz's proposal, as they showed that the rays were charged with negative electricity. In 1897, J. J. Thomson published supporting observations that the cathode rays consisted of particles which were a thousand times lighter than the hydrogen atom (Thomson 1897).

In 1859, an important theorem was proved about blackbody radiation by Gustav Kirchhoff. A blackbody is defined as an object that absorbs all the incident radiation energy and, as it does not reflect the incident light, therefore, it would appear black to any observer. A blackbody is not only a perfect energy absorber but also a perfect energy emitter. Kirchhoff proved that the energy emitted E depends only on the temperature T and the frequency ν of the emitted energy, i.e.

$$E = J(T, \nu) \quad (2.1)$$

The function J was not yet known to the physicists. Hence, they were challenged to find it.

In 1879, Josef Stefan, based on his experimental results, proposed that the total energy emission by a hot body was proportional to the fourth power of the temperature (Stefan

1879). In 1884, Ludwig Boltzmann achieved the same conclusion for blackbody radiation; however, his works based on theoretical derivations by means of thermodynamics and Maxwell's electromagnetic theory (Boltzmann 1884). These two results are now referred to as the Stefan-Boltzmann law. However, they do not fully address the question for specific wavelengths and thus, leave the Kirchhoff challenge unanswered.

In 1896, Wilhelm Wien proposed a solution for the Kirchhoff J function (Wien 1896). His solution agrees with experimental results for small values of wavelength; however, Rubens and Kurlbaum showed it to break down in the far infrared radiation (Rubens and Kurlbaum 1900).

In 1900, Planck successfully guessed the correct formula for the Kirchhoff J function based on Rubens' results. This presumption fitted experimental evidence at all wavelengths accompanied by accuracy; however, Planck was yet to be satisfied and tried to give a theoretical derivation of the formula (Planck 1900). In order to do that, he started his derivation by assuming that the total energy is composed of indistinguishable elements of energy, i.e. quanta of energy. He left the experimental proof of his hypothesis to the nature.

Planck himself credited Boltzmann's statistical method, but Planck's approach differed fundamentally. However, his basic assumption of presence of energy quanta had no experimental basis. Planck won the 1918 Nobel Prize for Physics for this work.

Ricci and Levi-Civita published an important book named *Absolute differential calculus* in 1901 (Ricci-Curbastro and Levi-Civita 1901). Ricci and Levi-Civita provided the

general formulation of a tensor. This mathematical work which was done by mathematicians appeared at the right moment for the quantum theory.

In 1905, Einstein investigated the photoelectric effect (Einstein 1905). The electromagnetic theory of light predicts results which are known to be non-physical. In order to resolve the problem, Einstein proposed a theory which was based on quantum theory of light. Henceforth, he stumbled upon the fact that Planck's theory made implicit use of the light quantum hypothesis. By 1906, Einstein positively deduced that energy changes arise in a quantum material oscillator changes in jumps which are multiples of $\hbar\nu$ where $\hbar$ is Planck's reduced constant and ν is the frequency. Einstein was awarded the 1921 Nobel Prize for Physics in 1922, for this work on the photoelectric effect.

In 1913, Niels Bohr wrote a ground-breaking paper on the hydrogen atom (Bohr 1913). He successfully proposed the key principles of the spectral lines. Bohr was awarded the 1922 Nobel Prize for Physics for this work.

In 1923, Arthur Compton derived relativistic kinematics for the scattering of a photon off an electron at rest (Compton 1923).

However, there were perceptions in the new quantum theory, which worried top physicists. Einstein, specifically, was unhappy about the element of 'chance' which had entered physics.

In 1924, Bohr, Kramers and Slater made significant theoretical proposals regarding the interaction of light and matter which rejected the photon (Bohr, Kramers et al. 1924).

Although the proposals were aimed in an incorrect path, they inspired crucial experimental work. Bohr addressed certain paradoxes in his work:

- (i) How can energy be conserved when some energy changes are continuous and some are discontinuous, i.e. change by quantum amounts?
- (ii) How does the electron know when to emit radiation?

Einstein had been confused by paradox (ii) and Pauli told Bohr that he did not believe his theory. Further experimental work shortly finished any struggle to the belief of the existence of electron. Further means had to be established to overcome the paradoxes.

Up to this phase, quantum theory was arranged in Euclidean space and used Cartesian tensors of linear and angular momentum. However, this was soon to be changed.

In 1924, another ground-breaking paper was written by Satyendra Nath Bose. It was originally rejected by a referee for publication. Bose then sent the paper to Einstein and he instantly noticed the significance of Bose's work and arranged for its publication. Bose suggested different states for the photon as well as the no conservation of their numbers (Bose 1924). Instead of statistical independence of particles, Bose put particles into cells and thus argued the statistical independence of cells. The progression of history proved all predictions by Bose to be right.

Counterpart to Bose's works, there were other fundamental researches being conducted; the doctoral thesis of Louis de Broglie was introduced. His work extended the particle-wave duality for light to all particles, specifically to electrons (De Broglie 1924).

In 1926, Erwin Schrödinger published a paper giving his equation for the hydrogen atom

and heralded the birth of wave mechanics (Schrödinger 1926). Schrödinger presented operators associated with each dynamical variable. It is important to note that the Schrödinger equation was deduced, but not derived, from the reasoning that if the wave function $\psi(\mathbf{r}, t)$ uniquely defines the state of a system, then according to the causality principle, the value of the wave function at any moment of time, and hence the state of the system, can be found from its initial value the $\psi(\mathbf{r}, t_0)$.

In 1926, Dirac proposed the complete solution of the derivation of Planck's law. In the same year, Born discarded the causality of traditional physics (Born 1926).

In 1925, Heisenberg wrote his first paper on quantum mechanics (Heisenberg 1925) and 2 years later, stated his uncertainty principle (Heisenberg 1927). It states that the process of measuring the position x of a particle disturbs the particle's momentum p , so that

$$\Delta x \Delta p \geq \hbar = h / 2\pi \quad (2.2)$$

where Δx is the uncertainty of the position, and Δp is the uncertainty of the momentum. Here h is Planck's constant and $\hbar$ is by and large called the 'reduced Planck's constant'.

Heisenberg used matrix methods in his works which were based on the work by Cayley on matrices 50 years earlier.

The works by Schrödinger and Heisenberg were not appropriately shown to be equivalent until the necessary mathematics was developed by Riesz about 25 years later (Riesz 1949).

Bohr, in 1927, stated that causality and space-time coordinates are complementary. Later, in 1940, Pauli identified that spin, one of the states proposed by Bose, corresponded to a new kind of tensor, one not covered by the Ricci and Levi-Civita work of 1901 (Pauli 1940). However, in 1913, the mathematics of this had been anticipated by Eli Cartan who introduced a 'spinor' as part of a much more general investigation (Cartan 1913).

In 1928, Dirac for the first time solved problem of expressing quantum theory in a Lorentz invariant form (Dirac 1928). In his solution, he tried to express the d'Alembert wave equation by means of operator algebra.

Up to the stage, not everyone could accept the uncertainty principle. Its most blunt challenger was Einstein. In 1930, at a conference which both Einstein and Bohr attended, Einstein devised a challenge to Niels Bohr. Einstein proposed a box which had a clock fitted in one side. The box was filled with radiation. The clock was designed to open a shutter and permit one photon to run off. The box may be weighed once more a little bit later. The photon energy and its time of runoff may both be measured with arbitrary accuracy (Einstein, Tolman et al. 1931). It should be noted that, this is only a 'thought experiment', not an actual one.

Niels Bohr could not address the challenge the day that Einstein proposed it. However, Niels Bohr had the final triumph, as the next day he had derived the solution. A compensation weight must be hung under the box to measure the mass of the box. Consequently, this imparts a momentum to the box and there will be an error in measuring the position due to it. But, according to the theory of special relativity, time is

not absolute and the error in measuring the position of the box transforms into an error in measuring the time.

After Bohr's explanation, Einstein was forced to accept the uncertainty principal; however, he was never happy with this notion.

In 1932, von Neumann presented quantum theory based on a concrete theoretical basis (von Neumann 1932). Some of the earlier works were short of mathematical rigour, but von Neumann presented the whole theory by means of the setting of operator algebra.

2.2 Quantum Field Theory

In the 1920s, physicists tried to create a quantum mechanical theory of the electromagnetic fields. The result of these efforts heralded the formation of what is called Quantum Field Theory (QFT).

QFT is a theoretical framework for creating quantum mechanical models of many-body systems or, equivalently, field-like systems. It is widely used in describing particle physics as well as condensed matter phenomena. The standard model of elementary particles and their interactions, as most of the theories in modern particle physics, is formulated as relativistic quantum field theories. In condensed matter physics, quantum

field theories are employed in many situations, especially those where the number of particles is allowed to change — i.e., in the BCS¹ theory of superconductivity.

In 1926, Max Born, Pascual Jordan, and Werner Heisenberg created a quantum field theory by working out quantum systems from any number of degrees of freedom (Born, Heisenberg et al. 1926). They described the field's internal degrees of freedom as an infinite set of harmonic oscillators and by employing the usual procedure for quantizing those oscillators. They assumed that no electric charges or currents were present.

In 1927, Paul Dirac created the first relatively complete theory of quantum electrodynamics (Dirac 1927). Dirac's theory included both the electromagnetic field and electrically charged particles, specifically electrons, as quantum mechanical objects. This quantum field theory could be used to model significant processes, for instance, the emission of a photon by an electron falling into a quantum state of lower energy, a process in which the *number of particles changes* — one atom, in the initial state, becomes an atom plus a photon in the final state. It is now understood that the ability to explain such processes is one of the most important features of QFT.

Another major motivation in developing the quantum field theory was the need to put together quantum mechanics and relativity. It was clear from the beginning that a proper quantum treatment of the electromagnetic field had to somehow incorporate Einstein's relativity theory. In 1928, Jordan and Wolfgang Pauli showed that quantum fields could

¹ BSC theory is named after its creators, Bardeen, Cooper and Schrieffer, and is used to explain superconductivity.

be made to behave in the way predicted by special relativity during coordinate transformations (Jordan and Pauli 1928). In 1933, Niels Bohr and Leon Rosenfeld explained that this result could be described as a limitation on the ability to measure fields at space-like separations, precisely as expected by relativity (Bohr and Rosenfeld 1933). A further inspiration for the quantum field theory occurred with the discovery of the Dirac equation, a relativistic formulation for quantum mechanics, when it was shown that several of its adverse properties, such as negative-energy states, could be removed by reformulating the Dirac equation as a quantum field theory. This work was performed by Wendell Furry (Furry and Carlson 1933), Robert Oppenheimer (Oppenheimer 1930; Oppenheimer and Plesset 1933), Vladimir Fock (Fock 1932), and others.

In spite of its early successes, quantum field theory confronted several critical theoretical difficulties. Many physical quantities, such as the energy shift of electron states as a result of the presence of the electromagnetic field, gave infinity, which is an unacceptable result, when computed using quantum field theory. During the 1940s, this divergence problem was resolved by Bethe, Tomonaga, Schwinger, Feynman, and Dyson, by means of the method known as renormalization (Tomonaga 1946; Bethe 1947; Schwinger 1948; Schwinger 1948; Dyson 1949; Dyson 1949; Feynman 1949; Feynman 1949; Schwinger 1949). Feynman, Schwinger and Tomonaga were awarded Nobel Prize for physics, for their fundamental work in quantum electrodynamics, in 1965.

This stage of progress ended with the creation of the modern theory of quantum electrodynamics (QED).

Beginning in the 1950s with the work of Yang and Mills (Yang and Mills 1954; Yang and Mills 1954B), QED was generalized to a class of quantum field theories known as gauge theories. During 1960s and 1970s, the standard model of particle physics was formulated as a gauge theory. This theory describes all known elementary particles as well as the interaction between them. Sheldon Glashow formulated the weak interaction part of the standard model with the Higgs mechanism added by Steven Weinberg and Abdus Salam (Gross 1992). The theory was shown to be consistent by Gerardus't Hooft and Martinus Veltman ('t Hooft and Veltman 1972; 't Hooft and Veltman 1972B).

As it was mentioned earlier, the standard model of particle physics is a quantum field theory developed between 1970 and 1973. This theory is able to explain three of the four known fundamental interactions between the elementary particles. This theory is in agreement with both quantum mechanics and special relativity. To date, nearly all experimental tests of the three forces described by the standard model have agreed with its predictions. However, the standard model is not considered a unifying theory, as it is short of inclusion of gravity, as the fourth known fundamental interaction. Another drawback from the standard model is the large number of numerical parameters such as masses and coupling constants. These constants and numerical parameters are not obtained from first principles, and must be put “by hand” into the theory.

Quantum field theory, as well as its broad applications, is still considered to be an ongoing research area. It continues to be one of the most central areas of theoretical physics today, offering a common language to various aspects of physics.

2.3 The Schrödinger Equation

The Schrödinger equation was first proposed by Erwin Schrödinger in 1926 (Schrödinger 1926). This equation mainly describes the space-time dependency of quantum mechanical systems. In quantum mechanics, the role of the Schrödinger equation is as important as the role of Newton's second law in classical mechanics.

Historically, the Schrödinger equation was deduced, and not derived, from the reasoning that if the wave function $\psi(\mathbf{x}, t)$ uniquely defines the state of the system, then, according to causality principle, the value of the wave function at any moment of time, and hence the state of the system at any instant of time can be found from its initial value $\psi(\mathbf{x}, t_0)$.

It is known that the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m_0} \nabla^2 + V(\mathbf{x}) \right) \psi(\mathbf{x}, t), \quad (2.3)$$

corresponds to a nonrelativistic energy relation in operator form,

$$\hat{E} = \frac{\hat{p}^2}{2m_0} + V(x) \quad (2.4)$$

where

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad p = -i\hbar \nabla \quad (2.5)$$

are the operators of energy and momentum, respectively (Greiner 1994). It is also known that the wave equation involved here is a nonrelativistic equation. In the following sections, some of the attempts to derive relativistic versions of the Schrödinger equation will be discussed.

2.3.1 The Paradox of Instantaneous Propagation in the Schrödinger Equation

In this section, the paradox of instantaneous energy propagation will be discussed. Consider (2.3), in three dimensional space, for a particle of mass $m_0 = \hbar$ and drop the potential term. In this case, one could write

$$-i \frac{\partial \psi}{\partial t} = \frac{1}{2} \nabla^2 \psi. \quad (2.6)$$

According to Strauss (Strauss 1992), the solutions of (2.6) that tend to zero as $|\mathbf{r}| \rightarrow \infty$, will be of the following form

$$\psi(\mathbf{r}, t) = \frac{1}{(2\pi i t)^{3/2}} \int_{\Omega} \exp\left(i \frac{|\mathbf{r} - \mathbf{r}'|^2}{2t}\right) \phi(\mathbf{r}') d\mathbf{r}' \quad (2.7)$$

where $\phi(\mathbf{r})$ is the initial condition $\psi(\mathbf{r}, 0)$ and Ω is the domain of solution (entire three dimensional space). It is evident from (2.7) that if the initial distribution of $\psi(\mathbf{r}, 0)$ is such that $\psi(\mathbf{r}, 0) \neq 0$ as $|\mathbf{r} - \mathbf{R}| \leq r_0$ and $\psi(\mathbf{r}, 0) = 0$ as $|\mathbf{r} - \mathbf{R}| > r_0$ then $\psi(\mathbf{r}, t) \neq 0$

for any value of r and t . In other words, no matter how short the duration of the process, the influence of the initial disturbance is felt everywhere inside the domain. This means that the information transport speed has to be infinite in order for the solution given by (2.7) to apply.

In Chapter 3, this paradox will be discussed in more detail. Consequently, some ways to overcome this paradox will also be proposed.

2.4 Hidden Variables theory and the Bell Inequality

From the early dates of development of quantum mechanics, many physicists noticed the paradox of instantaneous propagation of energy in quantum mechanical formulations. As mentioned before, as soon as the Schrödinger equation was proposed, physicists like Dirac, Klein, Gordon and several others attempted to overcome the paradox of instantaneous propagation of energy by introducing Lorentz invariant formulations of the Schrödinger equation.

Some physicists like Einstein did not believe in the descriptions that quantum mechanics provided. In 1935, Einstein, Podolsky and Rosen (Einstein and Rosen 1935) introduced a paradox which is now called the EPR paradox, after the names of its creators. They argued that quantum mechanics is not a complete theory and should be supplemented by additional variables. These additional variables had to be supplemented to reinstate causality and locality. The determination of these variables would predetermine the result of measuring any observables of the system. They assumed that the predictions of quantum mechanics to be correct but only as a result of statistical distributions of hidden

variables. In one of his papers, which seems to be a philosophical paper on physics and reality, Einstein reemphasizes the idea of hidden variables and incompleteness of quantum mechanics (Einstein 1936).

The proposal of the EPR paradox inspired a theory which is now called hidden variables theory. In 1964, John S. Bell published a fundamental paper in a journal named *Physics*, which changed the study of fundamentals of quantum mechanics (Bell 1964). In this paper, the idea of hidden variables is formulated mathematically². The paper proved, under conditions which were suppressed in a later work (Bell 1966; Clauser, Horne et al. 1969; Clauser and Horne 1974; Aspect 1983; Bell 1985; Bell 1987; Aspect 2000) “no physical theory, which is realistic and also local in a specified sense, can agree with all of the statistical implications of quantum mechanics.”

Several different versions and cases, with relative resemblances, were motivated by Bell’s paper (Bell 1964) and are considered under the statement, “Bell’s theorem”, being the shared name for the entire family.

A hidden variable theory must be a non-local theory to be consistent with quantum mechanics; otherwise, the existence of instantaneous causal relations between physically separated entities could not be preserved. In late 1920s, Louis de Broglie proposed the first hidden-variable theory which is known as the pilot wave theory. The best known hidden-variable theory, known to date, is the Bohmian mechanics, developed by the

² To see the complete proof of Bell’s inequality see Appendix A.

physicist and philosopher David Bohm, in 1952. This theory is a non-local hidden variable theory (Bohm 1952; Bohm 1952B).

However, nowadays, most of the physicists believe that the universe is not governed by a hidden variable theory. They doubt particles can have any additional information which is not included in their quantum mechanics explanation (Norsen 2006).

2.4.1 Experimental Tests of Bell's Theorem

Originally, Bell, in his ground-breaking paper, did not suggest any experimental solution for the disagreement between local realistic theories and quantum mechanics, in 1964. Possibly this was due to the fact that quantum mechanics has been precisely confirmed in many branches of physics.

However, it was uncertain that any of these many confirmations appeared in conditions of disagreement between quantum mechanics and Local Realistic Theories; as a result, a consistent experimental arbitration was needed. Clauser, Horne, Shimony, and Holt made suggestion to this effect (Clauser, Horne et al. 1969), which is from now on called CHSH. They proposed that the couple 1 and 2 be the photons created in an atomic cascade. The initial atomic state of the photons has an overall angular momentum of $J = 0$, while their transitional atomic state will be $J = 1$, and their ultimate atomic state $J = 0$. This is similar to an experiment conducted using calcium vapor purposed by Kocher and Commins (Kocher and Commins 1967). The benefits of the above mentioned setting of states, over other possible settings, are as follows:

1. The conservation of angular momentum implies that the photon pair, released in the cascade, has total angular momentum of 0. Therefore, if the photons are gathered, in cones of minute aperture alongside the z and $-z$ directions, the overall orbital angular momentum will be negligible. Consequently, the overall spin angular momentum is almost zero. Hence, the polarizations of the two photons are firmly correlated.
2. It is possible to rather precisely examine the polarization of the photons by means of the customary polarization analyzers. This is due to the fact that the photons are generated in a visible frequency range.
3. The two photons of the pair are emitted in a stochastic interval of ten nanoseconds. This time interval is negligible in contrast to the average time interval required between two productions of pairs. Hence, coupled photons 1 and 2 of the pair are clearly matched.

However, a drawback of this setting of states is that photo-detectors in the related frequency range are not very effective. Therefore, a supplementary assumption is required so as to make presumption from the statistics of the sub-ensemble of the pairs that is summed to the whole group of pairs released in the course of observation.

In the experiment that is suggested by CHSH, the polarization analysis is done by two polarization analyzers, namely analyzer I and II . The orientation angles of transmission axis of analyzer I are a and a' , while the orientation angles of transmission axis of analyzer II are b and b' . The outcomes $s = 1$ and $s = -1$ respectively specify passing and non-passing of photon 1 through analyzer I , and $t = 1$ and $t = -1$ respectively specify passing and non-passing of photon 2 through

analyzer *II*. Non-passing through the analyzer is therefore replaced for passing through the extraordinary ray. This simplification of the device initiates a noticeable difficulty: it is not possible to distinguish directly between a photon that does not succeed to pass through the analyzer and one which does pass through the analyzer but is not sensed due to the photo-detector errors.

Freedman and Clauser measured the linear polarization correlation of the photons emitted in an atomic cascade of calcium. They showed by a generalization of Bell's inequality that the existence of local hidden variables imposes restrictions on this correlation in conflict with the predictions of quantum mechanics. Their data, in agreement with quantum mechanics, violate these restrictions to a high statistical accuracy, thus providing strong evidence against local hidden-variable theories (Freedman and Clauser 1972).

Fry and Thompson also performed a variant of the Holt-Pipkin experiment, using a different isotope of mercury and a different cascade and exciting the atoms by radiation from a narrow-bandwidth tunable dye laser. Their results showed close agreement with the quantum mechanical predictions and clearly violated Bell's inequality. They collected data in no more than 80 minutes, due to the high excitation rate attained by the laser (Fry and Thompson 1976).

In the 1970s, Kasday-Ullman-Wu, Faraci-Gutkowski-Notarigo-Pennisi, Wilson-Lowe-Butt, and Bruno-d'Agostino-Maroni conducted four experiments by means of photon pairs created in positronium annihilation as an alternative for cascade photons. The results of all of these experiments except for that proposed by Faraci et al. showed fair

agreement with the quantum mechanical predictions and violated Bell's Inequalities. A review of these experiments is presented in the review article by Clauser and Shimony (Clauser and Shimony 1978). They deem these experiments as less credible than those using cascade photons, since they depend on stronger supplementary assumptions.

In the early 1980s, Aspect, Grangier, and Roger carried out the first experiment by means of polarization analyzers with two exit channels (Aspect, Grangier et al. 1982). The result of their experiment confirmed the predictions of quantum mechanics and violated those of local realistic theories more severely than any of its forerunners. Soon after that, Aspect, Dalibard, and Roger conducted an experiment to close the communication loophole (Aspect, Dalibard et al. 1982). The article by Aspect in 2000 reviews these experiments as well as the investigations carried out by Rarity and Tapster, by Shih and Alley, by Ou and Mandel, and by others, using photon pairs with correlated linear momenta produced by down-conversion in non-linear crystals (Aspect 2000). Some of the recent Bell tests are reviewed in an article by Zeilinger (Zeilinger 1999).

The most popular physical systems in conducting the Bell experiments are pairs of entangled photons. This is due to the fact they are fairly easy to generate and also analyze. However, some experiments have been proposed which employ other systems. Laméhi-Rachti and Mittag proposed experiments which involved measuring spin correlations in proton pairs which were prepared by low-energy scattering (Laméhi-Rachti and Mittag 1976). Their results showed reasonably good agreement with the quantum mechanical prediction and violated Bell's Inequality. However, they had to incorporate strong supplementary assumptions, similar to those in the positronium

annihilation experiments. In 2003, A. Go conducted a Bell experiment with B-mesons (Go 2003). Once more, the outcomes confirmed the quantum mechanical predictions.

The results of the Bell experiments present spectacular verifications of the self evident entanglement of many quantum states of systems consisting of two or more parts, and, consequently, of the existence of holism in physics at a primary level. In point of fact, the first verification of entanglement and holism was not first proposed by Bell's work, since Bohm and Aharonov (Bohm and Aharonov 1957) confirmed that the results of Wu and Shakhnov (Wu and Shakhnov 1950), Compton scattering of the photon pairs produced in positronium annihilation, previously represented the entanglement of the photon pairs.

2.5 Relativistic Quantum Mechanics³

The phenomena involving high energies or high speeds entail study of the relativistic wave equations. Up to now, there have been several efforts to put forward models which can explain these phenomena. In this section, some of these efforts will be explained.

The evolution from a nonrelativistic to a relativistic explanation involves that several notions of the nonrelativistic theory have to be reexamined, specifically:

Spatial and temporal coordinates have to be considered similarly within the theory.

1. since:

$$\Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_0 c} \quad (2.8),$$

One can deduce that a relativistic particle cannot be restricted more precisely than $\approx \hbar/m_0 c$; or else, pair creation happens for $E > 2m_0 c^2$. Therefore, the idea of a free particle can only seem right if the particle is not constrained by external limitation to a volume which is lesser than nearly the Compton wavelength

³ The content of this section is mainly taken from the following reference: Greiner, W. (2000). Relativistic quantum mechanics wave equations. Berlin; New York, Springer.

$\lambda_c = \hbar/m_0 c$. If not, the particle has companions as a result of particle-antiparticle creation.

2. If the location of the particle is not certain, i.e. if

$$\Delta x > \frac{\hbar}{m_0 c} \quad (2.9)$$

then the time is obliged to also not be certain, because

$$\Delta t \sim \frac{\Delta x}{c} > \frac{\hbar}{m_0 c} \quad (2.10)$$

In a nonrelativistic theory, Δt can turn out to be arbitrarily small, as $c \rightarrow \infty$.

Thus, the need to review the concept of probability density is clear.

$$w(x, y, z, t) \quad (2.11)$$

explains the probability of observing a particle at a specific location r and at a time t .

3. Pair creation and annihilation processes can take place at high relativistic energies, typically in the form of generation of particle-antiparticle pairs. Consequently, at relativistic energies, conservation of particles cannot be considered as a physical supposition. A relativistic theory should be capable of expressing pair creation, particle conservation, etc (Greiner 2000).

2.5.1 The Notation

To start the subject, the notation used should be introduced. Four-vectors by Minkowski's notation (Naber 1992; Goenner 1999) will be used in this section. The forth component is an imaginary component, e.g.

$$\begin{aligned}
 x &= \{x, y, z, ict\} && \text{(world vector),} \\
 p &= \{p_x, p_y, p_z, iE/c\} && \text{(four-momentum),} \\
 A &= \{A_x, A_y, A_z, iA_0\} && \text{(four-potential),} \\
 \nabla &= \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{i\partial(ct)} \right\} && \text{(four-gradient), etc.}
 \end{aligned} \tag{2.12}$$

The full four-vectors will be denoted by the letters x, p, A, ∇ . Some text books denote

them as $\vec{x}, \vec{p}, \vec{A}, \vec{\nabla}$, etc. but, provided that there will be no arousal of confusion, the first notation is chosen in this manuscript. At this moment, it is beneficial to introduce the metric tensor (covariant components)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.13}$$

The first index designates the row while the second index represents the column.

Therefore, the vector length $dx = \{dx^\mu\}$ can be written as $ds^2 = dx dx = g_{\mu\nu} dx^\mu dx^\nu$.

This is frequently used to define a relation for a metric tensor.⁴ The contravariant representation $g^{\mu\nu}$ of the metric tensor is preceded by the condition

$$g^{\mu\sigma} g_{\sigma\nu} = \delta_{\nu}^{\mu} \stackrel{def}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.14)$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2.15)$$

Notice that, for the special Lorentz metric, the contravariant and covariant metric tensors are identical:

$$g^{\mu\nu} = g_{\mu\nu} \quad [\text{for Lorentz metric}] \quad (2.16)$$

Henceforth, the following contravariant four-vector will be used for the description of space-time coordinates

$$x^{\mu} = \{x^0, x^1, x^2, x^3\} \equiv \{ct, x, y, z\}, \quad (2.17)$$

⁴ The notation used here is the same as Bjorken, J. D. and S. D. Drell (1964). Relativistic Quantum Mechanics. New York, McGraw Hill.

where the time-like component is designated as the zero component. One would acquire the covariant form of the four-vector by means of “lowering” the index μ by using the metric tensor, i.e.

$$x_\mu = g_{\mu\nu} x^\nu = \{ct, -x, -y, -z\} = \{x_0, x_1, x_2, x_3\} \quad (2.18)$$

Likewise, the indices can be raised to obtain the contravariant. This indicates that one can simply transform the contravariant form of a four-vector to its covariant form and vice versa. Unless otherwise stated, Einstein’s summation convention will be used.

Four-vectors will be represented by a common letter. Thus for example,

$$a = \{a_0, a_1, a_2, a_3\} \quad (2.19)$$

Contrary to this, three-vectors are represented by bold letters as in

$$\mathbf{a} = \{a_1, a_2, a_3\} \quad (2.20)$$

Notice that Greek indices always run from 0 to 3 while Latin indices run from 1 to 3.

The four-momentum operator will be defined as:

$$\begin{aligned} \hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu} &= \left\{ i\hbar \frac{\partial}{\partial(ct)}, i\hbar \frac{\partial}{\partial x_1}, i\hbar \frac{\partial}{\partial x_2}, i\hbar \frac{\partial}{\partial x_3} \right\} \\ &\equiv i\hbar \nabla^\mu = \left\{ i\hbar \frac{\partial}{\partial(ct)}, -i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z} \right\} \\ &\equiv i\hbar \left\{ \frac{\partial}{\partial(ct)}, -\nabla \right\} \end{aligned} \quad (2.21)$$

It obeys contravariant four-vector transformations, so that

$$\begin{aligned}
 \hat{p}^\mu \hat{p}_\mu &= -\hbar^2 \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} = -\hbar^2 \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right) \\
 &\equiv -\hbar^2 \square = -\hbar^2 \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right)
 \end{aligned} \tag{2.22}$$

In this equation, both the three-dimensional operator $\Delta = \nabla^2$ and the four-dimensional d'Alembertian which is $\square = \partial^2 / (c^2 \partial t^2) - \Delta$ are defined. Finally, the commutation relations of momentum and position are investigated using (2.21) and (2.17). Consequently, the following result is obtained

$$\begin{aligned}
 [\hat{p}^\mu, \hat{p}_\mu]_- &= i\hbar \left[\frac{\partial}{\partial x_\mu}, g^{\nu\sigma} x_\sigma \right] = i\hbar g^{\nu\sigma} \frac{\partial x_\sigma}{\partial x_\mu} \\
 &= i\hbar g^{\nu\sigma} \delta_\sigma^\mu = i\hbar g^{\nu\mu} = i\hbar g^{\mu\nu}
 \end{aligned} \tag{2.23}$$

The metric tensor $g^{\mu\nu}$ specifies the covariant representation of the commutation relation.

The four-potential of electromagnetic field is given by

$$A^\mu = \{A_0, \mathbf{A}\} = \{A_0, A_x, A_y, A_z\} = g^{\mu\nu} A_\nu \tag{2.24}$$

Here A^μ and A_μ are the contravariant and covariant components respectively. The electromagnetic field tensor can be written in the following familiar way by means of A^μ :

$$F^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu} \quad (2.25)$$

2.5.2 The Klein Gordon Equation

In the previous section, we briefly explained the nonrelativistic Schrödinger equation. The Klein-Gordon equation is a relativistic version of Schrödinger equation which is used to describe the state of the particles with zero spin.

It should be noted that there are no known stable elementary particles of spin zero; however, the π mesons and K mesons are quite stable candidates to use such an equation to describe their relativistic nature.

The following relativistic relation will be used to derive the relativistic wave equation:

$$p^\mu p_\mu = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p} = m_0^2 c^2. \quad (2.26)$$

The four-momentum p^μ will be replaced by the four-momentum operator

$$\hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu} = i\hbar \left\{ \frac{\partial}{\partial(ct)}, -\nabla \right\} = \{\hat{p}_0, \hat{\mathbf{p}}\}. \quad (2.27)$$

Following (2.18) and (2.21), the result is in line with (2.27). Therefore, one can write the Klein-Gordon equation for free particles,

$$\hat{p}^\mu \hat{p}_\mu = m_0^2 c^2 \psi \quad (2.28)$$

where m_0 is the rest mass of the particle and c is the speed of light in vacuum. By using (2.22), one can rewrite (2.28) in the following form

$$\left(\square + \frac{m_0^2 c^2}{\hbar^2} \right) \psi = \left(\frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2} \right) \psi = 0, \quad (2.29)$$

It is immediately proved that the Lorentz covariance of the Klein-Gordon equation, as $\hat{p}^\mu \hat{p}_\mu$ is in fact Lorentz invariant. It is also noted that (2.29) is the classical wave equation including the mass term $m_0^2 c^2 / \hbar^2$. Free solutions are of the form

$$\begin{aligned} \psi &= \exp \left(-\frac{i}{\hbar} p_\mu x^\mu \right) = \exp \left[-\frac{i}{\hbar} (p_0 x^0 - \mathbf{p} \cdot \mathbf{x}) \right], \\ &= \exp \left[+\frac{i}{\hbar} (\mathbf{p} \cdot \mathbf{x} - Et) \right] \end{aligned} \quad (2.30)$$

Undoubtedly, substituting (2.30) into (2.28) results in

$$\begin{aligned} \hat{p}^\mu \hat{p}_\mu \psi &= m_0^2 c^2 \psi \rightarrow p^\mu p_\mu \exp \left(-\frac{i}{\hbar} p_\mu x^\mu \right) = m_0^2 c^2 \exp \left(-\frac{i}{\hbar} p_\mu x^\mu \right), \\ &\rightarrow p^\mu p_\mu = m_0^2 c^2 \quad \text{or} \quad \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p} = m_0^2 c^2 \end{aligned} \quad (2.31)$$

which leads to

$$E = \pm \sqrt{m_0^2 c^2 + p^2}. \quad (2.32)$$

Therefore, one can obtain solutions for positive E as well as for negative E energies. Negative energies are physically related to antiparticles. Antiparticles can be detected in nature. This already represents the value of extending the nonrelativistic theory.

The four-current j_μ associated with (2.26) can be proposed now. Similar to the analogy which was mentioned in the case of the Schrödinger equation, one can assume a conservation law for the j_μ . One would start from (2.29), in the following way

$$\left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi = 0, \quad (2.33)$$

And consequently take the complex conjugate of this equation, i.e.

$$\left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi^* = 0. \quad (2.34)$$

Multiplying (2.33) by ψ^* and (2.34) by ψ , and subtracting the resulting two equations leads to

$$\psi^* \left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi - \psi \left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi^* = 0 \quad (2.35)$$

Or

$$\begin{aligned} -\psi^* \left(\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2\right) \psi - \psi \left(\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2\right) \psi^* &= 0 \\ \Rightarrow \nabla_\mu \left(\psi^* \nabla^\mu \psi - \psi \nabla^\mu \psi^*\right) &\equiv \nabla_\mu j^\mu = 0 \end{aligned} \quad (2.36)$$

Hence, the four-current density can be written as

$$j_\mu = \frac{i\hbar}{2m_0} (\psi^* \nabla_\mu \psi - \psi \nabla_\mu \psi^*) \quad (2.37)$$

The result is multiplied by $i\hbar/2m_0$, in order to make sure that the dimensions of the zero component j_0 and the probability density match. This makes sure that we get the right nonrelativistic limit. (2.36) can be written in a detailed form

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{i\hbar}{2m_0 c^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \right] \\ + \text{div} \left[-\frac{i\hbar}{2m_0 c^2} \left(\psi^* (\nabla \psi) - \psi (\nabla \psi^*) \right) \right] = 0 \end{aligned} \quad (2.38)$$

which is the continuity equation:

$$\frac{\partial w}{\partial t} + \text{div } \mathbf{j} = 0 \quad (2.39)$$

Integration over the entire configuration space yields

$$\int_V \frac{\partial w}{\partial t} d^3x = \frac{\partial}{\partial t} \int_V w d^3x = - \int_V \text{div } \mathbf{j} d^3x = - \int_V \mathbf{j} \cdot d\mathbf{F} = 0$$

Hence,

$$\int_V w d^3x = \text{const.}$$

i.e. $\int_V w d^3x$ is constant in time. Naturally, one would interpret

$$w = \frac{i\hbar}{2m_0c^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad (2.40)$$

as the probability density. However, there this interpretation leads to a difficulty: at a given time t , both ψ and $\partial\psi/\partial t$ may have arbitrary values; hence, $w(\mathbf{x}, t)$ in (2.40) can be either positive or negative. Hence, $w(\mathbf{x}, t)$ is not a positive definite function. This means that it cannot be considered as the probability density. This is due to the fact that the Klein-Gordon equation is of the second order in time; therefore both $\psi(\mathbf{x}, t)$ and $\partial\psi(\mathbf{x}, t)/\partial t$ must be known for a specified t . In addition, one could derive at solutions for the negative energies. These two difficulties were the main reason that the Klein-Gordon equation was considered as a nonphysical equation for a long time.

Finally, Dirac could find a relativistic wave equation of first order in time. Although his equation resulted in positive definite probability, it turned out that this equation also leads to negative energy solutions. Later it was found that these negative energy solutions are related to the existence of antiparticles.

2.5.3 A wave equation for Spin- $\frac{1}{2}$ Particles:

The Dirac Equation

In 1927, Dirac made an attempt to formulate a wave equation to describe emission and absorption of energy, in a relativistic sense (Dirac 1927).

He investigated relativistic covariant form of the wave equation of the Schrödinger form with positive definite density (Dirac 1928).

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad (2.41)$$

Back then, there existed uncertainties that the Klein-Gordon equation would not result in such a probability density. As the charge density description was not yet known this equation would not seem to be physical. This was due to the fact that π^+ and π^- mesons were yet to be discovered as charged particles.

By employing a point electron in an arbitrary electromagnetic field, he obtained an explanation for the simplest Hamiltonian which satisfies the requirements of both relativity and the general transformation theory of quantum mechanics, without any further assumption. Dirac devised the spinning electron model to account for the observed duplexity, which, as a first approximation, may be regarded as true.

As (2.41) is linear in its time derivative, it is possible to try to write a Hamiltonian which is linear in its spatial derivatives as well. Therefore, equation (2.41) is obliged to be written as

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{\hbar c}{i} \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) + \hat{\beta} m_0 c^2 \right] \psi = \hat{H}_i \psi. \quad (2.42)$$

It should be noted that coefficients $\hat{\alpha}_i$ cannot be regular numbers; or else (2.42) would not be form invariant with respect to simple spatial coordinates. One would suspect that the $\hat{\alpha}_i$ are matrices. This is denoted by the operator sign $\hat{\cdot}$. As a result, ψ cannot be a simple scalar, and it must be a column vector

$$\psi = \begin{pmatrix} \psi_1(\mathbf{x}, t) \\ \psi_2(\mathbf{x}, t) \\ \vdots \\ \psi_N(\mathbf{x}, t) \end{pmatrix} \quad (2.43)$$

from which a positive definite density of the form

$$w(x) = \psi^\dagger \psi(x) = (\psi_1^*, \psi_2^*, \dots, \psi_N^*) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} = \sum_{i=1}^N \psi_i^* \psi_i \quad (2.44)$$

can be constructed at once. It is yet to be shown that $w(x)$ is the temporal component of a current four-vector. The continuity equation for the mentioned four-vector must be such that the spatial integral $\int w d^3x$ befalls to be constant in time. Only then it is guaranteed that w has a probability interpretation. It is noted that the wave function ψ in (2.43) is a column vector similar to the spin wave functions of the Pauli equation (Shankar 1994). Therefore, they are named spinors. The dimension N of the spinor is not yet known but we will be able to decide on this soon. The coefficients $\hat{\alpha}_i$ and $\hat{\beta}$ must obviously be quadratic $N \times N$ (2.42). Thus the Schrödinger-like equation (2.41) and (2.43) represents a system of N coupled first-order differential equations for the spinor

components ψ_i , $i = 1, 2, \dots, N$. We also indicate this point in the notation and write (2.42) in the form

$$\begin{aligned} i\hbar \frac{\partial \psi_\sigma}{\partial t} &= \frac{\hbar c}{i} \sum_{\tau=1}^N \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right)_{\sigma\tau} \psi_\tau + m_0 c^2 \sum_{\tau=1}^N \hat{\beta}_{\sigma\tau} \psi_\tau \\ &\equiv \sum_{\tau=1}^N \left(\hat{H}_f \right)_{\sigma\tau} \psi_\tau \end{aligned} \quad (2.45)$$

Equation (2.42) is a short form of the equation (2.45), in which the four $N \times N$ matrices $(\hat{\alpha}_i)_{\sigma\tau}$ and $\hat{\beta}_{\sigma\tau}$ are expressed in the usual abbreviated form for matrices by $\hat{\alpha}_i$ and $\hat{\beta}$, respectively.

With proper definition of the Dirac matrices, the Dirac equation will become a Lorentz invariant formulation for the wave function⁵.

The problem with the Dirac equation is that it does not apply to spinless particles, which do not obey the Pauli principle, but which do have the problem of negative energy solutions, as one can see by plugging a plane wave solution into the Klein-Gordon equation (Shankar 1994).

⁵ For more information on Dirac matrices, the reader is suggested to see the following references:

1-Halzen, F. and A. D. Martin (1984). Quarks and leptons an introductory course in modern particle physics. New York, Wiley.

2-Shankar, R. (1994). Principles of quantum mechanics. New York, Plenum Press.

3-Greiner, W. (2000). Relativistic quantum mechanics wave equations. Berlin; New York, Springer.

4-Zee, A. (2003). Quantum field theory in a nutshell. Princeton, N.J., Princeton University Press.

2.6 Derivation of the Schrodinger equation from Newtonian Mechanics

In this section, the possible derivations of the Schrödinger equation from Newtonian mechanics will be discussed. Mainly, the original work by Nelson, in 1966, will be reviewed in detail (Nelson 1966).

The basic assumption in this theory is that each particle of mass m goes through a Brownian motion. The corresponding diffusion coefficient of this motion is $\hbar/2m$ and there will be no friction. The effect of the external field is described using Newton's second law, $\mathbf{F} = m\mathbf{a}$, as in the Ornstein-Uhlenbeck theory of the macroscopic Brownian motion with friction. This hypothesis paves the way to derive the Schrödinger equation. However, the physical interpretation of the proposed theory is entirely classical. The wave function constructed in this theory cannot be considered to give a complete description of the state. In spite of the conflict between the suggested theory and the quantum mechanics, an examination of the measurement process implies that, in a restricted framework, the two theories are equivalent.

Consider an electron in an external field. The electron is considered to be a classical particle of mass m . It will be assumed that this electron continuously goes through a Brownian motion. It should be noted that in the study of the macroscopic Brownian motion, the fluid viscosity is a key parameter. However, in the case of an electron, one has to assume zero viscosity so as to conserve Galilean covariance.

It will be helpful to describe the situation in more detail. Consider a hydrogen atom which is in the ground state. The random force causing the Brownian motion and the nucleus attractive Coulomb force keep the electron in a dynamical equilibrium. Its trajectory is very irregular. Generally, the electron is close to the nucleus. However, it may go farther away. This situation is fairly similar to that of a particle which is in a colloidal suspension, in dynamical equilibrium between the osmotic forces and gravity. However, the electron in this case may have other dynamical equilibrium states, at the typical discrete energy levels of the atom.

In this section, the derived equations are nonlinear. However, if the wave function ψ is introduced, in a similar way associated to the kinematical description of the motion, one can find that ψ satisfies the Schrödinger equation. Every solution of the Schrödinger equation can be derived in this way.

This proposed theory in this section cannot be considered as a causal theory at all. However, the probabilistic notions appear in a classical manner. Nonetheless, it is also known that the macroscopic Brownian motion enforces constraints on the accuracy of the measurements if the measuring instruments are depending on it (Barnes and Silverman 1934). If it is assumed that every system is depending on the Brownian motion, this imposes an absolute limit restricting some measurements.

It should be kept in mind that the discussion in this section is restricted to nonrelativistic mechanics theory of particles without spin, in the presence of external fields.

2.6.1 Stochastic mechanics

Many classical physical theories involve stochastic processes. Statistical mechanics is derived from a stationary stochastic process which is a measure-preserving flow in phase space. While the Ornstein-Uhlenbeck theory of the Brownian motion is expressed in terms of the Markov process in phase space (Uhlenbeck and Ornstein 1930; Wang and Uhlenbeck 1945), the Einstein-Smoluchowski theory of the Brownian motion involves the Markov process in coordinate space,.

Consider $x(t)$ to be a stochastic process, for instance, the x coordinate of a particle at time t . It is known that for several key processes, $x(t)$ is not differentiable. This is also true for the Wiener process (Doob 1953), which is involved in the Einstein theory of the Brownian motion. A similar difficulty arises in calculating the velocity in the Ornstein-Uhlenbeck theory. Therefore, one needs to define a proper way to derive the derivatives in stochastic processes.

The mean forward derivative $Dx(t)$ is defined as

$$Dx(t) = \lim_{\Delta t \rightarrow 0+} E_t \frac{x(t + \Delta t) - x(t)}{\Delta t}, \quad (2.46)$$

where E_t is the conditional expectation, for the corresponding state of the system at time t . Therefore, $Dx(t)$ is also a stochastic process. The symbol $0+$ is used in order to indicate that Δt tends to zero through positive values. In the same way, the mean backward derivative $D_*x(t)$ can be defined as

$$D_*x(t) = \lim_{\Delta t \rightarrow 0+} E_t \frac{x(t) - x(t - \Delta t)}{\Delta t}, \quad (2.47)$$

If $x(t)$ is differentiable, then evidently $Dx(t) = D_*x(t) = dx/dt$; however, normally, $D_*x(t)$ is not identical to $Dx(t)$.

2.6.2 The Ornstein-Uhlenbeck Theory

Consider the Ornstein-Uhlenbeck theory of the Brownian motion with friction in the presence of a potential V . $\mathbf{x}(t)$ designates the position of the Brownian particle at time t , $\mathbf{v}(t)$ its velocity, m its mass, and

$$\mathbf{K} = (1/m) \vec{\nabla} V, \quad (2.48)$$

the acceleration of the particle created by V . The system is assumed to be in equilibrium, with the Maxwell-Boltzmann distribution (Blatt 1992). β is considered to be the friction coefficient. Consequently, the Langevin equations (Reif 1965) are

$$d\mathbf{x}(t) = \mathbf{v}(t)dt \quad (2.49)$$

$$d\mathbf{v}(t) = -\beta\mathbf{v}(t)dt + \mathbf{K}(\mathbf{x}(t))dt + d\mathbf{B}(t), \quad (2.50)$$

where $\mathbf{B}(t)$ is a Wiener process which characterizes the residual random impacts at time t . The $d\mathbf{B}(t)$ are Gaussian with mean zero, mutually independent, and

$$Ed\mathbf{B}(t)^2 = 6(\beta kT/m)dt, \quad (2.51)$$

where E designates expectation value, k is the Boltzmann constant, and T is the absolute temperature. The $d\mathbf{B}(t)$ are independent of the $\mathbf{x}(s)$, $\mathbf{v}(s)$ for $s \leq t$ (see Doob's discussion (Doob 1953)). In this case, there is evidently an asymmetry in time. One could also rewrite the Langevin equations with (2.50) replaced by

$$d\mathbf{v}(t) = \beta\mathbf{v}(t) + \mathbf{K}(\mathbf{x}(t))dt + d\mathbf{B}_*(t), \quad (2.52)$$

where the $d\mathbf{B}_*(t)$ are independent of all $\mathbf{x}(s)$, $\mathbf{v}(s)$ for $s \geq t$. By applying the definitions (2.46), (2.47), one may find

$$D\mathbf{x}(t) = D_*\mathbf{x}(t) = \mathbf{v}(t). \quad (2.53)$$

By (2.51), $d\mathbf{B}(t)$ is of the order $dt^{1/2}$, so that $\mathbf{B}(t)$, and as a result $\mathbf{v}(t)$, is not differentiable. However, $D\mathbf{B}(t) = 0$, since $\mathbf{B}(t+\Delta t)$ for $\Delta t > 0$, is independent of the pair $\mathbf{x}(t)$, $\mathbf{v}(t)$ (the state of the system at time t), and has expectation zero. Since D is a linear operator, (2.50) means that

$$D\mathbf{v}(t) = -\beta\mathbf{v}(t) + \mathbf{K}(\mathbf{x}(t)). \quad (2.54)$$

In the same way, from (2.52), one would find that

$$D_*\mathbf{v}(t) = \beta\mathbf{v}(t) + \mathbf{K}(\mathbf{x}(t)) \quad (2.55)$$

It is noted that for a free particle ($\mathbf{K} = 0$), $D\mathbf{v}(t) = -D_*\mathbf{v}(t) = -\beta\mathbf{v}(t)$. Due to the damping effect of friction, the velocity has a natural tendency toward zero, regardless of which direction is taken for time.

(2.54) and (2.55) proceed by

$$\frac{1}{2}DD_*\mathbf{x}(t) + \frac{1}{2}D_*D\mathbf{x}(t) = \mathbf{K}(\mathbf{x}(t)). \quad (2.56)$$

The mean second derivative of the stochastic process is defined to be

$$a(t) = \frac{1}{2}DD_*x(t) + \frac{1}{2}D_*Dx(t). \quad (2.57)$$

$D\mathbf{x}(t)$, $D_*\mathbf{x}(t)$ and $\mathbf{a}(t)$ are called the mean forward velocity, the mean backward velocity, and the mean acceleration if $\mathbf{x}(t)$ is a position vector.

Therefore, in the Ornstein-Uhlenbeck theory of Brownian motion, the Newton second law, $\mathbf{F} = m\mathbf{a}$ is valid, if $\mathbf{F}$ is the external force and $\mathbf{a}$ is the mean acceleration.

2.6.3 Kinematics of the Markov Processes

A Wiener process $\mathbf{w}(t)$ can be used to describe the Brownian motion of a free particle in a fluid, if the time scale of the problem is large enough compare to the relaxation time β^{-1} . As it was previously mentioned, $d\mathbf{w}(t)$ are Gaussian with zero mean, mutually independent, and

$$E d\mathbf{w}_i(t) d\mathbf{w}_j(t) = 2v\delta_{ij} dt, \quad (2.58)$$

where v is the diffusion coefficient and is equal to $kT/m\beta$. The notation v is used in preference to the common notation D for the diffusion coefficient. This is to prevent any

confusion with the mean forward derivative. The position $\mathbf{x}(t)$ of the Brownian particle, in the presence of external forces or currents in the surrounding fluid, usually, satisfies

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t)dt + d\mathbf{w}(t), \quad (2.59)$$

where $\mathbf{b}$ is a vector valued function on space-time. Einstein and Smoluchowski applied this hypothesis to describe their approximate theory of Brownian motion. The limiting case of their theory is the Ornstein-Uhlenbeck theory for large values of β . The $d\mathbf{w}$ are independent of the $\mathbf{x}(s)$ with $s \leq t$, so, by (2.46), $\mathbf{b}$ is the mean forward velocity

$$D\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t). \quad (2.60)$$

This description is asymmetrical in time. One could also propose

$$d\mathbf{x}(t) = \mathbf{b}_*(\mathbf{x}(t), t) + d\mathbf{w}_* \quad (2.61)$$

where $\mathbf{w}_*$ has the same properties as $\mathbf{w}$ except that $d\mathbf{w}_*$ are independent of the $\mathbf{x}(t)$ with $s \geq t$. Thus

$$D_*\mathbf{x}(t) = \mathbf{b}_*(\mathbf{x}(t), t) \quad (2.62)$$

is the mean backward velocity.

This type of process will be consider in more detail, as this kinematical description is used for analyzing the motion of the electron.

$\rho(\mathbf{x}, t)$ will be considered to be the probability density of $\mathbf{x}(t)$. Subsequently, ρ satisfies the forward Focker-Planck equation

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\mathbf{b}\rho) + v\Delta\rho, \quad (2.63)$$

where Δ is the Laplacian operator (Korn and Korn 2000). Similarly, the backward Focker-Planck equation will be

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\mathbf{b}_*\rho) - v\Delta\rho \quad (2.64)$$

This equation was first proposed by Focker and Planck (Focker 1914; Planck 1917). The same equation was derived by A. N. Kolmogorov in his fundamental paper on the Markov processes (Kolmogorov 1931; Gikhman and Skorokhod 1979). Kolmogorov referred to it as the second fundamental equation.

The average of (2.63) and (2.64) leads in the continuity equation

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\mathbf{v}\rho), \quad (2.65)$$

where $\mathbf{v}$ is defined as

$$\mathbf{v} = \frac{1}{2}(\mathbf{b} + \mathbf{b}_*). \quad (2.66)$$

$\mathbf{v}$ is called the current velocity.

Consider f to be a function of $\mathbf{x}$ and t . To derive an expression for $Df(\mathbf{x}(t), t)$, one can expand f in a Taylor series up to the second order terms in $d\mathbf{x}(t)$:

$$df(\mathbf{x}(t), t) = \frac{\partial f}{\partial t}(\mathbf{x}(t), t)dt + d\mathbf{x}(t) \cdot \vec{\nabla} f(\mathbf{x}(t), t) + \frac{1}{2} \sum_{i,j} dx_i(t) dx_j(t) \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}(t), t), \quad (2.67)$$

By (2.59), one might substitute $dx_i(t)$ by $dw_i(t)$ in the last term. By taking the average, one might replace $d\mathbf{x}(t) \cdot \vec{\nabla} f(\mathbf{x}(t), t)$ by $\mathbf{b}(\mathbf{x}(t), t) \cdot \vec{\nabla} f(\mathbf{x}(t), t)$, since $d\mathbf{w}(t)$ is independent of $d\mathbf{x}(t)$ and has zero mean. By using (2.58), one can obtain

$$Df(\mathbf{x}(t), t) = \left(\frac{\partial}{\partial t} + \mathbf{b} \cdot \vec{\nabla} + v\Delta \right) f(\mathbf{x}(t), t). \quad (2.68)$$

Similarly, one can write

$$D_*f(\mathbf{x}(t), t) = \left(\frac{\partial}{\partial t} + \mathbf{b}_* \cdot \vec{\nabla} - v\Delta \right) f(\mathbf{x}(t), t). \quad (2.69)$$

$\rho d^3x dt$ has invariant distribution in space-time. Therefore, $\left(\frac{\partial}{\partial t} + \mathbf{b} \cdot \vec{\nabla} + v\Delta \right)$ and $\left(-\frac{\partial}{\partial t} - \mathbf{b}_* \cdot \vec{\nabla} + v\Delta \right)$ are adjoints of each other with respect to $\rho d^3x dt$. This means

$$\rho^{-1} \left(\frac{\partial}{\partial t} + \mathbf{b} \cdot \vec{\nabla} + v\Delta \right)^+ \rho = -\frac{\partial}{\partial t} - \mathbf{b}_* \cdot \vec{\nabla} + v\Delta, \quad (2.70)$$

where the plus sign designates the Lagrange adjoint. By computing the left hand side of (2.69) and using the forward Fokker-Planck equation (2.63), one can write

$$\mathbf{b}_* = \mathbf{b} - 2v \left(\frac{\vec{\nabla} \rho}{\rho} \right) \quad (2.71)$$

or

$$\mathbf{u} = v \left(\frac{\vec{\nabla} \rho}{\rho} \right) \quad (2.72)$$

where $\mathbf{u}$ is defined as

$$\mathbf{u} = \frac{1}{2} (\mathbf{b} - \mathbf{b}_*). \quad (2.73)$$

According to the Einstein theory of the Brownian motion (Einstein 1926), (2.72) is the velocity gained by a Brownian particle in equilibrium with regard to an external force, to balance the osmotic force. Therefore, $\mathbf{u}$ is called the osmotic velocity. Note that by deducting (2.63) from (2.64) one can write

$$0 = \vec{\nabla} \cdot (\mathbf{u} \rho) - v \Delta \rho = \vec{\nabla} \cdot [\mathbf{u} \rho - v \vec{\nabla} \rho], \quad (2.74)$$

which proceeds (2.72).

By (2.72),

$$\mathbf{u} = v \vec{\nabla} \ln \rho. \quad (2.75)$$

By means of the continuity equation(2.65), one may find

$$\frac{\partial \mathbf{u}}{\partial t} = -v \vec{\nabla} (\vec{\nabla} \cdot \mathbf{v}) - \vec{\nabla} (\mathbf{v} \cdot \mathbf{u}). \quad (2.76)$$

If (2.68) is applied to $\mathbf{b}_*$ and (2.69) to $\mathbf{b}$, and by recalling the definition (2.57) of the mean acceleration $\mathbf{a}$, one can find

$$\mathbf{a} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{b} + \mathbf{b}_*) + \frac{1}{2} (\mathbf{b} \cdot \vec{\nabla}) \mathbf{b}_* + \frac{1}{2} (\mathbf{b}_* \cdot \vec{\nabla}) \mathbf{b} - \frac{1}{2} v \Delta (\mathbf{b} - \mathbf{b}_*). \quad (2.77)$$

By (2.66) and (2.72), $\mathbf{b} = \mathbf{v} + \mathbf{u}$ and $\mathbf{b}_* = \mathbf{v} - \mathbf{u}$, so that (2.77) is equivalent to

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{a} - (\mathbf{v} \cdot \vec{\nabla}) \mathbf{v} + (\mathbf{u} \cdot \vec{\nabla}) \mathbf{v} + v \Delta \mathbf{u}. \quad (2.78)$$

2.6.4 The Hypothesis of Universal Brownian Motion

In this section, the hypothesis that the particles in empty space, undergo the Brownian motion will be investigated. However, it is known that macroscopic bodies do not show such behavior; therefore, it is assumed that the diffusion coefficient of the motion is inversely proportional to the mass, say

$$v = \frac{\hbar}{2m}. \quad (2.79)$$

The constant $\hbar$ has the dimensions of action; later, it will be shown that it can be considered to be the Planck constant divided by 2π . It should be noted that no friction can be associated with the space, otherwise, one could recognize absolute rest from

uniform motion. That is the Brownian motion will not be smooth, and the velocities will not exist. Therefore, the state of the particle cannot be described by a point in space. The motion will be explained using a Markov process in a coordinate space, similar to the Einstein-Smoluchowski theory. However, the diffusion coefficient is $v = \hbar/2m$.

In the Einstein-Smoluchowski theory, the mean acceleration, a , has no important effect on the dynamics of the particle. The above mentioned theory is applied for large friction limits; therefore, the external force $\mathbf{F}$ does not accelerate the particle; however, it simply exposes a velocity $\mathbf{F}/m\beta$ to the particle. In this discussion, the dynamics of the particle is described using Newton's second law, $\mathbf{F} = m\mathbf{a}$. This means that, to examine the Brownian motion, in a medium with no friction, the kinematics of the Einstein-Smoluchowski theory is assumed but by means of Newtonian dynamics as in the Ornstein-Uhlenbeck theory.

Now, consider a particle with mass m in an external force $\mathbf{F}$ which undergoes a Markov process. The location of the particle at time t_0 is given by a point $\mathbf{x}(t_0)$. However, in order to determine the motion of the particle, the characteristics of the Markov process must be known. This means that $\mathbf{b}(\mathbf{x}, t)$ and $\mathbf{b}_*(\mathbf{x}, t)$, or, equivalently, $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ must be determined for all t . On the contrary, $\mathbf{u}$ and $\mathbf{v}$ satisfy (2.76) and (2.78) and the term $\mathbf{a}$ in (2.78) is known; it is $\mathbf{F}/m$. Hence $\mathbf{u}$ and $\mathbf{v}$ satisfy

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\frac{\hbar}{2m}\right) \vec{\nabla}(\vec{\nabla} \cdot \mathbf{v}) - \vec{\nabla}(\mathbf{v} \cdot \mathbf{u}), \quad (2.80)$$

$$\frac{\partial \mathbf{v}}{\partial t} = \left(\frac{1}{m} \right) \mathbf{F} - (\mathbf{v} \cdot \vec{\nabla}) \mathbf{v} + (\mathbf{u} \cdot \vec{\nabla}) \mathbf{u} + \left(\frac{\hbar}{2m} \right) \Delta \mathbf{u}. \quad (2.81)$$

As a result, if $\mathbf{u}(\mathbf{x}, t_0)$ and $\mathbf{v}(\mathbf{x}, t_0)$ are known the Cauchy problem for the coupled nonlinear partial differential equations (2.80) and (2.81) can be solved, then the Markov process will be fully determined.

2.6.5 The Real Time Independent Schrödinger Equation

In the case that the force comes from a potential, $\mathbf{F} = -\vec{\nabla} V$, a solution of (2.80) and (2.81) is desired. Assume that $\mathbf{v} = 0$. This means that by (2.65) and (2.72), ρ and $\mathbf{u}$ are independent of t and the solution is stationary. Therefore, (2.80) leads to $\partial \mathbf{u} / \partial t = 0$ and (2.81) can be written as

$$\mathbf{u} \cdot \vec{\nabla} \mathbf{u} + \left(\frac{\hbar}{2m} \right) \Delta \mathbf{u} = \frac{1}{m} \vec{\nabla} V. \quad (2.82)$$

By (2.75), $\mathbf{u}$ is a gradient, so that $(\mathbf{u} \cdot \vec{\nabla}) \mathbf{u} = \frac{1}{2} \vec{\nabla} u^2$ and $\Delta \mathbf{u} = \vec{\nabla} (\vec{\nabla} \cdot \mathbf{u})$. Hence, (2.82)

can be written as

$$\vec{\nabla} \left(\frac{1}{2} u^2 + \frac{\hbar}{2m} \vec{\nabla} \cdot \mathbf{u} \right) = \frac{1}{m} \vec{\nabla} V, \quad (2.83)$$

or

$$\frac{1}{2} u^2 + \frac{\hbar}{2m} \vec{\nabla} \cdot \mathbf{u} = \frac{1}{m} V - \frac{1}{m} E, \quad (2.84)$$

where E has the dimensions of energy and is a constant. If (2.84) is multiplied by $m\rho$ and is integrated, one can may get

$$\int \frac{1}{2} m \mathbf{u}^2 \rho d^3x - \int (\mathbf{u} \cdot \vec{\nabla} \rho) d^3x = \int V \rho d^3x - E \quad (2.85)$$

By using (2.72), the left-hand side is $-\int \frac{1}{2} m \mathbf{u}^2 \rho d^3x$, so that

$$E = \int \frac{1}{2} m \mathbf{u}^2 \rho d^3x + \int V d^3x \quad (2.86)$$

Hence, E is the average value of $\frac{1}{2} m \left(\frac{dx}{dt} \right)^2$; therefore it can be described as the energy of the particle. In this case, note that $\mathbf{b} = -\mathbf{b}_* = \mathbf{u}$, so one would substitute either $\mathbf{b}$ or $-\mathbf{b}_*$ for $\mathbf{u}$ in computing the mean kinetic energy.

Although, (2.84) is a nonlinear equation, it can be transformed to a linear equation by a change of dependent variable. By (2.75),

$$R = \frac{1}{2} \ln \rho \quad (2.87)$$

is the potential of $m\mathbf{u}/\hbar$. Let

$$\psi = e^R. \quad (2.88)$$

Notice that ψ is real and $\rho = \psi^2$. It is clear that (2.84) is equivalent to the time-independent Schrödinger equation

$$\left[-\left(\frac{\hbar^2}{2m} \right) \Delta + V - E \right] \psi = 0 \quad (2.89)$$

for real ψ . As a result, the theory of Brownian motion for the hydrogen atom results in the proper energy levels for the bound states⁶ of the atom.

2.6.6 The Time-Dependent Schrödinger Equation

Other stationary solutions of (2.80) and (2.81) for which $\mathbf{v} \neq 0$ can be considered. Though, it is easier to discuss the time-dependent case in general.

An additional kinematical assumption is required at this stage. Consider that it should be assumed that $\vec{\nabla} \cdot \mathbf{u} = 0$. Given that $\mathbf{u}$ is always a gradient, one would write (2.81) as

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \vec{\nabla}) \mathbf{v} - \vec{\nabla} p. \quad (2.90)$$

(2.90) is the Euler equation. It is used to describe the motion of an inviscid incompressible fluid with unit density, in the presence of the pressure p . Therefore, this

⁶ Bound states are states in which a potential prevents the particle from escaping to infinity. To read more on bound states, see Shankar, R. (1994). Principles of quantum mechanics. New York, Plenum Press. chapter 5, or, Merzbacher, E. (1998). Quantum mechanics. New York, Wiley. chapter 5.

equation seems to be associated with a diffusion processes with currents flowing in them. This is not what is sought here. As a result, the opposite assumption is made which considers $\mathbf{v}$ to be the gradient. Now, set

$$\vec{\nabla} S = \frac{m}{\hbar} \mathbf{v} \quad (2.91)$$

Keep R as before [Eq. (2.87)], and set

$$\psi = e^{R+iS}. \quad (2.92)$$

It must be shown that ψ satisfies the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = i \frac{\hbar}{2m} \Delta \psi - i \frac{1}{\hbar} V \psi + i \alpha(t) \psi \quad (2.93)$$

This means that one must prove that

$$\left(\frac{\partial R}{\partial t} + i \frac{\partial S}{\partial t} \right) \psi = i \frac{\hbar}{2m} \left(\Delta R + i \Delta S + \left[\vec{\nabla} (R + iS) \right]^2 \right) \psi - i \frac{1}{\hbar} V \psi + i \alpha(t) \psi \quad (2.94)$$

If (2.94) is divided by ψ , the gradients are taken, and real and imaginary parts are separated, then one can conclude that (2.94) is identical to the pair of equations

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\hbar}{2m} \Delta \mathbf{v} - \vec{\nabla} (\mathbf{v} \cdot \mathbf{u}), \quad (2.95)$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\hbar}{2m} \Delta \mathbf{u} + \frac{1}{2} \vec{\nabla} (\mathbf{u}^2) - \frac{1}{2} \vec{\nabla} (\mathbf{v}^2) - \frac{1}{m} \vec{\nabla} V. \quad (2.96)$$

As $\mathbf{u}$ and $\mathbf{v}$ are gradients, (2.95) and (2.96) are equivalent to (2.80) and (2.81).

On the other hand, if any solution to the Schrödinger equation (2.93) is considered, and

normalized so that $\int |\psi|^2 d^3x = 1$, one could write $\psi = e^{R+iS}$, $\mathbf{u} = \hbar \frac{\vec{\nabla} R}{m}$, $\mathbf{v} = \hbar \frac{\vec{\nabla} S}{m}$,

$\mathbf{b} = \mathbf{u} + \mathbf{v}$, $\mathbf{b}_* = \mathbf{v} - \mathbf{u}$, and $\rho = |\psi|^2$. The Markov process with the diffusion

coefficient $\hbar/2m$, forward velocity $\mathbf{b}$, and backward velocity $\mathbf{b}_*$ has probability density

ρ and mean acceleration $\mathbf{a} = -\vec{\nabla} V/m$. This means that every solution of the

Schrödinger equation can be derived in this fashion.

2.7 Summary and Conclusion

In this chapter, a brief history of quantum mechanics and the major attempts and developments in the early days of quantum mechanics was presented. It was explained how quantum mechanics was developed. It was explained that the Schrödinger equation, the corner stone in quantum mechanics, was deduced and not derived from the basic laws of physics.

The paradox of instantaneous propagation of energy arising from the Schrödinger equation was also addressed. It was also underlined that the Schrödinger equation is not a relativistic formula. It was demonstrated how physicists tried to overcome the paradox of instantaneous propagation. They were, more or less, successful in developing theories, like QFT and hidden variables theory, which remove the paradox for certain problems. However, it should be noted that, despite of all the attempts, *no* unique and general formulation which removes the paradox and gives relativistic results in all cases has been proposed. The quest for a general and unique formulation still remains vital.

Then, the work by Nelson in deriving the Schrödinger equation from Newtonian mechanics was reviewed in detail. Nelson's hypotheses will be used in Chapter 3 to propose a new formulation of quantum mechanics.

Chapter 3 Theoretical Background

3.1 Introduction

In Chapter 2, we showed that the Schrödinger equation is not a relativistic equation. It allows instantaneous energy propagation. There have been several attempts to remove the paradox but, the problem is that, there is no unique relativistic equation. Theories like hidden variables theories cause paradoxes which conflict with probability theory. We also mentioned that these paradoxes have been observed experimentally.

In this chapter, we derive the Schrödinger in a heuristic way.

Then, the phase-lagged Schrödinger equation is derived from conservation of energy and the modified Fick's law.

We derive the Schrödinger equation using the same hypothesis in Nelson's work to derive the Schrodinger equation from Newtonian mechanics (Nelson 1966). We discussed Nelson's hypothesis in detail in section 2.6. Nelson's hypothesis is rather proof-based, which means that he tries to prove that, it is possible to derive the Schrödinger equation from Newtonian mechanics. However, in this section we derive the Schrödinger equation in a *constructive way*.

The basic assumptions behind this derivation indicate that, the Schrödinger equation allows for instantaneous information transport which should be corrected. We propose a way to overcome this paradox. We try to illustrate that, there must be a finite time lag between the onset of gradient and its corresponding induced flux.

In the end of this chapter we will discuss the possible consequences of these time lags in our formulations. We hope that our formulations are general. In case of assuming zero time lags, the model becomes identical to the classical Schrödinger equation. Non-zero time lags lead to some transformation of the sources that may describe some other fields. These transformed fields may be either gravitational fields or some other unknown fields.

3.2 Heuristic derivation of the Schrödinger equation

In this section, a heuristic derivation of the Schrödinger equation is provided.

As we mentioned in previous sections, historically, the Schrödinger equation was deduced, but not derived, from the reasoning that if the wave function $\psi(\mathbf{r}, t)$ uniquely defines the state of a system, then, according to the causality principle, the value of the wave function at any moment of time, and hence, the state of the system, can be found from its initial value $\psi(\mathbf{r}, t_0)$.

Consider the moment of time t after the initial moment t_0 , such that $\Delta t = t - t_0 \ll 1$.

Then,

$$\psi(\mathbf{r}, t) = \psi(\mathbf{r}, t_0) + \left. \frac{\partial \psi}{\partial t} \right|_{t=t_0} \Delta t + \dots \quad (3.1)$$

According to what is said previously, the value of $\left. \frac{\partial \psi}{\partial t} \right|_{t=t_0}$ can be found from $\psi(\mathbf{r}, t_0)$,

that is ,

$$\left. \frac{\partial \psi}{\partial t} \right|_{t=t_0} = \hat{T}(\mathbf{r}, t_0) \psi(\mathbf{r}, t_0) \quad (3.2)$$

where $\hat{T}$, usually called the time translation operator (Hilfer 2000), is a certain operation, to which $\psi(\mathbf{r}, t_0)$ is to be subjected, in order to obtain $\left. \frac{\partial \psi}{\partial t} \right|_{t=t_0}$.

Since the initial time moment has been arbitrarily chosen, it follows that

$$\frac{\partial \psi}{\partial t} = \hat{T}(\mathbf{r}, t) \psi(\mathbf{r}, t) \quad (3.3)$$

It follows from superposition of states that the operator $\hat{T}$ is linear. Furthermore, $\hat{T}$ is independent of the derivatives of ψ with respect to time. Indeed, if $\hat{T}$ depends on such derivatives, it means that it would be necessary to know their initial values to determine the state of the system following its initial value. In this case, however, the wave function $\psi(\mathbf{r}, t)$ does not uniquely define the state of the system, which contradicts the initial assumption. The presence of the integral in (3.3) would mean that the state of the system depends on a piece of the past history of this system. Hence, $\hat{T}$ can only contain t as a parameter.

Unfortunately, the form of the time translation operator cannot be found from the above argument. A correct choice of $\hat{T}$ is hinted by considering a free motion with the value of momentum p . The wave function in this case is the de Broglie wave, given by

$$\psi(\mathbf{r}, t) = N e^{-\frac{i}{\hbar}(Et - \mathbf{p} \cdot \mathbf{r})} \quad (3.4)$$

where $E = p^2 / 2m$ is the energy of the particle of mass m .

The wave function given by (3.4) satisfies

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \nabla^2 \psi. \quad (3.5)$$

The latter equation can be written as

$$\frac{\partial \psi}{\partial t} = \hat{T} \psi. \quad (3.6)$$

It follows that, in case of free motion, the time translation operator is given by

$$\hat{T} = -\frac{i\hbar}{2m} \nabla^2. \quad (3.7)$$

In quantum mechanics, one generalizes this particular result: it is *assumed* that the time translation operator is *always* given by (3.7).

3.3 Derivation of the Phase-Lagged Schrödinger Equation from Conservation of Energy

In this section, the phase-lagged Schrödinger equation is derived, assuming that a finite time lag exists between the onset of the energy density gradient and the corresponding energy flux. It is shown that the classical Schrödinger equation arises in the case of the zero lag.

It is now claimed in many courses of quantum mechanics that the Schrödinger equation cannot be derived (Eisberg and Resnick 1985; Blatt 1992). Yet, it has been shown that if a partial differential equation is to adequately describe physical processes, such an equation can be derived from the energy conservation equation (Kulish 2004). Usually, an auxiliary assumption is needed, because the conservation equation contains two unknown functions – energy density and energy flux. Thus, the final form of a partial differential equation depends not only on the conservation equation but also on the chosen form of the auxiliary equation, usually called the constitutive equation, of which examples are Fick's and Fourier's laws, assuming a direct proportionality between the energy flux and the energy density gradient. This is exactly why there are so many different partial differential equations. Each of them is valid only in those cases in which the corresponding constitutive equation is valid.

The Schrödinger equation is not an exception. A link between it and the conservation equation can be established only if an auxiliary equation of the Fickian type is to be employed. This circumstance makes the Schrödinger equation inexact and valid only at

relatively large scales. Consider a domain Ω , whose total energy content is initially E_0 .

Upon introducing the dimensionless energy $w = E / E_0$, the energy conservation equation can be written as

$$\frac{\partial w}{\partial t} + \vec{\nabla} \cdot \mathbf{j} = \Xi(\mathbf{r}, t) \quad (3.8)$$

where $\mathbf{j}$ is the flux and $\Xi(\mathbf{r}, t)$ denotes the source function.

Note that the dimensionless variable w is more convenient than E , because it can be normalized and treated as a distribution

$$\int_{\Omega} w d\mathbf{r} = 1 \quad (3.9)$$

In this sense, it is possible to speak of the probability to find energy at a certain location within the domain at a certain time.

To overcome the paradox of instantaneous propagation, it is assumed that there is a finite time lag, τ , between the onset of the energy density gradient and the corresponding flux, that is

$$\mathbf{j}(\mathbf{r}, t + \tau) = -D \nabla w(\mathbf{r}, t) \quad (3.10)$$

where D is a proportionality coefficient that can in general depend on $\mathbf{r}$ and t . However, for the sake of clarity but without loss in generality, D is assumed constant in this study.

Equation (3.10) in turn can be written as

$$\sum_{k=0}^{\infty} \frac{\tau^k}{k!} \frac{\partial^k \mathbf{j}(\mathbf{r}, t)}{\partial t^k} = -D \nabla w(\mathbf{r}, t) \quad (3.11)$$

where $\frac{\partial^0 \mathbf{j}(\mathbf{r}, t)}{\partial t^0} = \mathbf{j}(\mathbf{r}, t)$.

Upon applying the divergence operator to (3.11), the latter becomes

$$\sum_{k=0}^{\infty} \frac{\tau^k}{k!} \frac{\partial^k [\nabla \cdot \mathbf{j}(\mathbf{r}, t)]}{\partial t^k} = -D \nabla^2 w(\mathbf{r}, t) \quad (3.12)$$

Now, upon combining (3.12) and (3.8), one obtains

$$\begin{aligned} \frac{\partial w(\mathbf{r}, t)}{\partial t} + \sum_{k=1}^{\infty} \frac{\tau^k}{k!} \frac{\partial^{k+1} w(\mathbf{r}, t)}{\partial t^{k+1}} \\ = -D \nabla^2 w(\mathbf{r}, t) + \Xi(\mathbf{r}, t) + \sum_{k=1}^{\infty} \frac{\tau^k}{k!} \frac{\partial^k \Xi(\mathbf{r}, t)}{\partial t^k} \end{aligned} \quad (3.13)$$

which is the phase-lagged energy transport equation without the convective term (Kulish 2004). Note the presence of the apparent energy source in the right side of (3.13). This apparent energy source is responsible for the resonance states within the energy field. Its importance will be analyzed in our future work.

If now one denotes $w(\mathbf{r}, t) = \psi \psi^*$ and $\mathbf{j}(\mathbf{r}, t) = D(\psi \nabla \psi^* - \psi^* \nabla \psi)$, where ψ and ψ^* are complex conjugate functions of $\mathbf{r}$ and t , then, upon applying the Leibniz rule,

$$\frac{\partial^k (\psi \psi^*)}{\partial t^k} = \sum_{n=0}^k C_k^n \frac{\partial^n \psi}{\partial t^n} \frac{\partial^{k-n} \psi^*}{\partial t^{k-n}} \quad (3.14)$$

where $C_k^n = \frac{k!}{n!(k-n)!}$,

$$\begin{aligned} & \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} + \\ & \sum_{k=1}^{\infty} \tau^k \left[\sum_{n=0}^k \frac{k!}{n!(k+1-n)!} \frac{\partial^n \psi}{\partial t^n} \frac{\partial^{k+1-n} \psi^*}{\partial t^{k+1-n}} \right] = \\ & D(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) + \\ & \Xi(\mathbf{r}, t) + \sum_{k=1}^{\infty} \frac{\tau^k}{k!} \frac{\partial^k \Xi(\mathbf{r}, t)}{\partial t^k} \end{aligned} \quad (3.15)$$

where the identity $\nabla \cdot (\psi \nabla \psi^* - \psi^* \nabla \psi) = \psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi$ has been used to write the first two terms in the right hand side.

Equation (3.15) is the Taylor expansion of the phase-lagged Schrödinger equation. It accounts for the presence of a finite time lag between the onset of the energy gradient and the corresponding energy flux. As such, equation (3.15) reduces the paradox of instantaneous propagation hidden behind the classical Schrödinger equation. Therefore, the phase-lagged version of the Schrödinger equation presented here – unlike the classical Schrödinger equation – does not fully contradict the fact that energy propagation takes place at a finite speed.

It is noted that if we apply the Lorentz transformations to the equation (3.15), the mathematical form of this equation changes. This means that, although this equation reduces the instantaneous propagation of energy, it is still **not** a relativistic formulation.

Note also that, if $\tau = 0$ in (3.15), the latter reduces to the two classical Schrödinger equations, written for ψ and ψ^* , respectively, that is,

$$i \frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m} \nabla^2 \psi + \Phi(\mathbf{r}, t) \psi \quad (3.16)$$

and

$$-i \frac{\partial \psi^*}{\partial t} = -\frac{\hbar}{2m} \nabla^2 \psi^* + \Phi(\mathbf{r}, t) \psi^* \quad (3.17)$$

provided that one assumes $D = i\hbar / (2m)$ and $\Xi(\mathbf{r}, t) = 0$ where m is the mass of the system in question and $\Phi(\mathbf{r}, t)$ denotes the field acting on this system. Hence, the classical Schrödinger equation is but a limit case of (3.15).

3.4 Constructive Derivation of the Classical Schrödinger Equation from Newtonian Mechanics

In this section, the Schrödinger equation is derived, in a constructive way, from Newtonian mechanics. Nelson's hypothesis is followed and a free particle which undergoes a Markov process is considered.

All the definitions and equations in section 2.6 hold true in this section too.

Remember that $\rho(\mathbf{x}, t)$ is the probability density function and satisfies the Focker-Planck equations (2.63) and (2.64). Then it must satisfy the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\mathbf{v} \rho). \quad (3.18)$$

where $\mathbf{v}$ is the current velocity. One can denote

$$\mathbf{j} = -\mathbf{v} \rho, \quad (3.19)$$

so that (3.18) becomes

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \mathbf{j} \quad (3.20)$$

Remember from (2.75) that

$$\mathbf{u} = v \vec{\nabla} \ln \rho. \quad (3.21)$$

Where $\mathbf{u}$ is the osmotic velocity according to Einstein's theory of the Brownian motion (Einstein 1926) and v is the diffusion coefficient corresponding to the Brownian motion.

Notice that $\mathbf{u}$ is a gradient.

It was previously, in Eq. (2.87), defined

$$R = \frac{1}{2} \ln \rho, \quad (3.22)$$

or

$$\rho = e^{2R}. \quad (3.23)$$

From (3.21) and (3.22) one can conclude that

$$\mathbf{u} = 2v\vec{\nabla}R \quad (3.24)$$

$\mathbf{v}$ was assumed to be a gradient in equation (2.91):

$$\mathbf{v} = 2v\vec{\nabla}S \quad (3.25)$$

Now, one can assume that the particles, in empty space, are subject to the Brownian motion. It will be assumed that the diffusion coefficient v is as follows:

$$v = \frac{\hbar}{2m}. \quad (3.26)$$

Using this diffusion coefficient, equation (3.24) becomes

$$\mathbf{u} = \frac{\hbar}{m}\vec{\nabla}R, \quad (3.27)$$

and equation (3.25) becomes

$$\mathbf{v} = \frac{\hbar}{m}\vec{\nabla}S. \quad (3.28)$$

Now, one can write (3.19) as

$$\mathbf{j} = \frac{i}{2}\rho(2i\mathbf{v}). \quad (3.29)$$

Equation (3.29) is equivalent to

$$\mathbf{j} = \frac{i}{2} \rho \left(\mathbf{u} + i\mathbf{v} - (\mathbf{u} - i\mathbf{v}) \right). \quad (3.30)$$

Using the equations (3.23), (3.27) and (3.28), one can write (3.30) as

$$\mathbf{j} = \frac{i\hbar}{2m} e^{2R} \left(\vec{\nabla} (R + iS) - \vec{\nabla} (R - iS) \right). \quad (3.31)$$

It is noted that $e^{2R} = e^{R+iS} e^{R-iS}$, so (3.31) becomes

$$\mathbf{j} = \frac{i\hbar}{2m} \left(e^{R-iS} \left(\vec{\nabla} (R + iS) \right) e^{R+iS} - e^{R+iS} \left(\vec{\nabla} (R - iS) \right) e^{R-iS} \right). \quad (3.32)$$

One can rearrange (3.32) in the following way:

$$\mathbf{j} = \frac{i\hbar}{2m} \left(e^{R-iS} \left(\vec{\nabla} e^{R+iS} \right) - e^{R+iS} \left(\vec{\nabla} e^{R-iS} \right) \right). \quad (3.33)$$

Now, if it is assumed that $\psi = e^{R+iS}$, and $\psi^* = e^{R-iS}$, one can write (3.33) as

$$\mathbf{j} = \frac{i\hbar}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right). \quad (3.34)$$

Now, one can substitute (3.34) into (3.18) to get

$$\frac{\partial (\psi \psi^*)}{\partial t} = \vec{\nabla} \cdot \left(\frac{i\hbar}{2m} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \right). \quad (3.35)$$

The identity $\vec{\nabla} \cdot (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) = \psi^* \Delta \psi - \psi \Delta \psi^*$ can be used to rewrite equation (3.35)

in the following way

$$\frac{\partial(\psi\psi^*)}{\partial t} = \frac{i\hbar}{2m}(\psi^* \Delta \psi - \psi \Delta \psi^*) + \frac{i}{\hbar} V \psi \psi^* - \frac{i}{\hbar} V \psi \psi^*. \quad (3.36)$$

Equation (3.36) is equal to

$$\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} = \frac{i\hbar}{2m}(\psi^* \Delta \psi - \psi \Delta \psi^*) + \frac{i}{\hbar} V \psi \psi^* - \frac{i}{\hbar} V \psi \psi^*. \quad (3.37)$$

Equation (3.37) immediately leads to

$$\psi^* \left(\frac{\partial \psi}{\partial t} - \frac{i\hbar}{2m} \Delta \psi + \frac{i}{\hbar} V \psi \right) + \psi \left(\frac{\partial \psi^*}{\partial t} + \frac{i\hbar}{2m} \Delta \psi^* - \frac{i}{\hbar} V \psi^* \right) = 0. \quad (3.38)$$

Notice that $\psi = 0$ and $\psi^* = 0$ lead to trivial answers which are not desirable so, one might conclude

$$\frac{\partial \psi}{\partial t} = i \frac{\hbar}{2m} \Delta \psi - i \frac{1}{\hbar} V \psi, \quad (3.39)$$

and

$$\frac{\partial \psi^*}{\partial t} = -i \frac{\hbar}{2m} \Delta \psi^* + i \frac{1}{\hbar} V \psi^*. \quad (3.40)$$

Equation (3.39) is the Schrödinger equation for a free particle.

3.4.1 Notes on the Constructive derivation of the Schrödinger equation from Newtonian Mechanics

In this section, the Schrödinger equation was derived, from Newtonian mechanics, in a constructive way. The equation of continuity and a couple of constitutive eqns. (3.27) and (3.28) were used to derive the Schrödinger equation.

In this derivation, it becomes clear that the imaginary number i in the Schrödinger equation appears to take care for the phase of the wave function. It is also noticed that, diffusion coefficient of the process is $\hbar / 2m$.

The basic assumption that u and v are produced by potentials should be more emphasized here: equations of the type (3.27) and (3.28) allow for instantaneous information transport over the entire domain of the solution. The Einstein theory of relativity implies that no information, or energy, can propagate with infinite speed. In order to overcome the paradox of instantaneous propagation, it was proposed to use a finite time lag between the onset of the gradient and the corresponding flux induced by the gradient (Cattaneo 1958).

3.5 Derivation of the Coupled Phase-lagged Schrödinger Equation

In the previous section, it was mentioned that equations of the type (3.27) and (3.28) result in instantaneous propagation paradoxes. Hence, according to Cattaneo, one should use a finite time lag between the onset of the gradients and the corresponding fluxes induced by theses gradients (Cattaneo 1958). This means that (3.27) and (3.28) can be written as

$$\mathbf{u}(\mathbf{x}, t) = 2v\vec{\nabla}R(\mathbf{x}, t - \tau_1), \quad (3.41)$$

and

$$\mathbf{v}(\mathbf{x}, t) = 2v\vec{\nabla}S(\mathbf{x}, t - \tau_2). \quad (3.42)$$

Basically τ_1 and τ_2 do not need to be equal as they are produced by two different potentials and the settling times for the two different processes are not necessarily equal. However, $\mathbf{u}$ is introduced into the formulation in (3.30) arbitrarily, so $\mathbf{u}$ can be chosen such that one can assume $\tau_1 = \tau_2 = \tau$ to derive the phase-lagged Schrödinger equation. Hence, using (3.41) and (3.42), (3.30) results in

$$\begin{aligned} \mathbf{j}(\mathbf{x}, t) = & \frac{i\hbar}{2m} \rho(\mathbf{x}, t - \tau) \vec{\nabla} \left(R(\mathbf{x}, t - \tau) + iS(\mathbf{x}, t - \tau) \right) \\ & - \frac{i\hbar}{2m} \rho(\mathbf{x}, t - \tau) \vec{\nabla} \left(R(\mathbf{x}, t - \tau) - iS(\mathbf{x}, t - \tau) \right) \end{aligned} \quad (3.43)$$

It is noticed that ρ times any variable, gives the expectation value of that variable, so in order to keep the physical meaning of each term, the same time lag is introduced to ρ .

Now, by using equation (3.43) instead of (3.31), and repeating the procedure in the same was as from (3.32) to (3.35), one can derive the phase-lagged Schrödinger equation

$$\left. \frac{\partial(\psi\psi^*)}{\partial t} \right|_{(\mathbf{x}, t)} = \frac{i\hbar}{2m} \left(\psi^* \Delta\psi - \psi \Delta\psi^* \right) \Big|_{(\mathbf{x}, t - \tau)}. \quad (3.44)$$

If a source term is introduced in the continuity equation (3.18), one can write

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\mathbf{v} \rho) + \Xi(\mathbf{x}(t), t), \quad (3.45)$$

where $\Xi(\mathbf{x}(t), t)$ is the source term. Using this equation, and repeating the same procedure to derive the Schrödinger equation, one can get the complete coupled phase-lagged Schrödinger equation:

$$\left. \frac{\partial (\psi \psi^*)}{\partial t} \right|_{(\mathbf{x}, t)} = \frac{i\hbar}{2m} \left(\psi^* \Delta \psi - \psi \Delta \psi^* \right) \Big|_{(\mathbf{x}, t-\tau)} + \Xi(\mathbf{x}, t) \quad (3.46)$$

It is clear from (3.46) that, unlike the case of the zero time lag, the equation cannot be separated into two equations: one for the wave function and the other for its complex conjugate. This implies that duality – the coexistence of the two complementary features in any physical process – is fundamental.

3.6 Derivation of the classical Schrödinger Equation from an Electromagnetic Wave Point of View

In this section, the Schrödinger equation is derived from an electromagnetic point of view. A rather different derivation of the Schrödinger equation from the previous section is presented here.

It has been shown that, if a partial differential equation is to adequately describe a physical process, such an equation can be derived from continuity equation (Kulish 2004). In view of this, one must write the continuity equation for the wave function, ψ . It has been proposed to treat the wave function as an electromagnetic wave (Feynman, Leighton et al. January 1, 1989).

Consider a process, in which an electromagnetic wave, ψ , changes. The rate at which this electromagnetic wave changes can be related to the divergence of a certain wave flux, $\mathbf{k}$, and some source term, $\Phi\psi$, which produces the waves in the domain. These statements can be formulated as:

$$\frac{\partial\psi}{\partial t} + \vec{\nabla} \cdot \mathbf{k} = \Phi\psi \quad (3.47)$$

This equation cannot be solely solved, because it contains two unknown quantities, namely ψ and $\mathbf{k}$. One must consider an auxiliary equation (constitutive equation) to make this equation solvable; therefore, the following constitutive equation is assumed:

$$\mathbf{k} = -i \frac{\hbar}{2m} \vec{\nabla} \psi. \quad (3.48)$$

Now, if one substitutes (3.48) into (3.47), the classical Schrödinger equation can be derived:

$$\frac{\partial \psi}{\partial t} = i \frac{\hbar}{2m} \Delta \psi + \Phi \psi \quad (3.49)$$

One can assume the source term to be

$$\Phi = -i \frac{1}{\hbar} V, \quad (3.50)$$

Then equation (3.49) becomes

$$\frac{\partial \psi}{\partial t} = i \frac{\hbar}{2m} \Delta \psi - i \frac{1}{\hbar} V \psi, \quad (3.51)$$

which is identical to the equation (3.39).

3.7 Derivation of the Phase-lagged Schrödinger Equation from an Electromagnetic Wave Point of View

The equations of type (3.48), allow for instantaneous information transport over the entire domain of the solution. In order to overcome the paradox of instantaneous propagation, one has to assume a finite time lag between the onset of the wave gradient and the corresponding wave flux. This means that, (3.48) has to be modified in the following way:

$$\mathbf{k}(\mathbf{x}, t) = -i \frac{\hbar}{2m} \vec{\nabla} \psi(\mathbf{x}, t - \tau). \quad (3.52)$$

Notice that, the time lag, τ , in the equation (3.52) is not equal to the time lag in equations (3.41) and (3.42).

Substituting equation (3.52) into equation (3.47) will result in the phased-lagged Schrödinger equation:

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = i \frac{\hbar}{2m} \Delta \psi(\mathbf{x}, t - \tau) + \Phi(\mathbf{x}, t) \psi(\mathbf{x}, t) \quad (3.53)$$

It is clearly noticed that in case of the zero time lag, this equation becomes identical with the classical Schrödinger equation. Another important feature of this equation is that it is a linear equation.

It is noted that if we apply the Lorentz transformations to the equation (3.53), the mathematical form of this equation changes. This means that, although this equation reduces the instantaneous propagation of energy, it is still not a relativistic formulation.

The advantage of this formulation, compared to the available relativistic formulations and the classical Schrödinger equation, is its generality: it has no constraint on the nature of the particle. It reduces the instantaneous propagation; however, it fails to completely remove it.

3.8 Finiteness of the Time-Lag

As it was previously mentioned, the time-lag in the phase-lagged Schrödinger equation should be finite and greater than zero. It was mentioned that this is due to the quantum nature of the universe. This matter is shortly discussed in this section.

Consider that the characteristic length between two communicators of energy or information is proportional to the Planck distance

$$d \propto \sqrt{\frac{\hbar G}{c^3}} \quad (3.54)$$

where $\hbar$ is the reduced Planck constant, G is the gravitational constant, and c is the propagation speed of electro-magnetic waves in vacuum. Therefore, the minimum time required for the communication to occur between the communicators of energy will be proportional to the Planck time, which is:

$$\tau = \frac{d}{c} \propto \sqrt{\frac{\hbar G}{c^5}}. \quad (3.55)$$

This means that the finiteness of the time-lag is directly related to the finiteness of the Planck constant and, hence, directly related to foundations of the quantum theory.

If the time-lag is assumed to be zero, it will immediately require that either the propagation speed of the energy is infinite or the Planck constant is zero. While the former contradicts relativity, the latter contradicts the foundations of quantum mechanics.

3.9 Recursive Solution of the Phase-Lagged Schrödinger Equation

Consider the phase-lagged Schrödinger equation for a free particle of mass m in three dimensional space

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = i \frac{\hbar}{2m} \nabla^2 \psi(\mathbf{x}, t - \tau). \quad (3.56)$$

The history of the system is considered to be

$$\psi(\mathbf{x}, t - \tau) = H_1(\mathbf{x}, t) \quad \text{for} \quad 0 \leq t \leq \tau. \quad (3.57)$$

For the sake of simplicity one can denote $i \frac{\hbar}{2m}$ as D :

$$D = i \frac{\hbar}{2m}. \quad (3.58)$$

Now, one might rewrite (3.56) using (3.57) and (3.58) as

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = D \nabla^2 H_1(\mathbf{x}, t) \quad \text{for} \quad 0 \leq t \leq \tau \quad (3.59)$$

As (3.57) is known, it is possible to integrate (3.59) to find ψ :

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}, 0) + D \int_0^t \nabla^2 H_1(\mathbf{x}, t) dt \quad \text{for} \quad 0 \leq t \leq \tau \quad (3.60)$$

where $\psi(\mathbf{x}, 0)$ is the initial condition of the equation (3.56). Now we can introduce H_2 as follows

$$\psi(\mathbf{x}, t - \tau) = H_2(\mathbf{x}, t) \quad \text{for} \quad \tau \leq t \leq 2\tau \quad (3.61)$$

Hence, (3.56) can be written as

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = D \nabla^2 H_2(\mathbf{x}, t) \quad \text{for} \quad \tau \leq t \leq 2\tau \quad (3.62)$$

As (3.61) is known from the solution of the previous step, (3.62) can be integrated to find

$\psi(\mathbf{x}, t) \tau \leq t \leq 2\tau$ for $\tau \leq t \leq 2\tau$:

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}, \tau) + D \int_{\tau}^t \nabla^2 H_2(\mathbf{x}, t) dt \quad \text{for} \quad \tau \leq t \leq 2\tau \quad (3.63)$$

It is noted that solution for the equation (3.56) can be found recursively by using the values obtained in each corresponding previous step, in the same manner. After substituting (3.58) as the diffusion coefficient of the process, the general solution of the phase-lagged Schrödinger equation in recursive form, will be as follows:

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}, (n-1)\tau) + \frac{i\hbar}{2m} \int_{(n-1)\tau}^{n\tau} \nabla^2 H_n(\mathbf{x}, t) dt \quad \text{for} \quad (n-1)\tau \leq t \leq n\tau \quad (3.64)$$

where,

$$H_n(\mathbf{x}, t) = \psi(\mathbf{x}, t - \tau) \quad \text{for} \quad (n-1)\tau \leq t \leq n\tau \quad (3.65)$$

and n is a positive integer number. H_1 is known from (3.57) the initial history of the system and H_n is obtained from the previous step of the solution recursively.

This method is called the method of steps. If the history of a phase-lagged system is known, one can use the method of steps to solve the corresponding phase-lagged equation.

3.10 Summary and Conclusion

In this chapter, the derivation of the classical Schrödinger equation was provided based on three rather different viewpoints. For each of these viewpoints, the phase-lagged formulations were also derived. It was shown that, in the case of zero time lags, in all the 3 cases, the phase-lagged formulations are identical to the classical formulation of the Schrödinger equation.

Three possible phased-lagged derivations of the Schrödinger equation were proposed based on the assumption that there exists a *finite* time lag between the onset of gradients and the corresponding flux. The finiteness of the time lag was shown to be related to the finiteness of the Planck constant and the energy propagation speed. The extended version of the Schrödinger equation, therefore, helps to reduce the paradox of instantaneous propagation, intrinsic to the classical Schrödinger equation.

Furthermore, the equations presented, in this chapter, reduce the violation of relativity for, unlike the classical Schrödinger equation, they restrict the speed of energy propagation.

However, at this point, it should be underlined that although the phase-logged Schrödinger equation reduces the paradox of instantaneous propagation, it fails to remove it completely. The proposed phase-lagged Schrödinger equation is still not a

relativistic formulation: the mathematical form of this equation changes under the Lorentz transformations. Hence, the quest for general relativistic equation remains a vital task for future research work.

In the end of the chapter, a method to solve the phase-lagged Schrödinger equation was proposed. It was shown, that if the history of the quantum system is known the solution of the corresponding Schrödinger equation can be found recursively.

Chapter 4 Macroscopic Phenomena and Phase-Lagged Formulation of Quantum Mechanics

In Chapter 3, three approaches to derive phase-lagged formulations were discussed. In this chapter, it is attempted to bridge the phase-lagged Schrödinger equation and the macroscopic transport equations.

4.1 The Classical Schrödinger Equation and Diffusion Equation

The first step is to link the phase-lagged Schrödinger equation and the heat equation. In order to do so, a derivation of the diffusion equation from the classical Schrödinger equation for a classical particle and antiparticle is proposed.

Consider the classical Schrödinger equation for a particle with mass m in a potential V :

$$\frac{\partial \psi}{\partial t} = i \frac{\hbar}{2m} \Delta \psi - i \frac{1}{\hbar} V \psi \quad (4.1)$$

One can take the complex-conjugate of equation (4.1) to write

$$\frac{\partial \psi^*}{\partial t} = -i \frac{\hbar}{2m} \Delta \psi^* + i \frac{1}{\hbar} V \psi^* \quad (4.2)$$

If (4.1) and (4.2) are multiplied by ψ^* and ψ , respectively, they are summed up, and the results are rearranged, the outcome will be

$$\frac{\partial(\psi\psi^*)}{\partial t} = \frac{i\hbar}{2m}(\psi^* \Delta\psi - \psi \Delta\psi^*) \quad (4.3).$$

The identity $\psi^* \Delta\psi - \psi \Delta\psi^* = \nabla \cdot (\psi^* \nabla\psi - \psi \nabla\psi^*)$ can be used to rewrite (4.3) in the following way:

$$\frac{\partial(\psi\psi^*)}{\partial t} = \nabla \cdot \left(\frac{i\hbar}{2m} (\psi^* \nabla\psi - \psi \nabla\psi^*) \right) \quad (4.4).$$

Now, one can introduce $\mathbf{j}$ as

$$\mathbf{j} = \frac{i\hbar}{2m} (\psi^* \nabla\psi - \psi \nabla\psi^*) \quad (4.5).$$

If it is assumed

$$\psi = e^{R+iS} \quad (4.6),$$

and

$$\psi^* = e^{R-iS} \quad (4.7),$$

one can rewrite, and rearrange (4.5) as

$$\mathbf{j} = \frac{i\hbar}{2m} e^{2R} \left(\nabla(R+iS) - \nabla(R-iS) \right) \quad (4.8).$$

Now, the corresponding suitable Brownian motion can be constructed by introducing the following variables as in Chapter 3:

$$\rho = e^{2R} \quad (4.9)$$

which is the probability density function, and

$$\mathbf{u} = 2v\nabla R \quad (4.10)$$

as the osmotic velocity, and

$$\mathbf{v} = 2v\Delta S \quad (4.11)$$

as the current velocity. In (4.10) and (4.11), v represents the diffusion coefficient of the corresponding Brownian motion:

$$v = \frac{\hbar}{2m} \quad (4.12)$$

Now, using equations (4.9), (4.10), and (4.11), one can rewrite (4.8) as

$$\mathbf{j} = i\frac{1}{2}\rho\left(\mathbf{u} - i\mathbf{v} - (\mathbf{u} - i\mathbf{v})\right) \quad (4.13)$$

(4.13) can be further simplified to write

$$\mathbf{j} = -\frac{1}{2}\rho\mathbf{v} \quad (4.14)$$

Now, one can rewrite (4.4), using (4.6), (4.7) and (4.14) as

$$\frac{\partial \rho}{\partial t} = -\nabla(\mathbf{v}\rho) \quad (4.15)$$

which is the continuity equation for the probability density function. Now, as the osmotic velocity, current velocity, and the diffusion coefficient of the Brownian motion are known, according to the Einstein theory of Brownian motion (Einstein 1956), one can assume

$$\rho \mathbf{v} = v \nabla \rho \quad (4.16).$$

This leads to the following diffusion equation for the Brownian particle.

$$\frac{\partial \rho}{\partial t} = v \Delta \rho \quad (4.17).$$

4.2 The Phase-Lagged Schrödinger Equation and The Diffusion Equation

In this section, a derivation of the diffusion equation from the phase-lagged Schrödinger equation is proposed.

The first step is to write the phase-lagged Schrödinger equation for a free particle of mass m :

$$\left. \frac{\partial(\psi\psi^*)}{\partial t} \right|_{(\mathbf{x},t)} = \frac{i\hbar}{2m} \left(\psi^* \Delta \psi - \psi \Delta \psi^* \right) \Big|_{(\mathbf{x},t-\tau)} \quad (4.18)$$

One can use the identity $\psi^* \Delta \psi - \psi \Delta \psi^* = \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*)$ to rewrite (4.18)

in the following way

$$\left. \frac{\partial(\psi\psi^*)}{\partial t} \right|_{(x,t)} = \nabla \cdot \left(\frac{i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right) \Big|_{(x,t-\tau)} \quad (4.19)$$

At this point, a suitable Brownian motion corresponding to this equation will be proposed. One can introduce

$$\mathbf{j}(x,t) = \frac{i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \Big|_{(x,t-\tau)} \quad (4.20)$$

Now, under the assumptions (4.6), (4.7), and (4.9), one can write

$$\begin{aligned} \mathbf{j}(\mathbf{x},t) = & \frac{i\hbar}{2m} \rho(\mathbf{x},t-\tau) \vec{\nabla} \left(R(\mathbf{x},t-\tau) + iS(\mathbf{x},t-\tau) \right) \\ & - \frac{i\hbar}{2m} \rho(\mathbf{x},t-\tau) \vec{\nabla} \left(R(\mathbf{x},t-\tau) - iS(\mathbf{x},t-\tau) \right) \end{aligned} \quad (4.21).$$

Assuming

$$\mathbf{u}(\mathbf{x},t) = 2v \vec{\nabla} R(\mathbf{x},t-\tau) \quad (4.22)$$

as the osmotic velocity and

$$\mathbf{v}(\mathbf{x},t) = 2v \vec{\nabla} S(\mathbf{x},t-\tau) \quad (4.23)$$

to be the current velocity, where, v represents the diffusion coefficient of the corresponding Brownian motion, as in (4.12), one can substitute (4.22) and (4.23) into (4.21) to write

$$\mathbf{j} = \frac{i}{2} \rho \left(\mathbf{u} + i\mathbf{v} - (\mathbf{u} - i\mathbf{v}) \right) = \rho \mathbf{v} \quad (4.24)$$

Now, by using (4.6), (4.7), (4.9), (4.19), and (4.24), one can write

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v} \rho) \quad (4.25)$$

Up to this stage, the osmotic velocity, current velocity, and the diffusion coefficient of the Brownian motion are known. Therefore, according to the Einstein theory of Brownian motion (Einstein 1956), one can assume (4.16) to derive at

$$\frac{\partial \rho}{\partial t} = v \Delta \rho \quad (4.26)$$

It is noted that both the classical and the coupled phase-lagged Schrödinger equations end up in the classical diffusion equation, and not in the phase-lagged diffusion equation.

4.2.1 Transformation of the boundary and initial conditions for the classical Schrödinger equation for the diffusion equation

In this section, a method to transfer the initial and boundary conditions of the phase-lagged Schrödinger equation (4.18), to construct the boundary conditions of the diffusion equation (4.26), is proposed.

Fig 1 Domain of the solution of equation (4.18), where Ω is the domain and Γ is boundary of the domain

4.2.1.1 Transformation of the Initial Condition

Consider the initial condition of the equation (4.18) to be

$$\psi(x, 0) = \psi_0(x) \quad , \quad x \in \Omega \quad (4.27),$$

where Ω is the domain of definition of ψ which is a complex-variable function. One can assume that

$$\psi_0(x) = e^{R_0(x) + iS_0(x)} \quad (4.28),$$

where

$$R_0(x) = R(x, t), S_0(x) = S(x, t) \text{ where } x \in \Omega, t = 0 \quad (4.29).$$

Hence, using (4.9) and (4.28), one can conclude

$$\rho(x, 0) = e^{2R_0(x)} \quad (4.30).$$

Equation (4.30) represents the initial condition of the equation (4.26).

If ψ_0 is not given in polar representation, one can derive the initial condition of equation (4.26) by calculating e^{R_0} from the below equation.

$$e^{R_0(x)} = \sqrt{\text{Re}^2 \psi_0(x) + \text{Im}^2 \psi_0(x)} \quad (4.31)$$

Using (4.9) and (4.31), one can write

$$\rho(x, 0) = \text{Re}^2 \psi_0(x) + \text{Im}^2 \psi_0(x) \quad (4.32),$$

which is the initial condition for the equation (4.26).

4.2.1.2 Transformation of the Dirichlet Boundary Condition

In this section, a method to transform the Dirichlet boundary conditions is proposed. One may assume the following Dirichlet boundary condition for the phase-lagged Schrödinger equation:

$$\psi(x, t) = f(x, t), \quad x \in \Gamma, t > 0 \quad (4.33),$$

where f is an complex-valued function which defines the Dirichlet boundary condition on the boundary, Γ , and for positive values of time, t .

Based on the assumption (4.6), the definition (4.9) and (4.33), one can write

$$\rho(x, t) = \text{Re}^2 f(x, t) + \text{Im}^2 f(x, t), \quad x \in \Omega, \quad t > 0 \quad (4.34).$$

Equation (4.34) is the transformed Dirichlet boundary condition for the equation (4.26).

4.2.1.3 Transformation of the Neumann Boundary Condition

Transformation of the Neumann boundary condition is not as straightforward as the transformation of the initial and the Dirichlet conditions. This time, the derivation will not lead to an explicit solution.

Consider the following Neumann boundary condition for the Schrödinger equation, (4.18), is imposed:

$$\frac{\partial \psi(x, t)}{\partial n} = \psi_n(x, t), \quad x \in \Gamma, \quad t > 0 \quad (4.35),$$

where $\frac{\partial}{\partial n}$ represents the normal derivative operator and ψ_n is an complex-valued function which is defined on the boundary, Γ , and is defined for positive times, t .

Now, one can represent ψ by its real and imaginary parts:

$$\psi(x, t) = \operatorname{Re} \psi(x, t) + i \operatorname{Im} \psi(x, t) \quad (4.36).$$

It is possible to do the same procedure for $\psi_n(x, t)$ and represent it by its real and imaginary parts:

$$\psi_n(x, t) = \operatorname{Re} \psi_n(x, t) + i \operatorname{Im} \psi_n(x, t) \quad (4.37).$$

One would compare (4.35), (4.36) and (4.37) to conclude

$$\begin{aligned} \frac{\partial \operatorname{Re} \psi(x, t)}{\partial n} &= \operatorname{Re} \psi_n(x, t) \\ \frac{\partial \operatorname{Im} \psi(x, t)}{\partial n} &= \operatorname{Im} \psi_n(x, t) \end{aligned}, \quad x \in \Gamma, t > 0 \quad (4.38)$$

Remember from the definition of ρ that

$$\rho(x, t) = \operatorname{Re}^2 \psi(x, t) + \operatorname{Im}^2 \psi(x, t) \quad (4.39).$$

One would attempt to construct the Neumann boundary condition for ρ , in the equation (4.26), in the following way:

$$\begin{aligned} \frac{\partial \rho}{\partial n} &= \frac{\partial}{\partial n} (\operatorname{Re}^2 \psi + \operatorname{Im}^2 \psi) \\ &= 2 \left(\operatorname{Re} \psi \frac{\partial \operatorname{Re} \psi}{\partial n} + \operatorname{Im} \psi \frac{\partial \operatorname{Im} \psi}{\partial n} \right), \quad x \in \Gamma, t > 0 \end{aligned} \quad (4.40).$$

Using equation (4.38), the condition (4.40) becomes

$$\frac{\partial \rho(x, t)}{\partial n} = 2 (\operatorname{Re} \psi \operatorname{Re} \psi_n + \operatorname{Im} \psi \operatorname{Im} \psi_n) \quad (4.41).$$

It is noted that in condition (4.41), $\text{Re } \psi_n$ and $\text{Im } \psi_n$ are known from the Neumann condition (4.35), while $\text{Re } \psi$ and $\text{Im } \psi$ are unknown and they cannot be explicitly derived from (4.35). In order to specify these two unknowns, one has to solve an additional equation. However, based on the equation which is chosen to be solved to specify these values, the result will be different. Two equations are to be chosen between: the classical Schrödinger equation, (4.1), or the phase-lagged Schrödinger equation, (4.18).

4.3 The relationship between the fluid velocity and de Broglie's wave function

In this section, the shortcomings and the hidden assumptions behind a previously published work, by Kulish and Lage (Kulish and Lage 2002), will be critically discussed. In the above mentioned work, the authors demonstrate the existence of a close relationship between the Schrödinger equation and the continuity equation as well as the diffusion equation. This is done by investigating the relationship between the group velocity of the de Broglie waves and a particle velocity.

The relationship discussed, in their work, leads to the proportionality between the fluid velocity v and the corresponding de Broglie's wave at the same location. The authors propose that the velocity of the fluid particles can be written in the form $v = \nabla \tilde{\theta}$, where $\tilde{\theta}$ is a scalar function proportional to the phase of the de Broglie wave. This has a crucial implication which has not been known to the authors at the time: if the fluid velocity has such a relationship with the phase of de Broglie's wave, the fluid can be considered as a superfluid according to Leggett's definition of superfluidity (Leggett 1991). However, it is known that the ordinary fluids do not show superfluid behavior under normal conditions.

It has been recently shown that the ordinary fluids, in ultra-short processes, can demonstrate superfluid behavior (Kulish and Chan 2006; Kulish and Chan 2007). This urges the author of this manuscript to investigate the possible hidden assumptions and the situations which may relate the work by Kulish and Lage (Kulish and Lage 2002)

to the above mentioned works which investigate high temperature superfluidity in ordinary fluids.

At this point, it is necessary to discuss the work by Kulish and Lage, in more detail.

4.3.1 Wave Function

Any freely moving particle, which has energy E and momentum $\mathbf{p}$, can be associated with a flat wave

$$\psi(\mathbf{r}, t) = C \exp[i(\omega t - \mathbf{k}\mathbf{r})], \quad (4.42)$$

where $\mathbf{r}$ is the radius-vector of an arbitrary point in the space, C is the amplitude of the wave function, and t is the time (de Broglie 1923). The wave number, $\mathbf{k}$, of this wave and its frequency ω are related to the particle momentum $\mathbf{p}$ and energy E by the same equations which are valid for a photon, i.e.,

$$\mathbf{p} = \hbar \mathbf{k}, \quad (4.43)$$

$$E = \hbar \omega, \quad (4.44)$$

where $\hbar$ is the Planck constant. Equations (4.43) and (4.44) are called de Broglie's equations. By substituting (4.43) and (4.44) into (4.42), one can write

$$\psi(\mathbf{r}, t) = C \exp \left[i \left(\frac{Et}{\hbar} - \frac{\mathbf{p}\mathbf{r}}{\hbar} \right) \right]. \quad (4.45)$$

This wave is called the de Broglie wave. The function ψ is the wave function similar to the previous sections. As mentioned before, this function is used to determine the probability density of the presence of the particle in a certain state. The probability density is proportional to the squared modules of the wave function.

A given function ψ corresponds to a certain state of a particle, i.e. the velocity of the particle. It should be noted that, if the wave function $\psi(\mathbf{r}, t)$ is possible for a certain state of the particle, then the wave function $\psi_1(\mathbf{r}, t) = \psi(\mathbf{r}, t) \exp(i\theta)$ is also possible, provided that θ is real. This is due to the fact that the amplitudes of the two wave functions are the same so that the probability densities determined by these wave functions are the same. This implies that the two wave functions describe the same state.

4.3.2 Group Velocity

Consider a one dimensional de Broglie wave, propagating in the x -axis direction of a Cartesian coordinate system, i.e.

$$\psi(x, t) = C \exp \left[i(\omega t - kx) \right]. \quad (4.46)$$

The quantity $(\omega t - kx)$ is the phase of the wave. Now, consider a certain point, x , where the phase has a given value, α . The location of this point can be found from the equation

$$\alpha = \omega t - kx. \quad (4.47)$$

One can see that the phase value α moves in the course of time along the x -axis with the velocity, u . This velocity is called the phase velocity and its value can be determined by differentiating $\alpha = \omega t - kx$ with respect to time, and equating the result to zero. Hence,

$$\frac{d\alpha}{dt} = \omega - ku = 0 \quad \Rightarrow \quad u = \frac{\omega}{k}. \quad (4.48)$$

Now, consider a group of the de Broglie waves, i.e. a superposition of waves whose wave lengths belong to a very narrow interval. The wave function of this group can be written as

$$\psi(x, t) = \int_{k_0 - \Delta k}^{k_0 + \Delta k} C(k) \exp[i(\omega t - kx)] dk, \quad (4.49)$$

where k_0 is the wave number, around which the wave number of other waves, forming the group are located. The value Δk is considered to be very small.

The above assumption is at great debate. The importance and the consequences of this assumption are discussed in section 4.3.4.

The energy of a nonrelativistic particle can be written as

$$E = [m_0^2 c^4 + p^2 c^2]^{1/2} = m_0 c^2 + \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3 c^2} + \dots \quad (4.50)$$

where m_0 is the rest mass of the particle, c is the speed of light, and p is the momentum of the particle (Blokhintsev 1976).

In equation (4.50), it is noticed that $m_0 c^2$ is the part of energy relating to the mass of the particle while pc is the energy of the particle due to its translational momentum. Therefore, in this equation, none of the terms account for the rotational energy of the particle. This means that equation (4.50) is valid only for monatomic particles. On the other hand, this equation does not account for the interaction energy of the particles of the fluid. Hence, one other restriction for this equation is that it can only be used for the cases where the interaction energy is negligible. In other words, the use of this equation is also restricted to the case of dilute gasses.

Now, by substituting equation (4.50) into (4.44) and expressing p^2 through k^2 using equation (4.43), one can write

$$\omega = \frac{m_0 c^2}{\hbar} + \frac{\hbar k^2}{2m_0} - \frac{\hbar^3 k^5}{8m_0^3 c^2} + \dots \quad (4.51)$$

On the other hand, one can expand the frequency ω as a function of k using the Taylor series around k_0 :

$$\omega = \omega_0 + \left(\frac{d\omega}{dk} \right)_0 (k - k_0) + \dots, \quad k = k_0 + (k - k_0) \quad (4.52)$$

By introducing an auxiliary integration variable $\xi = k - k_0$ and assuming that $C(k)$ is a slowly varying function of k , one finds:

$$\psi(x, t) = C(k_0) \exp \left[i(\omega_0 t - k_0 x) \right] \int_{-\Delta k}^{\Delta k} \exp \left\{ i \left[\left(\frac{d\omega}{dk} \right)_0 t - x \right] \xi \right\} d\xi, \quad (4.53)$$

where $C(k_0)$ is the average value of $C(k)$ on the interval $[k_0 - \Delta k, k_0 + \Delta k]$. $C(k_0)$

can be considered as a constant. The result of the integration can be written as

$$\begin{aligned}\psi(x, t) &= 2C(k_0) \frac{\sin \left\{ \left[\left(\frac{d\omega}{dt} \right)_0 t - x \right] \Delta k \right\}}{\left(\frac{d\omega}{dt} \right)_0 t - x} \exp[i(\omega_0 t - k_0 x)] \\ &= C(x, t) \exp[i(\omega_0 t - k_0 x)]\end{aligned}\quad (4.54)$$

Notice that $C(x, t)$ is the wave amplitude corresponding to this group of the de Broglie waves. As Δk is assumed to be very small, $C(x, t)$ varies slowly with time and x . Hence, $C(x, t)$ can be considered as the amplitude of an almost monochromatic wave, whose phase is $(\omega_0 t - k_0 x)$. It is apparent that the maximum value of the amplitude is at $x = \left(\frac{d\omega}{dt} \right)_0 t$. This point is called the center of the wave group. It moves with velocity $U = d\omega / dk$. This velocity is the group velocity which is defined as

$$U \equiv \frac{d\omega}{dk}. \quad (4.55)$$

One can use equation (4.51) together with the above definition to find the group velocity of the wave group:

$$U = \frac{\hbar k}{m_0} - \frac{\hbar^3 k^3}{m_0^3} \frac{1}{2c^2} + \dots \quad (4.56)$$

On the other hand, by using the equation (4.43), and the definition of the linear momentum, $p = \hbar k = m_0 v$, one can write

$$U = v \left(1 - \frac{v^2}{2c^2} + \dots \right). \quad (4.57)$$

Therefore, in a nonrelativistic case, one can conclude that

$$U = v. \quad (4.58)$$

That is the group velocity of the wave packet, associated with the group of particles, which is equal to the velocity of this group of particles. This result, which was derived for a one dimensional case, can be easily extended to a multi-dimensional case (Blokhintsev 1976):

$$U_x = \frac{\partial \omega}{\partial k_x} = \frac{\partial E}{\partial p_x}, \quad U_y = \frac{\partial \omega}{\partial k_y} = \frac{\partial E}{\partial p_y}, \quad U_z = \frac{\partial \omega}{\partial k_z} = \frac{\partial E}{\partial p_z}, \quad (4.59)$$

or, in a vector form,

$$\vec{U} = \nabla_k \omega = \nabla_p E = \vec{v}. \quad (4.60)$$

4.3.3 Schrödinger's Equation and the Continuity Equation

In Chapter 3, the relationship between the Schrödinger equation and the Continuity equation was discussed in detail. However, as this relationship is of great importance in the discussion, this topic will be briefly reviewed.

As it was discussed earlier, the Schrödinger equation, for a non-relativistic particle of mass m , and a potential field $\Phi(\mathbf{x}, t)$ is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \Phi(\mathbf{x}, t) \psi. \quad (4.61)$$

One can write the Schrödinger equation for the complex-conjugate wave function

$\psi^* = C \exp(-i\theta)$, namely,

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi^* + \Phi(\mathbf{x}, t) \psi^* \quad (4.62)$$

By multiplying equation (4.61) by ψ^* , and (4.62) by ψ , and consequently subtracting the resulting equations one obtains:

$$i\hbar \left(\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \right) = -\frac{\hbar^2}{2m} (\psi^* \Delta \psi - \psi \Delta \psi^*) \quad (4.63)$$

This equation can be rewritten as

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (4.64)$$

where

$$\mathbf{j} = i \frac{\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*), \quad (4.65)$$

and

$$w = \psi \psi^*. \quad (4.66)$$

w is called the probability density and $\mathbf{j}$ is considered as the corresponding flux to this probability density. Hence, if w is thought to be the probability density of the particles, then $\mathbf{j}$ should be interpreted as the flux of the particles.

It should be noted that equation (4.64) is the most general form of the conservation equation. Multiplying this equation by any physical property will result in the continuity equation for that physical property. Therefore, if equation (4.64) is multiplied by the mass of the a particle, the resulting equation is

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \mathbf{j}_m = 0, \quad (4.67)$$

where ρ_m is the particle density and $\mathbf{j}_m$ is the density flux. Therefore equation (4.67) is nothing but the continuity equation.

Recall the expression of the wave function given by equation (4.42). This equation can be written in a more compact form

$$\psi(\mathbf{x}, t) = C \exp(i\theta), \quad (4.68)$$

where C is the real amplitude and θ is the real phase of the wave function. By substituting equation (4.68) into (4.65), one obtains

$$\mathbf{j} = C^2 \frac{\hbar}{m} \nabla \theta. \quad (4.69)$$

Since $C^2 = w$, then the quantity $(\hbar / m) \nabla \theta$ must be the velocity at the point $\mathbf{x}$. In other, words:

$$v(\mathbf{x}) = \frac{\hbar}{m} \nabla \theta = \nabla \tilde{\theta} \quad (4.70)$$

The conclusion drawn above is of great importance. Kulish and Lage overlooked the important consequences of this result (Kulish and Lage 2002). According to Leggett's definition of superfluidity, if the fluid velocity can be written in terms of the gradient of the phase of the de Broglie wave, the fluid is superfluid (Leggett 1991). They derive this result as if there are no assumptions involved and it can be considered as a general conclusion for any ordinary fluid. However, it has been shown that this result is correct only under very restrictive conditions.

4.3.4 Wave Packet Assumption

In section 4.3.2, a group of the de Broglie waves, i.e. a superposition of waves whose wave lengths belong to a very narrow interval, was considered. The wave function of this group was written as

$$\psi(x, t) = \int_{k_0 - \Delta k}^{k_0 + \Delta k} C(k) \exp[i(\omega t - kx)] dk, \quad (4.71)$$

where k_0 is the wave number, around which the wave numbers of other waves, forming the group are located. The value Δk was assumed to be very small.

The above lines have the following meaning: It is possible to, somehow, approximate the wave functions of a group of fluid particles with one single state $\psi(x, t)$. It is stated that this is possible due to the fact that the wave lengths of the wave functions

of these particles belong to a very narrow interval. However, this assumption is at great debate.

The velocity distribution of the ordinary fluid particles follows the Maxwell-Boltzmann law of velocity distributions (Feynman, Leighton et al. 1989). This means that, from a microscopic point of view, normally, the velocity of the fluid particles varies significantly from one particle to another. In other words, the wave function of a fluid particle is normally significantly different from the other fluid particles wave functions. Therefore, the assumption that it is possible to approximate the wave functions of a number of particles with only one single state is not physical.

This assumption is the key assumption which leads to the superfluidity conclusions. This assumption means that the Bose-Einstein Condensation (BEC) occurs. Let's review a few words by A. Leggett:

“Given that BEC occurs, that there exists one and only one single particle state $\psi(x, t)$ which is macroscopically occupied, the concept of superfluidity is quite simple” (Leggett 1999). Then he writes:

$$\psi(x, t) = C(x, t) \exp(i\theta). \quad (4.72)$$

Consequently he defines the superfluid velocity as

$$v \equiv \frac{\hbar}{m} \nabla \theta. \quad (4.73)$$

One would notice that equation (4.73) proposed by A. Leggett is identical to the equation (4.70). This is due to the fact that the assumptions behind both of these two works were identical: one and only one single particle state is macroscopically occupied.

As it was mentioned earlier, this assumption is not normally satisfied. However, in some special cases, this condition can be satisfied. The examples of this situation are when BEC occurs, i.e. in low temperature superfluidity (Leggett 1999), or in startup flows (Kulish and Chan 2006; Kulish and Chan 2007).

4.3.5 Review of the Assumptions and Conditions for the Presented Derivations

In this section, the vital assumptions, behind the conclusion (4.70), are reviewed. These derivations are valid under the following conditions and assumptions:

1. A group of particles can be considered as a wave packet. This assumption is not normally satisfied since the particles in a fluid have a random distribution according to the Maxwell-Boltzmann law of velocity distribution. However, under some restrictive conditions, this assumption may be satisfied, i.e., due to some boundary conditions for ultra-fast processes (Kulish and Chan 2006; Kulish and Chan 2007).
2. The rotational energy of the particles can be neglected, i.e., monatomic particles satisfy this condition.
3. The interaction energy of particles of the fluid is negligible. Dilute gasses satisfy this condition.

4. A nonrelativistic case is considered.

As it was mentioned earlier, in this section, these assumption were originally overlooked by Kulish and Lage in their work on the relationship between de Broglie's waves and the fluid velocity (Kulish and Lage 2002). In the mentioned work, it seemed that the results presented, in their paper, are generally true for any ordinary fluid. However, as it was presented here, their results are valid only under some very restrictive conditions. The importance of these conditions should not be overlooked, otherwise, the derivations, in the mentioned work (Kulish and Lage 2002), would result in unphysical conclusions.

4.4 Superfluid Decay

In this section, it is assumed that there exists a situation where the wave-packet formation is possible (e.g. a number of particles take the same velocity due to some boundary conditions.) The purpose of this section is to investigate how long it takes for the wave-packet to decay.

As previously mentioned, the particles belonging to the wave packet are believed to be in close states so that it can be thought that they belong to the same state. Due to this property they can be considered as superfluid particles according to Leggett (Leggett 1991). Based on the definition by Leggett, from now on, the particles belonging to such a wave-packet are referred to as superfluid particles.

It is assumed that the superfluid particles, or the wave-packet particles, are formed homogeneously throughout a normal fluid. The corresponding density of the superfluid is ρ_s , while the normal fluid density is ρ_n .

It is known, while all the superfluid particles occupy the same state, the normal fluid particles occupy random states. As it was discussed before, it is possible to associate a state with particle velocity. So, for the sake of simplicity, it is possible to assume that the superfluid particles have the same speed, v_s , while the normal fluid particles can take random velocities. The velocity of the normal fluid particles, v_n , will have a distribution according to the Maxwell-Boltzmann law of velocity distribution:

$$f(v_n) = 4\pi \left[\frac{M}{2\pi R T} \right]^{3/2} v_n^2 \exp \left[\frac{-M v_n^2}{2 R T} \right], \quad (4.74)$$

where $f(v_n)$ is the velocity distribution function.

As the normal fluid particles have random velocities, when they collide with superfluid particles, they will transfer random amounts of energy to the superfluid particles. Therefore, the velocity distribution among the superfluid particles which have undergone a collision with normal fluid particles will be a random distribution: they will no longer behave as superfluid particles. In order to estimate the decaying time of the superfluid (wave-packet), it is necessary to study the interaction of the superfluid and normal fluid particles in the course of time.

In this section, the formulation of superfluid decay will be presented.

$$n_s(t) = \alpha_\tau(t - \tau) n_s(t - \tau) \quad (4.75)$$

Equation (4.75) is called the superfluid decay equation. In this equation, α_τ is the remainder-factor. In general, it can be a function of τ , $n_s(t)$ and $n_n(t)$. It shows the portion of the superfluid particles, at time $t - \tau$, which will remain superfluid after the time τ .

The best situation is when α_τ is a function of τ only and not a function of $n_s(t)$. In order to do that, one should define τ properly so that the remainder-factor will become independent of time. In this case, one can rewrite (4.75) as:

$$n_s(t) = \alpha_\tau^{t/\tau} n_s(t=0) \quad (4.76)$$

In the following parts of this section, first, some attempts to define a references for τ are presented. Then, the remainder-factor, α_τ , is determined.

It is beneficial to start with the definition of the mean free path:

Relative Mean Free Path: The relative mean free path for a normal fluid relative to a superfluid, $l_{n/s}$, is defined as the average distance for a normal fluid particle to collide with a target (superfluid) particle. It is also possible to define $l_{s/n}$, which is the average distance for a superfluid particle to collide with a target (normal fluid) particle.

Fig 2 Effective collision area

In Fig 2, the effective collision area is shown. Based on the effective collision area, one can define the effective collision volume which is swept by the colliding particle.

Fig 3 Effective swept collision volume

In Fig 3, the effective swept collision volume is shown. In this picture, $\langle v_{rel} \rangle$ is the expectation value for the relative velocity of the corresponding particles for each definition of mean free path, and t is the traveling time of the particle.

It should be noted that if the relative velocity expectation is used for calculations, the target particles can be treated as stationary targets.

Based on these definitions, one can calculate mean free path of a projectile particle relative to a target particle as:

$$\begin{aligned}
 l_{p/tp} &= \frac{\langle v_p \rangle t}{\pi d^2 \langle v_{rel} \rangle t n_{tp}} \\
 &= \frac{\langle v_p \rangle}{\pi d^2 \langle v_{rel} \rangle n_{tp}}
 \end{aligned}
 \tag{4.77}$$

In equation (4.77), $\langle v_p \rangle t$ is the average distance traveled in time t by the projectile particle, and $\pi d^2 \langle v_{rel} \rangle t n_{tp}$ is the number of the target particles in the effective swept volume, while n_{tp} is the number of the target particles per unit of volume.

One needs an expression for $\langle v_{rel} \rangle$. The following figure may be helpful to illustrate the situation:

Fig 4 The particles have relative velocity.

Fig 4 shows that the particles of the fluid have relative velocity, so the target particles cannot be considered to be stationary unless the relative velocity is taken into account.

For normal fluid particles, as the projectile particles and target particles are of the same type, one can let

$$l = l_{n/n} \quad (4.78)$$

Thus, the relative velocity for two particles, namely, particle 1 and particle 2, will be

$$v_{rel} = v_1 - v_2. \quad (4.79)$$

where v_{rel} , v_1 and v_2 are vector quantities representing the relative velocity of the two particles, velocity of particle 1 and 2, respectively. Therefore, the corresponding expectation value of the relative velocity can be calculated as follows:

$$\begin{aligned}\langle v_{rel} \rangle^2 &= \langle v_{rel} \cdot v_{rel} \rangle \\ &= \langle v_1 \cdot v_1 - 2v_1 \cdot v_2 + v_2 \cdot v_2 \rangle \\ &= \langle v_1 \cdot v_1 \rangle - 2\langle v_1 \cdot v_2 \rangle + \langle v_2 \cdot v_2 \rangle\end{aligned}\quad (4.80)$$

However, as v_1 and v_2 have random magnitude and direction, the component $\langle v_1 \cdot v_2 \rangle$, which is the expectation value of the dot product of the two vectors, will be zero, and also one can say that:

$$\langle v_1 \cdot v_1 \rangle = \langle v_2 \cdot v_2 \rangle = \langle v \cdot v \rangle = \langle v \rangle^2 \quad (4.81)$$

Hence, the expectation value of the relative velocity will be

$$\begin{aligned}\langle v_{rel} \rangle &= \langle v_{n/n} \rangle \\ &= \sqrt{2} \langle v_n \rangle\end{aligned}\quad (4.82)$$

Using equations (4.77) and (4.82), one can calculate the mean free path of the normal fluid particles:

$$\begin{aligned}l_{n/n} &= \frac{\langle v_n \rangle}{\pi d^2 \sqrt{2} \langle v_n \rangle n_n} \\ &= \frac{1}{\sqrt{2} \pi d^2 n_n}\end{aligned}\quad (4.83)$$

Similarly, using equation (4.77), one can derive mean free paths:

$$l_{n/s} = \frac{\langle v_n \rangle}{\pi d^2 \langle v_{n/s} \rangle n_s} \quad (4.84)$$

$$l_{s/n} = \frac{\langle v_s \rangle}{\pi d^2 \langle v_{s/n} \rangle n_n} \quad (4.85)$$

In equations (4.84) and (4.85), $\langle v_{n/s} \rangle$ and $\langle v_{s/n} \rangle$ must be predetermined to find out the corresponding relative mean free paths, $l_{n/s}$ and $l_{s/n}$. One can determine these values using the same hypothesis as equation (4.80):

$$\begin{aligned} \langle v_{n/s}^2 \rangle &= \langle v_{n/s} \cdot v_{n/s} \rangle \\ &= \langle (v_n - v_s) \cdot (v_n - v_s) \rangle \\ &= \langle v_n^2 - 2v_n \cdot v_s + v_s^2 \rangle \end{aligned} \quad (4.86)$$

$$\begin{aligned} \langle v_{s/n}^2 \rangle &= \langle v_{s/n} \cdot v_{s/n} \rangle \\ &= \langle (v_s - v_n) \cdot (v_s - v_n) \rangle \\ &= \langle v_s^2 - 2v_n \cdot v_s + v_n^2 \rangle \end{aligned} \quad (4.87)$$

Considering equations (4.86) and (4.87), it is noted that

$$\begin{aligned} \langle v_{n/s}^2 \rangle &= \langle v_{s/n}^2 \rangle \\ &= \langle v_n^2 - 2v_n \cdot v_s + v_s^2 \rangle \\ &= \langle v_n^2 \rangle - 2\langle v_n \cdot v_s \rangle + \langle v_s^2 \rangle \end{aligned} \quad (4.88)$$

In equation (4.88), v_n , velocity of the normal fluid particle, is a vector whose direction and magnitude is random, thus, one can state that

$$\langle v_n \cdot v_s \rangle = 0 \quad (4.89)$$

Using equations (4.88) and (4.89), one can conclude

$$\begin{aligned}\langle v_{n/s}^2 \rangle &= \langle v_{s/n}^2 \rangle \\ &= \langle v_n^2 \rangle + \langle v_s^2 \rangle\end{aligned}\quad (4.90)$$

Taking into account equations (4.84), (4.85) and (4.90), it is possible to write:

$$l_{n/s} = \frac{\langle v_n \rangle}{\pi d^2 \sqrt{\langle v_n \rangle^2 + \langle v_s \rangle^2} n_s} \quad (4.91)$$

$$l_{s/n} = \frac{\langle v_s \rangle}{\pi d^2 \sqrt{\langle v_n \rangle^2 + \langle v_s \rangle^2} n_n} \quad (4.92)$$

4.4.1 Relative Mean Free Time

Up to this stage, relative mean free path idea has been discussed. It is beneficial, now, to discuss the idea of relative mean free time:

Relative mean free time is the average time required for a projectile particle to hit a target particle. For each relative mean free path, there exists a corresponding relative mean free time.

$$\tau_{p/tp} = \frac{l_{p/tp}}{\langle v_p \rangle} \quad (4.93)$$

In equation (4.93), $\tau_{p/tp}$ refers to the relative mean free time of a projectile particle, p , relative to a target particle, tp . Consequently, one can find the relative mean free times corresponding to relative mean free paths, (4.83), (4.91) and (4.92):

$$\tau_{n/n} = \frac{1}{\sqrt{2\pi d^2} \langle v_n \rangle n_n} \quad (4.94)$$

$$\tau_{n/s} = \frac{1}{\pi d^2 \sqrt{\langle v_n \rangle^2 + \langle v_s \rangle^2} n_s} \quad (4.95)$$

$$\tau_{s/n} = \frac{1}{\pi d^2 \sqrt{\langle v_n \rangle^2 + \langle v_s \rangle^2} n_n} \quad (4.96)$$

4.4.2 The Remainder Factor

As mentioned before, the remainder factor can be defined as follows

$$\alpha_\tau(t) \equiv \frac{n_s(t + \tau)}{n_s(t)}. \quad (4.97)$$

According to what was discussed in previous sections, all superfluid particles occupy the same state, while the normal fluid particles occupy random states. Hence, when a normal fluid particles collides with a superfluid particle, the state of the superfluid particle will become random and it will no longer be a superfluid particle. In this view, the remainder factor can be considered as the probability that a superfluid particle, as a target particle, **will not** be hit by **any** normal fluid particle in the time interval $(t, t + \tau)$. Therefore, it will be reasonable to find this probability in order to find α_τ . The next sections, are dedicated to estimate the remainder-factor, α_τ .

4.4.2.1 Correct Direction:

The purpose of this section is to find out the probability that if the target particle is at distance r from the normal fluid particle, the projectile particle will have the correct direction for collision towards the target.

Fig 5 Correct direction probability

Consider the situation when the target is a circular plate whose radius is equal to d (the diameter of the particles) and the projectile is initially at distance r from it. One can imagine a spherical surface which encloses the circular plate-target and its center coincides with the projectile center (Fig 5).

In order to calculate the correct direction probability towards the target, one should calculate the area ratio of the spherical cap shown in Fig 5, to the area of the entire sphere:

$$P_{cd}(r, d) = \frac{A_{sc}}{A_s} \quad (4.98)$$

where $P_{cd}(r, d)$ is the probability of having correct direction towards the target, A_{sc} is the area of the spherical cap shown in Fig 5, and A_s is the area of the entire sphere.

Hence, one can rewrite equation (4.98) as:

$$\begin{aligned} P_{cd}(r, d) &= \frac{2\pi Rh}{4\pi R^2} = \frac{h}{2R} \\ &= \frac{1}{2} - \frac{r}{2\sqrt{r^2 + d^2}} \end{aligned} \quad (4.99)$$

4.4.2.2 Probability of Having Sufficient Speed

Consider a particle which has a velocity probability distribution, given by equation (4.74). This particle is at distance l from the target. If the velocity of the particle is towards the target, it is desirable to determine the probability of the particle having sufficient speed to hit the target in less than τ time. This is equivalent to finding the

probability of the particle having a minimum speed of $v_{\min} = \frac{l}{\tau}$:

$$P_{ss}(l, \tau) = \int_{v_{\min}}^{\infty} f(v)dv \quad (4.100)$$

where $P_{ss}(l, \tau)$ is the probability of the particle having sufficient speed for hitting the target, from a distance, l , in less than a time, τ .

One can use equations (4.74) and (4.100) to write

$$P_{ss}(l, \tau) = \int_{v_{\min}}^{\infty} 4\pi \left[\frac{M}{2\pi R T} \right]^{3/2} v^2 \exp \left[\frac{-Mv^2}{2RT} \right] dv, \quad (4.101)$$

$$= 2 \sqrt{\frac{M}{2\pi R T}} v_{\min} \exp \left[\frac{-Mv_{\min}^2}{2RT} \right] + \operatorname{erfc} \left[v_{\min} \sqrt{\frac{M}{2RT}} \right]$$

where $\operatorname{erfc}[x]$ is the error function complement and is defined as

$$\operatorname{erfc}[x] \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \quad (4.102)$$

Mean Distance

It should be noted that as both the target and the projectile particles are considered to be spherical (monatomic) particles, the distance, l , used to calculate $v_{\min}$ in equation (4.101), may be different from the distance between the center point of the projectile particle and the circular target-plate, r , which is shown in Fig 5. Hence, at this point, it is desirable to estimate average value of l in terms of r and d .

One can consider the following figure in order to obtain the distance between the target and the projectile.

Fig 6 Distance between the target and projectile

In Fig 6, it is assumed that the projectile is a point with zero size, and the target is a spherical target. One can simplify the problem by reducing the three dimensional problem to a two-dimensional problem, by considering a circular target of radius d instead of the spherical target.

In the case of a circular target, the distance, l , between the projectile and the target, for each angle, θ , shown in Fig 6 is

$$l = \sqrt{(r - d \cos \theta)^2 + d^2 \sin^2 \theta} . \quad (4.103)$$

By introducing the auxiliary variable k as

$$k = \frac{r}{d}, \quad (4.104)$$

one can rewrite equation (4.103) as

$$l = d \sqrt{(k - \cos \theta)^2 + \sin^2 \theta} . \quad (4.105)$$

It should be noted that k must be always greater than one.

Equation (4.105) shows the distance between the projectile and the circular target, for any angle, θ , shown in Fig 6. However, it is more useful to calculate the mean distance between the projectile and the circular target.

$$\begin{aligned} \bar{l} &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d \sqrt{(k - \cos \theta)^2 + \sin^2 \theta} d\theta \\ &= \frac{2d}{\pi} \int_0^{\frac{\pi}{2}} \sqrt{(k - \cos \theta)^2 + \sin^2 \theta} d\theta \end{aligned} \quad (4.106)$$

Equation (4.106) shows the average distance, $\bar{l}$. The result of the integration in equation (4.106) will be in terms of hypergeometric functions. This will make our future derivations too complicated. However, the values of k , in our original problem, will be very large, because one of the basic assumptions, in this theory, is that the medium is a dilute gas. In the case of a dilute gas, the distance between the particles is orders of magnitudes larger than the diameter of the atoms. Therefore, it is reasonable to approximate the integral in equation (4.106) with a linear function. The following linear approximation is proposed:

$$\frac{\bar{l}}{d} \approx k + 0.64 \quad (4.107)$$

This approximation is used in the rest of the work in order to calculate the average distance. It was calculated using linear regression of the exact values of $\bar{l}$ in the range $1 < k < 10000$. The following figure shows the agreement between the exact values and the linear approximation for equations (4.106) and (4.107).

Fig 7 Comparison of the linear approximation and exact values of the dimensionless mean distance

Fig 7 shows the comparison between the linear approximation of the dimensionless mean distance, according to equation (4.107), and its corresponding exact values, given by equation (4.106). This figure shows the relatively excellent agreement between the proposed approximation, and the exact values of the mean distance.

Now, one can rewrite equation (4.101), in terms of r and τ , instead of l and τ :

$$\begin{aligned}
 P_{ss}(r, \tau) &= \int_{v_{\min}}^{\infty} 4\pi \left[\frac{M}{2\pi R T} \right]^{3/2} v^2 \exp \left[\frac{-Mv^2}{2 R T} \right] dv \\
 &= 2 \sqrt{\frac{M}{2\pi R T}} v_{\min} \exp \left[\frac{-Mv_{\min}^2}{2 R T} \right] + \operatorname{erfc} \left[v_{\min} \sqrt{\frac{M}{2 R T}} \right]
 \end{aligned} \tag{4.108}$$

where $v_{\min}$, using equation (4.107), is defined according to the following equation

$$v_{\min} = \frac{r + 0.64d}{\tau} \quad (4.109)$$

In the following derivations, equation (4.108) is used as the probability of the projectile particle having sufficient speed towards the target, if the center points of the projectile and target particles are at a distance r from each other, and the maximum time for collision is considered to be τ .

4.4.2.3 Probability of hitting a target particle at distance r in a time less than τ

Consider a stationary target (superfluid) particle, whose center has a distance r from the center of the projectile (normal fluid) particle. The probability distribution of the projectile particle is calculated according to the Maxwell-Boltzmann velocity distribution, equation (4.74).

It is desirable to find out what the probability of hitting the target in a time less than τ . This is equivalent to finding the probability of the projectile having a velocity with correct direction towards the target, and also sufficient speed to hit the target in the specified time. In this manuscript, this probability is denoted by P_{cdss} . The following equation is proposed to calculate this probability, since, P_{cd} and P_{ss} are clearly related to independent events.

$$P_{cdss}(r, \tau) = P_{cd}(r, d) \cdot P_{ss}(r, \tau) \quad (4.110)$$

In equation (4.110), $P_{cd}(r, d)$ is calculated by equation (4.99), while $P_{ss}(r, \tau)$ is determined by equation (4.108).

4.4.2.4 Collision of a Confined Projectile Particle with the Target Particle

Consider a stationary target particle. This target particle is assumed to be in the center of a spherical space. Consider that there exists a projectile particle, in this spherical space, whose location is not known. The probability of finding the projectile particle, at any location inside this sphere, at any time, is assumed to be identical to finding the projectile, at any other location, inside this sphere, at the same time.

Fig 8 cross section of the spherical differential volume inside the confining sphere

Fig 8 shows the cross section of a spherical differential volume inside the sphere where the particle is confined in. The probability of the presence of the projectile particle, in this differential volume, P_p , is the ratio of the volume of spherical differential volume to the volume of the entire volume of confinement which is the

volume of the confining sphere minus the volume of the target particle at the center of the confining sphere.

$$P_p = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r_c^3 - \frac{1}{6}\pi d^3} = \frac{3r^2 dr}{r_c^3 - \frac{1}{2}d^3} \quad (4.111)$$

The probability of the projectile being present in the differential volume is shown in Fig 8, and also colliding with the target particle at the center of this sphere in a period of time less than τ , according to equations (4.110) and (4.111), is

$$\begin{aligned} dP_{cop}(r, d, \tau) &= P_p(r, d) P_{cdss}(r, d, \tau) \\ &= \frac{3r^2 P_{cdss}(r, d, \tau) dr}{r_c^3 - d^3} \end{aligned} \quad (4.112)$$

Consider the union of all possible disjoint differential volumes over the entire sphere. As the probable events, corresponding to each of these volumes, are disjoint events, it is possible to say that the probability of the union of these events is the sum of the probability of each individual event. The probability of the union of these disjoint events is actually the probability of the collision of the projectile particle with the stationary target particle in a period of time less than τ . In order to sum all the disjoint probabilities, both sides of the equation (4.112) should be integrated to write the following equation.

$$P_{cop}(r_c, d, \tau) = \int_d^{r_c} \frac{3r^2 P_{cdss}(r, d, \tau) dr}{r_c^3 - d^3} \quad (4.113)$$

Equation (4.113) shows the probability, P_{cop} , of the target particle being hit by one projectile which is confined in a spherical space of radius r_c in a time less than τ . The integration result cannot be simply and analytically derived. This integral should be calculated numerically. However, as the integrant is close to zero, numerical integration must be carried out with great care.

Now, it is possible to calculate the probability of no collision of the projectile with the target particle in the specified time. This probability is denoted as P_{ncop} .

$$P_{ncop}(r_c, d, \tau) = 1 - P_{cop}(r_c, d, \tau) \quad (4.114)$$

4.4.2.5 Probability of total no collision with the target particle

In this section, it is desirable to find out the probability for no projectile hitting the target in the specified time τ . It is assumed that the probability of not hitting the target for each of the projectile particles is independent of the presence of the other particles, e.g. the gas is dilute. Hence, if a confinement sphere is considered around the target particle like in the previous section, it is possible to say that the probability for no particle hitting the target is

$$P_{tnc}(r_c, d, \tau) = \prod_i P_{ncop}(r_c, d, \tau)_i \quad (4.115)$$

where the subscript i indicate each projectile particle present in the confining sphere. As the probability for each projectile is independent of the presence of the other projectiles and is equal for all of them, equation (4.115) can be rewritten as

$$P_{tnc}(r_c, d, \tau) = \left(P_{ncop}(r_c, d, \tau) \right)^N \quad (4.116)$$

where N is total number of the projectile particles in the confining sphere.

As it was mentioned earlier, P_{tnc} is exactly the remainder factor corresponding to the time τ . However, it should be noted that P_{tnc} is a function of τ and r_c . This is due to the fact that if one changes r_c , both P_{ncop} and the number of particles which are present in the confined sphere will change.

Choosing a meaningful value for r_c not only depends on the physics of the problem but also on the choice of τ . If τ is too large, the projectile particles, from far away, will have enough time to reach the target. However, there is a high possibility that these projectile particles will hit other projectile particles on their way to reach the target. This will render the assumption that the probability of not hitting the target by each projectile particle is independent of the presence of the other projectile particles.

The first guess for τ would be to use relative mean free times like $\tau_{n/s}$ or $\tau_{s/n}$. Consequently one would try to use some factor times their corresponding mean free paths for the confinement radius. However, if these times and length are used, no specific trend is observed until the length is so large. As it was discussed, this will render the assumption of independence of the probabilities for each particle.

With a few assumptions and simplifications, one can be able to overcome this problem. The basic simplification will be

$$\langle v_s \rangle \ll \langle v_n \rangle \Rightarrow \langle v_{n/s} \rangle = \langle v_n \rangle. \quad (4.117)$$

τ was considered to be equal to $\tau_{n/n}$ given by equation (4.94). The confinement radius r_c can be made non-dimensional by introducing $r_c / l_{n/n}$. The mean free path $l_{n/n}$ is calculated according to (4.83). For several cases, P_{nc} was plotted against the introduced dimensional length as the confinement radius. Surprisingly, the following curve was obtained according to this collision time and it was independent of the particle size and physical state of the system, such as temperature or pressure. This is due to the fact that the confinement length is written in non-dimensional form, and the suitable characteristic length, $l_{n/n}$, is used.

Fig 9 Probability of Total No Collision with Target, in less than $\tau_{n/n}$ time, against $r_c / l_{n/n}$

Fig 9 shows P_{tnc} against $r_c / l_{n/n}$. As it was mentioned, the time of collision, τ , was considered to be equal to $\tau_{n/n}$. The curve obtained, in this figure, has the property that it is independent of the particle size, and physical properties of the normal fluid. It is also noted that, as $r_c / l_{n/n}$ reaches a value of almost equal to two, P_{tnc} does not change with the increase of radius of confinement, any more. This might be due to the fact that the particles, which are farther than $2l_{n/n}$, most probability, do not have sufficient speed to reach the target (superfluid) particle, in a time less than $\tau_{n/n}$.

As it was earlier discussed, P_{tnc} can be considered as the remainder-factor. Hence, one can write:

$$\alpha_{\tau_{n/n}} = \frac{1}{2}. \quad (4.118)$$

This result means that, in this case, the superfluid decay equation (4.76) can be written as

$$n_s(t) = 2^{-t/\tau_{n/n}} n_s(0). \quad (4.119)$$

Using equation (4.119), one can plot the dimensionless superfluid decay curve (Fig 1). This figure shows that, after approximately $10 \times \tau_{n/n}$, almost no superfluid particle will be left compared to the initial number of superfluid particles.

Fig 10 Dimensionless superfluid decay curve

However, it must be mentioned that the derived value for the remainder factor α can be considered as a rough estimate. This is due to the fact that this value was derived under very restrictive conditions, while some of these conditions may not be fully satisfied. For the sake of clarity, the assumptions used to determine the value of the remainder factor are listed below:

1. Superfluid formation is possible within a normal fluid.
2. Particles of the superfluid are dispersed homogenously among particles of the normal fluid
3. The normal fluid particles and superfluid particles are considered to be classical particles
4. Particles of superfluid and normal fluid are geometrically the same; they are spherical particles with diameter d .
5. The velocity of the superfluid particles is negligible compared to the velocity of the normal-fluid particles. This assumption can be suppressed in future studies. Consequently, its effect on the remainder factor can be studies.

6. The probability of hitting a superfluid by a normal fluid is independent of the presence of other superfluid and normal fluid particles. This assumption in future studies should be thoroughly addressed.

On the other hand, it should be noted that these conditions do not restrict the superfluid formation and they were merely considered to determine the value for the remainder factor. In future works, these conditions can be suppressed to determine the value for the remainder factor. The method used in this manuscript can be used in future works as a starting point to derive at a solution for the remainder factor.

4.4.3 Superfluid Decay and Phase-Lagged Formulations of Quantum Mechanics

In Chapter 3, it was shown that the time lag in the phase-lagged Schrödinger equation, is finite, and it is proportional to the Planck time. In this chapter, it was proposed that the characteristic time for the superfluid decay equation in ordinary fluids is $\tau_{n/n}$.

At this point, it is important to mention that if $\tau_{n/n}$ is of the order of the Planck time, the use of the corresponding classical Schrödinger equation related to this superfluid velocity is at great debate. However, if $\tau_{n/n}$ is orders of magnitude greater than the Planck time, then the classical Schrödinger equation can be used safely without considering the time lag in the phase-lagged Schrödinger equation.

It must be also noted that $\tau_{n/n}$ is the characteristic time for the specific conditions mentioned in the previous section, and the general case remains to be investigated in future works.

4.5 Summary and Conclusion

In this chapter, the relationship between the phase-lagged formulations of quantum mechanics and the macroscopic phenomena was discussed.

It was shown that the diffusion equation can be derived from both the classical and the phase-lagged Schrödinger equation. It was also demonstrated that, normally, both the classical and the phase-lagged formulations lead into the classical diffusion equation.

A method to transform the boundary and initial conditions of the phase-lagged Schrödinger equation to those of the diffusion equation was proposed. It was shown that using this method, one may get explicit transformations for the initial condition and the Dirichlet boundary conditions of the phase-lagged Schrödinger equation to those of the diffusion equation; however, this method will result in an implicit system of equations for the transformation of the Neumann boundary conditions of the Schrödinger equation to those of the diffusion equation.

Then, the relationship between the de Broglie waves and the fluid velocity was discussed. It was shown that an earlier work by Kulish and Lage (Kulish and Lage 2002) would result in superfluid predictions. It was proposed that the results of their work could be correct only under some restrictive conditions.

Consequently, it was discussed that other recent works predict superfluidity, in ordinary fluids, for ultra-short time scales using phase-lagged formulations of the Stokes equations (Kulish and Chan 2006; Kulish and Chan 2007). It was demonstrated that the wave packet assumption was the key assumption in predicting superfluid behavior.

The superfluid decay equation was proposed under restrictive conditions, and an estimate for the remainder factor was determined. It was argued that if the time scale of the superfluid decay is, in orders of magnitude, comparable to the Planck time, then the classical Schrödinger equation is at great debate and, hence, cannot be used to determine the evolution of de Broglie wave. In such cases, the phase-lagged Schrödinger equation must be used to determine the evolution of the de Broglie wave. However, if the time scale of the superfluid decay is orders of magnitude greater than the Planck time, the classical Schrödinger can be used safely.

Chapter 5 Summary, Conclusion, and Future Work

5.1 *Summary and Conclusion*

In Chapter 2, a brief history of quantum mechanics and the major attempts and developments in the early days of quantum mechanics were presented. It was explained how quantum mechanics was developed. It was argued that the Schrödinger equation, the corner stone in quantum mechanics, was deduced and not derived from the basic laws of physics.

The paradox of instantaneous propagation of energy arising from the Schrödinger equation was addressed. It was underlined that the Schrödinger equation allowed for infinite speed of energy propagation. Some of the attempts to overcome the paradox of instantaneous propagation were presented. It was shown that physicists were, more or less, successful in developing theories, like QFT and hidden variables theory, which remove the paradox for certain problems. It should be noted that, despite of all the attempts, *no* unique and general formulation which removes the paradox and gives relativistic results in all cases has been proposed. The quest for a general and unique formulation still remains vital.

Then, the work by Nelson in deriving the Schrödinger equation from Newtonian mechanics was reviewed in detail. Nelson's hypothesis was used, in Chapter 3, to propose a new formulation of quantum mechanics.

The derivation of the classical Schrödinger equation based on three rather different viewpoints was proposed in Chapter 3. For each of these viewpoints, the phase-lagged

formulations were also derived. It was shown that, in case of zero time lags, in all the three cases, the phase-lagged formulation is identical to the classical formulation of the Schrödinger equation.

Three possible phase-lagged derivations of the Schrödinger equation were derived based on the assumption that there exists a *finite* time lag between the onset of gradients and the corresponding fluxes. The extended version of the Schrödinger equation, therefore, helps preventing the paradox of instantaneous propagation, intrinsic to the classical Schrödinger equation.

Furthermore, the equations presented, in Chapter 3, reduce the violation of relativity for, unlike the classical Schrödinger equation, they restrict the speed of energy propagation.

It was further mentioned, in Chapter 3, that although the proposed phase-lagged formulations help to reduce the instantaneous propagation of energy, they unfortunately fail to completely remove it. This is due to the fact the mathematical form of these formulations change under the Lorentz transformations. This means that the phase-lagged formulations are also non-relativistic formulations. Therefore, the quest for general and relativistic formulations remains to be a vital scope for the future research in this field.

In Chapter 4, the relationship between the phase-lagged formulations of quantum mechanics and the macroscopic phenomena was discussed.

It was shown that the diffusion equation can be derived from both the classical and the phase-lagged Schrödinger equation. It was also demonstrated that, normally, both the classical and the phase-lagged formulations lead into the classical diffusion equation.

A method to transform the boundary and initial conditions of the phase-lagged Schrödinger equation to those of the diffusion equation was proposed.

Then, the relationship between the de Broglie waves and the fluid velocity was discussed. It was shown that an earlier work by Kulish and Lage (Kulish and Lage 2002) would result in superfluid predictions. It was proposed that the results of their work could be correct only under some restrictive conditions.

Consequently, it was discussed that other recent works predict superfluidity, in ordinary fluids, for ultra-short time scales, using phase-lagged formulations of the Stokes equations (Kulish and Chan 2006; Kulish and Chan 2007). It was demonstrated that the wave packet assumption was the key assumption in predicting superfluid behavior.

The superfluid decay equation was proposed under restrictive conditions, and an estimate for the remainder factor was determined. It was argued that if the time scale of the superfluid decay is, in orders of magnitude, comparable to the Planck time, the classical Schrödinger equation is at great debate and, hence, cannot be used to determine the evolution of de Broglie wave. In such cases, the phase-lagged Schrödinger equation must be used to determine the evolution of the de Broglie wave. However, if the time scale of the superfluid decay is orders of magnitude greater than the Planck time, the classical Schrödinger can be used safely.

5.2 *Future Work*

Up to this stage, a phase-lagged formulation of the quantum mechanics, and a method to solve it have been successfully proposed.

It has been also shown in this report that there are links between the Schrödinger equation and the macroscopic equations. The consequences of using the phase-lagged Schrödinger equation can be further studied in future.

The nature of the apparent source terms in the phase-lagged Schrödinger remains to be a goal for future research. Although the phase-lagged Schrödinger equation is not a relativistic equation, these apparent source terms can describe fields which might have been unnoticed until today.

Determining the remainder factor for the superfluid decay for a general case, or even other cases, can be considered as a scope for future research. This can shed light on the topic of high temperature superfluidity in future.

Appendix A. *Proof of Bell's Theorem*

In this section a detailed proof of Bell's theorem will be presented. Bell proposed a pioneering proof of the theorem that that is now called after him (Bell 1964). First, he suggested a theoretical framework, in which expectation values could be determined for the spin components of a pair of spin-half particles. Then he showed that, regardless of the choices that are made for specific undetermined functions that arise in the framework, the expectation values follow a particular inequality which is now called "Bell's Inequality". Nowadays, that term is frequently used to designate a particular type of inequalities derived in theoretical frameworks which are more general but similar to the framework proposed by Bell. Now, these inequalities are often called as "Inequalities of Bell's type". Each of these theoretical frameworks encompasses a certain type of hidden variables theory and follows a locality assumption. "Local Realistic Theory" is also a suitable name. Bell determined the expectation values for certain products of the form $(\sigma_1 \cdot \hat{a})(\sigma_2 \cdot \hat{e})$, where σ_1 and σ_2 are the Pauli spin operator vector for particle 1 and 2 respectively. Note that both of the particles have spin $\frac{1}{2}$. $\hat{a}$ and $\hat{e}$ are unit vectors in three-space. Subsequently, he showed that the quantum description of the setting violate Bell's Inequality. This violation establishes a specific case of Bell's theorem, as it shows that no Local Realistic Theory subsumed under the framework of Bell's 1964 paper can agree with all of the statistical predictions of quantum theory (Bell 1964).

In this short review, the Bell 1964 paper strategy will be followed. Though, a more general theoretical framework will be considered; and therefore, a more general inequality will be derived. Hence, the proof presented here results to a more general

theorem than the one derived by Bell in 1964. Other works which preceded the Bell 1964 explanation are (Clauser, Horne et al. 1969; Bell 1971; Clauser and Horne 1974; Aspect 2000).

First, an ensemble of pairs of systems will be postulated within the theoretical framework which will be used to describe the Bell inequality. Each systems within every pair is labeled as 1 and 2. Every pair is described by means of a complete state m . This state is assumed to contain all the properties of the pair at the time of their generation. The complete state may be different from one pair to another; however, the generation mode of the pairs determines a probability distribution ρ which is independent of the processes that each of the systems goes through after they separate. Different experiments can be conducted on every system. The experiments performed on 1 are denoted by a , a' , etc. and those on 2 by b , b' , etc.

Theoretically, one can assume a , a' etc. to contain characteristics of the measurement device as well. However, the dependence of the measurement outcome upon the microscopic features of the device is not experimentally measurable; therefore, only the macroscopic features of the device in their partly controllable setting should be introduced into the description of the measurement performed on the particle a , a' , etc., and similarly for b , b' , etc. The fair agreement between the quantum mechanical predictions and the outcomes of the experimental measurements of the Bell tests confirms the validity of the limiting considerations on the characteristics of the measuring device. The outcomes of the experiments on 1 and 2 are denoted by s and t respectively. The outcomes of the experiments can be integer numbers in the interval

$[-1, 1]$. In the original theorem by Bell, it was assumed that s and t are both bivalent, either -1 or 1 ; however, other ranges are considered in other versions of the theorem.

It is assumed that the following probabilities are predefined.

(A.1) $p_m^1(s | a, b, t)$ is defined as the probability that the result of the measurement carried out on 1 is s when m is the complete state, the measurements a and b are carried out on 1 and 2 respectively and the result of the experiment on 2 is t .

(A.2) $p_m^2(s | a, b, t)$ is defined as the probability that the result of the measurement carried out on 2 is t when the complete state is m , the measurements a and b are carried out on 1 and 2 respectively, and the result of the measurement 1 is s .

(A.3) $p_m(s, t | a, b)$ is defined as the probability that the outcomes of the joint measurements a and b , carried out on 1 and 2, are s and t respectively, while the complete state is m .

p is the probability function which is assumed to be real non-negative value and to sum to unity when the summation is taken over all allowed values of s and t .

An additional feature of the theoretical framework is its locality, which is described as the conjunction of the following Independence Conditions:

Remote Outcome Independence

$$p_m^1(s | a, b, t) \equiv p_m^1(s | a, b) \text{ is independent of } t, \quad (\text{A.4})$$

$$p_m^2(t | a, b, s) \equiv p_m^2(t | a, b) \text{ is independent of } s; \quad (\text{A.5})$$

Notice that conditions (A.4) and (A.5) do not rule out correspondence of the outcomes of the experiment a and b on 1 and 2 respectively; they show that if the complete state m is provided, the outcome s of the experiment on 1 gives no further information regarding the outcome of the experiment on 2, and vice versa.

Remote Context Independence:

$$p_m^1(s | a, b) \equiv p_m^1(s | a) \text{ is independent of } b, \quad (\text{A.6})$$

$$p_m^2(t | a, b) \equiv p_m^2(t | b) \text{ is independent of } a. \quad (\text{A.7})$$

Bell (Bell 1990) verified the equivalence of the combination of (A.4), (A.5) and (A.6), (A.7) to the factorization condition:

$$p_m(s, t | a, b) = p_m^1(s | a) p_m^2(t, b) \quad (\text{A.8})$$

and similarly for (a, b') , (a', b) , and (a', b') replaced for (a, b) .

The condition (A.8) is also frequently called the Bell locality. However, notice that, up to this point in of explanation, the Bell locality is simply a mathematical condition, within a theoretical framework, and no physical importance has been associated with it yet.

By using a simple lemma, one can derive the Bell inequality from his locality condition

$$\forall q, q', r, r' \in [-1, 1] \Rightarrow S \equiv qr + qr' + q'r - q'r' \in [-2, 2] \quad (\text{A.9})$$

Proof: As S is linear in all four variables q , q' , r and r' , it must have its extremum values at the corners of the domain of this four of variables. This means that the extremums of S happen at values where each of q , q' , r and r' is $+1$ or -1 . Therefore, S can merely be an integer between -4 and $+4$, at these corners. However, S can be rewritten as $(q + q')(r + r') - 2q'r'$. On the other hand, the two quantities in parentheses can only be 0 , 2 , or -2 , while the last term can only be -2 or 2 . Therefore, S cannot take on the values -3 , $+3$, -4 , or $+4$ at the corners. $\square$

It is now beneficial to introduce the expectation value of the product st of outcomes:

$$E_m(a, b) \equiv \sum_s \sum_t p_m(s, t | a, b) (st) \quad (\text{A.10})$$

The summation in (A.10) is taken over the entire acceptable values of s and t .

One can similarly replace (a, b') , (a', b) and (a', b') for (a, b) . It should be noted that all the quantities q , q' , r and r' of the above lemma (A.9), must be the single expectation values.

$$q = \sum_s s p_m^1(s | a) \quad (\text{A.11})$$

$$q' = \sum_s s p_m^1(s | a') \quad (\text{A.12})$$

$$r = \sum_t t p_m^2(t | b) \quad (\text{A.13})$$

$$r' = \sum_t t p_m^2(t|b') \quad (\text{A.14})$$

Then the lemma, together with (A.10), condition (A.8), and the bounds on s and t stated previous to (A.1), necessitates:

$$-2 \leq E_m(a, b) + E_m(a, b') + E_m(a', b) - E_m(a', b') \leq 2 \quad (\text{A.15})$$

Finally, one should go back to the fact the ensemble of interest is composed of pairs of systems which are governed by a mapping m . However, m is selected stochastically from a space M of mappings that a standard probability density ρ governs them. This means that $\rho(B)$ is a real non-negative number for every Borel subset B of M , $\rho(M) = 1$, and $\rho(U_j B_j) = \sum_j \rho(B_j)$ where the B_j 's are disjoint Borel subsets of M and $U_j B_j$ is the set-theoretical union of the B_j 's.

If one defines

$$p_\rho(s, t | a, b) \equiv \int_M p_m(s, t | a, b) d\rho, \quad (\text{A.16})$$

$$E_\rho(a, b) \equiv \int_M E_m(a, b) d\rho = \sum_s \sum_t \int_M p_m(s, t | a, b) (st) d\rho, \quad (\text{A.17})$$

And similarly when (a, b') , (a', b) and (a', b') are replaced for (a, b) into (A.15), (A.16), (A.17), applying the properties of ρ necessitates

$$-2 \leq E_\rho(a, b) + E_\rho(a, b') + E_\rho(a', b) - E_\rho(a', b') \leq 2 \quad (\text{A.18})$$

Inequality (A.18) is a Bell inequality. It is called the Bell-Clauser-Horne-Shimony-Holt (BCHSH) inequality.

The next step to derive the Bell theorem is to demonstrate a system, a quantum state, and a group of quantities for which the statistical description and predictions violate inequality (A.18). Consider a system comprising a pair of photons 1 and 2 which propagate in the z and $-z$ directions.

The two kets $|x\rangle_j$ and $|y\rangle_j$ form a polarization basis for photon j ($j = 1, 2$). In Dirac's notation (Feynman, Leighton et al. 1989), $|x\rangle_j$ and $|y\rangle_j$ designate, the states in which the photon j is linearly polarized in the x -direction or y -direction respectively. For the systems comprising a pair of photons, the four product kets $|x\rangle_1|x\rangle_2$, $|x\rangle_1|y\rangle_2$, $|y\rangle_1|x\rangle_2$, and $|y\rangle_1|y\rangle_2$ form a polarization basis. All the polarization states of two-photon systems can be described using this basis. Quantum entangled states are of special interest. These states can never be expressed as $|\varphi\rangle_1|\xi\rangle_2$, with $|\varphi\rangle$ and $|\xi\rangle$ single-photon states, e.g.

$$|\Phi\rangle = (1/\sqrt{2})\{|x\rangle_1|x\rangle_2 + |y\rangle_1|y\rangle_2\} \quad (\text{A.19})$$

Equation (A.19) is invariant under rotation of the x and y axis in the plane perpendicular to z . Neither photon 1 nor photon 2 is in a certain polarization state when the pair is in the state $|\psi\rangle$. However, according to Heisenberg (Heisenberg 1958), their potentialities are interrelated: any information on the potentiality of photon 1 which is

polarized along the x or y direction, will determine the potentiality of the photon 2, and vice versa. At this point, consider that photons 1 and 2 are emitted on the sides of birefringent⁷ crystal polarization analyzers I and II respectively, with the input side of each analyzer at right angles to z . Each analyzer separates the incident light into two output non-parallel rays, namely the ordinary and the extraordinary rays. If the crystal is assumed to be ideal, a photon which is polarized along the transmission axis of the analyzer will appear in the ordinary ray, while a photon which is polarized in a direction perpendicular to both the z -axis and the transmission axis of the analyzer will appear in the extraordinary ray. See Fig A-1.

Fig A-1 Photon pairs are released from the source and enter the Analyzers

Depending on the orientation angles a and b of the analyzers, one can compute the probability of the emergence of the photons 1 and 2 in specific rays according to the

⁷ Birefringence, or double refraction, is the decomposition of a ray of light into two rays (the *ordinary ray* and the *extraordinary ray*) when it passes through certain types of material. The materials which have this property to decompose a ray of light into two rays are called birefringent materials. For more information see Born, M. and E. Wolf (2006). Principles of optics electromagnetic theory of propagation, interference and diffraction of light. Cambridge [u.a.], Cambridge Univ. Pr.

laws of quantum mechanics. a and b are the angles between the transmission axes of analyzers I and II and an arbitrary fixed a fixed direction in the $x-y$ plane.

$$\text{prob}_{\Phi}(s, t | a, b) = \left| \langle \Phi | \theta_s \rangle_1 | \varphi_t \rangle_2 \right|^2 \quad (\text{A.20})$$

where s and t are quantum numbers related to the rays, into which photons 1 and 2 appear respectively. $+1$ and -1 denote the appearance of the photon in the ordinary and the extraordinary ray respectively when a and b are given. $|\theta_s\rangle_1 |\varphi_t\rangle_2$ is the ket denoting the quantum state of photons 1 and 2 with the quantum numbers s and t respectively.

The relevant probabilities from (A.20) can be more simply calculated by using the invariance renowned after (A.19), and rewriting $|\Phi\rangle$ as

$$|\Phi\rangle = (1/\sqrt{2}) \{ |\theta_1\rangle_1 |\theta_1\rangle_2 + |\theta_{-1}\rangle_1 |\theta_{-1}\rangle_2 \} \quad (\text{A.21})$$

Equation (A.21) can be derived from (A.19) by substituting the transmission axis of analyzer I for x and the direction perpendicular to both z and this transmission axis for y .

Since $|\theta_{-1}\rangle_1$ is orthogonal to $|\theta_1\rangle_1$, merely the first term of (A.21) contributes to the inner product in (A.20) if $s = t = 1$; and as the inner product of $|\theta_1\rangle_1$ with itself is unity because of normalization, (A.20) reduces for $s = t = 1$ to

$$\text{prob}_{\Phi}(1, 1|a, b) = \left(\frac{1}{2}\right) \left| {}_2\langle \theta_1 | \varphi_1 \rangle_2 \right|^2 \quad (\text{A.22})$$

As a final step, the right hand side expression of (A.22) is calculated by using the Malus law (Collett 2005). This law must be conserved when dealing with the quantum mechanical polarization states. If a photon is polarized in a direction n and the transmission axis of the analyzer is n' , one can introduce σ to be the angle between n and n' . Then, according to Malus law, the probability of the photon to pass through the mentioned analyzer must be

$$\text{prob}_{\Phi}(1, 1|a, b) = \left(\frac{1}{2}\right) \cos^2 \sigma. \quad (\text{A.23})$$

Similarly, one can write

$$\text{prob}_{\Phi}(-1, -1|a, b) = \left(\frac{1}{2}\right) \cos^2 \sigma \quad (\text{A.24})$$

and

$$\text{prob}_{\Phi}(1, -1|a, b) = \text{prob}_{\Phi}(-1, 1|a, b) = \left(\frac{1}{2}\right) \sin^2 \sigma \quad (\text{A.25})$$

Therefore, the following expectation value can be found to be as

$$\begin{aligned} E_{\Phi}(a, b) &= \text{prob}_{\Phi}(1, 1|a, b) + \text{prob}_{\Phi}(-1, -1|a, b) \\ &\quad - \text{prob}_{\Phi}(1, -1|a, b) - \text{prob}_{\Phi}(-1, 1|a, b) \\ &= \cos^2 \sigma - \sin^2 \sigma \\ &= \cos 2\sigma \end{aligned} \quad (\text{A.26})$$

At this point, the orientation angles of the transmission axis can be chosen to be

$$\begin{aligned}
 a &= \pi / 4, \\
 a' &= 0, \\
 b &= \pi / 8, \\
 b' &= 3 \pi / 8
 \end{aligned}
 \tag{A.27}$$

Then,

$$\begin{aligned}
 E_{\Phi}(a, b) &= \cos 2(-\pi / 8) = 0.707, \\
 E_{\Phi}(a, b') &= \cos 2(\pi / 8) = 0.707 \\
 E_{\Phi}(a', b) &= \cos 2(\pi / 8) = 0.707 \\
 E_{\Phi}(a', b') &= \cos 2(3\pi / 8) = -0.707
 \end{aligned}
 \tag{A.28}$$

Hence:

$$S_{\Phi} \equiv E_{\Phi}(a, b) + E_{\Phi}(a, b') + E_{\Phi}(a', b) - E_{\Phi}(a', b') = 2.828 \tag{A.29}$$

Equation (A.29) proves that there are situations where the quantum mechanical predictions violate the BCHSH inequality. This completes the proof of the Bell theorem. However, it should be noted that all entangled quantum states result in predictions which violate inequality (A.18), as Gisin (Gisin 1991) and Popescu and Rohrlich (Popescu and Rohrlich 1992) have independently showed. Popescu and Rohrlich (1992) also show that a quantum state of the utmost degree of entanglement, represented by $|\Phi\rangle$ of (A.19), can maximally violate the Bell inequality.

Now, it is crucial to explain the importance of the Bell theorem from an entirely theoretical viewpoint. The Bell theorem proves that quantum mechanics has an incomplete structure within the theoretical framework of the derivation of the Bell inequality. The most important consequence of the Bell theorem is the way that it can

explains the EPR argument. That argument considers an entangled quantum state and demonstrates that a necessary condition for avoiding action-at-a-distance between measurement outcomes of correlated properties of the two subsystems is the acknowledgment of “elements of physical reality” related to the interrelated properties to each subsystem without reference to the other. The Bell theorem shows that such an acknowledgment leads to statistical consequences which violate the quantum mechanical predictions.

References

- 't Hooft, G. and M. Veltman (1972B). "COMBINATORICS OF GAUGE FIELDS."
Nucl. Phys **50**(1).
- 't Hooft, G. and M. J. G. Veltman (1972). "REGULARIZATION AND
RENORMALIZATION OF GAUGE FIELDS." Nuclear Physics B **44**(1): 189-
213.
- Aspect, A. (1983). Three Experimental Tests of Bell's Inequalities by Measurement of
Polarization Correlation of Photons. France: 372p.
- Aspect, A. (2000). Bell's Theorem: the naïve view of an experimentalist. Quantum
[Un]speakables, Springer.
- Aspect, A., J. Dalibard, et al. (1982). "Experimental Test of Bell's Inequalities Using
Time- Varying Analyzers." Physical Review Letters **49**(25): 1804.
- Aspect, A., P. Grangier, et al. (1982). "Experimental Realization of Einstein-Podolsky-
Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities."
Physical Review Letters **49**(2): 91.
- Barnes, R. B. and S. Silverman (1934). "Brownian Motion as a Natural Limit to all
Measuring Processes." Reviews of Modern Physics **6**(3): 162.
- Bell, J. S. (1964). "On the Einstein Podolsky Rosen paradox." Physics **1**: 195-200.

- Bell, J. S. (1966). "On the problem of hidden variables in quantum mechanics." Reviews of Modern Physics **38**: 447-452.
- Bell, J. S. (1971). "Introduction to the hidden-variable question." Foundations of Quantum Mechanics: 171-181.
- Bell, J. S. (1985). "The theory of local beables." Dialectica **39**: 97-102.
- Bell, J. S. (1987). Speakable and Unspeakable in Quantum Mechanics, Cambridge University Press, Cambridge U.K.
- Bell, J. S. (1990). La nouvelle cuisine. Between Science and Technology, Elsevier.
- Bethe, H. A. (1947). "The Electromagnetic Shift of Energy Levels." Physical Review **72**(4): 339-341.
- Bjorken, J. D. and S. D. Drell (1964). Relativistic Quantum Mechanics. New York, McGraw Hill.
- Blatt, F. J. (1992). Modern Physics, McGraw-Hill.
- Blokhintsev, D. I. (1976). Elementary Quantum Mechanics. Moscow, Nauka.
- Bohm, D. (1952). "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I." Physical Review **85**(2): 166-179.
- Bohm, D. (1952B). "A Suggested Interpretation of the Quantum Theory in Terms of 'Hidden variables', I and II." Physical Review **85**(2): 180-193.

- Bohm, D. and Y. Aharonov (1957). "Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky." Physical Review **108**(4): 1070.
- Bohr, N. (1913). "On the constitution of atoms and molecules (Part I)." Philosophical Magazine **26**(6): 1–25.
- Bohr, N., H. A. Kramers, et al. (1924). "The quantum theory of radiation." Philosophical Magazine **47**(78): 69-87.
- Bohr, N. H. D. and L. Rosenfeld (1933). Zur Frage der Messbarkeit der elektromagnetischen Feldgrößen, Levin und Munksgaard.
- Boltzmann, L. (1884). "Ableitung des Stefan'schen Gesetzes, betreffend die Abhängigkeit der Wärmestrahlung von der Temperatur aus der electromagnetischen Lichttheorie." Annalen der Physik **258**(6): 291-294.
- Born, M. (1926). "Quantum mechanics of collision processes." Zeitschrift für Physik **38**: 803-827.
- Born, M., W. Heisenberg, et al. (1926). "Zur Quantenmechanik. II." Zeitschrift für Physik A Hadrons and Nuclei **35**(8): 557-615.
- Born, M. and E. Wolf (2006). Principles of optics electromagnetic theory of propagation, interference and diffraction of light. Cambridge [u.a.], Cambridge Univ. Pr.
- Bose, S. N. (1924). "Wärmegleichgewicht im Strahlungsfeld bei Anwesenheit von Materie." Zeitschrift für Physik A Hadrons and Nuclei **27**(1): 384-393.

- Cartan, E. (1913). "Les groupes projectifs qui ne laissent invariante aucune multiplicité plane." Bull. Soc. Math. France **41**(1): 53-96.
- Cattaneo, C. (1958). "A form of heat conduction equation which eliminates the paradox of instantaneous propagation." Compte Rendus **247**(4): 431-433.
- Christianto, V. and F. Smarandache (2008). "An Exact Mapping from Navier-Stokes Equation to Schrodinger Equation via Riccati Equation." PROGRESS IN PHYSICS.
- Clauser, J. F. and M. A. Horne (1974). "Experimental consequences of objective local theories." Physical Review D **10**(2): 526.
- Clauser, J. F., M. A. Horne, et al. (1969). "Proposed experiment to test local hidden-variable theories." Physical Review Letters **23**(15): 880-4.
- Clauser, J. F. and A. Shimony (1978). "REVIEW: Bell's theorem. Experimental tests and implications." Reports on Progress in Physics **41**(12): 1881-1927.
- Collett, E. (2005). Field guide to polarization. Bellingham, Wash., SPIE Press.
- Compton, A. H. (1923). "A Quantum Theory of the Scattering of X-rays by Light Elements." Physical Review **21**(5): 483.
- de Broglie, L. V. (1923). "Ondes et quanta." Comptes rendus de l'Académie des Sciences **177**: 507-510.

- De Broglie, L. V. (1924). *Recherche sur la Theorie des Quanta* (Research on the Quantum Theory), Doctoral thesis, Univ. of Paris. Paris: Masson & Cie.
- Dirac, P. A. M. (1927). "The Quantum Theory of the Emission and Absorption of Radiation." Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character **114**(767): 243-265.
- Dirac, P. A. M. (1928). "Quantum theory of the electron." Proceedings of the Royal Society of London **117**: 610-624.
- Doob, J. L. (1953). Stochastic Processes. New York, John Wiley & Sons, Inc.
- Dyson, F. J. (1949). "The Radiation Theories of Tomonaga, Schwinger, and Feynman." Physical Review **75**(3): 486-502.
- Dyson, F. J. (1949). "The S Matrix in Quantum Electrodynamics." Physical Review **75**(11): 1736-1755.
- Einstein, A. (1905). "On a heuristic point of view concerning the production and transformation of light." Annalen der Physik **17**: 132-148.
- Einstein, A. (1926). Investigations on the Theory of the Brownian Movement. London, Methuen and Company, Ltd.
- Einstein, A. (1936). "Physics and reality." Journal of the Franklin Institute **221**(3): 349-382.

- Einstein, A. (1956). "Investigations on the Theory of Brownian Movement." New York: Dover.
- Einstein, A. and N. Rosen (1935). "The Particle Problem in the General Theory of Relativity." Physical Review **48**(1): 73.
- Einstein, A., R. C. Tolman, et al. (1931). "Knowledge of Past and Future in Quantum Mechanics." Physical Review **37**(6): 780.
- Eisberg, R. and R. Resnick (1985). Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, John Wiley & Sons New York.
- Feynman, R. P. (1949). "Space-Time Approach to Quantum Electrodynamics." Physical Review **76**(6): 769-789.
- Feynman, R. P. (1949). "The Theory of Positrons." Physical Review **76**(6): 749-759.
- Feynman, R. P., R. B. Leighton, et al. (1989). The Feynman Lectures on Physics, Addison-Wesley.
- Feynman, R. P., R. B. Leighton, et al. (January 1, 1989). The Feynman Lectures on Physics, Addison-Wesley.
- Fock, V. (1932). "Konfigurationsraum und zweite Quantelung." Zeitschrift für Physik A Hadrons and Nuclei **75**(9): 622-647.
- Fock, A. D. (1914). "Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld." Annalen der Physik(43): 810-820.

- Freedman, S. J. and J. F. Clauser (1972). "Experimental Test of Local Hidden-Variable Theories." Physical Review Letters **28**(14): 938-941.
- Fry, E. S. and R. C. Thompson (1976). "Experimental Test of Local Hidden-Variable Theories." Physical Review Letters **37**(8): 465.
- Furry, W. H. and J. F. Carlson (1933). "On The Production of Positive Electrons by Electrons." Physical Review **44**(3): 237.
- Gikhman, I. I. and A. V. Skorokhod (1979). The theory of stochastic processes, Springer.
- Gisin, N. (1991). "Bell's inequality holds for all non-product states." Physics Letters A **154**(5-6): 201-202.
- Go, A. (2003). "Observation of Bell Inequality violation in B mesons." Arxiv preprint quant-ph/0310192.
- Goenner, H. (1999). The expanding worlds of general relativity. Boston, Birkhauser.
- Greiner, W. (1994). Quantum Mechanics - An Introduction. Berlin, Heidelberg, Springer.
- Greiner, W. (2000). Relativistic quantum mechanics wave equations. Berlin; New York, Springer.
- Gross, D. J. (1992). "Gauge Theory-Past, Present, and Future?" CHINESE JOURNAL OF PHYSICS **30**(7).

- Halzen, F. and A. D. Martin (1984). Quarks and leptons an introductory course in modern particle physics. New York, Wiley.
- Heisenberg, W. (1925). "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen." Zeitschrift für Physik A Hadrons and Nuclei **33**(1): 879-893.
- Heisenberg, W. (1927). "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik." Zeitschrift für Physik A Hadrons and Nuclei **43**(3): 172-198.
- Heisenberg, W. (1958). Physics and Philosophy. New York, Harper.
- Hilfer, R. (2000). Applications of fractional calculus in physics, World Scientific Singapore.
- Jordan, P. and W. Pauli (1928). "Zur Quantenelektrodynamik ladungsfreier Felder." Zeitschrift für Physik A Hadrons and Nuclei **47**(3): 151-173.
- Kocher, C. A. and E. D. Commins (1967). "Polarization Correlation of Photons Emitted in an Atomic Cascade." Physical Review Letters **18**(15): 575-577.
- Kolmogorov, A. N. (1931). "Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung." Math. Ann.(104): 415-458.
- Korn, G. A. and T. M. Korn (2000). Mathematical handbook for scientists and engineers definitions, theorems, and formulas for reference and review. Mineola, N.Y., Dover Publications.

- Kulish, V. and W. Chan (2008). "The Paradox of Instantaneous Propagation in the Schrödinger Equation." Physica A (submitted for publication).
- Kulish, V. V. (2004). "An Analytical Solution of the Generalized Equation of Energy Transport in One-Dimensional Semi-infinite Domains." Math. Probl. Eng.(3): 185-195.
- Kulish, V. V. (2004). Partial Differential Equations, Prentice Hall.
- Kulish, V. V. and W. Chan (2006). "Superfluid Behavior of Start-Up Flows in Nano-Channels." INTERNATIONAL JOURNAL OF TRANSPORT PHENOMENA 8(3): 211-222.
- Kulish, V. V. and W. Chan (2007). "On the Phase-Lagged Model of Superfluidity: High-Temperature Superfluidity Versus Superfluidity at Low Temperatures." INTERNATIONAL JOURNAL OF TRANSPORT PHENOMENA 9(2): 113-118.
- Kulish, V. V. and J. L. Lage (2002). "On the Relationship between Fluid Velocity and de Broglie's Wave Function and the Implications to the Navier-Stokes Equation." Int. J. Fluid Mechanics Research 29: 39-54.
- Lamehi-Rachti, M. and W. Mittig (1976). "Quantum mechanics and hidden variables: A test of Bell's inequality by the measurement of the spin correlation in low-energy proton-proton scattering." Physical Review D 14(10): 2543.

- Leggett, A. (1991). Topics in superfluidity and superconductivity. Low Temperature Physics: 1-92.
- Leggett, A. J. (1999). "Superfluidity." Reviews of Modern Physics **71**(2): S318.
- Merzbacher, E. (1998). Quantum mechanics. New York, Wiley.
- Naber, G. L. (1992). The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity. New York, Springer.
- Nelson, E. (1966). "Derivation of the Schrödinger Equation from Newtonian Mechanics." Physical Review **150**(4): 1079.
- Norsen, T. (2006). "Comment on ``Experimental realization of Wheeler's delayed-choice GedankenExperiment'". Arxiv preprint quant-ph/0611034.
- Oppenheimer, J. R. (1930). "Note on the Theory of the Interaction of Field and Matter." Physical Review **35**(5): 461.
- Oppenheimer, J. R. and M. S. Plesset (1933). "On the Production of the Positive Electron." Physical Review **44**(1): 53-55.
- Pauli, W. (1940). "The Connection Between Spin and Statistics." Physical Review **58**(8): 716-722.
- Perrin, J. B. (1896). "New experiments on the cathode rays." Nature **53**: 298-299.

- Planck, M. (1900). "Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum." Verhandlungen der Deutschen Physikalischen Gesellschaft 2(17): 237-245.
- Planck, M. (1917). "Ueber einen Satz der statistischen Dynamik und eine Erweiterung in der Quantumtheorie." Sitzungsberichte der Preussischen Akademie der Wissenschaften: 324-341.
- Popescu, S. and D. Rohrlich (1992). "Generic quantum nonlocality." Physics Letters A 166(5-6): 293-7.
- Reif, F. (1965). Fundamentals of Statistical and Thermal Physics. New York, McGraw Hill.
- Ricci-Curbastro, G. and T. Levi-Civita (1901). "Methodes de calcul différentiel absolu et leurs applications." Mathematische Annalen 54: 125.
- Riesz, M. (1949). "L'intégrale de Riemann-Liouville et le problème de Cauchy." Acta Mathematica 81(1): 1-222.
- Rubens, H. and F. Kurlbaum (1900). "Ü die Emission langwelliger Wärmestrahlen durch den schwarzen Körper bei verschiedenen Temperaturen." Sitzungsberichte der Preussischen Akademie der Wissenschaften: 929-941.
- Schrödinger, E. (1926). "Quantisierung als Eigenwertproblem." Annalen der Physik 385(13): 437-490.

- Schwinger, J. (1948). "On Quantum-Electrodynamics and the Magnetic Moment of the Electron." Physical Review **73**(4): 416.
- Schwinger, J. (1948). "Quantum Electrodynamics. I. A Covariant Formulation." Physical Review **74**(10): 1439-1461.
- Schwinger, J. (1949). "Quantum Electrodynamics. II. Vacuum Polarization and Self-Energy." Physical Review **75**(4): 651-679.
- Shankar, R. (1994). Principles of quantum mechanics. New York, Plenum Press.
- Stefan, J. (1879). "Über die Beziehung zwischen der Wärmestrahlung und der Temperatur." Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der kaiserlichen Akademie der Wissenschaften, Bd 79: 391-428.
- Strauss, W. A. (1992). Partial differential equations an introduction. New York, Wiley.
- Thomson, J. J. (1897). "Cathode Rays." Philosophical Magazine **44**: 293-316.
- Tomonaga, S. (1946). "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields." Progress of Theoretical Physics **1**(2): 27-42.
- Uhlenbeck, G. E. and L. S. Ornstein (1930). "On the Theory of the Brownian Motion." Physical Review **36**(5): 823.
- von Neumann, J. (1932). "Mathematische Grundlagen der Quantentheorie." Springer.
- Wang, M. C. and G. E. Uhlenbeck (1945). "On the Theory of the Brownian Motion II." Reviews of Modern Physics **17**(2-3): 323.

- Wien, W. (1896). "On the Division of Energy in the Emission-Spectrum of a Black Body." translated in Philosophical Magazine **43**(1897): 214–20.
- Wu, C. S. and I. Shaknov (1950). "The Angular Correlation of Scattered Annihilation Radiation." Physical Review **77**(1): 136.
- Yang, C. N. and R. Mills (1954B). "Isotopic spin conservation and a generalized gauge invariance." Phys. Rev **95**(1): 631.
- Yang, C. N. and R. L. Mills (1954). "Conservation of Isotopic Spin and Isotopic Gauge Invariance." Physical Review **96**(1): 191.
- Zee, A. (2003). Quantum field theory in a nutshell. Princeton, N.J., Princeton University Press.
- Zeilinger, A. (1999). "Experiment and the foundations of quantum physics." Reviews of Modern Physics **71**(2): S288.